# One-Variable Fragments of First-Order Many-Valued Logics

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vorgelegt von

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# Contents

# Introduction

This thesis is concerned with the study of one-variable fragments of first-order logics, in particular, Łukasiewicz, Abelian, and intermediate logics, and their connection to many-valued modal logics. In this introduction, we discuss our motivations for studying these fragments and give an overview of the chapters to come.

## Propositional Many-Valued Logics

Classical logic is concerned with reasoning about statements that are either true or false. It deals with propositions such as "5 is a prime number" or "I am 28 years old". It is natural to wonder why we would not consider more than these two traditional truth values 0 (falsity) and 1 (truth). Indeed, there exist plenty of examples in natural language of propositions that are not necessarily true or false: saying that "Olfe is tall" might be 'more true' than "Féluna is tall", but there might be someone even taller. And what does 'more true' mean in this case? Such examples, and many others like them, motivate extending the scope of classical logic by considering a set of truth values that is larger than the usual $\{0, 1\}$. This new set can be finite or infinite and, in most cases, will bear some order structure, making it a poset, a lattice, or a chain. These logics are commonly collected together under the umbrella term *many-valued logics*.

A formal mathematical study of such logics originates with the work of Łukasiewicz and Post in the 1920s [97, 129]. The former introduced a logic, denoted by $Ł_3$, that has three truth values: "false", "true", and an additional value "undetermined". He proposed this third value to deal with future contingents like "It will rain tomorrow". Such an approach to future contingents is generally considered to be unsuccessful,[1] but Łukasiewicz's ideas provided a basis for the mathematical study of many-valued logics. Further examples of finite-valued logics may be found in the works of Post [129], Bochvar [30], Kleene [90, 91], Belnap [18], and Dunn [64].

Of the interesting infinite-valued logics, we would like to point out three in particular that have a prominent place in this thesis. Firstly, Łukasiewicz himself generalized his three-valued logic $Ł_3$ to first an $n$-valued logic $Ł_n$ for any $n \geq 2$, and then to a logic $Ł$ with truth values in the real unit interval $[0, 1]$ [98]. We refer to $Ł$ as *Łukasiewicz logic*. Secondly, in [74] Gödel defined implicitly a family of $n$-valued logics, usually denoted by $G_n$ ($n \geq 2$), which was extended to an infinite-valued logic, denoted by $G$, by Dummett [63]. We refer to $G$ as *Gödel–Dummett logic*, or simply *Gödel logic*. Although Dummett defined $G$ with truth values in $\omega^+$, it can also be viewed as a logic with truth

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[1]Nowadays, such future contingents are dealt with, for example, using temporal logic, see, e.g., [121] for details.

values in $[0, 1]$; in fact, we could take any infinite closed subset of $[0, 1]$ that contains both 0 and 1 as the set of truth values.[2] This idea of using the real unit interval as a set of truth values has since been extended and extensively studied by Hájek. With his work on Basic logic BL as the logic of continuous t-norms and their residuals, he was one of the main initiators of and contributors to the field of *mathematical fuzzy logic*, that studies truth-functional logics with truth values in $[0, 1]$ [76].[3] His work has lead to numerous generalizations, such as the study of logics of left-continuous t-norms [68, 88] and of left-continuous uninorms [104].

A third many-valued logic with an infinite set of truth values that we consider in this thesis is *Abelian logic*, denoted by A. It was introduced independently by Meyer and Slaney as a relevance logic [109] and by Casari as a comparative logic [43]. For Meyer and Slaney, it had the set of the integers $\mathbb{Z}$ as truth values, whereas Casari considered it the logic of *lattice-ordered abelian groups*, that is, abelian groups with an underlying lattice structure, where the group addition distributes over the lattice operations. Since the ordered additive groups of the integers and that of the real numbers both generate the variety of lattice-ordered abelian groups, we can consider Abelian logic A to be the logic of the reals. There exists a deep connection between Abelian and Łukasiewicz logic, already mentioned by Meyer and Slaney, and explored further in, e.g., [107].

## First-Order Many-Valued Logics

Let us now consider first-order extensions of many-valued logics. In first-order classical logic, the quantifiers $(\forall x)$ and $(\exists x)$ are interpreted as "for all" and "there exists", respectively. To generalize this to a many-valued setting, we adopt the Mostowski–Rasiowa–Hájek tradition (see [55, 76, 118, 135]). That is, if the set of truth values admits a lattice structure, we assign as a truth value to formulas $(\forall x)\alpha(x)$ and $(\exists x)\alpha(x)$ the infimum and supremum, respectively, of all relevant truth values of $\alpha(x)$. Let us consider three first-order many-valued logics that will run like a thread through this thesis.

**First-Order Łukasiewicz Logic**    The first-order extension of Łukasiewicz logic has been extensively studied. Unfortunately, the set of its valid formulas turned out to be not recursively enumerable [139]; in fact, its validity problem is $\Pi_2$-complete [134]. It is this validity of logics that is of primary interest to us; if we speak of the complexity of a logic, we mean the complexity of its validity problem. Despite its undecidability, various proof systems for first-order Łukasiewicz logic have been provided [9, 16, 17, 76, 82], all of which include some rule with infinitely many premises. Skolemization and an (approximate) Herbrand theorem have been proved [9], and various better-behaved fragments have been studied. For instance, of its monadic fragment, where only unary predicates are considered, satisfiability is known to be $\Pi_1$-complete [134], and validity was shown to be undecidable by Bou in unpublished work. The complexity of the latter problem still remains open. Hájek investigated a decidable fragment corresponding to a fuzzy description logic in [77]. The one-variable fragment, consisting of formulas containing only a single variable, was extensively studied by Rutledge in his PhD thesis, who gave

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[2]This is true if one considers only the valid formulas of (propositional) G, as we do here. If one considers consequences, or first-order extensions, the situation is more complex, as we will see.

[3]Note that we consider here the so-called "narrow" view on fuzzy logic. The broader sense would include fuzzy set theory, which is outside the scope of this thesis.

an axiomatization, proved completeness, and showed decidability of both its validity and satisfiability problem [138]. We return to such fragments shortly.

**First-Order Abelian Logic**   A first-order extension of Abelian logic has (as far as we know) not been considered yet in the literature. We argue however that it is interesting for a variety of reasons: its semantics is based on structures studied in both algebra and computer science, that is, lattice-ordered abelian groups; there exists a natural separation between the group and lattice fragments of the logic; and its language is rich enough to interpret other logics. In particular, we can interpret first-order Łukasiewicz logic in first-order Abelian logic. In [107], Metcalfe et al. give an intuitive interpretation of propositional Łukasiewicz logic into propositional Abelian logic. We extend this interpretation to their first-order counterparts in Section 1.2. As first-order Łukasiewicz logic is not recursively enumerable, it immediately follows that first-order Abelian logic is not recursively enumerable either. Chapter 4 of this thesis is a first investigation into first-order Abelian logic, and some fragments in particular.

**First-Order Gödel Logic**   Unlike first-order Łukasiewicz and Abelian logic, first-order Gödel logic defined over $[0, 1]$ is recursively enumerable. Indeed, Horn provided a recursive axiomatization in [86]. As we noted for propositional Gödel logic, we can consider any closed subset $A$ of $[0, 1]$ that contains 0 and 1 as the set of truth values; such a set $A$ is called a *Gödel set*. However, as opposed to propositional Gödel logic, the first-order Gödel logics of two different infinite Gödel sets do not necessarily coincide. A full classification of all such first-order Gödel logics, in terms of recursive enumerability, is given by Baaz et al. in [11], where they provide axiomatizations for those first-order Gödel logics that are recursively enumerable. An important factor here is that (first-order) Gödel logic is "order-based", that is, only the order type of the set of truth values matters and not the individual distance between any two values. In that sense, Gödel logic differs from Łukasiewicz and Abelian logic.

Gödel logic moreover contrasts with Łukasiewicz and Abelian logic in another significant way: it is an *intermediate* (or *super-intuitionistic*) logic, that is, it lies between intuitionistic logic and classical logic. Such intermediate logics, both propositional and first-order, form an extensive area of study, see, for instance, [46] for an introduction. They are primarily studied semantically, usually either via an algebraic semantics or via intuitionistic Kripke models.[4] The latter were introduced by Kripke in [94] and consist of a set of worlds equipped with a binary relation, in this case a partial order. Intuitively, formulas are then interpreted locally (at a particular world) as in classical logic, whereas the interpretation of the implication and possible quantifiers depends on all worlds accessible according to the relation. As is to be expected, the situation for first-order intermediate logics is much more intricate than that of their propositional counterparts. Nevertheless, a plethora of completeness, non-completeness and other results have been obtained in the first-order case, see, e.g., $[14, 50, 110, 122, 126, 143]$. In particular, it follows from completeness results by Minari [110], Takano [144], and Horn that first-order Gödel logic is the logic determined by all linearly ordered intuitionistic Kripke frames with so-called constant domains. This connection between first-order

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[4]See [25] for an extensive survey on the different types of semantics for propositional intermediate logics.

Gödel logic and first-order intuitionistic logic is generalized further by Beckmann and Preining in [13], where they match each first-order Gödel logic with truth values in a Gödel set $A$ to a first-order intermediate logic defined over a particular (countable) linearly ordered intuitionistic Kripke frame with constant domains, and vice versa.

## One-Variable Fragments

Although the leap to the first-order setting greatly increases expressivity, it comes with a number of disadvantages, one of which is lack of decidability. The validity problem of first-order classical logic is undecidable by the famous result of Church [49]. That is, no effective algorithm can decide whether a formula is valid in first-order classical logic. Similarly, first-order Łukasiewicz, Gödel and Abelian logic are all undecidable. This is in contrast with their propositional counterparts, whose validity problems are decidable.

One way to overcome this lack of decidability while preserving some of the expressive power of first-order logic is to consider only fragments of the first-order logic, that is, consider only formulas that are of a particular form. We have already seen some examples related to first-order Łukasiewicz logic. For instance, recall that the *monadic fragment* concerns those formulas that contain only unary predicates. Other examples include *prenex fragments*, consisting of formulas of the form $(Qx_1)\dots(Qx_n)\alpha$, where $\alpha$ does not contain any quantifiers and $(Qx_1)\dots(Qx_n)$ is some fixed sequence of quantifiers, or *guarded fragments*, where the type of quantification is restricted. In this thesis, we focus our attention on the fragments where the number of variables that occur in a formula is restricted. To obtain a decidable fragment, the maximum number of variables to consider is rather small: the two-variable fragment of first-order classical logic is decidable [114], but its three-variable fragment is not [142]. For first-order intuitionistic logic, the two-variable fragment is already undecidable [93], whereas its one-variable fragment is decidable [36]. For first-order Łukasiewicz and Gödel logic, the one-variable fragments were proved to be decidable in [138] and [38], respectively; for the two-variable fragments of either logic, decidability remains an interesting open problem. A decidable fragment of first-order Łukasiewicz logic corresponding to a fuzzy description logic was studied in [77].

For a one-variable fragment, it suffices to consider only unary predicates, and they can hence be viewed as particular monadic fragments. We can therefore study one-variable fragments under a different guise: unary predicates $P(x)$ can be viewed as propositional variables $p$, and quantifiers $(\forall x)$ and $(\exists x)$ can be replaced with unary operators $\square$ and $\lozenge$, respectively. This allows for a study of the one-variable fragment as a particular *modal logic*. For example, the one-variable fragment of first-order classical logic corresponds to the well-known modal logic $\mathsf{S5}$, as first axiomatized by Wajsberg in [152], and the one-variable fragment of first-order intuitionistic logic corresponds to the modal logic $\mathsf{MIPC}$, as shown by Bull [36]. A great advantage of this notational switch is that we can apply the well-developed theory of modal logic. This extensive area of research, instigated by Lewis in [96], studies *modalities*, that is, operators that express, e.g., obligation, belief, or knowledge (for an introduction to modal logic, see, e.g., [27]). It became particularly popular with the introduction of modal Kripke semantics in the late 1950s and early 1960s. As with intuitionistic Kripke semantics, modal Kripke semantics is concerned with a set of possible worlds and a binary relation, but this binary relation need not be a partial order. Moreover, only the interpretation of the modalities depends on the

accessible worlds; any other connectives are interpreted locally.

Many-valued generalizations of this modal Kripke semantics have also been considered. Rather than locally interpreting formulas using classical logic, we can equip each world with a many-valued interpretation, and even generalize the binary relation to be many-valued. Such many-valued Kripke semantics have been used to study, e.g., modal extensions of Łukasiewicz logic [34, 76, 80, 100], Abelian logic [61], and Gödel logic [38, 41, 42, 106]. When studying some one-variable fragment in its modal guise, the modalities will satisfy certain (translations of) quantifier laws. In such cases, the accessibility relation of the associated modal Kripke semantics is always a (possibly many-valued) equivalence relation. Indeed, the modal variants $\mathsf{S5}(\mathbf{Ł})^{\mathsf{C}}$ and $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$ of the one-variable fragments of first-order Łukasiewicz logic and of first-order Gödel logic, respectively, have such a many-valued Kripke semantics; in fact, in both these cases the accessibility relation is *crisp*, that is, two-valued.

This interplay between one-variable fragments and many-valued modal logics raises a number of interesting questions. One could ask, given the one-variable fragment of some first-order logic, to which modal logic it corresponds, and possibly which (many-valued) Kripke semantics this modal logic has. Conversely, one could consider a many-valued modal logic defined by some many-valued Kripke semantics with a (possibly many-valued) equivalence relation, and ask to which one-variable fragment it corresponds. We try to answer some of these questions in this thesis, while simultaneously answering questions about completeness, decidability, and complexity for such one-variable fragments and many-valued modal logics.

We focus on three classes of first-order logics in particular. In Chapter 3, we consider particular first-order intermediate logics, and their connection to modal Gödel logics. In Chapter 4, we consider first-order Abelian logic and its one-variable fragment. In Chapter 2, we take a much more general perspective and launch an investigation into algebraic semantics of one-variable fragments. Let us now provide some more detail for each of these chapters.

**First-Order Intermediate Logics**   We study first-order intermediate logics defined over particular classes of intuitionistic Kripke frames. Recall that the first-order intermediate logic defined over all linearly ordered intuitionistic Kripke frames with constant domains coincides with first-order Gödel logic. In particular, their respective one-variable fragments coincide. The modal Gödel logic $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$ that corresponds to these one-variable fragments was axiomatized and studied in [42, 78].

The first-order intermediate logic defined over all linearly ordered intuitionistic Kripke frames (without assuming constant domains) was axiomatized by Corsi in [56]. A first axiomatization of its one-variable fragment is provided in this thesis. To do so, we match this fragment to the modal Gödel logic $\mathsf{S5}(\mathbf{G})$ defined over a class of many-valued Kripke frames based on the standard Gödel logic over $[0, 1]$, as considered in [42]. We then extend Beckmann and Preining's result from [13] by matching each one-variable fragment of the first-order intermediate logic defined over a particular (countable) linearly ordered intuitionistic Kripke frame to a modal Gödel logic $\mathsf{S5}(\mathbf{A})$ with truth values in some Gödel set $A$. These modal Gödel logics, as opposed to $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$, have a non-crisp accessibility relation. However, we prove that for any Gödel set $A$, we can interpret $\mathsf{S5}(\mathbf{A})$ in the corresponding modal logic defined by modal Kripke frames with a crisp accessibility relation. For these modal Gödel logics, we are able to prove a finite model property

with respect to an alternative "relativized" semantics. This leads to decidability and complexity results for a large class of these modal Gödel logics and, consequently, for a large class of one-variable fragments of first-order intermediate logics.

**First-Order Abelian Logic**   In Chapter 4, we launch an investigation into first-order Abelian logic and its one-variable fragment. This one-variable fragment is matched to a modal Abelian logic $\mathsf{S5}(\mathbf{R})^{\mathsf{C}}$. We then make use of the natural separation between the lattice and group fragments of Abelian logic and study the group fragment of the modal logic $\mathsf{S5}(\mathbf{R})^{\mathsf{C}}$. Using a partial Herbrand theorem for the first-order Abelian logic and a linear programming argument, we prove completeness for this group fragment. In fact, this partial Herbrand theorem can also be used to establish decidability of $\mathsf{S5}(\mathbf{R})^{\mathsf{C}}$, and hence of the one-variable fragment of first-order Abelian logic. We then prove completeness for the full logic $\mathsf{S5}(\mathbf{R})^{\mathsf{C}}$, via algebraic means. Recall that the one-variable fragment of first-order Łukasiewicz logic was axiomatized by Rutledge in [138]. His methods were also algebraic in nature, defining a variety of so-called monadic MV-algebras (monadic Chang algebras in Rutledge's terminology) as the algebraic semantics for the one-variable fragment. We define a variety of monadic abelian $\ell$-groups, and show that they form the algebraic semantics for the one-variable fragment of first-order Abelian logic. Our methods here are based on an alternative to Rutledge's completeness proof from [45].

## An Algebraic Perspective

Rutledge's completeness proof points out an additional advantage of studying one-variable fragments: they allow for an algebraic semantics. The algebraic study of one-variable fragments can be traced back to Halmos, who defined monadic Boolean algebras as the algebraic semantics for the one-variable fragment of first-order classical logic [79]. Various generalizations of such algebras have since been considered in the literature, usually under the name "monadic". Rutledge's monadic MV-algebras form such an example, but other examples include the one-variable fragment of first-order intuitonistic logic, which was captured algebraically by Monteiro and Varsavsky in the form of monadic Heyting algebras [113], and monadic Gödel algebras introduced by Hájek to capture the one-variable fragment of first-order Gödel logic [78] (see also [42]).

All these algebras have more in common than the adjective "monadic"; for example, their modalities all satisfy common identities that correspond to particular quantifier laws. Chapter 2 is a first general algebraic investigation into such commonalities. In order to carry out such an investigation, we work in the rather general framework of *(first-order) substructural logics*, in particular those whose Gentzen-style proof system admit the rule of exchange. This framework allows us to capture various first-order logics, including first-order classical, intuitionistic, Łukasiewicz, Gödel, and Abelian logic. For more on substructural logics, we refer to [71]. Propositional substructural logics that admit exchange have as their algebraic semantics *commutative pointed residuated lattices* (or $\mathsf{FL}_e$-*algebras*). We define monadic $\mathsf{FL}_e$-algebras to capture the algebraic semantics for the one-variable fragments of any first-order substructural logic that admits the rule of exchange. We prove a number of interesting properties: we obtain an alternative representation in terms of "relatively complete" subalgebras, as well as a characterization of the congruences. Moreover, although completeness for the whole variety of these monadic residuated lattices remains an open problem, we obtain some

Chapter 1

Chapter 2        Chapter 3        §4.1 and §4.2

§4.3
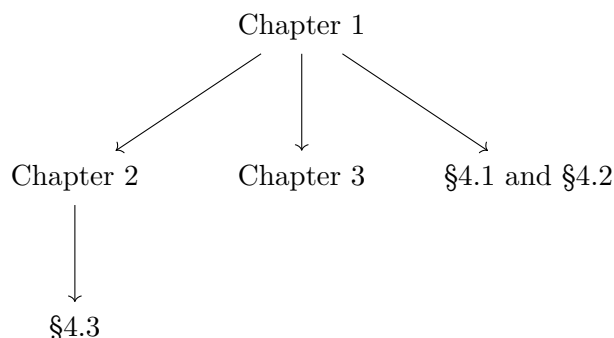
Figure 1: Dependencies between the chapters and sections of this thesis

form of completeness for some particular subvarieties, including the varieties of monadic abelian $\ell$-groups, monadic MV-algebras, and monadic Gödel algebras.

## Outline of the Thesis

Let us now give a detailed section-by-section outline of the contents of this thesis. In Figure 1, we have outlined the dependencies between the different chapters and sections.

Chapter 1 is a preliminary chapter. We define the notions necessary for the reading of this thesis, while placing them in historical context and recalling the appropriate literature. In Section 1.1, we define all the relevant propositional logics, i.e., propositional classical, intuitionistic, Łukasiewicz, Gödel, and Abelian logic. We then capture all these logics in the framework of substructural logics, using commutative pointed residuated lattices (or $\mathsf{FL}_e$-algebras). In Section 1.2, we define appropriate first-order semantics for first-order substructural logics, and recall the necessary results on first-order Łukasiewicz, Abelian, and Gödel logic. We also define intuitionistic Kripke frames to interpret first-order intermediate logics. Lastly, in Section 1.3, we focus on one-variable fragments, and their matching modal logics. We define an appropriate many-valued Kripke semantics, again based on $\mathsf{FL}_e$-algebras, and discuss the literature on one-variable fragments of first-order intermediate, Gödel, Łukasiewicz, and Abelian logic.

In Chapter 2, we give an algebraic account of one-variable fragments. In Section 2.1, we define monadic $\mathsf{FL}_e$-algebras, generalizing the notion of, in particular, monadic Boolean, Heyting, Gödel, and MV-algebras. We then show how these newly defined monadic $\mathsf{FL}_e$-algebras fit into the existing literature of the aforementioned monadic algebras. We also prove a soundness result, showing that monadic $\mathsf{FL}_e$-algebras are necessary (but not necessarily sufficient) to algebraically interpret the one-variable fragment of any first-order substructural logic. In Section 2.2, we give an alternative representation of any monadic $\mathsf{FL}_e$-algebra in terms of a particular "relatively complete" subalgebra. Section 2.3 is a study of the congruences of monadic $\mathsf{FL}_e$-algebras, where we give an equivalent characterization, and show that they are completely determined by this relatively complete subalgebra. Finally, in Section 2.4 we put these characterizations to use and show that varieties of monadic $\mathsf{FL}_e$-algebras satisfying certain conditions admit functional representations. The results of this section will be used in Chapter 4 to obtain completeness for $\mathsf{S5}(\mathbf{R})^{\mathsf{C}}$ and hence for the one-variable fragment of first-order Abelian

logic.

Chapter 3 is dedicated to the study of the one-variable fragments of first-order intermediate logics defined over linearly ordered intuitionistic Kripke frames, and the modal Gödel logics $S5(\mathbf{A})$. In Sections 3.1 and 3.2, we extend Beckman and Preining's result from [13], matching each such one-variable fragment to a modal Gödel logic $S5(\mathbf{A})$ defined using the many-valued Kripke semantics from Chapter 1. While doing so, we solve an open problem by axiomatizing the one-variable fragment of the first-order intermediate logic defined over all linearly ordered intuitionistic Kripke models. The rest of the chapter is dedicated to these modal Gödel logics $S5(\mathbf{A})$. In Section 3.3, we give an interpretation of each $S5(\mathbf{A})$ in the corresponding modal Gödel logic $S5(\mathbf{A})^{\mathsf{C}}$ defined over a crisp many-valued Kripke semantics. Section 3.4 establishes a finite model property for these crisp logics $S5(\mathbf{A})^{\mathsf{C}}$ using an alternative semantics. Finally, in Section 3.5, we put this finite model property to use and establish decidability and complexity for a large class of these modal Gödel logics $S5(\mathbf{A})^{\mathsf{C}}$.

In Chapter 4, we study first-order Abelian logic and its one-variable fragment, or rather its modal equivalent $S5(\mathbf{R})^{\mathsf{C}}$. We first prove a partial Herbrand theorem in Section 4.1 and use this to prove decidability of $S5(\mathbf{R})^{\mathsf{C}}$. We also prove a finite model property. In Section 4.2, the multiplicative fragment of $S5(\mathbf{R})^{\mathsf{C}}$ is investigated, that is, the fragment not containing the lattice connectives. A completeness theorem is proved using the partial Herbrand theorem, a normal form theorem, and linear programming methods. In Section 4.3, we give an axiomatization for $S5(\mathbf{R})^{\mathsf{C}}$ and show completeness. This completeness proof is algebraic in nature and makes extensive use of the results on monadic $\mathsf{FL}_e$-algebras obtained in Chapter 2.

## Sources for the Thesis

A majority of the results presented in this thesis have been obtained in collaboration with other researchers:

- Chapter 1 is written for the purpose of this thesis;

- Chapter 2 is independent work that has not appeared in print;

- Chapter 3 is based on the papers [39,40], joint with Xavier Caicedo, George Metcalfe, and Ricardo Rodríguez;

- Chapter 4 is based on the paper [108], joint with George Metcalfe.

This thesis is not meant to be self-contained. We assume some familiarity with basic concepts from universal algebra; all the needed standard definitions and results can be found in, e.g., [19,37].

# CHAPTER 1

## The Logics

In this thesis, we consider a plethora of logics, defined over several different languages. This chapter serves as a preliminary chapter. We introduce all these logics, together with the definitions and methods used in the following chapters, and recall the necessary results from the literature. In Section 1.1, we define our notion of a logic and a proof system, and go on to introduce the logics that are most prominent in this thesis: intuitionistic logic, Gödel logic, Łukasiewicz logic, and Abelian logic. We moreover introduce the general framework of substructural logics, defined over $\mathsf{FL}_e$-algebras (or commutative pointed residuated lattices). It is then briefly shown that all mentioned propositional logics can be interpreted using this framework. In Section 1.2, we define a general semantics for first-order substructural logics, using $\mathsf{FL}_e$-algebras, and recount relevant results from the literature for first-order Gödel, Łukasiewicz and Abelian logic. We in particular prove that first-order Łukasiewicz logic can be interpreted in first-order Abelian logic. Additionally, we define an intuitionistic Kripke semantics to interpret first-order intuitionistic logic and recall some of the literature on this subject. Finally, in Section 1.3, we introduce the main topic of this thesis: the interaction between (many-valued) modal logics and one-variable fragments of first-order logics. Such a one-variable fragment of a first-order logic, concerned with formulas that contain only a single variable $x$, can be viewed as a modal logic if we replace the quantifiers $(\forall x)$ and $(\exists x)$ with modalities $\Box$ and $\Diamond$, respectively, and vice versa. To interpret the modal logics, we define many-valued modal Kripke models with truth values in an arbitrary $\mathsf{FL}_e$-algebra and where the accessibility relation is a many-valued equivalence relation. We recall some of the literature on such many-valued Kripke semantics and many-valued modal logics, and prove a number of properties. We conclude the chapter by matching the one-variable fragments of some of the discussed first-order logics to appropriate (many-valued) modal logics.

## 1.1 Propositional Logics

A *(propositional) language* is a set $\mathcal{L}$ of function symbols, referred to as *(propositional) connectives*, such that to each $\star \in \mathcal{L}$ a non-negative integer is associated, which we call the *arity* of $\star$. If the arity of an $\star \in \mathcal{L}$ is $n$, we say that $\star$ is an *n-ary connective*. We let $\mathrm{Fm}(\mathcal{L})$ denote the set of *propositional formulas* $\varphi, \psi, \ldots$ built inductively over a countable set $\{p_i\}_{i \in \mathbb{N}}$ of propositional variables using the connectives in $\mathcal{L}$, where each $n$-ary $\star \in \mathcal{L}$

takes $n$ arguments. The *length* of a formula $\varphi \in \mathrm{Fm}(\mathcal{L})$ is the number of connectives occurring in $\varphi$ that have non-zero arity. An *algebra for a language* $\mathcal{L}$ is a set $A$ together with functions $\star^{\mathbf{A}} \colon A^n \to A$ for each $n$-ary $\star \in \mathcal{L}$, denoted by $\mathbf{A}$. If the algebra is clear from the context, we write $\star$ for $\star^{\mathbf{A}}$.

A *(logical) matrix* for a language $\mathcal{L}$ is a pair $\mathrm{M} = \langle \mathbf{A}, D \rangle$ containing an algebra $\mathbf{A}$ of language $\mathcal{L}$ and a subset $D \subseteq A$, called a *filter*, whose elements are called *designated values*. Elements of $\mathbf{A}$ will often be referred to as *truth values*. A *logic* $\mathsf{L}$ is a pair $\langle \mathcal{L}, \mathcal{K} \rangle$ consisting of a language $\mathcal{L}$ and a class $\mathcal{K}$ of matrices for $\mathcal{L}$. If $\mathcal{K}$ contains only a single matrix $\mathrm{M}$, we write simply $\langle \mathcal{L}, \mathrm{M} \rangle$ for $\langle \mathcal{L}, \{\mathrm{M}\} \rangle$. Given a logic $\mathsf{L} = \langle \mathcal{L}, \mathcal{K} \rangle$, an $\mathsf{L}$-*valuation* for some $\langle \mathbf{A}, D \rangle \in \mathcal{K}$ is a map $V \colon \{p_i\}_{i \in \mathbb{N}} \to A$ that is extended to a map $\bar{V} \colon \mathrm{Fm}(\mathcal{L}) \to A$ inductively as follows for all n-ary $\star \in \mathcal{L}$:

$$\bar{V}(p_i) = V(p_i)$$
$$\bar{V}(\star(\varphi_1, \ldots, \varphi_n)) = \star^{\mathbf{A}}(\bar{V}(\varphi_1), \ldots, \bar{V}(\varphi_n)).$$

For $\Sigma \cup \{\varphi\} \subseteq \mathrm{Fm}(\mathcal{L})$, we say that $\varphi$ is an $\mathsf{L}$-*consequence* of $\Sigma$, denoted by $\Sigma \models_{\mathsf{L}} \varphi$, if for all $\mathsf{L}$-valuations $V$ for matrices $\langle \mathbf{A}, D \rangle \in \mathcal{K}$, whenever $\bar{V}(\psi) \in D$ for all $\psi \in \Sigma$, then $\bar{V}(\varphi) \in D$. If $\emptyset \models_{\mathsf{L}} \varphi$, written simply $\models_{\mathsf{L}} \varphi$, we say that $\varphi$ is $\mathsf{L}$-*valid*. In this thesis, we are interested in validity of formulas, not in consequences. We will therefore often identify a logic $\mathsf{L}$ with the set of $\mathsf{L}$-valid formulas.

Since some logics might be different simply due to their representation, we also introduce a notion of equivalence for logics. Consider two logics $\mathsf{L}_1 = \langle \mathrm{Fm}(\mathcal{L}_1), \mathcal{K}_1 \rangle$ and $\mathsf{L}_2 = \langle \mathrm{Fm}(\mathcal{L}_2), \mathcal{K}_2 \rangle$. A map $\rho \colon \mathrm{Fm}(\mathcal{L}_1) \to \mathrm{Fm}(\mathcal{L}_2)$ is called *grammatical* if for each $n$-ary connective $\star \in \mathcal{L}_1$, there exists $\varphi_\star(p_1, \ldots, p_n) \in \mathrm{Fm}(\mathcal{L}_2)$ using propositional variables among $p_1, \ldots, p_n$ such that for all $\varphi_1, \ldots, \varphi_n$,

$$\rho(\star(\varphi_1, \ldots, \varphi_n)) \models_{\mathsf{L}_2} \varphi_\star(\rho(\varphi_1), \ldots, \rho(\varphi_n))$$
$$\varphi_\star(\rho(\varphi_1), \ldots, \rho(\varphi_n)) \models_{\mathsf{L}_2} \rho(\star(\varphi_1, \ldots, \varphi_n)).$$

Here, $\varphi_\star(\varphi_1, \ldots, \varphi_n)$ denotes the formula where each propositional variable $p_i$ occurring in $\varphi(p_1, \ldots, p_n)$ is simultaneously replaced by $\varphi_i$. Typically, these formulas $\varphi_\star$ will be defined inductively. We then say that $\mathsf{L}_1$ and $\mathsf{L}_2$ are *validity-equivalent*, denoted by $\mathsf{L}_1 \sim \mathsf{L}_2$, if there exist grammatical maps $\rho \colon \mathrm{Fm}(\mathcal{L}_1) \to \mathrm{Fm}(\mathcal{L}_2)$ and $\tau \colon \mathrm{Fm}(\mathcal{L}_2) \to \mathrm{Fm}(\mathcal{L}_1)$ such that for all $\varphi_1 \in \mathrm{Fm}(\mathcal{L}_1)$, $\varphi_2 \in \mathrm{Fm}(\mathcal{L}_2)$,

$$\models_{\mathsf{L}_1} \varphi_1 \iff \models_{\mathsf{L}_2} \rho(\varphi_1) \qquad \text{and} \qquad \models_{\mathsf{L}_2} \varphi_2 \iff \models_{\mathsf{L}_2} \rho(\tau(\varphi_2)).$$

Such maps $\rho$ and $\tau$ will be referred to as *translators*. Note that it follows from these two conditions that for all $\varphi_2 \in \mathrm{Fm}(\mathcal{L}_2)$, $\models_{\mathsf{L}_2} \varphi_2$ if and only if $\models_{\mathsf{L}_1} \tau(\varphi_2)$, and for all $\varphi_1 \in \mathrm{Fm}(\mathcal{L}_1)$, $\models_{\mathsf{L}_1} \varphi_1$ if and only if $\models_{\mathsf{L}_1} \tau(\rho(\varphi_1))$.[1]

A different approach to logic and validity comes in the form of *proof systems*. We present the definitions as in [103]. Let $\Sigma$ be some set of *structures*; in this thesis, such structures are primarily formulas: propositional formulas, but also modal formulas or first-order formulas as discussed later in this chapter. They can also be *sequents*, that is, ordered pairs consisting of finite multisets of formulas, or even more complicated structures. A *finitary inference*, or simply *inference*, is a pair $\langle \{\sigma_1, \ldots, \sigma_n\}, \sigma \rangle$ consisting

---

[1]For more on such translations between logics, see, e.g., [67, 116]. This notion is also closely related to that of equivalence of consequence relations from Abstract Algebraic Logic, see, e.g., [72].

of a finite (possibly empty) set $\{\sigma_1, \ldots, \sigma_n\} \subseteq \Sigma$ of *premises* and a single *conclusion* $\sigma \in \Sigma$. We can write such an inference as $\sigma_1, \ldots, \sigma_n/\sigma$ or

$$\frac{\sigma_1, \ldots, \sigma_n}{\sigma}$$

An inference with an empty set of premises is called an *axiom*, and is usually written without the premises, that is, $\sigma$ instead of $\emptyset/\sigma$. A *(finitary) inference rule* $r$ for $\Sigma$ (*rule* for short) is a set of inferences for $\Sigma$, called *instances* of $r$. A *(finitary) proof system* $\mathcal{C}$ is now a pair $\langle \Sigma, R \rangle$ consisting of a set of structures $\Sigma$ and set of rules $R$ for $\Sigma$. For another set of rules $R'$ for $\Sigma$, we write $\mathcal{C} + R'$ to denote the proof system $\langle \Sigma, R \cup R' \rangle$. If $\Sigma$ is a set of formulas, $\mathcal{C}$ is called a *finitary Hilbert-style system*, which will be referred to simply as a *Hilbert-style system*. For such a Hilbert-style system, inference rules $r$ are typically written schematically as a *formula schema*, that is, a rule containing placeholder variables to be uniformly replaced by arbitrary members of $\Sigma$. If $r$ is an axiom, we speak of an *axiom schema*. If $\Sigma$ consists of sequents, we call $\mathcal{C}$ a *sequent-style system*.

Given a proof system $\mathcal{C} = \langle \Sigma, R \rangle$, a *proof* of some $\sigma \in \Sigma$ in $\mathcal{C}$ is a finite sequence $\sigma_1, \ldots, \sigma_n$ of elements of $\Sigma$ such that $\sigma_n = \sigma$ and each $\sigma_i$ is either an axiom of $\mathcal{C}$ or is derived from previous members of the sequence by a rule $r \in R$, that is, $\sigma_i$ is the conclusion of $r$ and all premises of $r$ occur in the sequence prior to $\sigma_i$. If a proof exists for a $\sigma \in S$ in $\mathcal{C}$, we say that $\sigma$ is $\mathcal{C}$-*derivable* and write $\vdash_{\mathcal{C}} \sigma$.

Establishing a connection between a logic $\mathsf{L}$ and some proof system $\mathcal{C}$ is an important problem. A Hilbert-style system $\mathcal{C} = \langle \mathrm{Fm}(\mathcal{L}), R \rangle$ is known as an *axiomatization* for a logic $\mathsf{L} = \langle \mathcal{L}, M \rangle$ if for all $\varphi \in \mathrm{Fm}(\mathcal{L})$,

$$\vdash_{\mathcal{C}} \varphi \quad \Longleftrightarrow \quad \models_{\mathsf{L}} \varphi.$$

The left-to-right direction of this equivalence is often referred to as *soundness* or *correctness*, whereas the other direction is called *completeness*.[2]

Using the connection between a logic $\mathsf{L}$ and its (sound and complete) proof system $\mathcal{C}$, it is possible to establish results about the logic. For instance, to determine whether a formula is $\mathsf{L}$-valid, it suffices to establish whether or not some derivation exists in $\mathcal{C}$. In some cases, it can be decided whether or not a given formula is $\mathcal{C}$-derivable or not. This establishes the decidability of the *validity problem* in the corresponding logic, that is, the problem of determining whether a formula $\varphi$ is $\mathsf{L}$-valid or not. A closer inspection of the proof system can even lead to bounds on the complexity of this validity problem. For an introduction to the subject of decidability and complexity, see, e.g., [2]. We will often speak of the decidability of a logic, by which we mean the decidability of the validity problem in the logic.

**Example 1.1.** Let $\mathcal{L}_{\mathsf{CL}}$ be the language consisting of binary connectives $\wedge$ and $\vee$, a unary connective $\neg$, and constants $\bot$ and $\top$. Consider the algebra

$$\mathbf{2} := \langle \{0, 1\}, \wedge^{\mathbf{2}}, \vee^{\mathbf{2}}, \neg^{\mathbf{2}}, \bot^{\mathbf{2}}, \top^{\mathbf{2}} \rangle,$$

where $\wedge^{\mathbf{2}} = \min$, $\vee^{\mathbf{2}} = \max$, $\neg^{\mathbf{2}} a = 1 - a$, $\bot^{\mathbf{2}} = 0$, and $\top^{\mathbf{2}} = 1$. We define *classical (propositional) logic* to be $\mathsf{CL} = \langle \mathcal{L}_{\mathsf{CL}}, \langle \mathbf{2}, \{1\} \rangle \rangle$. For more on classical logic, we refer to, e.g., [59, 66].

---

[2]In some literature, this is what is referred to as *weak* soundness and completeness, and the terms of soundness and completeness (also called *strong* soundness and completeness) are reserved for the equivalence of consequences. Since our interest lies with validity, our terminology suffices.

$$
\begin{aligned}
&(1) \quad \varphi \to (\psi \to \varphi) \\
&(2) \quad (\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi)) \\
&(3) \quad (\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \vee \psi) \to \chi)) \\
&(4) \quad \varphi \to (\psi \to (\varphi \wedge \psi)) \\
&(5) \quad \bot \to \varphi \\
&(6) \quad (\varphi \wedge \psi) \to \varphi \\
&(7) \quad (\varphi \wedge \psi) \to \psi \\
&(8) \quad \varphi \to (\varphi \vee \psi) \\
&(9) \quad \psi \to (\varphi \vee \psi)
\end{aligned}
$$

$$
\frac{\varphi \quad \varphi \to \psi}{\psi} \ (\mathrm{mp})
$$

Figure 1.1: Proof system $\mathcal{IPC}$

On the syntactic side, proof systems for classical logic have been widely studied. Many (equivalent) systems have been given, an overview of which can be found in [59]. We pick any Hilbert-style system axiomatizing $\mathsf{CL}$ and denote it by $\mathcal{CPC}$. Although the decidability of classical logic can be investigated using a proof system, it is easily established using truth tables, and shown to be co-NP-complete via the famous Cook-Levin Theorem (see, e.g., [2, Theorem 2.10]).

Algebraically, $\mathbf{2}$ generates the variety of Boolean algebras. A *Boolean algebra* is an algebra $\mathbf{B} = \langle B, \wedge, \vee, \neg, \bot, \top \rangle$ such that $\langle B, \wedge, \vee, \bot, \top \rangle$ is a bounded distributive lattice and $\mathbf{B}$ satisfies

$$
x \vee \neg x \approx \top \ \text{ and } \ x \wedge \neg x \approx \bot.
$$

Using the identity maps as translators, it follows that

$$
\mathsf{CL} \sim \langle \mathcal{L}_{\mathsf{CL}}, \{\langle \mathbf{B}, \{\top^{\mathbf{B}}\}\rangle \mid \mathbf{B} \text{ a Boolean algebra}\}\rangle.
$$

**Example 1.2.** Let $\mathcal{L}_{\mathsf{IL}}$ be the language containing binary connectives $\wedge$, $\vee$, and $\to$, and constants $\bot$ and $\top$. A *Heyting algebra* is then an algebra $\mathbf{H} = \langle H, \wedge, \vee, \to, \bot, \top \rangle$ such that $\langle H, \wedge, \vee, \bot, \top \rangle$ is a bounded lattice with bottom and top elements $\bot$ and $\top$, respectively, and for all $a, b, c \in H$,

$$
a \wedge b \leq c \iff a \leq b \to c,
$$

where the lattice order is defined as $a \leq b :\Leftrightarrow a \wedge b = a$ for $a, b \in H$. We define *intuitionistic (propositional) logic* $\mathsf{IL}$ to be $\langle \mathcal{L}_{\mathsf{IL}}, \{\langle \mathbf{H}, \{\top^{\mathbf{H}}\}\rangle \mid \mathbf{H} \text{ a Heyting algebra}\}\rangle$. For background on intuitionistic logic, see, e.g., [112, 150].

An axiomatization of intuitionistic logic was first provided by Heyting in [83]. Since then, many different equivalent axiomatizations have been given, see for example [20, 73]. We consider the Hilbert-style axiomatization as given in Figure 1.1, denoted by $\mathcal{IPC}$. Although $\mathsf{IL}$ is, like $\mathsf{CL}$, decidable, it has higher complexity. Indeed, it was shown by Statman in [141] that $\mathsf{IL}$ is PSPACE-complete.

Let us now focus our attention on a class of logics more traditionally associated with the term many-valued logics: the logics associated with continuous t-norms. For an

extensive survey of such logics, we refer to [76]. A *triangular norm* (*t-norm* for short) is a binary operation $*\colon [0,1]^2 \to [0,1]$ that is commutative and associative, satisfies $1 * x = x$ for all $x \in [0,1]$, and is non-decreasing in both arguments, i.e., for all $x, y, z \in [0,1]$,

$$x \le y \implies x * z \le y * z \text{ and } z * x \le z * y.$$

We say that $*$ is a *continuous t-norm* if $*$ is a t-norm that is a continuous function, in the usual sense. It is easy to see that any continuous t-norm $*$ is *residuated*, that is, there exists a binary operation $\to\colon [0,1]^2 \to [0,1]$, called the *residual* of $*$, such that for all $x, y, z \in [0,1]$,

$$x * y \le z \iff x \le y \to z.$$

Let $\mathcal{L}_{ct}$ denote the language containing binary connectives $\wedge$, $\vee$, $*$, $\to$, and constants $\perp$ and $\top$. For each continuous t-norm $*$, we can define an algebra of the language $\mathcal{L}_{ct}$ as

$$[\mathbf{0},\mathbf{1}]_* \coloneqq \langle [0,1], \wedge^{[\mathbf{0},\mathbf{1}]_*}, \vee^{[\mathbf{0},\mathbf{1}]_*}, *, \to, \perp^{[\mathbf{0},\mathbf{1}]_*}, \top^{[\mathbf{0},\mathbf{1}]_*} \rangle,$$

where $\wedge^{[\mathbf{0},\mathbf{1}]_*} = \min$, $\vee^{[\mathbf{0},\mathbf{1}]_*} = \max$, $\perp^{[\mathbf{0},\mathbf{1}]_*} = 0$, and $\top^{[\mathbf{0},\mathbf{1}]_*} = 1$. We let $\mathsf{BL}$ denote the logic of all continuous t-norms, that is,

$$\mathsf{BL} = \langle \mathcal{L}_{ct}, \{\langle [\mathbf{0},\mathbf{1}]_*, \{\top^{[\mathbf{0},\mathbf{1}]_*}\} \rangle \mid * \text{ a continuous t-norm}\} \rangle.$$

We consider three pivotal examples of continuous t-norms:

(1) The *Łukasiewicz t-norm* $*^{\text{Ł}}$ where $x *^{\text{Ł}} y = \max\{0, x + y - 1\}$ and $x \to^{\text{Ł}} y = \min\{1, 1 - x + y\}$;

(2) The *Gödel t-norm* $*^{\text{G}}$ where $x *^{\text{G}} y = \min\{x, y\}$ and

$$x \to^{\text{G}} y = \begin{cases} 1 & x \le y \\ y & x > y; \end{cases}$$

(3) The *product t-norm* $*^{\text{P}}$ where $x *^{\text{P}} y = x \cdot y$ (the usual product in the reals) and

$$x \to^{\text{P}} y = \begin{cases} 1 & x \le y \\ \frac{y}{x} & x > y. \end{cases}$$

These three examples are in some sense exhaustive; indeed, any continuous t-norm can be constructed using just these three t-norms, as shown by Mostert and Shields in [117]. More formally, for any indexed set $\{*_i\}_{i \in I}$ of continuous t-norms with $*_i \in \{*^{\text{Ł}}, *^{\text{P}}\}$ for all $i \in I$ and family $\{(a_i, b_i)\}_{i \in I}$ of pairwise disjoint open intervals $(a_i, b_i) \subseteq [0,1]$, the following function $*\colon [0,1]^2 \to [0,1]$ is a continuous t-norm:

$$x * y \coloneqq \begin{cases} c_i + (d_i - c_i)(\frac{a - c_i}{d_i - c_i} *_i \frac{b - c_i}{d - c_i}) & \text{if } a, b \in [c_i, d_i] \\ a *^{\text{G}} b & \text{otherwise.} \end{cases}$$

Conversely, any continuous t-norm can be constructed in such a way.

## Gödel Logic(s)

In 1932, Gödel introduced a family of finite-valued logics to prove that intuitionistic logic cannot be given by a finite logical matrix [74]. Dummett extended on his ideas and presented the infinite-valued version in [63], which is now referred to as Gödel-Dummett logic, or simply Gödel logic.

Gödel logic is usually introduced over the language $\mathcal{L}_{\mathsf{IL}}$. We define the *standard Gödel algebra* over $\mathcal{L}_{\mathsf{IL}}$ as

$$\mathbf{G} = \langle [0,1], \wedge^{\mathbf{G}}, \vee^{\mathbf{G}}, \rightarrow^{\mathbf{G}}, \bot^{\mathbf{G}}, \top^{\mathbf{G}} \rangle,$$

where $\wedge^{\mathbf{G}} = \min$, $\vee^{\mathbf{G}} = \max$, $\bot^{\mathbf{G}} = 0$, and $\top^{\mathbf{G}} = 1$. Note that $\mathbf{G}$ is simply $[\mathbf{0},\mathbf{1}]_{*^{\mathsf{G}}}$ without $*^{\mathsf{G}}$ in the signature. In fact, $\mathbf{G}$ and $[\mathbf{0},\mathbf{1}]_{*^{\mathsf{G}}}$ are term-equivalent, since $*^{\mathsf{G}} = \min$ and can hence be recovered. We define Gödel logic $\mathsf{G}$ to be $\langle \mathcal{L}_{\mathsf{IL}}, \langle \mathbf{G}, \{1\} \rangle \rangle$.

Aside from the standard Gödel algebra, there is a large family of interesting subalgebras of $\mathbf{G}$ that can be considered. We call a subset $A \subseteq [0,1]$ a *Gödel set* if it is closed in the usual topology and contains 0 and 1. We can then consider the corresponding subalgebra of $\mathbf{G}$

$$\mathbf{A} = \langle A, \min, \max, \rightarrow^{\mathbf{G}}, 0, 1 \rangle.$$

Note that since $A$ is closed, $\mathbf{A}$ is necessarily *complete*, that is, for all $X \subseteq A$ the infimum $\bigwedge X$ and supremum $\bigvee X$ of $X$ exist in $A$. In fact, this gives an equivalent definition of a Gödel set: a subset $\{0,1\} \subseteq A \subseteq [0,1]$ is a Gödel set if and only if $\mathbf{A}$ is complete. Beside the *standard Gödel set* $G \coloneqq [0,1]$, notable Gödel sets include the finite Gödel sets $G_n \coloneqq \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$ (for $n \in \mathbb{N}^+$), as well as $G_\downarrow \coloneqq \{\frac{1}{n} \mid n \in \mathbb{N}^+\} \cup \{0\}$ and $G_\uparrow \coloneqq \{\frac{n-1}{n} \mid n \in \mathbb{N}^+\} \cup \{1\}$.

