Flexibility properties and homology of Gromov-Vaserstein fibres

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Introduction

The aim of this thesis consists in the study of a very concrete class of affine algebraic varieties, i.e. the fibres of the so-called *Gromov-Vaserstein fibration,* which are of importance in the modern rapidly developing area of *elliptic holomorphic geometry.* The study of these fibres has been initiated by Ivarsson and Kutzschebauch (see [IK]), who proved that such fibres are Oka manifolds, by showing that they are elliptic in the sense of Gromov. This was crucial in proving that the Gromov-Vaserstein fibration is *stratified elliptic,* which is in turn crucial for Ivarsson and Kutzschebauch's (see [IK]) solution, in 2012, of the following Gromov-Vaserstein problem, posed by Gromov in 1989 (see [G]). Here is Gromov-Vaserstein problem (see also Chapter 2 of the present work).

Gromov–Vaserstein problem. Let $f : \mathbb{C}^n \to \mathrm{SL}_m(\mathbb{C})$ be a holomorphic map. Does f factorise as a finite product of holomorphic maps sending $\mathbb{C}^n$ into unipotent subgroups in $\mathrm{SL}_m(\mathbb{C})$?

Such a problem was affirmatively solved by Ivarsson and Kutzschebauch, who proved the following

Theorem. Let X be a finite dimensional reduced Stein space and $f : X \to \mathrm{SL}_k(\mathbb{C})$ be a holomorphic mapping that is *null-homotopic.* Then there exist a natural number N and holomorphic mappings $G_1, \dots, G_N : X \to \mathbb{C}^{k(k-1)/2}$ such that, for every $x \in X$,

$$f(x) = M_1(G_1(x)) \cdots M_N(G_N(x)),$$

where, if $Z_K = \{z_{ij,K}\}$ is the accordingly defined multiple variable vector,

$$M_{2l}(Z_{2l}) = \begin{pmatrix} 1 & z_{12,2l} & \cdots & z_{1k,2l} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & z_{(k-1)k,2l} \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

and

$$M_{2l-1}(Z_{2l-1}) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ z_{21,2l-1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ z_{k1,2l-1} & \cdots & z_{k(k-1),2l-1} & 1 \end{pmatrix}.$$

So as to prove it, Ivarsson and Kutzschebauch define the map $\Psi_N : (\mathbb{C}^{k(k-1)/2})^N \to \mathrm{SL}_k(\mathbb{C})$ as

$$\Psi_N(Z_1, \dots, Z_N) = M_1(Z_1)^{-1} \cdots M_N(Z_N)^{-1}$$

and make use of the fibres (which we call *Gromov-Vaserstein fibres*) of $\pi_k \circ \Psi_N =: \Phi_N$, where π_k denotes the projection to the k-th (i.e. to the last) row of a given matrix in $\mathrm{SL}_k(\mathbb{C})$.

In Chapter 4 of the present work, we focus upon the case $k = 2$, indicate by $n := N \geq 3$ the number of variables involved (i.e. the dimension of the ambient space $\mathbb{C}^n$) and

- compute the homology with integer coefficients of the fibres (seen as hypersurfaces of $\mathbb{C}^{n-1}$) of the maps $\Phi_n = \pi_2 \circ \Psi_n$ over suitable subsets of $\mathbb{C}^2 \setminus \{0\}$ (the map $\pi_2 : \mathrm{SL}_2(\mathbb{C}) \to \mathbb{C}^2 \setminus \{0\}$ denotes the projection to the second row);
- show that certain *smooth* Gromov-Vaserstein fibres enjoy the (algebraic) volume density property (for a certain algebraic volume form) modulo vector fields which, after being inserted into the volume form (via the so-called interior product), generate the second highest de Rham cohomology class of the fibers. For those fibers for which we are able to calculate the second highest cohomology, we are able to prove the volume density property.

Proving these results will involve some technical topological notions and preliminary facts, which shall be explained in Chapter 3. Moreover, all background information needed to present a complete picture of the objects of our study is provided in Chapter 1.

The first interesting Gromov-Vaserstein fibre is provided by the smooth surface given by the equation

$$x + y + xyz = 1$$

whose simple connectedness has been studied by Ivarsson and Kutzschebauch in [IKN] in order to determine the optimal bound for the number of unipotent matrices needed in the Gromov-Vaserstein factorisation in the case of a 2-dimensional Stein space. In the *holomorphic* case as well as in the *continuous* case, as explained in [IK], there exists an upper bound which depends only on the dimension m of the space X and the size k of the matrix (while such a uniform bound does not exist in the *algebraic* case, as shown in [vdK]). As in [IK], let us denote by $N_{\mathcal{C}}(m, k)$ the number of matrices needed to factorise any null-homotopic map from a Stein space of dimension m into $\mathrm{SL}_k(\mathbb{C})$ by *continuous* triangular matrices and by $N_{\mathcal{O}}(m, k)$ the number needed in the *holomorphic* setting. Not much about $N_{\mathcal{C}}(m, k)$ and $N_{\mathcal{O}}(m, k)$ is known so far and to determine them constitutes a very hard problem, strongly related to algebraic K-theory. For sure we have $N_{\mathcal{C}}(m, k) \leq N_{\mathcal{O}}(m, k)$ and, if $k = 2$, the proof in [IK] yields $N_{\mathcal{O}}(m, 2) \leq N_{\mathcal{C}}(m, 2) + 4$. The latter inequality has been considerably improved in [IKN] to

$$N_{\mathcal{O}}(m, 2) \leq N_{\mathcal{C},\,\mathcal{O}}(m, 2) + 2,$$

where $N_{\mathcal{C},\,\mathcal{O}}(m, 2)$ denotes the minimal number s such that every null-homotopic *holomorphic* map from a Stein space of dimension m into $\mathrm{SL}_2(\mathbb{C})$ factorises as a product of s *continuous* unipotent matrices (starting with a lower triangular one). Also, in [IKN] (see Theorem 5.2), Ivarsson and Kutzschebauch obtain some exact estimates on the number of factors, namely

$$N_{\mathcal{O}}(1, 2) = 4, \quad N_{\mathcal{O}}(2, 2) = 5.$$

Other bounds are found, in case $m = 1, 2$ for any k, by A. Brudnyi (see Theorem 1.1 and Proposition 1.3 of [Br], where the K-theory notion of *Bass stable rank* is used).

Let us say a little bit more about the complex surface given by the equation $x+y+xyz = 1$. Its algebraic automorphism group is finite while, since the surface $X_{p,q}$ in $\mathbb{C}^3$ given by the equation

$$p(x) + q(y) + xyz = 1$$

(where p and q are polynomials with $p(0) = q(0) = 0$ such that $1 - p(x)$ and $1 - q(y)$ have simple roots only) enjoys the AVDP with respect to the form $\omega = \frac{1}{xy}\, dx \wedge dy$ (see sect. 7 of [KKAV]), its (ω-preserving) holomorphic automorphism group acts infinitely transitively on it and is «big» in a very concrete way (for further examples see also subsection 1.3.1 of

this thesis).
As we stated above, in the present work we consider the case $k = 2$ and extend the study of the surface given by the equation $x + y + xyz = 1$ to all Gromov-Vaserstein fibres X_n^c (generic and special) for any number of factors. Our computation of their integral homology groups led to the following (as for notations and definitions, see Chapter 2).

Theorem. The integral homology groups of the smooth generic Gromov-Vaserstein fibres are as follows.

- For fixed $c \in \mathbb{C} \setminus \{0\}$, we have $H_0(X_3^c) = \mathbb{Z}$, $H_1(X_3^c) = 0$, $H_2(X_3^c) = \mathbb{Z}$, $H_k(X_3^c) = 0$ for $k \geq 3$.
- For fixed $c \in \mathbb{C} \setminus \{0, 1\}$, we have $H_0(X_4^c) = \mathbb{Z}$, $H_1(X_4^c) = 0$ $H_2(X_4^c) = \mathbb{Z}$, $H_3(X_4^c) = \mathbb{Z}^2$, $H_k(X_4^c) = 0$ for $k \geq 4$.
- If $n \geq 5$ is odd and $c \in \mathbb{C} \setminus \{0\}$ is fixed, for $0 \leq j \leq n-1$ the homology groups $H_j(X_n^c)$ are alternately $\mathbb{Z}$ or 0, starting from $H_0(X_n^c) = \mathbb{Z}$. In particular $H_{n-2}(X_n^c) = 0$.

For the integral homology groups of the singular special Gromov-Vaserstein fibre X_4^1, the following holds.

- $H_0(X_4^1) = \mathbb{Z}$, $H_1(X_4^1) = 0$, $H_2(X_4^1) = \mathbb{Z}$, $H_3(X_4^1) = \mathbb{Z}$, $H_k(X_4^1) = 0$ for $k \geq 4$.

As regards integral homology groups of the smooth special Gromov-Vaserstein fibres (X_n^0 for $n \geq 3$), we have that $H_k(X_n^0) = \mathbb{Z}$ $\forall k \in \{0, \dots, n-1\}$ and $H_k(X_n^0) = 0$ $\forall k \geq n$.

Of crucial importance to the above computations is the concept of *affine modification,* introduced by Kaliman and Zaidenberg in their paper [KZ], which has constituted a valuable source of inspiration for the present work.

We remark that, in order to find a bound for $N_{\mathcal{C}}(m, 2)$, $m \geq 3$, (see above for def.), it would suffice to determine a topological section of the above fibration. We hope that our study of the topology of the fibres will allow us to prove the existence of a global topological section in certain cases.
In fact our results seem to suggest that the optimal bound for factorising holomorphic mappings from an n-dimensional Stein space to $\mathrm{SL}_2(\mathbb{C})$ (by continuous and holomorphic unipotent matrices) tends to infinity in the limit $n \to \infty$.

As we stated above, our second result (to which the criterion established in section 3 of [KKAV] has been of particular importance) yields that certain *smooth* Gromov-Vaserstein fibres enjoy the (algebraic) *volume density property.* So as to extend this result to the singular special case in future studies, it could be necessary to use some of the tools in the paper [KLL] by Kutzschebauch, Liendo and Leuenberger.

Originated from Andersén and Lempert's groundbreaking works [A] and [AL], density properties (see Def. 7 of this thesis) and the construction of holomorphic automorphisms of Stein manifolds with prescribed behaviour on compact subsets constitute the core of *Andersén-Lempert theory.* Let us remark that relatively few manifolds are known to enjoy density properties so far (see, as mentioned above, subsection 1.3.1) and that our results *substantially* (our fibres, as it follows from their integral homology, are *not* homeomorphic to previously known examples) enlarge the class of manifolds known to have them. Also, as explained in subsection 1.3.3, the above results make our manifolds ideal target spaces for a positive solution of

Lárusson's question. Does every Stein manifold admit an *acyclic* (i.e. being a homotopy equivalence) *embedding* into an Oka manifold?

Also, last but not least, our manifolds are *strongly universal for proper holomorphic embedding and immersions* (see Theorem 18, subsect. 1.3.3).

So as to improve readability, let us recall schematically how the present work is structured.

- *Chapter 1.* We start by defining *Stein* and *Oka manifolds*, providing some properties of theirs, along with some examples, and describing some crucial features of their interplay, via *algebraic* and *holomorphic flexibility* (sects. 1.1 and 1.2). We then introduce the *density properties* and provide an exhaustive list of objects known to enjoy them, together with some important results (due to Varolin, Kaliman and Kutzschebauch) about the product of such objects (subsect. 1.3.1). In subsect. 1.3.2 we recall and explain two strong crucial criteria (due to Kaliman and Kutzschebauch) for establishing (algebraic) density property and (algebraic) volume density property respectively, the latter being of particular importance to the present work. We conclude the first introductory chapter by providing various important consequences and applications of density properties, concerning the connections between such properties and
  - holomorphic flexibility;
  - Gromov-ellipticity;
  - infinite transitivity of the action of the holomorphic automorphism group of a Stein manifold on the Stein manifold itself;
  - strong universality of a Stein manifold for proper holomorphic embeddings and immersions;
  - existence of Fatou-Bieberbach domains of first and second kind.
- *Chapter 2.* We introduce the Gromov-Vaserstein problem, fibration and fibres, along with all notations we will use later on;
- *Chapter 3.* We introduce various notions and preliminary facts which will prove themselves to be of great importance to our work, constituting fundamental ingredients in the proof of our results;
- *Chapter 4.* We state and prove our results.

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Chapter 1

Flexibility properties

1.1 Stein and Oka manifolds

The following definition is due to K. Stein (see [S]).

Definition 1. *A complex manifold X of dimension n is said to be a* Stein manifold *(or a* holomorphically complete manifold*) if the following hold:*

1. *For every pair of distinct points $x \neq y$ in X there is a holomorphic function $f \in \mathcal{O}(X)$ such that $f(x) \neq f(y)$;*

2. *For every point $p \in X$ there exist functions $f_1, \ldots, f_n \in \mathcal{O}(X)$, whose differentials $\mathrm{d}f_j$ are $\mathbb{C}$-linearly independent at p (which is to say, local charts at each point are provided by global holomorphic functions);*

3. *X is* holomorphically convex, *i.e. is such that for every compact set $K \subset X$ its $\mathcal{O}(X)$-hull*
$$\widehat{K}_{\mathcal{O}(X)} = \{p \in X : \ \forall f \in \mathcal{O}(X) \ |f(p)| \leq \max_{x \in K} |f(x)|\}$$
is also compact.

Here are some remarks and examples:

- Domains in $\mathbb{C}$ and open Riemann surfaces (see [SB], [SB1]) are Stein manifolds;
- An open set in $\mathbb{C}^n$ is Stein if and only if it is a domain of holomorphy;
- A Stein manifold does not contain any compact complex subvariety of positive dimension;
- The Cartesian product $X \times Y$ of a pair of Stein manifolds is Stein;
- A closed complex submanifold X of $\mathbb{C}^N$ is Stein (more generally, every closed complex submanifold of a Stein manifold is Stein);
- If $E \to X$ is a holomorphic vector bundle over a Stein base X then the total space E is also Stein.

We now provide some equivalent characterizations of Stein manifolds.

Theorem 1 (Remmert-Bishop-Narasimhan)**.** *A complex manifold is Stein if and only if it is biholomorphic to a closed complex submanifold of some $\mathbb{C}^N$. In fact, if n is the dimension of X, then N can be taken to be $2n + 1$.*

This theorem, roughly speaking, implies the existence, for a Stein manifold X, of many holomorphic maps $X \to \mathbb{C}$. Another characterization (see [FH] p. 52) uses coherent sheaf cohomology.

Theorem 2 (Cartan-Serre Theorem B)**.** *A complex manifold X is Stein iff for every coherent analytic sheaf $\mathcal{F}$ over X we have*

$$H^k(X, \mathcal{F}) = 0 \textit{ for all } k \geq 1.$$

Applying this with sheaves Ω^p of holomorphic p-forms yields that every $\overline{\partial}$-problem on a Stein manifold is solvable, i.e. for every differential form f with $\overline{\partial} f = 0$ there exists a form u solving $\overline{\partial} u = f$.

Note further that the function $\rho_a : \mathbb{C}^N \to \mathbb{R}_+$ given by $\rho_a = |z - a|^2$ is strongly plurisubharmonic on any complex subvariety $X \subset \mathbb{C}^N$; if X is closed then this is an exhaustion function on X. Furthermore, if X is smooth, then $\rho_a|_X$ is a Morse function on X for most choices of the point $a \in \mathbb{C}^N$. Thus, every Stein manifold admits many smooth strongly plurisubharmonic exhaustion functions. Also, the converse holds true and it provides yet another useful characterization of Stein manifolds (see [FH] p. 51).

Theorem 3 (Grauert-Docquier-Narasimhan)**.** *A complex manifold X is Stein iff it admits a strongly plurisubharmonic exhaustion function $\rho : X \to \mathbb{R}_+$.*

Note that, in any local coordinate system on X, such a function is strongly subharmonic on each complex line L, i.e. $\Delta(\rho|_L) > 0$. Critical points of such ρ have Morse index at most $n = \dim X$. This implies the following (see [FH] p. 96).

Theorem 4 (Lefschetz-Milnor)**.** *A Stein manifold of complex dimension n is homotopy equivalent to a CW-complex of dimension at most n.*

Let us turn to another notion formalising, roughly speaking, the existence of many holomorphic maps $\mathbb{C} \to Y$. Let us give the following

Definition 2 (OP)**.** *A complex manifold Y is an* Oka manifold *if it satisfies the so-called* Oka property (OP)*, namely if, for every Stein manifold X, every compact $\mathcal{O}(X)$-convex subset K of X and every continuous map $f_0 : X \to Y$ which is holomorphic in an open neighborhood of K, there exists a homotopy of continuous maps $f_t : X \to Y$ ($t \in [0,1]$) such that for every $t \in [0,1]$ the map f_t is holomorphic in a neighborhood of K and uniformly close to f_0 on K, and the map $f_1 : X \to Y$ is holomorphic.*

It easily follows that Oka manifolds enjoy the following approximation property.

Definition 3 (CAP)**.** *A complex manifold Y satisfies the* convex approximation property (CAP) *if any holomorphic map from a neighborhood of a compact convex set $K \subset \mathbb{C}^m$ to Y can be approximated uniformly on K by entire maps $\mathbb{C}^m \to Y$.*

In fact, as shown by Forstnerič in sect. 3 of [FR], *the Oka property is equivalent to the convex approximation property,* thus providing an equivalent definition of Oka manifold (for other equivalent definitions, we refer the reader to sect. 5.15 of Forstnerič monograph [FH]). We now introduce the notion of dominating holomorphic spray on a complex manifold Y (see [FH], def. 5.5.11).