To each Gödel set $A$ we can associate a logic $\mathsf{G}(A) \coloneqq \langle \mathcal{L}_{\mathsf{IL}}, \langle \mathbf{A}, \{1\} \rangle \rangle$. In his work from 1959, Dummett showed that $\mathsf{G}(G_\downarrow)$ can be axiomatized by extending $\mathcal{IPC}$ with the prelinearity axiom schema (pre) $(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$. That is, for any $\varphi \in \mathrm{Fm}(\mathcal{L}_{\mathsf{IL}})$,

$$\models_{\mathsf{G}(G_\downarrow)} \varphi \quad \Longleftrightarrow \quad \vdash_{\mathcal{IPC}+(\mathrm{pre})} \varphi.$$

It is not hard to verify that the same holds for any infinite Gödel set $A$, including the standard Gödel set, giving an axiomatization of $\mathsf{G}$. This is in contrast with the first-order Gödel logics defined over these Gödel sets or even the one-variable fragments of such logics: there are infinitely many of them, as we shall see in Section 1.2. As for the finite Gödel sets, for each $n \in \mathbb{N}^+$ the logic $\mathsf{G}(G_n)$ can be axiomatized, as shown in [147], by extending $\mathcal{IPC}$ with the prelinearity axiom schema and the axiom schema

$$(\mathrm{fin}_n) \ (\top \rightarrow \varphi_1) \vee (\varphi_1 \rightarrow \varphi_2) \vee \cdots \vee (\varphi_{n-1} \rightarrow \varphi_n) \vee (\varphi_n \rightarrow \bot).$$

In fact, it is this chain of logics that Gödel proved in [74] to exist between $\mathsf{IL}$ and $\mathsf{CL}$. The intersection of this family is again the standard Gödel logic $\mathsf{G}$, that is,

$$\models_{\mathsf{G}} \varphi \quad \Longleftrightarrow \quad \models_{\mathsf{G}(G_n)} \varphi \ \text{ for all } n \in \mathbb{N}^+.$$

It can be regarded a folklore result that $\mathbf{G}$-validity is decidable. A proof showing that its complexity is co-NP-complete can be found in [8].

Following Dummett's axiomatization, we obtain directly an equivalent algebraic formulation of $\mathsf{G}$. We say that a Heyting algebra $\mathbf{H}$ is a *Gödel algebra* if it satisfies $(x \rightarrow y) \vee (y \rightarrow x) \approx \top$. It then follows directly from Dummett's completeness result for $\mathsf{G}$ and Heyting's axiomatization of $\mathsf{IL}$ that $\mathsf{G}$ is validity-equivalent to $\langle \mathcal{L}_{\mathsf{IL}}, \{\langle \mathbf{H}, \{\top^{\mathbf{H}}\} \rangle \mid \mathbf{H}$ a Gödel algebra$\} \rangle$.

## Łukasiewicz Logic

Łukasiewicz was the first to consider a proper many-valued logic. In the 1920s, he proposed a three-valued logic, where an additional truth value could be interpreted as "undetermined", next to the two traditional truth values being interpreted as "true" and "false" [97]. Later in the decade, this idea was expanded to $n$-valued logics for finite numbers $n \geq 2$, and then to the infinite-valued version in [98]. This infinite-valued logic is what nowadays is called Łukasiewicz logic.

Łukasiewicz logic is traditionally defined over the language $\mathcal{L}_{\text{Ł}}$ with a binary connective $\supset$ and a unary connective $\sim$. Additionally, the following connectives are defined:

$$\bar{1} := \varphi \supset \varphi \qquad\qquad \bar{0} := \sim \bar{1}$$
$$\varphi \otimes \psi := \sim(\varphi \supset \sim \psi) \qquad\qquad \varphi \oplus \psi := \sim \varphi \supset \psi$$
$$\varphi \wedge \psi := \varphi \otimes (\varphi \supset \psi) \qquad\qquad \varphi \vee \psi := (\varphi \supset \psi) \supset \psi.$$

We then consider the *standard Łukasiewicz algebra*

$$\mathbf{Ł} := \langle [0,1], \supset^{\mathbf{Ł}}, \sim^{\mathbf{Ł}} \rangle$$

where $a \supset^{\mathbf{Ł}} b = a \rightarrow^{\text{Ł}} b = \min\{1, 1 - a + b\}$ and $\sim^{\mathbf{Ł}} a = 1 - a$. For the defined connectives, we obtain

$$\bar{1}^{\mathbf{Ł}} = 1 \qquad\qquad \bar{0}^{\mathbf{Ł}} = 0$$
$$a \otimes^{\mathbf{Ł}} b = \max\{0, a + b - 1\} \qquad\qquad a \oplus^{\mathbf{Ł}} b = \min\{1, a + b\}$$
$$a \wedge^{\mathbf{Ł}} b = \min\{a, b\} \qquad\qquad a \vee^{\mathbf{Ł}} b = \max\{a, b\}.$$

With the defined connectives, it is clear how to recover the algebra $[\mathbf{0}, \mathbf{1}]_{*\text{Ł}}$ over $\mathcal{L}_{ct}$ if we define $a \rightarrow b := a \supset b$. Conversely, we can define $\sim a := a \rightarrow \bot$ and $a \supset b := a \rightarrow b$, establishing the term-equivalence between $\mathbf{Ł}$ and $[\mathbf{0}, \mathbf{1}]_{*\text{Ł}}$. We define *Łukasiewicz logic* Ł as the pair $\langle \mathcal{L}_{\text{Ł}}, \langle \mathbf{Ł}, \{1\} \rangle \rangle$.

Łukasiewicz proposed an axiomatization in [98], denoted here by $\mathcal{HŁ}$, which can be found in Figure 1.2.[3] This axiomatization was proved to be complete in the 1930s by Wajsberg, but his proof was not published. The first published proof is by Rose and Rosser in [137]. It was shown by Mundici in [120] that Ł-validity is co-NP-complete.

Algebraically, Łukasiewicz logic is studied using the class of *MV-algebras*, historically named for "many-valued" algebras, introduced by Chang in [47]. Let $\mathcal{L}_{MV}$ denote the language containing a binary connective $\oplus$, a unary connective $\sim$, and a constant 0. An MV-algebra is an algebra $\langle A, \oplus, \sim, 0 \rangle$ over the language $\mathcal{L}_{MV}$ satisfying

| | | | |
|---|---|---|---|
| (MV1) | $x \oplus (y \oplus z) \approx (x \oplus y) \oplus z$ | (MV4) | $x \oplus y \approx y \oplus x$ |
| (MV2) | $x \oplus 0 \approx x$ | (MV5) | $\sim \sim x \approx x$ |
| (MV3) | $x \oplus \sim 0 = \sim 0$ | (MV6) | $\sim(\sim x \oplus y) \oplus y \approx \sim(\sim y \oplus x) \oplus x$ |

The variety of MV-algebras can be generated by the algebra $\langle [0,1], \oplus, \sim, 0 \rangle$, where $a \oplus b = \min\{a + b, 1\}$ and $\sim a = 1 - a$, i.e., the algebra that, using the translations given in this section, is term-equivalent to $[\mathbf{0}, \mathbf{1}]_{*\text{Ł}}$ and $\mathbf{Ł}$. Defining the corresponding translators, it follows that Ł is validity-equivalent to $\langle \mathcal{L}_{\text{Ł}}, \{ \langle \mathbf{A}, \{\sim 0\} \rangle \mid \mathbf{A} \text{ a MV-algebra} \} \rangle$.

---

[3]Łukasiewicz's axiomatization included an additional axiom $((\varphi \supset \psi) \supset (\psi \supset \varphi)) \supset (\psi \supset \varphi)$, which was proved to be redundant by Meredith [102] and Chang [48] independently.

$$
\begin{array}{ll}
(1) & \varphi \supset (\psi \supset \varphi) \\
(2) & (\varphi \supset \psi) \supset ((\psi \supset \chi) \supset (\varphi \supset \chi)) \\
(3) & ((\varphi \supset \psi) \supset \psi) \supset ((\psi \supset \varphi) \supset \varphi) \\
(4) & (\sim\!\varphi \supset \sim\!\psi) \supset (\psi \supset \varphi)
\end{array}
$$

$$
\frac{\varphi \quad \varphi \supset \psi}{\psi} \;(\mathrm{mp})
$$

Figure 1.2: Proof system $\mathcal{H}\text{\L}$

## Abelian Logic

A third many-valued logic that plays a central role in this thesis is Abelian logic. It does not fit the framework of the continuous t-norm logics, since it is defined over all real numbers and not just the interval $[0,1]$. We let $\mathcal{L}_\mathsf{A}$ denote the language containing binary connectives $\wedge$, $\vee$, and $+$, a unary connective $-$, and a constant $\bar{0}$. For $\varphi, \psi \in \mathrm{Fm}(\mathcal{L}_\mathsf{A})$, we define $0\varphi := \bar{0}$ and $(n+1)\varphi := n\varphi + \varphi$ for any $n \in \mathbb{N}$. Moreover, we write $\psi - \varphi$ or $\varphi \to \psi$ for $\psi + (-\varphi)$. We define the algebra

$$
\mathbf{R} = \langle \mathbb{R}, \wedge^{\mathbf{R}}, \vee^{\mathbf{R}}, +, -, 0 \rangle
$$

with $\wedge^{\mathbf{R}} = \min$ and $\vee^{\mathbf{R}} = \max$ and $\langle \mathbb{R}, +, -, 0 \rangle$ the usual additive group of the reals. We then define *Abelian logic* $\mathsf{A}$ as $\langle \mathcal{L}_\mathsf{A}, \langle \mathbf{R}, \mathbb{R}^{\geq 0} \rangle \rangle$, where $\mathbb{R}^{\geq 0} := \{r \in \mathbb{R} \mid r \geq 0\}$.

Abelian logic was introduced independently by Meyer and Slaney in [109] as a relevance logic and by Casari in [43] as a comparative logic, introduced syntactically in both cases. Meyer and Slaney showed that their axiomatization was sound and complete with respect to the ordered group of the integers $\mathbf{Z} := \langle \mathbb{Z}, \min, \max, +, -, 0 \rangle$. Casari showed completeness with respect to the variety of *lattice-ordered abelian groups*. A lattice-ordered abelian group (*abelian $\ell$-group* for short) is an algebra $\langle A, \wedge, \vee, +, -, 0 \rangle$ such that $\langle A, +, -, 0 \rangle$ is an abelian group, $\langle A, \wedge, \vee \rangle$ is a lattice, and $+$ is compatible with the lattice order, that is, for all $a, b, c \in A$,

$$
a \leq b \implies a + c \leq b + c.
$$

For a survey on abelian $\ell$-groups, we refer to, e.g., [1]. Both $\mathbf{R}$ and $\mathbf{Z}$ generate the variety of lattice-ordered abelian groups. Hence, we obtain validity-equivalent logics

$$
\begin{aligned}
\mathsf{A} \;\sim\; & \langle \mathcal{L}_\mathsf{A}, \{ \langle \mathbf{A}, \{a \in G \mid 0^{\mathbf{A}} \leq a\} \rangle \mid \mathbf{A} \text{ an abelian } \ell\text{-group} \} \rangle \\
\sim\; & \langle \mathcal{L}_\mathsf{A}, \langle \mathbf{Z}, \mathbb{N} \rangle \rangle.
\end{aligned}
$$

A Hilbert-style proof system for $\mathsf{A}$, denoted by $\mathcal{H}\mathcal{A}$, is given in Figure 1.3. By Meyer and Slaney's completeness result,[4] we have for all $\varphi \in \mathrm{Fm}(\mathcal{L}_\mathsf{A})$,

$$
\vdash_{\mathcal{H}\mathcal{A}} \varphi \iff \models_\mathsf{A} \varphi.
$$

Validity in Abelian logic is co-NP-complete [153].

---

[4]The proof system $\mathcal{H}\mathcal{A}$ is not the same as the one given by Meyer and Slaney. In fact, here we give a single-constant version of multiplicative additive intuitionistic linear logic extended with the axiom schema (A), but it is not hard to show that this system is equivalent to that of Meyer and Slaney.

$$
\begin{array}{ll}
\text{(B)} & (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)) \\
\text{(C)} & (\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi)) \\
\text{(I)} & \varphi \to \varphi \\
\text{(A)} & ((\varphi \to \psi) \to \psi) \to \varphi \\
\text{(+1)} & \varphi \to (\psi \to (\varphi + \psi)) \\
\text{(+2)} & (\varphi \to (\psi \to \chi)) \to ((\varphi + \psi) \to \chi) \\
\text{($\bar{0}$1)} & \bar{0} \\
\text{($\bar{0}$2)} & \varphi \to (\bar{0} \to \varphi) \\
\text{($\wedge$1)} & (\varphi \wedge \psi) \to \varphi \\
\text{($\wedge$2)} & (\varphi \wedge \psi) \to \psi \\
\text{($\wedge$3)} & ((\varphi \to \psi) \wedge (\varphi \to \chi)) \to (\varphi \to (\psi \wedge \chi)) \\
\text{($\vee$1)} & \varphi \to (\varphi \vee \psi) \\
\text{($\vee$2)} & \psi \to (\varphi \vee \psi) \\
\text{($\vee$3)} & ((\varphi \to \chi) \wedge (\psi \to \chi)) \to ((\varphi \vee \psi) \to \chi)
\end{array}
$$

$$
\frac{\varphi \quad \varphi \to \psi}{\psi} \ (\text{mp}) \qquad \frac{\varphi \quad \psi}{\varphi \wedge \psi} \ (\text{adj})
$$

Figure 1.3: Proof system $\mathcal{HA}$

There exists a deep connection between Łukasiewicz logic Ł and Abelian logic A. On the algebraic side, we have a categorical correspondence between MV-algebras and abelian $\ell$-groups **A** with a *strong unit*, that is, an element $u \in A$ such that $0^{\mathbf{A}} \leq u$ and for all $a \in A$, there exists $n \in \mathbb{N}$ such that $a \leq nu$ [119]. For a more detailed exposition, we refer to [52]. A syntactic connection was studied already by Meyer and Slaney, relating a fragment of A to a fragment of Ł, which was generalized in [107] to show that the whole Ł corresponds to a fragment of A. In [107] also a more intuitive translation is given, which we will return to in the next section.

### FL$_e$-Algebras

As noted, both Gödel and Łukasiewicz fit into the setting of continuous t-norms and the logic BL, but Abelian logic does not. To capture all three logics in a common framework, we need to generalize the setting of continuous t-norms. One such generalization considers *left-continuous t-norms*, that is, t-norms that are left-continuous, giving rise to the logic MTL [68]. A further generalization is considered by Metcalfe and Montagna in [104] in the form of left-continuous *uninorms*, giving rise to the logic UL. A uninorm, originally introduced in [154], is a function $*\colon [0,1]^2 \to [0,1]$ that is commutative, associative, order-preserving in both arguments and such that for some $e_* \in [0,1]$, $e_* * x = x$ for all $x \in [0,1]$.[5] To incorporate Abelian logic, we generalize the underlying real unit interval $[0,1]$ to an arbitrary lattice. This lands us in the realm of *substructural logics*. This class of logics, whose name was coined by Došen and Schroeder-Heister in 1990, contains logics whose proof systems generally lack some "structural" rule. Examples include relevance logic [65], linear logic [149], and (as we will see) many-valued logics. Through the work of Ono, Tsinakis and others, substructural logics now have a solid algebraic basis using

---

[5]Note that if $e_* = 1$, $*$ is a t-norm.

*(pointed) residuated lattices*, also called FL-*algebras*. An extensive survey can be found in [71]. We define the subclass of FL-algebras relevant to us, over the language $\mathcal{L}_{\mathsf{FL}_e}$ containing binary connectives $\wedge$, $\vee$, $\cdot$, and $\rightarrow$ and constants $f$ and $e$. We define an additional binary connective $\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$.

**Definition 1.3.** An $\mathsf{FL}_e$-*algebra* (or *commutative pointed residuated lattice*) is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \rightarrow, f, e \rangle$ over $\mathcal{L}_{\mathsf{FL}_e}$ such that

(i) $\langle A, \cdot, e \rangle$ is a commutative monoid;

(ii) $\langle A, \wedge, \vee \rangle$ is a lattice;

(iii) $\cdot$ is residuated with residual $\rightarrow$, that is, for all $a, b, c \in A$,

$$a \cdot b \leq c \quad \Longleftrightarrow \quad a \leq b \rightarrow c.$$

We call $\mathbf{A}$ *integral* if the lattice $\langle A, \wedge, \vee \rangle$ has $e$ as the top element, *linear* (or an $\mathsf{FL}_e$-*chain*) if it is a chain, and an $\mathsf{FL}_{eo}$-*algebra* if it has $f$ as the bottom element. For two $\mathsf{FL}_e$-algebras $\mathbf{A}$, $\mathbf{B}$, a map $h \colon A \rightarrow B$ is called an $\mathsf{FL}_e$-*homomorphism* (or simply a *homomorphism*) if it preserves all the operations of $\mathbf{A}$. That is, for $* \in \{\wedge, \vee, \cdot, \rightarrow\}$ and $a, b \in A$, $h(a *^{\mathbf{A}} b) = h(a) *^{\mathbf{B}} h(b)$, $h(e^{\mathbf{A}}) = e^{\mathbf{B}}$, and $h(f^{\mathbf{A}}) = f^{\mathbf{B}}$. We sometimes write $h \colon \mathbf{A} \rightarrow \mathbf{B}$ to stress that it is a homomorphism.

For modal purposes later, we are particularly interested in complete $\mathsf{FL}_e$-algebras. An $\mathsf{FL}_e$-algebra $\mathbf{A}$ is called *complete* if for all $X \subseteq A$, the infimum $\bigwedge X$ and supremum $\bigvee X$ of $X$ exist in $A$. In particular, every complete $\mathsf{FL}_e$-algebra $\mathbf{A}$ has a bottom and top element. Note that these top and bottom elements do not have to coincide with $e$ or $f$.

We let $\mathsf{FL}_e$ denote the (substructural) logic

$$\langle \mathcal{L}_{\mathsf{FL}_e}, \{ \langle \mathbf{A}, \{ a \in A \mid e^{\mathbf{A}} \leq a \} \rangle \mid \mathbf{A} \text{ is an } \mathsf{FL}_e\text{-algebra} \} \rangle.$$

A Hilbert-style proof system for $\mathsf{FL}_e$, denoted by $\mathcal{HFL}_e$, can be found in [71, 2.5.1].

Under appropriate translations, all the algebras discussed thus far are term-equivalent to some $\mathsf{FL}_e$-algebra. Phrased differently, all the logics discussed so far are substructural logics. We recount the particular term-equivalences here.

**Example 1.4.** Boolean algebras, as defined in Example 1.1, are term-equivalent to $\mathsf{FL}_{eo}$-algebras satisfying $x \cdot y \approx x \wedge y$ and $(x \rightarrow y) \rightarrow y \approx x \vee y$. Indeed, given such an $\mathsf{FL}_e$-algebra $\mathbf{B}$, defining $\neg a := a \rightarrow f$, $\bot := f$, and $\top := e$ gives a Boolean algebra $\langle B, \wedge, \vee, \neg, \bot, \top \rangle$. Conversely, for a Boolean algebra $\mathbf{B}$ over $\mathcal{L}_{\mathsf{CL}}$, we define $a \cdot b := a \wedge b$, $a \rightarrow b := \neg a \vee b$, $e := \top$, and $f := \bot$ to obtain an $\mathsf{FL}_{eo}$-algebra $\langle B, \wedge, \vee, \cdot, \rightarrow, f, e \rangle$ satisfying the mentioned identities.

**Example 1.5.** Heyting algebras, as defined in Example 1.2, are term-equivalent to integral $\mathsf{FL}_{eo}$-algebras satisfying $x \cdot y \approx x \wedge y$. Removing $\cdot$ from the signature and defining $\bot := f$ and $\top := e$ transforms such an $\mathsf{FL}_e$-algebra into a Heyting algebra. For the converse direction, it suffices to define $a \cdot b := a \wedge b$, $e := \top$, and $f := \bot$.

**Example 1.6.** For any continuous t-norm $*$, the algebra $[\mathbf{0}, \mathbf{1}]_*$ is an $\mathsf{FL}_e$-algebra, and vice versa, if we identify constants $f$ and $e$ in $\mathcal{L}_{\mathsf{FL}_e}$ with the constants $\bot$ and $\top$ in $\mathcal{L}_{ct}$,

respectively. In light of what was discussed previously, this also establishes the term-equivalence for both Gödel and MV-algebras. Indeed, Gödel algebras are term-equivalent to integral $\mathsf{FL}_{eo}$-algebras satisfying $x \cdot y \approx x \wedge y$ and $(x \to y) \vee (y \to x) \approx e$, whereas MV-algebras are term-equivalent to $\mathsf{FL}_{eo}$-algebras satisfying $x \vee y \approx (x \to y) \to y$.

**Example 1.7.** Abelian $\ell$-groups are term-equivalent to $\mathsf{FL}_e$-algebras satisfying $e \approx f$ and $x \cdot (x \to e) \approx e$. For such an $\mathsf{FL}_e$-algebra, we define $a + b := a \cdot b$, $-a := a \to f$, and $0 := e$ giving an abelian $\ell$-group $\langle A, \wedge, \vee, +, -, 0 \rangle$. Conversely, given an abelian $\ell$-group **A**, we define $a \cdot b := a + b$, $a \to b := b - a$, and $e = f := 0$. Then $\langle A, \wedge, \vee, \cdot, \to, e, f \rangle$ is an $\mathsf{FL}_e$-algebra satisfying the aforementioned identities.

As proved in [28], the class of $\mathsf{FL}_e$-algebras forms a variety. Indeed, their defining identities consist of the defining identities for lattices and commutative monoids, together with the identities

$$x \cdot (y \vee z) \approx (x \cdot y) \vee (x \cdot z)$$
$$x \to y \leq x \to (y \vee z)$$
$$x \cdot (x \to y) \leq y \leq x \to (x \cdot y).$$

It follows immediately that all classes of $\mathsf{FL}_e$-algebras as defined in Examples 1.4-1.7 above are also varieties.

## 1.2  First-Order Logics

In this section, we focus on first-order extensions of the discussed propositional logics. We present a rather general semantics for first-order many-valued logics, inspired by the algebraic treatment of first-order substructural logics by, e.g., Ono [125], and Hájek's presentation in his book [76]. We then give a survey of some known results from the literature on first-order intuitionistic, Gödel, Łukasiewicz and Abelian logics.

Let $\mathcal{L}$ be a propositional language. A *term $t$* will generally be a variable $x$ from a countably infinite set of variables whose elements are denoted $x, y, z, \ldots$. We will not consider arbitrary function symbols in this thesis. In Chapter 4 however, we do need to consider 0-ary function symbols $c$, called *object constants*, from a countably infinite set whose elements are denoted $c, d, \ldots$. Unless stated otherwise, the set of terms Term consists only of all variables $x, y, z, \ldots$. We fix a countably infinite set of predicates $P, Q, \ldots$ of each finite arity. The set $\mathrm{Fm}_{\forall\exists}(\mathcal{L})$ of first-order formulas $\alpha, \beta, \ldots$ is then defined inductively as follows, with $P$ an $n$-ary predicate, $\star \in \mathcal{L}$ an $m$-ary propositional connective, $t_1, \ldots, t_n \in$ Term, and $x$ a variable,

$$\alpha ::= P(t_1, \ldots, t_n) \mid \star(\alpha, \ldots, \alpha) \mid (\forall x)\alpha \mid (\exists x)\alpha.$$

The occurrence of a variable $x$ in a formula $\alpha \in \mathrm{Fm}_{\forall\exists}(\mathcal{L})$ is called *bound* if it is under the scope of $(\forall x)$ or $(\exists x)$, and *free* otherwise. A term $t$ is free for a variable $x$ in $\alpha$ if $x$ does not occur free in $\alpha$ within the scope of $(\forall y)$ where $y$ is any variable occurring in $t$. If the free variables of a formula $\alpha \in \mathrm{Fm}_{\forall\exists}(\mathcal{L})$ are among $x_1, \ldots, x_n$, we sometimes write $\alpha(x_1, \ldots, x_n)$ for $\alpha$. For terms $t_1, \ldots, t_n$ that are free for $x_1, \ldots, x_n$, respectively, we write $\alpha(t_1, \ldots, t_n)$ to denote the formula where for each $i = 1, \ldots, n$, every free occurrence of $x_i$ is simultaneously replaced by $t_i$. The *length* of a formula $\alpha \in \mathrm{Fm}_{\forall\exists}(\mathcal{L})$ is again defined

inductively as the number of $n + 1$-ary connectives occurring in $\alpha$, where we consider $(\forall x)$ and $(\exists x)$ to be unary connectives.

**Definition 1.8.** Let $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \rightarrow, f, e \rangle$ be an $\mathsf{FL}_e$-algebra. An $\mathbf{A}$-*structure* is a pair $\mathfrak{M} = \langle D, \mathcal{I} \rangle$ consisting of a non-empty set $D$, called the *domain*, and for each $n$-ary predicate $P$, an $n$-ary $A$-valued relation $\mathcal{I}(P) \colon D^n \rightarrow A$. An $\mathfrak{M}$-*evaluation* is a map $v \colon \mathrm{Term} \rightarrow D$.[6] For $\mathfrak{M}$-evaluation $v$, variable $x$ and $a \in D$, we write $v(x = a)$ to denote the $\mathfrak{M}$-evaluation $v'$ such that $v'(x) = a$ and $v'(t) = v(t)$ for all terms $t \neq x$. Then $\mathfrak{M}$ and $v$ define a value $\|\alpha\|_{\mathfrak{M},v}^{\mathbf{A}}$ for each $\alpha \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{FL}_e})$ inductively, with $\star \in \{\wedge, \vee, \cdot, \rightarrow\}$, $t_1, \ldots, t_n \in \mathrm{Term}$,

$$\|P(t_1, \ldots, t_n)\|_{\mathfrak{M},v}^{\mathbf{A}} = \mathcal{I}(P)(v(t_1), \ldots, v(t_n))$$
$$\|f\|_{\mathfrak{M},v}^{\mathbf{A}} = f$$
$$\|e\|_{\mathfrak{M},v}^{\mathbf{A}} = e$$
$$\|\alpha \star \beta\|_{\mathfrak{M},v}^{\mathbf{A}} = \|\alpha\|_{\mathfrak{M},v}^{\mathbf{A}} \star \|\beta\|_{\mathfrak{M},v}^{\mathbf{A}}$$
$$\|(\forall x)\alpha\|_{\mathfrak{M},v}^{\mathbf{A}} = \bigwedge\{\|\alpha\|_{\mathfrak{M},v(x=a)}^{\mathbf{A}} \mid a \in D\}$$
$$\|(\exists x)\alpha\|_{\mathfrak{M},v}^{\mathbf{A}} = \bigvee\{\|\alpha\|_{\mathfrak{M},v(x=a)}^{\mathbf{A}} \mid a \in D\}.$$

If the supremum or infimum does not exist, we take the value of the quantified formula in question and all formulas in which it occurs as a subformula to be undefined. An $\mathbf{A}$-structure $\mathfrak{M}$ is called *safe* if $\|\alpha\|_{\mathfrak{M},v}^{\mathbf{A}}$ is defined for all $\alpha \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{FL}_e})$ and $\mathfrak{M}$-evaluations $v$. A formula $\alpha \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{FL}_e})$ is then called $\mathbf{A}$-*valid*, denoted by $\models_{\mathbf{A}}^{\forall\exists} \alpha$, if $\|\alpha\|_{\mathfrak{M},v}^{\mathbf{A}} \geq e$ for each safe $\mathbf{A}$-structure $\mathfrak{M}$ and $\mathfrak{M}$-evaluation $v$.

We say that $\alpha \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{FL}_e})$ is $\mathsf{QFL}_e$-*valid*, denoted by $\models_{\mathsf{QFL}_e} \alpha$, if $\alpha$ is $\mathbf{A}$-valid for all $\mathsf{FL}_e$-algebras $\mathbf{A}$. This notion of $\mathsf{QFL}_e$-validity has been studied extensively in the context of first-order substructural logics. A (non-Hilbert-style) proof system for it can be found in, e.g., [15]. A Hilbert-style system $\mathcal{HQFL}_e$ can be found in [62].[7] A Henkin-style proof shows that for all $\alpha \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{FL}_e})$,

$$\vdash_{\mathcal{HQFL}_e} \alpha \quad \Longleftrightarrow \quad \models_{\mathsf{QFL}_e} \alpha.$$

Ono improves on this result in [125] by showing that for all $\alpha \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{FL}_e})$,

$$\vdash_{\mathcal{HQFL}_e} \alpha \iff \alpha \text{ is } \mathbf{A}\text{-valid for all complete } \mathsf{FL}_e\text{-algebras } \mathbf{A}.$$

In Examples 1.4–1.7, we discussed term-equivalences for a number of classes of $\mathsf{FL}_e$-algebras. We use these term-equivalences to somewhat abuse the notation introduced above. That is, if an algebra $\mathbf{A}$ over $\mathcal{L}$ is term-equivalent to some $\mathsf{FL}_e$-algebra $\mathbf{A}'$, we say that $\alpha \in \mathrm{Fm}_{\forall\exists}(\mathcal{L})$ is $\mathbf{A}$-valid if the corresponding formula in $\mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{FL}_e})$ is $\mathbf{A}'$-valid.

**Example 1.9.** Recall the algebra $\mathbf{2}$ from Example 1.1. It is not hard to see that any formula $\alpha \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{CL}})$ is $\mathbf{2}$-valid if and only if it is valid in classical first-order logic. In fact, any $\mathbf{2}$-structure is exactly a first-order structure as classically defined. Famously, Church showed in [49] that first-order classical logic is undecidable. For more on first-order classical logic, see, e.g., [66].

---

[6]Although this is not the standard way of defining variable assignments for first-order structures, it suffices for our purposes. Indeed, we never consider $n + 1$-ary function symbols for any $n \in \mathbb{N}$.

[7]Došen introduces his Hilbert-style proof system over the language $\mathcal{L}_{\mathsf{FL}_e}$, extended with a unary negation $\neg$. This connective can be recovered in $\mathcal{L}_{\mathsf{FL}_e}$ however, by defining $\neg\alpha := \alpha \rightarrow f$.

$$
\begin{array}{ll}
(\forall x)\alpha(x) \to \alpha(t) & (\text{term } t \text{ is free for } x \text{ in } \alpha) \\
\alpha(t) \to (\exists x)\alpha(x) & (\text{term } t \text{ is free for } x \text{ in } \alpha) \\
(\forall x)(\beta \to \alpha) \to (\beta \to (\forall x)\alpha) & (x \text{ is not free in } \beta) \\
(\forall x)(\alpha \to \beta) \to ((\exists x)\alpha \to \beta) & (x \text{ is not free in } \beta)
\end{array}
$$

$$
\frac{\alpha}{(\forall x)\alpha} \; (\text{gen})
$$

Figure 1.4: Additional axiom and rule schemas for the proof system $\mathcal{IQC}$

It is worth noting that for any $\alpha \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{FL}_e})$ and $\mathsf{FL}_e$-algebra $\mathbf{A}$, $\alpha$ is $\mathbf{A}$-valid if and only if $(\forall x_1)\ldots(\forall x_n)\alpha$ is $\mathbf{A}$-valid, where $x_1,\ldots,x_n$ are all the free variables that occur in $\alpha$. Since we are primarily interested in validity, this means we can often restrict ourselves to *sentences*, that is, formulas $\alpha \in \mathrm{Fm}_{\forall\exists}(\mathcal{L})$ that contain no free variables.

### First-Order Intuitionistic Logic

First formulated by Heyting in [83], first-order intuitionistic logic was introduced as the intuitionistic version of first-order classical logic. A Hilbert-style axiomatization was given by, e.g., Kleene in [91]. This axiomatization, denoted by $\mathcal{IQC}$, consists of the axiom schemata from $\mathcal{IPC}$ extended with the axiom and rule schemas from Figure 1.4. It was shown by Rasiowa and Sikorski that a formula $\alpha \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{IL}})$ is $\mathcal{IQC}$-derivable if and only if $\alpha$ is $\mathbf{H}$-valid for all complete Heyting algebras $\mathbf{H}$ [136].

Rather than via an algebraic semantics, first-order intuitionistic logic is nowadays mostly presented via Kripke semantics, introduced by Kripke in his pioneering work [94]. A similar equivalent semantics was introduced by Beth in [21]. We present Kripke's semantics here.

**Definition 1.10.** A *frame* is a non-empty poset $\mathbf{K} = \langle K, \preceq \rangle$ and is said to be *linear* if $\preceq$ is a linear order. An *intuitionistic Kripke model* (or $\mathsf{IK}$-*model* for short) based on $\mathbf{K}$ is a 4-tuple

$$
\mathfrak{M} = \langle K, \preceq, \{D_k\}_{k \in K}, \{\mathcal{I}_k\}_{k \in K} \rangle,
$$

such that for all $k \in K$, each $D_k$ is a non-empty set (called the *domain* of $k$), and each $\mathcal{I}_k$ is a function mapping each $n$-ary predicate $P$ to $\mathcal{I}_k(P) \subseteq D_k^n$, satisfying

$$
k \preceq l \implies D_k \subseteq D_l \text{ and } \mathcal{I}_k(P) \subseteq \mathcal{I}_l(P).
$$

For $k \in K$, an $\mathfrak{M}$-*evaluation for $k$* is a map $\nu$ assigning to each variable $x$ an element $d \in D_k$. For $a \in D_k$ and variable $x$, we write $\nu(x = a)$ to denote the $\mathfrak{M}$-evaluation $\nu'$ for $k$ such that $\nu'(x) = a$ and $\nu'(y) = \nu(y)$ for all variables $y \neq x$. For any $k \in K$ and

$\mathfrak{M}$-evaluation $\nu$ for $k$, satisfaction in $\mathfrak{M}$ is defined inductively as follows:

$$
\begin{array}{lll}
\mathfrak{M}, k \models^{\nu} \bot & \Longleftrightarrow & \text{never} \\
\mathfrak{M}, k \models^{\nu} \top & \Longleftrightarrow & \text{always} \\
\mathfrak{M}, k \models^{\nu} P(x_1, \ldots, x_n) & \Longleftrightarrow & \langle \nu(x_1), \ldots, \nu(x_n) \rangle \in \mathcal{I}_k(P) \\
\mathfrak{M}, k \models^{\nu} \alpha \wedge \beta & \Longleftrightarrow & \mathfrak{M}, k \models^{\nu} \alpha \ \text{ and } \ \mathfrak{M}, k \models^{\nu} \beta \\
\mathfrak{M}, k \models^{\nu} \alpha \vee \beta & \Longleftrightarrow & \mathfrak{M}, k \models^{\nu} \alpha \ \text{ or } \ \mathfrak{M}, k \models^{\nu} \beta \\
\mathfrak{M}, k \models^{\nu} \alpha \to \beta & \Longleftrightarrow & \text{for all } l \succeq k, \ \mathfrak{M}, l \models^{\nu} \alpha \text{ implies } \mathfrak{M}, l \models^{\nu} \beta \\
\mathfrak{M}, k \models^{\nu} (\forall x)\alpha & \Longleftrightarrow & \text{for all } l \succeq k \text{ and } b \in D_l, \ \mathfrak{M}, l \models^{\nu(x=b)} \alpha \\
\mathfrak{M}, k \models^{\nu} (\exists x)\alpha & \Longleftrightarrow & \text{for some } b \in D_k, \ \mathfrak{M}, k \models^{\nu(x=b)} \alpha.
\end{array}
$$

A formula $\alpha \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{IL}})$ is said to be *valid* in $\mathfrak{M}$ if $\mathfrak{M}, k \models^{\nu} \alpha$ for all $k \in K$ and $\mathfrak{M}$-evaluations $\nu$ for $k$. We say that $\alpha \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{IL}})$ is $\mathsf{IK}$-*valid*, denoted by $\models_{\mathsf{IK}} \alpha$, if it is valid in all $\mathsf{IK}$-models.

Due to their connection with first-order Gödel logic, we are particularly interested in $\mathsf{IK}$-models satisfying a couple of conditions. Firstly, we are interested in those $\mathsf{IK}$-models $\mathfrak{M} = \langle K, \preceq, \{D_k\}_{k \in K}, \{\mathcal{I}_k\}_{k \in K} \rangle$ such that $\mathbf{K} = \langle K, \preceq \rangle$ is linear. In that case, we call $\mathfrak{M}$ an $\mathsf{IKL}$-model. Secondly, we are interested in those $\mathsf{IK}$-models $\mathfrak{M}$ that have constant domains, that is, $D_k = D_l$ for all $k, l \in K$. If that is the case, we call $\mathfrak{M}$ an $\mathsf{CDIK}$-model. If both conditions are satisfied, we call $\mathfrak{M}$ an $\mathsf{CDIKL}$-model. Given $\mathsf{L} \in \{\mathsf{IKL}, \mathsf{CDIK}, \mathsf{CDIKL}\}$, we say that $\alpha \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{IL}})$ is $\mathsf{L}$-*valid*, denoted by $\models_{\mathsf{L}} \alpha$, if it is valid in all $\mathsf{L}$-models.

First-order intuitionistic logics defined over some class of intuitionistic Kripke models fall under the umbrella term of *(first-order) intermediate logics*, a class of (first-order) logics whose set of valid formulas lies between the sets of intuitionistically valid formulas and classically valid formulas.[8] Let us recall some well-known completeness results. Firstly, Kripke showed completeness of $\mathsf{IK}$-validity with respect to $\mathcal{IQC}$ in [94]. An axiomatization for $\mathsf{CDIK}$-validity was found independently by three different authors, namely by Görnemann in [75], by Klemke in [92], and by Gabbay in [70]. $\mathsf{IKL}$-validity was axiomatized by Corsi in [56], and $\mathsf{CDIKL}$-validity by Minari [110]. To formulate these completeness results, we define the following *prelinearity* (pre) and *constant domain* (cd) axiom schemas

$$
\begin{array}{ll}
\text{(pre)} & (\alpha \to \beta) \vee (\beta \to \alpha) \\
\text{(cd)} & (\forall x)(\alpha \vee (\forall x)\beta) \to ((\forall x)\alpha \vee (\forall x)\beta).
\end{array}
$$

The mentioned completeness results can now be summarized as follows, for $\alpha \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{IL}})$,

$$
\begin{array}{lll}
\models_{\mathsf{IK}} \alpha & \Longleftrightarrow & \vdash_{\mathcal{IQC}} \alpha \\
\models_{\mathsf{IKL}} \alpha & \Longleftrightarrow & \vdash_{\mathcal{IQC}+(\text{pre})} \alpha \\
\models_{\mathsf{CDIK}} \alpha & \Longleftrightarrow & \vdash_{\mathcal{IQC}+(\text{cd})} \alpha \\
\models_{\mathsf{CDIKL}} \alpha & \Longleftrightarrow & \vdash_{\mathcal{IQC}+(\text{pre})+(\text{cd})} \alpha.
\end{array}
$$

Other notable work on first-order intermediate logics includes, e.g., [111, 122, 151].

We are also interested in first-order intuitionistic logics based on a single frame. For a frame $\mathbf{K} = \langle K, \preceq \rangle$, we say that an $\mathsf{IK}$-model ($\mathsf{CDIK}$-model) $\langle K, \preceq, \{D_k\}_{k \in K}, \{\mathcal{I}_k\}_{k \in K} \rangle$

---

[8]Note that there are intermediate logics that cannot be characterized by a particular class of intuitionistic Kripke models, as shown by Ono in [123].

is an $\mathsf{IK}(\mathbf{K})$-model ($\mathsf{CDIK}(\mathbf{K})$-model). If $\mathbf{K}$ is linear, we call it an $\mathsf{IKL}$-model ($\mathsf{CDIKL}(\mathbf{K})$-model). For any $\mathsf{L} \in \{\mathsf{IK}, \mathsf{IKL}, \mathsf{CDIK}, \mathsf{CDIKL}\}$, we say that $\alpha \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{IL}})$ is $\mathsf{L}(\mathbf{K})$-valid if $\alpha$ is $\mathfrak{M}$-valid for every $\mathsf{L}(\mathbf{K})$-model $\mathfrak{M}$.

**Example 1.11.** Consider the trivial frame $\mathbf{K} = \langle\{*\}, =\rangle$. It is then clear that for a formula $\alpha \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{IL}})$, $\alpha$ is $\mathsf{IKL}(\mathbf{K})$-valid if and only if $\alpha$ is $\mathsf{CDIKL}(\mathbf{K})$-valid if and only if $\alpha$ is valid in first-order classical logic.

**Example 1.12.** Consider the linear frame $\mathbf{Q} := \langle\mathbb{Q}, \leq\rangle$. Takano showed completeness of $\mathcal{IQC} + (\mathrm{pre}) + (\mathrm{cd})$ with respect to $\mathsf{CDIKL}(\mathbf{Q})$-models in [145]. Combined with Minari's completeness result from [110], it follows that for all $\alpha \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{IL}})$,

$$\models_{\mathsf{CDIKL}} \alpha \iff \models_{\mathsf{CDIKL}(\mathbf{Q})} \alpha \iff \vdash_{\mathcal{IQC}+(\mathrm{pre})+(\mathrm{cd})} \alpha.$$

Moreover, Corsi showed in [56] that $\mathsf{IKL}$-validity coincides with $\mathsf{IKL}(\mathbf{Q})$-validity. Hence, for all $\alpha \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{IL}})$,

$$\models_{\mathsf{IKL}} \alpha \iff \models_{\mathsf{IKL}(\mathbf{Q})} \alpha \iff \vdash_{\mathcal{IQC}+(\mathrm{pre})} \alpha.$$

Let us finally note that first-order classical logic can be interpreted in first-order intuitionistic logic via Gödel's double negation translation (see, e.g., [148]). That is, for any $\alpha \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{IL}})$,

$$\models_{\mathsf{IK}} \neg\neg\alpha \iff \alpha \text{ is valid in first-order classical logic,}$$

where $\neg\alpha := \alpha \to \bot$. From Church's undecidability result it follows that first-order intuitionistic logic is also undecidable.

## First-Order Gödel Logics

Standard first-order Gödel logic, here to be understood as the set of formulas in $\mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{IL}})$ that are $\mathbf{G}$-valid, was first properly investigated by Horn in 1969. He provided a completeness proof for this logic in [86], showing that for all $\alpha \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{IL}})$,

$$\models_{\mathbf{G}}^{\forall\exists} \alpha \iff \vdash_{\mathcal{IQC}+(\mathrm{pre})+(\mathrm{cd})} \alpha.$$

In fact, he also shows that $\mathbf{G}$-validity coincides with being $\mathbf{H}$-valid for each linearly ordered Heyting algebra $\mathbf{H}$ or, equivalently, being $\mathbf{A}$-valid for each Gödel set $A$. A different axiomatization was given by Takeuti and Titani in [146]. Their system incorporated a density rule, expressing that the truth value set is dense, which was later proven to be redundant by Takano in [144].

Using Horn's completeness result and the completeness result for $\mathsf{CDIKL}$-models, we obtain a relation between first-order Gödel logic and intuitionistic Kripke models. To be precise, we obtain that for all $\alpha \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{IL}})$,

$$\models_{\mathbf{G}}^{\forall\exists} \alpha \iff \models_{\mathsf{CDIKL}} \alpha.$$

Note that it follows that first-order Gödel logic is in fact an intermediate logic.

Similarly to our account of propositional Gödel logics, we can investigate first-order Gödel logics for different Gödel sets, that is, $\mathbf{A}$-validity for Gödel sets $A$. However, whereas in the propositional case the logics $\mathsf{G}(A)$ all coincide for any infinite Gödel set $A$, there are infinitely many different first-order Gödel logics, considered as sets of valid formulas. In fact, there are exactly countably infinitely many, as shown in [12]. We state this result for future reference.

**Theorem 1.13** (cf. [12])**.** *The set of first-order Gödel logics (viewed as sets of **A**-valid formulas, where A ranges over all Gödel sets) is countably infinite.*

The connection with first-order intuitionistic logic is extended further and characterized by Beckmann and Preining in [13]. They show a one-to-one correspondence between the class of sets of **A**-valid formulas for Gödel sets $A$ and the class of sets of CDIKL(**K**)-valid formulas for countable frames **K**. To be more precise, they show that for every Gödel set $A$, there exists a countable linear frame $\mathbf{K}_A$ such that for all $\alpha \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{IL}})$,

$$\models_{\mathbf{A}}^{\forall\exists} \alpha \quad \Longleftrightarrow \quad \models_{\mathsf{CDIKL}(\mathbf{K}_A)} \alpha,$$

and, conversely, for every countable linear frame **K**, there exists a Gödel set $A_K$ such that for all $\alpha \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{IL}})$,

$$\models_{\mathbf{A}_K}^{\forall\exists} \alpha \quad \Longleftrightarrow \quad \models_{\mathsf{CDIKL}(\mathbf{K})} \alpha.$$

A full characterization in terms of recursive enumerability for this class is given by Baaz, Preining, and Zach in [11]; that is, they characterize exactly for which Gödel sets $A$, **A**-validity is recursively enumerable, and provide axiomatizations for those that are. To state this characterization, we recall some topological terminology. For further details see, e.g., [115]. A point $x \in \mathbb{R}$ is called a *limit point* if every open neighbourhood of $x$ contains a point $y \neq x$, and *isolated* if it is not a limit point. A subset $X \subseteq \mathbb{R}$ is called *perfect* if it is closed and all points in $X$ are limit points in its relative topology. By a result of Cantor, every non-empty perfect set is necessarily uncountable. A proof of the following classical theorem can be found in [115].

**Theorem 1.14** (Cantor-Bendixson)**.** *If A is a closed subset of R, then it can be (uniquely) written as $A = X \cup C$ for some perfect set $X$ and countable set $C$ such that $X \cap C = \emptyset$. The set X is called the* perfect kernel *of A and the set C is called the* scattered part *of A.*

In [11], it was shown that for a Gödel set $A$, **A**-validity is recursively axiomatizable if and only if one of the following cases hold

(a)  $A$ is finite;

(b)  $A$ is uncountable and 0 is contained in the perfect kernel of $A$;

(c)  $A$ is uncountable and 0 is isolated.

To be more precise, if $A = G_n$ for some $n \in \mathbb{N}^+$, then $\mathbf{G}_n$-validity is axiomatized using the first-order version of the axiom schema $(\mathrm{fin}_n)$, i.e., for all $\alpha \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{IL}})$,

$$\models_{\mathbf{G}_n}^{\forall\exists} \alpha \quad \Longleftrightarrow \quad \vdash_{\mathcal{IQC}+(\mathrm{pre})+(\mathrm{cd})+(\mathrm{fin}_n)} \alpha.$$

If $A$ is uncountable and 0 is contained in the perfect kernel of $A$, then **A**-validity coincides with **G**-validity. Lastly, if $A$ is uncountable and 0 is isolated, then **A**-validity is axiomatized using the axiom schema $(\mathrm{iso}_0)$

$$(\mathrm{iso}_0) \ \ (\forall x)\neg\neg\alpha \to \neg\neg(\forall x)\alpha,$$

expressing exactly that 0 is isolated in $A$. That is, for all $\alpha \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{IL}})$,

$$\models_{\mathbf{A}}^{\forall\exists} \alpha \quad \Longleftrightarrow \quad \vdash_{\mathcal{IQC}+(\mathrm{pre})+(\mathrm{cd})+(\mathrm{iso}_0)} \alpha.$$

In all other cases, the set of **A**-valid formulas is not recursively enumerable. Via Gödel's double negation translation, it follows from Church's undecidability result that also first-order Gödel logic is undecidable. The proof methods employed in [13], [11], and [12] will be used in Chapter 3, when considering fragments of first-order Gödel logics.