Definition 4. *Let Y be a complex manifold.*

1. *A (holomorphic)* spray *on Y is a triple (E, π, s) consisting of a holomorphic vector bundle $\pi : E \to Y$ (a spray bundle) and a holomorphic map $s : E \to Y$ (a spray map) such that for each $y \in Y$ we have $s(0_y) = y$, where 0_y denotes the zero of the fibre over y. A spray (E, π, s) is algebraic if $\pi : E \to Y$ is an algebraic vector bundle over an algebraic manifold Y and $s : E \to Y$ is an algebraic map.*

2. *A spray* (E, π, s) *on* Y *is* dominating on a subset $U \subset Y$ *if the differential*

$$\mathrm{d}s_{0_y} : T_{0_y}E \to T_yY$$

maps the vertical subspace E_y *(i.e. the fibre over* y*) of* $T_{0_y}E$ *surjectively onto* T_yY *for every* $y \in U$*. The spray map* s *is* dominating *if this holds for all* $y \in Y$*.*

3. *A complex manifold* Y *is called* elliptic*, in the sense of Gromov, if it admits a dominating spray.*

In [G], Gromov shows the following

Theorem 5. *Every elliptic manifold satisfies* CAP *and therefore is an Oka manifold.*

A partial converse to this theorem is given by the following result, due to Lárusson (see [L]).

Theorem 6. *A Stein manifold which is Oka must be elliptic.*

Here are some examples of Oka manifolds (others will come later on, when the density properties are introduced):

- $\mathbb{C}^n$, $\mathbb{P}^n$, complex Lie groups and their homogeneous spaces;
- $\mathbb{C}^n \setminus A$, where A is an algebraic subvariety of codimension at least 2;
- $\mathbb{P}^n \setminus A$, where A is a subvariety of codimension at least 2;
- Complex torus of dimension at least 2 with finitely many points removed, or blown up at finitely many points.

We remark that the fact that, for $n \geq 2$, $\mathbb{C}^n \setminus A$, where A is an algebraic subvariety of codimension at least 2, is an Oka manifold is crucial, in [EG], to Eliashberg, Gromov and Schürmann's proof of *Forster's conjecture,* which, for each $n \geq 2$, identifies the smallest $N(n) = n + [n/2] + 1$ such that every n-dimensional Stein manifold embeds into $\mathbb{C}^{N(n)}$.

1.2 Algebraic flexibility and holomorphic (volume) flexibility

The following definition of (algebraic) flexibility may be found in [AKZ] (p. 2). We denote by X_{reg} the smooth part of a complex algebraic variety X.

Definition 5 (Algebraic flexibility)**.** *We say that a point* $x \in X_{\mathrm{reg}}$ *is* flexible *if the tangent space* T_xX *is spanned by the tangent vectors to the orbits* $H.x$ *of one-parameter unipotent subgroups* H *(i.e. subgroups isomorphic to the additive group* $\mathbb{C}_+$*) of* $\mathrm{Aut}(X)$*. The variety* X *is called flexible if every point* $x \in X_{\mathrm{reg}}$ *is.*

Clearly, X is algebraically flexible if one point of X_{reg} is and the group $\mathrm{Aut}(X)$ acts transitively on X_{reg}. The following holds (see [AKZ] p. 2).

Theorem 7. *Denoted by* $\mathrm{SAut}(X)$ *the subgroup of* $\mathrm{Aut}(X)$ *generated by all algebraic one-parameter unipotent subgroups of* $\mathrm{Aut}(X)$*, for an irreducible affine variety* X *of dimension at least* 2*, the following conditions are equivalent.*

1. *The group* $\mathrm{SAut}(X)$ *acts transitively on* X_{reg}*.*
2. *The group* $\mathrm{SAut}(X)$ *acts infinitely transitively on* X_{reg}*.*
3. X *is a flexible variety.*

For a complex manifold X, we have the following definition of holomorphic flexibility and of holomorphic volume flexibility, in case X is equipped with a volume form ω (implied by the above algebraic version if X is an algebraic variety).

Definition 6 (Holomorphic (volume) flexibility)**.** *We say that a point x in a complex manifold X is* holomorphically flexible *(resp.* holomorphically volume flexible*) if the values at x of completely integrable holomorphic vector fields on X (resp. of ω-divergence free completely integrable holomorphic vector fields on X) span the tangent space T_xX. The manifold X is holomorphically flexible (resp. holomorphically volume flexible) if every point $x \in X$ is.*

We remark that holomorphic flexibility (resp. holomorphic volume flexibility) of X is equivalent to transitivity of the action of $\mathrm{Aut}_{\mathrm{hol}}(X)$ (resp. $\mathrm{Aut}^{\omega}_{\mathrm{hol}}(X)$, the group of ω-preserving holomorphic automorphisms) on X, while the equivalence to infinite transitivity is not known (see [Kfl]).
We require the following, whose proof may be found in [Kfl] (Lemma 25, p. 14).

Lemma 1. *If a Stein manifold X is holomorphically flexible, then there are finitely many completely integrable holomorphic vector fields which span the tangent space T_xX at every point $x \in X$.*

We now prove the following (see [Kfl], Example 24, p. 14).

Theorem 8. *Every holomorphically flexible Stein manifold is elliptic.*

Proof. By the above lemma, there exist $\theta_1, \theta_2, \dots, \theta_N \in \mathrm{IVF}_{\mathrm{hol}}(X)$ s.t.

$$\forall x \in X \quad \mathrm{Span}\{\theta_1(x), \theta_2(x), \dots, \theta_N(x)\} = T_xX.$$

Let $\psi^i : \mathbb{C} \times X \to X$, given by $(t, x) \mapsto \psi^i_t(x)$, denote the corresponding one-parameter subgroups. Then the map $s : \mathbb{C}^N \times X \to X$ given by

$$((t_1, t_2, \dots, t_N), x) \mapsto \psi^N_{t_N} \circ \psi^{N-1}_{t_{N-1}} \circ \cdots \circ \psi^1_{t_1}(x)$$

is of full rank at $t = 0$ for any x. Hence, s is a dominating spray map for the trivial bundle $X \times \mathbb{C}^N \to X$, i.e. X is elliptic. □

Thanks to Theorem 5 one proves the following (see [FH] p. 206).

Corollary 1. *Every holomorphically flexible Stein manifold is an Oka manifold.*

1.3 Density properties and flexibility

1.3.1 Definitions and examples

The main feature of Andersén–Lempert theory, based on the groundbreaking works [A] and [AL], is that local injective holomorphic maps on holomorphically convex compact subsets can be approximated by global holomorphic automorphisms. Such a property was formalised via the concepts of (*algebraic*) *density property* and (*algebraic*) *volume density property*, introduced by Varolin ([V1]) and defined, according to [KKPS], as follows:

Definition 7. *A complex manifold X has the* density property *if, in the compact-open topology, the Lie algebra $\mathrm{Lie}_{\mathrm{hol}}(X)$ generated by completely integrable holomorphic vector fields on X is* dense *in the Lie algebra $\mathrm{VF}_{\mathrm{hol}}(X)$ of all holomorphic vector fields on X. An affine algebraic manifold X has the* algebraic density property *if the Lie algebra $\mathrm{Lie}_{\mathrm{alg}}(X)$ generated by completely integrable algebraic vector fields on it coincides with the Lie algebra $\mathrm{AVF}(X)$ of all algebraic vector fields on it. On the other hand, suppose a complex manifold X is equipped with a holomorphic* volume form ω. *We say that X has the* volume density property with respect to ω *(or the ω-volume density property) if, in the compact-open topology, the Lie* *algebra* $\mathrm{Lie}^{\omega}_{\mathrm{hol}}(X)$ *generated by completely integrable holomorphic vector fields* ν *such that* $\nu(\omega) = 0$ *is* dense *in the Lie algebra* $\mathrm{VF}^{\omega}_{\mathrm{hol}}(X)$ *of all holomorphic vector fields annihilating* ω*. If* X *is affine algebraic, we say that* X *has the* algebraic volume density property with respect to an algebraic volume form ω *(or the* ω*-algebraic volume density property) if the Lie algebra* $\mathrm{Lie}^{\omega}_{\mathrm{alg}}(X)$ *generated by completely integrable algebraic vector fields* ν *such that* $\nu(\omega) = 0$ *coincides with the Lie algebra* $\mathrm{AVF}_{\omega}(X)$ *of all algebraic vector fields annihilating* ω*.*

We remark that *algebraic density property and algebraic volume density property imply density property and volume density property respectively* (for a proof of the second implication we refer to Prop. 4.1 of [KKAVDP]).
Relatively few manifolds are known to enjoy the density property or the volume density property so far. Here are some examples.

Apart from $\mathbb{C}^n$ (which has, by a reformulation of Andersén's theorem in [A] to be found in Prop. 4.9.3 of [FH], the volume density property with respect to the standard volume form for all n), the example $M^2 := \mathbb{C}^2 \setminus \{xy = 1\}$ equipped with $\omega = \frac{1}{xy-1}\, dx \wedge dy$ is due to Varolin (see [V]), along with $(\mathbb{C}^*)^n$, which enjoys the volume density property for all n (see theorem 9(5)), and $\mathrm{SL}_2(\mathbb{C})$, which has the DP, and the VDP with respect to any left invariant holomorphic 3-form.

Andersén-Lempert's theorem (see [AL]) states that $\mathbb{C}^n$ has the density property for $n \geq 2$, while $\mathbb{C}$ does *not* have the density property, as every complete vector field on $\mathbb{C}$ is affine linear. Further, $\mathbb{C}^*$ does *not* have DP and it is *not* known (see [V1]) whether $(\mathbb{C}^*)^n$ for $n \geq 2$ has the DP while, given [Acompl], it does *not* have ADP, as pointed out in [KKPS].
D. Varolin and A. Toth showed in [VT] that every complex semi-simple Lie group has the DP. Also, the algebraic density property has been proven for the following manifolds: $\mathbb{C}^k \times (\mathbb{C}^*)^l$, with $k \geq 1$ and $k + l \geq 2$, the space $\mathrm{SL}_n(\mathbb{C})$, all complex linear algebraic groups whose connected components are not tori $(\mathbb{C}^*)^n$ or $\mathbb{C}$ and homogeneous spaces of the form $X = G/R$ where G is a linear algebraic group and R is a closed proper reductive subgroup such that X has connected components different from tori $(\mathbb{C}^*)^n$ and $\mathbb{C}$ (see [DDK]).
R. Andrist (see [Acm]) just established the (algebraic) density property for the so-called *Calogero-Moser space* $\mathfrak{C}_n$ ($n \in \mathbb{N}$), which describes the (completed) phase space of a certain system of n particles in classical physics and is defined as follows: *let us consider the set* $\widetilde{\mathfrak{C}_n} := \{(X, Y) \in \mathrm{Mat}(n \times n, \mathbb{C}) \times \mathrm{Mat}(n \times n, \mathbb{C}) : \ \mathrm{rank}([X, Y] + \mathrm{id}) = 1\}$ *on which* $\mathrm{GL}_n(\mathbb{C})$ *acts by simultaneous conjugation in both factors, namely by* $G \cdot (X, Y) := (GXG^{-1}, GYG^{-1})$*. Then* $\mathfrak{C}_n := \widetilde{\mathfrak{C}_n} /\!/ \mathrm{GL}_n(\mathbb{C})$.

On the other hand, Kaliman and Kutzschebauch proved the algebraic volume density property in [KKAVDP] for $\mathrm{SL}_2(\mathbb{C})$, for $\mathrm{PSL}_2(\mathbb{C})$, for all semi-simple groups and for all linear algebraic groups with respect to the invariant volume form (see, for instance, Theorem 2 of [KKAVDP]). Further, they extended this result to every connected affine homogenous space of a linear algebraic group G over $\mathbb{C}$ which admits a G-invariant volume form (see Theorem 1.3 of [KKH]). They also showed that the hypersurface

$$X' := \{P(u, v, \bar{x}) = uv - p(\bar{x}) = 0\} \subset \mathbb{C}^{n+2}_{u,v,\bar{x}},$$

which is called *Danielewski surface* if $n = 1$, enjoys, under some technical assumptions on the polynomial p, the algebraic volume density property with respect to an ω' s.t. $\omega' \wedge \mathrm{d}P = \Omega|_{X'}$, where Ω is the standard volume form on $\mathbb{C}^{n+2}$ (see [KKAVDP], Theorem 1). The holomorphic version of this result is due to Ramos-Peon (see [P], page 2).

Besides, the surface $X_{p,q}$ in $\mathbb{C}^3$ given by the equation

$$p(x) + q(y) + xyz = 1$$

(where p and q are polynomials with $p(0) = q(0) = 0$ such that $1 - p(x)$ and $1 - q(y)$ have simple roots only) enjoys the AVDP with respect to the form $\omega = \frac{1}{xy}\, dx \wedge dy$ while, in case $p(x) = x$ and $q(y) = y$, it does *not* have ADP (see sect. 7 of [KKAV] and sect. 8 of [KKPS], resp.), Danilov-Gizatullin surfaces have ADP (see [Don]), the Koras-Russel cubic threefold, hypersurface of $\mathbb{C}^4$ given by the equation

$$x + x^2 y + z^2 + t^3 = 0,$$

enjoys the ADP, and the AVDP with respect to the volume form $\omega = \frac{dx}{x^2} \wedge dz \wedge dt$ (see [Le]).

The following is due to Varolin (see [V1]). As a preliminary remark, we note that, as stated in [V1], for a complex Lie group G, one can choose the holomorphic volume form among the *left (or right) invariant* ones, and that the Lie algebra $\mathrm{VF}^{\omega_G}_{\mathrm{hol}}(G)$ of all holomorphic vector fields on G annihilating ω_G is *independent* of the choice of left invariant form ω_G. Hence, we will omit reference to the left invariant holomorphic volume form in question, when speaking of complex Lie groups enjoying the volume density property.

Theorem 9. *The following statements hold true.*

1. *If X and Y are Stein manifolds with the density property then so is $X \times Y$.*
2. *If a Stein manifold X enjoys the density property, then so do $X \times \mathbb{C}$ and $X \times \mathbb{C}^*$.*
3. *If (X, ω) is a Stein manifold with holomorphic volume element such that $(X \times \mathbb{C}, \omega \wedge dz)$ has the* volume density property*, then $X \times \mathbb{C}$ has the* density property.
4. *For any complex Lie group G, $G \times \mathbb{C}$ has the* volume density property. *If G is Stein and of positive dimension, then $G \times \mathbb{C}$ has the* density property.
5. *For any complex Lie group G enjoying the volume density property, $G \times \mathbb{C}^*$ has the volume density property (in particular $(\mathbb{C}^*)^n$ enjoys the volume density property for all n).*

On the other hand, the following result is due to Kaliman and Kutzschebauch (see proposition 4.3 of [KKAVDP]).

Theorem 10. *Let X and Y be affine algebraic manifolds equipped with algebraic volume forms ω_X and ω_Y respectively, such that X (resp. Y) enjoys the algebraic volume density property with respect to ω_X (resp. ω_Y). Then $X \times Y$ enjoys the algebraic volume density property with respect to $\omega_X \times \omega_Y$.*

1.3.2 Criteria for the density properties

A sufficient condition for an affine algebraic manifold X to enjoy the *algebraic density property* is provided by the following criterion (see Theorem 7 in [KKPS]). We shall require the following (see [KKPS]).

Definition 8. *Let X be an affine algebraic manifold and $x_0 \in X$ a point. A finite subset F of $T_{x_0}X$ is called a* generating subset of $T_{x_0}X$ *if the span of the orbit of F under the action of the isotropy group $(\mathrm{Aut}(X))_{x_0}$ coincides with $T_{x_0}X$.*

The criterion follows.

Theorem 11. *Let X be an affine algebraic manifold with a transitive group* $\mathrm{Aut}(X)$ *of algebraic automorphisms and let $\mathbb{C}[X]$ be its algebra of regular functions. Suppose that there is a* submodule L of the $\mathbb{C}[X]$-module T of all algebraic vector fields on X *such that $L \subset \mathrm{Lie}_{\mathrm{alg}}(X)$ and the fibre of L at some point $x_0 \in X$ contains a generating subset of $T_{x_0}X$. Then X has the algebraic density property.*

Hence, so as to prove that an affine algebraic manifold enjoys the algebraic density property, one has to locate a *nontrivial $\mathbb{C}[X]$-module L*, $L \subset \mathrm{Lie}_{\mathrm{alg}}(X)$. In order to provide a sufficient condition for such a nontrivial $\mathbb{C}[X]$-module to exist, we give the following (see definitions 2.2 of [KKAV], 1 of [Kfl] and 3.2 of [KKPS]).

Definition 9. *Let a smooth irreducible complex affine algebraic variety X be given.*

1. *A holomorphic vector field ξ on X is called* completely integrable *(or complete) if the Cauchy problem*
$$\frac{d}{dt}\varphi(x,t) = \xi(\varphi(x,t)), \quad \varphi(x,0) = x$$
has a solution $\varphi(x,t)$ defined for all complex times $t \in \mathbb{C}$ and all starting points $x \in X$. Such a solution gives a complex one-parameter subgroup in the holomorphic automorphism group $\mathrm{Aut}_{\mathrm{hol}}(X)$.
Equivalently, a holomorphic vector field ξ on X is called completely integrable (or complete) if there exists a holomorphic $\mathbb{C}_+$-action $\varphi : \mathbb{C} \times X \to X$ *(called the* phase flow *of ξ) such that*
$$\forall f \in \mathbb{C}[X] \quad \xi(f) = \frac{d}{dt} f \circ \varphi(t,*)|_{t=0}.$$
If φ is an algebraic $\mathbb{C}_+$-action, *the field ξ is called a* locally nilpotent derivation (LND), *while if φ factors through $\mathbb{C}^* \times X$ and induces an* algebraic $\mathbb{C}^*$-action, *ξ is called a* semisimple derivation.

2. *Let σ be a LND on X and δ be either a LND or semisimple.*
The pair (σ, δ) is called semicompatible *if the the span of* $\mathrm{Ker}\,\sigma \cdot \mathrm{Ker}\,\delta$ *contains a* nonzero ideal of $\mathbb{C}[X]$.
A semicompatible pair (σ, δ) is called compatible *if one of the following conditions holds, where the degree of $a \in \mathbb{C}[X]$ with respect to σ is defined as* $\deg_\sigma(a) := \min\{n - 1 : \sigma^n(a) = 0\}$:
   - *there exists $a \in \mathbb{C}[X]$ such that $a \in \mathrm{Ker}\,\delta$ and $\sigma(a) \in \mathrm{Ker}\,\sigma \setminus \{0\}$, i.e.,* $\deg_\sigma(a) = 1$.
   - *both σ and δ are LNDs and there exists $a \in \mathbb{C}[X]$ such that* $\deg_\sigma(a) = 1 = \deg_\delta(a)$.