### First-Order Łukasiewicz and Abelian Logic

As opposed to first-order Gödel logic, first-order Łukasiewicz logic was shown to not be recursively enumerable by Scarpellini in [139]. Here, we consider first-order Łukasiewicz logic to be the set of formulas $\alpha \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{IL}})$ that are **Ł**-valid. Proof systems for it have nevertheless been provided by a number of authors, e.g., Hay [82], Belluce and Chang [16, 17], Hájek [76], and Baaz and Metcalfe [9]. All their proof systems contain some infinitary rule, that is, a rule with infinitely many premises.[9]

As mentioned before, a deep connection exists between Łukasiewicz and Abelian logic. This connection extends to the first-order versions of both logics. Let us hence consider first-order Abelian logic, that is, the set of formulas from $\mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{A}})$ that are **R**-valid. Note that **R** is not complete: the real numbers are unbounded and so $\bigvee \mathbb{R}$ and $\bigwedge \mathbb{R}$ do not exist. It is here that we really need to work with safe **R**-structures. Note that for a safe **R**-structure $\mathfrak{M} = \langle D, \mathcal{I} \rangle$, the map from $D$ to $\mathbb{R}$ defined by $d \mapsto \mathcal{I}(P)(d_1, \ldots, d_{i-1}, d, d_{i+1}, \ldots, d_n)$ is *bounded* for each $n$-ary predicate $P$, $1 \leq i \leq n$, and $d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_n \in D$. Here, a function $f \colon X \to \mathbb{R}$ is called bounded if there exists $r \in \mathbb{R}$ such that $|f(x)| \leq r$ for all $x \in X$. In fact, this is an equivalent definition of safe **R**-structures. Indeed, consider any **R**-structure $\mathfrak{M} = \langle D, \mathcal{I} \rangle$ such that all maps $d \mapsto \mathcal{I}(P)(d_1, \ldots, d_{i-1}, d, d_{i+1}, \ldots, d_n)$ are bounded. We can then show by induction on formula length that for any formula $\alpha(x)$ with (possibly free) variable $x$ and $\mathfrak{M}$-evaluation $v$, $\|\alpha(x)\|_{\mathfrak{M},v}^{\mathbf{R}}$ exists and the map $a \mapsto \|\alpha(x)\|_{\mathfrak{M},v(x=a)}^{\mathbf{R}}$ from $D$ to $\mathbb{R}$ is bounded. We will often implicitly assume that an **R**-structure is safe.

We can now formulate an interpretation of first-order Łukasiewicz logic into first-order Abelian logic, making the connection between Łukasiewicz and Abelian logic concrete. The interpretation given here is an extension of the intuitive interpretation given in [107]. Let us fix a unary predicate $P_0$ and define $\bot := (\forall x) P_0(x) \wedge \neg (\forall x) P_0(x)$ in $\mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{A}})$, a constant that will be interpreted as the same non-positive real number under all $\mathfrak{M}$-evaluations $v$ for an **R**-structure $\mathfrak{M}$. We let $\mathrm{Fm}_{\forall\exists}^0(\mathcal{L}_{\mathsf{Ł}})$ denote the set of formulas from $\mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{Ł}})$ that do not contain $P_0$. We define the following map from the set $\mathrm{Fm}_{\forall\exists}^0(\mathcal{L}_{\mathsf{Ł}})$ to $\mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{A}})$:

$$
\begin{aligned}
P(t_1, \ldots, t_n)^{\bullet} &= (P(t_1, \ldots, t_n) \wedge \bar{0}) \vee \bot & ((\forall x)\alpha)^{\bullet} &= (\forall x)\alpha^{\bullet} \\
(\alpha \supset \beta)^{\bullet} &= (\alpha^{\bullet} \to \beta^{\bullet}) \wedge \bar{0} & ((\exists x)\alpha)^{\bullet} &= (\exists x)\alpha^{\bullet}. \\
(\sim \alpha)^{\bullet} &= \alpha^{\bullet} \to \bot
\end{aligned}
$$

We show that $(-)^{\bullet}$ preserves validity between first-order Łukasiewicz logic and first-order Abelian logic by identifying the value of $\alpha \in \mathrm{Fm}_{\forall\exists}^0(\mathcal{L}_{\mathsf{Ł}})$ in $[0, 1]$ with the value of $\alpha^{\bullet} \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{A}})$ in $[\bot, 0]$.

---

[9]With the current definition of a (finitary) proof system, a rule with infinitely many premises is not permitted. To adapt it, one needs to for example define a proof to be a (possibly infinite) tree. Since we will not use infinitary rules in this thesis, we assume the reader can extrapolate what is meant here.

**Theorem 1.15.** *For all $\alpha \in \mathrm{Fm}^0_{\forall\exists}(\mathcal{L}_{\text{Ł}})$,*

$$\models^{\forall\exists}_{\text{Ł}} \alpha \iff \models^{\forall\exists}_{\mathbf{R}} \alpha^\bullet.$$

*Proof.* Suppose first that $\alpha$ is not valid in some $\mathbf{Ł}$-structure $\mathfrak{M} = \langle D, \mathcal{I} \rangle$ with $\mathfrak{M}$-evaluation $v$, i.e., $\|\alpha\|^{\text{Ł}}_{\mathfrak{M},v} < 1$. We consider the $\mathbf{R}$-structure $\mathfrak{M}' = \langle D, \mathcal{I}' \rangle$ where $\mathcal{I}'(P_0)(d) = -1$ for all $d \in D$, and $\mathcal{I}'(P)(d_1, \ldots, d_n) = \mathcal{I}(P)(d_1, \ldots, d_n) - 1$ for each $n$-ary predicate $P$ and $d_1, \ldots, d_n \in D$. It easily follows that $|\mathcal{I}'(P)(d_1, \ldots, d_n)| \leq 1$ for all $n$-ary predicates $P$ and $d_1, \ldots, d_n \in D$, and hence $\mathfrak{M}'$ is a safe $\mathbf{R}$-structure. Moreover, any $\mathfrak{M}$-evaluation is an $\mathfrak{M}'$-evaluation and vice versa, and $\|\bot\|^{\mathbf{R}}_{\mathfrak{M}',v} = -1$. It suffices to prove that $\|\beta^\bullet\|^{\mathbf{R}}_{\mathfrak{M}',v'} = \|\beta\|^{\text{Ł}}_{\mathfrak{M},v'} - 1$ for any $\beta \in \mathrm{Fm}^0_{\forall\exists}(\mathcal{L}_{\text{Ł}})$ and $\mathfrak{M}$-evaluation $v'$, since then $\|\alpha^\bullet\|^{\mathbf{R}}_{\mathfrak{M}',v} = \|\alpha\|^{\text{Ł}}_{\mathfrak{M},v} - 1 < 0$ and so $\alpha^\bullet$ is not $\mathbf{R}$-valid. We proceed by induction on the length of $\beta$. The base case follows by definition. For the inductive step we obtain, we first consider the case when $\beta$ is $\beta_1 \supset \beta_2$. Then, using the induction hypothesis,

$$
\begin{aligned}
\|(\beta_1 \supset \beta_2)^\bullet\|^{\mathbf{R}}_{\mathfrak{M}',v_1} &= \|(\beta_1^\bullet \to \beta_2^\bullet) \wedge \bar{0}\|^{\mathbf{R}}_{\mathfrak{M}',v_1} \\
&= \min\{\|\beta_2^\bullet\|^{\mathbf{R}}_{\mathfrak{M}',v_1} - \|\beta_1^\bullet\|^{\mathbf{R}}_{\mathfrak{M}',v_1}, 0\} \\
&= \min\{\|\beta_2^\bullet\|^{\text{Ł}}_{\mathfrak{M},v_1} - 1 - (\|\beta_1^\bullet\|^{\text{Ł}}_{\mathfrak{M},v_1} - 1), 0\} \\
&= \min\{\|\beta_2^\bullet\|^{\text{Ł}}_{\mathfrak{M},v_1} - \|\beta_1^\bullet\|^{\text{Ł}}_{\mathfrak{M},v_1}, 0\} \\
&= \min\{1 - \|\beta_1^\bullet\|^{\text{Ł}}_{\mathfrak{M},v_1} - \|\beta_2^\bullet\|^{\text{Ł}}_{\mathfrak{M},v_1}, 1\} - 1 \\
&= \|\alpha \supset \beta\|^{\text{Ł}}_{\mathfrak{M},v_1} - 1.
\end{aligned}
$$

The case where $\beta$ is $\sim\beta_1$ is very similar. Now consider the case where $\beta$ is $(\forall x)\beta_1$. Then

$$
\begin{aligned}
\|(\forall x)\beta_1)^\bullet\|^{\mathbf{R}}_{\mathfrak{M}',v_1} &= \|(\forall x)\beta_1^\bullet\|^{\mathbf{R}}_{\mathfrak{M}',v_1} \\
&= \bigwedge\{\|\beta_1^\bullet\|^{\mathbf{R}}_{\mathfrak{M}',v_1(x=a)} \mid a \in D\} \\
&= \bigwedge\{\|\beta_1\|^{\text{Ł}}_{\mathfrak{M},v_1(x=a)} - 1 \mid a \in D\} \\
&= \bigwedge\{\|\beta_1\|^{\text{Ł}}_{\mathfrak{M},v_1(x=a)} \mid a \in D\} - 1 \\
&= \|(\forall x)\beta_1\|^{\text{Ł}}_{\mathfrak{M},v_1} - 1.
\end{aligned}
$$

The case where $\beta$ is $(\exists x)\beta_1$ is analogous.

Suppose now conversely that $\alpha^\bullet$ is not valid in an $\mathbf{R}$-structure $\mathfrak{M} = \langle D, \mathcal{I} \rangle$ with $\mathfrak{M}$-evaluation $v$, that is, $\|\alpha^\bullet\|^{\mathbf{R}}_{\mathfrak{M},v} < 0$. Observe first that if $\|(\forall x)P_0(x)\|^{\mathbf{R}}_{\mathfrak{M},v} = 0$, then $\|\bot\|^{\mathbf{R}}_{\mathfrak{M},v'} = 0$ for all $\mathfrak{M}$-evaluations $v'$. With a simple induction on the size of $\beta \in \mathrm{Fm}^0_{\forall\exists}(\mathcal{L}_{\text{Ł}})$ we obtain $\|\beta^\bullet\|^{\mathbf{R}}_{\mathfrak{M},v'} = 0$ for all $\mathfrak{M}$-evaluations $v'$, a contradiction with $\|\alpha^\bullet\|^{\mathbf{R}}_{\mathfrak{M},v} < 0$. Hence $\|(\forall x)P_0(x)\|^{\mathbf{R}}_{\mathfrak{M},v} \neq 0$. Moreover, by scaling (that is, dividing each $\mathcal{I}(P)(d_1, \ldots, d_n)$ by $\|(\forall x)P_0(x)\|^{\mathbf{R}}_{\mathfrak{M},v}$ for each $n$-ary predicate $P$ and $d_1, \ldots, d_n \in D$), we may assume that $\|\bot\|^{\mathbf{R}}_{\mathfrak{M},v'} = -1$ for all $\mathfrak{M}$-evaluations $v'$. We now consider the $\mathbf{Ł}$-structure $\mathfrak{M}' = \langle D, \mathcal{I}' \rangle$ where

$$\mathcal{I}'(P)(d_1, \ldots, d_n) = \max\{0, \min\{\mathcal{I}(P)(d_1, \ldots, d_n) + 1, 1\}\}$$

for each $n$-ary predicate $P$ and $d_1, \ldots, d_n \in D$. Note that since any $\mathbf{Ł}$-structure is safe, so is $\mathfrak{M}'$. It now suffices to show that $\|\beta\|^{\text{Ł}}_{\mathfrak{M}',v'} = \|\beta^\bullet\|^{\mathbf{R}}_{\mathfrak{M},v'} + 1$ for all $\beta \in \mathrm{Fm}^0_{\forall\exists}(\mathcal{L}_{\text{Ł}})$ and $\mathfrak{M}$-evaluations $v'$ by an easy induction on the length of $\beta$. $\qquad\square$

This interpretation allows for the study of first-order Łukasiewicz logic inside first-order Abelian logic. The advantages of studying first-order Abelian logic include a semantics based on structures that has been studied intensively, and a natural separation between the multiplicative (group) fragment and the additive (lattice) fragment of this logic. Such a separation is exploited in [61], where related modal Abelian logics are studied. A fragment of first-order Abelian logic is the main topic of Chapter 4.

## 1.3 One-Variable Fragments

In this section, we can finally introduce the main topic of this thesis: one-variable fragments. As should be clear from the previous section, first-order logics are, in general, computationally complicated. Most well-studied first-order logics are undecidable, or even not recursively enumerable. For this reason, fragments of first-order logics have been studied that are computationally easier and still have a high expressive power. Examples include *prenex* fragments, where only formulas of the form $(Qx_1)\dots(Qx_n)\alpha$ are considered with $\alpha$ a formula not containing quantifiers and $(Qx_1)\dots(Qx_n)$ some fixed sequence of quantifiers; *guarded* fragments, where the type of quantification is restricted; and the *monadic* fragment, where only formulas containing unary predicates are considered. Another approach is to limit the number of variables that are allowed to occur in a formula. The maximum number of variables to consider for a computationally desirable fragment is rather small: the two-variable fragment of first-order classical logic is decidable [114], but its three-variable fragment is undecidable [142]. When considering first-order intuitionistic logic, already the two-variable fragment is undecidable [93].

In this thesis, we are particularly interested in first-order formulas containing a single variable. Note that if $\alpha$ contains only a single variable $x$, and $\mathrm{Fm}_{\forall\exists}(\mathcal{L})$ is defined without function symbols, it suffices to consider only unary predicates.[10] For this reason, we define the set $\mathrm{Fm}_1(\mathcal{L})$ of first-order formulas $\alpha, \beta, \dots$ built from a countably infinite set of unary predicates $\{P_i\}_{i\in\mathbb{N}}$, propositional connectives $\star \in \mathcal{L}$, a single variable $x$, and quantifiers $\forall, \exists$.

**Remark 1.16.** Note that there is a difference between the one-variable fragment of a first-order logic and the monadic fragment. The latter concerns a restriction of the first-order language to only unary predicate letters, but no restriction on the number of variables. For first-order classical logic, these two fragments coincide up to equivalence of formulas. Note that the one-variable fragment is obviously included in the monadic fragment. Conversely, for first-order classical logic each sentence in the monadic fragment is equivalent to a formula in the one-variable fragment. For details, see, e.g., [84]. In general, they do not coincide. For example, the validity problem of the monadic fragment of first-order Gödel logic is undecidable [5], whereas it is co-NP-complete for the one-variable fragment, as proved in [38]. Despite the difference between the one-variable and monadic fragment, the literature is not always precise about the distinction. We shall be as precise as possible. In fact, all fragments studied in this thesis are one-variable fragments, even though they might be referred to as monadic in some of the literature.

For a one-variable first-order formula, a quantifier $(\forall x)$ or $(\exists x)$ can be viewed as a *modality*, that is, an operator that expresses statements like "it is necessary that", or "it

---

[10]Indeed, any occurrence of an $n$-ary predicate $P(x,\dots,x)$ for $n \geq 2$ in a formula can be replaced with $P'(x)$ for a unary predicate $P'$.

is possible that". Logics containing such modalities, usually referred to as modal logics, have been studied in much more generality. For more on modal logics, see, e.g., [27, 46].

Given a propositional language $\mathcal{L}$, we let $\mathrm{Fm}_{\square\Diamond}(\mathcal{L})$ denote the set of modal formulas $\varphi, \psi, \ldots$ built inductively over the set of propositional variables $\{p_i\}_{i\in\mathbb{N}}$, propositional connectives in $\mathcal{L}$, and unary modal connectives $\square, \Diamond$. The *length* of a formula $\varphi \in \mathrm{Fm}_{\square\Diamond}(\mathcal{L})$ is again defined inductively as the number of $n+1$-ary connectives occurring in $\varphi$. There exists a natural correspondence between the sets $\mathrm{Fm}_{\square\Diamond}(\mathcal{L})$ and $\mathrm{Fm}_1(\mathcal{L})$. We define standard translation functions $(-)^*$ and $(-)^\circ$ as follows for $n$-ary $\star \in \mathcal{L}$ and $i \in \mathbb{N}$

$$(P_i(x))^* = p_i \qquad\qquad\qquad p_i^\circ = P_i(x)$$
$$\star(\alpha_1, \ldots, \alpha_n)^* = \star(\alpha_1^*, \ldots, \alpha_n^*) \qquad \star(\varphi_1, \ldots, \varphi_n)^\circ = \star(\varphi_1^\circ, \ldots, \varphi^\circ)$$
$$((\forall x)\alpha)^* = \square\alpha^* \qquad\qquad\qquad (\square\varphi)^\circ = (\forall x)\varphi^\circ$$
$$((\exists x)\alpha)^* = \Diamond\alpha^* \qquad\qquad\qquad (\Diamond\varphi)^\circ = (\exists x)\varphi^\circ.$$

Clearly $(\alpha^*)^\circ = \alpha$ for any $\alpha \in \mathrm{Fm}_1(\mathcal{L})$ and $(\varphi^\circ)^* = \varphi$ for any $\varphi \in \mathrm{Fm}_{\square\Diamond}(\mathcal{L})$, so we may alternate between the first-order and modal notations as convenient.

Correspondences between one-variable fragments of first-order logics and modal logics form the main focus of this thesis. Here the line between one-variable and monadic fragments seems to blur in the literature. Modal logics that correspond to the one-variable fragment of a first-order logic are often referred to as monadic, despite the fact that they do not correspond to the monadic fragment. Of course, any one-variable fragment is included in the monadic fragment, so it is harmless speak about a monadic modal logic. We shall be precise in the naming of the first-order fragments. As first examples, we consider the one-variable fragments of first-order classical and intuitionistic logic.

**Example 1.17.** The one-variable fragment of first-order classical logic was first axiomatized by Wajsberg in [152]. This fragment has since been extensively studied; in particular as the modal logic of Kripke models based on an equivalence relation, denoted by S5. An axiomatization of S5 can be obtained by adding to $\mathcal{CPC}$ the axiom schemas and rules from Figure 1.5. We call the resulting proof system $\mathcal{S}5$. Axioms (4) and (B) could be replaced by a single axiom schema $\Diamond\varphi \to \square\Diamond\varphi$. It then follows that for any $\alpha \in \mathrm{Fm}_1(\mathcal{L}_{\mathsf{CL}})$,

$$\alpha \text{ is valid in first-order classical logic} \quad \Longleftrightarrow \quad \vdash_{\mathcal{S}5} \alpha^*.$$

Note that this completeness result is really an improvement on the completeness result with respect to $\mathcal{CQC}$. Indeed, a proof of a formula $\alpha \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{CL}})$ in $\mathcal{CQC}$ can possibly use multiple variables, something that is not possible in $\mathcal{S}5$. The validity problem for S5 is, similarly to CL, co-NP-complete [95].

**Example 1.18.** The one-variable fragment of first-order intuitionistic logic was axiomatized by Prior and Bull. Prior gave an axiomatization in the modal language, extending the intuitionistic calculus $\mathcal{IPC}$ by a number of rules [132]. Here we give an equivalent proof system called $\mathcal{MIPC}$, as taken from [26], defined to consist of all intuitionistic axiom schema from $\mathcal{IPC}$ extended with the axiom and rule schemas from Figure 1.6. In the presence of the other axioms, $(\square\varphi \land \square\psi) \to \square(\varphi \land \psi)$ is equivalent to the famous Kripke axiom schema (K) from $\mathcal{S}5$. Bull then subsequently showed in [36] that this is

$$
\begin{array}{rl}
\text{(K)} & \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \\
\text{(Dual)} & (\Diamond\varphi \to \neg\Box\neg\varphi) \wedge (\neg\Box\neg\varphi \to \Diamond\varphi) \\
\text{(T)} & \Box\varphi \to \varphi \\
\text{(4)} & \Box\varphi \to \Box\Box\varphi \\
\text{(B)} & \varphi \to \Box\Diamond\varphi
\end{array}
$$

$$\frac{\varphi}{\Box\varphi}\ \text{(nec)}$$

Figure 1.5: Additional axiom and rule schemas for proof system $\mathcal{S}5$

$$
\begin{array}{rl}
\text{(1)} & \Box\varphi \to \varphi \\
\text{(2)} & \Diamond\varphi \to \Box\Diamond\varphi \\
\text{(3)} & (\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi) \\
\text{(4)} & \Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi) \\
\text{(5)} & \varphi \to \Diamond\varphi \\
\text{(6)} & \Diamond\Box\varphi \to \Box\varphi \\
\text{(7)} & \Diamond(\varphi \vee \psi) \to (\Diamond\varphi \vee \Diamond\psi)
\end{array}
$$

$$\frac{\varphi}{\Box\varphi}\ \text{(nec)}$$

Figure 1.6: Additional axiom and rule schemas for proof system $\mathcal{MIPC}$

indeed an axiomatization of the one-variable fragment of first-order intuitionistic logic, by proving that for every $\alpha \in \mathrm{Fm}_1(\mathcal{L}_{\mathsf{IL}})$,

$$\models_{\mathsf{IK}} \alpha \quad \Longleftrightarrow \quad \vdash_{\mathcal{MIPC}} \alpha^*.$$

Although $\mathcal{MIPC}$-derivability is decidable, there is a significant jump in complexity compared to propositional $\mathsf{IL}$-validity: it is co-NEXPTIME-complete [101].

If we are only interested in formulas containing a single variable, we can simplify the notation for intuitionistic Kripke models. Indeed, for $\mathsf{L} \in \{\mathsf{IK}, \mathsf{IKL}, \mathsf{CDIK}, \mathsf{CDIKL}\}$, an $\mathsf{L}$-model $\mathfrak{M} = \langle K, \preceq, \{D_k\}_{k \in K}, \{\mathcal{I}_k\}_{k \in K}\rangle$ is called an $\mathsf{L}_1$-*model* if it is restricted to one-variable formulas. In particular, any $\mathcal{I}_k$ is restricted to unary predicates and any $\mathfrak{M}$-evaluation for $k \in K$ is a map $\nu\colon \{x\} \to D_k$. We then write $\mathfrak{M}, k \vDash^a \alpha$ for $\mathfrak{M}, k \vDash^\nu \alpha$ if $\nu_k(x) = a$. With this notation, we can leave out $\nu$ when considering satisfaction in $\mathfrak{M}$. We say that $\alpha \in \mathrm{Fm}_1(\mathcal{L}_{\mathsf{IL}})$ is $\mathsf{L}_1$-valid, written $\models_{\mathsf{L}_1} \alpha$, if it is valid in all $\mathsf{L}_1$-models. Note that it is immediate that

$$\models_{\mathsf{L}} \alpha \quad \Longleftrightarrow \quad \models_{\mathsf{L}_1} \alpha.$$

Despite the equivalence, we will often speak of $\mathsf{L}_1$-validity to stress that we are considering the one-variable fragment.

We now introduce a Kripke semantics to interpret the monadic many-valued logics studied in this thesis, inspired by approaches taken in [33, 34, 41, 69, 76].

**Definition 1.19.** Let $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \to, f, e\rangle$ be an $\mathsf{FL}_e$-algebra. An $\mathsf{S5}(\mathbf{A})$-*frame* is an $A$-valued equivalence relation: a pair $\langle W, R\rangle$ consisting of a non-empty set $W$ and a map $R\colon W \times W \to A$ satisfying for all $u, v, w \in W$,

(i)   $Rww = e$   (*reflexivity*);

(ii)  $Rwv = Rvw$   (*symmetry*);

(iii) $Rwv \cdot Rvu \leq Rwu$   (*transitivity*).

It is called *crisp* if $Rwv \in \{f, e\}$ for all $w, v \in W$. If $Rwv = e$ for all $w, v \in W$, we call $\langle W, R \rangle$ *universal*.

**Definition 1.20.** Let $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \rightarrow, f, e \rangle$ be an $\mathsf{FL}_e$-algebra. An $\mathsf{S5}(\mathbf{A})$-*model* is a triple $\mathcal{M} = \langle W, R, V \rangle$ consisting of an $\mathsf{S5}(\mathbf{A})$-frame $\langle W, R \rangle$ and a map $V \colon \{p_i\}_{i \in \mathbb{N}} \times W \rightarrow A$. The map $V$ is extended inductively to $\bar{V} \colon \mathrm{Fm}_{\square\Diamond}(\mathcal{L}_{\mathsf{FL}_e}) \times W \rightarrow A$ as follows, where $\star \in \{\wedge, \vee, \cdot, \rightarrow\}$:

$$\bar{V}(\bot, w) = f$$
$$\bar{V}(\top, w) = e$$
$$\bar{V}(\varphi \star \psi, w) = \bar{V}(\varphi, w) \star \bar{V}(\psi, w)$$
$$\bar{V}(\square\varphi, w) = \bigwedge\{Rwv \rightarrow \bar{V}(\varphi, v) \mid v \in W\}$$
$$\bar{V}(\Diamond\varphi, w) = \bigvee\{Rwv \cdot \bar{V}(\varphi, v) \mid v \in W\}.$$

If the supremum or infimum does not exist at a world, we take the value of the modal formula and all those formulas containing it at that world to be undefined. We call $\mathcal{M}$ *safe* if $\bar{V}(\varphi, w)$ exists for all $w \in W$ and $\varphi \in \mathrm{Fm}_{\square\Diamond}(\mathcal{L}_{\mathsf{FL}_e})$. We say that $\mathcal{M}$ is an $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-*model* if $\langle W, R \rangle$ is crisp, and *universal* if $\langle W, R \rangle$ is universal. A formula $\varphi \in \mathrm{Fm}_{\square\Diamond}(\mathcal{L}_{\mathsf{FL}_e})$ is said to be *valid* in $\mathcal{M}$ if $\bar{V}(\varphi, w) \geq e$ for all $w \in W$, and $\mathsf{S5}(\mathbf{A})$-*valid*, written $\models_{\mathsf{S5}(\mathbf{A})} \varphi$, if it is valid in all safe $\mathsf{S5}(\mathbf{A})$-models. We also say that $\varphi \in \mathrm{Fm}_{\square\Diamond}(\mathcal{L}_{\mathsf{FL}_e})$ is $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-*valid*, written $\models_{\mathsf{S5}(\mathbf{A})^{\mathsf{C}}} \varphi$, if it is valid in all safe $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-models.

Note that if $\mathbf{A}$ is a complete $\mathsf{FL}_e$-algebra, then any $\mathsf{S5}(\mathbf{A})$-model is safe. Moreover, if $\mathbf{A}$ is an $\mathsf{FL}_{eo}$-algebra or $f^{\mathbf{A}} = e^{\mathbf{A}}$, the inductive definitions for the modal formulas in a safe $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-model simplify to

$$\bar{V}(\square\varphi, w) = \bigwedge\{V(\varphi, v) \mid Rwv\}$$
$$\bar{V}(\Diamond\varphi, w) = \bigvee\{V(\varphi, v) \mid Rwv\},$$

since for all $a \in A$, $e \rightarrow a = a$ and, if $\mathbf{A}$ is an $\mathsf{FL}_{eo}$-algebra, $f \cdot a = f$ holds, and $f \rightarrow a = f \rightarrow f$ is the top element of $A$. It then follows that a formula $\varphi \in \mathrm{Fm}_{\square\Diamond}(\mathcal{L}_{\mathsf{FL}_e})$ is $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-valid if and only if $\varphi$ is valid in all safe universal $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-models. For a universal $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-model, we usually drop $R$ from the signature.

Similarly to our approach to many-valued first-order logics, we can use the term-equivalences from Examples 1.4–1.7 to abuse notation. If some algebra $\mathbf{A}$ over language $\mathcal{L}$ is term-equivalent to some $\mathsf{FL}_e$-algebra $\mathbf{A}'$, we say that a $\varphi \in \mathrm{Fm}_{\square\Diamond}(\mathcal{L})$ is $\mathsf{S5}(\mathbf{A})$-*valid* if its equivalent formula in $\mathrm{Fm}_{\square\Diamond}(\mathcal{L}_{\mathsf{FL}_e})$ is $\mathsf{S5}(\mathbf{A}')$-valid. Moreover, we abuse our definition of a logic by sometimes speaking of the logic $\mathsf{S5}(\mathbf{A})$ ($\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$), by which we mean the set of $\mathsf{S5}(\mathbf{A})$-valid ($\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-valid) formulas.

The naming of the defined models is derived from their connection to the modal logic S5. Indeed, $\mathsf{S5}(\mathbf{2})$-models and, equivalently, $\mathsf{S5}(\mathbf{2})^{\mathsf{C}}$-models are exactly Kripke models based on equivalence relations as considered in classical modal logic, see, e.g., [27]. That is,

$\varphi \in \mathrm{Fm}_{\square\lozenge}(\mathcal{L}_{\mathsf{CL}})$ is $\mathsf{S5}(\mathbf{2})^{\mathsf{C}}$-valid if and only if $\varphi$ is derivable in $\mathcal{S}5$. In fact, $\mathsf{S5}(\mathbf{A})$-models in general satisfy many of the axioms classically associated with modal logic $\mathsf{S5}$. Note that, unlike in $\mathsf{S5}$, $\square$ and $\lozenge$ are not interdefinable in $\mathsf{S5}(\mathbf{A})$-models in general.

**Lemma 1.21.** *Let* $\mathbf{A}$ *be an* $\mathsf{FL}_e$-*algebra. Then the following formulas are* $\mathsf{S5}(\mathbf{A})$-*valid for any* $\varphi, \psi \in \mathrm{Fm}_{\square\lozenge}(\mathcal{L}_{\mathsf{FL}_e})$:

| | | | |
|---|---|---|---|
| (i) | $\square\varphi \to \varphi$ | (vi) | $\lozenge\square\varphi \to \varphi$ |
| (ii) | $\varphi \to \lozenge\varphi$ | (vii) | $\lozenge\varphi \to \square\lozenge\varphi$ |
| (iii) | $\square\varphi \to \square\square\varphi$ | (viii) | $\lozenge\square\varphi \to \square\varphi$ |
| (iv) | $\lozenge\lozenge\varphi \to \lozenge\varphi$ | (ix) | $(\square\varphi \wedge \square\psi) \to \square(\varphi \wedge \psi)$ |
| (v) | $\varphi \to \square\lozenge\varphi$ | (x) | $\lozenge(\varphi \vee \psi) \to (\lozenge\varphi \vee \lozenge\psi)$. |

*Proof.* Consider any safe $\mathsf{S5}(\mathbf{A})$-model $\langle W, R, V \rangle$ and $w \in W$. Note that $e \leq \bar{V}(\varphi \to \psi, w)$ is equivalent to $\bar{V}(\varphi, w) \leq \bar{V}(\psi, w)$.

(i) Since $Rww = e$,

$$\begin{aligned}
\bar{V}(\square\varphi, w) &= \bigwedge\{Rwv \to \bar{V}(\varphi, v) \mid v \in W\} \\
&\leq Rww \to \bar{V}(\varphi, w) \\
&= e \to \bar{V}(\varphi, w) \\
&= \bar{V}(\varphi, w).
\end{aligned}$$

(ii) This follows similarly to (i).

(iii) Note that for all $u, v \in W$, by the transitivity of $R$,

$$\begin{aligned}
Rwv \cdot Rvu \cdot \bar{V}(\square\varphi, w) &\leq Rwu \cdot \bar{V}(\square\varphi, w) \\
&\leq Rwu \cdot (Rwu \to \bar{V}(\varphi, u)) \\
&\leq \bar{V}(\varphi, u).
\end{aligned}$$

It follows that $Rwv \cdot \bar{V}(\square\varphi, w) \leq \bar{V}(\square\varphi, v)$ for all $v \in W$, and so $\bar{V}(\square\varphi, w) \leq \bar{V}(\square\square\varphi, w)$ as required.

(iv) This follows similarly to (iii).

(v) Note that for all $v \in W$, $Rwv \cdot \bar{V}(\varphi, w) = Rvw \cdot \bar{V}(\varphi, w) \leq \bar{V}(\lozenge\varphi, v)$. By residuation, we obtain $\bar{V}(\varphi, w) \leq \bar{V}(\square\lozenge\varphi, w)$.

(vi) The proof is analogous to the proof of (v).

(vii) By the symmetry and transitivity of $R$, we have $Rwu \cdot Rwv \leq Rvu$ for all $v, u \in W$. Hence for all $v \in W$,

$$\begin{aligned}
\bar{V}(\lozenge\varphi, w) \cdot Rwv &= \bigvee\{Rwu \cdot \bar{V}(\varphi, u) \mid u \in W\} \cdot Rwv \\
&= \bigvee\{Rwu \cdot Rwv \cdot \bar{V}(\varphi, u) \mid u \in W\} \\
&\leq \bigvee\{Rvu \cdot \bar{V}(\varphi, u) \mid u \in W\} \\
&= \bar{V}(\lozenge\varphi, v).
\end{aligned}$$

Residuation then implies $\bar{V}(\lozenge\varphi, w) \leq Rwv \to \bar{V}(\lozenge\varphi, v)$ for all $v \in W$ and so $\bar{V}(\lozenge\varphi, w) \leq \bar{V}(\square\lozenge\varphi, w)$.

(viii) Consider any $u, v \in W$. By symmetry and transitivity of $R$, we have $Rwv \cdot Rwu \leq Rvu$, as well as $\bar{V}(\Box\varphi, v) \leq Rvu \rightarrow \bar{V}(\varphi, u)$. It follows that $Rvu \cdot \bar{V}(\Box\varphi, v) \leq \bar{V}(\varphi, u)$, and so

$$Rwv \cdot Rwu \cdot \bar{V}(\Box\varphi, v) \leq Rvu \cdot \bar{V}(\Box\varphi, v) \leq \bar{V}(\varphi, u).$$

This gives $Rwv \cdot \bar{V}(\Box\varphi, v) \leq Rwu \rightarrow \bar{V}(\varphi, u)$ for all $u, v \in W$, and so $\bar{V}(\Diamond\Box\varphi, w) \leq \bar{V}(\Box\varphi, w)$.

(ix) This follows from the fact that $a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c)$ for all $a, b, c \in A$.

(x) This follows from the fact that $a \cdot (b \vee c) = (a \cdot b) \vee (a \cdot c)$ for all $a, b, c \in A$.    □

**Remark 1.22.** Something to note is that, although it is $\mathcal{S}5$- and $\mathcal{MIPC}$-derivable, the Kripke axiom $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$ is not $\mathsf{S5}(\mathbf{A})$-valid in general. For example, consider the three-element Łukasiewicz chain $\mathbf{Ł}_3$, and the $\mathsf{S5}(\mathbf{Ł_3})$-model $\langle W, R, V \rangle$ where $W = \{w, v\}$, $Rww = Rvv = 1$, $Rwv = Rvw = \frac{1}{2}$, $V(p_1, w) = V(p_2, w) = 1$, $V(p_1, v) = \frac{1}{2}$, and $V(p_2, v) = 0$. It then follows that $\bar{V}(\Box(p_1 \rightarrow p_2), w) = 1$, $\bar{V}(\Box p_1, w) = 1$, and $\bar{V}(\Box p_2, w) = \frac{1}{2}$, and so

$$\bar{V}(\Box(p_1 \rightarrow p_2) \rightarrow (\Box p_1 \rightarrow \Box p_2), w) = \frac{1}{2} < 1.$$

## Monadic Gödel Logics

Let us now discuss $\mathsf{S5}(\mathbf{A})$- and $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-validity when $\mathbf{A}$ is the algebra corresponding to a Gödel set $A$. If $A$ is the standard Gödel set $G$, $\mathsf{S5}(\mathbf{A})$- and $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-validity has been studied extensively. Caicedo and Rodríguez provided axiomatizations for both $\mathsf{S5}(\mathbf{G})$- and $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$-validity in [42]. We define the modal versions (pre) and (cd$_\Box$) of the prelinearity and the constant domain schema, respectively, by

(pre) $(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$ and (cd$_\Box$) $\Box(\varphi \vee \Box\psi) \rightarrow (\Box\varphi \vee \Box\psi)$.

The results by Caicedo and Rodríguez can then be written as: for every $\varphi \in \mathrm{Fm}_{\Box\Diamond}(\mathcal{L}_{\mathsf{IL}})$,

$$\models_{\mathsf{S5}(\mathbf{G})} \varphi \iff \vdash_{\mathcal{MIPC}+(\mathrm{pre})} \varphi$$
$$\models_{\mathsf{S5}(\mathbf{G})^{\mathsf{C}}} \varphi \iff \vdash_{\mathcal{MIPC}+(\mathrm{pre})+(\mathrm{cd}_\Box)} \varphi.$$

In [38], it was shown that $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$-validity is co-NP-complete. Moreover, $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$ corresponds to the one-variable fragment of first-order Gödel logic: $\mathbf{G}$-structures $\mathfrak{M}$ with an $\mathfrak{M}$-evaluation restricted to one-variable formulas correspond exactly to the $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$-models. Combining this fact with the axiomatization for $\mathsf{CDIKL}$-validity by Minari and Takano discussed earlier, we obtain that for all $\alpha \in \mathrm{Fm}_1(\mathcal{L}_{\mathsf{IL}})$,

$$\models_{\mathsf{CDIKL}_1} \alpha \iff \vdash_{\mathcal{MIPC}+(\mathrm{pre})+(\mathrm{cd}_\Box)} \alpha^*.$$

Building on Bull's completeness proof for $\mathcal{MIPC}$, Ono showed in [124] that for all $\alpha \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{IL}})$,

$$\models_{\mathsf{CDIK}_1} \alpha \iff \vdash_{\mathcal{MIPC}+(\mathrm{cd}_\Box)} \alpha^*.$$

In Chapter 3, we solve an open problem by showing that for each $\alpha \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{IL}})$,

$$\models_{\mathsf{IKL}_1} \alpha \iff \vdash_{\mathcal{MIPC}+(\mathrm{pre})} \alpha^*.$$

We also further investigate $\mathsf{S5}(\mathbf{A})$- and $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-validity for Gödel sets $A$, connecting them to $\mathsf{IKL}_1(\mathbf{K})$- and $\mathsf{CDIKL}_1(\mathbf{K})$-validity for linear frames $\mathbf{K}$, similarly to the correspondences obtained by Beckmann and Preining in [13] for the full first-order Gödel logics. We additionally establish decidability and complexity results for a large class of them.

Recall that there are exactly countably infinitely many first-order Gödel logics (Theorem 1.13). Since for each Gödel set $A$, $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ corresponds to the one-variable fragment of the first-order Gödel logic based on $\mathbf{A}$, this implies that there are at most countably infinitely many different logics $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$. We show that there are at least infinitely many different logics $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ and $\mathsf{S5}(\mathbf{A})$, giving the precise number of logics $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ and paving the way to obtain a similar result for the logics $\mathsf{S5}(\mathbf{A})$ in Chapter 3.

Let us say that an element $a$ of a Gödel set $A$ is a *right accumulation point of $A$* if $a < 1$ and for all $b \in A$ such that $a < b$, there exists $c \in A$ such that $a < c < b$. Left accumulation points of $A$ are defined analogously. Let $\mathsf{R}(A)$ and $\mathsf{L}(A)$ denote the sets of right and left accumulation points of $A$, respectively. We use the following formula to detect right accumulation points of $A$:

$$\chi(p_i) := \Box((p_i \rightarrow \Box p_i) \rightarrow \Box p_i) \rightarrow \Box p_i.$$

**Lemma 1.23.** *Let $A$ be any Gödel set and let $\langle W, R, V \rangle$ be an $\mathsf{S5}(\mathbf{A})$-model with $w \in W$. If $\bar{V}(\chi(p_i), w) < 1$, then $\bar{V}(\Box p_i, w)$ is a right accumulation point of $A$.*

*Proof.* Suppose that $\bar{V}(\chi(p_i), w) < 1$. To prove that $\bar{V}(\Box p_i, w)$ is a right accumulation point of $A$, it suffices to show that $\bar{V}(\Box p_i, w) < Rwv \rightarrow V(p_i, v)$ for all $v \in W$. For a contradiction, suppose that there exists $v \in W$ such that $\bar{V}(\Box p_i, w) = Rwv \rightarrow V(p_i, v)$. From $\bar{V}(\Box p_i, w) = \bar{V}(\chi(p_i), w) < 1$ it follows that $\bar{V}(\Box p_i, w) = V(p_i, v) < Rwv$. By the symmetry and transitivity of $R$, we have $Rwv \wedge Rvu = Rwv \wedge Rwu$ for all $u \in W$, so

$$
\begin{aligned}
Rwv &\rightarrow \bar{V}(\Box p_i, v) \\
&= Rwv \rightarrow \bigwedge \{Rvu \rightarrow V(p_i, u) \mid u \in W\} \\
&= Rwv \rightarrow \bigwedge \{(Rwv \wedge Rvu) \rightarrow V(p_i, u) \mid u \in W\} \\
&= Rwv \rightarrow \bigwedge \{(Rwv \wedge Rwu) \rightarrow V(p_i, u) \mid u \in W\} \\
&= Rwv \rightarrow \bigwedge \{Rwu \rightarrow V(p_i, u) \mid u \in W\} \\
&= Rwv \rightarrow \bar{V}(\Box p_i, w).
\end{aligned}
$$

This yields $1 > \bar{V}(\Box p_i, w) = Rwv \rightarrow \bar{V}(\Box p_i, w) = Rwv \rightarrow \bar{V}(\Box p_i, v)$ and hence $\bar{V}(\Box p_i, v) = \bar{V}(\Box p_i, w) = V(p_i, v) < Rwv$. Now note that from $\bar{V}(\chi(p_i), w) < 1$, we obtain $\bar{V}(\Box((p_i \rightarrow \Box p_i) \rightarrow \Box p_i, w) > V(\Box p_i, w)$ and so $\bar{V}(\Box p_i, w) < Rwu \rightarrow \bar{V}((p_i \rightarrow \Box p_i) \rightarrow \Box p_i, u)$ for all $u \in W$. We obtain a contradiction

$$
\begin{aligned}
\bar{V}(\Box p_i, w) &< Rwv \rightarrow \bar{V}((p_i \rightarrow \Box p_i) \rightarrow \Box p_i, v) \\
&= Rwv \rightarrow \bar{V}(\Box p_i, v) \\
&= \bar{V}(\Box p_i, v) \\
&= \bar{V}(\Box p_i, w). \qquad \Box
\end{aligned}
$$

**Proposition 1.24.** *The sets of logics $\mathsf{S5}(\mathbf{A})$ and $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ (considered as sets of valid formulas), where $A$ ranges over infinite Gödel sets, are both infinite.*

*Proof.* It suffices to show that infinitely many of the logics $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ can be distinguished by formulas. For each $n \in \mathbb{N}^+$, let

$$\chi_n := \bigvee_{j=1}^{n} \chi(p_j) \vee \bigvee_{i=1}^{n-1} (\square p_{i+1} \to \square p_i).$$

We prove first that $\models_{\mathsf{S5}(\mathbf{A})^{\mathsf{C}}} \chi_n$ implies $|\mathsf{R}(A)| < n$. Suppose that $A$ has $n$ distinct right accumulation points $a_1 < \cdots < a_n$. Then for each $j \in \{1, \ldots, n\}$, there exists a strictly descending sequence $(c_n^j)_{n \in \mathbb{N}} \subseteq (a_j, 1] \cap A$ converging to $a_j$. We define an $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-model $\mathcal{M} = \langle \mathbb{N}, V \rangle$ with $V(p_j, m) = c_m^j$ for all $m \in \mathbb{N}$ and $j \in \{1, \ldots, n\}$. Then $\bar{V}(\square p_j, m) = a_j < c_m^j = V(p_j, m)$ for all $j \in \{1, \ldots, n\}$ and $m \in \mathbb{N}$, which implies $\bar{V}(\chi(p_j), 0) = a_j$ for all $j \in \{1, \ldots, n\}$. Moreover, $\bar{V}(\square p_{i+1} \to \square p_i, 0) = \bar{V}(\square p_i, 0) = a_i$ for all $i \in \{1, \ldots, n-1\}$. Hence $\bar{V}(\chi_n, 0) = a_n < 1$ and $\not\models_{\mathsf{S5}(\mathbf{A})^{\mathsf{C}}} \chi_n$.

Next, we prove that $|\mathsf{R}(A)| < n$ implies $\models_{\mathsf{S5}(\mathbf{A})} \chi_n$. Suppose that $\bar{V}(\chi_n, w) < 1$ for some $\mathsf{S5}(\mathbf{A})$-model $\langle W, R, V \rangle$ and $w \in W$. It follows that $\bar{V}(\square p_1, w) < \cdots < \bar{V}(\square p_n, w)$ and $\bar{V}(\chi(p_j), w) < 1$ for all $j \in \{1, \ldots, n\}$. By Lemma 1.23, each of the $\bar{V}(\square p_j, w)$ is a right accumulation point of $A$ and so $|\mathsf{R}(A)| \geq n$.

Since also $\models_{\mathsf{S5}(\mathbf{A})} \chi_n$ implies $\models_{\mathsf{S5}(\mathbf{A})^{\mathsf{C}}} \chi_n$, we have that $\models_{\mathsf{S5}(\mathbf{A})^{\mathsf{C}}} \chi_n$ if and only if $\models_{\mathsf{S5}(\mathbf{A})} \chi_n$ if and only if $|\mathsf{R}(A)| < n$. Hence the sets of logics $\mathsf{S5}(\mathbf{A})$ and $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$, where $A$ ranges over infinite Gödel sets, are both infinite. $\square$

Similarly, we can detect left accumulation points. For each $n \in \mathbb{N}^+$, let

$$\theta_n := \bigvee_{i=1}^{n} (\Diamond(\Diamond p_i \to p_i)) \vee \bigvee_{i=1}^{n-1} (\Diamond p_{i+1} \to \Diamond p_i).$$

It can then be verified that for any Gödel set $A$ and $n \in \mathbb{N}^+$,

$$\models_{\mathsf{S5}(\mathbf{A})^{\mathsf{C}}} \theta_n \iff \models_{\mathsf{S5}(\mathbf{A})} \theta_n \iff |\mathsf{L}(A) \setminus \{1\}| < n.$$

Since there are at most countably infinitely many first-order Gödel logics, it follows from previous results that there are exactly countably infinitely many logics $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$. An upper bound on the number of logics $\mathsf{S5}(\mathbf{A})$ will be obtained in Section 3.3.