For $n \geq 2$ we consider, for $1 \leq i \leq n$, the vector fields on $\mathbb{C}^n$ given by $\delta_i = \frac{\partial}{\partial x_i}$. Noticing that
$$\mathbb{C}^{[n]} = \mathrm{Span}(\mathrm{Ker}\,\delta_1 \cdot \mathrm{Ker}\,\delta_2),$$
it is easy to show that $\mathrm{Lie}_{\mathrm{alg}}(\mathbb{C}^n)$ contains the nontrivial $\mathbb{C}[\mathbb{C}^n]$-module generated by δ_2 whence, by Theorem 11, $\mathbb{C}^n$ enjoys the algebraic density property for $n \geq 2$.
The same argument for (σ, δ) shows the following.

Proposition 1. *The existence of a compatible pair on a smooth irreducible complex affine algebraic variety X yields the existence of a nontrivial $\mathbb{C}[X]$-module in* $\mathrm{Lie}_{\mathrm{alg}}(X)$.

Hence, Theorem 11 implies the following (see [KKPS]).

Theorem 12 (Criterion for ADP)**.** *Let X be a smooth affine algebraic variety with a transitive automorphism group* $\mathrm{Aut}(X)$. *Suppose that there are finitely many pairs of compatible vector fields $\{\sigma_i, \delta_i\}$ such that at some point $x_0 \in X$ vectors $\{\delta_i(x_0)\}$ form a generating subset of $T_{x_0}X$. Then X enjoys the algebraic density property.*

As regards algebraic volume density property, given an affine algebraic manifold X of dimension n equipped with an algebraic volume form ω, the criterion established in [KKAV] is based upon the search for a $\mathbb{C}[X]$-module in the space ${\mathrm{d}_{n-1}}^{-1} \circ \Theta(\mathrm{Lie}^{\omega}_{\mathrm{alg}}(X))$, where

- the map $\mathrm{d}_k : \mathcal{C}_{k-1}(X) \to \mathcal{B}_k(X)$ is the *outer differentiation operator* sending each algebraic differential $(k-1)$-form $\alpha \in \mathcal{C}_{k-1}(X)$ to the exact k-form $\mathrm{d}_k\alpha \in \mathcal{B}_k(X)$;

- the map $\Theta : \mathrm{AVF}_\omega(X) \to \mathcal{Z}_{n-1}(X)$ the *isomorphism* sending each ω-divergence-free algebraic vector field $\xi \in \mathrm{AVF}_\omega(X)$ to the closed $(n-1)$-form $\Theta(\xi) = \iota_\xi\omega \in \mathcal{Z}_{n-1}(X)$, defined by means of the interior product ι of ξ and ω;

- the algebra $\mathrm{Lie}^{\omega}_{\mathrm{alg}}(X)$ is the Lie algebra generated by the set $\mathrm{IVF}_{\omega}(X)$ of complete ω-divergence-free algebraic vector fields on X.

Further, one requires to extend Definition 9(2) of semicompatible pairs to pairs of complete vector fields as follows.

Definition 10. *Let ξ and η be nontrivial* complete *algebraic vector fields on an affine algebraic manifold X. We say that the pair (ξ, η) is* semicompatible *if the the span of* $\mathrm{Ker}\,\xi \cdot \mathrm{Ker}\,\eta$ *contains a* nonzero ideal of $\mathbb{C}[X]$*. The largest ideal in the above span will be called the ideal associated to the pair (ξ, η).*

The connection between the existence of semi-compatible pairs and the existence of $\mathbb{C}[X]$-modules in ${\mathrm{d}_{n-1}}^{-1} \circ \Theta(\mathrm{Lie}^{\omega}_{\mathrm{alg}}(X))$ is provided by what follows (see Prop. 3.1 of [KKAV]).

Proposition 2. *Let Y be a complex manifold equipped with a volume form ω and let ξ and η be ω-divergence-free holomorphic vector fields on Y. Then*

$$\iota_{[\xi,\eta]}\omega = \mathrm{d}\iota_{\xi}\iota_{\eta}\omega.$$

Given ξ, $\eta \in \mathrm{IVF}_{\omega}(X)$, $f \in \mathrm{Ker}\,\xi$ and $g \in \mathrm{Ker}\,\eta$, the last formula yields, for the complete fields $f\xi$ and $g\eta$, $(fg)\iota_{\xi}\iota_{\eta}\omega \in {\mathrm{d}_{n-1}}^{-1} \circ \Theta(\mathrm{Lie}^{\omega}_{\mathrm{alg}}(X))$, whence we obtain the following

Corollary 2. *Let X be a variety equipped with an algebraic volume form ω and let ξ and η be semicompatible divergence-free vector fields on X. Then ${\mathrm{d}_{n-1}}^{-1} \circ \Theta(\mathrm{Lie}^{\omega}_{\mathrm{alg}}(X))$ contains a nontrivial $\mathbb{C}[X]$-submodule L of the module $\mathcal{C}_{n-2}(X)$.*

We notice that, as Kaliman and Kutzschebauch proved in Prop. 3.12 of [KKAV], a sufficient condition in order the equality $L = \mathcal{C}_{n-2}(X)$ (and hence the inclusion $\Theta(\mathrm{Lie}^{\omega}_{\mathrm{alg}}(X)) \supset \mathcal{B}_{n-1}(X)$) to be true is that condition $(*)$ of the next proposition holds.

Proposition 3. *Let X be a variety equipped with an algebraic volume form ω and let $(\xi_j, \eta_j)_{j=1}^{k}$ be pairs of divergence-free semicompatible vector fields. Let I_j be the ideal associated with (ξ_j, η_j), and let $I_j(x) = \{f(x) : f \in I_j\}$ for $x \in X$. Suppose that, for every $x \in X$,*

the set $\{I_j(x)\xi_j(x) \wedge \eta_j(x)\}_{j=1}^{k}$ generates the fiber $T_xX \wedge T_xX$ of $TX \wedge TX$ over x. $(*)$

Then $\Theta(\mathrm{Lie}^{\omega}_{\mathrm{alg}}(X)) \supset \mathcal{B}_{n-1}(X)$.

From this last proposition, we obtain the following criterion for establishing algebraic volume density property (see theorem 1 of [KKAV]).

Theorem 13 (Criterion for AVDP)**.** *Let X be a variety equipped with an algebraic volume form ω and pairs of divergence-free semicompatible vector fields satisfying condition $(*)$ from* Proposition 3*, namely let X be a variety for which the inclusion* $\Theta(\mathrm{Lie}^{\omega}_{\mathrm{alg}}(X)) \supset \mathcal{B}_{n-1}(X)$ *holds true. Suppose also that the following condition is true:*

the image of $\Theta(\mathrm{Lie}^{\omega}_{\mathrm{alg}}(X))$ under the De Rham homomorphism $\Phi_{n-1} : \mathcal{Z}_{n-1}(X) \to H^{n-1}_{\mathrm{dR}}(X, \mathbb{C})$ coincides with $H^{n-1}_{\mathrm{dR}}(X, \mathbb{C})$. $(**)$

Then $\Theta(\mathrm{Lie}^{\omega}_{\mathrm{alg}}(X)) = \mathcal{Z}_{n-1}(X)$ *and therefore* $\mathrm{Lie}^{\omega}_{\mathrm{alg}}(X) = \mathrm{AVF}_{\omega}(X)$*, i.e., X has the algebraic volume density property.*

1.3.3 Consequences and applications of density properties

We now provide various applications and consequences of the density properties we introduced in previous sections, starting from a stronger version, by Forstnerič and Rosay, of the following theorem, due to Andersén and Lempert (see [A] and [AL]), theorem of crucial importance in understanding the automorphism groups $\mathrm{Aut}(\mathbb{C}^n)$ for $n \geq 2$ and in constructing holomorphic automorphisms of Stein manifolds with prescribed behaviour on compact subsets.

Theorem 14. *Let $n \geq 2$.*

1. *Every holomorphic automorphism of $\mathbb{C}^n$ can be approximated uniformly on compacts by compositions of shears (holomorphic automorphisms of $\mathbb{C}^n$ given, for $z = (z', z_n) \in \mathbb{C}^n$, $t \in \mathbb{C}$, $f \in \mathcal{O}(\mathbb{C}^{n-1})$, by $\Phi_t(z) = (z', z_n + tf(z'))$ and $\Psi_t(z) = (z', e^{tf(z')} z_n)$).*

2. *Every automorphism of $\mathbb{C}^n$ with Jacobian 1 can be approximated uniformly on compacts by compositions of shears of the kind $\Phi_t(z) = (z', z_n + tf(z'))$, where f is a* polynomial map.

Theorem 14 can be generalised as follows (see [FRos] and [KKPS]) in a form much better adapted to application of density properties.

Theorem 15. *Let X be a Stein manifold with the density (resp. volume density) property and let Ω be an open subset of X (resp. with $H^{n-1}_{\mathrm{dR}}(\Omega, \mathbb{C}) = 0$). Suppose that $\Phi : [0,1] \times \Omega \to X$ is a C^1-smooth map such that*

1. *$\Phi_t : \Omega \to X$ is holomorphic and injective (and resp. volume preserving) for every $t \in [0,1]$;*

2. *$\Phi_0 : \Omega \to X$ is the natural embedding of Ω into X;*

3. *$\Phi_t(\Omega)$ is a Runge subset of X (an open subset U of X is Runge if any holomorphic function on U can be approximated by global holomorphic functions on X in the compact-open topology) for every $t \in [0,1]$.*

Then for each $\varepsilon > 0$ and every compact subset $K \subset \Omega$ there is a continuous family $\alpha : [0,1] \to \mathrm{Aut}_{\mathrm{hol}}(X)$ of holomorphic (resp. holomorphic volume-preserving) automorphisms of X such that

$$\alpha_0 = \Phi_0 \quad \text{and} \quad |\alpha_t - \Phi_t|_K < \varepsilon.$$

for every $t \in [0,1]$.

Further, the approximating maps may be required to fulfil some even more particular features as next theorem shows (see [V1II]).

Theorem 16. *Let X be a Stein manifold of dimension $n \geq 2$ with the density (resp. volume density) property, K be a compact in X , and x, $y \in X$ be two points outside the $\mathcal{O}(X)$-hull $\widehat{K}_{\mathcal{O}(X)}$ of K. Suppose that $x_1, \ldots, x_m \in K$. Then there exists a (resp. volume-preserving) holomorphic automorphism Ψ of X such that $\Psi(x_i) = x_i$ for every $i = 1, \ldots, m$, $\Psi|_K : K \to X$ is as close to the natural embedding as we wish and $\Psi(y) = x$.*

This theorem implies that one can always move one point from a given $(m+1)$-tuple to another point, while keeping the other m points fixed. Hence we have the following

Corollary 3. *Let X be a Stein manifold of dimension at least 2 enjoying the density property (resp. the ω-volume density property). Then $\mathrm{Aut}_{\mathrm{hol}}(X)$ (resp. $\mathrm{Aut}^{\omega}_{\mathrm{hol}}(X)$) acts* infinitely transitively *on X.*

Also, the connection between density properties and holomorphic flexibility is provided by the following (see Theorem 4 in [KKDP] and Lemma 4.1 of [KKPS]).

Proposition 4. *Let X be a Stein manifold enjoying the density property (resp. the ω-volume density property, for ω holomorphic volume form on X). Then there exist finitely many vector fields $\delta_1, \delta_2, \ldots, \delta_N \in \mathrm{IVF}_{\mathrm{hol}}(X)$ (resp. finitely many vector fields $\delta_1, \delta_2, \ldots, \delta_N \in \mathrm{IVF}^{\omega}_{\mathrm{hol}}(X)$) such that* $\mathrm{Span}\{\delta_i(x) : \ i = 1, \ldots, N\} = T_x X$ *for every point $x \in X$.*

Hence, proceeding as in the proof of Theorem 8, one shows

Theorem 17. *Let X be a Stein manifold enjoying the density property or the volume density property. Then X admits a dominating spray, i.e. is elliptic in the sense of Gromov, and hence is an Oka manifold.*

Besides, let us give the following (see Def. 3.1 of [F]).

Definition 11. *Let Y be a Stein manifold.*

1. *Y is* universal for proper holomorphic embeddings *(i.e. for proper injective holomorphic immersions) if every Stein manifold X with $2 \dim X < \dim Y$ admits a proper holomorphic embedding $X \hookrightarrow Y$.*
2. *Y is* strongly universal for proper holomorphic embeddings *if, under the assumptions in (1), every continuous map $f_0 : X \to Y$ which is holomorphic in a neighborhood of a compact $\mathcal{O}(X)$-convex set $K \subset X$ is homotopic to a proper holomorphic embedding $f_0 : X \hookrightarrow Y$ by a homotopy $f_t : X \to Y$ $(t \in [0, 1])$ such that f_t is holomorphic and arbitrarily close to f_0 on K for every $t \in [0, 1]$.*
3. *Y is* (strongly) universal for proper holomorphic immersions *if condition (1) (resp. (2)) holds for proper holomorphic immersions $X \to Y$ from any Stein manifold X satisfying $2 \dim X \leq \dim Y$.*

The following theorem has been proven by Andrist and Wold (see [AW]), if X is an open Riemann surface, by Andrist et al. (see [AF], Theorems 1.1-1.2) for embeddings and by Forstnerič (see [F2], Theorem 1.1) for immersions in the double dimension.

Theorem 18. *Let X be a Stein manifold with the density or the volume density property. Such X is* strongly universal for proper holomorphic embeddings and immersions*.*

The above makes Stein manifolds with volume density property ideal target spaces for a positive solution of

Lárusson's question. Does every Stein manifold admit an *acyclic* (i.e. being a homotopy equivalence) *embedding* into an Oka manifold?

In other words, we are looking for a generalisation of the Remmert-Bishop-Narasimhan embedding theorem (see Theorem 1).
We note T. Ritter has recently proven that every open Riemann surface acyclically embeds into an elliptic manifold. In case the surface is an annulus, he showed the wished target can be $\mathbb{C} \times \mathbb{C}^*$ (see [T], [T1]).

Andersén-Lempert theory, along with density properties, has been applied to many natural geometric issues, some of which we illustrate below (see also sect. 2 of [KKPS]).

Varolin (see [V1II]), P. G. Dixon and J. Esterle (see [DE]), the latter in the volume preserving case, showed that *if X is a Stein manifold of dimension at least 2 enjoying the density property (or the volume density property) then for each point $x \in X$ there is an injective but not surjective holomorphic map $f : X \to X$ with $f(x) = x$.* The images of such maps are called *Fatou-Bieberbach domains of the second kind* (we remark that in the algebraic setting there can be no such morphisms, by Ax's theorem (see [Ax]), according to which any injective morphism of a scheme of a finite type into itself it automatically surjective). Further, *if X is a Stein manifold of dimension $n \geq 2$ which has the density property, then for each point $x \in X$ there is an injective non-surjective equidimensional holomorphic map $f : \mathbb{C}^n \to X$ with $f(0) = x$* (see again [V1II]). Such maps are called Fatou-Bieberbach maps of the first kind and their images are named *Fatou-Bieberbach domains of the first kind*, i.e., proper subdomains of $\mathbb{C}^n$ that are biholomorphic to $\mathbb{C}^n$.
Note that this last result does *not* have an analogue in the volume preserving case, namely the volume density property does not guarantee the existence of Fatou-Bieberbach domains of the first kind, and in fact it is unknown whether $(\mathbb{C}^*)^n$, for $n \geq 2$, contains such a domain.

Chapter 2

The Gromov-Vaserstein fibration and its fibres

One of the most spectacular applications of modern Oka theory which is of great importance to our work is provided by the following.

Gromov–Vaserstein problem (see [G]). Let $f : \mathbb{C}^n \to \mathrm{SL}_m(\mathbb{C})$ be a holomorphic map. Does f factorise as a finite product of holomorphic maps sending $\mathbb{C}^n$ into unipotent subgroups in $\mathrm{SL}_m(\mathbb{C})$?

Naturally one can ask this question not only for holomorphic maps. For continuous or polynomial maps $f : X \to \mathrm{SL}_m(\mathbb{C})$ (X being a topological space resp. an affine algebraic variety) it has been studied as well. Since products of unipotent subgroups are homotopic to the constant identity map, a necessary condition for the existence of a factorization is the null-homotopy of f.

We remark that the *continuous* version of this theorem was proven by Vaserstein in [Vas]. The *algebraic* version of it (for a polynomial map f of n variables), apart from the trivial case $n = 1$ and Cohn's well-known counterexample in case $n = k = 2$ (see [C]), for $k \geq 3$ and any n is based upon a deep result of Suslin (see [Sus]): any matrix in $\mathrm{SL}_k(\mathbb{C}[\mathbb{C}^n])$ decomposes as a finite product of unipotent (and equivalently elementary) matrices.
In [IK] Ivarsson and Kutzschebauch affirmatively solve it in the holomorphic case, by proving the following.

Theorem 19. *Let X be a finite dimensional reduced Stein space and $f : X \to \mathrm{SL}_k(\mathbb{C})$ be a holomorphic mapping that is null-homotopic. Then there exist a natural number N and holomorphic mappings $G_1, \dots, G_N : X \to \mathbb{C}^{k(k-1)/2}$ such that, for every $x \in X$,*

$$f(x) = M_1(G_1(x)) \cdots M_N(G_N(x)),$$

where, if $Z_K = \{z_{ij,K}\}$ is the accordingly defined multiple variable vector,

$$M_{2l}(Z_{2l}) = \begin{pmatrix} 1 & z_{12,2l} & \dots & z_{1k,2l} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & z_{(k-1)k,2l} \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

and

$$M_{2l-1}(Z_{2l-1}) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ z_{21,2l-1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ z_{k1,2l-1} & \dots & z_{k(k-1),2l-1} & 1 \end{pmatrix}.$$

So as to prove it, Ivarsson and Kutzschebauch define $\Psi_N : (\mathbb{C}^{k(k-1)/2})^N \to \mathrm{SL}_k(\mathbb{C})$ as

$$\Psi_N(Z_1, \dots, Z_N) = M_1(Z_1)^{-1} \cdots M_N(Z_N)^{-1}$$

and they wish to show the existence of a holomorphic map

$$G = (G_1, \dots, G_N) : X \to (\mathbb{C}^{k(k-1)/2})^N$$

such that

$$\begin{array}{ccc} & (\mathbb{C}^{k(k-1)/2})^N & \\ {}^{G}\nearrow & & \searrow^{\Psi_N} \\ X & \xrightarrow{\quad f \quad} & \mathrm{SL}_k(\mathbb{C}) \end{array}$$

is commutative.
Furthermore, they choose to concentrate upon

$$\begin{array}{ccc} & (\mathbb{C}^{k(k-1)/2})^N & \\ {}^{F}\nearrow & & \searrow^{\pi_k \circ \Psi_N} \\ X & \xrightarrow{\quad \pi_k \circ f \quad} & \mathbb{C}^k \setminus \{0\} \end{array}$$

and to make use of the fibres of $\pi_k \circ \Psi_N =: \Phi_N$, where π_k denotes the projection to the k-th (i.e. to the last) row of a given matrix in $\mathrm{SL}_k(\mathbb{C})$.
In the present work we focus upon the case $k = 2$, indicate by $n := N \geq 3$ the number of variables involved (i.e. the dimension of the ambient space $\mathbb{C}^n$) and study the fibres (which we call *Gromov-Vaserstein fibres*) of the maps $\Phi_n = \pi_2 \circ \Psi_n$ over suitable subsets of $\mathbb{C}^2 \setminus \{0\}$, where $\pi_2 : \mathrm{SL}_2(\mathbb{C}) \to \mathbb{C}^2 \setminus \{0\}$ is the projection to the second row.

We now introduce the notations we will employ for the study of these fibres in Chapter 4, where we will see them as hypersurfaces of $\mathbb{C}^n$, for $n \geq 3$.

For $n \geq 1$ consider the family $\{M_n\}$ of 2×2 matrices defined inductively as follows:

$M_1 = \begin{pmatrix} 1 & 0 \\ z_1 & 1 \end{pmatrix}$,

$M_2 = \begin{pmatrix} 1 & 0 \\ z_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & z_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & z_2 \\ z_1 & z_1 z_2 + 1 \end{pmatrix}$

$M_3 = \begin{pmatrix} 1 & 0 \\ z_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & z_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z_3 & 1 \end{pmatrix} = \begin{pmatrix} 1 + z_2 z_3 & z_2 \\ z_1 + z_3(z_1 z_2 + 1) & z_1 z_2 + 1 \end{pmatrix}$,

$M_4 = M_3 \begin{pmatrix} 1 & z_4 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 + z_2 z_3 & z_4(1 + z_2 z_3) + z_2 \\ z_1 + z_3(z_1 z_2 + 1) & z_4(z_1 + z_3(z_1 z_2 + 1)) + z_1 z_2 + 1 \end{pmatrix}$,

. . .

$M_n = M_{n-1} \begin{pmatrix} 1 & 0 \\ z_n & 1 \end{pmatrix}$ if n is odd,

$M_n = M_{n-1} \begin{pmatrix} 1 & z_n \\ 0 & 1 \end{pmatrix}$ if n is even.

Further, let us consider the family of polynomials $\{p_n\}_{n\geq 1}$, where $p_n(z_1,\ldots,z_n) = (M_n)_{2,1}$ if n is odd, $p_n(z_1,\ldots,z_n) = (M_n)_{2,2}$ if n is even. We have

$$p_1(z_1) = z_1$$

$$p_2(z_1,z_2) = 1 + z_1 z_2$$

$$p_3(z_1,z_2,z_3) = z_1 + z_3 + z_1 z_3 z_2,$$

$$p_4(z_1,\ldots,z_4) = z_1 z_2 + 1 + z_4(z_1 + z_3 + z_1 z_3 z_2),$$

and, inductively, for $n \geq 5$,

$$p_n(z_1,\ldots,z_n) = p_{n-2}(z_1,\ldots,z_{n-2}) + z_n p_{n-1}(z_1,\ldots,z_{n-1}). \tag{2.1}$$

For $c \in \mathbb{C}$ and $n \geq 3$, let X_n^c denote the hypersurface of $\mathbb{C}^n$ given by the equation $p_n(z_1,\ldots,z_n) = c$.
The hypersurfaces we focus upon in the present work can be subdivided into three classes:

1. *smooth generic* fibres $X_n^c \subset \mathbb{C}^n$, with $c \in \mathbb{C} \setminus \{0\}$ if n is odd, with $c \in \mathbb{C} \setminus \{0,1\}$ if n is even;

2. *singular special* fibres $X_n^1 \subset \mathbb{C}^n$, with $c = 1$ and n even (singular at the origin);

3. *smooth special* fibres $X_n^0 \subset \mathbb{C}^n$, with $c = 0$ and n even or odd.

Lemma 2. *The smooth special fibers X_n^0 are related by $X_n^0 \simeq \mathbb{C}^{n-1} \setminus X_{n-1}^0$, $n \geq 2$.*

Proof. The induction formula (2.1) for p_n shows that on X_n^0 the polynomial p_{n-1} does not vanish. Indeed, if for a point $z = (z_1,\ldots,z_n)$ we have $p_n(z) = p_{n-1}(z) = 0$ we get $p_{n-2}(z) = 0$ by (2.1). Inductively we end up with $1 + z_1 z_2 = p_2(z_1,z_2) = 0$ and $p_1(z_1) = z_1 = 0$. These two equations have no common solution. Thus on X_n^0 we can express

$$z_n = -\frac{p_{n-2}(z_1,\ldots,z_{n-2})}{p_{n-1}(z_1,\ldots,z_{n-1})}.$$

This realizes X_n^0 as a graph over $\mathbb{C}^{n-1} \setminus X_{n-1}^0$, giving the desired isomorphism. □

Above we have stated that these hypersurfaces originate as fibres of the maps $\Phi_n = \pi_2 \circ \Psi_n$ over suitable subsets of $\mathbb{C}^2 \setminus \{0\}$, where $\pi_2 : \mathrm{SL}_2(\mathbb{C}) \to \mathbb{C}^2 \setminus \{0\}$ is the projection to the second row. This is easy to show if one notices that, for instance, in the generic case, the subsets of $\mathbb{C}^2 \setminus \{0\}$ one has to consider are:

- for odd n and $c \in \mathbb{C} \setminus \{0\}$, the set $A^c = \{(c, a_2) : \ a_2 \in \mathbb{C}\}$;
- for even n and $c \in \mathbb{C} \setminus \{0,1\}$, the set $B^c = \{(a_1, c) : \ a_1 \in \mathbb{C}\}$

and that we have $(M_n)_{2,2} = p_{n-1}$ if n is odd, while $(M_n)_{2,1} = p_{n-1}$ if n is even.
One could also notice that, in case n is odd (similarly for n even),

- if $a_2 \in \mathbb{C} \setminus \{0\}$ and $c \in \mathbb{C} \setminus \{0\}$, the two equations $p_{n-1} = a_2$ and $p_n = c$ *reduce* to $p_{n-1} = a_2$ (with $z_n = \frac{c - p_{n-2}}{a_2}$);
- if $a_2 = 0$ and $c \in \mathbb{C} \setminus \{0\}$, the two equations $p_{n-1} = 0$ and $p_n = c$ *reduce* to $p_{n-2} = c$ (with $z_{n-1} = -c^{-1} p_{n-3}$).

Chapter 3

Topological interlude

In this chapter we introduce various notions and preliminary facts which will prove themselves to be of great importance to our work.

Modification of an affine space along a divisor with centre at a codimension-two complete intersection. Let us set $Y = \mathbb{C}^r$ and $I = (f, g)$, where $f, g \in \mathbb{C}^{[r]}$ are non-constant polynomials without common factor. The *affine modification* of Y along the divisor $D = \{\overline{x} \in \mathbb{C}^r : f(\overline{x}) = 0\} \subset Y$ with centre at $C = \{f(\overline{x}) = g(\overline{x}) = 0\} \subset D$ is the hypersurface $X \subset \mathbb{C}^{r+1}$ given by $X = \{(\overline{x}, y) \in \mathbb{C}^{r+1} : f(\overline{x})y - g(\overline{x}) = 0\}$, obtained from Y by means of the blow-up morphism $\sigma_I : X \to Y$ along the center, which is the restriction to X of the natural projection $\mathbb{C}^{r+1} \to \mathbb{C}^r$, $(\overline{x}, y) \to \overline{x}$, together with the removal of the proper transform of the divisor D. To D we associate the divisor $A \subset X$ given by $A = C \times \mathbb{C}$. The *Euler characteristic* of an affine modification can be easily computed by means of the following (see Example 3.1 of [KZ]).

Lemma 3. *Denoted by e the Euler characteristic, we have, the hypersurface $X \subset \mathbb{C}^{r+1}$ being the affine modification, via the projection $\sigma : X \to Y$, of $Y = \mathbb{C}^r$ along the divisor $D \subset Y$ with centre at $C \subset D$, that the following equality holds true*

$$e(X) = 1 + e(C) - e(D).$$

Also, the following proposition yields a sufficient condition in order *the first integral homology group* (and the fundamental group) to be preserved under a modification (cf. Prop. 3.1 of [KZ]).

Proposition 5. *In the above notation, let the hypersurface $X \subset \mathbb{C}^{r+1}$ be the affine modification, via the projection $\sigma : X \to Y$, of $Y = \mathbb{C}^r$ along the divisor $D \subset Y$ with centre at $C \subset D$, to which the divisor $A = C \times \mathbb{C}$ is associated. Let the divisors D and A admit finite decompositions into irreducible components $D = \cup_{i=1}^{n} D_i$ and $A = \cup_{j=1}^{n'} A_j$ respectively. Suppose that $\sigma^*(D_i) = \sum_{j=1}^{n'} m_{ij} A_j$ and that $\sigma(A_j) \cap \operatorname{reg} D_i \neq \emptyset$ as soon as $m_{ij} > 0$. Then, if the lattice vectors $b_j = (m_{1j}, \ldots, m_{nj}) \in \mathbb{Z}^n$, $j = 1, \ldots, n'$, generate the lattice $\mathbb{Z}^n$, the induced map $\sigma_* : H_1(X, \mathbb{Z}) \to H_1(Y, \mathbb{Z}) = 0$ is an isomorphism.*
Assume further that there is a disjoint partition $\{1, \ldots, n'\} = J_1 \sqcup \cdots \sqcup J_n$ such that $\sigma^*(D_i) = \sum_{j \in J_i} m_{ij} A_j \neq 0$, $i = 1, \ldots, n$. Let us set $d_i = \gcd(m_{ij} : j \in J_i)$, $i = 1, \ldots, n$. Then, if $d_1 = \cdots = d_n = 1$, the induced map $\sigma_* : \pi_1(X) \to \pi_1(Y) = 0$ is an isomorphism.

Homological Thom isomorphism theorem. The following may be found in [MilS].

Theorem 20 (Homological Thom isomorphism theorem)**.** *Let $\pi : E \to B$ be a real vector bundle of rank r. Denoted by $E_0 \subset E$ the image of the zero section, there is a relative* *cohomology class* $u \in H^r(E, E \setminus E_0)$, *called the* Thom class*, such that the composition*

$$H_{r+j}(E, E \setminus E_0) \xrightarrow{\vartheta_\cap} H_j(E) \xrightarrow{\pi_*} H_j(B)$$

is an isomorphism *for every integer* j*, where* $\vartheta_\cap(\eta) = u \cap \eta$ *is the so-called* Thom isomorphism *(defined by means of the cap product* $\cap$ *of an* r*-cohomology class and an* $(r+j)$*-homology class) and* π_* *is the homology isomorphism induced by the projection* π*.*

A combination of this last theorem and the excision theorem yields the following, which is sometimes referred to as the Thom isomorphism theorem itself (see sect. 3 of [KZ]).

Proposition 6. *Let* Y *be a reduced connected complex space and* $X \subset Y$ *a* closed hypersurface*. Then, for every integer* j*,*

$$H_{j+2}(Y, Y \setminus X) \cong H_j(X).$$

Gysin sequences. In the context of De Rham cohomology, the following lemma holds, a proof of which may be found in [Mon].

Lemma 4. *Let* Y *be an irreducible smooth variety and* $X \subset Y$ *a smooth hypersurface. Then the following sequence, called the* De-Rham-cohomology Gysin sequence *is exact:*

$$\cdots \longrightarrow H^k_{\mathrm{dR}}(Y) \longrightarrow H^k_{\mathrm{dR}}(Y \setminus X) \xrightarrow{\mathfrak{R}} H^{k-1}_{\mathrm{dR}}(X) \longrightarrow H^{k+1}_{\mathrm{dR}}(Y) \longrightarrow \cdots .$$

We note that, if $Y = \mathbb{C}^r$, by exactness the map $\mathfrak{R}$ (the so-called *residue map*) is an isomorphism for every $k \geq 1$ and that, as explained in [Sch], the inverse $\rho : H^{k-1}_{\mathrm{dR}}(X) \to H^k_{\mathrm{dR}}(\mathbb{C}^r \setminus X)$ of $\mathfrak{R}$ is such that $\rho([\alpha]_{\mathrm{dR}}) = [\frac{\mathrm{d}f}{f} \wedge \alpha]_{\mathrm{dR}}$, where $X = \{x \in \mathbb{C}^r :\ f(x) = 0\}$ is a smooth hypersurface of $\mathbb{C}^r$.
Further, we remark that, by combining the excision theorem and the Thom isomorphism theorem, one shows, as in [D] and in [D1] (chap. 2), that the (exact) Gysin sequence exists in the homological context, too, whence setting $Y = \mathbb{C}^r$ yields, also in this case, an isomorphism. Hence we can formulate the following.

Proposition 7. *Let* $X \subset \mathbb{C}^r$ *be a smooth hypersurface. Then, for every* $k \geq 1$*,*

$$H_k(\mathbb{C}^r \setminus X) \cong H_{k-1}(X) \quad \text{and} \quad H^k_{\mathrm{dR}}(\mathbb{C}^r \setminus X) \cong H^{k-1}_{\mathrm{dR}}(X).$$

Non-vanishing regular functions on a connected simply connected variety. Given a connected *simply connected* variety X, let us suppose $\mathfrak{f}$ is a non-constant non-vanishing regular function on X. By simple connectedness, $\mathfrak{f} = \exp(\mathfrak{g})$, where $\mathfrak{g}$ is a non-constant holomorphic function on X whose image is cofinite in $\mathbb{C}$, whence every fibre of $\mathfrak{f}$ has infinitely many connected components, which leads to a contradiction. Therefore one has the following well-known fact, an alternative proof of which, not involving the factorisation of $\mathfrak{f}$ through the exponential map, may be found in [Fuj] (see cor. 1.20).

Theorem 21. *Any non-vanishing regular function on a connected and simply connected variety is constant.*

It follows at once that, given a connected simply connected variety X endowed with an algebraic volume form ω, such a volume form is uniquely determined up to multiplication by a non-zero constant factor.

Chapter 4

Homology and volume density property for the Gromov-Vaserstein fibres

In this chapter we present our results about the fibres X_n^c (hypersurfaces of $\mathbb{C}^n$) for $c \in \mathbb{C}$ and $n \geq 3$. We shall employ the notations established in Chapter 2.

Proposition 7 (see chap. 3) together with Lemma 2 immediately leads to the following lemma concerning the homology of the smooth special fibres $X_n^0 \simeq \mathbb{C}^{n-1} \setminus X_{n-1}^0$.

Lemma 5. *For the variety* $X_n^0 = \{p_n = 0\} \subset \mathbb{C}^n$, *we have* $H_k(X_n^0) = \mathbb{Z} \ \forall k \in \{0, \ldots, n-1\}$ *and* $H_k(X_n^0) = 0 \ \forall k \geq n$.

Proof. $H_k(X_n^0) = H_k(\mathbb{C}^{n-1} \setminus X_{n-1}^0) \cong H_{k-1}(X_{n-1}^0)$ leads after finite induction to $H_k(X_n^0) = H_0(X_{n-k}^0) \cong \mathbb{Z}$. □

As we will see, our generic fibres $X_n^c \subset \mathbb{C}^n_{z_1,\ldots,z_n}$ are affine modifications of $\mathbb{C}^{n-1}$ via the projection σ which is the restriction of the projection $(z_1, z_2, \ldots, z_n) \mapsto (z_1, z_2, \ldots, z_{n-1})$ to X_n^c. Let $D_n \subset \mathbb{C}^{n-1}$ denote the divisor (independent of c) and $C_n^c \subset D_n$ the center of the modification. Also let A_n^c denote the inverse image $A_n^c = \sigma^{-1}(C_n^c) \simeq C_n^c \times \mathbb{C}_{z_n}$.

The main tool in our calculations of the homology groups are the commuting sequences of the pair $(X_n^c, X_n^c \setminus A_n^c)$ (upper row) and the pair $(\mathbb{C}^{n-1}, \mathbb{C}^{n-1} \setminus D_n)$ (lower row). The vertical arrows are induced by σ. The indicated isomorphisms are due to the fact that σ induces an isomorphism between the varieties $X_n^c \setminus A_n^c$ and $\mathbb{C}^{n-1} \setminus D_n$.