### Monadic Łukasiewicz and Abelian Logic

Recall that it was shown by Scarpellini in [139] that first-order Łukasiewicz logic is not recursively axiomatizable. However, its one-variable fragment can be axiomatized, as was done by Rutledge in his PhD thesis [138]. He provides an algebraic semantics of $\mathbf{Ł}$-validity in the form of monadic MV-algebras, or monadic Chang algebras in Rutledge's terminology, from which a Hilbert-style axiomatization can be extracted. We define one such axiomatization, denoted $\mathcal{MHŁ}$, as $\mathcal{HŁ}$ extended with the axiom and rule schemas from Figure 1.7. As universal $\mathsf{S5}(\mathbf{Ł})^{\mathsf{C}}$-models correspond exactly to $\mathbf{Ł}$-structures $\mathfrak{M}$ with an $\mathfrak{M}$-evaluation, it follows immediately that for $\varphi \in \mathrm{Fm}_{\square\Diamond}(\mathcal{L}_{\mathbf{Ł}})$,

$$\models_{\mathsf{S5}(\mathbf{Ł})^{\mathsf{C}}} \varphi \iff \vdash_{\mathcal{MHŁ}} \varphi.$$

This fragment was shown to be decidable already by Rutledge in [138].

$$
\begin{array}{ll}
(1) & \varphi \to \Diamond\varphi \\
(2) & (\Diamond\varphi \lor \Diamond\psi) \to \Diamond(\varphi \lor \psi) \\
(3) & \Diamond\Box\varphi) \to \Box\varphi \\
(4) & \Diamond(\Diamond\varphi \oplus \Diamond\psi) \to (\Diamond\varphi \oplus \Diamond\psi) \\
(5) & (\Diamond\varphi \otimes \Diamond\varphi) \to \Diamond(\varphi \otimes \varphi) \\
(6) & (\Diamond\varphi \oplus \Diamond\varphi) \to \Diamond(\varphi \oplus \varphi)
\end{array}
$$

$$
\frac{\varphi}{\Box\varphi} \ (\text{nec})
$$

Figure 1.7: Additional axiom and rule schemas for proof system $\mathcal{MHL}$

The interpretation $(-)^\bullet$ from first-order Łukasiewicz logic into first-order Abelian logic restricts to the one-variable fragments of both logics. To be precise, for each $\alpha \in \mathrm{Fm}_1(\mathcal{L}_\text{Ł})$ not containing $P_0$,

$$
\models_{\text{Ł}}^{\forall\exists} \alpha \iff \models_{\mathbf{R}}^{\forall\exists} \alpha^\bullet.
$$

Now consider any $\mathsf{S5}(\mathbf{R})^\mathsf{C}$-model $\mathcal{M} = \langle W, R, V \rangle$. Using the term-equivalence from Example 1.7, $\mathcal{M}$ is by definition universal, since $Rwv = 0$ for all $w, v \in W$. We can hence drop $R$ from the signature. It then follows that each (safe) $\mathsf{S5}(\mathbf{R})^\mathsf{C}$-model corresponds exactly to a (safe) $\mathbf{R}$-structure $\mathfrak{M}$ together with an $\mathfrak{M}$-evaluation. Therefore, for each $\alpha \in \mathrm{Fm}_1(\mathcal{L}_\mathsf{A})$,

$$
\models_{\mathbf{R}}^{\forall\exists} \alpha \iff \models_{\mathsf{S5}(\mathbf{R})^\mathsf{C}} \alpha^*.
$$

Combining these two equivalences, we obtain an interpretation of $\mathsf{S5}(\text{Ł})^\mathsf{C}$ into $\mathsf{S5}(\mathbf{R})^\mathsf{C}$. Chapter 4 is dedicated to the study of $\mathsf{S5}(\mathbf{R})^\mathsf{C}$. We provide a proof system for which we prove completeness with respect to $\mathsf{S5}(\mathbf{R})^\mathsf{C}$-validity using algebraic means, and prove completeness of a system for the multiplicative fragment of $\mathsf{S5}(\mathbf{R})^\mathsf{C}$, that is, the fragment of $\mathsf{S5}(\mathbf{R})^\mathsf{C}$ not containing $\land$ or $\lor$, via syntactic arguments.

**Remark 1.25.** We can give a slightly different characterization of a safe $\mathsf{S5}(\mathbf{R})^\mathsf{C}$-model, similarly to the alternative characterization of safe $\mathbf{R}$-structures given at the end of Section 1.2. For a safe $\mathsf{S5}(\mathbf{R})^\mathsf{C}$-model $\langle W, R, V \rangle$, the map $V_i \colon W \to \mathbb{R}$, $w \mapsto V(p_i, w)$, is bounded for each $i \in \mathbb{N}$. In fact, each $\mathsf{S5}(\mathbf{R})^\mathsf{C}$-model $\langle W, R, V \rangle$ for which these $V_i$ are bounded for each $i \in \mathbb{N}$, is safe.

# Monadic Algebras

In this chapter we are concerned with algebraic semantics for one-variable fragments of various first-order substructural logics. Such an algebraic study was initiated by Halmos for classical logic. In the 1950s, he defined the notion of a monadic Boolean algebra as the algebraic semantics for the one-variable fragment, or equivalently, the monadic fragment, of first-order classical logic [79]. Since then, algebraic semantics for one-variable fragments of other first-order logics have been considered in the literature, usually under the name "monadic".[1] Indeed, Halmos' ideas were first extended by Monteiro and Varsavsky in [113], who defined monadic Heyting algebras as the algebraic semantics for the one-variable fragment of first-order intuitionistic logic. More generally, algebraic semantics for one-variable fragments of first-order intermediate logics have been studied in, e.g., [22, 126]. Rutledge gave an algebraic semantics for the one-variable fragment of first-order Łukasiewicz logic in [138] by defining monadic MV-algebras (or monadic Chang algebras in Rutledge's terminology), and an algebraic semantics for $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$, i.e., the one-variable fragment of first-order Gödel logic, was given by Hájek [78] (see also [42]). The one-variable fragment of a first-order extension of $\mathsf{BL}$ was studied algebraically by Hájek in [78],[2] which was more recently revisited in [44, 45].

In this chapter, we introduce monadic $\mathsf{FL}_e$-algebras as a first step towards a general approach of axiomatizing one-variable fragments of first-order substructural, and hence also intermediate, logics. In Section 2.1, we define monadic $\mathsf{FL}_e$-algebras as $\mathsf{FL}_e$-algebras with two modalities $\square$ and $\lozenge$. We prove a number of properties of them and show that this notion encompasses all the mentioned monadic algebras. We moreover prove a soundness result: we show that the algebraic semantics of the one-variable fragment of a first-order substructural logic as defined in Section 1.2 necessarily consists of monadic $\mathsf{FL}_e$-algebras. In Section 2.2, we focus on so-called relatively complete subalgebras of an $\mathsf{FL}_e$-algebra, a notion that was already considered by Halmos for monadic Boolean algebras in [79]. We show that the image of the modality $\square$ for a monadic $\mathsf{FL}_e$-algebra forms such a relatively complete subalgebra and, conversely, that any such a relatively complete subalgebra of an $\mathsf{FL}_e$-algebra determines a monadic $\mathsf{FL}_e$-algebra. Section 2.3 investigates the congruences of monadic $\mathsf{FL}_e$-algebras. We give an alternative characterization of the congruences, and

---

[1] As noted in Remark 1.16, these algebras are named thus since they generalize the notion of a monadic Boolean algebra, not because they correspond to the monadic fragment of a first-order logic.

[2] The completeness proof given in this paper contains an error; for more, see [45].

use it to show that the congruences are completely determined by the congruences of the image of $\Box$, i.e., of the relatively complete subalgebra. Section 2.4 focuses on varieties of monadic $\mathsf{FL}_e$-algebras that satisfy some particular properties. Examples will include the varieties of monadic MV-algebras, of crisp monadic Gödel algebras, and of monadic abelian $\ell$-groups. For such varieties, we prove that their members admit some functional representation.

## 2.1   Monadic $\mathsf{FL}_e$-algebras

Recall the definition of an $\mathsf{FL}_e$-algebra from Definition 1.3. We define the monadic extension of such algebras by adding two unary operators $\Box$ and $\Diamond$ that satisfy certain properties of the universal and existential quantifier, respectively.

**Definition 2.1.** A *monadic $\mathsf{FL}_e$-algebra* is an algebra $\langle A, \wedge, \vee, \cdot, \rightarrow, f, e, \Box, \Diamond \rangle$ such that $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \rightarrow, f, e \rangle$ is an $\mathsf{FL}_e$-algebra and for all $a, b, c \in A$,

| | | | |
|---|---|---|---|
| (L1) | $\Box a \le a$ | (L5) | $\Box e = e$ |
| (L2) | $\Box\Diamond a = \Diamond a$ | (L6) | $\Box(a \rightarrow \Box b) = \Diamond a \rightarrow \Box b$ |
| (L3) | $\Box(a \wedge b) = \Box a \wedge \Box b$ | (L7) | $\Box(\Box a \rightarrow b) = \Box a \rightarrow \Box b.$ |
| (L4) | $\Box f = f$ | | |

We often write $\langle \mathbf{A}; \Box, \Diamond \rangle$ to denote the monadic $\mathsf{FL}_e$-algebra $\langle A, \wedge, \vee, \cdot, \rightarrow, f, e, \Box, \Diamond \rangle$ where $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \rightarrow, f, e \rangle$. We refer to $\mathbf{A}$ as the $\mathsf{FL}_e$-*reduct* of $\langle \mathbf{A}; \Box, \Diamond \rangle$. For two monadic $\mathsf{FL}_e$-algebras $\langle \mathbf{A}; \Box, \Diamond \rangle$ and $\langle \mathbf{B}; \Box, \Diamond \rangle$, a homomorphism $h \colon A \rightarrow B$ will sometimes be called a *modal homomorphism* to stress that it preserves $\Box$ and $\Diamond$ and distinguish it from an $\mathsf{FL}_e$-homomorphism.

Recall that the class of $\mathsf{FL}_e$-algebras is a variety. It follows from the definition above that the class of monadic $\mathsf{FL}_e$-algebras, denoted $\mathcal{MFL}_e$, is also a variety. Let us now summarize and prove a number of properties of monadic $\mathsf{FL}_e$-algebras.

**Lemma 2.2.** *Let $\langle \mathbf{A}; \Box, \Diamond \rangle$ be a monadic $\mathsf{FL}_e$-algebra. Then for all $a, b \in A$,*

| | | | |
|---|---|---|---|
| (L8) | $a \le \Diamond a$ | (L16) | $\Diamond e = e$ |
| (L9) | *if $a \le b$, then $\Box a \le \Box b$* | (L17) | $\Diamond f = f$ |
| (L10) | *if $a \le b$, then $\Diamond a \le \Diamond b$* | (L18) | $\Diamond(\Diamond a \cdot \Diamond b) = \Diamond a \cdot \Diamond b$ |
| (L11) | $\Box a = \Diamond\Box a$ | (L19) | $\Box(\Box a \cdot \Box b) = \Box a \cdot \Box b$ |
| (L12) | $\Box(\Diamond a \rightarrow b) = \Diamond a \rightarrow \Box b$ | (L20) | $\Diamond(a \cdot \Diamond b) = \Diamond a \cdot \Diamond b$ |
| (L13) | $\Box(a \rightarrow \Diamond b) = \Diamond a \rightarrow \Diamond b$ | (L21) | $\Box(\Diamond a \vee \Diamond b) = \Diamond a \vee \Diamond b$ |
| (L14) | $\Box\Box a = \Box a$ | (L22) | $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$ |
| (L15) | $\Diamond\Diamond a = \Diamond a$ | (L23) | $\Diamond(a \rightarrow \Diamond b) \le \Box a \rightarrow \Diamond b$ |

*Proof.* (L8) By (L2), $\Diamond a \le \Box\Diamond a$ and so $e \le \Diamond a \rightarrow \Box\Diamond a$. Therefore by (L1) and (L6), $e \le \Diamond a \rightarrow \Box\Diamond a = \Box(a \rightarrow \Box\Diamond a) \le a \rightarrow \Box\Diamond a = a \rightarrow \Diamond a$.

(L9) Suppose $a \le b$. Then $a = a \wedge b$, hence $\Box a = \Box(a \wedge b) = \Box a \wedge \Box b$ by (L3). Therefore, $\Box a \le \Box b$.

(L10) Suppose that $a \le b$. By (L8), $a \le b \le \Diamond b$ and so $e \le a \rightarrow \Diamond b$. Applying (L9) and using (L5) and (L6) gives $e = \Box e \le \Box(a \rightarrow \Diamond b) = \Diamond a \rightarrow \Diamond b$, so $\Diamond a \le \Diamond b$.

(L11) By (L8), $\Box a \leq \Diamond\Box a$. Furthermore, $e = \Box e \leq \Box(\Box a \to \Box a) = \Diamond\Box a \to \Box a$ by (L5), (L6) and (L9), hence $\Diamond\Box a \leq \Box a$.

(L12) By (L2) and (L7), $\Box(\Diamond a \to b) = \Box(\Box\Diamond a \to b) = \Box\Diamond a \to \Box b = \Diamond a \to \Box b$.

(L13) By (L2) and (L6), $\Box(a \to \Diamond b) = \Box(a \to \Box\Diamond b) = \Diamond a \to \Box\Diamond b = \Diamond a \to \Diamond b$.

(L14) By (L1), $\Box\Box a \leq \Box a$. Moreover, $e = \Box e \leq \Box(\Box a \to \Box a) = \Box a \to \Box\Box a$ using (L5), (L7), and (L9), hence $\Box a \leq \Box\Box a$.

(L15) By (L9), $\Diamond a \leq \Diamond\Diamond a$. Moreover, $e = \Box e \leq \Box(\Diamond a \to \Diamond a) = \Diamond\Diamond a \to \Diamond a$ using (L5), (L9), and (L13), and so $\Diamond\Diamond a \leq \Diamond a$.

(L16) Note that $\Diamond e = \Diamond\Box e = \Box e = e$ by (L5) and (L11).

(L17) Note that $\Diamond f = \Diamond\Box f = \Box f = f$ by (L4) and (L11).

(L18) Firstly note that by (L1), $\Box(\Diamond a \cdot \Diamond b) \leq \Diamond a \cdot \Diamond b$. Moreover, residuation yields $e \leq \Diamond a \to (\Diamond b \to (\Diamond a \cdot \Diamond b))$. Using (L12) and (L6) then gives

$$
\begin{aligned}
e = \Box\Box e &\leq \Box\Box(\Diamond a \to (\Diamond b \to (\Diamond a \cdot \Diamond b))) \\
&= \Box(\Diamond a \to \Box(\Diamond b \to (\Diamond a \cdot \Diamond b))) \\
&= \Box(\Diamond a \to (\Diamond b \to \Box(\Diamond a \cdot \Diamond b))) \\
&= \Box((\Diamond a \cdot \Diamond b) \to \Box(\Diamond a \cdot \Diamond b)) \\
&= \Diamond(\Diamond a \cdot \Diamond b) \to \Box(\Diamond a \cdot \Diamond b) \\
&\leq \Diamond(\Diamond a \cdot \Diamond b) \to (\Diamond a \cdot \Diamond b).
\end{aligned}
$$

Additionally, $\Diamond a \cdot \Diamond b \leq \Diamond(\Diamond a \cdot \Diamond b)$ by (L8).

(L19) By (L11), (L2), and (L18),

$$
\begin{aligned}
\Box(\Box a \cdot \Box b) = \Box(\Diamond\Box a \cdot \Diamond\Box b) = \Box\Diamond(\Diamond\Box a \cdot \Diamond\Box b) \\
= \Diamond(\Diamond\Box a \cdot \Diamond\Box b) \\
= \Diamond\Box a \cdot \Diamond\Box b \\
= \Box a \cdot \Box b.
\end{aligned}
$$

(L20) Since $a \leq \Diamond a$ by (L8), $a \cdot \Diamond b \leq \Diamond a \cdot \Diamond b$. Therefore, $\Diamond(a \cdot \Diamond b) \leq \Diamond(\Diamond a \cdot \Diamond b) = \Diamond a \cdot \Diamond b$ by (L10) and (L18). Moreover, since $a \cdot \Diamond b \leq \Diamond(a \cdot \Diamond b)$ by (L8),

$$
\begin{aligned}
e = \Box e &\leq \Box((a \cdot \Diamond b) \to \Diamond(a \cdot \Diamond b)) \\
&= \Box(\Diamond b \to (a \to \Diamond(a \cdot \Diamond b))) \\
&= \Diamond b \to \Box(a \to \Diamond(a \cdot \Diamond b)) \\
&= \Diamond b \to (\Diamond a \to \Diamond(a \cdot \Diamond b)) \\
&= (\Diamond a \cdot \Diamond b) \to \Diamond(a \cdot \Diamond b)
\end{aligned}
$$

follows using (L5), (L12), (L13) and so $\Diamond a \cdot \Diamond b \leq \Diamond(a \cdot \Diamond b)$.

(L21) By (L1), $\Box(\Diamond a \lor \Diamond b) \leq \Diamond a \lor \Diamond b$. Moreover, $\Diamond a \leq \Diamond a \lor \Diamond b$ implies $\Diamond a = \Box\Diamond a \leq \Box(\Diamond a \lor \Diamond b)$ by (L9) and (L2). Similarly, $\Diamond b \leq \Box(\Diamond a \lor \Diamond b)$ and so $\Diamond a \lor \Diamond b \leq \Box(\Diamond a \lor \Diamond b)$.

(L22) By (L8), we have $a \lor b \leq \Diamond a \lor \Diamond b$, so $\Diamond(a \lor b) \leq \Diamond(\Diamond a \lor \Diamond b)$. Moreover, since $\Diamond a \lor \Diamond b = \Box(\Diamond a \lor \Diamond b)$ by (L21), $\Diamond(\Diamond a \lor \Diamond b) = \Diamond\Box(\Diamond a \lor \Diamond b) = \Box(\Diamond a \lor \Diamond b) = \Diamond a \lor \Diamond b$ by (L11). It follows that $\Diamond(a \lor b) \leq \Diamond a \lor \Diamond b$. For the other direction, note that $a \leq a \lor b$ and $b \leq a \lor b$ and so $\Diamond a \leq \Diamond(a \lor b)$ and $\Diamond b \leq \Diamond(a \lor b)$ by (L10), hence $\Diamond a \lor \Diamond b \leq \Diamond(a \lor b)$.

(L23) Since $\Box a \leq a$, $a \to \Diamond b \leq \Box a \to \Diamond b$. Hence, by (L5), (L9), (L11), (L6), and (L13),

$$
\begin{aligned}
e = \Box e &\leq \Box((a \to \Diamond b) \to (\Box a \to \Diamond b)) \\
&= \Box((a \to \Diamond b) \to (\Diamond \Box a \to \Diamond b)) \\
&= \Box((a \to \Diamond b) \to \Box(\Box a \to \Diamond b)) \\
&= \Diamond(a \to \Diamond b) \to \Box(\Box a \to \Diamond b) \\
&= \Diamond(a \to \Diamond b) \to (\Diamond \Box a \to \Diamond b) \\
&= \Diamond(a \to \Diamond b) \to (\Box a \to \Diamond b)
\end{aligned}
$$

and so $\Diamond(a \to \Diamond b) \leq \Box a \to \Diamond b$. □

We now consider a number of examples. In particular, we show that most well-known monadic algebras fit into the framework of monadic $\mathsf{FL}_e$-algebras presented here.

**Example 2.3.** As the algebraic counterpart of the one-variable fragment of first-order classical logic, Halmos defined the variety of *monadic Boolean algebras* [79]. A monadic Boolean algebra is an algebra $\langle B, \wedge, \vee, \neg, \bot, \top, \Diamond \rangle$ such that $\langle B, \wedge, \vee, \neg, \bot, \top \rangle$ is a Boolean algebra, as defined in Example 1.1, and $\Diamond$ is a unary operator satisfying for all $a, b \in B$,

$$
\begin{aligned}
&\text{(Q1)} \quad \Diamond \bot = \bot \\
&\text{(Q2)} \quad a \leq \Diamond a \\
&\text{(Q3)} \quad \Diamond(a \wedge \Diamond b) = \Diamond a \wedge \Diamond b.
\end{aligned}
$$

We define $\Box a \coloneqq \neg \Diamond \neg a$ for all $a \in B$ and let $\mathcal{MBA}$ to denote the variety of all monadic Boolean algebras. Halmos' completeness result, together with Wajsberg's axiomatization from [152], now states that for all $\varphi \in \mathrm{Fm}_{\Box \Diamond}(\mathcal{L}_{\mathsf{CL}})$,

$$
\begin{aligned}
\vdash_{\mathcal{S}5} \varphi &\iff \varphi^\circ \text{ is valid in first-order classical logic} \\
&\iff \mathcal{MBA} \models \varphi \approx \top.
\end{aligned}
$$

It is straightforward to verify that a monadic Boolean algebra is term-equivalent, using the same term-equivalence as in Example 1.4, to a monadic $\mathsf{FL}_e$-algebra $\langle \mathbf{B}; \Box, \Diamond \rangle$ where the $\mathsf{FL}_e$-reduct $\mathbf{B}$ is term-equivalent to a Boolean algebra. It is worth noting that although $\Box$ and $\Diamond$ are interdefinable in this case, this is not the case in general.

**Example 2.4.** Monteiro and Varsavsky introduced monadic Heyting algebras as the algebraic semantics of $\mathcal{MIPC}$ [113]. A *monadic Heyting algebra*, as presented in [22], is an algebra $\langle H, \wedge, \vee, \to, \bot, \top, \Box, \Diamond \rangle$ ($\langle \mathbf{H}; \Box, \Diamond \rangle$ for short) such that $\mathbf{H} = \langle H, \wedge, \vee, \to, \top, \bot \rangle$ is a Heyting algebra and for all $a, b \in H$,

$$
\begin{array}{llll}
\text{(H1)} & \Box a \leq a & \text{(H5)} & (\Box a \wedge \Box b) \leq \Box(a \wedge b) \\
\text{(H2)} & a \leq \Diamond a & \text{(H6)} & \Diamond(a \vee b) \to (\Diamond a \vee \Diamond b) \\
\text{(H3)} & \Diamond a \leq \Box \Diamond a & \text{(H7)} & \Box(a \to b) \leq \Diamond a \to \Diamond b. \\
\text{(H4)} & \Diamond \Box a \leq \Box a
\end{array}
$$

We let $\mathcal{MHA}$ denote the variety of all monadic Heyting algebras. By the completeness result of Monteiro and Varsavsky, as well as Bull's completeness result [36], we have for all $\varphi \in \mathrm{Fm}_{\Box \Diamond}(\mathcal{L}_{\mathsf{IL}})$,

$$
\begin{aligned}
\vdash_{\mathcal{MIPC}} \varphi &\iff \varphi^\circ \text{ is valid in first-order intuitionistic logic} \\
&\iff \mathcal{MHA} \models \varphi \approx \top.
\end{aligned}
$$

One can verify that monadic Heyting algebras are term-equivalent to monadic $\mathsf{FL}_e$-algebras $\langle \mathbf{A}; \Box, \Diamond \rangle$ where the $\mathsf{FL}_e$-reduct $\mathbf{A}$ is (term-equivalent to) a Heyting algebra. A detailed study of monadic Heyting algebras can be found in [22, 23].

**Example 2.5.** A *monadic MV-algebra*, as introduced by Rutledge in [138], is an algebra $\langle A, \oplus, \sim, 0, \Diamond \rangle$ such that $\langle A, \oplus, \sim, 0 \rangle$ is an MV-algebra satisfying for all $a, b \in A$,

| | | | |
|---|---|---|---|
| (P1) | $a \leq \Diamond a$ | (P4) | $\Diamond(\Diamond a \oplus \Diamond b) = \Diamond a \oplus \Diamond b$ |
| (P2) | $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$ | (P5) | $\Diamond(a \oplus a) = \Diamond a \oplus \Diamond a$ |
| (P3) | $\Diamond \sim \Diamond a = \sim \Diamond a$ | (P6) | $\Diamond(a \otimes a) = \Diamond a \otimes \Diamond a.$ |

We let $\Box a$ denote $\sim \Diamond \sim a$ for all $a \in A$, and let $\mathcal{MMV}$ denote the variety of all monadic MV-algebras. Rutledge proved that monadic MV-algebras form the algebraic semantics for the one-variable fragment of first-order Łukasiewicz logic. To be precise, Rutledge shows that for all $\varphi \in \mathrm{Fm}_{\Box\Diamond}(\mathcal{L}_\text{Ł})$,

$$\vdash_{\mathcal{MHL}} \varphi \iff \varphi^\circ \text{ is } \text{Ł-valid}$$
$$\iff \mathcal{MMV} \models \varphi \approx \sim 0.$$

Rutledge's completeness proof is rather involved. Recently, Castaño et al. in [45] gave a simpler proof of the slightly different statement that for all $\varphi \in \mathrm{Fm}_{\Box\Diamond}(\mathcal{L}_\text{Ł})$,

$$\vdash_{\mathcal{MHL}} \varphi \iff \varphi^\circ \text{ is } \mathbf{A}\text{-valid for all linearly ordered}$$
$$\text{MV-algebras } \mathbf{A}.^3$$

We can verify that a monadic MV-algebra is term-equivalent to a monadic $\mathsf{FL}_e$-algebra $\langle \mathbf{A}; \Box, \Diamond \rangle$ where $\mathbf{A}$ is term-equivalent to an MV-algebra and the identities $\Box(x \vee \Box y) \approx \Box x \vee \Box y$ and $\Diamond(x \cdot x) \approx \Diamond x \cdot \Diamond x$ are satisfied. As with monadic Boolean algebras, the modalities $\Box$ and $\Diamond$ are interdefinable. More on monadic MV-algebras can be found in [53, 54, 60].

**Example 2.6.** We say that any monadic Heyting algebra $\langle \mathbf{H}; \Box, \Diamond \rangle$ is a *monadic Gödel algebra* if $\mathbf{H}$ is a Gödel algebra, as defined in Section 1.1. We write $\mathcal{MGA}$ for the variety of monadic Gödel algebras. These algebras are clearly term-equivalent to monadic $\mathsf{FL}_e$-algebras $\langle \mathbf{A}; \Box, \Diamond \rangle$ whose $\mathsf{FL}_e$-reduct $\mathbf{A}$ is term-equivalent to a Gödel algebra. As a consequence of the completeness results in [42], these form the algebraic semantics for the logic $\mathsf{S5(G)}$. That is, for all $\varphi \in \mathrm{Fm}_{\Box\Diamond}(\mathcal{L}_\text{IL})$,

$$\models_{\mathsf{S5(G)}} \varphi \iff \mathcal{MGA} \models \varphi \approx \top.$$

Monadic Gödel algebras appear in [42] under the name reflexive transitive symmetric bi-modal Gödel algebras. In Chapter 3 we add to this result by showing a connection with first-order intermediate logics. In particular, we show that for all $\varphi \in \mathrm{Fm}_{\Box\Diamond}(\mathcal{L}_\text{IL})$,

$$\models_{\mathsf{S5(G)}} \varphi \iff \models_{\mathsf{IKL}_1} \varphi^\circ.$$

---

[3] In fact, Castaño et al. prove a more general result. They prove a strong completeness result with respect to all consequences, not just validities. Their result is hence both weaker and stronger than Rutledge's result: it is weaker since they prove completeness with respect to all linearly ordered MV-algebras rather than the standard Łukasiewicz algebra $\text{Ł}$, and it is stronger since they prove strong completeness for all consequences.

**Example 2.7.** A monadic Gödel algebra is called a *crisp monadic Gödel algebra* if it additionally satisfies $\Box(x \lor \Box y) \approx \Box x \lor \Box y$. Let $c\mathcal{MGA}$ denote the variety of crisp monadic Gödel algebras. These algebras form the algebraic semantics for the one-variable fragment of first-order Gödel logic, as a consequence of the completeness result in [78]. Combined with the completeness result from [42], we obtain that for each $\varphi \in \mathrm{Fm}_{\Box\Diamond}(\mathcal{L}_{\mathsf{IL}})$,

$$\vdash_{\mathcal{MIPC}+(\mathrm{pre})+(\mathrm{cd}_\Box)} \varphi \quad \Longleftrightarrow \quad \varphi^\circ \text{ is } \mathbf{G}\text{-valid}$$
$$\Longleftrightarrow \quad c\mathcal{MGA} \models \varphi \approx \top.$$

**Example 2.8.** A monadic version of abelian $\ell$-groups was introduced by Cimadamore and Díaz Varela in [53]. The category of such algebras was shown to be categorically equivalent to that of monadic MV-algebras, and their monadic abelian $\ell$-groups therefore included a strong unit. In this thesis we adopt an alternative definition that does not include a strong unit: a *monadic abelian $\ell$-group* is an algebra $\langle A, +, -, 0, \Box \rangle$ such that $\mathbf{A} = \langle A, +, -, 0 \rangle$ is an abelian $\ell$-group with defined operator $\Diamond a := -\Box - a$ satisfying for all $a, b \in A$,

| | | | |
|---|---|---|---|
| (M1) | $\Box(a+b) \leq \Box a + \Diamond b$ | (M4) | $\Box(a \land b) = \Box a \land \Box b$ |
| (M2) | $\Box a \leq a$ | (M5) | $\Box(a \lor \Box b) = \Box a \lor \Box b$ |
| (M3) | $\Diamond a = \Box \Diamond a$ | (M6) | $\Box(a+a) = \Box a + \Box a.$ |

In Remark 4.15 we show that this definition is in fact equivalent to the definition given in [53], aside from the absence of a strong unit. Moreover, in Chapter 4 we show that monadic abelian $\ell$-groups as defined here form the algebraic semantics for the one-variable fragment of first-order Abelian logic. That is, for each $\varphi \in \mathrm{Fm}_{\Box\Diamond}(\mathcal{L}_{\mathsf{A}})$,

$$\models_{\mathsf{S5(R)}^{\mathsf{c}}} \varphi \quad \Longleftrightarrow \quad \varphi^\circ \text{ is } \mathbf{R}\text{-valid}$$
$$\Longleftrightarrow \quad \mathcal{MA\ell G} \models 0 \leq \varphi,$$

where $\mathcal{MA\ell G}$ denotes the variety of monadic abelian $\ell$-groups. It is not hard to check that monadic abelian $\ell$-groups are term-equivalent to monadic $\mathsf{FL}_e$-algebras $\langle \mathbf{A}; \Box, \Diamond \rangle$ such that $\mathbf{A}$ is term-equivalent to an abelian $\ell$-group and the identities $\Box(x \lor \Box y) \approx \Box x \lor \Box y$ and $\Diamond(x \cdot x) \approx \Diamond x \cdot \Diamond x$ are satisfied.

In the examples above, the identities $\Box(x \lor \Box y) \approx \Box x \lor \Box y$ (the algebraic formulation of the constant domain axiom $(\mathrm{cd}_\Box)$ and $\Diamond(x \cdot x) \approx \Diamond x \cdot \Diamond x$ play a significant role. It is therefore useful to see some distinguishing examples.

**Example 2.9.** Consider the Gödel algebra with defined modalities depicted in Figure 2.1. It is routine to check that this is a monadic Gödel algebra. However, note that $\Box(b \lor \Box c) = \top$, whereas $\Box b \lor \Box c = c$. This is hence an example of a monadic $\mathsf{FL}_e$-algebra that refutes $\Box(x \lor \Box y) \approx \Box x \lor \Box y$.

**Example 2.10.** Consider the three-element Łukasiewicz chain $\mathbf{Ł}_3$ (over the language $\mathcal{L}_{\mathsf{FL}_e}$) with added modalities depicted in Figure 2.2. It is again routine to check that this is a monadic $\mathsf{FL}_e$-algebra. Note that $\Diamond(\frac{1}{2} \cdot \frac{1}{2}) = \Diamond 0 = 0$ and $\Diamond\frac{1}{2} \cdot \Diamond\frac{1}{2} = 1 \cdot 1 = 1$, and so the given monadic $\mathsf{FL}_e$-algebra refutes $\Diamond(x \cdot x) \approx \Diamond x \cdot \Diamond x$.
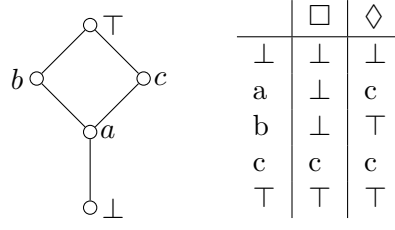
Figure 2.1: Monadic residuated lattice $\langle \mathbf{H}; \square, \lozenge \rangle$
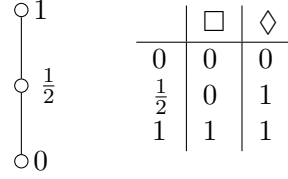


Figure 2.2: Monadic residuated lattice $\langle \mathbf{\text{Ł}}_3; \square, \lozenge \rangle$

We now focus on how monadic $\mathsf{FL}_e$-algebras fit within the setting of first-order many-valued logics as defined in Section 1.2. We first verify the folklore result that the first-order equivalents of conditions (L1)–(L7) are $\mathbf{A}$-valid for each $\mathsf{FL}_e$-algebra $\mathbf{A}$. Recall that we defined $\alpha \leftrightarrow \beta = (\alpha \to \beta) \wedge (\beta \to \alpha)$ for $\alpha, \beta \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{FL}_e})$.

**Proposition 2.11.** *Let $\mathbf{A}$ be an $\mathsf{FL}_e$-algebra. Then for $\alpha, \beta \in \mathrm{Fm}_1(\mathcal{L}_{\mathsf{FL}_e})$, the following formulas are $\mathbf{A}$-valid:*

(i) $(\forall x)\alpha \to \alpha$

(ii) $(\exists x)\alpha \leftrightarrow (\forall x)(\exists x)\alpha$

(iii) $(\forall x)(\alpha \wedge \beta) \leftrightarrow ((\forall x)\alpha \wedge (\forall x)\beta)$

(iv) $f \leftrightarrow (\forall x)f$

(v) $e \leftrightarrow (\forall x)e$

(vi) $((\exists x)\alpha \to (\forall x)\beta) \leftrightarrow (\forall x)(\alpha \to (\forall x)\beta)$

(vii) $(\forall x)((\forall x)\alpha \to \beta) \leftrightarrow ((\forall x)\alpha \to (\forall x)\beta)$.

*Proof.* Consider any safe $\mathbf{A}$-structure $\mathfrak{M} = \langle D, \mathcal{I} \rangle$ and $\mathfrak{M}$-evaluation $v$. The $\mathbf{A}$-validity of formulas (i)–(v) is easy to check. We give the proofs for (vi) and (vii).

(vi) The $\mathbf{A}$-validity of $((\exists x)\alpha \to (\forall x)\beta) \to (\forall x)(\alpha \to (\forall x)\beta)$ follows from the fact that for all $a, b, c \in A$, $a \leq b$ implies $b \to c \leq a \to c$. For the converse formula, note that for all $a \in D$,

$$\|\alpha\|^{\mathbf{A}}_{\mathcal{M}, v(x=a)} \cdot \bigwedge \{\|\alpha\|^{\mathbf{A}}_{\mathfrak{M}, v(x=b)} \to \|(\forall x)\beta\|^{\mathbf{A}}_{\mathfrak{M}, v(x=b)} \mid b \in D\}$$
$$\leq \|\alpha\|^{\mathbf{A}}_{\mathcal{M}, v(x=a)} \cdot (\|\alpha\|^{\mathbf{A}}_{\mathfrak{M}, v(x=a)} \to \|(\forall x)\beta\|^{\mathbf{A}}_{\mathfrak{M}, v(x=a)})$$
$$\leq \|(\forall x)\beta\|^{\mathbf{A}}_{\mathfrak{M}, v(x=a)}$$
$$= \|(\forall x)\beta\|^{\mathbf{A}}_{\mathfrak{M}, v}.$$

It follows that

$$\|(\exists x)\alpha\|^{\mathbf{A}}_{\mathcal{M},v} \cdot \bigwedge\{\|\alpha \to (\forall x)\beta\|^{\mathbf{A}}_{\mathfrak{M},v(x=a)} \mid a \in D\} \leq \|(\forall x)\beta\|^{\mathbf{A}}_{\mathfrak{M},v},$$

and so by residuation, $\mathbf{A}$-validity of $(\forall x)(\alpha \to (\forall x)\beta) \to ((\exists x)\alpha \to (\forall x)\beta)$ follows.

(vii) The $\mathbf{A}$-validity of $(\forall x)((\forall x)\alpha \to \beta) \to ((\forall x)\alpha \to (\forall x)\beta)$ follows since for every $a,b,c \in A$, $a \leq b$ implies $c \to a \leq c \to b$. For the converse, note that for all $a \in D$,

$$\begin{aligned}
&\|(\forall x)\alpha\|^{\mathbf{A}}_{\mathcal{M},v} \cdot \bigwedge\{\|(\forall x)\alpha \to \beta\|^{\mathbf{A}}_{\mathfrak{M},v(x=b)} \mid b \in D\} \\
&= \|(\forall x)\alpha\|^{\mathbf{A}}_{\mathcal{M},v(x=a)} \cdot \bigwedge\{\|(\forall x)\alpha \to \beta\|^{\mathbf{A}}_{\mathfrak{M},v(x=b)} \mid b \in D\} \\
&\leq \|(\forall x)\alpha\|^{\mathbf{A}}_{\mathcal{M},v(x=a)} \cdot \|(\forall x)\alpha \to \beta\|^{\mathbf{A}}_{\mathfrak{M},v(x=a)} \\
&\leq \|(\forall x)\alpha\|^{\mathbf{A}}_{\mathcal{M},v(x=a)} \cdot (\|(\forall x)\alpha \to \beta\|^{\mathbf{A}}_{\mathfrak{M},v(x=a)} \to \|\beta\|^{\mathbf{A}}_{\mathfrak{M},v(x=a)}) \\
&\leq \|\beta\|^{\mathbf{A}}_{\mathfrak{M},v(x=a)}.
\end{aligned}$$

It follows that

$$\|(\forall x)\alpha\|^{\mathbf{A}}_{\mathcal{M},v} \cdot \bigwedge\{\|(\forall x)\alpha \to \beta\|^{\mathbf{A}}_{\mathfrak{M},v(x=b)} \mid b \in D\} \leq \|(\forall x)\beta\|^{\mathbf{A}}_{\mathfrak{M},v}.$$

Now $\mathbf{A}$-validity of $(\forall x)((\forall x)\alpha \to \beta) \to ((\forall x)\alpha \to (\forall x)\beta)$ follows by residuation.  □

The result from Proposition 2.11 reveals one of the main motivations of this particular definition of $\mathsf{FL}_e$-algebras. It can be interpreted as a type of soundness result: the algebraic semantics for the one-variable fragment of the first-order substructural logic $\mathsf{QFL}_e$, defined over all $\mathsf{FL}_e$-algebras, consists of monadic $\mathsf{FL}_e$-algebras.

**Corollary 2.12.** *For all $\varphi \in \mathrm{Fm}_{\square\Diamond}(\mathcal{L}_{\mathsf{FL}_e})$,*

$$\mathcal{MFL}_e \models e \leq \varphi \implies \models_{\mathsf{QFL}_e} \varphi^{\circ}.$$

Whether the converse direction holds is currently an open problem. In Section 2.4, we study this problem for particular subvarieties of $\mathcal{MFL}_e$.

**Remark 2.13.** Note that the monadic $\mathsf{FL}_e$-algebra as defined here differ from the monadic residuated lattices as defined by Rachůnek and Šalounová in [133]. The monadic residuated lattices in that paper have less conditions on their $\mathsf{FL}_e$-reduct, since they do not assume commutativity of the monoidal operation $\cdot$, and more conditions on the modal part, since they satisfy both $\square(x \vee \square y) \approx \square x \vee \square y$ and $\Diamond(x \cdot x) \approx \Diamond x \cdot \Diamond x$. As is clear from Examples 2.9 and 2.10, monadic $\mathsf{FL}_e$-algebras satisfy neither of those identities. In fact, the monadic residuated lattices from [133] that are commutative coincide exactly with the monadic $\mathsf{FL}_e$-algebras as defined here that satisfy these two additional identities.

## 2.2   Relatively Complete Subalgebras

Halmos already noticed in [79] that the set of modal values $\{\Diamond b \mid b \in B\}$ for a monadic Boolean algebra $\langle \mathbf{B}; \Diamond \rangle$ was significant. He showed that the range of the modality $\Diamond$ is a so-called "relatively complete" subalgebra that uniquely determines the modality. Since then, similar results have been obtained for, e.g., monadic Heyting algebras and monadic MV-algebras. In this section, we apply similar methods to monadic $\mathsf{FL}_e$-algebras. In

particular, we show that the range of $\square$ for a monadic $\mathsf{FL}_e$-algebra $\langle \mathbf{A}; \square, \Diamond \rangle$ captures exactly this notion of relative completeness.

Let $\langle \mathbf{A}; \square, \Diamond \rangle$ be a monadic $\mathsf{FL}_e$-algebra. Let $\square A$ and $\Diamond A$ denote the sets $\{\square a \mid a \in A\}$ and $\{\Diamond a \mid a \in A\}$, respectively. By (L2) and (L11), it follows that $\square A = \Diamond A$. Combining this with (L14) and (L15) additionally implies that for any $c \in \square A$, $\square c = \Diamond c = c$.

**Lemma 2.14.** *Let $\langle \mathbf{A}; \square, \Diamond \rangle$ be a monadic $\mathsf{FL}_e$-algebra. Then $\square A = \Diamond A$ is a subuniverse of $\mathbf{A}$.*

*Proof.* We show that $\square A$ is closed under all operations of $\mathbf{A}$. Firstly, note that $f, e \in \square A$ by (L4) and (L5), respectively. Now consider $c, d \in \square A$. Then $c \wedge d = \square c \wedge \square d = \square(c \wedge d) \in \square A$ by (L3), so $\square A$ is closed under $\wedge$. Similarly, $c \vee d = \Diamond c \vee \Diamond d = \Diamond(c \vee d) \in \Diamond A$ by (L22) and so $\square A = \Diamond A$ is closed under $\vee$. By (L20), we obtain $c \cdot d = c \cdot \Diamond d = \Diamond(c \cdot \Diamond d) \in \Diamond A$ and so $\square A = \Diamond A$ is closed under $\cdot$. Lastly, by (L6), $c \rightarrow d = \Diamond c \rightarrow \square d = \square(c \rightarrow \square d) \in \square A$ and so $\square A = \Diamond A$ is closed under $\rightarrow$. $\qquad\square$

We let $\square \mathbf{A}$ denote the subalgebra of $\mathbf{A}$ with universe $\square A$. For any $a \in A$, note that

$$\square a = \max\{c \in \square A \mid c \leq a\} \ \text{ and } \ \Diamond a = \min\{c \in \square A \mid a \leq c\}.$$

To see this, note that by (L1), $\square a \in \{c \in \square A \mid c \leq a\}$. Now consider any $c \in A$ such that $c \leq a$. Then $c = \square c \leq \square a$ by (L9) and so $\square a = \max\{c \in \square A \mid c \leq a\}$. The fact that $\Diamond a = \min\{c \in \square A \mid a \leq c\}$ follows similarly. This leads us to the following definition, first introduced for monadic Boolean algebras in [79].

**Definition 2.15.** Let $\mathbf{A}$ be an $\mathsf{FL}_e$-algebra. A subalgebra $\mathbf{A}_0$ of $\mathbf{A}$ is called *relatively complete* if for any $a \in A$, the sets $\{c \in A_0 \mid c \leq a\}$ and $\{c \in A_0 \mid a \leq c\}$ contain a greatest and least element, respectively.

As shown above, $\square \mathbf{A}$ is a relatively complete subalgebra of any monadic $\mathsf{FL}_e$-algebra $\langle \mathbf{A}; \square, \Diamond \rangle$. Conversely, any relatively complete subalgebra $\mathbf{A}_0$ of an $\mathsf{FL}_e$-algebra $\mathbf{A}$ determines two modalities. We define modalities $\square_0$ and $\Diamond_0$ such that for each $a \in A$,

$$\square_0 a := \max\{c \in A_0 \mid c \leq a\} \ \text{ and } \ \Diamond_0 a := \min\{c \in A_0 \mid a \leq c\}.$$

Using these definitions, it follows that, as for monadic Boolean algebras, the modalities determine a relatively complete subalgebra, and vice versa.

**Lemma 2.16.** *Let $\mathbf{A}_0$ be a relatively complete subalgebra of an $\mathsf{FL}_e$-algebra $\mathbf{A}$. Then $\langle \mathbf{A}; \square_0, \Diamond_0 \rangle$ is a monadic $\mathsf{FL}_e$-algebra.*

*Proof.* We show that $\langle \mathbf{A}; \square_0, \Diamond_0 \rangle$ satisfies the properties (L1)-(L7).

(L1) This follows by definition of $\square_0$.

(L2) Since $\Diamond_0 a \in A_0$ by definition, we have $\square_0 \Diamond_0 a = \max\{c \in A_0 \mid c \leq \Diamond_0 a\} = \Diamond_0 a$.

(L3) Since $\square_0 a \leq a$ and $\square_0 b \leq b$, we have $\square_0 a \wedge \square_0 b \leq a \wedge b$ and so $\square_0 a \wedge \square_0 b \in \{c \in A_0 \mid c \leq a \wedge b\}$. We claim that $\square_0 a \wedge \square_0 b = \max\{c \in A_0 \mid c \leq a \wedge b\}$. Consider any $c \in A_0$ such that $c \leq a \wedge b$, i.e. $c \leq a$ and $c \leq b$. Since $c \in A_0$, we have $c \leq \square_0 a$ and $c \leq \square_0 b$, hence $c \leq \square_0 a \wedge \square_0 b$. So $\square_0(a \wedge b) = \square_0 a \wedge \square_0 b$.

(L4) This follows since $f \in A_0$, as $\mathbf{A}_0$ is a subalgebra.

(L5) This follows since $e \in A_0$, as $\mathbf{A}_0$ is a subalgebra.

(L6) Since $a \leq \Diamond_0 a$ by definition of $\Diamond_0$, we have $\Diamond_0 a \to \Box_0 b \leq a \to \Box_0 b$, so $\Diamond_0 a \to \Box_0 b \in \{c \in A_0 \mid c \leq a \to \Box_0 b\}$. Now consider any $c \in A_0$ such that $c \leq a \to \Box_0 b$. Then by residuation, $a \leq c \to \Box_0 b$. As $\mathbf{A}_0$ is a subalgebra of $\mathbf{A}$, we have $c \to \Box_0 b \in A_0$ and so by the definition of $\Diamond_0$, we obtain $\Diamond_0 a \leq c \to \Box_0 b$. Applying residuation again gives $c \leq \Diamond_0 a \to \Box_0 b$. Therefore, $\Diamond_0 a \to \Box_0 b = \max\{c \in A_0 \mid c \leq a \to \Box_0 b\}$, as required.

(L7) Since $\Box_0 b \leq b$ by definition of $\Box_0$, we have $\Box_0 a \to \Box_0 b \leq \Box_0 a \to b$, i.e., $\Box_0 \to \Box_0 b \in \{c \in A_0 \mid c \leq \Box_0 a \to b\}$. Now take any $c \in A_0$ such that $c \leq \Box_0 a \to b$, which gives $c \cdot \Box_0 a \leq b$. Since $\mathbf{A}_0$ is a subalgebra of $\mathbf{A}$, it follows that $c \cdot \Box_0 a \in A_0$, and so we obtain $c \cdot \Box_0 a \leq \max\{d \in A_0 \mid d \leq b\} = \Box_0 b$. Hence, $c \leq \Box_0 a \to \Box_0 b$ and so $\Box_0 a \to \Box_0 b = \Box_0 (\Box_0 a \to b)$. $\qquad\qquad\square$

Putting together these results, we obtain the following characterization of monadic $\mathsf{FL}_e$-algebras.

**Theorem 2.17.** *There exists a one-to-one correspondence between*

(1) *monadic $\mathsf{FL}_e$-algebras $\langle \mathbf{A}; \Box, \Diamond \rangle$;*

(2) *pairs $\langle \mathbf{A}, \mathbf{A}_0 \rangle$ of $\mathsf{FL}_e$-algebras where $\mathbf{A}_0$ is a relatively complete subalgebra of $\mathbf{A}$,*

*witnessed by the maps $\langle \mathbf{A}; \Box, \Diamond \rangle \mapsto \langle \mathbf{A}, \Box\mathbf{A} \rangle$ and $\langle \mathbf{A}, \mathbf{A}_0 \rangle \mapsto \langle \mathbf{A}; \Box_0, \Diamond_0 \rangle$.*

*Proof.* First, let $\langle \mathbf{A}; \Box, \Diamond \rangle$ be a monadic $\mathsf{FL}_e$-algebra. As noted, $\Box a = \max\{c \in \Box A \mid c \leq a\}$ and $\Diamond a = \min\{c \in \Box A \mid a \leq c\}$, and so by Lemma 2.14, $\Box\mathbf{A}$ is a relatively complete subalgebra of $\mathbf{A}$. Conversely, consider a pair $\langle \mathbf{A}, \mathbf{A}_0 \rangle$ of $\mathsf{FL}_e$-algebras where $\mathbf{A}_0$ is a relatively complete subalgebra of $\mathbf{A}$. By Lemma 2.16, $\langle \mathbf{A}; \Box_0, \Diamond_0 \rangle$ is a monadic $\mathsf{FL}_e$-algebra.