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H_k(X_n^c \setminus A_n^c) & \longrightarrow & H_k(X_n^c) & \longrightarrow & H_k(X_n^c, X_n^c \setminus A_n^c) & \longrightarrow & H_{k-1}(X_n^c \setminus A_n^c) & \longrightarrow \cdots \\
 & \cong \downarrow & & \downarrow & & \downarrow & & \cong \downarrow & \\
\cdots \longrightarrow & H_k(\mathbb{C}^{n-1} \setminus D_n) & \longrightarrow & H_k(\mathbb{C}^{n-1}) & \longrightarrow & H_k(\mathbb{C}^{n-1}, \mathbb{C}^{n-1} \setminus D_n) & \longrightarrow & H_{k-1}(\mathbb{C}^{n-1} \setminus D_n) & \longrightarrow \cdots
\end{array}$$

If in addition we use the Thom isomorphism, which provides $H_k(X_n^c, X_n^c \setminus A_n^c) \cong H_{k-2}(A_n^c)$ and $H_k(\mathbb{C}^{n-1}, \mathbb{C}^{n-1} \setminus D_n) \cong H_{k-2}(D_n)$, the fact that $A_n^c \cong X_{n-2}^c \times \mathbb{C}$ and the definition of $D_n = X_{n-1}^0$, the sequences read

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H_k(X_n^c \setminus A_n^c) & \longrightarrow & H_k(X_n^c) & \longrightarrow & H_{k-2}(X_{n-2}^c \times \mathbb{C}) & \longrightarrow & H_{k-1}(X_n^c \setminus A_n^c) & \longrightarrow \cdots \\
 & \cong \downarrow & & \downarrow & & \downarrow \tau_*^{k-2} \circ \sigma_* & & \cong \downarrow & \\
\cdots \longrightarrow & H_k(\mathbb{C}^{n-1} \setminus X_{n-1}^0) & \longrightarrow & H_k(\mathbb{C}^{n-1}) & \longrightarrow & H_{k-2}(X_{n-1}^0) & \longrightarrow & H_{k-1}(\mathbb{C}^{n-1} \setminus X_{n-1}^0) & \rightarrow \cdots
\end{array}$$

Here the map $\tau_*^{k-2} : H_{k-2}(X_{n-2}^c) \to H_{k-2}(X_{n-1}^0)$ (we suppress the dependence on c) is the map induced by the inclusion of the centre X_{n-2}^c into the divisor X_{n-1}^0. This last inclusion can be seen as the inclusion of a fibre into a fibration over $\mathbb{C}^*$ as follows. We use the isomorphism $X_{n-1}^0 \simeq \mathbb{C}^{n-2} \setminus X_{n-2}^0$. The latter is fibred over $\mathbb{C}^*$ by the restriction of $z \mapsto p_{n-2}(z)$ to the complement of the zero fiber and X_{n-2}^c is just the fibre in this fibration over the point $c \in \mathbb{C}^*$.

In the case that n is odd (and thus $n-2$ is odd) this fibration is trivial. Let us prove this.

Lemma 6. *For n odd the fibration $\mathbb{C}^n \setminus X_n^0 \to \mathbb{C}^\star$, given by $z \mapsto p_n(z)$ is trivial. More precisely it is isomorphic to the fibration $\mathbb{C}^* \times X_n^1 \to \mathbb{C}^*$ with projection to the first factor. In particular all fibers X_n^c for $c \neq 0$ are isomorphic.*

Proof. The triviality of the fibration comes from the $\mathbb{C}^*$-action on the ambient space $\mathbb{C}^n$ given by $\mathbb{C}_\lambda^* \times \mathbb{C}_{z_1,z_2,\dots,z_n}^n \mapsto (\lambda z_1, \lambda^{-1} z_2, \lambda z_3, \dots, \lambda^{(-1)^{i+1}} z_i, \dots, \lambda z_n)$. This action (for n odd) moves the fiber X_n^1 to the fibres X_n^λ. □

Remark 1. *The same $\mathbb{C}^*$-action on $\mathbb{C}^n$ as in the last proof restricts to a $\mathbb{C}^*$-action on X_n^0 for odd n and for even n it restricts to a $\mathbb{C}^*$-action on each of the varieties X_n^c $c \in \mathbb{C}$, instead of moving fibre to fibre over $\mathbb{C}^*$ as in the odd case. We do not know whether the varieties X_n^c, for $c \in \mathbb{C} \setminus \{0,1\}$ and n even, are biholomorphic to each other. Moreover the the so-called singular special fibre is for sure not biholomorphic to these fibers. This makes the study of this fibration difficult and we have been unable in general to understand the map τ_*^k for even n.*

For odd n this is simpler.

Lemma 7. *For odd n the following holds:*

(a) If $H_{k-1}(X_n^1) = 0$ then the map $\tau_*^k : H_k(X_n^1) \to H_k(X_{n+1}^0)$ is an isomorphism.

(b) If $\tau_*^{k-1} : H_{k-1}(X_n^1) \to H_{k-1}(X_{n+1}^0)$ is an isomorphism, then $\tau_*^k : H_k(X_n^1) \to H_k(X_{n+1}^0)$ is the zero map.

Proof. By the above lemma $X_{n+1}^0 \simeq \mathbb{C}^* \times X_n^1$. The Künneth formula for the product yields $H_k(\mathbb{C}^* \times X_n^1) \cong H_k(X_n^1) \otimes H_0(\mathbb{C}^*) \oplus H_{k-1}(X_n^1) \otimes H_1(\mathbb{C}^*) \cong H_k(X_n^1) \otimes \mathbb{Z} \oplus H_{k-1}(X_n^0) \otimes \mathbb{Z}$. To prove the statement (a) observe that the second summand is zero by our assumption. For (b) observe that the assumption forces the first summand to be zero. □

Let us finally remark that the long exact sequence associated to the pair $(\mathbb{C}^{n-1}, \mathbb{C}^{n-1} \setminus D_n) = (\mathbb{C}^{n-1}, \mathbb{C}^{n-1} \setminus X_{n-1}^0)$ (lower row) consisting, for $2 \leq j \leq n$, of blocks of the kind

$$0 = H_j(\mathbb{C}^{n-1}) \to H_j(\mathbb{C}^{n-1}, \mathbb{C}^{n-1} \setminus X_{n-1}^0) \overset{\cong}{\to} H_{j-1}(\mathbb{C}^{n-1} \setminus X_{n-1}^0) \to H_{j-1}(\mathbb{C}^{n-1}) = 0,$$

is by Lemma 5 isomorphic to blocks of the kind

$$0 \to \mathbb{Z} \overset{\cong}{\to} \mathbb{Z} \to 0.$$

4.1 Generic fibres

In this section we deal with the generic fibres X_n^c, where $c \in \mathbb{C} \setminus \{0\}$ for odd n and $c \in \mathbb{C} \setminus \{0,1\}$ for even n.

4.1.1 Results in the generic case

Theorem 22. *The following statements hold true:*

1a) Let $c \in \mathbb{C} \setminus \{0\}$ *be fixed. The variety* $X_3^c = \{z_1 + z_3 + z_1 z_3 z_2 = c\} \subset \mathbb{C}^3_{z_1,z_2,z_3}$ *is the affine modification, via the projection* $\sigma : X_3^c \to \mathbb{C}^2_{z_1,z_3}$*, of* $\mathbb{C}^2$ *along the divisor* $\Delta = \{z_1 z_3 = 0\} \subset \mathbb{C}^2$ *with centre at* $C_3^c = \{z_1 z_3 = 0,\ z_1 + z_3 = c\} = \{(c, 0), (0, c)\} \subset \operatorname{reg} \Delta \subset \Delta$.

1b) Let $c \in \mathbb{C} \setminus \{0, 1\}$ *be fixed. The variety* $X_4^c = \{z_1 z_2 + 1 + z_4(z_1 + z_3 + z_1 z_3 z_2) = c\} \subset \mathbb{C}^4_{z_1,z_2,z_3,z_4}$ *is the affine modification, via the projection* $\sigma : X_4^c \to \mathbb{C}^3_{z_1,z_2,z_3}$*, of* $\mathbb{C}^3$ *along the smooth divisor* $X_3^0 = \{z_1 + z_3 + z_1 z_3 z_2 = 0\} \subset \mathbb{C}^3$ *with centre at* $C_4^c = \{z_1 z_2 + 1 = c,\ z_1 + z_3 + z_1 z_2 z_3 = 0\} = \{(z_1, \frac{c-1}{z_1}, -\frac{z_1}{c}) : z_1 \in \mathbb{C}^*_{z_1}\} \simeq \mathbb{C}^*$, $C_4^c \subset X_3^0$.

1c) If $n \geq 5$ *is odd, suppose* $c \in \mathbb{C} \setminus \{0\}$ *is fixed. Then the variety* $X_n^c \subset \mathbb{C}^n_{z_1,\ldots,z_n}$ *is the affine modification, via the projection* $\sigma : X_n^c \to \mathbb{C}^{n-1}_{z_1,\ldots,z_{n-1}}$*, of* $\mathbb{C}^{n-1}$ *along the smooth divisor* $X_{n-1}^0 = \{p_{n-1}(z_1, \ldots, z_{n-1}) = 0\} \subset \mathbb{C}^{n-1}$ *with centre at* $C_n^c = \{p_{n-2}(z_1, \ldots, z_{n-2}) = c,\ p_{n-1}(z_1, \ldots, z_{n-1}) = 0\} \subset X_{n-1}^0$*, where* $X_{n-1}^0 \simeq \mathbb{C}^{n-2} \setminus X_{n-2}^0$ *and* $C_n^c \simeq X_{n-2}^c$.
If $n \geq 6$ *is even, suppose* $c \in \mathbb{C} \setminus \{0, 1\}$ *is fixed. Then the variety* $X_n^c \subset \mathbb{C}^n_{z_1,\ldots,z_n}$ *is the affine modification, via the projection* $\sigma : X_n^c \to \mathbb{C}^{n-1}_{z_1,\ldots,z_{n-1}}$*, of* $\mathbb{C}^{n-1}$ *along the smooth divisor* $X_{n-1}^0 = \{p_{n-1}(z_1, \ldots, z_{n-1}) = 0\} \subset \mathbb{C}^{n-1}$ *with centre at* $C_n^c = \{p_{n-2}(z_1, \ldots, z_{n-2}) = c,\ p_{n-1}(z_1, \ldots, z_{n-1}) = 0\} \subset X_{n-1}^0$*, where* $X_{n-1}^0 \simeq \mathbb{C}^{n-2} \setminus X_{n-2}^0$ *and* $C_n^c \simeq X_{n-2}^c$.

In what follows, let $H_j(\cdot)$ *denote the* j*-th homology group with integer coefficients.*

2a) For fixed $c \in \mathbb{C} \setminus \{0\}$*, we have* $H_0(X_3^c) = \mathbb{Z}$, $H_1(X_3^c) = 0$, $H_2(X_3^c) = \mathbb{Z}$, $H_k(X_3^c) = 0$ *for* $k \geq 3$.

2b) For fixed $c \in \mathbb{C} \setminus \{0, 1\}$*, we have* $H_0(X_4^c) = \mathbb{Z}$, $H_1(X_4^c) = 0$ $H_2(X_4^c) = \mathbb{Z}$, $H_3(X_4^c) = \mathbb{Z}^2$, $H_k(X_4^c) = 0$ *for* $k \geq 4$.

2c) If $n \geq 5$ *is odd and* $c \in \mathbb{C} \setminus \{0\}$ *is fixed, for* $0 \leq j \leq n - 1$ *the homology groups* $H_j(X_n^c)$ *are alternately* $\mathbb{Z}$ *or* 0*, starting from* $H_0(X_n^c) = \mathbb{Z}$*. In particular* $H_{n-2}(X_n^c) = 0$.

Further, let ω denote the unique (up to multiplication by a nonzero constant) algebraic volume form on X_n^c (see, in subsection 4.1.4, Lemma 8 and the remark following it). Furthermore, let us denote by $\Theta : \mathrm{AVF}_\omega(X_n^c) \to \mathcal{Z}_{n-2}(X_n^c)$ the isomorphism sending the ω-divergence-free algebraic vector field ξ to the closed $(n-2)$-form $\iota_\xi \omega$, defined via the interior product ι of ξ and ω, by $\mathrm{Lie}^\omega_{\mathrm{alg}}(X_n^c)$ the Lie algebra generated by the set $\mathrm{IVF}_\omega(X_n^c)$ of complete ω-divergence-free algebraic vector fields on X_n^c and by $\mathcal{B}_{n-2}(X_n^c)$ the space of exact algebraic $(n-2)$-forms on X_n^c. We will also show the following.

Theorem 23. *For each* $n \geq 3$ *the variety* $X_n^c \subset \mathbb{C}^n$*, where* $c \in \mathbb{C} \setminus \{0\}$ *if* n *is odd and* $c \in \mathbb{C} \setminus \{0, 1\}$ *if* n *is even, is such that* $\Theta(\mathrm{Lie}^\omega_{\mathrm{alg}}(X_n^c)) \supset \mathcal{B}_{n-2}(X_n^c)$*, i.e. for every algebraic* $(n-3)$*-form* α *on* X_n^c *there exists* $\xi \in \mathrm{Lie}^\omega_{\mathrm{alg}}(X_n^c)$ *such that* $\Theta(\xi) = \iota_\xi \omega = \mathrm{d}\alpha$*. Moreover for odd* $n \geq 3$ *as well as for* $n = 4$ *the equality* $\mathrm{Lie}^\omega_{\mathrm{alg}}(X_n^c) = \mathrm{AVF}_\omega(X_n^c)$ *holds, i.e.* X_n^c *has the algebraic volume density property.*

In the following subsection we will study the initial case ($n = 3$) and the case $n = 4$ together with the base case ($n = 5$) necessary to our induction.

4.1.2 Base cases

Let us prove the following

Proposition 8. *Let $c \in \mathbb{C} \setminus \{0\}$ be fixed. Let $X_3^c = \{z_1 + z_3 + z_1 z_3 z_2 = c\} \subset \mathbb{C}^3_{z_1,z_2,z_3}$ be the affine modification, via the projection $\sigma : X_3^c \to \mathbb{C}^2_{z_1,z_3}$, of $\mathbb{C}^2$ along the divisor $\Delta = \{z_1 z_3 = 0\} \subset \mathbb{C}^2$ with centre at $C_3^c = \{z_1 z_3 = 0,\ z_1 + z_3 = c\} = \{(c,0),(0,c)\} \subset \operatorname{reg}\Delta \subset \Delta$. We have*

$$H_0(X_3^c) = \mathbb{Z},\ \ H_1(X_3^c) = 0,\ \ H_2(X_3^c) = \mathbb{Z},\ \ H_k(X_3^c) = 0 \text{ for } k \geq 3.$$

Proof. Since X_3^c is connected, we have $H_0(X_3^c) = \mathbb{Z}$. Note that $\mathbb{C}^2 \setminus \Delta \simeq \mathbb{C}^* \times \mathbb{C}^*$, which is a torus. Consider the divisors Δ and $A_3^c = \sigma^*(\Delta) = \sigma^{-1}(C_3^c) = C_3^c \times \mathbb{C} \simeq \mathbb{C} \sqcup \mathbb{C}$. By Prop. 3.1 of [KZ] one concludes that $H_1(X_3^c) \cong H_1(\mathbb{C}^2) = 0$. As $H_*(X_3^c \setminus A_3^c) \overset{\sigma_*}{\cong} H_*(\mathbb{C}^2 \setminus \Delta)$, denoting by e the Euler characteristic, we have $e(X_3^c) = 1 + e(C_3^c) - e(\Delta)$ and, since $e(C_3^c) = 2$ and $e(\Delta) = 1$, it follows that $e(X_3^c) = 2$ and consequently that $H_2(X_3^c) = \mathbb{Z}$, as we can have no torsion in the maximal-dimension homology group. □

Remark 2. Consider, as in [KKAV] (sect. 7), the variety $X_{p,q} = \{p(x) + q(y) + xyz = 1\} \subset \mathbb{C}^3_{x,y,z}$, where p and q are polynomials such that $p(0) = q(0) = 0$, $1 - p(x)$ and $1 - q(y)$ have simple roots only, namely $x_1, \ldots, x_k$ and $y_1, \ldots, y_l$. Thanks to Prop. 3.1 of [KZ] one easily shows that $H_1(X_{p,q}) \cong H_1(\mathbb{C}^2) = 0$. Since we have $e(X_{p,q}) = k + l$, we get $H_2(X_{p,q}) = \mathbb{Z}^{k+l-1}$, as we can have no torsion in the maximal-dimension homology group. In particular we obtain the homology with integer coefficients of X_3^1 if we set $p(x) = x$, $q(y) = y$.

Remark 3. From Lemma 6 we know that for $c \in \mathbb{C} \setminus \{0\}$, the variety X_3^c is diffeomorphic to X_3^1 via the map sending (z_1, z_2, z_3) to $(cz_1,\ c^{-1}z_2,\ cz_3)$, thus the homology of X_3^c is the same as that of X_3^1, studied in [DV].

The next result is already contained in Lemma 5, we show how it can be derived from diagram chasing as well. This is a good introduction to our approach in the general case.