Secondly, we need that this correspondence between (1) and (2) is one-to-one. For a monadic $\mathsf{FL}_e$-algebra $\langle \mathbf{A}; \Box, \Diamond \rangle$, if we let $A_0 := \Box A$, we have $\Box a = \Box_0 a$ and $\Diamond a = \Diamond_0 a$ for all $a \in A$. Also, for a pair $\langle \mathbf{A}, \mathbf{A}_0 \rangle$ of $\mathsf{FL}_e$-algebras with $\mathbf{A}_0$ a relatively complete subalgebra of $\mathbf{A}$, it suffices to note that $A_0 = \Box_0 A$ by definition of $\Box_0$. $\qquad\square$

Note that this theorem reduces the study of a monadic $\mathsf{FL}_e$-algebra to the study of some particular pair of $\mathsf{FL}_e$-algebras. In many cases, the $\mathsf{FL}_e$-algebras are well-studied, allowing us to apply results for these algebras in the study of monadic $\mathsf{FL}_e$-algebras.

**Example 2.18.** For any $\mathsf{FL}_e$-algebra $\mathbf{A}$, $\mathbf{A}$ is trivially a relatively complete subalgebra of $\mathbf{A}$. Therefore $\langle \mathbf{A}, \mathbf{A} \rangle$ corresponds to a monadic $\mathsf{FL}_e$-algebra, where $\Box$ and $\Diamond$ are identity maps.

**Example 2.19.** Recall the monadic $\mathsf{FL}_e$-algebra $\langle \mathbf{H}; \Box, \Diamond \rangle$ from Figure 2.1. The relatively complete subalgebra corresponding to the modalities is the set $H_0 = \{\bot, c, \top\}$.

**Remark 2.20.** For another characterization of relatively complete subalgebras, we say that for two posets $\langle P, \leq_P \rangle$ and $\langle Q, \leq_Q \rangle$, an order-preserving function $h: P \to Q$ has a *right adjoint* $\Box_h: Q \to P$ and *left adjoint* $\Diamond_h: Q \to P$ if for all $a \in P$, $b \in Q$,

$$a \leq_P \Box_h b \iff h(a) \leq_Q b \quad \text{and} \quad \Diamond_h b \leq_P a \iff b \leq_Q h(a),$$

respectively. One can then show that for a subalgebra $\mathbf{A}_0$ of an $\mathsf{FL}_e$-algebra $\mathbf{A}$, the inclusion map $i: A_0 \to A$ has a right adjoint if and only if $\{c \in A_0 \mid c \leq a\}$ contains a greatest element for each $a \in A$, and $i$ has a left adjoint if and only if $\{c \in A_0 \mid a \leq c\}$ contains a least element for each $a \in A$. It follows that $\mathbf{A}_0$ is relatively complete if and only if $i: A_0 \to A$ has a right and left adjoint.

We would like to extend the characterization of Theorem 2.17 to any class $\mathcal{K}$ of monadic $\mathsf{FL}_e$-algebras. If $\mathcal{K}$ is a variety defined over $\mathcal{MFL}_e$ using identities that do not contain $\square$ or $\lozenge$, such a generalization is straightforward. Examples include monadic Boolean algebras, monadic Heyting algebras and monadic Gödel algebras. We let $\mathcal{K}_{\mathsf{FL}_e}$ denote the class $\{\mathbf{A} \mid \langle \mathbf{A}; \square, \lozenge \rangle \in \mathcal{K}\}$ of $\mathsf{FL}_e$-reducts of members of $\mathcal{K}$. Note that if $\mathcal{K}$ is defined over $\mathcal{MFL}_e$ using identities that do not contain $\square$ or $\lozenge$, then $\mathcal{K}_{\mathsf{FL}_e}$ is itself a variety.[4]

**Corollary 2.21.** *Let $\mathcal{V}$ be a subvariety of $\mathcal{MFL}_e$ that is defined over $\mathcal{MFL}_e$ using identities that do not contain $\square$ or $\lozenge$. Then there exists a one-to-one correspondence between*

(1) *members $\langle \mathbf{A}; \square, \lozenge \rangle$ of $\mathcal{V}$;*

(2) *pairs $\langle \mathbf{A}, \mathbf{A}_0 \rangle$ of members of $\mathcal{V}_{\mathsf{FL}_e}$ where $\mathbf{A}_0$ is a relatively complete subalgebra of $\mathbf{A}$,*

*witnessed by the maps $\langle \mathbf{A}; \square, \lozenge \rangle \mapsto \langle \mathbf{A}, \square\mathbf{A} \rangle$ and $\langle \mathbf{A}, \mathbf{A}_0 \rangle \mapsto \langle \mathbf{A}; \square_0, \lozenge_0 \rangle$.*

Such a characterization is not valid for all classes $\mathcal{K}$ of monadic $\mathsf{FL}_e$-algebras. For example, consider the monadic residuated lattice $\langle \mathbf{Ł}_3; \square, \lozenge \rangle$ as given in Figure 2.2. The set $A_0 = \{0, 1\}$ is the relatively complete subuniverse of $\mathbf{Ł}_3$ that corresponds to the given modalities $\square$ and $\lozenge$. However, $\langle \mathbf{Ł}_3; \square, \lozenge \rangle = \langle \mathbf{Ł}_3; \square_0, \lozenge_0 \rangle$ is not (term-equivalent to) a monadic MV-algebra, as it does not satisfy $\lozenge(x \cdot x) \approx \lozenge x \cdot \lozenge x$. A characterization like those given in Theorem 2.17 and Corollary 2.21 therefore does not hold for the variety of monadic MV-algebras, since these algebras by definition satisfy $\lozenge(x \cdot x) \approx \lozenge x \cdot \lozenge x$.

To extend the result of Theorem 2.17 to all classes $\mathcal{K}$ of monadic $\mathsf{FL}_e$-algebras, we extend the notion of relative completeness. For any $\mathsf{FL}_e$-algebra $\mathbf{A}$, a subalgebra $\mathbf{A}_0$ of $\mathbf{A}$ is called $\mathcal{K}$-*relatively complete* if it is relatively complete and $\langle \mathbf{A}; \square_0, \lozenge_0 \rangle$ is a member of $\mathcal{K}$.

**Theorem 2.22.** *Let $\mathcal{K}$ be a class of $\mathsf{FL}_e$-algebras. Then there exists a one-to-one correspondence between*

(1) *monadic $\mathsf{FL}_e$-algebras $\langle \mathbf{A}; \square, \lozenge \rangle \in \mathcal{K}$;*

(2) *pairs $\langle \mathbf{A}, \mathbf{A}_0 \rangle$ of $\mathsf{FL}_e$-algebras where $\mathbf{A}_0$ is a $\mathcal{K}$-relatively complete subalgebra of $\mathbf{A}$,*

*witnessed by the maps $\langle \mathbf{A}; \square, \lozenge \rangle \mapsto \langle \mathbf{A}, \square\mathbf{A} \rangle$ and $\langle \mathbf{A}, \mathbf{A}_0 \rangle \mapsto \langle \mathbf{A}; \square_0, \lozenge_0 \rangle$.*

*Proof.* For any monadic $\mathsf{FL}_e$-algebra $\langle \mathbf{A}; \square, \lozenge \rangle \in \mathcal{K}$, $\mathbf{A}_0 := \square\mathbf{A}$ is a $\mathcal{K}$-relatively completely subalgebra of $\mathbf{A}$, since $\langle \mathbf{A}; \square, \lozenge \rangle = \langle \mathbf{A}; \square_0, \lozenge_0 \rangle$. Conversely, for a $\mathcal{K}$-relatively complete subalgebra $\mathbf{A}_0$ of an $\mathsf{FL}_e$-algebra $\mathbf{A}$, we have $\langle \mathbf{A}; \square_0, \lozenge_0 \rangle$ by definition. That the correspondence is one-to-one follows as in the proof of Theorem 2.17. $\square$

Although $\mathcal{K}$-relative completeness is a rather abstract notion, it becomes easier to digest in particular cases. We discuss an interesting example.

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[4]It is an open problem whether $\mathcal{V}_{\mathsf{FL}_e}$ is a variety for all varieties $\mathcal{V}$ of monadic $\mathsf{FL}_e$-algebras. We conjecture this not to be the case.

**Example 2.23.** Consider the variety $\mathcal{MMV}$ of monadic MV-algebras (over the language $\mathcal{L}_{\mathsf{FL}_e}$). Note that $\mathcal{MMV}_{\mathsf{FL}_e}$ consists of all MV-algebras. For any such $\mathbf{A} \in \mathcal{MMV}_{\mathsf{FL}_e}$, a relatively complete subalgebra $\mathbf{A}_0$ of $\mathbf{A}$ is $\mathcal{MMV}$-relatively complete if $\langle \mathbf{A}; \square_0, \Diamond_0 \rangle \in \mathcal{MMV}$. This amounts to the condition that for all $a, b \in A$,

$$\square_0(a \vee \square_0 b) = \square_0 a \vee \square_0 b \ \text{ and } \ \Diamond_0(a \cdot a) = \Diamond_0 a \cdot \Diamond_0 a.$$

This is in turn equivalent to the notion of *m-relative completeness*, as introduced in [60]. An equivalent formulation, as taken from [44], defines a subalgebra $\mathbf{A}_0$ to be m-relatively complete if it is relatively complete and for all $a \in A$, $c_1, c_2 \in A_0$,

(1) if $c_1 \leq c_2 \vee a$, there exists $c_3 \in A_0$ such that $c_1 \leq c_2 \vee c_3$ and $c_3 \leq a$;

(2) if $a \cdot a \leq c_1$, there exists $c_3 \in A_0$ such that $a \leq c_3$ and $c_3 \cdot c_3 \leq c_1$.

Condition (1) here is equivalent to $\square_0(a \vee \square_0 b) = \square_0 a \vee \square_0 b$ holding for all $a, b \in A$, whereas condition (2) is equivalent to $\Diamond_0(a \cdot a) = \Diamond_0 a \cdot \Diamond_0 a$ holding for all $a \in A$.

**Remark 2.24.** Similarly to Remark 2.20, there exists an additional characterization of $\mathcal{K}$-relative completeness in terms of adjoints, for classes $\mathcal{K}$ of monadic $\mathsf{FL}_e$-algebras. For $\mathsf{FL}_e$-algebras $\mathbf{A}$ and $\mathbf{B}$, we say that an order-preserving function $h \colon A \to B$ has a *right $\mathcal{K}$-adjoint* $\square_h \colon B \to A$ and *left $\mathcal{K}$-adjoint* $\Diamond_h \colon B \to A$ if $\square_h$ is a left-adjoint of $h$, $\Diamond_h$ is a right-adjoint of $h$, and $\langle \mathbf{B}; \square_h, \Diamond_h \rangle$ is a member of $\mathcal{K}$. It can then be proved that $\mathbf{A}_0$ is a $\mathcal{K}$-relatively complete subalgebra of $\mathbf{A}$ if and only if the inclusion map $i \colon A_0 \to A$ has left and right $\mathcal{K}$-adjoints.

**Remark 2.25.** In [22], Bezhanishvili extends the correspondence between monadic Heyting algebras and pairs consisting of a Heyting algebra and a relatively complete subalgebra to a categorical equivalence (for an introduction to category theory, see, e.g., [3]). A similar equivalence can be found for any class $\mathcal{K}$ of monadic $\mathsf{FL}_e$-algebras. Indeed, on one side we have the category $\mathsf{K}$ of members of $\mathcal{K}$ as objects and modal homomorphisms as morphisms. On the other side we have the category $\mathsf{K}^2_{\mathsf{FL}_e}$ of pairs $\langle \mathbf{A}, \mathbf{A}_0 \rangle$ of $\mathsf{FL}_e$-algebras such that $\mathbf{A} \in \mathcal{K}_{\mathsf{FL}_e}$ and $\mathbf{A}_0$ is a $\mathcal{K}$-relatively complete subalgebra of $\mathbf{A}$ as objects or, equivalently, the inclusion map $i_A \colon A_0 \to A$ has left and right $\mathcal{K}$-adjoints $\square_A$ and $\Diamond_A$ respectively. As morphisms, it has pairs $\langle f, f_0 \rangle$ of functions $f \colon A \to B$ and $f_0 \colon A_0 \to B_0$ such that

(i) $f$ is a homomorphism;

(ii) $f \circ i_A = i_B \circ f_0$;

(iii) $\square_B \circ f = f_0 \circ \square_A$;

(iv) $\Diamond_B \circ f = f_0 \circ \Diamond_A$.

Conditions (ii)–(iv) are equivalent to the commutativity of the diagram in Figure 2.3. Categories $\mathsf{K}$ and $\mathsf{K}^2_{\mathsf{FL}_e}$ can be shown to be isomorphic. We leave the details to the reader.

$$
\begin{array}{ccc}
 & \square_A & \\
A_0 \xrightarrow{\hookleftarrow i_A} & & A \\
 & \diamondsuit_A & \\
f_0 \downarrow & & \downarrow f \\
 & \square_B & \\
B_0 \xrightarrow{\hookleftarrow i_B} & & B \\
 & \diamondsuit_B & \\
\end{array}
$$
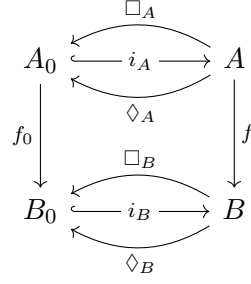
Figure 2.3: Commutative diagram defining morphisms for $\mathsf{K}^2_{\mathsf{FL}_e}$

## 2.3 Congruences

We now turn our attention to the study of congruences. The congruences of $\mathsf{FL}_e$-algebras have been well-studied (see, e.g., [28, 71, 89]). In this section we apply similar methods to monadic $\mathsf{FL}_e$-algebras. Moreover, we show that the congruence structure of a monadic $\mathsf{FL}_e$-algebra $\langle \mathbf{A}; \square, \diamondsuit \rangle$ is determined by the relatively complete subalgebra $\square \mathbf{A}$.

Let us first fix notation and recall the necessary notions from universal algebra and the theory of $\mathsf{FL}_e$-algebras. For any algebra $\mathbf{A}$, a *congruence* is an equivalence relation $\Theta \subseteq A^2$ compatible with the operations of $A$. For $a \in A$, we write $[a]_\Theta$ to denote the equivalence class $\{b \in A \mid \langle a, b \rangle \in \Theta\}$, and let $\mathbf{A}/\Theta$ denote the quotient algebra with universe $A/\Theta := \{[a]_\Theta \mid a \in A\}$ and the induced operations. The lattice of congruences of $\mathbf{A}$, ordered by inclusion, is denoted by $\mathsf{Con}(\mathbf{A})$. Now let $\mathbf{A}$ be an $\mathsf{FL}_e$-algebra. We let $A^-$ denote the set $\{a \in A \mid a \leq e\}$ of *negative* elements of $\mathbf{A}$. Let $T \subseteq A$. We call a subset $H \subseteq A$ $T$-*convex* (or simply *convex*, if $T$ is clear from the context) if for all $a, b \in H$, $c \in T$, $a \leq c \leq b$ implies $c \in H$. We say that $H \subseteq A$ is an $f$-*free subuniverse* of $\mathbf{A}$ if $H$ is closed under the operations $\wedge, \vee, \cdot, \rightarrow$ and $e \in H$, but not necessarily $f \in H$. If $f \in H$, we sometimes pedantically call $H$ a *pointed subuniverse* of $\mathbf{A}$. We let $\mathcal{C}(\mathbf{A})$ denote the lattice of all $A$-convex $f$-free subuniverses of $\mathbf{A}$ ordered by inclusion. For an $f$-free subuniverse $H$ of $A$ and $a \in A$, we write $H(a)$ to denote the smallest $f$-free subuniverse that contains $H \cup \{a\}$, which we can alternatively write as

$$
H(a) = \{b \in A \mid h \cdot a^n \leq b \leq (h \cdot a^n) \rightarrow e \text{ for some } h \in H \text{ and } n \in \mathbb{N}\}.
$$

We extend these definitions to the setting of monadic $\mathsf{FL}_e$-algebras: for a monadic $\mathsf{FL}_e$-algebra $\langle \mathbf{A}; \square, \diamondsuit \rangle$, we say that $H \subseteq A$ is an $f$-*free subuniverse* of $\langle \mathbf{A}; \square, \diamondsuit \rangle$ if $H$ is an $f$-free subuniverse of $\mathbf{A}$ closed under $\square$ and $\diamondsuit$. If $f \in H$, we call $H$ a *pointed subuniverse* of $\langle \mathbf{A}; \square, \diamondsuit \rangle$. We let $\mathcal{C}(\langle \mathbf{A}; \square, \diamondsuit \rangle)$ denote the lattice of all $A$-convex $f$-free subuniverses of $\langle \mathbf{A}; \square, \diamondsuit \rangle$ ordered by inclusion.

To give a characterization of congruences for monadic $\mathsf{FL}_e$-algebras, we need a couple of lemmas. For the bulk of the proofs, we refer to [71]. We give only the parts relevant to the modalities.

**Lemma 2.26.** *Let $\langle \mathbf{A}; \square, \diamondsuit \rangle$ be a monadic $\mathsf{FL}_e$-algebra. If $\Theta$ is a congruence on $\langle \mathbf{A}; \square, \diamondsuit \rangle$, then $H_\Theta := [e]_\Theta$ is an $A$-convex $f$-free subuniverse of $\langle \mathbf{A}; \square, \diamondsuit \rangle$.*

*Proof.* Note that for $a \in [e]_\Theta$, $\square a \Theta \square e = e$ and $\diamondsuit a \Theta \diamondsuit e = e$, hence $\square a, \diamondsuit a \in [e]_\Theta$. The rest of the proof follows from [71, Theorem 3.47]. $\qquad\square$

**Lemma 2.27.** *Let $\langle \mathbf{A}; \square, \Diamond \rangle$ be a monadic residuated lattice. If $H$ is an $A$-convex $f$-free subuniverse of $\langle \mathbf{A}; \square, \Diamond \rangle$, then*

$$\Theta_H := \{\langle a, b \rangle \in A^2 \mid a \cdot h \leq b \text{ and } b \cdot h \leq a \text{ for some } h \in H\}$$

*is a congruence on $\langle \mathbf{A}; \square, \Diamond \rangle$.*

*Proof.* Consider $\langle a, b \rangle \in \Theta_H$. By definition, there exist $h \in H$ such that $a \cdot h \leq b$ and $b \cdot h \leq a$. To show that $\langle \square a, \square b \rangle \in \Theta$, note that from $a \cdot h \leq b$ and (L1), we have $a \cdot \square h \leq a \cdot h \leq b$, and similarly $b \cdot \square h \leq b \cdot h \leq a$. Then $a \leq \square h \to b$ and $b \leq \square h \to a$, so by (L9) and (L7), we obtain $\square a \leq \square(\square h \to b) = \square h \to \square b$, i.e. $\square a \cdot \square h \leq \square b$ and similarly, $\square b \cdot \square h \leq \square a$. Since $\square h \in H$, we have $\langle \square a, \square b \rangle \in \Theta_H$. For compatibility with $\Diamond$, we have $a \cdot h \leq b \leq \Diamond b$ and $b \cdot h \leq a \leq \Diamond a$ by (L8), so $a \leq h \to \Diamond b$ and $b \leq h \to \Diamond a$. Applying (L10) and (L23) gives $\Diamond a \leq \Diamond(h \to \Diamond b) \leq \square h \to \Diamond b$, i.e. $\Diamond a \cdot \square h \leq \Diamond b$ and similarly, $\Diamond b \cdot \square h \leq \Diamond a$. Since $\square h \in H$, $\langle \Diamond a, \Diamond b \rangle \in \Theta_H$.

The rest of the proof follows from [71, Theorem 3.47]. $\qquad\qquad\square$

We summarize the characterization of congruences in the following theorem. Its proof is based on the one found in [71, Theorem 3.47] and we will not recall it here.

**Theorem 2.28.** *Let $\langle \mathbf{A}; \square, \Diamond \rangle$ be a monadic $\mathsf{FL}_e$-algebra. Then*

$$\mathsf{Con}(\langle \mathbf{A}; \square, \Diamond \rangle) \cong \mathcal{C}(\langle \mathbf{A}; \square, \Diamond \rangle),$$

*as witnessed by the following lattice isomorphisms*

$$\mathsf{Con}(\langle \mathbf{A}; \square, \Diamond \rangle) \to \mathcal{C}(\langle \mathbf{A}; \square, \Diamond \rangle);$$
$$\Theta \mapsto H_\Theta := [e]_\Theta$$
$$\mathcal{C}(\langle \mathbf{A}; \square, \Diamond \rangle) \to \mathsf{Con}(\langle \mathbf{A}; \square, \Diamond \rangle);$$
$$H \mapsto \Theta_H := \{(a, b) \in A^2 \mid h \cdot a \leq b \text{ and } h \cdot b \leq a \text{ for some } h \in H\}.$$

**Remark 2.29.** For $\mathsf{FL}_e$-algebras, other alternative characterizations of congruences have been considered. We indicate here how two such characterizations can be extended to the setting of monadic $\mathsf{FL}_e$-algebras. Firstly, recall that a subset $M \subseteq A$ is called a *submonoid* of an $\mathsf{FL}_e$-algebra $\mathbf{A}$ if $e \in M$ and $M$ is closed under $\cdot$. We extend this definition to the monadic setting by defining $M \subseteq A$ to be a *monadic submonoid* of a monadic $\mathsf{FL}_e$-algebra $\langle \mathbf{A}; \square, \Diamond \rangle$ if $M$ is a submonoid of $\mathbf{A}$ and $M$ is closed under $\square$. One can then prove that $\mathsf{Con}(\langle \mathbf{A}; \square, \Diamond \rangle)$ is isomorphic to the lattice of all subsets $M \subseteq A^-$ that are $A^-$-convex submonoids of $\langle \mathbf{A}; \square, \Diamond \rangle$.

Secondly, for an $\mathsf{FL}_e$-algebra $\mathbf{A}$, a *deductive filter* is a subset $F \subseteq A$ that is upwards closed, $e \in F$, $a \in F$ implies $a \wedge e \in F$, and $a, b \in F$ implies $a \cdot b \in F$. This characterization can be extended to the setting of monadic $\mathsf{FL}_e$-algebras. Indeed, congruences of a monadic $\mathsf{FL}_e$-algebra $\langle \mathbf{A}; \square, \Diamond \rangle$ correspond to deductive filters $F \subseteq A$ of $\mathbf{A}$ that are closed under $\square$. The details are left to the reader.

We now use the characterization of congruences described in Theorem 2.28 to study the prevalent role that the relatively complete subalgebra $\square \mathbf{A}$ plays in the context of monadic $\mathsf{FL}_e$-algebras. In fact, we will prove that the congruences of a monadic $\mathsf{FL}_e$-algebra are determined by $\square \mathbf{A}$. To show this, we first need a pair of lemmas.

**Lemma 2.30.** *Let $H$ be an $A$-convex $f$-free subuniverse of a monadic $\mathsf{FL}_e$-algebra $\langle \mathbf{A}; \square, \lozenge \rangle$. Then $H \cap \square A$ is a $\square A$-convex $f$-free subuniverse of $\square \mathbf{A}$.*

*Proof.* As both $H$ and $\square A$ are $f$-free subuniverses of $\langle \mathbf{A}; \square, \lozenge \rangle$, so is $H \cap \square A$. The $\square A$-convexity of $H \cap \square A$ follows by the $A$-convexity of $H$. $\qquad\square$

**Lemma 2.31.** *Let $\langle \mathbf{A}; \square, \lozenge \rangle$ be a monadic $\mathsf{FL}_e$-algebra. If $H$ is a $\square A$-convex $f$-free subuniverse of $\square \mathbf{A}$, then*

$$\{a \in A \mid h \leq a \leq h \to e \text{ for some } h \in H\} = \{a \in A \mid \square a \in H \text{ and } \lozenge a \in H\}$$

*is an $A$-convex $f$-free subuniverse of $\langle \mathbf{A}; \square, \lozenge \rangle$.*

*Proof.* We first show that $\{a \in A \mid \square a \in H$ and $\lozenge a \in H\} = \{a \in A \mid h \leq a \leq h \to e$ for some $h \in H\}$. Consider $a \in A$ such that $h \leq a \leq h \to e$ for some $h \in H$. It then follows from $h, h \to e \in \square A$ and property (L9) that $h = \square h \leq \square a \leq \square(h \to e) = h \to e$. Similarly, $h = \lozenge h \leq \lozenge a \leq \lozenge(h \to e) = h \to e$, and so $\square a, \lozenge a \in H$. Conversely, consider $a \in A$ such that $\square a, \lozenge a \in H$. Let $h := \square a \wedge (\lozenge a \to e) \in H$. Then

$$a \cdot h \leq a \cdot (\lozenge a \to e) \leq \lozenge a \cdot (\lozenge a \to e) \leq e,$$

and $h \leq \square a \leq a$, so $h \leq a \leq h \to e$.

A proof showing that $K := \{a \in A \mid h \leq a \leq h \to e$ for some $h \in H\}$ is an $A$-convex $f$-free subuniverse of $\mathbf{A}$ can be found in [71, Theorem 3.47], so it suffices to show closure under $\square$ and $\lozenge$. Let $a \in K$. Then $\square a, \lozenge a \in H$. Since $\square \square a = \lozenge \square a = \square a$ and $\square \lozenge a = \lozenge \lozenge a = \lozenge a$, it follows that $\square a, \lozenge a \in K$. $\qquad\square$

**Theorem 2.32.** *Let $\langle \mathbf{A}; \square, \lozenge \rangle$ be a monadic $\mathsf{FL}_e$-algebra. Then*

$$\mathsf{Con}(\langle \mathbf{A}; \square, \lozenge \rangle) \cong \mathsf{Con}(\square \mathbf{A}),$$

*as witnessed by the following isomorphisms*

$$\mathcal{C}(\langle \mathbf{A}; \square, \lozenge \rangle) \to \mathcal{C}(\square \mathbf{A});$$
$$\qquad H \mapsto H \cap \square A$$
$$\mathcal{C}(\square \mathbf{A}) \to \mathcal{C}(\langle \mathbf{A}; \square, \lozenge \rangle);$$
$$\qquad H \mapsto \{a \in A \mid h \leq a \leq h \to e \text{ for some } h \in H\} = \{a \in A \mid \square a \in H \text{ and } \lozenge a \in H\}.$$

*Proof.* If we show that the proposed maps are indeed isomorphisms, it follows by Theorem 2.28 that $\mathsf{Con}(\langle \mathbf{A}; \square, \lozenge \rangle) \cong \mathsf{Con}(\square \mathbf{A})$. The maps are obviously order-preserving and are well-defined by Lemmas 2.30 and 2.31. We show that they compose to identity maps, from which it follows that they are bijections.

Firstly, let $H$ be an $A$-convex $f$-free subuniverse of $\langle \mathbf{A}; \square, \lozenge \rangle$. Then $H = \{a \in A \mid \square a \in H \cap \square A$ and $\lozenge a \in H \cap \square A\}$. The left-to-right inclusion follows since $H$ is closed under $\square$ and $\lozenge$. For the converse inclusion, consider $a \in A$ such that $\square a \in H \cap \square A$ and $\lozenge a \in H \cap \square A$. Since $\square a \leq a \leq \lozenge a$, $a \in H$ by the $A$-convexity of $H$.

Secondly, consider a $\square A$-convex $f$-free subuniverse $H$ of $\square \mathbf{A}$. Since $c = \square c = \lozenge c$ for any $c \in H$, it easily follows that $H = \{a \in A \mid \square a \in H$ and $\lozenge a \in H\} \cap \square A$. $\qquad\square$

This result reduces the study of congruences of some monadic $\mathsf{FL}_e$-algebra $\langle \mathbf{A}; \square, \lozenge \rangle$ to the study of the congruences of the $\mathsf{FL}_e$-algebra $\square\mathbf{A}$. In many cases, the congruences of (subalgebras of) the $\mathsf{FL}_e$-reduct are easier to study or well-studied already, for example in the case of monadic Heyting algebras, monadic MV-algebras, monadic Gödel algebras or monadic abelian $\ell$-groups. We take full advantage of this fact in the next section, where we study particular classes of monadic $\mathsf{FL}_e$-algebras.

**Corollary 2.33.** *Let* $\langle \mathbf{A}; \square, \lozenge \rangle$ *be a monadic* $\mathsf{FL}_e$*-algebra. Then* $\langle \mathbf{A}; \square, \lozenge \rangle$ *is subdirectly irreducible (simple) if and only if* $\square\mathbf{A}$ *is subdirectly irreducible (simple).*

## 2.4   Functional Completeness

Recall that in Corollary 2.12, we showed a soundness result for the variety $\mathcal{MFL}_e$ with respect to the first-order substructural logic $\mathsf{QFL}_e$. As mentioned, the converse direction of completeness is an open problem. In this section, we aim to investigate this problem of completeness for some subvarieties of $\mathcal{MFL}_e$. To express exactly what kind of completeness results we obtain, we first require some additional terminology.

Let $\mathbf{B}$ be an $\mathsf{FL}_e$-algebra and $W$ a non-empty set. We write $\mathbf{B}^W$ to denote the direct power of $\mathbf{B}$ over $W$, that is, the $\mathsf{FL}_e$-algebra with the universe $B^W$ of functions $f\colon W \to B$ with operations defined pointwise.

**Definition 2.34.** Let $\mathbf{B}$ be an $\mathsf{FL}_e$-algebra and $W$ a non-empty set. For each $f \in B^W$ we call $\bigwedge\{f(v) \mid v \in W\}$ and $\bigvee\{f(v) \mid v \in W\}$ the *lower* and *upper limit point* of $f$, if they exist, respectively. Now let $\mathbf{A}$ be any subalgebra of $\mathbf{B}^W$ such that the lower and upper limit point exist for each $f \in A$. We define for each $f \in A$, $w \in W$,

$$\square f(w) \coloneqq \bigwedge\{f(v) \mid v \in W\} \text{ and } \lozenge f(w) \coloneqq \bigvee\{f(v) \mid v \in W\}.$$

By Proposition 2.11, $\langle \mathbf{A}; \square, \lozenge \rangle$ is a monadic $\mathsf{FL}_e$-algebra. Any monadic $\mathsf{FL}_e$-algebra $\langle \mathbf{A}; \square, \lozenge \rangle$ constructed in such a way is called *functional*. A variety $\mathcal{V}$ of monadic $\mathsf{FL}_e$-algebras is called *functionally complete* if $\mathcal{V}$ is generated by its functional members.

Note that the mentioned completeness problem for $\mathcal{MFL}_e$ reduces to showing that $\mathcal{MFL}_e$ is functionally complete. For $\mathcal{MFL}_e$, this is again an open problem, but it has been solved for a number of subvarieties of $\mathcal{MFL}_e$. For example, Bezhanishvili and Harding proved functional completeness for the variety of monadic Heyting algebras in [24]. Additionally, the standard completeness results imply that the varieties of monadic MV-algebras and crisp monadic Gödel algebras are generated by the functional algebras $\langle \mathbf{Ł}^{\mathbb{N}}; \square, \lozenge \rangle$ and $\langle \mathbf{G}^{\mathbb{N}}; \square, \lozenge \rangle$, respectively. Slightly weaker results for these varieties were obtained by Castaño et al. in [45] through different methods.

In this section we apply similar methods to varieties of monadic $\mathsf{FL}_e$-algebras that satisfy some particular properties. The goal is not functional completeness itself, but rather a slightly weaker generation result. Indeed, we are interested in the following generalization of functional monadic $\mathsf{FL}_e$-algebras.

**Definition 2.35.** A monadic $\mathsf{FL}_e$-algebra $\langle \mathbf{A}; \square, \lozenge \rangle$ is called *relatively functional* if $\mathbf{A}$ is a subalgebra of $\mathbf{B}^W$ for some $\mathsf{FL}_e$-algebra $\mathbf{B}$ and non-empty set $W$, and $\square A$ consists of only constant functions.

Let $\langle \mathbf{A}; \Box, \Diamond \rangle$ be a relatively functional monadic $\mathsf{FL}_e$-algebra. Then for each $f \in A$,

$$\Box f(w) = \bigvee \{g \in \Box A \mid g(v) \leq f(v) \text{ for all } v \in W\}$$
$$\Diamond f(w) = \bigwedge \{g \in \Box A \mid f(v) \leq g(v) \text{ for all } v \in W\}.$$

Moreover, this is indeed a generalization of a functional monadic $\mathsf{FL}_e$-algebra: any relatively functional monadic $\mathsf{FL}_e$-algebra $\langle \mathbf{A}; \Box, \Diamond \rangle$ is functional if the lower and upper limit points exist for each $f \in A$ and

$$\Box A = \{f_b \mid b \text{ a lower or upper limit point of some } g \in A\},$$

where for each $b \in B$, $f_b \colon W \to B$ is the constant function mapping each $w$ to $b$. In Section 4.3 we apply the methods outlined in this section to obtain proper functional completeness for the variety of monadic abelian $\ell$-groups.

Let us now introduce the varieties of monadic $\mathsf{FL}_e$-algebras that we will study in this section. We are interested in three properties in particular. Firstly, we consider varieties $\mathcal{V}$ such that the members of $\mathcal{V}_{\mathsf{FL}_e}$ are *semilinear*. An $\mathsf{FL}_e$-algebra is called semilinear if it a subdirect product of $\mathsf{FL}_e$-chains. It was shown in [28] (see also [89]) that an $\mathsf{FL}_e$-algebra is semilinear if and only if it satisfies the identity

$$e \approx ((x \to y) \wedge e) \vee ((y \to x) \wedge e).$$

An alternative axiomatization was given in [81] using the two identities

$$e \leq (x \to y) \vee (y \to x) \text{ and } e \wedge (x \vee y) \approx (e \wedge x) \vee (e \wedge y).$$

We say that a class of $\mathsf{FL}_e$-algebras $\mathcal{K}$ is *semilinear* if all its members are. We write $\mathsf{lin}(\mathcal{K})$ to denote the class of all linearly ordered members of $\mathcal{K}$.

Secondly, we consider varieties $\mathcal{V}$ of monadic $\mathsf{FL}_e$-algebras such that the class $\mathsf{lin}(\mathcal{V}_{\mathsf{FL}_e})$ has the *amalgamation property*. A class $\mathcal{K}$ of algebras is said to have the amalgamation property if for any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and embeddings $f \colon \mathbf{A} \to \mathbf{B}$ and $g \colon \mathbf{A} \to \mathbf{C}$, there exist $\mathbf{D} \in \mathcal{K}$ (called the *amalgam*) and embeddings $f' \colon \mathbf{B} \to \mathbf{D}$ and $g' \colon \mathbf{C} \to \mathbf{D}$ such that $f' \circ f = g' \circ g$. That is, the following diagram commutes.
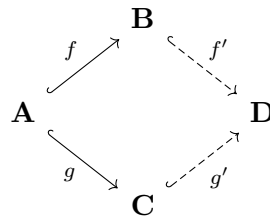


We say that $\mathcal{K}$ has the *generalized amalgamation property* if for any $\mathbf{A} \in \mathcal{K}$, any family of algebras $\{\mathbf{B}_i\}_{i \in I} \subseteq \mathcal{K}$ and embeddings $f_i \colon \mathbf{A} \to \mathbf{B}_i$, there exist $\mathbf{C} \in \mathcal{K}$ and embeddings $g_i \colon \mathbf{B}_i \to \mathbf{C}$ such that $g_i \circ f_i = g_j \circ f_j$ for all $i, j \in I$. For $\mathsf{lin}(\mathcal{V})$, where $\mathcal{V}$ is any variety of $\mathsf{FL}_e$-algebras, the two notions coincide.[5]

**Lemma 2.36** (cf. [45, Theorem 3.3]). *Let $\mathcal{V}$ be a variety of $\mathsf{FL}_e$-algebras. If $\mathsf{lin}(\mathcal{V})$ has the amalgamation property, then it has the generalized amalgamation property.*

---

[5]In fact, it is shown in [45] that the two notions coincide if $\mathcal{K}$ is elementary (as a class of first-order structures).

Finally, we consider monadic $\mathsf{FL}_e$-algebras satisfying the equality $\Box(x \vee \Box y) \approx \Box x \vee \Box y$, i.e., the algebraic formulation of the constant domain axiom $(\mathrm{cd}_\Box)$. The main theorem of this section (Theorem 2.43) concerns varieties $\mathcal{V}$ of monadic $\mathsf{FL}_e$-algebras such that

(i) $\mathcal{V}_{\mathsf{FL}_e}$ is semilinear;

(ii) the class $\mathsf{lin}(\mathcal{V}_{\mathsf{FL}_e})$ has the generalized amalgamation property;

(iii) $\mathcal{V} \models \Box(x \vee \Box y) \approx \Box x \vee \Box y$.

Although these seem like rather heavy restrictions, it is worth noting that the varieties of monadic MV-algebras, crisp monadic Gödel algebras, and monadic abelian $\ell$-groups all satisfy these three conditions. Indeed, they satisfy the identity $\Box(x \vee \Box y) \approx \Box x \vee \Box y$ by definition. Moreover, MV-algebras, Gödel algebras and abelian $\ell$-groups are well-known to be semilinear. Finally, the class of linearly ordered abelian $\ell$-groups has the amalgamation property, as shown by Pierce in [128]. Mundici's correspondence between linearly ordered abelian $\ell$-groups and linearly ordered MV-algebras then implies the same for the class of linearly ordered MV-algebras; for a concrete proof, see, e.g., [105, Proposition 62]. The class of linearly ordered Gödel algebras was shown to have the amalgamation property by Maksimova in [99]. Lemma 2.36 then implies that all these classes have the generalized amalgamation property.

Let us first focus on a variety $\mathcal{V}$ of monadic $\mathsf{FL}_e$-algebras whose $\mathsf{FL}_e$-reducts are semilinear. It is useful to recall a number of results from the literature regarding semilinear $\mathsf{FL}_e$-algebras. Let $\mathbf{A}$ be an $\mathsf{FL}_e$-algebra. We call an $A$-convex $f$-free subuniverse $H \subseteq A$ of $\mathbf{A}$ *prime* if it is a meet-prime element in the lattice $\mathcal{C}(\mathbf{A})$, i.e., if $I \cap J \subseteq H$ implies $I \subseteq H$ or $J \subseteq H$, for all $I, J \in \mathcal{C}(\mathbf{A})$. We write $\mathbf{A}/H$ to denote the quotient $\mathbf{A}/\Theta_H$, where $\Theta_H$ is the congruence corresponding to $H$ via the isomorphisms given in Theorem 2.28, and write $a/H$ to denote $[a]_{\Theta_H}$. In the presence of semilinearity, we can give a useful equivalent characterization of prime convex $f$-free subuniverses. We also recall a property of semilinear $\mathsf{FL}_e$-algebras that is useful in the proofs to follow.

**Lemma 2.37** ([32, Lemma 4.2]). *Let $\mathbf{A}$ be a semilinear $\mathsf{FL}_e$-algebra. Then for any $A$-convex $f$-free subuniverse $H$ of $\mathbf{A}$, the following are equivalent:*

(1) *$H$ is a prime $A$-convex $f$-free subuniverse of $\mathbf{A}$;*

(2) *for all $a, b \in A$, $(a \rightarrow b) \wedge e \in H$ or $(b \rightarrow a) \wedge e \in H$;*

(3) *$\mathbf{A}/H$ is linearly ordered.*

*Proof.* The equivalence between (1) and (2) can be found in [32, Lemma 4.2]. For the equivalence between (2) and (3), first suppose that (2) holds. Let $a, b \in A$ and suppose without loss of generality that $(a \rightarrow b) \wedge e \in H$. Note that

$$(a \rightarrow b) \wedge e \le (a \rightarrow b) \wedge e \le ((a \rightarrow b) \wedge e) \rightarrow e,$$

and so $\langle (a \rightarrow b) \wedge e, e \rangle \in \Theta_H$. We can conclude that $[a]_{\Theta_H} \le [b]_{\Theta_H}$. Conversely, suppose that (3) holds. Consider $a, b \in A$ and assume without loss of generality that $[a]_{\Theta_H} \le [b]_{\Theta_H}$. Then $\langle (a \rightarrow b) \wedge e, e \rangle \in \Theta_H$, that is, $h \le (a \rightarrow b) \wedge e \le h \rightarrow e$ for some $h \in H$. By convexity of $H$, it follows that $(a \rightarrow b) \wedge e \in H$. $\qquad\square$

**Remark 2.38.** As noted in [32], if $\mathbf{A}$ is not semilinear, then (1) and (2) are not equivalent. Indeed, if $\mathbf{A}$ is not semilinear, then there exist $a, b \in A$ such that

$$c := ((a \to b) \wedge e) \vee ((b \to a) \wedge e) < e.$$

Let $P$ be an $A$-convex $f$-free subuniverse of $\mathbf{A}$ that is maximal with respect to not containing $c$, which exists by Zorn's Lemma and is necessarily meet-prime in $\mathcal{C}(\mathbf{A})$. Then neither $(a \to b) \wedge e$ nor $(b \to a) \wedge e$ are elements of $P$ by convexity. So $P$ is a prime $A$-convex $f$-free subuniverse of $\mathbf{A}$ not satisfying (2).

**Lemma 2.39** (cf. [28, Proposition 6.13]). *Let $\mathbf{A}$ be a semilinear $\mathsf{FL}_e$-algebra, and $H$ an $A$-convex $f$-free subuniverse of $\mathbf{A}$. Then for each $a, b \in A$,*

$$H((a \to b) \wedge e) \cap H((b \to a) \wedge e) = H(((a \to b) \wedge e) \vee ((b \to a) \wedge e)) = H.$$

The following lemma shows that prime convex $f$-free subuniverses exist in the context of semilinear $\mathsf{FL}_e$-algebras. In fact, any convex $f$-free subuniverse can be extended to a prime one. The result generalizes the prime ideal theorem for Boolean algebras.

**Lemma 2.40.** *Let $\mathbf{A}$ be a semilinear $\mathsf{FL}_e$-algebra, $H \subseteq A$ an $A$-convex $f$-free subuniverse of $\mathbf{A}$, and $a \in A \setminus H$. Then there exists a prime $A$-convex $f$-free subuniverse $P$ of $\mathbf{A}$ such that $H \subseteq P$ and $a \notin P$.*

*Proof.* Consider the set $\mathcal{D} = \{I \in \mathcal{C}(\mathbf{A}) \mid H \subseteq I, a \notin I\}$ ordered by set-inclusion. Then $H \in \mathcal{D}$ and for any chain $\{I_x\}_{x \in X} \subseteq \mathcal{D}$, we have $\bigcup_{x \in X} I_x \in \mathcal{D}$. By Zorn's Lemma, $\mathcal{D}$ contains a maximal element $P$.

We argue that $P$ is prime. For a contradiction, suppose that there exist elements $b, c \in A$ such that $(b \to c) \wedge e, (c \to b) \wedge e \notin P$. We consider the smallest convex $f$-free subuniverses $P((b \to c) \wedge e)$ and $P((c \to b) \wedge e)$ generated by $P \cup \{(b \to c) \wedge e\}$ and $P \cup \{(c \to b) \wedge e\}$, respectively. Since $P$ is maximal and $P \subseteq P((b \to c) \wedge e), P((c \to b) \wedge e)$, it follows that $a \in P((b \to c) \wedge e) \cap P((c \to b) \wedge e)$. By Lemma 2.39, we deduce that $a \in P$, contradicting $P \in \mathcal{D}$. $\qquad\square$

Unlike the variety of semilinear $\mathsf{FL}_e$-algebras, the variety of monadic $\mathsf{FL}_e$-algebras whose $\mathsf{FL}_e$-reducts are semilinear is not necessarily generated by its linearly ordered members. By Lemma 2.40 above, we do however obtain a slightly weaker result. Let us call a monadic $\mathsf{FL}_e$-algebra $\langle \mathbf{A}; \square, \Diamond \rangle$ *chain-monadic* if $\square\mathbf{A}$ is linearly ordered. We show that a variety $\mathcal{V}$ of monadic $\mathsf{FL}_e$-algebras such that $\mathcal{V}_{\mathsf{FL}_e}$ is semilinear is generated by its chain-monadic members.

**Theorem 2.41.** *Let $\mathcal{V}$ be a variety of monadic $\mathsf{FL}_e$-algebras such that $\mathcal{V}_{\mathsf{FL}_e}$ is semilinear. Then every $\langle \mathbf{A}; \square, \Diamond \rangle \in \mathcal{V}$ is a subdirect product of chain-monadic $\mathsf{FL}_e$-algebras in $\mathcal{V}$.*

*Proof.* Consider a monadic $\mathsf{FL}_e$-algebra $\langle \mathbf{A}; \square, \Diamond \rangle$ such that $\mathbf{A}$ is semilinear. Let $S$ be the set of all prime $\square A$-convex $f$-free subuniverses $P$ of $\square\mathbf{A}$. By Theorem 2.32, each $P \in S$ corresponds to a (not necessarily prime) $A$-convex $f$-free subuniverse $P^{\square\Diamond} = \{a \in A \mid \square a \in P \text{ and } \Diamond a \in P\}$ of $\langle \mathbf{A}; \square, \Diamond \rangle$. By Lemma 2.40, for each $r \in \square A \setminus \{e\}$, there exists $P \in S$ such that $r \notin P$. Hence, $\bigcap S = \{e\}$. Now consider any $a \in A \setminus e$. Then $\square a \neq e$ or $\Diamond a \neq e$, since if $\square a = e = \Diamond a$, $e = \square a \leq a \leq \Diamond a = e$. It follows that there exists $P \in S$ such that $\square a \notin P$ or $\Diamond a \notin P$, and so $a \notin P^{\square\Diamond}$. Hence, $\bigcap\{P^{\square\Diamond} \mid P \in S\} = \{e\}$ and we obtain a subdirect embedding $\langle \mathbf{A}; \square, \Diamond \rangle \hookrightarrow \prod_{P \in S} \langle \mathbf{A}; \square, \Diamond \rangle / P^{\square\Diamond}$, defined by $a \mapsto (P \mapsto a/P^{\square\Diamond})$, such that each $\square\mathbf{A}/P^{\square\Diamond} = \square\mathbf{A}/P$ is linearly ordered. $\qquad\square$

We have now established that to study a variety of monadic $\mathsf{FL}_e$-algebras whose $\mathsf{FL}_e$-reducts are semilinear, it suffices to study those members $\langle \mathbf{A}; \square, \lozenge \rangle$ such that $\square\mathbf{A}$ is linearly ordered. Let us now consider such algebras that additionally satisfy the identity $\square(x \vee \square y) \approx \square x \vee \square y$. For $a \in A$, we write $|a|$ to denote $a \wedge (a \to e) \wedge e$.

**Lemma 2.42.** *Let $\langle \mathbf{A}; \square, \lozenge \rangle$ be a chain-monadic $\mathsf{FL}_e$-algebra satisfying the identity $\square(x \vee \square y) \approx \square x \vee \square y$. Then for each $a \in A \setminus \{e\}$, there exists a prime $A$-convex $f$-free subuniverse $P$ of $\mathbf{A}$ such that $a \notin P$ and $P \cap \square A = \{e\}$.*

*Proof.* We apply Zorn's Lemma to the set $\mathcal{D} = \{H \in \mathcal{C}(\mathbf{A}) \mid \text{for all } c \in \square A \setminus \{e\}, |a| \vee |c| \notin H\}$. It it easy to check that for any chain $\{H_i\}_{i \in I} \subseteq \mathcal{D}$, also $\bigcup_{i \in I} H_i \in \mathcal{D}$. We show that $\{e\} \in \mathcal{D}$. Consider any $c \in \square A \setminus \{e\}$. For a contradiction, suppose that $|a| \vee |c| = e$. Then, using the identity $\square(x \vee \square y) \approx \square x \vee \square y$ and that $c \in \square A$, we obtain

$$e = \square e = \square(|a| \vee |c|) = \square(|a| \vee \square|c|) = \square|a| \vee \square|c| = \square|a| \vee |c|.$$

Moreover, we have $e = \lozenge e = \lozenge(|a| \vee |c|) = \lozenge|a| \vee \lozenge|c| = \lozenge|a| \vee |c|$. Since $\square\mathbf{A}$ is linearly ordered, it follows that $|c| = e$ or $\square|a| = \lozenge|a| = e$. The former case gives a contradiction with $c \neq e$ and the latter implies $a = e$, again a contradiction. So $\mathcal{D}$ is non-empty, and Zorn's Lemma gives a maximal element $P$ of $\mathcal{D}$.