Proposition 9. *Let $X_3^0 = \{z_1 + z_3 + z_1 z_3 z_2 = 0\} \subset \mathbb{C}^3_{z_1,z_2,z_3}$ be the affine modification, via the projection $\sigma : X_3^0 \to \mathbb{C}^2_{z_1,z_3}$, of $\mathbb{C}^2$ along the divisor $\Delta = \{z_1 z_3 = 0\} \subset \mathbb{C}^2$ with centre at $C_3^0 = \{z_1 z_3 = -z_1 - z_3 = 0\} = \{(0,0)\} = \operatorname{sing}\Delta \subset \Delta$. We have*

$$H_0(X_3^0) = \mathbb{Z},\ \ H_1(X_3^0) = \mathbb{Z},\ \ H_2(X_3^0) = \mathbb{Z},\ \ H_k(X_3^0) = 0 \text{ for } k \geq 3.$$

Proof. Since X_3^0 is connected, we have $H_0(X_3^0) = \mathbb{Z}$. Note that $\mathbb{C}^2 \setminus \Delta \simeq \mathbb{C}^* \times \mathbb{C}^*$, which is a torus. Consider the divisors Δ and $A_3^0 = \sigma^*(\Delta) = \sigma^{-1}(C_3^0) = C_3^0 \times \mathbb{C} = \{(0,0)\} \times \mathbb{C}$. Proceeding as in the proof of Prop. 3.1 of [KZ], one concludes that $H_1(X_3^0) = \mathbb{Z}$. As $H_*(X_3^0 \setminus A_3^0) \overset{\sigma_*}{\cong} H_*(\mathbb{C}^2 \setminus \Delta)$, denoting by e the Euler characteristic, we have $e(X_3^0) = 1 + e(C_3^0) - e(\Delta)$ and, since $e(C_3^0) = e(\Delta) = 1$, it follows that $e(X_3^0) = 1$ and that $H_2(X_3^0) = \mathbb{Z}$, as we can have no torsion in the maximal-dimension homology group. □

Now we study the case $n = 4$ by proving the following

Proposition 10. *Let $c \in \mathbb{C} \setminus \{0, 1\}$ be fixed. Let $X_4^c = \{z_1 z_2 + 1 + z_4(z_1 + z_3 + z_1 z_3 z_2) = c\} \subset \mathbb{C}^4_{z_1,z_2,z_3,z_4}$ be the affine modification, via the projection $\sigma : X_4^c \to \mathbb{C}^3_{z_1,z_2,z_3}$, of $\mathbb{C}^3$ along the smooth divisor $X_3^0 = \{z_1 + z_3 + z_1 z_3 z_2 = 0\} \subset \mathbb{C}^3$ with centre at $C_4^c = \{z_1 z_2 + 1 = c,\ z_1 + z_3 + z_1 z_2 z_3 = 0\} = \{(z_1, \frac{c-1}{z_1}, -\frac{z_1}{c}) : z_1 \in \mathbb{C}^*_{z_1}\} \simeq \mathbb{C}^*$, $C_4^c \subset X_3^0$. We have*

$$H_0(X_4^c) = \mathbb{Z},\ \ H_1(X_4^c) = 0,\ \ H_2(X_4^c) = \mathbb{Z},\ \ H_3(X_4^c) = \mathbb{Z}^2,\ \ H_k(X_4^c) = 0 \text{ for } k \geq 4.$$

Proof. Since X_4^c is connected, we have $H_0(X_4^c) = \mathbb{Z}$. Let us consider the two long exact sequences associated to the couples $(\mathbb{C}^3, \mathbb{C}^3 \setminus X_3^0)$ and $(X_4^c, X_4^c \setminus A_4^c)$, where $A_4^c = \sigma^*(X_3^0) = \sigma^{-1}(C_4^c) = C_4^c \times \mathbb{C} \simeq \mathbb{C}^* \times \mathbb{C}$.
As both divisors X_3^0 and A_4^c are smooth and irreducible, thanks to Prop. 3.1 of [KZ] we can conclude that $H_1(X_4^c) \overset{\sigma_*}{\cong} H_1(\mathbb{C}^3) = 0$.
Also, thanks to Thom Isomorphism theorem, we obtain $H_2(X_4^c, X_4^c \setminus A_4^c) \cong H_0(A_4^c) = \mathbb{Z}$

and $H_2(\mathbb{C}^3, \mathbb{C}^3 \setminus X_3^0) \cong H_0(X_3^0) = \mathbb{Z}$, as well as $H_3(X_4^c, X_4^c \setminus A_4^c) \cong H_1(A_4^c) = \mathbb{Z}$ and $H_3(\mathbb{C}^3, \mathbb{C}^3 \setminus X_3^0) \cong H_1(X_3^0) = \mathbb{Z}$.
The main point is to study the map $\tau_*^1 : H_1(X_2^c) \to H_1(X_3^0)$ coming from the inclusion of the fibre bundle into the fibration discussed before Lemma 6.

The loop generating $H_1(C_4^c) \cong H_1(X_2^c)$ is given by $((c-1)\mathrm{e}^{it}, \mathrm{e}^{-it})$, $c \in \mathbb{C} \setminus \{1, 0\}$. It can be moved, for $\varepsilon \in \mathbb{C}^*$ with ε near 0 to the loop $(\varepsilon \mathrm{e}^{it}, \mathrm{e}^{-it}) \in X_2^{1+\varepsilon}$ via the lifting of any path γ in $\mathbb{C} \setminus \{-1, 0\}$ (just multiply the first coordinate with $\gamma(s)$). This loop can be contracted to the singular point $(0,0) \in X_2^1$ by means of the deformation multiplying both coordinates by $s \in [0,1]$. Hence the vertical map in our diagram $\tau_*^1 : H_1(C_4^c) \to H_1(X_3^0)$ is the *zero map.* Consider the following diagram, where we applied Thom isomorphism theorem to relative homology groups $H_*(X_4^c, X_4^c \setminus A_4^c)$ and $H_*(\mathbb{C}^3, \mathbb{C}^3 \setminus X_3^0)$.

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & H_1(X_2^c) & \xrightarrow[\alpha]{0} & H_2(X_4^c \setminus A_4^c) & \xrightarrow{\cong} & H_2(X_4^c) & \xrightarrow[\gamma]{0} & H_0(X_2^c) & \longrightarrow \cdots \\
& {\scriptstyle 0}\big\downarrow{\scriptstyle \tau_*^1} & & {\scriptstyle \cong}\big\downarrow & & {\scriptstyle 0}\big\downarrow & & {\scriptstyle \cong}\big\downarrow{\scriptstyle \tau_*^0} & \\
\cdots \longrightarrow & H_1(X_3^0) & \xrightarrow{\cong} & H_2(\mathbb{C}^3 \setminus X_3^0) & \longrightarrow & H_2(\mathbb{C}^3) & \longrightarrow & H_0(X_3^0) & \longrightarrow \cdots
\end{array}
$$

The upper sequence immediately yields $H_2(X_4^c) = \mathbb{Z}$ (the maps α and γ are zero by diagram commutativity). Now, denoting by e the Euler characteristic, we have $e(X_4^c) = 1 + e(C_4^c) - e(X_3^0)$. Since $e(C_4^c) = e(\mathbb{C}^*) = 0$ and $e(X_3^0) = 1$, it follows that $e(X_4^c) = 0$ and consequently that $H_3(X_4^c) = \mathbb{Z}^2$, as we can have no torsion in the maximal-dimension homology group. □

The base case $n = 5$ (for the induction on odd n) is dealt with in the next proposition.

Proposition 11. *Let* $c \in \mathbb{C} \setminus \{0\}$ *be fixed. Let* $X_5^c = \{z_1 + z_3 + z_1 z_3 z_2 + z_5(z_1 z_2 + 1 + z_4(z_1 + z_3 + z_1 z_3 z_2)) = c\} \subset \mathbb{C}^5_{z_1, \ldots, z_5}$ *be the affine modification, via the projection* $\sigma : X_5^c \to \mathbb{C}^4_{z_1, \ldots, z_4}$, *of* $\mathbb{C}^4$ *along the smooth divisor* $X_4^0 = \{z_1 z_2 + 1 + z_4(z_1 + z_3 + z_1 z_3 z_2) = 0\} \simeq \mathbb{C}^3 \setminus X_3^0$, $X_4^0 \subset \mathbb{C}^4$, *with centre at* $C_5^c = \{z_1 + z_3 + z_1 z_3 z_2 = c,\ z_1 z_2 + 1 + z_4(z_1 + z_3 + z_1 z_3 z_2) = 0\} = \{(z_1, z_2, z_3, z_4) : (z_1, z_2, z_3) \in X_3^c,\ z_4 = -c^{-1}(z_1 z_2 + 1)\} \simeq X_3^c$, $C_5^c \subset X_4^0$. *We have*

$$H_0(X_5^c) = \mathbb{Z},\ H_1(X_5^c) = 0,\ H_2(X_5^c) = \mathbb{Z},\ H_3(X_5^c) = 0,\ H_4(X_5^c) = \mathbb{Z}$$

and, for $k \geq 5$,

$$H_k(X_5^c) = 0.$$

Proof. Since X_5^c is connected, we have $H_0(X_5^c) = \mathbb{Z}$. Let us consider the two long exact sequences associated to the couples $(\mathbb{C}^4, \mathbb{C}^4 \setminus X_4^0)$ and $(X_5^c, X_5^c \setminus A_5^c)$, where $A_5^c = \sigma^*(X_4^0) = \sigma^{-1}(C_5^c) = C_5^c \times \mathbb{C} \simeq X_3^c \times \mathbb{C}$.
As both divisors X_4^0 and A_5^c are smooth and irreducible, thanks to Prop. 3.1 of [KZ] we can conclude that $H_1(X_5^c) \overset{\sigma_*}{\cong} H_1(\mathbb{C}^4) = 0$. By Lemma 5 we have $H_j(X_4^0) = \mathbb{Z}$ for $j = 0, \ldots, 3$, as X_4^0 is connected, and also $H_j(\mathbb{C}^4 \setminus X_4^0) = \mathbb{Z}$ for $j = 0, \ldots, 3$, as $\mathbb{C}^4 \setminus X_4^0$ is connected. We then apply the Thom isomorphism theorem as explained and finally a piece of our diagram looks as follows

$$
\begin{array}{cccccccccc}
H_1(X_3^c) = 0 & \xrightarrow[\beta]{} & \mathbb{Z} & \longrightarrow & H_2(X_5^c) & \xrightarrow[\alpha]{} & H_0(X_3^c) & \longrightarrow & H_1(X_5^c \setminus A_5^c) & \longrightarrow \cdots \\
\big\downarrow{\scriptstyle 0} & & {\scriptstyle \cong}\big\downarrow & & \big\downarrow & & {\scriptstyle \cong}\big\downarrow{\scriptstyle \tau_*^0} & & {\scriptstyle \cong}\big\downarrow & \\
\mathbb{Z} & \xrightarrow{\cong} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cong} & \mathbb{Z} & \longrightarrow \cdots
\end{array}
$$

By diagram commutativity both the maps α and β have to be the zero map. Therefore the map between them induces an isomorphism $H_2(X_5^c) \cong \mathbb{Z}$.

Now we shift our attention to the left in the commuting diagram and get (the two entries at the right are the left two entries from the diagram above)

$$\begin{array}{ccccccccccc}
H_2(X_3^c) = \mathbb{Z} & \xrightarrow{\beta} & \mathbb{Z} & \xrightarrow{\gamma} & H_3(X_5^c) & \xrightarrow{\alpha} & H_1(X_3^c) & \xrightarrow{0} & \mathbb{Z} \cong H_2(X_5^c \setminus A_5^c) & \longrightarrow & \cdots \\
\downarrow \tau_*^2 & & \cong \downarrow & & \downarrow & & 0 \downarrow \tau_*^1 & & \cong \downarrow & & \\
\mathbb{Z} & \xrightarrow{\cong} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cong} & \mathbb{Z} & \longrightarrow & \cdots
\end{array}$$

Now by Lemma 7 we conclude from $H_1(X_3^c) = 0$ that the map τ_*^2 is an isomorphism. By commutativity β is an isomorphism and thus γ the zero map.

It follows that α induces an isomorphism between $H_3(X_5^c)$ and $H_1(X_3^c) = 0$.

Denoting by e the Euler characteristic, we have $e(X_5^c) = 1 + e(C_5^c) - e(X_4^0) = 1 + 2 - 0 = 3$, given that $e(C_5^c) = e(X_3^c) = 2$ and $e(X_4^0) = e(\mathbb{C}^3 \setminus X_3^0) = 0$, from which it follows that $H_4(X_5^c) = \mathbb{Z}$, as we can have no torsion in the maximal-dimension homology group. $\square$

4.1.3 Induction

We prove our Theorem 22, which we restate, so as to improve readability.

Theorem 24. *The following statements hold true:*

1a) Let $c \in \mathbb{C} \setminus \{0\}$ be fixed. The variety $X_3^c = \{z_1 + z_3 + z_1 z_3 z_2 = c\} \subset \mathbb{C}^3_{z_1,z_2,z_3}$ is the affine modification, via the projection $\sigma : X_3^c \to \mathbb{C}^2_{z_1,z_3}$, of $\mathbb{C}^2$ along the divisor $\Delta = \{z_1 z_3 = 0\} \subset \mathbb{C}^2$ with centre at $C_3^c = \{z_1 z_3 = 0,\ z_1 + z_3 = c\} = \{(c, 0), (0, c)\} \subset \operatorname{reg} \Delta \subset \Delta$.

1b) Let $c \in \mathbb{C} \setminus \{0, 1\}$ be fixed. The variety $X_4^c = \{z_1 z_2 + 1 + z_4(z_1 + z_3 + z_1 z_3 z_2) = c\} \subset \mathbb{C}^4_{z_1,z_2,z_3,z_4}$ is the affine modification, via the projection $\sigma : X_4^c \to \mathbb{C}^3_{z_1,z_2,z_3}$, of $\mathbb{C}^3$ along the smooth divisor $X_3^0 = \{z_1 + z_3 + z_1 z_3 z_2 = 0\} \subset \mathbb{C}^3$ with centre at $C_4^c = \{z_1 z_2 + 1 = c,\ z_1 + z_3 + z_1 z_2 z_3 = 0\} = \{(z_1, \frac{c-1}{z_1}, -\frac{z_1}{c}) : z_1 \in \mathbb{C}^*_{z_1}\} \simeq \mathbb{C}^*$, $C_4^c \subset X_3^0$.

1c) If $n \geq 5$ is odd, suppose $c \in \mathbb{C} \setminus \{0\}$ is fixed. Then the variety $X_n^c \subset \mathbb{C}^n_{z_1,\dots,z_n}$ is the affine modification, via the projection $\sigma : X_n^c \to \mathbb{C}^{n-1}_{z_1,\dots,z_{n-1}}$, of $\mathbb{C}^{n-1}$ along the smooth divisor $X_{n-1}^0 = \{p_{n-1}(z_1, \dots, z_{n-1}) = 0\} \subset \mathbb{C}^{n-1}$ with centre at $C_n^c = \{p_{n-2}(z_1, \dots, z_{n-2}) = c,\ p_{n-1}(z_1, \dots, z_{n-1}) = 0\} \subset X_{n-1}^0$, where $X_{n-1}^0 \simeq \mathbb{C}^{n-2} \setminus X_{n-2}^0$ and $C_n^c \simeq X_{n-2}^c$.
If $n \geq 6$ is even, suppose $c \in \mathbb{C} \setminus \{0, 1\}$ is fixed. Then the variety $X_n^c \subset \mathbb{C}^n_{z_1,\dots,z_n}$ is the affine modification, via the projection $\sigma : X_n^c \to \mathbb{C}^{n-1}_{z_1,\dots,z_{n-1}}$, of $\mathbb{C}^{n-1}$ along the smooth divisor $X_{n-1}^0 = \{p_{n-1}(z_1, \dots, z_{n-1}) = 0\} \subset \mathbb{C}^{n-1}$ with centre at $C_n^c = \{p_{n-2}(z_1, \dots, z_{n-2}) = c,\ p_{n-1}(z_1, \dots, z_{n-1}) = 0\} \subset X_{n-1}^0$, where $X_{n-1}^0 \simeq \mathbb{C}^{n-2} \setminus X_{n-2}^0$ and $C_n^c \simeq X_{n-2}^c$.

In what follows, let $H_j(\cdot)$ denote the j-th homology group with integer coefficients.

2a) For fixed $c \in \mathbb{C} \setminus \{0\}$, we have $H_0(X_3^c) = \mathbb{Z}$, $H_1(X_3^c) = 0$, $H_2(X_3^c) = \mathbb{Z}$, $H_k(X_3^c) = 0$ for $k \geq 3$.

2b) For fixed $c \in \mathbb{C} \setminus \{0, 1\}$, we have $H_0(X_4^c) = \mathbb{Z}$, $H_1(X_4^c) = 0$ $H_2(X_4^c) = \mathbb{Z}$, $H_3(X_4^c) = \mathbb{Z}^2$, $H_k(X_4^c) = 0$ for $k \geq 4$.

2c) If $n \geq 5$ is odd and $c \in \mathbb{C} \setminus \{0\}$ is fixed, for $0 \leq j \leq n - 1$ the homology groups $H_j(X_n^c)$ are alternately $\mathbb{Z}$ or 0, starting from $H_0(X_n^c) = \mathbb{Z}$. In particular $H_{n-2}(X_n^c) = 0$.

Proof. Statements 1a), 1b), 2a) and 2b) have been proven in Propositions 8 and 10. The base case for statements 1c) and 2c) is dealt with in Proposition 11. Statement 1c) can be easily proved by induction on n as $p_n = p_{n-2} + z_n p_{n-1}$. In particular, as regards the centre, we need to show that $C_{n+1}^c = \{p_{n-1} = c,\ p_n = 0\} \simeq X_{n-1}^c$. We may write $p_n = p_{n-2} + z_n p_{n-1} = p_{n-2} + c z_n$ and $C_{n+1}^c = \{p_{n-1} = c,\ z_n = -c^{-1} p_{n-2}\} \simeq X_{n-1}^c$.
Let us prove 2c) by induction on n for odd n.

Suppose that the homology of X_{n-2}^c is as claimed. Let us compute the homology of X_n^c. Suppose n is *odd* and $c \in \mathbb{C} \setminus \{0\}$ is fixed. Since X_n^c is connected, we have $H_0(X_n^c) = \mathbb{Z}$. Let us consider the two long exact sequences associated to the couples $(\mathbb{C}^{n-1}, \mathbb{C}^{n-1} \setminus X_{n-1}^0)$ and $(X_n^c, X_n^c \setminus A_n^c)$, where $A_n^c = \sigma^{-1}(C_n^c) = C_n^c \times \mathbb{C} \simeq X_{n-2}^c \times \mathbb{C}$, and $X_{n-1}^0 \simeq \mathbb{C}^{n-2} \setminus X_{n-2}^0$. The first homology group is preserved under σ, as both divisors X_{n-1}^0 and A_n^c are smooth and irreducible. Thus, thanks to Prop. 3.1 of [KZ], we can conclude that $H_1(X_n^c) \cong H_1(\mathbb{C}^{n-1}) = 0$. We look again at our diagram starting at the same place as we did for proving the case $n = 5$:

$$
\begin{array}{ccccccccccc}
H_1(X_{n-2}^c) = 0 & \xrightarrow{\beta} & \mathbb{Z} & \longrightarrow & H_2(X_n^c) & \xrightarrow{\alpha} & H_0(X_{n-2}^c) & \longrightarrow & H_1(X_n^c \setminus A_n^c) & \longrightarrow & \cdots \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \cong} & & \downarrow & & \downarrow{\scriptstyle \cong\ \tau_*^0} & & \downarrow{\scriptstyle \cong} & & \\
\mathbb{Z} & \xrightarrow{\cong} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cong} & \mathbb{Z} & \longrightarrow & \cdots
\end{array}
$$

By diagram commutativity both the maps α and β have to be the zero map. Therefore the arrow between them induces an isomorphism $H_2(X_n^c) \cong \mathbb{Z}$.

Now we shift our attention to the left in the commuting diagram and get (the two columns at the right are the left two columns from the diagram above)

$$\begin{array}{ccccccccccc}
H_2(X_{n-2}^c) = \mathbb{Z} & \xrightarrow[\beta]{} & \mathbb{Z} & \xrightarrow[\gamma]{} & H_3(X_n^c) & \xrightarrow[\alpha]{} & H_1(X_{n-2}^c) & \xrightarrow{0} & \mathbb{Z} \cong H_2(X_n^c \setminus A_n^c) & \longrightarrow & \cdots \\
\downarrow{\scriptstyle \tau_*^2} & & {\scriptstyle \cong}\downarrow & & \downarrow & & {\scriptstyle 0}\downarrow{\scriptstyle \tau_*^1} & & {\scriptstyle \cong}\downarrow & & \\
\mathbb{Z} & \xrightarrow{\cong} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cong} & \mathbb{Z} & \longrightarrow & \cdots
\end{array}$$

By Lemma 7 we conclude from $H_1(X_{n-2}^c) = 0$ that the map τ_*^2 is an isomorphism. By commutativity β is an isomorphism and thus γ the zero map.

It follows that α induces an isomorphism between $H_3(X_n^c)$ and $H_1(X_{n-2}^c) = 0$.

Going further left we can make induction over k looking at the diagram at the following place (the right two columns and the properties of the maps between them follow by induction on k):

$$\begin{array}{ccccccccccc}
H_k(X_{n-2}^c) = 0 & \xrightarrow[\beta]{} & \mathbb{Z} & \longrightarrow & H_{k+1}(X_n^c) & \xrightarrow[\alpha]{} & H_{k-1}(X_{n-2}^c) & \longrightarrow & H_k(X_n^c \setminus A_n^c) & \longrightarrow & \cdots \\
{\scriptstyle 0}\downarrow{\scriptstyle \tau_*^k} & & {\scriptstyle \cong}\downarrow & & \downarrow & & {\scriptstyle \cong}\downarrow{\scriptstyle \tau_*^{k-1}} & & {\scriptstyle \cong}\downarrow & & \\
\mathbb{Z} & \xrightarrow{\cong} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cong} & \mathbb{Z} & \longrightarrow & \cdots
\end{array}$$

Since $\tau_*^{k-1} : H_{k-1}(X_{n-2}^c) \to \mathbb{Z}$ is an isomorphism we conclude by Lemma 7 that τ_*^k is the zero map. By diagram commutativity both the maps α and β have to be the zero map. Therefore the between them induces an isomorphism $H_{k+1}(X_n^c) \cong \mathbb{Z}$.

Now we shift our attention to the left in the commuting diagram and get (the two columns at the right are the left two columns from the diagram above)

$$\begin{array}{ccccccccccc}
H_{k+1}(X_{n-2}^c) = \mathbb{Z} & \xrightarrow[\beta]{} & \mathbb{Z} & \xrightarrow[\gamma]{} & H_{k+2}(X_n^c) & \xrightarrow[\alpha]{} & H_k(X_{n-2}^c) & \xrightarrow{0} & \mathbb{Z} \cong H_{k+1}(X_n^c \setminus A_n^c) & \rightarrow & \cdots \\
\downarrow{\scriptstyle \tau_*^{k+1}} & & {\scriptstyle \cong}\downarrow & & \downarrow & & {\scriptstyle 0}\downarrow{\scriptstyle \tau_*^k} & & {\scriptstyle \cong}\downarrow & & \\
\mathbb{Z} & \xrightarrow{\cong} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cong} & \mathbb{Z} & \longrightarrow & \cdots
\end{array}$$

By Lemma 7 we conclude from $H_k(X_{n-2}^c) = 0$ that the map τ_*^{k+1} is an isomorphism. By commutativity β is an isomorphism and thus γ the zero map.

It follows that α induces an isomorphism between $H_{k+2}(X_n^c)$ and $H_k(X_{n-2}^c) = 0$.

We remark that an alternative way of obtaining $H_{n-1}(X_n^c) = \mathbb{Z}$ when n is odd consists in noticing that, as an easy induction shows, $e(X_n^c) = 1 + e(X_{n-2}^c) - e(X_{n-1}^0) = 1 + (n - 2 - \frac{n-3}{2}) - 0 = \frac{n+1}{2}$, where e denotes the Euler characteristic. This ends the proof of point 2c). $\square$

Remark 4. We thank the referee for pointing out to us another simpler approach to statement 2c) by using the Künneth formula in the same way as in Lemma 7.

4.1.4 Volume forms

We prove the following

Lemma 8. *For each $n \geq 3$ the variety $X_n^c = \{p_n = c\}$, where $c \in \mathbb{C} \setminus \{0\}$ if n is odd and $c \in \mathbb{C} \setminus \{0, 1\}$ if n is even, can be equipped with an algebraic volume form ω.*

Proof. Let us express each variable in terms of the others. We obtain

$$z_j = \zeta_j(z_1, \ldots, \widehat{z_j}, \ldots, z_n) = \frac{q_{cj}(z_1, \ldots, \widehat{z_j}, \ldots, z_n)}{\partial_{z_j} p_n},$$

where the polynomial q_{cj} is linear in each of its variables. Let us consider the cover $\{U_i\}_{i=1}^n$ of X_n^c with $U_i = X_n^c \setminus K_i$, where $K_i = \{\underline{z} \in X_n^c : \partial_{z_i} p_n(\underline{z}) = \frac{\partial p_n}{\partial z_i}(\underline{z}) = 0\}$, and the set $\{(\omega_i, U_i)\}_{i=1}^n$, where $\omega_i = \frac{(-1)^i}{\partial_{z_i} p_n} \, dz_1 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_n$ is a volume form defined on $\varphi_i^{-1}(U_i) = \mathbb{C}^{n-1}_{z_1,\ldots,\widehat{z_i},\ldots,z_n} \setminus H_i$, $H_i = \{\underline{z} \in \mathbb{C}^{n-1} : \partial_{z_i} p_n(\underline{z}) = 0\} \subset \mathbb{C}^{n-1}_{z_1,\ldots,\widehat{z_i},\ldots,z_n}$ and $\varphi_i : \mathbb{C}^{n-1}_{z_1,\ldots,\widehat{z_i},\ldots,z_n} \setminus H_i \to U_i$ is defined via

$$\varphi_i(z_1, \ldots, \widehat{z_i}, \ldots, z_n) = (z_1, \ldots, z_i = \zeta_i(z_1, \ldots, \widehat{z_i}, \ldots, z_n), \ldots, z_n),$$

while $\varphi_i^{-1} : U_i \to \mathbb{C}^{n-1}_{z_1,\ldots,\widehat{z_i},\ldots,z_n} \setminus H_i$ takes $(z_1, \ldots, z_n)$ to $(z_1, \ldots, \widehat{z_i}, \ldots, z_n)$. For $i \neq j$ the map $\varphi_i^{-1} \circ \varphi_j : \varphi_j^{-1}(U_i \cap U_j) \to \varphi_i^{-1}(U_i \cap U_j)$ is such that $(\varphi_i^{-1} \circ \varphi_j)^* \omega_i = \omega_j$. Let us make this computation more explicit. We have, supposing without loss of generality $i < j$,

$$\varphi_i^{-1} \circ \varphi_j : (z_1, \ldots, \widehat{z_j}, \ldots, z_n) \overset{\varphi_j}{\mapsto} (z_1, \ldots, z_j = \zeta_j, \ldots, z_n) \overset{\varphi_i^{-1}}{\mapsto} (z_1, \ldots, \widehat{z_i}, \ldots, z_j = \zeta_j, \ldots, z_n)$$

and we can write, for some polynomial $s_{cj} = s_{cj}(z_1, \ldots, \widehat{z_j}, \ldots, z_n)$,

$$\begin{aligned}
(\varphi_i^{-1} \circ \varphi_j)^* \omega_i &= \frac{(-1)^i}{\partial_{z_i} p_n(z_1, \ldots, \widehat{z_i}, \ldots, z_j = \zeta_j(z_1, \ldots, \widehat{z_j}, \ldots, z_n), \ldots, z_n)} \\
&\qquad dz_1 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge d\zeta_j \wedge \cdots \wedge dz_n = \\
&= \frac{(-1)^i (-1)^{j-1-i}}{\partial_{z_i} p_n(z_1, \ldots, \widehat{z_i}, \ldots, z_j = \zeta_j(z_1, \ldots, \widehat{z_j}, \ldots, z_n), \ldots, z_n)} \, \frac{\partial_{z_i} q_{cj} \partial_{z_j} p_n - q_{cj} \partial_{z_i} \partial_{z_j} p_n}{{\partial_{z_j}}^2 p_n} \\
&\qquad dz_1 \wedge \cdots \wedge dz_i \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n = \\
&= \frac{(-1)^j \partial_{z_j} p_n}{s_{cj}(z_1, \ldots, \widehat{z_j}, \ldots, z_n)} \, \frac{\partial_{z_i} q_{cj} \partial_{z_j} p_n - q_{cj} \partial_{z_i} \partial_{z_j} p_n}{{\partial_{z_j}}^2 p_n} \\
&\qquad dz_1 \wedge \cdots \wedge dz_i \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n = \\
&= \frac{(-1)^j}{\partial_{z_j} p_n} \, dz_1 \wedge \cdots \wedge dz_i \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n = \omega_j,
\end{aligned}$$

The above shows that the local forms (ω_i, U_i) are pairwise compatible. Glued together, they constitute a volume form ω on X_n^c. □

Now we make the following

Remark 5. The algebraic volume form ω on X_n^c (simply connected variety) we constructed in Lemma 8 is *unique* up to multiplication by a nonzero constant, i.e., according to the notations in the above proof, there exists (modulo multiplication by a nonzero constant) one and only one volume form defined on $\varphi_i^{-1}(U_i)$, namely ω_i, which extends regularly to X_n^c. In fact, let us suppose ω' is another volume form on X_n^c. Since we can write $\omega' = \mathfrak{f}\omega$, where $\mathfrak{f}$ is a non-vanishing regular function on X_n^c, our statement follows from Theorem 21 (see chap. 3).

Let ω denote the unique (up to multiplication by a nonzero constant) algebraic volume form on X_n^c we constructed above. Let us consider, for $i \neq j \in \{1, \ldots, n\}$, the following complete algebraic vector fields δ_{ij} on X_n^c, which, as an easy computation shows, are ω-divergence-free:

$$\delta_{ij} = \frac{\partial p_n}{\partial z_i} \frac{\partial}{\partial z_j} - \frac{\partial p_n}{\partial z_j} \frac{\partial}{\partial z_i}.$$

So as to prove that X_n^c for odd $n \geq 3$ has the algebraic volume density property, it suffices to show, since for each odd $n \geq 3$ we have $H_{n-2}(X_n^c) = 0$, that

$$\Theta(\mathrm{Lie}^{\omega}_{\mathrm{alg}}(X_n^c)) \supset \mathcal{B}_{n-2}(X_n^c),$$

where $\Theta : \mathrm{AVF}_{\omega}(X_n^c) \to \mathcal{Z}_{n-2}(X_n^c)$ is the isomorphism sending the ω-divergence-free algebraic vector field ξ to the closed $(n-2)$-form $\iota_\xi \omega$, defined by means of the interior product

ι of ξ and ω, $\mathrm{Lie}^{\omega}_{\mathrm{alg}}(X^c_n)$ is the Lie algebra generated by the set $\mathrm{IVF}_{\omega}(X^c_n)$ of complete ω-divergence-free algebraic vector fields on X^c_n and $\mathcal{B}_{n-2}(X^c_n)$ is the space of exact algebraic $(n-2)$-forms on X^c_n.
Now, the above inclusion does hold, as an easy proof shows. Let us go through its key steps for arbitrary $n \geq 3$ (as above, $c \in \mathbb{C} \setminus \{0\}$ for odd n and $c \in \mathbb{C} \setminus \{0,1\}$ for even n are fixed). First of all, by Grothendieck's theorem, we have $\mathcal{B}_{n-2}(X^c_n) \subset \Theta(\mathrm{AVF}_{\omega}(X^c_n))$, i.e. for every algebraic $(n-3)$-form α on X^c_n there exists $\xi \in \mathrm{AVF}_{\omega}(X^c_n)$ such that $\Theta(\xi) = \iota_{\xi}\omega = \mathrm{d}\alpha$. It suffices to show ξ can be chosen in $\mathrm{Lie}^{\omega}_{\mathrm{alg}}(X^c_n)$. Each coefficient of α is a regular function on X^c_n, thus being the sum of monomials of the kind $z^{r_1}_{i_1}$, $z^{r_1}_{i_1} z^{r_2}_{i_2}, \ldots,$ $z^{r_1}_{i_1} z^{r_2}_{i_2} \cdots z^{r_{n-1}}_{i_{n-1}}$, where $i_1 < i_2 < \cdots < i_{n-1}$ are elements of the set $\{1, 2, \ldots, n\}$. We further have that the isomorphism Λ induced by the map $\alpha \to \xi$ is such that, up to constant factors, if $i, j \neq 1$ (otherwise we obtain a similar result), $\Lambda^{-1}(\delta_{ij}) = z_1 \, dz_2 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n$, as $\mathrm{Ker}(\delta_{ij}) = \mathbb{C}[z_1, \ldots, \widehat{z_i}, \ldots, \widehat{z_j}, \ldots, z_n]$. The other summands of the kind z_{i_1} and $z^{r_1}_{i_1}$ are dealt with similarly. As regards summands of the kind $z_{i_1} z_{i_2}$, in case they were not already covered by the previous step, the Lie bracket $[\delta_{ij}, \delta_{kl}] \in \mathrm{Lie}^{\omega}_{\mathrm{alg}}(X^c_n)$, $i \neq j, k \neq l$ is sent to a $(n-3)$-form having coefficients of the kind $z_{i_1} z_{i_2}$. Monomials $z^{r_1}_{i_1} z^{r_2}_{i_2}$ are treated similarly. As for summands of the kind $z_{i_1} z_{i_2} z_{i_3}$, in case they were not already covered by the previous steps, without loss of generality we can suppose that the extra variable is not in the kernel of both vector fields δ_{ij} and δ_{kl}. Then the product of the extra variable with the vector field annihilating it, bracketed with the other vector field belongs to $\mathrm{Lie}^{\omega}_{\mathrm{alg}}(X^c_n)$ and is sent to a $(n-3)$-form having coefficients of the kind $z_{i_1} z_{i_2} z_{i_3}$. One proceeds similarly for $z^{r_1}_{i_1} z^{r_2}_{i_2} z^{r_3}_{i_3}$, for $z_{i_1} z_{i_2} \cdots z_{i_m}$ and for $z^{r_1}_{i_1} z^{r_2}_{i_2} \cdots z^{r_m}_{i_m}$, $4 \leq m \leq n-1$, by considering one extra variable at a time. Hence $\xi \in \mathrm{Lie}^{\omega}_{\mathrm{alg}}(X^c_n)$ and the following theorem holds true.

Theorem 25. *For each $n \geq 3$ the variety $X^c_n \subset \mathbb{C}^n$, where $c \in \mathbb{C} \setminus \{0\}$ if n is odd and $c \in \mathbb{C} \setminus \{0,1\}$ if n is even, is such that $\Theta(\mathrm{Lie}^{\omega}_{\mathrm{alg}}(X^c_n)) \supset \mathcal{B}_{n-2}(X^c_n)$, i.e. for every algebraic $(n-3)$-form α on X^c_n there exists $\xi \in \mathrm{Lie}^{\omega}_{\mathrm{alg}}(X^c_n)$ such that $\Theta(\xi) = \iota_{\xi}\omega = \mathrm{d}\alpha$. Hence, for odd $n \geq 3$, $\mathrm{Lie}^{\omega}_{\mathrm{alg}}(X^c_n) = \mathrm{AVF}_{\omega}(X^c_n)$, i.e. X^c_n has the algebraic volume density property.*

In order to prove the volume density property for the fibers X^c_4, $c \in \mathbb{C} \setminus \{0,1\}$, we have to find a complete ω-divergence free vector field ξ on X^c_4 such that

$$\Theta(\xi) = \alpha,$$

where $\Theta : \mathrm{AVF}_{\omega}(X^c_4) \to \mathcal{Z}_2(X^c_4)$ is the isomorphism sending the ω-divergence-free algebraic vector field ξ on X^c_4 to a closed 2-form α which is a generator of $H^2_{\mathrm{dR}}(X^c_4) \cong \mathbb{C}$. This is exactly the vector field induced by the $\mathbb{C}^*$-action on X^c_4 mentioned after Lemma 6. Thus, we have proven Theorem 23.

4.2 A singular special fibre

In the following proposition we compute the homology of the singular special fibre X_4^1.