We show that $P$ is prime. For a contradiction, suppose not. Then there exist $b, c \in A$ such that $(b \to c) \wedge e \notin P$ and $(c \to b) \wedge e \notin P$. By maximality of $P$, neither $P((b \to c) \wedge e)$ nor $P((c \to b) \wedge e)$ are members of $\mathcal{D}$, so there exist $p, q \in \square A \setminus \{e\}$ such that $|a| \vee |p| \in P((b \to c) \wedge e)$ and $|a| \vee |q| \in P((c \to b) \wedge e)$. As $\square\mathbf{A}$ is linearly ordered, we can assume without loss of generality that $|p| \leq |q|$, and so $|a| \vee |p| \leq |a| \vee |q| \leq e$. Convexity then implies that $|a| \vee |q| \in P((b \to c) \wedge e)$. But then $|a| \vee |q| \in P((b \to c) \wedge e) \cap P((c \to b) \wedge e) = P$ by Lemma 2.39, a contradiction with $P \in \mathcal{D}$. $\qquad\square$

We now have all the ingredients necessary to prove the promised functional representation for all members of varieties of monadic $\mathsf{FL}_e$-algebras satisfying the properties (i)–(iii) mentioned at the start of this section.

**Theorem 2.43.** *Let $\mathcal{V}$ be a variety of monadic $\mathsf{FL}_e$-algebras satisfying $\square(x \vee \square y) \approx \square x \vee \square y$ such that $\mathcal{V}_{\mathsf{FL}_e}$ is semilinear and $\mathsf{lin}(\mathcal{V}_{\mathsf{FL}_e})$ has the amalgamation property. Then for any chain-monadic $\langle \mathbf{A}; \square, \lozenge \rangle \in \mathcal{V}$, there exist a linearly ordered $\mathbf{B} \in \mathcal{V}_{\mathsf{FL}_e}$, a non-empty set $I$, and an embedding $\rho \colon \mathbf{A} \to \mathbf{B}^I$ such that $\rho(\square A)$ consists only of constant functions and $\rho(\square A)$ is a $\mathcal{V}$-relatively complete subuniverse of $\rho(\mathbf{A})$. In particular, for all $a \in A$, $i \in I$,*

$$\rho(\square a)(i) = \bigvee \{\rho(c)(i) \mid c \in \square A, \rho(c) \leq \rho(a)\}$$
$$\rho(\lozenge a)(i) = \bigwedge \{\rho(c)(i) \mid c \in \square A, \rho(a) \leq \rho(c)\}.$$

*Proof.* Let $I$ be the family of all prime $A$-convex $f$-free subuniverses $P$ of $\mathbf{A}$ such that $P \cap \square A = \{e\}$. For each $a \in A \setminus \{e\}$, by Lemma 2.42 there exists $P \in I$ such that $a \notin P$ and so $\bigcap_{P \in I} P = \{e\}$. Hence, we obtain a subdirect representation of $\langle \mathbf{A}; \square, \lozenge \rangle$ given by $\sigma \colon A \to \prod_{P \in I} A/P$. Since $P \cap \square A = \{e\}$ for each $P \in I$, the maps $\{(\pi_P \circ \sigma)|_{\square A} \colon \square A \to A/P\}_{P \in I}$ are embeddings.

Using amalgamation, we obtain an amalgam $\mathbf{B} \in \mathcal{V}_{\mathsf{FL}_e}$ with embeddings $\gamma_P \colon A/P \to B$ such that $\gamma_P \circ \pi_P \circ \sigma|_{\Box A} = \gamma_Q \circ \pi_Q \circ \sigma|_{\Box A}$ for all $P, Q \in I$. Defining

$$\gamma := \prod_{P \in I} \gamma_P \colon \prod_{P \in I} A/P \to B^I$$

yields an embedding $\rho := \gamma \circ \sigma \colon \mathbf{A} \to \mathbf{B}^I$. Moreover, for any $c \in \Box A$ and $P, Q \in I$, we have

$$\rho(c)(P) = \gamma_P(\sigma(c)(P)) = \gamma_P(\pi_P(\sigma(c))) = \gamma_Q(\pi_Q(\sigma(c))) = \gamma_Q(\sigma(c)(Q)) = \rho(c)(Q).$$

That is, $\rho(c)$ is a constant function. To show that $\rho(\Box A)$ is a $\mathcal{V}$-relatively completely subuniverse of $\rho(\mathbf{A})$, it suffices to show that for all $a \in A$,

$$\rho(\Box a) = \bigvee \{\rho(c) \mid c \in \Box A, \rho(c) \leq \rho(a)\}$$
$$\rho(\Diamond a) = \bigwedge \{\rho(c) \mid c \in \Box A, \rho(a) \leq \rho(c)\}.$$

Indeed, this would imply that $\rho$ is a map between $\langle \mathbf{A}; \Box, \Diamond \rangle$ and the monadic $\mathsf{FL}_e$-algebra corresponding to $\langle \rho(\mathbf{A}), \rho(\Box \mathbf{A}) \rangle$ preserving $\Box$ and $\Diamond$, and hence prove that $\rho$ is an isomorphism between $\langle \mathbf{A}, \Box \mathbf{A} \rangle$ and $\langle \rho(\mathbf{A}), \rho(\Box \mathbf{A}) \rangle$.

Firstly note that for $c \in \Box A$, $\rho(c) \leq \rho(a)$ if and only if $c \leq a$. Indeed, $c \leq a$ implies $\rho(c) \leq \rho(a)$ since $\rho$ is an $\mathsf{FL}_e$-homomorphism. Conversely, $\rho(c)(P) = \gamma_P(\pi_P(\sigma(c)))$ and $\rho(a)(P) = \gamma_P(\pi_P(\sigma(a)))$, so since $\gamma_P$ is an embedding, $\rho(c) \leq \rho(a)$ implies that

$$c/P = \pi_P(\sigma(c)) \leq \pi_P(\sigma(a)) = a/P$$

for all $P \in I$. Since $\bigcap_{P \in I} P = \{e\}$, this implies $c \leq a$, as required.

Now note that, as $\Box a \leq a$, we have

$$\rho(\Box a)(P) \in \{\rho(c)(P) \mid c \in \Box A, \rho(c) \leq \rho(a)\}.$$

Moreover, consider any $c \in \Box A$ such that $\rho(c) \leq \rho(a)$. As shown above, this is equivalent to $c \leq a$. Then $c = \Box c \leq \Box a$, and so $\rho(c) \leq \rho(\Box a)$. This yields

$$\rho(\Box a)(P) = \bigvee \{\rho(c)(P) \mid c \in \Box A, \rho(c) \leq \rho(a)\}.$$

The proof of $\rho(\Diamond a)(P) = \bigwedge \{\rho(c)(P) \mid c \in \Box A, \rho(a) \leq \rho(c)\}$ follows analogously. $\square$

**Corollary 2.44.** *Let $\mathcal{V}$ be a variety of monadic $\mathsf{FL}_e$-algebras satisfying $\Box(x \vee \Box y) \approx \Box x \vee \Box y$ such that $\mathcal{V}_{\mathsf{FL}_e}$ is semilinear and $\mathsf{lin}(\mathcal{V}_{\mathsf{FL}_e})$ has the amalgamation property. Then $\mathcal{V}$ is generated by its relatively functional chain-monadic members.*

As mentioned at the start of this section, there are a number of varieties to which this result can be applied. We summarize them in the following corollary.

**Corollary 2.45.** *The following varieties of monadic $\mathsf{FL}_e$-algebras are generated by their relatively functional chain-monadic members:*

- *the variety of monadic MV-algebras;*

- *the variety of crisp monadic Gödel algebras;*

- *the variety of monadic abelian $\ell$-groups.*

It is worth pointing out again that for monadic MV-algebras and crisp monadic Gödel algebras, stronger results are known already. Indeed, Rutledge showed in [138] that the variety of monadic MV-algebras is generated by the single functional algebra $\langle \mathbf{Ł}^{\mathbb{N}}; \Box, \Diamond \rangle$, and the variety of crisp monadic Gödel algebras is generated by the single functional algebra $\langle \mathbf{G}^{\mathbb{N}}; \Box, \Diamond \rangle$ [42]. In Section 4.3, we use the methods from this section to show that the variety of monadic abelian $\ell$-groups is generated by the functional algebra $\langle \mathbf{A}; \Box, \Diamond \rangle$ whose universe consists of all bounded functions $f \colon \mathbb{N} \to \mathbb{R}$.

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# CHAPTER 3

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# Monadic Gödel Logics

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This chapter focuses on monadic Gödel logics. In particular, we consider the modal Gödel logics $\mathsf{S5}(\mathbf{A})$ and $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ for Gödel sets $A$ (see Definition 1.20). These modal Gödel logics are then matched to one-variable fragments $\mathsf{IKL}_1(\mathbf{K})$ of first-order intermediate logics for countable linear frames $\mathbf{K}$ (see Definition 1.10). In Section 3.1, we define for each countable linear frame $\mathbf{K}$ a Gödel set $A$ such that for all $\alpha \in \mathrm{Fm}_1(\mathcal{L}_{\mathsf{IL}})$,

$$\models_{\mathsf{IKL}_1(\mathbf{K})} \alpha \iff \models_{\mathsf{S5}(\mathbf{A})} \alpha^*.$$

In particular, we show that for all $\alpha \in \mathrm{Fm}_1(\mathcal{L}_{\mathsf{IL}})$,

$$\models_{\mathsf{IKL}_1} \alpha \iff \models_{\mathsf{S5}(\mathbf{G})} \alpha^*.$$

In Section 3.2, we prove the converse: for each Gödel set $A$, there exists a countable linear frame $\mathbf{K}$ such that for all $\alpha \in \mathrm{Fm}_1(\mathcal{L}_{\mathsf{IL}})$,

$$\models_{\mathsf{IKL}_1(\mathbf{K})} \alpha \iff \models_{\mathsf{S5}(\mathbf{A})} \alpha^*.$$

Note that in [13], Beckmann and Preining investigated similar questions: they matched $\mathsf{CDIKL}(\mathbf{K})$-validity for countable linear frames $\mathbf{K}$ to $\mathbf{A}$-validity of certain Gödel sets $A$. The results obtained in the first two sections of this chapter are both more general, as we drop the constant domain condition, and less general, as we only consider one-variable fragments. In Section 3.3, we show that the one-variable fragment of an intermediate logic defined over a linear frame can be interpreted in the one-variable fragment of the constant domain logic over the same frame. In light of the previous sections, this also gives an interpretation of $\mathsf{S5}(\mathbf{A})$ into $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ for any Gödel set $A$. Finally, we turn to decidability and complexity. In Section 3.4, we prove a finite model property for $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$. To this end, we introduce an alternative semantics, as $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ does not have the finite model property with respect to its $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-models. In particular, we deduce that the variety of crisp monadic Gödel algebras — the algebraic semantics of $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$ (see Example 2.7) — is generated by its finite members. This alternative semantics is investigated further in Section 3.5 to obtain decidability and complexity results for $\mathsf{S5}(\mathbf{A})$ and $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ for a large class of Gödel sets $A$.

The results in this chapter have appeared in [39] and [40] with co-authors Caicedo, Metcalfe, and Rodríguez.

## 3.1    From Linear Frames to Gödel Sets

In this section, we match the one-variable fragment of an intermediate logic $\mathsf{IKL}(\mathbf{K})$ defined over a single countable linear frame $\mathbf{K}$ to a corresponding logic $\mathsf{S5}(\mathbf{A})$ for some Gödel set $A$. In particular, we match the one-variable fragment of the intermediate logic defined over $\langle \mathbb{Q}, \leq \rangle$ to the standard Gödel modal logic $\mathsf{S5}(\mathbf{G})$. If we recall from Example 1.12 that $\mathsf{IKL}(\langle \mathbb{Q}, \leq \rangle)$-validity coincides with $\mathsf{IKL}$-validity, this proves the correspondence between $\mathsf{IKL}_1$ and $\mathsf{S5}(\mathbf{G})$.

Let $\mathbf{K} = \langle K, \preceq \rangle$ be any countable linear frame. A subset $U \subseteq K$ is called an *upset* of $\mathbf{K}$ if whenever $k \in U$, $l \in K$, and $k \preceq l$, also $l \in U$. For each $k \in K$, we denote the upset $\{l \in K \mid k \preceq l\}$ by $[k)$. Now let $\mathrm{Up}(\mathbf{K})$ be the set of all upsets of $\mathbf{K}$. Then $\langle \mathrm{Up}(\mathbf{K}), \subseteq \rangle$ is a complete linearly ordered set with greatest and least elements $K$ and $\emptyset$, respectively. Moreover, since $K$ is countable, there exists a complete (i.e., preserving all suprema and infima) order-embedding of $\langle \mathrm{Up}(\mathbf{K}), \subseteq \rangle$ into $\langle [0,1], \leq \rangle$ (see [86]). Hence we may identify $\mathrm{Up}(\mathbf{K})$ with a Gödel set and obtain an $\mathsf{S5}(\mathbf{Up}(\mathbf{K}))$-model based on the Gödel algebra, $\mathbf{Up}(\mathbf{K}) \coloneqq \langle \mathrm{Up}(\mathbf{K}), \cap, \cup, \rightarrow, \emptyset, K \rangle$, where

$$X \rightarrow Y \coloneqq \begin{cases} K & \text{if } X \subseteq Y; \\ Y & \text{otherwise.} \end{cases}$$

Now let $\mathfrak{M} = \langle K, \preceq, \{D_k\}_{k \in K}, \{\mathcal{I}_k\}_{k \in K} \rangle$ be any $\mathsf{IKL}_1(\mathbf{K})$-model. We define for all $a, b \in \bigcup_{k \in K} D_k$ and $i \in \mathbb{N}$,

$$W \coloneqq \bigcup_{k \in K} D_k;$$
$$U(a) \coloneqq \{k \in K \mid a \in D_k\};$$
$$Rab \coloneqq \begin{cases} K & a = b \\ U(a) \cap U(b) & a \neq b; \end{cases}$$
$$V(p_i, a) \coloneqq \{k \in K \mid a \in \mathcal{I}_k(P_i)\}.$$

Note that each $V(p_i, a)$ is an upset of $\mathbf{K}$ since $k \preceq l$ implies $\mathcal{I}_k(P_i) \subseteq \mathcal{I}_l(P_i)$. Moreover, $Raa = K$, $Rab = Rba$, and $Rab \cap Rbc \subseteq Rac$ for all $a, b, c \in W$. Hence $\mathcal{M}_{\mathfrak{M}} \coloneqq \langle W, R, V \rangle$ is an $\mathsf{S5}(\mathbf{Up}(\mathbf{K}))$-model. Moreover, if $\mathfrak{M}$ is a $\mathsf{CDIKL}_1(\mathbf{K})$-model, then $\mathcal{M}_{\mathfrak{M}}$ is universal. We now prove that the definition of $V$ extends to all formulas, i.e., for all $\varphi \in \mathrm{Fm}_{\square\Diamond}(\mathcal{L}_{\mathsf{IL}})$, $a \in W$,

$$\bar{V}(\varphi, a) = \{k \in K \mid \mathfrak{M}, k \models^a \varphi^\circ, a \in D_k\}.$$

**Lemma 3.1.** *Let* $\mathbf{K} = \langle K, \preceq \rangle$ *be a countable linear frame and* $\mathfrak{M} = \langle K, \preceq, \{D_k\}_{k \in K}, \{\mathcal{I}_k\}_{k \in K} \rangle$ *any* $\mathsf{IKL}_1(\mathbf{K})$-*model over* $\mathbf{K}$ *with* $\mathcal{M}_{\mathfrak{M}} = \langle W, R, V \rangle$. *Then for any* $\varphi \in \mathrm{Fm}_{\square\Diamond}(\mathcal{L}_{\mathsf{IL}})$, $k \in K$, *and* $a \in D_k$,

$$\mathfrak{M}, k \models^a \varphi^\circ \quad \Longleftrightarrow \quad k \in \bar{V}(\varphi, a).$$

*Proof.* We begin with the following useful observation. If $a \in D_k$ and $b \in W$, then $b \in D_k$ if and only if $k \in Rab$. Just note that if $b = a$, this is trivial, and if $b \neq a$, then $k \in U(a) \cap U(b)$ if and only if $k \in U(b)$, i.e., $b \in D_k$.

We prove the claim by induction on the length of $\varphi$. The base cases for $\bot$, $\top$, and $p_i$ are immediate from the definitions, and the cases for $\wedge$ and $\vee$ are straightforward, so we just consider the cases for $\to$, $\Box$, and $\Diamond$.

- Suppose that $\varphi = \psi_1 \to \psi_2$. Then

$$
\begin{aligned}
\mathfrak{M}, k \models^a (\psi_1 \to \psi_2)^\circ &\iff \mathfrak{M}, l \models^a \psi_1^\circ \text{ implies } \mathfrak{M}, l \models^a \psi_2^\circ \text{ for all } l \succeq k \\
&\iff l \in \bar{V}(\psi_1, a) \text{ implies } l \in \bar{V}(\psi_2, a) \text{ for all } l \succeq k \\
&\iff [k) \cap \bar{V}(\psi_1, a) \subseteq \bar{V}(\psi_2, a) \\
&\iff [k) \subseteq (\bar{V}(\psi_1, a) \to \bar{V}(\psi_2, a)) \\
&\iff k \in \bar{V}(\psi_1 \to \psi_2, a).
\end{aligned}
$$

- Suppose that $\varphi = \Box\psi$. Then

$$
\begin{aligned}
\mathfrak{M}, k \models^a (\Box\psi)^\circ &\iff \mathfrak{M}, l \models^b \psi^\circ \text{ for all } l \succeq k \text{ and } b \in D_l \\
&\iff l \in \bar{V}(\psi, b) \text{ for all } l \succeq k, \ b \in W \text{ with } l \in Rab \\
&\iff [k) \cap Rab \subseteq \bar{V}(\psi, b) \text{ for all } b \in W \\
&\iff [k) \subseteq (Rab \to \bar{V}(\psi, b)) \text{ for all } b \in W \\
&\iff k \in \bigcap \{Rab \to \bar{V}(\psi, b) \mid b \in W\} \\
&\iff k \in \bar{V}(\Box\psi, a).
\end{aligned}
$$

- Suppose that $\varphi = \Diamond\psi$. Then

$$
\begin{aligned}
\mathfrak{M}, k \models^a (\Diamond\psi)^\circ &\iff \mathfrak{M}, k \models^b \psi^\circ \text{ for some } b \in D_k \\
&\iff k \in \bar{V}(\psi, b) \text{ and } k \in Rab \text{ for some } b \in W \\
&\iff k \in \bigcup \{Rab \cap \bar{V}(\psi, b) \mid b \in W\} \\
&\iff k \in \bar{V}(\Diamond\psi, a). \qquad \Box
\end{aligned}
$$

From this lemma, it follows that $\not\models_{\mathsf{IKL}_1(\mathbf{K})} \varphi^\circ$ implies that $\not\models_{\mathsf{S5}(\mathbf{Up}(\mathbf{K}))} \varphi$ for any $\varphi \in \mathrm{Fm}_{\Box\Diamond}(\mathcal{L}_{\mathsf{IL}})$. For the converse direction, let $\mathcal{M} = \langle W, R, V \rangle$ be any $\mathsf{S5}(\mathbf{Up}(\mathbf{K}))$-model and fix $w_0 \in W$. We define for each $k \in K$ and $i \in \mathbb{N}$,

$$
\begin{aligned}
D_k &:= \{v \in W \mid k \in Rw_0 v\}; \\
\mathcal{I}_k(P_i) &:= \{v \in W \mid k \in V(p_i, v)\} \cap D_k.
\end{aligned}
$$

It is easily checked that if $k \preceq l$, then $D_k \subseteq D_l$ and $\mathcal{I}_k(P_i) \subseteq \mathcal{I}_l(P_i)$ for each $i \in \mathbb{N}$. Hence we obtain an $\mathsf{IKL}_1$-model $\mathfrak{M}_{\mathcal{M},w_0} := \langle K, \preceq, \{D_k\}_{k \in K}, \{\mathcal{I}_k\}_{k \in K} \rangle$. Moreover, if $\mathcal{M}$ is universal, then $\mathfrak{M}_{\mathcal{M},w_0}$ is a $\mathsf{CDIKL}_1(\mathbf{K})$-model.

**Lemma 3.2.** *Let $\mathbf{K} = \langle K, \preceq \rangle$ be a countable linear frame and let $\mathcal{M} = \langle W, R, V \rangle$ be an $\mathsf{S5}(\mathbf{Up}(\mathbf{K}))$-model with $w_0 \in W$ and $\mathfrak{M}_{\mathcal{M},w_0} = \langle K, \preceq, \{D_k\}_{k \in K}, \{\mathcal{I}_k\}_{k \in K} \rangle$. For any $\varphi \in \mathrm{Fm}_{\Box\Diamond}(\mathcal{L}_{\mathsf{IL}})$, $k \in K$, and $v \in D_k$,*

$$
\mathfrak{M}_{\mathcal{M},w_0}, k \models^v \varphi^\circ \iff k \in \bar{V}(\varphi, v).
$$

*Proof.* Note first that if $v \in D_k$, then $k \in Rw_0 v$ and for any $l \succeq k$ and $u \in W$,

$$l \in Rw_0 u \implies l \in Ruw_0 \cap Rw_0 v \subseteq Ruv$$
$$l \in Ruv \implies l \in Rw_0 v \cap Rvu \subseteq Rw_0 u;$$

that is, $Rw_0 u \cap [k) = Rvu \cap [k)$.

We prove the claim by induction on the length of $\varphi$. The base cases for $\bot$, $\top$, and $p_i$ are immediate from the definitions and the cases for $\wedge$ and $\vee$ are straightforward, so we just consider the cases for $\rightarrow$, $\square$, and $\Diamond$.

- Suppose that $\varphi = \psi_1 \rightarrow \psi_2$. Then $\mathfrak{M}_{\mathcal{M},w_0}, k \models^v (\psi_1 \rightarrow \psi_2)^\circ$

$$\iff \mathfrak{M}_{\mathcal{M},w_0}, l \models^v \psi_1^\circ \text{ implies } \mathfrak{M}_{\mathcal{M},w_0}, l \models^v \psi_2^\circ \text{ for all } l \succeq k$$
$$\iff l \in \bar{V}(\psi_1, v) \text{ implies } l \in \bar{V}(\psi_2, v) \text{ for all } l \succeq k$$
$$\iff [k) \cap \bar{V}(\psi_1, v) \subseteq \bar{V}(\psi_2, v)$$
$$\iff [k) \subseteq (\bar{V}(\psi_1, v) \rightarrow \bar{V}(\psi_2, v))$$
$$\iff k \in \bar{V}(\psi_1 \rightarrow \psi_2, v).$$

- Suppose that $\varphi = \square\psi$. Then $\mathfrak{M}_{\mathcal{M},w_0}, k \models^v (\square\psi)^\circ$

$$\iff \mathfrak{M}_{\mathcal{M},w_0}, l \models^u \psi^\circ \text{ for all } l \succeq k \text{ and } u \in D_l$$
$$\iff l \in \bar{V}(\psi, u) \text{ for all } l \succeq k, \ u \in W \text{ with } l \in Rw_0 u$$
$$\iff l \in \bar{V}(\psi, u) \text{ for all } l \succeq k, \ u \in W \text{ with } l \in Rvu$$
$$\iff [k) \cap Rvu \subseteq \bar{V}(\psi, u) \text{ for all } u \in W$$
$$\iff [k) \subseteq (Rvu \rightarrow \bar{V}(\psi, u)) \text{ for all } u \in W$$
$$\iff k \in (Rvu \rightarrow \bar{V}(\psi, u)) \text{ for all } u \in W$$
$$\iff k \in \bigcap\{Rvu \rightarrow \bar{V}(\psi, u) \mid u \in W\}$$
$$\iff k \in \bar{V}(\square\psi, v).$$

- Suppose that $\varphi = \Diamond\psi$. Then $\mathfrak{M}_{\mathcal{M},w_0}, k \models^v (\Diamond\psi)^\circ$

$$\iff \mathfrak{M}_{\mathcal{M},w_0}, k \models^u \psi^\circ \text{ for some } u \in D_k$$
$$\iff k \in \bar{V}(\psi, u) \text{ for some } u \in W \text{ such that } k \in Rw_0 u$$
$$\iff k \in \bar{V}(\psi, u) \text{ for some } u \in W \text{ such that } k \in Rvu$$
$$\iff k \in \bar{V}(\psi, u) \cap Rvu \text{ for some } u \in W$$
$$\iff k \in \bigcup\{\bar{V}(\psi, u) \cap Rvu \mid u \in W\}$$
$$\iff k \in \bar{V}(\Diamond\psi, v). \qquad \square$$

We now put these two lemmas together to obtain the desired correspondence, recalling that the result for the constant domain case is implicit in [13].

**Theorem 3.3.** *For any countable linear frame* $\mathbf{K} = \langle K, \preceq \rangle$ *and* $\alpha \in \mathrm{Fm}_1(\mathcal{L}_{\mathsf{IL}})$,

$$\models_{\mathsf{IKL}_1(\mathbf{K})} \alpha \quad \Longleftrightarrow \quad \models_{\mathsf{S5}(\mathbf{Up}(\mathbf{K}))} \alpha^*$$

$$\models_{\mathsf{CDIKL}_1(\mathbf{K})} \alpha \quad \Longleftrightarrow \quad \models_{\mathsf{S5}(\mathbf{Up}(\mathbf{K}))^{\mathsf{C}}} \alpha^*.$$

*Proof.* Suppose first that $\not\models_{\mathsf{IKL}_1(\mathbf{K})} \alpha$. Then there exists an $\mathsf{IKL}_1(\mathbf{K})$-model $\mathfrak{M} = \langle K, \preceq, \{D_k\}_{k \in K}, \{\mathcal{I}_k\}_{k \in K} \rangle$, $k \in K$, and $a \in D_k$ such that $\mathfrak{M}, k \not\models^a \alpha$. An application of Lemma 3.1 with $\mathcal{M}_{\mathfrak{M}} = \langle W, R, V \rangle$ yields $k \notin \bar{V}(\alpha^*, a)$. Hence $\bar{V}(\alpha^*, a) \neq K$ and $\not\models_{\mathsf{S5}(\mathbf{Up}(\mathbf{K}))} \alpha^*$.

Suppose now that $\not\models_{\mathsf{S5}(\mathbf{Up}(\mathbf{K}))} \alpha^*$. Then there exists an $\mathsf{S5}(\mathbf{Up}(\mathbf{K}))$-model $\mathcal{M} = \langle W, R, V \rangle$ and $w_0 \in W$ such that $\bar{V}(\alpha^*, w_0) \neq K$. Let $k \in K \setminus \bar{V}(\alpha^*, w_0)$. Then Lemma 3.2 yields $\mathfrak{M}_{\mathcal{M}, w_0}, k \not\models^{w_0} \alpha$, so $\not\models_{\mathsf{IKL}_1(\mathbf{K})} \alpha$.

The second equivalence follows from the fact that if $\mathfrak{M}$ is a $\mathsf{CDIKL}_1(\mathbf{K})$-model, then $\mathcal{M}_{\mathfrak{M}}$ is universal, and, conversely, if $\mathcal{M}$ is universal, then $\mathfrak{M}_{\mathcal{M}, w_0}$ is a $\mathsf{CDIKL}_1(\mathbf{K})$-model. $\square$

By choosing suitable linear frames, we obtain the corresponding Gödel modal logics defined over certain notable Gödel sets.

**Corollary 3.4.** *For any formula* $\alpha \in \mathrm{Fm}_1(\mathcal{L}_{\mathsf{IL}})$ *and* $n \in \mathbb{N}^+$,

$$\models_{\mathsf{IKL}_1(\langle \mathbb{N}, \leq \rangle)} \alpha \iff \models_{\mathsf{S5}(\mathbf{G}_\downarrow)} \alpha^* \qquad \models_{\mathsf{CDIKL}_1(\langle \mathbb{N}, \leq \rangle)} \alpha \iff \models_{\mathsf{S5}(\mathbf{G}_\downarrow)^{\mathsf{C}}} \alpha^*$$

$$\models_{\mathsf{IKL}_1(\langle \mathbb{N}, \geq \rangle)} \alpha \iff \models_{\mathsf{S5}(\mathbf{G}_\uparrow)} \alpha^* \qquad \models_{\mathsf{CDIKL}_1(\langle \mathbb{N}, \geq \rangle)} \alpha \iff \models_{\mathsf{S5}(\mathbf{G}_\uparrow)^{\mathsf{C}}} \alpha^*$$

$$\models_{\mathsf{IKL}_1(\langle \{1,\dots,n\}, \leq \rangle)} \alpha \iff \models_{\mathsf{S5}(\mathbf{G}_n)} \alpha^* \quad \models_{\mathsf{CDIKL}_1(\langle \{1,\dots,n\}, \leq \rangle)} \alpha \iff \models_{\mathsf{S5}(\mathbf{G}_n)^{\mathsf{C}}} \alpha^*.$$

For the logic $\mathsf{S5}(\mathbf{G})$, however, the obvious choice of a countable linear frame $\mathbf{Q} = \langle \mathbb{Q}, \leq \rangle$ produces a Gödel set $\mathrm{Up}(\mathbf{Q})$ that is not order-isomorphic to $[0, 1]$. Indeed, as explained in [13], the Gödel set $\mathrm{Up}(\mathbf{Q})$ is isomorphic to the Cantor set $\mathcal{C}_{[0,1]}$. In the next section, we will show that there exists a matching countable linear frame for every Gödel set $A$, but first we give here a construction that directly relates $\mathsf{S5}(\mathbf{G})$-validity to $\mathsf{IKL}_1(\mathbf{Q})$-validity.

For technical reasons, we begin by showing that we can restrict our attention to a particular class of $\mathsf{S5}(\mathbf{G})$-models. We say that an $\mathsf{S5}(\mathbf{G})$-model $\mathcal{M} = \langle W, R, V \rangle$ is *irrational* if $\bar{V}(\varphi, w)$ is irrational, 0, or 1 for all $\varphi \in \mathrm{Fm}_{\square\Diamond}(\mathcal{L}_{\mathsf{IL}})$ and $w \in W$.

**Lemma 3.5.** *For any countable* $\mathsf{S5}(\mathbf{G})$*-model* $\mathcal{M} = \langle W, R, V \rangle$, *there exists an irrational* $\mathsf{S5}(\mathbf{G})$*-model* $\mathcal{M}' = \langle W, R', V' \rangle$ *such that for all* $\varphi, \psi \in \mathrm{Fm}_{\square\Diamond}(\mathcal{L}_{\mathsf{IL}})$ *and* $w, v \in W$,

$$\bar{V}(\varphi, w) < \bar{V}(\psi, v) \iff \bar{V}'(\varphi, w) < \bar{V}'(\psi, v).$$

*Proof.* By [86, Lemma 3.7], there exists a complete order-embedding $f$ from the countable set

$$S = \{\bar{V}(\varphi, w) \mid w \in W, \varphi \in \mathrm{Fm}_{\square\Diamond}(\mathcal{L}_{\mathsf{IL}})\} \cup R[W \times W]$$

into $\mathbb{Q} \cap [0, 1]$. For each $q \in \mathbb{Q} \cap [0, 1]$, define

$$g(q) := \begin{cases} \frac{\pi}{3} q & q \leq \frac{1}{2}; \\ \frac{\pi}{6} + (2 - \frac{\pi}{3})(q - \frac{1}{2}) & q > \frac{1}{2}. \end{cases}$$

Then $g$ is a complete order-embedding from $\mathbb{Q} \cap [0,1]$ into $([0,1] \setminus \mathbb{Q}) \cup \{0,1\}$ with $g(0) = 0$ and $g(1) = 1$. So $h = g \circ f$ is a complete order-embedding from $S$ into $([0,1] \setminus \mathbb{Q}) \cup \{0,1\}$ with $h(0) = 0$ and $h(1) = 1$. Now let $\mathcal{M}' = \langle W, R', V' \rangle$ where $R'wv := h(Rwv)$ and $V'(p_i, w) := h(V(p_i, w))$ for $w, v \in W$ and $i \in \mathbb{N}$. A straightforward induction on formula length yields $\bar{V}'(\varphi, w) = h(\bar{V}(\varphi, w))$ for all $\varphi \in \text{Fm}_{\Box\Diamond}(\mathcal{L}_{\mathsf{IL}})$ and $w \in W$ and the claim follows immediately. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now let $(0,1)_{\mathbb{Q}} := (0,1) \cap \mathbb{Q}$ and $(0,1)_{\mathbf{Q}} := \langle (0,1)_{\mathbb{Q}}, \geq \rangle$. Given any irrational $\mathsf{S5}(\mathbf{G})$-model $\mathcal{M} = \langle W, R, V \rangle$ and $w_0 \in W$, we define for $q \in (0,1)_{\mathbb{Q}}$ and $i \in \mathbb{N}$,

$$D_q := \{v \in W \mid Rw_0v \geq q\}$$
$$\mathcal{I}_q(P_i) := \{v \in W \mid V(p_i, v) \geq q\} \cap D_q.$$

It is easily checked that if $q \geq r$, then $D_q \subseteq D_r$ and $\mathcal{I}_q(P_i) \subseteq \mathcal{I}_r(P_i)$ for each $i \in \mathbb{N}$ and $q, r \in (0,1)_{\mathbb{Q}}$, so we obtain an $\mathsf{IKL_1}((0,1)_{\mathbf{Q}})$-model

$$\mathfrak{M}^i_{\mathcal{M}, w_0} := \langle (0,1)_{\mathbb{Q}}, \geq, \{D_q\}_{q \in (0,1)_{\mathbb{Q}}}, \{\mathcal{I}_q\}_{q \in (0,1)_{\mathbb{Q}}} \rangle.$$

Moreover, if $\mathcal{M}$ is universal, then $\mathfrak{M}^i_{\mathcal{M}, w_0}$ is a $\mathsf{CDIKL_1}((0,1)_{\mathbf{Q}})$-model.

**Lemma 3.6.** *Let* $\mathcal{M} = \langle W, R, V \rangle$ *be an irrational* $\mathsf{S5}(\mathbf{G})$*-model with* $w_0 \in W$ *and* $\mathfrak{M}^i_{\mathcal{M}, w_0} = \langle (0,1)_{\mathbb{Q}}, \geq, \{D_q\}_{q \in (0,1)_{\mathbb{Q}}}, \{\mathcal{I}_q\}_{q \in (0,1)_{\mathbb{Q}}} \rangle$. *For any* $\varphi \in \text{Fm}_{\Box\Diamond}(\mathcal{L}_{\mathsf{IL}})$, $q \in (0,1)_{\mathbb{Q}}$, *and* $w \in D_q$,

$$\mathfrak{M}^i_{\mathcal{M}, w_0}, q \models^w \varphi^\circ \quad \Longleftrightarrow \quad \bar{V}(\varphi, w) \geq q.$$

*Proof.* We prove the claim by induction on the length of $\varphi$. The base cases for $\bot$, $\top$, and $p_i$ are immediate from the definitions and the cases for $\wedge$, $\vee$, and $\rightarrow$ are straightforward, so we just consider the cases for $\Box$ and $\Diamond$.

- For $\varphi = \Box\psi$, observe first that

$$\mathfrak{M}^i_{\mathcal{M}, w_0}, q \models^w (\forall x)\psi^\circ \Longleftrightarrow \mathfrak{M}^i_{\mathcal{M}, w_0}, r \models^v \psi^\circ \text{ for all } r \leq q \text{ and } v \in D_r$$
$$\Longleftrightarrow \bar{V}(\psi, v) \geq r \text{ for all } r \leq q \text{ and } v \in D_r$$

  and

$$\bar{V}(\Box\psi, w) \geq q \Longleftrightarrow \bigwedge\{Rwv \rightarrow \bar{V}(\psi, v) \mid v \in W\} \geq q$$
$$\Longleftrightarrow Rwv \rightarrow \bar{V}(\psi, v) \geq q \text{ for all } v \in W$$
$$\Longleftrightarrow \bar{V}(\psi, v) \geq q \wedge Rwv \text{ for all } v \in W.$$

  For the left-to-right direction suppose that $\bar{V}(\psi, v) \geq r$ for all $r \leq q$ and $v \in D_r$. By assumption, $w \in D_q$, so $Rw_0w \geq q$. Let $v \in W$. If $q \leq Rwv$, then, by symmetry and transitivity, $Rw_0v \geq q$, i.e., $v \in D_q$, and hence $\bar{V}(\psi, v) \geq q = q \wedge Rwv$ as required. Suppose now that $q > Rwv$. Then $Rw_0w \geq q > Rwv$ and, by transitivity, $Rwv = Rw_0w \wedge Rwv \leq Rw_0v$. But also, if $Rwv < Rw_0v$, then, by symmetry and transitivity, $Rwv < Rw_0v \wedge Rw_0w = Rww_0 \wedge Rw_0v \leq Rwv$, a contradiction. So $Rw_0v = Rwv$. It follows that for any $r \in (0,1)_{\mathbb{Q}}$ satisfying $r \leq Rw_0v$, we have $v \in D_r$ and hence $\bar{V}(\psi, v) \geq r$. Finally, since $(0,1)_{\mathbb{Q}}$ is dense in $(0,1) \setminus \mathbb{Q}$, we have $\sup\{r \in (0,1)_{\mathbb{Q}} \mid Rw_0v \geq r\} = Rw_0v$, so $\bar{V}(\psi, v) \geq Rw_0v = q \wedge Rwv$.

  For the right-to-left direction, suppose that $\bar{V}(\psi, v) \geq q \wedge Rwv$ for every $v \in W$. Let $r \leq q$ and $v \in D_r$. Then $Rw_0v \geq r$. Since $w \in D_q$, also $Rw_0w \geq q \geq r$, and by symmetry and transitivity, $Rwv \geq r$. So $\bar{V}(\psi, v) \geq q \wedge Rwv \geq r$.

- For $\varphi = \Diamond\psi$, observe first that since $\mathfrak{M}$ is irrational and $q \in (0,1)_{\mathbb{Q}}$, $\bar{V}(\psi, w) \geq q$ if and only if $\bar{V}(\psi, w) > q$. Now observe that

$$\mathfrak{M}^i_{\mathcal{M}, w_0}, q \models^w (\exists x)\psi^\circ \iff \mathfrak{M}^i_{\mathcal{M}, w_0}, q \models^v \psi^\circ \text{ for some } v \in D_q$$
$$\iff \bar{V}(\psi, v) \geq q \text{ for some } v \in D_q$$

and

$$\bar{V}(\Diamond\psi, w) \geq q \iff \bigvee\{Rwv \wedge \bar{V}(\psi, v) \mid v \in W\} \geq q$$
$$\iff \bigvee\{Rwv \wedge \bar{V}(\psi, v) \mid v \in W\} > q$$
$$\iff Rwv \wedge \bar{V}(\psi, v) \geq q \text{ for some } v \in W.$$

For the left-to-right direction, suppose that $\bar{V}(\psi, v) \geq q$ for some $v \in D_q$. Since $w, v \in D_q$, by transitivity, $Rwv \geq q$ and hence $Rwv \wedge \bar{V}(\psi, v) \geq q$. For the right-to-left direction, suppose that there exists $v \in W$ such that $Rwv \wedge \bar{V}(\psi, v) \geq q$, i.e., $Rwv \geq q$ and $\bar{V}(\psi, v) \geq q$. Since $w \in D_q$, also $Rw_0v \geq q$, so $v \in D_q$ and $\bar{V}(\psi, v) \geq q$. $\qquad\square$

We can now use this last lemma to prove the desired result, noting that the constant domain case was already proved in [145].

**Theorem 3.7.** *For any $\alpha \in \mathrm{Fm}_1(\mathcal{L}_{\mathsf{IL}})$,*

$$\models_{\mathsf{S5(G)}} \alpha^* \iff \models_{\mathsf{IKL}_1} \alpha \iff \models_{\mathsf{IKL}_1(\mathbf{Q})} \alpha$$
$$\models_{\mathsf{S5(G)}^c} \alpha^* \iff \models_{\mathsf{CDIKL}_1} \alpha \iff \models_{\mathsf{CDIKL}_1(\mathbf{Q})} \alpha.$$

*Proof.* Clearly, $\models_{\mathsf{IKL}_1} \alpha$ implies $\models_{\mathsf{IKL}_1(\mathbf{Q})} \alpha$. Suppose now that $\not\models_{\mathsf{IKL}_1} \alpha$. This gives a countable linear frame $\mathbf{K} = \langle K, \preceq\rangle$ and an $\mathsf{IKL}_1(\mathbf{K})$-model $\mathfrak{M} = \langle K, \preceq, \{D_k\}_{k \in K}, \{\mathcal{I}_k\}_{k \in K}\rangle$, $k \in K$, and $a \in D_k$ such that $\mathfrak{M}, k \not\models^a \alpha$. An application of Lemma 3.1 with $\mathcal{M}_{\mathfrak{M}} = \langle W, R, V\rangle$ yields $k \notin \bar{V}(\alpha^*, a)$. Hence $\bar{V}(\alpha^*, a) \neq K$ and, since there exists a complete embedding of $\langle \mathrm{Up}(\mathbf{K}), \subseteq\rangle$ into $\langle [0,1], \leq\rangle$, we obtain $\not\models_{\mathsf{S5(G)}} \alpha^*$.

Now suppose that $\not\models_{\mathsf{S5(G)}} \alpha^*$. It follows that there exist a countable $\mathsf{S5(G)}$-model $\mathcal{M} = \langle W, R, V\rangle$ and $w \in W$ such that $\bar{V}(\alpha^*, w) < 1$. By Lemma 3.5, there exist an irrational $\mathsf{S5(G)}$-model $\mathcal{M}' = \langle W, R', V'\rangle$ and $r \in (0,1)_{\mathbb{Q}}$ such that $\bar{V}'(\alpha^*, w) < r < 1$. But then Lemma 3.6 gives an $\mathsf{IKL}_1((0,1)_{\mathbf{Q}})$-model $\mathfrak{M}^i_{\mathcal{M}', w}$ such that $\mathfrak{M}^i_{\mathcal{M}', w}, r \not\models^w \alpha$. So $\not\models_{\mathsf{IKL}_1((0,1)_{\mathbf{Q}})} \alpha$ and since $(0,1)_{\mathbf{Q}}$ is order-isomorphic to $\mathbf{Q}$, also $\not\models_{\mathsf{IKL}_1(\mathbf{Q})} \alpha$.

Finally, for the second chain of equivalences, it suffices to recall that if $\mathfrak{M}$ is a $\mathsf{CDIKL}_1(\mathbf{K})$-model, then $\mathcal{M}_{\mathfrak{M}}$ is universal, and if $\mathcal{M}$ is universal, then $\mathfrak{M}^i_{\mathcal{M}, w_0}$ is a $\mathsf{CDIKL}_1((0,1)_{\mathbf{Q}})$-model. $\qquad\square$

## 3.2 From Gödel Sets to Linear Frames

In the previous section, we proved that for every countable linear frame $\mathbf{K}$, there exists a Gödel set $A$ such that the $\mathsf{IKL}_1(\mathbf{K})$-validity of any $\alpha \in \mathrm{Fm}_1(\mathcal{L}_{\mathsf{IL}})$ corresponds to the $\mathsf{S5(A)}$-validity of $\alpha^*$ (Theorem 3.3). In this section, we prove the converse: for any Gödel set $A$, there exists a countable linear frame $\mathbf{K}$ such that the $\mathsf{S5(A)}$-validity of any $\varphi \in \mathrm{Fm}_{\Box\Diamond}(\mathcal{L}_{\mathsf{IL}})$ corresponds to the $\mathsf{IKL}_1(\mathbf{K})$-validity of $\varphi^\circ$ (Theorem 3.12).

We follow the strategy used in [13] to establish a correspondence between first-order Gödel logics and constant domain logics defined over a countable linear frame, making the necessary adjustments to accommodate many-valued relations and increasing domains. First we show that for any countable Gödel set $A$, the algebra $\mathbf{A}$ is isomorphic to $\mathbf{Up}(\mathbf{K})$ for some linear frame $\mathbf{K}$. Then, for the general case, we partition any Gödel set $A$ into a countable part and an uncountable part. Using this partition and the result for countable Gödel sets, we show that $\models_{\mathsf{S5(A)}}$ coincides with $\models_{\mathsf{S5(B)}}$ for some Gödel set B such that $\mathbf{B}$ is isomorphic to some $\mathbf{Up}(\mathbf{K})$. Theorem 3.3 then gives the desired result.

Recall from Section 1.2 the definition of a limit point and perfect set, as well as the Cantor-Bendixson Theorem. We also recall a useful lemma proved in [130].

**Lemma 3.8** ([130, Section 5.4.1])**.** *Let $C \subseteq [0,1]$ be a countable set and $X \subseteq [0,1]$ a perfect set. Then there exists an order-embedding $h$ from $C$ into $X$ preserving all existing suprema and infima, and satisfying $h(\inf C) = \inf X$ if $\inf C \in C$.*

We first consider the case of countable Gödel sets.

**Lemma 3.9.** *For any countable Gödel set $A$, there exists a countable linear frame $\mathbf{K}$ such that $\mathbf{Up}(\mathbf{K})$ and $\mathbf{A}$ are isomorphic.*

*Proof.* We call $a \in A$ *left isolated in $A$* if $a \notin \mathsf{L}(A)$, i.e., if $\sup\{b \in A \mid b < a\} < a$, and define $K := \{a \in A \mid a \text{ left isolated in } A\}$. Note that $K$ is non-empty, since otherwise $A$ would be perfect and thus uncountable. Let $\mathbf{K} := \langle K, \geq \rangle$ and consider the map $h\colon \mathrm{Up}(\mathbf{K}) \to A; U \mapsto \sup U$. Since $A$ is closed, $h$ is well-defined. First we show that $h$ is an order-embedding. Suppose that $U \subsetneq U'$ for some $U, U' \in \mathrm{Up}(\mathbf{K})$, and let $a \in U' \setminus U$. Since $a$ is left isolated in $A$, we have $h(U) = \sup U < a \leq \sup U' = h(U')$. It remains to prove that $h$ is surjective. Given $a \in A$, we consider the upset $U_a := \{b \in K \mid b \leq a\}$ of $\mathbf{K}$. Note that $h(U_a) \leq a$. Suppose for a contradiction that $h(U_a) < a$. Then $a \notin K$, since if $a \in K$, clearly $h(U_a) = a$. So $a$ is not left isolated in $A$, i.e., $\sup\{b \in A \mid b < a\} = a$, and $[h(U_a), a] \cap A$ contains infinitely many points. Moreover, for any $c \in A$ such that $h(U_a) < c < a$, the set $[c,a] \cap A$ is again infinite and contains no left isolated points. But then $[c,a] \cap A$ is perfect and hence uncountable, a contradiction.

It follows that $h$ is an order-isomorphism and since $h(\emptyset) = 0$ and $h(K) = 1$, $h$ is an isomorphism between the Gödel algebras $\mathbf{Up}(\mathbf{K})$ and $\mathbf{A}$.                                □

For an uncountable Gödel set $A$, we obtain a partition of $A$ into a non-empty (uncountable) perfect kernel $X$ and a countable set $C$, by the Cantor-Bendixson Theorem (Theorem 1.14). To deal with such uncountable Gödel sets, we prove the following lemma, noting that the case for $\mathsf{S5(A)}^\mathsf{C}$ follows already from results in [13].