Proposition 12. *Let* $X_4^1 = \{z_1z_2 + 1 + z_4(z_1 + z_3 + z_1z_3z_2) = 1\} \subset \mathbb{C}^4_{z_1,z_2,z_3,z_4}$ *be the affine modification, via the projection* $\sigma : X_4^1 \to \mathbb{C}^3_{z_1,z_2,z_3}$*, of* $\mathbb{C}^3$ *along the smooth divisor* $X_3^0 = \{z_1 + z_3 + z_1z_3z_2 = 0\} \subset \mathbb{C}^3$ *with centre at* $C_4^1 = \{z_1z_2 = z_1 + z_3 + z_1z_2z_3 = 0\} = \{(0, z_2, 0) : z_2 \in \mathbb{C}_{z_2}\} \cup \{(z_1, 0, -z_1) : z_1 \in \mathbb{C}_{z_1}\} \subset X_3^0$. *We have*

$$H_0(X_4^1) = \mathbb{Z},\ \ H_1(X_4^1) = 0,\ \ H_2(X_4^1) = \mathbb{Z},\ \ H_3(X_4^1) = \mathbb{Z},\ \ H_k(X_4^1) = 0 \textit{ for } k \geq 4.$$

Proof. Since X_4^1 is connected, we have $H_0(X_4^1) = \mathbb{Z}$. Let us consider the two long exact sequences associated to the couples $(\mathbb{C}^3, \mathbb{C}^3 \setminus X_3^0)$ and $(X_4^1, X_4^1 \setminus A_4^1)$, where $A_4^1 = \sigma^*(X_3^0) = \sigma^{-1}(C_4^1) = C_4^1 \times \mathbb{C}$. Thanks to Prop. 3.1 of [KZ], one concludes that $H_1(X_4^1) \cong H_1(\mathbb{C}^3) = 0$. Consider the following diagram, where we applied Thom isomorphism theorem to relative homology groups $H_*(X_4^1, X_4^1 \setminus A_4^1)$ and $H_*(\mathbb{C}^3, \mathbb{C}^3 \setminus X_3^0)$.

$$\begin{array}{ccccccccc}
\cdots \longrightarrow H_1(C_4^1)=0 & \longrightarrow & H_2(X_4^1\setminus A_4^1) & \xrightarrow{\cong} & H_2(X_4^1) & \xrightarrow{0} & H_0(C_4^1) & \xrightarrow{\cong} & H_1(X_4^1\setminus A_4^1)\\
 & & \cong\downarrow & & & & \cong\downarrow \tau^0_* & & \cong\downarrow \\
\cdots \longrightarrow H_1(X_3^0) & \longrightarrow & H_2(\mathbb{C}^3\setminus X_3^0) & \longrightarrow & H_2(\mathbb{C}^3) & \longrightarrow & H_0(X_3^0) & \xrightarrow[\cong]{} & H_1(\mathbb{C}^3\setminus X_3^0)
\end{array}$$

The upper sequence immediately yields $H_2(X_4^1) = \mathbb{Z}$. Since, denoted by e the Euler characteristic, we have $e(X_4^1) = 1 + e(C_4^1) - e(X_3^0) = e(C_4^1) = e(X_3^0) = 1$, we obtain $H_3(X_4^1) = \mathbb{Z}$, as we can have no torsion in the maximal-dimension homology group. □

4.3 The smooth special fibres

In the present section we turn our attention to the smooth special fibres $X_n^0 = \{(z_1, \dots, z_n) \in \mathbb{C}^n : \ p_n = p_{n-2} + z_n p_{n-1} = 0\}$, where $n \geq 3$. Further we will need, for $i \neq j \in \{1, \dots, n\}$, the following complete algebraic vector fields δ_{ij} on X_n^0:

$$\delta_{ij} = \frac{\partial p_n}{\partial z_i} \frac{\partial}{\partial z_j} - \frac{\partial p_n}{\partial z_j} \frac{\partial}{\partial z_i}.$$

In case $n = 3$, the following holds.

Proposition 13. *Let $X_3^0 = \{z_1 + z_3 + z_1 z_3 z_2 = 0\} \subset \mathbb{C}^3_{z_1,z_2,z_3}$ be the affine modification, via the projection $\sigma : X_3^0 \to \mathbb{C}^2_{z_1,z_3}$, of $\mathbb{C}^2$ along the divisor $\Delta = \{z_1 z_3 = 0\} \subset \mathbb{C}^2$ with centre at $C_3^0 = \{z_1 z_3 = -z_1 - z_3 = 0\} = \{(0,0)\} = \operatorname{sing} \Delta \subset \Delta$. Such a variety can be equipped with an algebraic volume form ω with respect to which it has the algebraic volume density property.*

Proof. Proceeding as in the proof of Lemma 8 in Section 4.1.4, one can easily show that the local forms

$$\omega_1 = \frac{-1}{1 + z_2 z_3}\, dz_2 \wedge dz_3, \ \ \omega_2 = \frac{1}{z_1 z_3}\, dz_1 \wedge dz_3, \ \ \omega_3 = \frac{-1}{1 + z_1 z_2}\, dz_1 \wedge dz_2$$

are pairwise compatible and constitute a volume form ω on X_3^0.
We remark that a volume form on X_3^0, which is *not* simply connected as $H_1(X_3^0) = \mathbb{Z}$, is *not* uniquely determined up to multiplication by non-zero constant factors. In fact, for $s,\ t \in \mathbb{Z}$, $\omega_2^{s,t} = z_1^s z_3^t\, dz_1 \wedge dz_3$ extends to X_3^0 if and only if $s + t = -2$. All the same, a straightforward computation shows that the vector fields δ_{ij} have $\omega_2^{s,t}$-divergence 0 if and only if $s = t = -1$, couple corresponding to our volume form ω.
So as to show that X_3^0 has the algebraic volume density property, we apply the criterion established in [KKAV] (see Section 3). First of all we prove the inclusion

$$\Theta(\mathrm{Lie}^{\omega}_{\mathrm{alg}}(X_3^0)) \supset \mathcal{B}_1(X_3^0),$$

where $\Theta : \mathrm{AVF}_\omega(X_3^0) \to \mathcal{Z}_1(X_3^0)$ is the isomorphism sending the ω-divergence-free algebraic vector field ξ to the closed 1-form $\iota_\xi \omega$, defined by means of the interior product ι of ξ and ω, $\mathrm{Lie}^{\omega}_{\mathrm{alg}}(X_3^0)$ is the Lie algebra generated by the set $\mathrm{IVF}_\omega(X_3^0)$ of complete ω-divergence-free algebraic vector fields on X_3^0 and $\mathcal{B}_1(X_3^0)$ is the space of exact algebraic 1-forms on X_3^0. To this end, we note that by Grothendieck's theorem we have $\mathcal{B}_1(X_3^0) \subset \Theta(\mathrm{AVF}_\omega(X_3^0))$, i.e. for every smooth function f on X_3^0 there exists $\xi \in \mathrm{AVF}_\omega(X_3^0)$ such that $\Theta(\xi) = \iota_\xi \omega = \mathrm{d}f$. It suffices to show ξ can be chosen in $\mathrm{Lie}^{\omega}_{\mathrm{alg}}(X_3^0)$. Each regular fuction f on X_3^0 is the sum of monomials of the kind $z_{i_1}^{r_1},\ z_{i_1}^{r_1} z_{i_2}^{r_2}$, where $i_1 < i_2$ are elements of the set $\{1, 2, 3\}$. We further have that the isomorphism Λ induced by the map $f \to \xi$ is such that

$$\Lambda^{-1}(\delta_{12}) = -z_3, \ \ \Lambda^{-1}(\delta_{13}) = z_2, \ \ \Lambda^{-1}(\delta_{23}) = -z_1.$$

Summands of the kind $z_{i_1}^{r_1}$ are dealt with similarly. The Lie bracket $[\delta_{ij}, \delta_{kl}] \in \mathrm{Lie}^{\omega}_{\mathrm{alg}}(X_3^0),\ i \neq j,\ k \neq l$ is sent to the regular function $z_{i_1} z_{i_2}$. Monomials $z_{i_1}^{r_1} z_{i_2}^{r_2}$ are treated similarly and we obtain the desired conclusion.
Finally we need to show that $\exists \xi \in \mathrm{Lie}^{\omega}_{\mathrm{alg}}(X_3^0)$ s.t. $[\Theta(\xi)]_{\mathrm{dR}}$ generates $H^1_{\mathrm{dR}}(X_3^0, \mathbb{C}) = \mathbb{C}$ as a vector space. We note that $[\alpha_2]_{\mathrm{dR}}$, where

$$\alpha_2 = \frac{1}{z_1}\, dz_1 - \frac{1}{z_3}\, dz_3$$

is a closed non-exact 1-form which extends to a closed non-exact 1-form on X_3^0, generates $H^1_{\mathrm{dR}}(X_3^0, \mathbb{C}) = \mathbb{C}$ and that $\Theta(\xi_2) = \iota_{\xi_2} \omega_2 = \alpha_2$, where

$$\xi_2 = -z_1 \frac{\partial}{\partial z_1} - z_3 \frac{\partial}{\partial z_3}$$

is a complete ω_2-divergence-free algebraic vector field on $\mathbb{C}^*_{z_1} \times \mathbb{C}^*_{z_3} \subset X_3^0$ which extends to a complete ω-divergence-free algebraic vector field on X_3^0. Therefore the criterion in [KKAV] can be applied, whence X_3^0 has the algebraic volume density property. $\square$

We are ready to prove the following

Theorem 26. *Let $X_n^0 = \{p_n = p_{n-2} + z_n p_{n-1} = 0\} \simeq \mathbb{C}^{n-1} \setminus X_{n-1}^0$, for $n \geq 4$, be the smooth hypersurface of $\mathbb{C}^n$ whose homology has been computed in the proof of Lemma 5. Such variety enjoys the AVDP with respect to the algebraic volume form $\omega_n = \frac{(-1)^n}{\partial_{z_n} p_n} dz_1 \wedge \cdots \wedge dz_{n-1} = \frac{(-1)^n}{p_{n-1}} dz_1 \wedge \cdots \wedge dz_{n-1}$.*

Proof. So as to show that X_n^0 has the algebraic volume density property, we apply the criterion established in [KKAV] (see Section 3).
First of all, by induction on $n \geq 4$, we find a generator of $H^{n-2}_{\mathrm{dR}}(X_n^0) \cong H^{n-3}_{\mathrm{dR}}(X_{n-1}^0) = \mathbb{C}$, namely we show that, for every $n \geq 4$,

$$H^{n-2}_{\mathrm{dR}}(X_n^0) = \langle [\alpha_{n-2}]_{\mathrm{dR}} \rangle, \text{ where } \alpha_{n-2} = \sum_{i=1}^{n-1} \frac{z_i}{p_{n-1}} \, dz_{n-1} \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_1. \quad (*)$$

For this we shall use the fact that the isomorphism $\rho : H^{n-3}_{\mathrm{dR}}(X_{n-1}^0) \to H^{n-2}_{\mathrm{dR}}(X_n^0)$ (inverse of the so-called *residue map*) is such that $\rho([\alpha]_{\mathrm{dR}}) = [\frac{\mathrm{d}p_{n-1}}{p_{n-1}} \wedge \alpha]_{\mathrm{dR}}$.
Suppose $n = 4$. The generator $[\alpha_1]_{\mathrm{dR}}$, where $\alpha_1 := \frac{z_2}{1+z_1 z_2} dz_1 + \frac{z_1}{1+z_1 z_2} dz_2$, of $H^1_{\mathrm{dR}}(X_3^0)$ (note that, if pulled back to the suitable chart, it corresponds to the 1-form α_2 found in the proof of the last proposition) is sent by ρ to the de Rham cohomology class of α_2, where

$$\begin{aligned} \alpha_2 := \frac{\mathrm{d}p_3}{p_3} \wedge \alpha_1 = \frac{-z_1}{(z_1 z_2 + 1)(z_1 + z_3 + z_1 z_2 z_3)} dz_2 \wedge dz_1 + \frac{z_2}{z_1 + z_3 + z_1 z_2 z_3} dz_3 \wedge dz_1 + \\ + \frac{z_1}{z_1 + z_3 + z_1 z_2 z_3} dz_3 \wedge dz_2 = \\ = \frac{z_3}{p_3} dz_2 \wedge dz_1 + \frac{z_2}{p_3} dz_3 \wedge dz_1 + \frac{z_1}{p_3} dz_3 \wedge dz_2, \end{aligned}$$

generator of $H^2_{\mathrm{dR}}(X_4^0)$, whence the base case is proved. Now, let statement $(*)$ hold true for some $n \geq 4$. Let us prove it for $n+1$. We have, as a standard computation shows,

$$\begin{aligned} \frac{\mathrm{d}p_n}{p_n} \wedge \alpha_{n-2} = \sum_{i=1}^{n-1} \frac{z_i}{p_{n-1}} \frac{\mathrm{d}p_n}{p_n} \wedge dz_{n-1} \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_1 = \frac{\sum_{i=0}^{n-2} (-1)^i z_{n-1-i} \partial_{z_{n-1-i}} p_n}{p_{n-1} p_n} dz_{n-1} \wedge \cdots \wedge dz_1 + \\ + \sum_{i=1}^{n-1} \frac{z_i}{p_n} dz_n \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_1 \overset{(**)}{=} \\ = \frac{z_n}{p_n} dz_{n-1} \wedge \cdots \wedge dz_1 + \\ + \sum_{i=1}^{n-1} \frac{z_i}{p_n} dz_n \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_1 = \sum_{i=1}^{n} \frac{z_i}{p_n} dz_n \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_1 =: \alpha_{n-1}, \end{aligned}$$

where the equality $(**)$ comes from the fact that, if $n+1$ is odd, we have

$$\sum_{i=0}^{n-2} (-1)^i z_{n-1-i} \partial_{z_{n-1-i}} p_n = z_n p_{n-1}$$

and, if $n+1$ is even, we have

$$\sum_{i=0}^{n-2} (-1)^i z_{n-1-i} \partial_{z_{n-1-i}} p_n = -p_{n-2}$$

and we change the variable $z_n = -\frac{p_{n-2}}{p_{n-1}}$, corresponding to the equation $p_n = p_{n-2}+z_n p_{n-1} = 0$. Whence $(*)$ is proved by induction.

Now, for every $n \geq 4$, let us find a complete ω_n-divergence-free vector field ξ_{n-1} on $\mathbb{C}^{n-1} \setminus X^0_{n-1} \simeq X^0_n$ such that, up to sign,

$$\Theta(\xi_{n-1}) = \alpha_{n-2},$$

where $\Theta : \mathrm{AVF}_{\omega_n}(X^0_n) \to \mathcal{Z}_{n-2}(X^0_n)$ is the isomorphism sending the ω_n-divergence-free algebraic vector field ξ on X^0_n to the closed $(n-2)$-form $\iota_\xi \omega_n$, defined by means of the interior product ι of ξ and ω_n. It is straightforward to verify that one can choose

$$\xi_{n-1} = \sum_{i=1}^{n-1} (-1)^i z_i \frac{\partial}{\partial z_i}$$

in order to obtain, up to sign, $\Theta(\xi_{n-1}) = \iota_{\xi_{n-1}}\omega_n = \alpha_{n-2}$.
So as to apply the criterion established in [KKAV] (see Section 3), one needs to show the inclusion

$$\Theta(\mathrm{Lie}^{\omega_n}_{\mathrm{alg}}(X^0_n)) \supset \mathcal{B}_{n-2}(X^0_n),$$

where $\mathrm{Lie}^{\omega_n}_{\mathrm{alg}}(X^0_n)$ is the Lie algebra generated by the set $\mathrm{IVF}_{\omega_n}(X^0_n)$ of complete ω_n-divergence-free algebraic vector fields on X^0_n and $\mathcal{B}_{n-2}(X^0_n)$ is the space of exact algebraic $(n-2)$-forms on X^0_n. To begin with, for every $n \geq 4$, by Grothendieck's theorem we have

$$\Theta(\mathrm{AVF}_{\omega_n}(X^0_n)) \supset \mathcal{B}_{n-2}(X^0_n),$$

i.e. for every algebraic $(n-3)$-form α on X^0_n there exists $\xi \in \mathrm{AVF}_{\omega_n}(X^0_n)$ such that $\Theta(\xi) = \iota_\xi \omega_n = \mathrm{d}\alpha$. It suffices to show ξ can be chosen in $\mathrm{Lie}^{\omega_n}_{\mathrm{alg}}(X^0_n)$. To this end we note that

- each coefficient of α is a regular function on X^0_n, thus being the sum of monomials of the kind $z^{r_1}_{i_1}$, $z^{r_1}_{i_1} z^{r_2}_{i_2}, \ldots,$ $z^{r_1}_{i_1} z^{r_2}_{i_2} \cdots z^{r_{n-1}}_{i_{n-1}}$, where $i_1 < i_2 < \cdots < i_{n-1}$ are elements of the set $\{1, 2, \ldots, n\}$;

- the isomorphism Λ induced by the map $\alpha \to \xi$ is such that, up to constant factors, if $i, j \neq 1$ (otherwise the result is similar),

$$\Lambda^{-1}(\delta_{ij}) = z_1\, dz_2 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n,$$

  as $\mathrm{Ker}(\delta_{ij}) = \mathbb{C}[z_1, \ldots, \widehat{z_i}, \ldots, \widehat{z_j}, \ldots, z_n]$ (the other summands of the kind z_{i_1} and $z^{r_1}_{i_1}$ are treated similarly);

- as regards summands of the kind $z_{i_1} z_{i_2}$, in case they were not already covered by the previous step, the Lie bracket $[\delta_{ij}, \delta_{kl}] \in \mathrm{Lie}^{\omega_n}_{\mathrm{alg}}(X^0_n)$, $i \neq j$, $k \neq l$ is sent by Λ^{-1} to a $(n-3)$-form having coefficients of the kind $z_{i_1} z_{i_2}$ (monomials $z^{r_1}_{i_1} z^{r_2}_{i_2}$ are treated similarly);

- regarding summands of the kind $z_{i_1} z_{i_2} z_{i_3}$, in case they were not already covered by the previous steps, without loss of generality we can suppose that the extra variable is not in the kernel of both vector fields δ_{ij} and δ_{kl}. Then the product of the extra variable with the vector field annihilating it, bracketed with the other vector field belongs to $\mathrm{Lie}^{\omega_n}_{\mathrm{alg}}(X^0_n)$ and is sent by Λ^{-1} to a $(n-3)$-form having coefficients of the kind $z_{i_1} z_{i_2} z_{i_3}$ (one proceeds similarly for $z^{r_1}_{i_1} z^{r_2}_{i_2} z^{r_3}_{i_3}$, for $z_{i_1} z_{i_2} \cdots z_{i_m}$ and for $z^{r_1}_{i_1} z^{r_2}_{i_2} \cdots z^{r_m}_{i_m}$, $4 \leq m \leq n-1$, by considering one extra variable at a time).

Therefore $\xi \in \mathrm{Lie}^{\omega_n}_{\mathrm{alg}}(X^0_n)$ and, thanks to the criterion established in sect. 3 of [KKAV], we conclude that, for every $n \geq 4$, the variety X^0_n enjoys the (algebraic) volume density property. □

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Erklärung

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Leiter der Arbeit:	Prof. Dr. Frank Kutzschebauch

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