**Lemma 3.10.** *Let $A$ be a Gödel set with a non-empty perfect kernel $X$, and let $B := A \cup [\inf X, 1]$. Then for all $\varphi \in \mathrm{Fm}_{\square\Diamond}(\mathcal{L}_{\mathsf{IL}})$,*

$$\models_{\mathsf{S5(A)}} \varphi \iff \models_{\mathsf{S5(B)}} \varphi \quad and \quad \models_{\mathsf{S5(A)}^\mathsf{C}} \varphi \iff \models_{\mathsf{S5(B)}^\mathsf{C}} \varphi.$$

*Proof.* The right-to-left-direction of both statements follows from the fact that $A \subseteq B$. For the other direction, suppose that $\bar{V}_B(\varphi, w) < 1$ for some $\mathsf{S5(B)}$-model $\mathcal{M}_B = \langle W, R_B, V_B \rangle$ and $w \in W$. For each subformula $\square\psi$ or $\Diamond\psi$ of $\varphi$, there exists a countable subset $W_{\square\psi}$ or $W_{\Diamond\psi}$ of $W$ such that, respectively, $\bar{V}_B(\square\psi, w) = \bigwedge\{Rwv \to \bar{V}(\psi, v) \mid v \in W_{\square\psi}\}$ or $\bar{V}_B(\Diamond\psi, w) = \bigvee\{Rwv \wedge \bar{V}(\psi, v) \mid v \in W_{\Diamond\psi}\}$. An easy induction yields

$\bar{V}'_B(\varphi, w) < 1$ when $R'_B$ and $V'_B$ are $R_B$ and $V_B$ restricted to $W' = \{w\} \cup \bigcup\{W_{\psi'} \mid$ $\psi'$ is a subformula $\Box\psi$ or $\Diamond\psi$ of $\varphi\}$. We may therefore assume that $W$ is countable and hence also that $C := \{\bar{V}_B(\psi, v) \mid \psi$ a subformula of $\varphi, v \in W\}$ is countable. So, as $B$ is uncountable, there exists $b \in B \setminus C$ such that $\bar{V}_B(\varphi, w) < b < 1$. By Lemma 3.8, there exists an order-embedding $h$ from $[\inf X, b] \cap (C \cup \{b\})$ into $X$. We define the following function $k_b \colon B \to A$ such that for every $a \in B$,

$$k_b(a) := \begin{cases} a & a < \inf X; \\ h(a) & \inf X \le a \le b; \\ 1 & \text{otherwise.} \end{cases}$$

Now let $\mathcal{M}_A := \langle W, R_A, V_A \rangle$ be the $\mathsf{S5(A)}$-model where $R_A vu := k_b(R_B vu)$, $V_A(p_i, v) :=$ $k_b(V_B(p_i, v))$ for all $u, v \in W$ and each $p_i$ that occurs in $\varphi$ and $V_A(p_j, v) := 1$ for all other propositional variables $p_j$.[1]

We claim that this valuation extends to all subformulas of $\varphi$; that is, $\bar{V}_A(\psi, v) = k_b(\bar{V}_B(\psi, v))$ for every subformula $\psi$ of $\varphi$ and $v \in W$. It follows from this claim that $\bar{V}_A(\varphi, w) < 1$, since

$$\begin{aligned} either \quad & \bar{V}_B(\varphi, w) < \inf X \text{ and } \bar{V}_A(\varphi, w) = \bar{V}_B(\varphi, w) < b < 1 \\ or \quad & \inf X \le \bar{V}_B(\varphi, w) < b \text{ and } \bar{V}_A(\varphi, w) = h(\bar{V}_B(\varphi, w)) \\ & < h(b) \le 1. \end{aligned}$$

So $\not\models_{\mathsf{S5(A)}} \varphi$. Moreover, if $\mathcal{M}_B$ is universal, then so is $\mathcal{M}_A$, so we also obtain that $\not\models_{\mathsf{S5(B)^c}} \varphi$ implies $\not\models_{\mathsf{S5(A)^c}} \varphi$.

We prove the claim by induction on the length of a subformula $\psi$ of $\varphi$. The base cases follow by definition and the cases for the propositional connectives are straightforward, using the fact that $k_b(c \star d) = k_b(c) \star k_b(d)$ for all $c, d \in B$ and $\star \in \{\wedge, \vee, \to\}$. For a subformula $\Box\psi$ of $\varphi$, we have

$$\begin{aligned} \bar{V}_A(\Box\psi, v) &= \bigwedge\{R_A vu \to \bar{V}_A(\psi, u) \mid u \in W\} \\ &= \bigwedge\{k_b(R_B vu) \to k_b(\bar{V}_B(\psi, u)) \mid u \in W\} \\ &= \bigwedge\{k_b(R_B vu \to \bar{V}_B(\psi, u)) \mid u \in W\} \\ &= k_b(\bigwedge\{R_B vu \to \bar{V}_B(\psi, u) \mid u \in W\}) \qquad (3.1) \\ &= k_b(\bar{V}_B(\Box\psi, v)). \end{aligned}$$

To prove (3.1), there are three cases to consider:

(i) $\bar{V}_B(\Box\psi, v) < \inf X$. Then $k_b(\bar{V}_B(\Box\psi, v)) = \bar{V}_B(\Box\psi, v)$. Moreover, $U := \{u \in W \mid R_B vu_0 \to \bar{V}_B(\psi, u_0) < \inf X\} \ne \emptyset$ and, by definition, $k_b(R_B vu \to \bar{V}_B(\psi, u)) = R_B vu \to \bar{V}_B(\psi, u)$ for all $u \in U$.

(ii) $\inf X \le \bar{V}_B(\Box\psi, v) \le b$. By the choice of $b$, we have $\bar{V}_B(\Box\psi, v) < b$. So $\inf X \le R_B vu \to \bar{V}_B(\psi, u)$ for all $u \in W$ and $R_B vt \to \bar{V}_B(\psi, t) < b$ for some $t \in W$. It follows that $\bigwedge\{k_b(R_B vu \to \bar{V}_B(\psi, u)) \mid u \in W\} = \bigwedge\{h(R_B vu \to \bar{V}_B(\psi, u)) \mid u \in W, R_B vu \to \bar{V}_B(\psi, u) < b\}$ and the fact that $h$ preserves infima concludes the case.

---

[1] Note that this function differs slightly from the one used for the constant domain case in [12].

(iii) $b < \bar{V}_B(\Box\psi, v)$. Then $b \le R_B vu \to \bar{V}_B(\psi, u)$ for all $u \in W$ and hence $k_b(R_B vu \to \bar{V}_B(\psi, u)) = 1 = k_b(\bar{V}_B(\Box\psi, v))$ for all $u \in W$.

Next, for a subformula $\Diamond\psi$, we have

$$
\begin{aligned}
\bar{V}_A(\Diamond\psi, v) &= \bigvee\{R_A vu \wedge \bar{V}_A(\psi, u) \mid u \in W\} \\
&= \bigvee\{k_b(R_B vu) \wedge k_b(\bar{V}_B(\psi, u)) \mid u \in W\} \\
&= \bigvee\{k_b(R_B vu \wedge \bar{V}_B(\psi, u)) \mid u \in W\} \\
&= k_b(\bigvee\{R_B vu \wedge \bar{V}_B(\psi, u) \mid u \in W\}) \\
&= k_b(\bar{V}_B(\Diamond\psi, v)).
\end{aligned}
\tag{3.2}
$$

To prove (3.2), there are again three cases to consider:

(i) $\bar{V}_B(\Diamond\psi, v) < \inf X$. Then $k_b(\bar{V}_B(\Diamond\psi, v)) = \bar{V}_B(\Diamond\psi, v)$ and, since $R_B vu \wedge \bar{V}_B(\psi, u) < \inf X$ for all $u \in W$, also $k_b(R_B vu \wedge \bar{V}_B(\psi, u)) = R_B vu \wedge \bar{V}_B(\psi, u)$ for all $u \in W$, yielding (3.2).

(ii) $\inf X \le \bar{V}_B(\Diamond\psi, v) \le b$. If $\inf X \le R_B vt \wedge \bar{V}_B(\psi, t)$ for some $t \in W$, then (3.2) follows since $h$ preserves existing suprema. Otherwise $R_B vu \wedge \bar{V}_B(\psi, u) < \inf X$ for all $u \in W$, and so $\bar{V}_B(\Diamond\psi, v) = \inf X$. But then $k_b(R_B vu \wedge \bar{V}_B(\psi, u)) = R_B vu \wedge \bar{V}_B(\psi, u)$ for all $u \in W$, and their join is $\inf X$. The equality (3.2) then follows from the fact that $h(\inf X) = \inf X$.

(iii) $b < \bar{V}_B(\Diamond\psi, v)$. Then there exists $u \in W$ such that $b < R_B vu \wedge \bar{V}_B(\psi, u)$, i.e., $k_b(R_B vu \wedge \bar{V}_B(\psi, u)) = 1 = k_b(\bar{V}_B(\Diamond\psi, v))$. $\qquad\square$

We will also make use of the following lemma from [13] for composing Gödel sets and linear frames.

**Lemma 3.11** ([13, Lemma 24]). *Let $A_1$ and $A_2$ be Gödel sets and let $\mathbf{K}_1 = \langle K_1, \preceq_1 \rangle$ and $\mathbf{K}_2 = \langle K_2, \preceq_2 \rangle$ be linear frames with $K_1 \cap K_2 = \emptyset$ such that $\mathbf{Up}(\mathbf{K}_1) \cong \mathbf{A}_1$ and $\mathbf{Up}(\mathbf{K}_2) \cong \mathbf{A}_2$. Define $\mathbf{K} := \langle K, \preceq \rangle$, where $K := K_1 \cup K_2$ and*

$$
\preceq := \preceq_1 \cup \preceq_2 \cup \{\langle k_2, k_1 \rangle \mid k_2 \in K_2, k_1 \in K_1\},
$$

*and for any $\rho \in (0, 1)$, the Gödel set*

$$
A := \rho A_1 \cup ((1 - \rho)A_2 + \rho).
$$

*Then $\mathbf{Up}(\mathbf{K}) \cong \mathbf{A}$.*

We are now able to prove the main theorem of this section.

**Theorem 3.12.** *For each Gödel set $A$, there exists a countable linear frame $\mathbf{K}$ such that for all $\varphi \in \mathrm{Fm}_{\Box\Diamond}(\mathcal{L}_{\mathsf{IL}})$,*

$$
\begin{aligned}
\models_{\mathsf{S5(A)}} \varphi &\iff \models_{\mathsf{IKL_1(K)}} \varphi^\circ \\
\models_{\mathsf{S5(A)^c}} \varphi &\iff \models_{\mathsf{CDIKL_1(K)}} \varphi^\circ.
\end{aligned}
$$

*Proof.* Let $A$ be a Gödel set. By the Cantor-Bendixson Theorem (Theorem 1.14), there exists a partition of $A$ into a countable set $C$ and a perfect set $X$. If $A$ is countable, then $X = \emptyset$ and so by Lemma 3.9 and Theorem 3.3, we are done. Now suppose that $A$ is uncountable and so $X \neq \emptyset$. We define

$$A_1 := A \cup [\inf X, 1] \quad \text{and} \quad A_2 := (A \cap [0, \inf X]) \cup \mathcal{C}_{[\inf X, 1]},$$

where $\mathcal{C}_{[\inf X, 1]}$ is the middle third Cantor set on the interval $[\inf X, 1]$. Note that the perfect kernel $X_2$ of $A_2$ is $\mathcal{C}_{[\inf X, 1]}$ and so $A_2 \cup [\inf X_2, 1] = A_1$. By Lemma 3.10, for all $\varphi \in \text{Fm}_{\square \lozenge}(\mathcal{L}_{\mathsf{IL}})$,

$$\models_{\mathsf{S5}(\mathbf{A})} \varphi \iff \models_{\mathsf{S5}(\mathbf{A}_1)} \varphi \iff \models_{\mathsf{S5}(\mathbf{A}_2)} \varphi$$
$$\models_{\mathsf{S5}(\mathbf{A})^{\mathsf{C}}} \varphi \iff \models_{\mathsf{S5}(\mathbf{A}_1)^{\mathsf{C}}} \varphi \iff \models_{\mathsf{S5}(\mathbf{A}_2)^{\mathsf{C}}} \varphi.$$

If $\inf X = 0$, then $A_1 = [0, 1]$, in which case $\mathsf{S5}(\mathbf{A})$ coincides with $\mathsf{S5}(\mathbf{G})$ and $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ coincides with $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$. If $\inf X > 0$, we can write $A_2 = \rho B_1 \cup ((1 - \rho)B_2 + \rho)$, where $\rho = \inf X$, $B_1 = (1/\rho)(A \cap [0, \rho])$, and $B_2 = \mathcal{C}_{[0,1]}$. Since $A \cap [0, \rho] \subseteq C \cup \{\inf X\}$, $B_1$ is countable. Therefore, by Lemma 3.9, $\mathbf{B}_1$ is isomorphic to $\mathbf{Up}(\mathbf{K}_1)$ for some linear frame $\mathbf{K}_1 = \langle K_1, \preceq_1 \rangle$. Moreover, $\mathbf{B}_2$ is isomorphic to $\mathbf{Up}((0, 1)_{\mathbf{Q}})$. So by Lemma 3.11, we obtain a linear frame $\mathbf{K} = \langle K, \preceq \rangle$ such that $\mathbf{A}_2$ is isomorphic to $\mathbf{Up}(\mathbf{K})$. Theorem 3.3 then completes the proof. $\square$

## 3.3 An Interpretation Theorem

In this section, we provide an interpretation of the one-variable fragment $\mathsf{IKL}_1(\mathbf{K})$ defined over a linear frame $\mathbf{K}$ in the one-variable fragment of the corresponding constant domain logic $\mathsf{CDIKL}_1(\mathbf{K})$, thereby obtaining also an interpretation of $\mathsf{S5}(\mathbf{A})$ in $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ for any Gödel set $A$. The key idea is to describe the domains of an $\mathsf{IKL}_1$-model using a distinguished unary predicate $P_0$ for the corresponding $\mathsf{CDIKL}_1$-model. To this end, let $\text{Fm}_1^0(\mathcal{L}_{\mathsf{IL}}) \subseteq \text{Fm}_1(\mathcal{L}_{\mathsf{IL}})$ denote the set of one-variable first-order formulas not containing $P_0$. An $\mathsf{IKL}_1^0(\mathbf{K})$-*model*, based on a linear frame $\mathbf{K} = \langle K, \preceq \rangle$, is an $\mathsf{IKL}_1(\mathbf{K})$-model $\mathfrak{M} = \langle K, \preceq, \{D_k\}_{k \in K}, \{\mathcal{I}_k\}_{k \in K} \rangle$ such that the functions $\{\mathcal{I}_k\}_{k \in K}$ are restricted to $\{P_i\}_{i \in \mathbb{N}^+}$.

Now let $\mathbf{K} = \langle K, \preceq \rangle$ be any linear frame and let $\mathfrak{M} = \langle K, \preceq, \{D\}, \{\mathcal{I}_k\}_{k \in K} \rangle$ be a $\mathsf{CDIKL}_1(\mathbf{K})$-model satisfying

$$\bigcap_{k \in K} \mathcal{I}_k(P_0) \neq \emptyset.$$

Define for each $k \in K$ and $i \in \mathbb{N}^+$,

$$D_k := \mathcal{I}_k(P_0) \quad \text{and} \quad \mathcal{I}_k^0(P_i) := \mathcal{I}_k(P_i) \cap D_k.$$

Then $\mathfrak{M}^0 := \langle K, \preceq, \{D_k\}_{k \in K}, \{\mathcal{I}_k^0\}_{k \in K} \rangle$ is an $\mathsf{IKL}_1^0(\mathbf{K})$-model. Indeed, $\mathfrak{M} \mapsto \mathfrak{M}^0$ is a surjective map from $\mathsf{CDIKL}_1(\mathbf{K})$-models to $\mathsf{IKL}_1^0(\mathbf{K})$-models.

For each $\alpha \in \text{Fm}_1^0(\mathcal{L}_{\mathsf{IL}})$, we define $\alpha^c \in \text{Fm}_1(\mathcal{L}_{\mathsf{IL}})$ inductively by relativizing quantifiers to the unary predicate $P_0$. That is, $(P_i(x))^c := P_i(x)$ for each $i \in \mathbb{N}^+$, $\perp^c := \perp$, $\top^c := \top$, $(\alpha \star \beta)^c := \alpha^c \star \beta^c$ for $\star \in \{\wedge, \vee, \rightarrow\}$, and

$$((\forall x)\alpha)^c := (\forall x)(P_0(x) \rightarrow \alpha^c)$$
$$((\exists x)\alpha)^c := (\exists x)(P_0(x) \wedge \alpha^c).$$

**Lemma 3.13.** *Let $\mathbf{K} = \langle K, \preceq \rangle$ be a linear frame and let $\mathfrak{M} = \langle K, \preceq, \{D\}, \{\mathcal{I}_k\}_{k \in K} \rangle$ be a $\mathsf{CDIKL_1(K)}$-model satisfying $\bigcap_{k \in K} \mathcal{I}_k(P_0) \neq \emptyset$. Then for any $\alpha \in \mathrm{Fm}_1^0(\mathcal{L}_{\mathsf{IL}})$, $k \in K$, and $a \in \mathcal{I}_k(P_0)$,*

$$\mathfrak{M}^0, k \models^a \alpha \iff \mathfrak{M}, k \models^a \alpha^c.$$

*Proof.* We prove the claim by induction on the length of $\alpha$. For the base case, using the assumption that $a \in \mathcal{I}_k(P_0) = D_k$, we have for each $i \in \mathbb{N}^+$,

$$\mathfrak{M}^0, k \models^a P_i(x) \iff a \in \mathcal{I}_k^0(P_i) \iff a \in \mathcal{I}_k(P_i) \iff \mathfrak{M}, k \models^a P_i(x).$$

The cases for the propositional connectives follow easily using the induction hypothesis and the definition of $\alpha^c$, so we just check the cases for the quantifiers:

$$
\begin{aligned}
\mathfrak{M}^0, k \models^a (\forall x)\beta &\iff \mathfrak{M}^0, l \models^b \beta \text{ for all } l \succeq k \text{ and } b \in D_l \\
&\iff \mathfrak{M}, l \models^b \beta^c \text{ for all } l \succeq k \text{ and } b \in \mathcal{I}_l(P_0) \\
&\iff (\mathfrak{M}, l \models^b P_0(x) \Rightarrow \mathfrak{M}, l \models^b \beta^c) \text{ for all } l \succeq k,\, b \in D \\
&\iff \mathfrak{M}, l \models^b P_0(x) \to \beta^c \text{ for all } l \succeq k \text{ and } b \in D \\
&\iff \mathfrak{M}, k \models^a (\forall x)(P_0(x) \to \beta^c) \\
&\iff \mathfrak{M}, k \models^a ((\forall x)\beta)^c;
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{M}^0, k \models^a (\exists x)\beta &\iff \mathfrak{M}^0, k \models^b \beta \text{ for some } b \in D_k \\
&\iff \mathfrak{M}, k \models^b \beta^c \text{ for some } b \in \mathcal{I}_k(P_0) \\
&\iff (\mathfrak{M}, k \models^b P_0(x) \text{ and } \mathfrak{M}, k \models^b \beta^c) \text{ for some } b \in D \\
&\iff \mathfrak{M}, k \models^b P_0(x) \wedge \beta^c \text{ for some } b \in D \\
&\iff \mathfrak{M}, k \models^a (\exists x)(P_0(x) \wedge \beta^c) \\
&\iff \mathfrak{M}, k \models^a ((\exists x)\beta)^c. \qquad \square
\end{aligned}
$$

**Theorem 3.14.** *For any linear frame $\mathbf{K} = \langle K, \preceq \rangle$ and formula $\alpha \in \mathrm{Fm}_1^0(\mathcal{L}_{\mathsf{IL}})$,*

$$\models_{\mathsf{IKL_1(K)}} (\forall x)\alpha \iff \models_{\mathsf{CDIKL_1(K)}} ((\forall x)\alpha)^c.$$

*Proof.* ($\Rightarrow$) Suppose that $\not\models_{\mathsf{CDIKL_1(K)}} ((\forall x)\alpha)^c$, i.e., $\mathfrak{M}_1, k_0 \not\models^a \alpha^c$ for some $\mathsf{CDIKL_1(K)}$-model $\mathfrak{M}_1 = \langle K, \preceq, \{D\}, \{\mathcal{I}_k\}_{k \in K} \rangle$, $k_0 \in K$, and $a \in \mathcal{I}_{k_0}(P_0)$. Let $\mathbf{K}_0 := \langle [k_0), \preceq \rangle$. Then also $\mathfrak{M}_2, k_0 \not\models^a \alpha^c$, where $\mathfrak{M}_2$ is the $\mathsf{CDIKL_1(K_0)}$-model $\langle [k_0), \preceq, \{D\}, \{\mathcal{I}_k\}_{k \in [k_0)} \rangle$ satisfying $\bigcap_{k \in [k_0)} \mathcal{I}_k(P_0) \neq \emptyset$. An application of Lemma 3.13 yields $\mathfrak{M}_2^0, k_0 \not\models^a \alpha$. We can then extend $\mathfrak{M}_2^0$ to an $\mathsf{IKL_1^0(K)}$-model by defining $D_l := D_{k_0}$ and $\mathcal{I}_l(P_i) := \mathcal{I}_{k_0}(P_i)$ for all $l \in K$ such that $l \prec k_0$, giving $\not\models_{\mathsf{IKL_1(K)}} (\forall x)\alpha$ as required.

($\Leftarrow$) Suppose that $\not\models_{\mathsf{IKL_1(K)}} (\forall x)\alpha$, i.e., $(\forall x)\alpha$ is not valid in some $\mathsf{IKL_1(K)}$-model $\mathfrak{M}$. Since $\alpha$ does not contain $P_0$, we can assume that $\mathfrak{M}$ is an $\mathsf{IKL_1^0(K)}$-model. Because the map $(-)^0$ is surjective, there exists a $\mathsf{CDIKL_1(K)}$-model $\mathfrak{N}$ such that $\mathfrak{M} = \mathfrak{N}^0$. By Lemma 3.13, the formula $((\forall x)\alpha)^c$ is not valid in $\mathfrak{N}$ and hence $\not\models_{\mathsf{CDIKL_1(K)}} ((\forall x)\alpha)^c$ as required. $\qquad \square$

Now let $\mathrm{Fm}_{\square\Diamond}^0(\mathcal{L}_{\mathsf{IL}}) \subseteq \mathrm{Fm}_{\square\Diamond}(\mathcal{L}_{\mathsf{IL}})$ denote the set of modal formulas over $\mathcal{L}_{\mathsf{IL}}$ not containing $p_0$. For each $\varphi \in \mathrm{Fm}_{\square\Diamond}^0(\mathcal{L}_{\mathsf{IL}})$, we define $\varphi^c \in \mathrm{Fm}_{\square\Diamond}(\mathcal{L}_{\mathsf{IL}})$ inductively by relativizing modalities to $p_0$. That is, $(p_i)^c := p_i$ for each $i \in \mathbb{N}^+$, $\bot^c := \bot$, $\top^c := \top$, $(\varphi \star \psi)^c := \varphi^c \star \psi^c$ for $\star \in \{\wedge, \vee, \to\}$, $(\square\varphi)^c := \square(p_0 \to \varphi^c)$, and $(\Diamond\varphi)^c := \Diamond(p_0 \wedge \varphi^c)$.

**Theorem 3.15.** *For any formula $\varphi \in \mathrm{Fm}^0_{\Box\Diamond}(\mathcal{L}_{\mathsf{IL}})$ and Gödel set $A$,*

$$\models_{\mathsf{S5(A)}} \varphi \quad \Longleftrightarrow \quad \models_{\mathsf{S5(A)^C}} (\Box\varphi)^c.$$

*Proof.* Consider any Gödel set $A$. By Theorem 3.12, there exists a countable linear frame $\mathbf{K}$ such that both $\models_{\mathsf{S5(A)}} \varphi$ if and only if $\models_{\mathsf{IKL_1(K)}} \varphi^\circ$, and $\models_{\mathsf{S5(A)^C}} \varphi$ if and only if $\models_{\mathsf{CDIKL_1(K)}} \varphi^\circ$ hold. Note that the translations $(-)^\circ$ and $(-)^c$ commute on formulas $\varphi \in \mathrm{Fm}^0_{\Box\Diamond}(\mathcal{L}_{\mathsf{IL}})$. Combining this with Theorem 3.14 gives for every $\varphi \in \mathrm{Fm}^0_{\Box\Diamond}(\mathcal{L}_{\mathsf{IL}})$,

$$
\begin{aligned}
\models_{\mathsf{S5(A)}} \varphi &\Longleftrightarrow \models_{\mathsf{S5(A)}} \Box\varphi \\
&\Longleftrightarrow \models_{\mathsf{IKL_1(K)}} (\Box\varphi)^\circ \\
&\Longleftrightarrow \models_{\mathsf{CDIKL_1(K)}} ((\Box\varphi)^\circ)^c \\
&\Longleftrightarrow \models_{\mathsf{CDIKL_1(K)}} ((\Box\varphi)^c)^\circ \\
&\Longleftrightarrow \models_{\mathsf{S5(A)^C}} (\Box\varphi)^c. \qquad\qquad \Box
\end{aligned}
$$

Recall that in Proposition 1.24, we showed that the sets of logics $\mathsf{S5(A)}$ and $\mathsf{S5(A)^C}$, where $A$ ranges over infinite Gödel sets, are infinite. Moreover, recall from Theorem 1.13 that there are at most countably infinitely many first-order Gödel logics, and so there are only countably infinitely many logics $\mathsf{S5(A)^C}$. It follows now by Theorem 3.15 that there are exactly countably infinitely many different logics $\mathsf{S5(A)}$.

**Corollary 3.16.** *The sets of logics $\mathsf{S5(A)}$ and $\mathsf{S5(A)^C}$ (considered as sets of valid formulas), where $A$ ranges over all Gödel sets, are countably infinite.*

**Remark 3.17.** The distinguished unary predicate $P_0$ used in the interpretation $(-)^c$ corresponds exactly to the existence predicate as considered by Iemhoff in the context of Scott logics in [87]. It is moreover closely related to the possibilistic semantics for the modal Gödel logic $\mathsf{KD45(G)}$ studied in [35]. The exact nature of this relation is still to be determined.

## 3.4   A Finite Model Property

In this section, we establish a finite model property for the logic $\mathsf{S5(A)^C}$ for any Gödel set $A$. Using the interpretation from Theorem 3.15, this also establishes the finite model property for $\mathsf{S5(A)}$. Crucially, however, this property does not hold in general with respect to the $\mathsf{S5(A)^C}$-models as defined in Section 1.3. Indeed, for any Gödel set $A$ containing at least one right accumulation point $c$, the formula $\Diamond(p_1 \to \Box p_1)$ is valid in all finite $\mathsf{S5(A)^C}$-models, but not in any infinite universal $\mathsf{S5(A)^C}$-model $\langle \mathbb{N}^+, V \rangle$ satisfying $V(p_1, n) \in A \cap (c, c + \frac{1}{n}]$ for each $n \in \mathbb{N}^+$. We introduce an alternative semantics here, related to the semantics proposed in [38].[2]

Let $P \subseteq \{p_i\}_{i \in \mathbb{N}}$ be a set of propositional variables and let $\mathrm{Fm}_{\Box\Diamond}(\mathcal{L}_{\mathsf{IL}}, P)$ denote the set of formulas in $\mathrm{Fm}_{\Box\Diamond}(\mathcal{L}_{\mathsf{IL}})$ with variables in $P$. A *relativized universal $\mathsf{S5(A)^C}$-model over $P$* (for short, $\mathsf{ruS5(A)^C}$*-model over $P$*) based on a Gödel set $A$ is a triple $\mathcal{M} = \langle W, V, T \rangle$

---

[2]This paper contains a flawed proof that all logics $\mathsf{S5(A)^C}$ have the finite model property with respect to an alternative semantics. More precisely, Lemma 23 of [38] is false unless $T_\Box = T_\Diamond$; this restriction does not cause any problems for $\mathsf{S5(G)^C}$, but is not sufficient for other cases.

consisting of finite non-empty sets $W$ and $T$ satisfying $\{0,1\} \subseteq T \subseteq A$, and a map $V \colon P \times W \to A$. The map $V$ is extended inductively to $\bar{V} \colon \mathrm{Fm}_{\square\lozenge}(\mathcal{L}_{\mathsf{IL}}, P) \times W \to A$ as follows, where $\star \in \{\wedge, \vee, \to\}$:

$$\bar{V}(\bot, w) = 0$$
$$\bar{V}(\top, w) = 1$$
$$\bar{V}(\varphi \star \psi, w) = \bar{V}(\varphi, w) \star \bar{V}(\psi, w)$$
$$\bar{V}(\square\varphi, w) = \bigvee\{r \in T \mid r \leq \bigwedge\{\bar{V}(\varphi, v) \mid v \in W\}\}$$
$$\bar{V}(\lozenge\varphi, w) = \bigwedge\{r \in T \mid r \geq \bigvee\{\bar{V}(\varphi, v) \mid v \in W\}\}.$$

We say that $\varphi \in \mathrm{Fm}_{\square\lozenge}(\mathcal{L}_{\mathsf{IL}}, P)$ is *valid* in $\mathcal{M}$ if $\bar{V}(\varphi, w) = 1$ for all $w \in W$.

Note that since $W$ and $T$ are finite, $\bar{V}(\square\varphi, w), \bar{V}(\lozenge\varphi, w) \in T$ for all $\square\varphi, \lozenge\varphi \in \mathrm{Fm}_{\square\lozenge}(\mathcal{L}_{\mathsf{IL}}, P)$ and $w \in W$, and these values are independent of $w$. Moreover, a simple induction on the length of $\varphi \in \mathrm{Fm}_{\square\lozenge}(\mathcal{L}_{\mathsf{IL}}, P)$ shows that always

$$\bar{V}(\varphi, w) \in B_{\mathcal{M}} := \{V(p_i, v) \mid p_i \in P, v \in W\} \cup T.$$

Indeed, $\mathcal{M}$ may also be viewed as an $\mathsf{ruS5}(\mathbf{B}_{\mathcal{M}})^{\mathsf{C}}$-model over $P$; that is, we may assume that $V$ is a function from $P \times W$ to $B_{\mathcal{M}}$. In particular, if $P$ is finite, then $\mathcal{M}$ is a truly finite object.

Recall that $\mathsf{R}(A)$ and $\mathsf{L}(A)$ denote the sets of right and left accumulation points, respectively, of a Gödel set $A$. An $\mathsf{ruS5}(\mathbf{A})^{\mathsf{C}}$-model $\mathcal{M} = \langle W, V, T \rangle$ over $P \subseteq \{p_i\}_{i \in \mathbb{N}}$ is called $\Sigma$-*normal* for $\Sigma \subseteq \mathrm{Fm}_{\square\lozenge}(\mathcal{L}_{\mathsf{IL}}, P)$ if for all $\square\varphi, \lozenge\psi \in \Sigma$ and $w \in W$,

$$\bar{V}(\square\varphi, w) \notin \mathsf{R}(A) \implies \bar{V}(\square\varphi, w) = \bar{V}(\varphi, v) \text{ for some } v \in W$$
$$\bar{V}(\lozenge\psi, w) \notin \mathsf{L}(A) \implies \bar{V}(\lozenge\psi, w) = \bar{V}(\psi, v) \text{ for some } v \in W.$$

Let us also call $\Sigma \subseteq \mathrm{Fm}_{\square\lozenge}(\mathcal{L}_{\mathsf{IL}})$ a *fragment* if it is closed under subformulas. The next lemma shows that (roughly speaking) for a finite fragment, validity in a (possibly infinite) universal $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-model can be matched to validity in a corresponding $\mathsf{ruS5}(\mathbf{A})^{\mathsf{C}}$-model that is normal for the fragment.

**Lemma 3.18.** *Let $A$ be a Gödel set and let $\mathcal{M} = \langle W, V \rangle$ be a universal $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-model with $w \in W$. For any $P \subseteq \{p_i\}_{i \in \mathbb{N}}$ and finite fragment $\Sigma \subseteq \mathrm{Fm}_{\square\lozenge}(\mathcal{L}_{\mathsf{IL}}, P)$, there exists a $\Sigma$-normal $\mathsf{ruS5}(\mathbf{A})^{\mathsf{C}}$-model $\mathcal{M}' = \langle W', V', T \rangle$ over $P$ with $w \in W' \subseteq W$, $|W'| \leq |\Sigma|$, and $|B_{\mathcal{M}'}| \leq |\Sigma|^2$, satisfying $\bar{V}'(\varphi, v) = \bar{V}(\varphi, v)$ for all $\varphi \in \Sigma$ and $v \in W'$.*

*Proof.* We define

$$T := \{\bar{V}(\square\varphi, w) \mid \square\varphi \in \Sigma\} \cup \{\bar{V}(\lozenge\varphi, w) \mid \lozenge\varphi \in \Sigma\} \cup \{0, 1\}$$

and write $T = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$. Then for each $\square\varphi \in \Sigma$, we have $\bar{V}(\square\varphi, w) = t_i$ for some $0 \leq i \leq n$ and we choose a witness $v_{\square\varphi} \in W$ satisfying

$$t_i \in \mathsf{R}(A) \implies \bar{V}(\varphi, v_{\square\varphi}) \in [t_i, t_{i+1}) \cap A$$
$$t_i \notin \mathsf{R}(A) \implies \bar{V}(\square\varphi, w) = \bar{V}(\varphi, v_{\square\varphi}) = t_i.$$

Similarly, for each $\Diamond\varphi \in \Sigma$, we have $\bar{V}(\Diamond\varphi, w) = t_i$ for some $0 \le i \le n$, and we choose a witness $v_{\Diamond\varphi} \in W$ satisfying

$$
\begin{aligned}
t_i \in \mathsf{L}(A) &\implies \bar{V}(\varphi, v_{\Diamond\varphi}) \in (t_{i-1}, t_i] \cap A \\
t_i \notin \mathsf{L}(A) &\implies \bar{V}(\Diamond\varphi, w) = \bar{V}(\varphi, v_{\Diamond\varphi}) = t_i.
\end{aligned}
$$

We now define

$$
W' := \{w\} \cup \{v_{\Box\varphi} \mid \Box\varphi \in \Sigma\} \cup \{v_{\Diamond\varphi} \mid \Diamond\varphi \in \Sigma\}
$$

and $V'(p_i, v) := V(p_i, v)$ for all $p_i \in P$ and $v \in W'$. Then by construction, $\mathcal{M}' := \langle W', V', T \rangle$ is a $\Sigma$-normal $\mathsf{ruS5}(\mathbf{A})^{\mathsf{C}}$-model over $P$ and clearly $|W'| \le |\Sigma|$. It follows by induction on formula length that $\bar{V}'(\varphi, v) = \bar{V}(\varphi, v)$ for all $\varphi \in \Sigma$, $v \in W'$. The base cases and the cases of the propositional connectives are straightforward. If $\varphi = \Box\psi$, then $\bar{V}(\Box\psi, w) = t_i$ for some $0 \le i \le n$, and we have two cases. If $t_i \in \mathsf{R}(A)$, then

$$
\begin{aligned}
\bar{V}(\Box\psi, w) &= \bigwedge\{\bar{V}(\psi, v) \mid v \in W\} \\
&\le \bigwedge\{\bar{V}(\psi, v) \mid v \in W'\} \\
&\le \bar{V}(\psi, v_{\Box\psi}) < t_{i+1},
\end{aligned}
$$

and if $t_i \notin \mathsf{R}(A)$, then

$$
\begin{aligned}
\bar{V}(\Box\psi, w) &= \bigwedge\{\bar{V}(\psi, v) \mid v \in W\} \\
&\le \bigwedge\{\bar{V}(\psi, v) \mid v \in W'\} \\
&\le \bar{V}(\psi, v_{\Box\psi}) = t_i.
\end{aligned}
$$

Together with the induction hypothesis, this gives

$$
\begin{aligned}
\bar{V}(\Box\psi, w) &= \bigvee\{r \in T \mid r \le \bigwedge\{\bar{V}(\psi, v) \mid v \in W'\}\} \\
&= \bigvee\{r \in T \mid r \le \bigwedge\{\bar{V}'(\psi, v) \mid v \in W'\}\} \\
&= \bar{V}'(\Box\psi, w).
\end{aligned}
$$

The case $\varphi = \Diamond\psi$ is very similar. It easily follows also that $|B_{\mathcal{M}'}| \le |\Sigma|^2$.  $\qquad\square$

The second crucial lemma proceeds in the other direction; it shows that (roughly speaking) validity for a fragment in an $\mathsf{ruS5}(\mathbf{A})^{\mathsf{C}}$-model can be matched to validity in a corresponding universal $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-model. The key idea here is to approximate values in the set $T$ taken by formulas $\Box\varphi$ and $\Diamond\varphi$ by taking multiple copies of the set of worlds and choosing elements in $A$ that get closer and closer to the values in $T$ from the left or right as appropriate.

**Lemma 3.19.** *Let $\mathcal{M} = \langle W, V, T \rangle$ be a $\Sigma$-normal $\mathsf{ruS5}(\mathbf{A})^{\mathsf{C}}$-model over a finite set $P \subseteq \{p_i\}_{i\in\mathbb{N}}$ for a fragment $\Sigma \subseteq \mathrm{Fm}_{\Box\Diamond}(\mathcal{L}_{\mathsf{IL}}, P)$. Then there exists a (countable) universal $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-model $\mathcal{M}' = \langle W', V' \rangle$ such that $W \subseteq W'$ and $\bar{V}(\varphi, w) = \bar{V}'(\varphi, w)$ for all $\varphi \in \Sigma$ and $w \in W$.*

*Proof.* Let $T = \{0 = t_0 < t_1 < \cdots < t_N = 1\}$. For each $t_i \in \mathsf{R}(A)$, we fix a strictly descending sequence $(r_n^i)_{n\in\mathbb{N}^+} \subseteq A \cap (t_i, t_{i+1})$ such that $t_i < r_n^i < t_i + \frac{1}{n}$ for each $n \in \mathbb{N}^+$.

Similarly, for each $t_i \in \mathsf{L}(A)$, we fix a strictly ascending sequence $(s_n^i)_{n \in \mathbb{N}^+} \subseteq A \cap (t_{i-1}, t_i)$ such that $t_i - \frac{1}{n} < s_n^i < t_i$ for each $n \in \mathbb{N}^+$. For each $0 \leq i < N$, we write

$$[t_i, t_{i+1}] \cap B_{\mathcal{M}} = \{t_i = b_0^i < b_1^i < \cdots < b_{k_i}^i < b_{k_i+1}^i = t_{i+1}\}.$$

Note that $B_{\mathcal{M}} = \bigcup_{0 \leq i < N}([t_i, t_{i+1}] \cap B_{\mathcal{M}})$.

We now define a map $h_n \colon B_{\mathcal{M}} \to A$ for each $n \in \mathbb{N}$, where

(i) $h_0 \colon B_{\mathcal{M}} \to A$ is the identity embedding;

(ii) if $n > 0$ is odd, then $h_n(t_N) := t_N$ and for each $i \in \{0, 1, \ldots, N-1\}$, we define $h_n(t_i) := t_i$ and for each $j \in \{1, \ldots, k_i\}$,

$$h_n(b_j^i) := \begin{cases} r_{n+k_i-j}^i & t_i \in \mathsf{R}(A); \\ b_j^i & t_i \notin \mathsf{R}(A); \end{cases}$$

(iii) if $n > 0$ is even, then $h_n(t_N) := t_N$ and for each $i \in \{0, 1, \ldots, N-1\}$, we define $h_n(t_i) := t_i$, and for each $j \in \{1, \ldots, k_i\}$,

$$h_n(b_j^i) := \begin{cases} s_{n+j}^{i+1} & t_{i+1} \in \mathsf{L}(A); \\ b_j^i & t_{i+1} \notin \mathsf{L}(A). \end{cases}$$

Note that each $h_n \colon B_{\mathcal{M}} \to A$ is a strictly order-preserving embedding that fixes $T$.

For each $n \in \mathbb{N}$, let $W_n$ denote a disjoint copy of $W$ with elements $w_n \in W_n$ corresponding to the element $w \in W$, with $W_0 = W$. Now for each $p_i \in P$, $w \in W$, and $n \in \mathbb{N}$, define

$$W' := \bigcup_{n \in \mathbb{N}} W_n \quad \text{and} \quad V'(p_i, w_n) := h_n(V(p_i, w)).$$

Defining also $V'(p_j, w_n) := 0$ for $p_j \notin P$ and $n \in \mathbb{N}$, we obtain a universal $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-model $\mathcal{M}' := \langle W', V' \rangle$.

We prove by induction on formula length that $\bar{V}'(\varphi, w_n) = h_n(\bar{V}(\varphi, w))$ for all $\varphi \in \Sigma$, $w \in W$, and $n \in \mathbb{N}$. The base cases follow by definition and the fact that each $h_n$ fixes 0 and 1. The cases for propositional connectives follow from the fact that each $h_n$ is a strictly order-preserving embedding fixing 0 and 1.

Now consider $\varphi = \Box\psi \in \Sigma$ with $\bar{V}(\Box\psi, w) = t_i$. Then $\bar{V}(\Box\psi, w) \leq \bar{V}(\psi, v)$ and so $h_n(\bar{V}(\Box\psi, w)) \leq h_n(\bar{V}(\psi, v))$ for all $v \in W$. We consider two cases. If $t_i \notin \mathsf{R}(A)$, then since $\mathcal{M}$ is $\Sigma$-normal, there exists $v \in W$ such that $\bar{V}(\Box\psi, w) = \bar{V}(\psi, v)$ and so $h_n(\bar{V}(\psi, v)) = t_i$ for all $n \in \mathbb{N}$. If $t_i \in \mathsf{R}(A)$, then $i < N$ and there exists $v \in W$ such that $\bar{V}(\psi, v) \in [t_i, t_{i+1}) \cap B_{\mathcal{M}}$. Then by construction, $h_n(\bar{V}(\psi, v)) \in [t_i, r_n^i] \subseteq [t_i, t_i + \frac{1}{n})$ for each odd $n \in \mathbb{N}$. In both cases,

$$t_i \leq \bigwedge \{h_n(\bar{V}(\psi, v)) \mid v \in W; n \in \mathbb{N}\}$$
$$\leq \bigwedge \{h_n(\bar{V}(\psi, v)) \mid v \in W; n \in \mathbb{N} \text{ odd}\}$$
$$= t_i = \bar{V}(\Box\psi, w).$$

Applying the induction hypothesis then gives

$$\bar{V}'(\Box\psi, w) = \bigwedge\{\bar{V}'(\psi, w_n) \mid w \in W; n \in \mathbb{N}\}$$
$$= \bigwedge\{h_n(\bar{V}(\psi, w)) \mid w \in W; n \in \mathbb{N}\}$$
$$= \bar{V}(\Box\psi, w).$$

The case for $\varphi = \Diamond\psi \in \Sigma$ is similar. So we have $\bar{V}'(\varphi, w_n) = h_n(\bar{V}(\varphi, w))$ for all $\varphi \in \Sigma$, $w \in W$, and $n \in \mathbb{N}$. Taking $n = 0$ then gives $\bar{V}'(\varphi, w) = V(\varphi, w)$ for all $\varphi \in \Sigma$ and $w \in W$. $\qquad\square$

Let $P_\varphi$ denote the (finite) set of propositional variables occurring in a formula $\varphi \in \mathrm{Fm}_{\Box\Diamond}(\mathcal{L}_{\mathsf{IL}})$, and let $\Sigma_\varphi$ denote the fragment of subformulas in $\varphi$. The following theorem expresses the desired finite model property $\mathsf{S5(A)^C}$, recalling that an $\mathsf{ruS5(A)^C}$-model $\mathcal{M}$ over a finite set of variables not only has a finite set of worlds, but may be considered a finite object if $\mathbf{A}$ is replaced by $\mathbf{B}_\mathcal{M}$.

**Theorem 3.20.** *Let A be a Gödel set. For any $\varphi \in \mathrm{Fm}_{\Box\Diamond}(\mathcal{L}_{\mathsf{IL}})$,*

$$\models_{\mathsf{S5(A)^C}} \varphi \iff \varphi \text{ is valid in all } \Sigma_\varphi\text{-normal } \mathsf{ruS5(A)^C}\text{-models over } P_\varphi.$$

*Proof.* If $\not\models_{\mathsf{S5(A)^C}} \varphi$, then there exists a universal $\mathsf{S5(A)^C}$-model $\mathcal{M} = \langle W, V \rangle$ and $w \in W$ such that $\bar{V}(\varphi, w) < 1$. By Lemma 3.18, there exists a $\Sigma_\varphi$-normal $\mathsf{ruS5(A)^C}$-model $\mathcal{M}' = \langle W', V', T \rangle$ over $P_\varphi$ such that $\bar{V}'(\varphi, w) = \bar{V}(\varphi, w) < 1$.

Conversely, if $\bar{V}(\varphi, w) < 1$ for some $w \in W$ in a $\Sigma_\varphi$-normal $\mathsf{ruS5(A)^C}$-model $\langle W, V, T \rangle$ over $P_\varphi$, then, by Lemma 3.19, there exists a universal $\mathsf{S5(A)^C}$-model $\langle W', V' \rangle$ such that $\bar{V}'(\varphi, w) = \bar{V}(\varphi, w) < 1$. $\qquad\square$

It is worth noting that Theorem 3.20 also establishes an algebraic finite model property. Recall the definition of a crisp monadic Gödel algebra from Example 2.7, and let $c\mathcal{MGA}$ denote the variety of all crisp monadic Gödel algebras. Consider the functional crisp monadic Gödel algebra $\langle \mathbf{G}^W; \Box, \Diamond \rangle$; that is, for each $f \in [0,1]^W$,

$$\Box f(w) = \bigwedge\{f(v) \mid v \in W\} \text{ and } \Diamond f(w) = \bigvee\{f(v) \mid v \in W\}.$$

In light of Theorem 2.22, this functional crisp monadic Gödel algebra corresponds exactly to the pair consisting of the Gödel algebra $\mathbf{G}^W$ and the $c\mathcal{MGA}$-relatively complete subalgebra with universe $\{f_r \mid r \in [0,1]\}$, where for each $r \in [0,1]$, $f_r \colon W \to [0,1]$ is the constant function mapping each $w \in W$ to $r$. Since this functional crisp monadic Gödel algebra together with an evaluation $e \colon \{p_i\}_{i \in \mathbb{N}} \to G^W$ corresponds exactly to the universal $\mathsf{S5(G)^C}$-model $\langle W, V \rangle$ where $V(p_i, w) := e(p_i)(w)$ for all $i \in \mathbb{N}$ and $w \in W$, it follows that

$$\models_{\mathsf{S5(G)^C}} \varphi \iff \langle \mathbf{G}^W; \Box, \Diamond \rangle \models \top \approx \varphi \text{ for all sets } W.$$

Theorem 3.20 now establishes an algebraic finite model property for $\mathsf{S5(A)^C}$. Indeed, an $\mathsf{ruS5(G)^C}$-model $\mathcal{M} = \langle W, V, T \rangle$ corresponds to the pair consisting of the Gödel algebra $\mathbf{B}_\mathcal{M}^W$ and the $c\mathcal{MGA}$-relatively complete subalgebra with universe $\{f_r \mid r \in T\}$, together with an evaluation $e \colon \{p_i\}_{i \in \mathbb{N}} \to B_\mathcal{M}^W$ where $e(p_i)(w) := V(p_i, w)$ for all $i \in \mathbb{N}$ and $w \in W$.

**Corollary 3.21.** *The variety $c\mathcal{MGA}$ of crisp monadic Gödel algebras has the finite model property, i.e., it is generated by its finite members.*

## 3.5   Decidability and Complexity

The finite model property established in Theorem 3.20 does not directly yield decidability of $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-validity for an arbitrary Gödel set $A$. In order to check the normality condition for an $\mathsf{ruS5}(\mathbf{A})^{\mathsf{C}}$-model, we require some representation of the sets $\mathsf{R}(A)$ and $\mathsf{L}(A)$, which in general, might not even be recursive. We resolve this issue here by specifying sufficient conditions on a Gödel set $A$ that ensure the decidability and even co-NP-completeness of $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-validity, and hence also of $\mathsf{S5}(\mathbf{A})$-validity and the corresponding one-variable fragments of first-order Gödel logics and intermediate logics with or without constant domains.

Observe first that to determine the $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-validity of a formula $\varphi \in \mathrm{Fm}_{\square\Diamond}(\mathcal{L}_{\mathsf{IL}})$ it suffices, by Lemmas 3.18 and 3.19, to check validity in $\Sigma_\varphi$-normal $\mathsf{ruS5}(\mathbf{A})^{\mathsf{C}}$-models $\mathcal{M} = \langle W, V, T \rangle$ over $P_\varphi$. Indeed, as remarked in the previous section, such an $\mathcal{M}$ may be viewed as an $\mathsf{ruS5}(\mathbf{B}_{\mathcal{M}})^{\mathsf{C}}$-model, where $B_{\mathcal{M}}$ is finite. Let us also note that the property of $\Sigma_\varphi$-normality of $\mathcal{M}$ is determined by the sets $T_r := T \cap \mathsf{R}(A)$ and $T_l := T \cap \mathsf{L}(A)$. It therefore follows that the $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-validity of a formula $\varphi \in \mathrm{Fm}_{\square\Diamond}(\mathcal{L}_{\mathsf{IL}})$ of length $n$ is determined by structures of the form

$$\langle W, V, B, \leq, 0, 1, T, T_r, T_l \rangle$$

satisfying the following conditions:

(i) $|W|, |T|, |T_r|, |T_l| \leq n$ and $|B|, |V| \leq n^2$;

(ii) $\{0, 1\}, T_r, T_l \subseteq T \subseteq B$ and $0 \notin T_l$, $1 \notin T_r$;

(iii) $\leq \; \subseteq B^2$ is a linear order with top and bottom elements 1 and 0, respectively;

(iv) $\langle W, V, T \rangle$ is an $\mathsf{ruS5}(\mathbf{B}_{\mathcal{M}})^{\mathsf{C}}$-model over $P_\varphi$ such that for all $\square\psi, \Diamond\psi \in \Sigma_\varphi$ and $w \in W$,

$$\bar{V}(\square\psi, w) \notin T_r \implies \bar{V}(\square\psi, w) = \bar{V}(\psi, v) \text{ for some } v \in W$$
$$\bar{V}(\Diamond\psi, w) \notin T_l \implies \bar{V}(\Diamond\psi, w) = \bar{V}(\psi, v) \text{ for some } v \in W;$$

(v) the finite structure $\langle B, \leq, 0, 1, T, T_r, T_l \rangle$ is *consistent with $A$*; that is, there exists an order-embedding $f \colon \langle B, \leq, 0, 1 \rangle \to \langle A, \leq, 0, 1 \rangle$ preserving 0 and 1 such that

$$f[T_r] = f[T] \cap \mathsf{R}(A) \quad \text{and} \quad f[T_l] = f[T] \cap \mathsf{L}(A).$$

**Theorem 3.22.** *Let $A$ be a Gödel set. Then $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-validity and $\mathsf{S5}(\mathbf{A})$-validity are decidable (co-NP-complete) relative to the problem of checking the consistency of finite structures $\langle B, \leq, 0, 1, T, T_r, T_l \rangle$ with $A$.*

*Proof.* Consider the following procedure to check the non-validity of a formula $\varphi \in \mathrm{Fm}_{\square\Diamond}(\mathcal{L}_{\mathsf{IL}})$ of length $n$ in $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$, where we may assume that all sets involved are subsets of $\{0, \ldots, n^2\}$:

(1) Guess a structure $\langle W, V, B, \leq, 0, 1, T, T_r, T_l \rangle$ satisfying (i), (ii), (iii), and (iv);

(2) Check that $\bar{V}(\varphi, i) < 1$ for some $i \in W$;

(3) Check that $\langle B, \leq, 0, 1, T, T_r, T_l \rangle$ is consistent with $A$.

It is easy to see that (1) and (2) are problems with complexity in NP, and hence that the complexity of the full procedure is decidable (in NP) relative to step (3). Co-NP-hardness follows from the fact that propositional classical logic $\mathsf{CL}$ can be interpreted in $\mathsf{S5(A)^C}$. Finally, the same result for $\mathsf{S5(A)}$-validity follows from the interpretation of $\mathsf{S5(A)}$ in $\mathsf{S5(A)^C}$ provided by Theorem 3.15. $\qquad\square$

Determining the consistency with respect to a Gödel set $A$ of some structure $\langle B, \leq, 0, 1, T, T_r, T_l \rangle$ is trivial for some Gödel sets. For example, such a structure is consistent with $A = G$ if and only if $T_r = T \setminus \{0\}$ and $T_l = T \setminus \{1\}$, with $A = G_\uparrow$ if and only if $T_r = \emptyset$ and $T_l = \{1\}$, and with $A = G_\downarrow$ if and only if $T_r = \{0\}$ and $T_l = \emptyset$. Determining consistency with respect to other Gödel sets may be more complicated, however. The following observation simplifies the problem.

A finite structure $\langle B, \leq, 0, 1, T, T_r, T_l \rangle$ satisfying (ii) and (iii) may be coded via a finite word in the alphabet $\{a, t, r, l, d\}$, where each letter represents the "status" of an element of $B$ with respect to their membership in $T$, $T_r$, and $T_l$:

| | |
|---|---|
| $a$ for an element of $B \setminus T$; | $r$ for an element of $T_r \setminus T_l$; |
| $d$ for an element of $T_r \cap T_l$; | $l$ for an element of $T_l \setminus T_r$. |
| $t$ for an element of $T \setminus (T_r \cup T_l)$; | |

We say that a finite word in the alphabet $\{a, t, r, l, d\}$ is *consistent with a Gödel set $A$* if this is true of the corresponding finite structure. In light of Theorem 3.22, to determine the decidability of $\mathsf{S5(A)^C}$-validity it suffices to determine all words consistent with the Gödel set $A$. Note that these words necessarily start with $t$ or $r$ (the possible status of 0) and end with $t$ or $l$ (the possible status of 1). For two Gödel sets $A$ and $B$, we write $A \oplus B$ to denote the ordered sum of $A$ and $B$, identifying the top element of $A$ and bottom element of $B$, $A^\smile$ to denote the Gödel set $A$ with the ordering reversed, and $\bigoplus_\omega A$ to denote the (countably) infinite ordered sum of $A$ with itself adding a new top element. If additionally $A$ is countable, we write $A \times_{\mathsf{lex}} B$ to denote the lexicographic product of $A$ and $B$. Through squeezing, stretching, and shifting, we can harmlessly assume that the result of these operations are again Gödel sets.[3] We state a number of examples of Gödel sets and their class of consistent words in Table 3.1.

Note that all these classes of words consistent with the respective Gödel sets in Table 3.1 form regular sets of words and are therefore decidable in linear time (for background on regular languages, see, e.g., [85]). It is not difficult to check that this property is preserved by the operations mentioned. That is, if the sets of words consistent with Gödel sets $A$ and $B$ are regular, then so are the sets of words consistent with $A \oplus B$, $A^\smile$, $\bigoplus_\omega A$, and, if $A$ is countable, $A \times_{\mathrm{lex}} B$. This gives a large family of Gödel sets with a linearly decidable consistency problem. Note that $A \oplus G_1$ adds a new top element and $G_1 \oplus A$ adds a new bottom element to $A$. Hence, the disjoint ordered sum can be defined as $A \oplus^d B := A \oplus G_1 \oplus B$.

**Corollary 3.23.** $\mathsf{S5(A)^C}$ *and* $\mathsf{S5(A)}$ *are co-NP-complete for $A = G$, $A = G_\uparrow$, $A = G_\downarrow$, $A = G_n$ for any $n \in \mathbb{N}^+$, and all finite combinations of these Gödel sets by $\oplus$, $(-)^\smile$, $\bigoplus_\omega$, and $\times_{\mathsf{lex}}$ if the first argument is countable.*

---

[3]For example, we can formally define the ordered sum $A \oplus B$ as follows: let $f \colon [0, 1] \to [1, 2]$ and $g \colon [0, 2] \to [0, 1]$ be the maps such that $x \mapsto x + 1$ and $x \mapsto \frac{x}{2}$, respectively. Then $A \oplus B$ can be defined as $g[A \cup f[B]]$.

| Gödel set $A$ | Words consistent with $A$ |
|---|---|
| $G$ | $\{rwl \mid w \in \{a,d\}^*\}$ |
| $G_n \ (n \in \mathbb{N}^+)$ | $\{twt \mid w \in \{a,t\}^* \text{ of length at most } n-1\}$ |
| $G_\uparrow$ | $\{twl \mid w \in \{a,t\}^*\}$ |
| $G_\downarrow$ | $\{rwt \mid w \in \{a,t\}^*\}$ |
| $G_\uparrow \oplus G_\uparrow$ | $\{twl \mid w \in \{a,t\}^*\} \cup \{twlw'l \mid w,w' \in \{a,t\}^*\}$ |
| $G_\downarrow \oplus G_\uparrow$ | $\{twt \mid w \in \{a,t\}^*\} \cup \{twdw't \mid w,w' \in \{a,t\}^*\}$ |
| $\bigoplus_\omega G_\uparrow$ | $\{twl \mid w \in \{a,t,l\}^*\}$ |
| $\bigoplus_\omega G_\downarrow$ | $\{rwl \mid w \in \{a,t,r\}^*\}$ |
| $G_\uparrow \times_{\text{lex}} G_\uparrow$ | $\{twl \mid w \in \{a,t,l\}^*\}$ |
| $G_\downarrow \times_{\text{lex}} G_\uparrow$ | $\{rwl \mid w \in \{a,t\}^*\}$ |

Table 3.1: Examples of Gödel sets and corresponding consistent words (i.e. structures).

In light of Theorem 3.3, this also yields decidability results for various one-variable fragments $\mathsf{CDIKL}_1(\mathbf{K})$ and $\mathsf{IKL}_1(\mathbf{K})$. Note that $\mathrm{Up}(\omega)$ and $\mathrm{Up}(\omega^\smile)$ can be viewed as the Gödel sets $G_\downarrow$ and $G_\uparrow$, respectively, where $\omega$ and $\omega^\smile$ should be read as the ordinal $\omega$ with the usual and reverse ordering, respectively. In general, for any ordinal $\alpha$, $\mathrm{Up}(\alpha)$ and $\mathrm{Up}(\alpha^\smile)$ have the same order structure as the ordinals $(\alpha+1)^\smile$ and $\alpha+1$, respectively. Using Cantor's normal form, any such successor ordinal $2 \le \alpha+1 < \omega^\omega$ (and its reverse $(\alpha+1)^\smile$) can be viewed as a finite combination of $G_\uparrow$ by $\oplus$, $(-)^\smile$, $\bigoplus_\omega$, and $\times_{\text{lex}}$. For example, for any $n \ge 2$,

$$\omega + n = G_\uparrow \oplus G_n$$
$$\omega^2 + 1 = \bigoplus_\omega G_\uparrow$$
$$\omega^2 + \omega + 1 = G_\uparrow \times_{\text{lex}} G_\uparrow = (\bigoplus_\omega G_\uparrow) \oplus G_\uparrow$$
$$\omega^3 + \omega 2 + 5 = \bigoplus_\omega (\bigoplus_\omega G_\uparrow) \oplus G_\uparrow \oplus G_\uparrow \oplus G_5.$$

Moreover, for any pair of linear frames $\mathbf{K}$ and $\mathbf{L}$ we have $\mathrm{Up}(\mathbf{K}^\smile) = \mathrm{Up}(\mathbf{K})^\smile$, $\mathrm{Up}(\mathbf{K} \oplus^d \mathbf{L}) = \mathrm{Up}(\mathbf{L}) \oplus \mathrm{Up}(\mathbf{K})$, and if $\bigoplus_\omega^d \mathbf{K}$ denotes the (countably) infinite disjoint ordered sum of $\mathbf{K}$ with itself, then $\mathrm{Up}(\bigoplus_\omega^d \mathbf{K}) = \bigoplus_\omega \mathrm{Up}(\mathbf{K})$. These observations together with Theorem 3.3 yield the following decidability results.

**Corollary 3.24.** $\mathsf{IKL}_1(\mathbf{K})$ *and* $\mathsf{CDIKL}_1(\mathbf{K})$ *are co-NP-complete if* $\mathbf{K}$ *is any finite combination of countable ordinals below* $\omega^\omega$ *by* $(-)^\smile$, $\oplus^d$, *and* $\bigoplus_\omega^d$.

This notion of consistency can also be used to compare logics.

**Theorem 3.25.** *Let* $A_1$ *and* $A_2$ *be two Gödel sets. Suppose that any finite structure* $\langle B, \le, 0, 1, T, T_r, T_l \rangle$ *satisfying (ii) and (iii) is consistent with* $A_1$ *if and only if it is consistent with* $A_2$. *Then for all* $\varphi \in \mathrm{Fm}_{\square\Diamond}(\mathcal{L}_{\mathsf{IL}})$,

$$\models_{\mathsf{S5}(\mathbf{A}_1)^c} \varphi \quad \Longleftrightarrow \quad \models_{\mathsf{S5}(\mathbf{A}_2)^c} \varphi.$$

*Proof.* Suppose that $\not\models_{\mathsf{S5(A_1)^C}} \varphi$ for some $\varphi \in \mathrm{Fm}_{\square\Diamond}(\mathcal{L}_{\mathsf{IL}})$. Then by Lemma 3.18, there exists a $\Sigma_\varphi$-normal $\mathsf{ruS5(A_1)^C}$-model $\mathcal{M} = \langle W, V, T \rangle$ over $P_\varphi$ such that $\bar{V}(\varphi, w) < 1$ for some $w \in W$. Then the finite structure $\langle B_{\mathcal{M}}, \leq, 0, 1, T, T \cap \mathsf{R}(A_1), T \cap \mathsf{L}(A_1) \rangle$ is consistent with $A_1$, so by assumption it is also consistent with $A_2$. We may therefore assume that $B_{\mathcal{M}} \subseteq A_2$, $T \cap \mathsf{R}(A_1) = T \cap \mathsf{R}(A_2)$, and $T \cap \mathsf{L}(A_1) = T \cap \mathsf{L}(A_2)$. By Lemma 3.19, $\mathcal{M}$ can be extended to a universal $\mathsf{S5(A_2)^C}$-model $\mathcal{M}' = \langle W', V' \rangle$ such that $\bar{V}'(\varphi, w) < 1$ and so $\not\models_{\mathsf{S5(A_2)^C}} \varphi$. The other direction follows by symmetry. $\qquad\square$

Even undecidable Gödel sets can have a decidable consistency problem. For example, consider any countable limit ordinal $\alpha \geq \omega^2$. Then the words consistent with $\alpha + 1$ are all *twl* with $w \in \{a, t, l\}^*$. The same holds for $\omega^2 + 1$, so by Theorem 3.25, we obtain for any $\varphi \in \mathrm{Fm}_{\square\Diamond}(\mathcal{L}_{\mathsf{IL}})$,

$$\models_{\mathsf{S5}(\alpha+1)^C} \varphi \quad \Longleftrightarrow \quad \models_{\mathsf{S5}(\omega^2+1)^C} \varphi.$$

As undecidable countable ordinals $\alpha \geq \omega^2$ exist, such as the Church-Kleene ordinal, there are logics $\mathsf{S5(A)^C}$ (and corresponding one-variable fragments) for which $A$ is undecidable, but have a decidable validity problem. In contrast, none of the full first-order Gödel logics determined by these ordinals are recursively enumerable [11].

**Remark 3.26.** In a currently unpublished manuscript, Caicedo has extended the results in this section. He has provided a full classification of all logics $\mathsf{S5(A)^C}$, showing that for any Gödel set $A$, $\mathsf{S5(A)^C}$-validity coincides with $\mathsf{S5(B)^C}$-validity for some countable Gödel set $B$ obtained as in Corollary 3.23. It then follows that the logics $\mathsf{S5(A)^C}$ and $\mathsf{S5(A)}$ are co-NP-complete for any Gödel set $A$ and so also, by Theorems 3.3 and 3.12, that the one-variable fragments $\mathsf{CDIKL_1(K)}$ and $\mathsf{IKL_1(K)}$ are co-NP-complete for any countable linear frame $\mathbf{K}$.

# Monadic Abelian Logic

In this chapter we focus on the one-variable fragment of first-order Abelian logic, defined over the ordered additive group of the reals. Recall that (first-order) Abelian logic is closely connected to (first-order) Łukasiewicz logic. Indeed, in Theorem 1.15 we gave a translation $(-)^\bullet$ such that for any $\alpha \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\text{Ł}})$ that does not contain the unary predicate $P_0$,

$$\models^{\forall\exists}_{\text{Ł}} \alpha \iff \models^{\forall\exists}_{\mathbf{R}} \alpha^\bullet.$$

This translation restricts to the one-variable fragment. Under the modal translation, this gives an interpretation of $\mathsf{S5}(\text{Ł})^{\mathsf{C}}$ into $\mathsf{S5}(\mathbf{R})^{\mathsf{C}}$.

One advantage of studying Abelian logic over Łukasiewicz logic is that in Abelian logic, there exists a natural separation between the multiplicative (group) and additive (lattice) fragments. Recall that the propositional language of Abelian logic $\mathcal{L}_{\mathsf{A}}$ contains binary connectives $\wedge$, $\vee$, and $+$, a unary connective $-$, and a constant $\bar{0}$. The multiplicative fragment, where we consider the language $\mathcal{L}^{\mathsf{m}}_{\mathsf{A}} := \mathcal{L}_{\mathsf{A}} \setminus \{\wedge, \vee\}$, is the topic of Section 4.2. We propose an axiomatization for the one-variable fragment of this multiplicative first-order Abelian logic or, equivalently, the multiplicative fragment of $\mathsf{S5}(\mathbf{R})^{\mathsf{C}}$, and prove completeness syntactically. We make use of a normal form theorem, as well as a Herbrand theorem. The Herbrand theorem is proved in Section 4.1. In that section, we also use the Herbrand theorem to prove decidability of (full) $\mathsf{S5}(\mathbf{R})^{\mathsf{C}}$, and we prove a finite model property. In Section 4.3 we further focus on the full logic $\mathsf{S5}(\mathbf{R})^{\mathsf{C}}$. We propose an axiomatization and prove completeness via algebraic means. To be precise, we consider the variety of monadic abelian $\ell$-groups, as defined in Example 2.8, for which we prove a functional representation theorem. This representation theorem is a strengthening of the result from Corollary 2.45, making use of some additional properties of monadic abelian $\ell$-groups. We then establish completeness with respect to the real-valued semantics via a partial embedding lemma for linearly ordered abelian $\ell$-groups.

## 4.1 A Herbrand Theorem

In this section, we prove some general properties of first-order Abelian logic. In particular, we prove a Herbrand theorem for first-order Abelian logic. This Herbrand Theorem will prove useful in proving completeness for the multiplicative fragment of $\mathsf{S5}(\mathbf{R})^{\mathsf{C}}$ in

Section 4.2. Moreover, we use it to prove decidability of the one-variable fragment of first-order Abelian logic or, equivalently, of $\mathsf{S5}(\mathbf{R})^\mathsf{C}$. To formulate the Herbrand Theorem, we need to consider object constants. For the remainder of this chapter, we assume that the set of terms contains, in addition to object variables $x, y, z, \dots$, object constants $c, d, \dots$. We call a formula $\alpha \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_\mathsf{A})$ *constant-free* if $\alpha$ contains no such object constants, and *variable-free* if $\alpha$ contains no variables. For convenience, we often write $\bar{x}$ and $\bar{c}$ to denote an $n$-tuple of object variables or constants, respectively. Given tuples $\bar{c} = c_1, \dots, c_n$ and $\bar{d} = d_1, \dots, d_m$ of object constants, we write $\bar{d} \subseteq \bar{c}$ for $\{d_1, \dots, d_m\} \subseteq \{c_1, \dots, c_n\}$. We can now prove the Herbrand Theorem.

**Theorem 4.1.** *For any quantifier-free formula $\alpha \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_\mathsf{A})$ with free variables in $\bar{x} = x_1, \dots, x_m$ and constants in $\bar{c} = c_1, \dots, c_n$ with $n \in \mathbb{N}^+$,*

$$\models_{\mathbf{R}}^{\forall\exists} (\exists x_1) \dots (\exists x_m) \alpha(\bar{x}) \iff \models_{\mathbf{R}}^{\forall\exists} \bigvee \{\alpha(\bar{d}) \mid \bar{d} \subseteq \bar{c}\}.$$

*Proof.* The right-to-left direction follows using the easily-verified fact that $\beta(c) \to (\exists y)\beta(y)$ is $\mathbf{R}$-valid for any $\beta \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_\mathsf{A})$ and constant $c$. For the converse, we suppose contrapositively that $\bigvee\{\alpha(\bar{d}) \mid \bar{d} \subseteq \bar{c}\}$ is not $\mathbf{R}$-valid. Then there exists an $\mathbf{R}$-structure $\mathfrak{M} = \langle D, \mathcal{I} \rangle$ and $\mathfrak{M}$-evaluation $v$ such that $\|\alpha(\bar{d})\|_{\mathfrak{M},v}^{\mathbf{R}} < 0$ for all $\bar{d} \subseteq \bar{c}$. Consider now the $\mathbf{R}$-structure $\mathfrak{M}' := \langle D', \mathcal{I}' \rangle$ and $\mathfrak{M}'$-evaluation $v'$ such that $D' := \{v(c_1), \dots, v(c_n)\}$, $\mathcal{I}'$ maps each $P$ to the restriction of $\mathcal{I}(P)$ to $D'$, and $v'$ coincides on $c_1, \dots, c_n$ with $v$. Then

$$\|(\exists x_1) \dots (\exists x_m)\alpha(\bar{x})\|_{\mathfrak{M}',v'}^{\mathbf{R}} = \bigvee\{\|\alpha(\bar{d})\|_{\mathfrak{M},v}^{\mathbf{R}} \mid \bar{d} \subseteq \bar{c}\} < 0.$$

So $\not\models_{\mathbf{R}}^{\forall\exists} (\exists x_1) \dots (\exists x_m)\alpha(\bar{x})$. $\qquad\square$

**Remark 4.2.** We have shown this Herbrand theorem to hold for existential sentences $(\exists x_1) \dots (\exists x_m)\alpha(\bar{x})$ that contain only object variables and object constants. First-order Abelian logic with arbitrary function symbols does not admit a Herbrand theorem. It does however, as in the case of first-order Łukasiewicz logic, admit an "approximate" Herbrand theorem. For details on the Łukasiewicz case, see [10]. The proof in that paper can be adapted to the first-order Abelian logic that allows arbitrary function symbols. In fact, it also admits Skolemization.

For one-variable formulas, Theorem 4.1 can be used to prove decidability. To see this, note firstly that $\mathbf{R}$-validity is preserved by all quantifier shifts; that is, for all $\alpha, \beta \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_\mathsf{A})$, variable $x$ that does not occur free in $\beta$, and $\star \in \{\wedge, \vee, +\}$,

$$\models_{\mathbf{R}}^{\forall\exists} (\forall x)(\alpha \star \beta) \leftrightarrow ((\forall x)\alpha \star \beta) \qquad \models_{\mathbf{R}}^{\forall\exists} (\exists x)(\alpha \to \beta) \leftrightarrow ((\forall x)\alpha \to \beta)$$
$$\models_{\mathbf{R}}^{\forall\exists} (\exists x)(\alpha \star \beta) \leftrightarrow ((\exists x)\alpha \star \beta) \qquad \models_{\mathbf{R}}^{\forall\exists} (\forall x)(\beta \to \alpha) \leftrightarrow (\beta \to (\forall x)\alpha)$$
$$\models_{\mathbf{R}}^{\forall\exists} (\forall x)(\alpha \to \beta) \leftrightarrow ((\exists x)\alpha \to \beta) \qquad \models_{\mathbf{R}}^{\forall\exists} (\exists x)(\beta \to \alpha) \leftrightarrow (\beta \to (\exists x)\alpha),$$

where we recall that we write $\alpha \to \beta$ for $\beta + -\alpha$ and $\alpha \leftrightarrow \beta$ for $(\alpha \to \beta) \wedge (\beta \to \alpha)$. Now consider any one-variable formula $\alpha \in \mathrm{Fm}_1(\mathcal{L}_\mathsf{A})$. First, we replace all free occurences of $x$ with a single new object constant $c$. Then we iteratively replace each positive occurrence of a subformula $(\forall x)\beta(x)$ and negative occurrence of a subformula $(\exists x)\beta(x)$ with $\beta(d)$ for a new object constant $d$. Here, the occurrence of a subformula $\chi$ in $\alpha$ is called *positive* if $\chi$ occurs under an even number of negations $-$, and *negative* otherwise. Note that in this step, it is crucial that $\alpha$ is a one-variable formula. Indeed, for any subformula $(\forall x)\beta(x)$ or

$(\exists x)\beta(x)$ of $\alpha$, $\beta(x)$ does not contain any free variable other than $x$. Finally, we rename the bound variables and apply the quantifier shifts, pushing all remaining quantifiers outwards. This yields a sentence $\chi = (\exists x_1)\dots(\exists x_m)\chi'$ such that $\chi'$ is quantifier-free and

$$\models_{\mathbf{R}}^{\forall\exists} \alpha \iff \models_{\mathbf{R}}^{\forall\exists} \chi.$$

Theorem 4.1 then tells us that $\alpha$ is $\mathbf{R}$-valid if and only if a particular quantifier-free sentence containing some object constants is $\mathbf{R}$-valid. Checking the validity of such quantifier-free sentences in $\mathbf{R}$ is decidable via standard linear programming arguments. We obtain the following decidability result.

**Corollary 4.3.** *The one-variable fragment of first-order Abelian logic, and hence also* $\mathsf{S5}(\mathbf{R})^{\mathsf{C}}$, *is decidable.*

Finally, we show that, similar to $\mathsf{S5}(\mathbf{Ł})^{\mathsf{C}}$ but unlike $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$, $\mathsf{S5}(\mathbf{R})^{\mathsf{C}}$ enjoys a finite model property.

**Proposition 4.4.** *For any* $\varphi \in \mathrm{Fm}_{\square\lozenge}(\mathcal{L}_{\mathsf{A}})$, $\models_{\mathsf{S5}(\mathbf{R})^{\mathsf{C}}} \varphi$ *if and only if* $\varphi$ *is* $\mathsf{S5}(\mathbf{R})^{\mathsf{C}}$*-valid in all universal* $\mathsf{S5}(\mathbf{R})^{\mathsf{C}}$*-models* $\langle W, V \rangle$ *where* $W$ *is finite.*

*Proof.* It suffices to prove that for any universal $\mathsf{S5}(\mathbf{R})^{\mathsf{C}}$-model $\langle W, V \rangle$, finite set of formulas $\Sigma$, finite set $X \subseteq W$ and $\varepsilon > 0$, there exists a universal $\mathsf{S5}(\mathbf{R})^{\mathsf{C}}$-model $\langle W', V' \rangle$ such that $X \subseteq W' \subseteq W$, $W'$ is finite and for all $\varphi \in \Sigma$, $w \in W'$, $\left| \bar{V}'(\varphi, w) - \bar{V}(\varphi, w) \right| < \varepsilon$. Indeed, if some $\varphi \in \mathrm{Fm}_{\square\lozenge}(\mathcal{L}_{\mathsf{A}})$ is not $\mathsf{S5}(\mathbf{R})^{\mathsf{C}}$-valid, there exists a universal $\mathsf{S5}(\mathbf{R})^{\mathsf{C}}$-model $\langle W, V \rangle$ and $w \in W$ such that $\bar{V}(\varphi, w) < 0$. We can then apply the claim with $\Sigma = \{\varphi\}$, $X = \{w\}$, and $\varepsilon = \left| \bar{V}(\varphi, w) \right|$. We prove the claim by induction on the sum of the length of formulas in $\Sigma$.

For the base case, $\Sigma$ contains only propositional variables or $\bar{0}$. We can then define the universal $\mathsf{S5}(\mathbf{R})^{\mathsf{C}}$-model $\langle X, V' \rangle$ where $V'(p_i, w) := V(p_i, w)$ for all $i \in \mathbb{N}$, $w \in X$. It follows directly that $\bar{V}(\varphi, w) = \bar{V}'(\varphi, w)$ for all $\varphi \in \Sigma$, $w \in X$.

Now suppose that $\Sigma = \Sigma'' \cup \{\varphi + \psi\}$. We apply the induction hypothesis to $\Sigma' := \Sigma'' \cup \{\varphi, \psi\}$, $X$ and $\varepsilon/2$ to obtain a universal $\mathsf{S5}(\mathbf{R})^{\mathsf{C}}$-model $\langle W', V' \rangle$ such that $X \subseteq W'$ and $\left| \bar{V}'(\chi, w) - \bar{V}(\chi, w) \right| < \varepsilon/2$ for all $\chi \in \Sigma'$, $w \in W'$. Then

$$\begin{aligned}
\left| \bar{V}'(\varphi + \psi, w) - \bar{V}(\varphi + \psi, w) \right| &= \left| \bar{V}'(\varphi, w) - \bar{V}(\varphi, w) + \bar{V}'(\psi, w) - \bar{V}(\psi, w) \right| \\
&\leq \left| \bar{V}'(\varphi, w) - \bar{V}(\varphi, w) \right| + \left| \bar{V}'(\psi, w) - \bar{V}(\psi, w) \right| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

The case for $-\varphi$ is similar. Suppose that $\Sigma = \Sigma'' \cup \{\varphi \wedge \psi\}$. We apply the induction hypothesis to $\Sigma' := \Sigma'' \cup \{\varphi, \psi\}$, $X$ and $\varepsilon$ to obtain an appropriate universal $\mathsf{S5}(\mathbf{R})^{\mathsf{C}}$-model $\langle W', V' \rangle$. Consider $w \in W'$ and assume without loss of generality that $\bar{V}(\varphi, w) \leq \bar{V}(\psi, w)$. If $\bar{V}'(\varphi, w) \leq \bar{V}'(\psi, w)$, then

$$\left| \bar{V}'(\varphi \wedge \psi, w) - \bar{V}(\varphi \wedge \psi, w) \right| = \left| \bar{V}'(\varphi, w) - \bar{V}(\varphi, w) \right| < \varepsilon.$$

If $\bar{V}'(\psi, w) \leq \bar{V}'(\varphi, w)$, it follows that $\left| \bar{V}'(\psi, w) - \bar{V}(\varphi, w) \right| < \varepsilon$ from $\bar{V}(\varphi, w) \leq \bar{V}(\psi, w)$, $\left| \bar{V}'(\varphi, w) - \bar{V}(\varphi, w) \right| < \varepsilon$, and $\left| \bar{V}'(\psi, w) - \bar{V}(\psi, w) \right| < \varepsilon$.

Finally, we can assume that $\Sigma$ consists only of formulas $\Box\varphi_1, \ldots, \Box\varphi_n, \Diamond\psi_1, \ldots, \Diamond\psi_m$ and propositional variables. For each $\Box\varphi_i$ and $\Diamond\psi_j$, we pick $v_i \in W$ and $u_j \in W$ such that

$$\left|\bar{V}(\Box\varphi_i, v_i) - \bar{V}(\varphi, v_i)\right| < \frac{\varepsilon}{4} \text{ and } \left|\bar{V}(\Diamond\psi_j, u_j) - \bar{V}(\psi_j, u_j)\right| < \frac{\varepsilon}{4},$$

respectively. We then apply the induction hypothesis to

$$\Sigma' \coloneqq (\Sigma \setminus \{\Box\varphi_1, \ldots, \Box\varphi_n, \Diamond\psi_1, \ldots, \Diamond\psi_m\}) \cup \{\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_m\},$$

$X' \coloneqq X \cup \{v_1, \ldots, v_n, u_1, \ldots, u_m\}$ and $\varepsilon/4$ to obtain an appropriate $\mathsf{S5(R)^C}$-model $\langle W', V' \rangle$. Since $W'$ is finite, there exists $v_i' \in W'$ such that $\bar{V}'(\Box\varphi_i, v_i') = \bar{V}'(\varphi_i, v_i')$ for each $i \leq n$, and $u_j' \in W'$ such that $\bar{V}'(\Diamond\psi_j, u_j') = \bar{V}'(\psi_j, u_j')$ for each $j \leq m$. Therefore, for each $i \leq n$ and $w \in W'$,

$$\begin{aligned}
\left|\bar{V}'(\Box\varphi_i, w) - \bar{V}(\Box\varphi_i, w)\right| &= \left|\bar{V}'(\varphi_i, v_i') - \bar{V}(\Box\varphi_i, w)\right| \\
&\leq \left|\bar{V}'(\varphi_i, v_i') - \bar{V}(\varphi_i, v_i')\right| + \left|\bar{V}(\varphi_i, v_i') - \bar{V}(\Box\varphi, w)\right| \\
&< \frac{\varepsilon}{4} + \left|\bar{V}(\varphi_i, v_i') - \bar{V}(\Box\varphi, w)\right| \\
&\leq \frac{\varepsilon}{4} + \left|\bar{V}(\varphi_i, v_i) - \bar{V}(\Box\varphi, w)\right| + \left|\bar{V}(\varphi_i, v_i') - \bar{V}(\varphi_i, v_i)\right| \\
&< \frac{\varepsilon}{2} + \left|\bar{V}(\varphi_i, v_i') - \bar{V}(\varphi_i, v_i)\right|.
\end{aligned}$$

To finish the approximation, we suppose for a contradiction that $\left|\bar{V}(\varphi_i, v_i') - \bar{V}(\varphi_i, v_i)\right| \geq \varepsilon/2$. Note that by definition we have $\bar{V}(\Box\varphi_i, w) \leq \bar{V}(\varphi_i, v_i')$ and $\bar{V}(\Box\varphi_i, w) \leq \bar{V}(\varphi_i, v_i)$, so the assumption and $\left|\bar{V}(\Box\varphi_i, w) - \bar{V}(\varphi_i, v_i)\right| < \varepsilon/4$ yields that $\bar{V}(\varphi_i, v_i) < \bar{V}(\varphi_i, v_i')$. Therefore, as $\left|\bar{V}'(\varphi_i, v_i) - \bar{V}(\varphi_i, v_i)\right| < \varepsilon/4$, $\left|\bar{V}'(\varphi_i, v_i') - \bar{V}(\varphi_i, v_i')\right| < \varepsilon/4$ and our assumption, we obtain $\bar{V}'(\varphi_i, v_i) < \bar{V}'(\varphi_i, v_i') = \bar{V}'(\Box\varphi_i, w)$, a contradiction. It follows that

$$\begin{aligned}
\left|\bar{V}'(\Box\varphi_i, w) - \bar{V}(\Box\varphi_i, w)\right| &< \frac{\varepsilon}{2} + \left|\bar{V}(\varphi_i, v_i') - \bar{V}(\varphi_i, v_i)\right| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Similarly, we can show that $\left|\bar{V}'(\Diamond\psi_j, w) - \bar{V}(\Diamond\psi_j, w)\right| < \varepsilon$ for any $j \leq m, w \in W$. $\qquad\square$

Although this finite model property does not (directly) provide an alternative proof of decidability, it does show that to determine $\mathsf{S5(R)^C}$-validity, it suffices to consider $\mathsf{S5(R)^C}$-models $\langle W, V \rangle$ that are "witnessed". That is, for each $\varphi \in \mathrm{Fm}_{\Box\Diamond}(\mathcal{L}_\mathsf{A})$,

$$\begin{aligned}
\bar{V}(\Box\varphi, w) &= \min\{\bar{V}(\varphi, v) \mid v \in W\} \\
\bar{V}(\Diamond\varphi, w) &= \max\{\bar{V}(\varphi, v) \mid v \in W\}.
\end{aligned}$$

In Section 4.3, we provide an algebraic proof of this fact.

## 4.2 The Multiplicative Fragment

In this section, we focus on the *multiplicative* fragment of first-order Abelian logic, that is, the fragment concerned with formulas not containing the lattice connectives $\wedge$ and $\vee$.[1] We let $\mathcal{L}_A^m$ denote the language $\mathcal{L}_A \setminus \{\wedge, \vee\}$. The goal of this section is to prove completeness for the multiplicative fragment of the one-variable fragment of first-order Abelian logic or, equivalently, the multiplicative fragment of $\mathsf{S5}(\mathbf{R})^\mathsf{C}$. Note that such a multiplicative fragment has been considered for a weaker modal logic in [61]. To be more precise, in [61] the authors consider a modal logic $\mathsf{K}(\mathbf{R})$ defined by a real-valued Kripke semantics that is similar to $\mathsf{S5}(\mathbf{R})^\mathsf{C}$-models, but where the relation $R$ is a classical two-valued relation, without any additional properties. Completeness for the multiplicative fragment of $\mathsf{K}(\mathbf{R})$ is proved in [61], but completeness for the full logic is still an open problem. In the next section, we prove completeness for the (full) stronger logic $\mathsf{S5}(\mathbf{R})^\mathsf{C}$.

Let us first consider the proof system $\mathcal{HA}_m$ by removing the axiom and rule schema for $\wedge$ and $\vee$ from the system $\mathcal{HA}$ presented in Figure 1.3. This proof system is complete with respective to the multiplicative fragment of propositional Abelian logic defined as $\mathsf{A}_m = \langle \mathcal{L}_A^m, \langle \langle \mathbb{R}, +, -, 0 \rangle, \mathbb{R}^{\geq 0} \rangle \rangle$. A proof can be found in, e.g., [43].

**Proposition 4.5** ([43]). *For all $\varphi \in \mathrm{Fm}(\mathcal{L}_A^m)$,*

$$\vdash_{\mathcal{HA}_m} \varphi \iff \models_{\mathsf{A}_m} \varphi.$$

We note the following useful property of $\mathsf{A}_m$.

**Lemma 4.6.** *Let $\varphi \in \mathrm{Fm}(\mathcal{L}_A^m)$. Then $\varphi$ is $\mathsf{A}_m$-valid if and only if $\bar{V}(\varphi) = 0$ for all $\mathsf{A}_m$-valuations $V$.*

*Proof.* The right-to-left direction follows directly. For the other direction, suppose that $\varphi$ is $\mathsf{A}_m$-valid and, for a contradiction, assume that $\bar{V}(\varphi) > 0$ for some $\mathsf{A}_m$-valuation $V$. Then define the $\mathsf{A}_m$-valuation $V'$ with $V'(p_i) := -V(p_i)$ for each $i \in \mathbb{N}$. A straightforward induction on the length of $\varphi$ then shows that $\bar{V}'(\varphi) = -\bar{V}(\varphi) < 0$, contradicting the $\mathsf{A}_m$-validity of $\varphi$. $\qquad\square$

This result extends to quantifier-free formulas $\alpha \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_A^m)$ and formulas $\varphi \in \mathrm{Fm}_{\square\Diamond}(\mathcal{L}_A^m)$ that contain no modalities. The absence of the lattice connectives as well as the quantifiers and modalities, respectively, is essential here. Indeed, consider the $\mathsf{S5}(\mathbf{R})^\mathsf{C}$-valid formulas $(p_1 \wedge p_2) \to p_1$ and $\square p_1 \to p_1$. Now consider any $\mathsf{S5}(\mathbf{R})^\mathsf{C}$-model $\langle W, V \rangle$ where $W = \{w_1, w_2\}$, $V(p_1, w_1) = 2$, and $V(p_2, w_1) = V(p_1, w_2) = 1$. It follows that $\bar{V}((p_1 \wedge p_2) \to p_1, w_1) = \bar{V}(\square p_1 \to p_1, w_1) = 1 > 0$.

Now let $\mathcal{S}5(\mathcal{A}_m)$ denote the proof system that extends $\mathcal{HA}_m$ with the modal axiom and rule schema from Figure 4.1, as well the as rule schema

$$\frac{n\varphi}{\varphi} \ (\mathrm{con}_n) \quad (n \geq 2).$$

Soundness of this system is easy to check.

**Lemma 4.7.** *Let $\varphi \in \mathrm{Fm}_{\square\Diamond}(\mathcal{L}_A^m)$. If $\vdash_{\mathcal{S}5(\mathcal{A}_m)} \varphi$, then $\models_{\mathsf{S5}(\mathbf{R})^\mathsf{C}} \varphi$.*

---

[1]Note that we follow here standard terminology from the linear and substructural logic literature in referring to the *multiplicative* fragment of Abelian logic, even though the group multiplication for the real numbers is in fact addition.

$$
\begin{array}{ll}
(\text{K}) & \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \\
(\text{T}) & \Box\varphi \to \varphi \\
(5) & \Diamond\varphi \to \Box\Diamond\varphi \\
(\text{M}) & \Box(\varphi + \varphi) \to (\Box\varphi + \Box\varphi)
\end{array}
$$

$$
\frac{\varphi}{\Box\varphi} \ (\text{nec})
$$

Figure 4.1: Modal Axiom and Rule Schema

To prove completeness, we require some additional properties. First, we show that occurrences of $\Box$ and $\Diamond$ can be shifted inwards, showing that each formula $\varphi \in \mathrm{Fm}_{\Box\Diamond}(\mathcal{L}_{\mathsf{A}}^{\mathsf{m}})$ is provably equivalent in $\mathcal{S}5(\mathcal{A}_{\mathsf{m}})$ to a formula without nested modalities. For $\varphi, \psi \in \mathrm{Fm}_{\Box\Diamond}(\mathcal{L}_{\mathsf{A}}^{\mathsf{m}})$, let us write $\vdash_{\mathcal{S}5(\mathcal{A}_{\mathsf{m}})} \varphi \equiv \psi$ to denote that $\vdash_{\mathcal{S}5(\mathcal{A}_{\mathsf{m}})} \varphi \to \psi$ and $\vdash_{\mathcal{S}5(\mathcal{A}_{\mathsf{m}})} \psi \to \varphi$.

**Lemma 4.8.** *For any $\varphi, \psi \in \mathrm{Fm}_{\Box\Diamond}(\mathcal{L}_{\mathsf{A}}^{\mathsf{m}})$,*

(i) $\vdash_{\mathcal{S}5(\mathcal{A}_{\mathsf{m}})} \Box(\varphi + \Box\psi) \equiv \Box\varphi + \Box\psi$

(ii) $\vdash_{\mathcal{S}5(\mathcal{A}_{\mathsf{m}})} \Box(\varphi + \Diamond\psi) \equiv \Box\varphi + \Diamond\psi$

(iii) $\vdash_{\mathcal{S}5(\mathcal{A}_{\mathsf{m}})} \Box\Box\varphi \equiv \Box\varphi$

(iv) $\vdash_{\mathcal{S}5(\mathcal{A}_{\mathsf{m}})} \Box\Diamond\varphi \equiv \Diamond\varphi$

(v) $\vdash_{\mathcal{S}5(\mathcal{A}_{\mathsf{m}})} \Box n\varphi \equiv n\Box\varphi$ *for all $n \in \mathbb{N}$*

(vi) $\vdash_{\mathcal{S}5(\mathcal{A}_{\mathsf{m}})} \Box(\varphi + \psi) \to (\Box\varphi + \Diamond\psi)$

(vii) $\vdash_{\mathcal{S}5(\mathcal{A}_{\mathsf{m}})} \Diamond(\varphi + \psi) \to (\Diamond\varphi + \Diamond\psi)$.

*Proof.* Derivations for (i)–(iv) are obtained, similarly to other "S5" logics, using the modal axiom schema (K), (T), and (5), and are omitted here. For (v), we note first that $n\Box\varphi \to \Box n\varphi$ is derivable in $\mathcal{S}5(\mathcal{A}_{\mathsf{m}})$ for $n \in \mathbb{N}$ using (nec) and (K) together with the axioms of $\mathcal{H}\mathcal{A}_m$. For the converse, observe that $\Box(2^k)\varphi \to (2^k)\Box\varphi$ is derivable in $\mathcal{S}5(\mathcal{A}_{\mathsf{m}})$ for $k \in \mathbb{N}$ using repeated applications of (M), (mp), and the axiom $(+1)$. But then also for any $n \geq 1$, we can choose $k \in \mathbb{N}$ such that $2^k \geq n$ and observe that $(\Box n\varphi + (2^k - n)\Box\varphi) \to \Box(2^k)\varphi$ and hence $(\Box n\varphi + (2^k - n)\Box\varphi) \to (2^k)\Box\varphi$ are derivable in $\mathcal{S}5(\mathcal{A}_{\mathsf{m}})$. Since $(((2^k - n)\Box\varphi) \to ((2^k - n)\Box\varphi)) \to \bar{0}$ is derivable in $\mathcal{S}5(\mathcal{A}_{\mathsf{m}})$, also $\Box n\varphi \to n\Box\varphi$ is derivable in $\mathcal{S}5(\mathcal{A}_{\mathsf{m}})$ as required. Finally, for the case $n = 0$ just note that $\Box\bar{0} \to \bar{0}$ is an instance of (T).

A derivation for (vi) is obtained using (K) and the axioms of $\mathcal{H}\mathcal{A}_m$. For a derivation of (vii), note that since $\varphi \to \Diamond\varphi$ is $\mathcal{S}5(\mathcal{A}_{\mathsf{m}})$-derivable, so is $(\varphi + \psi) \to (\Diamond\varphi + \Diamond\psi)$. It follows using (nec) and (K) that $\Diamond(\varphi + \psi) \to \Diamond(\Diamond\varphi + \Diamond\psi)$ is $\mathcal{S}5(\mathcal{A}_{\mathsf{m}})$-derivable. Using that $\vdash_{\mathcal{S}5(\mathcal{A}_{\mathsf{m}})} \Diamond(\Diamond\varphi + \Diamond\psi) \equiv \Diamond\varphi + \Diamond\psi$, we obtain a derivation for $\Diamond(\varphi + \psi) \to (\Diamond\varphi + \Diamond\psi)$. $\quad\square$

Let us write $\sum_{i=1}^{n} \varphi_i$ to denote $\varphi_1 + \ldots + \varphi_n$ for any $\varphi_1, \ldots, \varphi_n \in \mathrm{Fm}_{\Box\Diamond}(\mathcal{L}_{\mathsf{A}})$. An easy induction on formula length using Lemma 4.8 (i)–(iv) yields the following normal form property for formulas in $\mathrm{Fm}_{\Box\Diamond}(\mathcal{L}_{\mathsf{A}}^{\mathsf{m}})$.

**Lemma 4.9.** *Let $\varphi \in \mathrm{Fm}_{\square\lozenge}(\mathcal{L}_{\mathsf{A}}^{\mathsf{m}})$. Then there exist formulas $\varphi_0, \ldots, \varphi_n, \psi_1, \ldots, \psi_m \in \mathrm{Fm}(\mathcal{L}_{\mathsf{A}}^{\mathsf{m}})$ such that*

$$\vdash_{\mathcal{S}5(\mathcal{A}_{\mathsf{m}})} \varphi \equiv \varphi_0 + \sum_{i=1}^{n} \square\varphi_i + \sum_{j=1}^{m} \lozenge\psi_j.$$

**Remark 4.10.** It is not possible to obtain a similar normal form property for all $\varphi \in \mathrm{Fm}_{\square\lozenge}(\mathcal{L}_{\mathsf{A}})$ simply by shifting boxes; e.g., $\square(p_1 \vee (p_2 + \square p_3))$ is not equivalent to a formula that has no nested modalities.

Secondly, we make use of a well-known duality principle for linear programming stating that either one or another linear system has a solution, but not both (see, e.g., [58, p. 136]). More precisely, we use a result known as Gordan's Theorem, that states that for any $M \in \mathbb{Z}^{m \times n}$, either $y^T M < \mathbf{0}$ for some $y \in \mathbb{R}^m$ or $Mx = \mathbf{0}$ for some $x \in \mathbb{N}^n \backslash \{\mathbf{0}\}$. The following lemma is a consequence of this principle.

**Lemma 4.11.** *For any formulas $\alpha_1, \ldots, \alpha_n \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{A}}^{\mathsf{m}})$ that are quantifier- and variable-free,*

$$\models_{\mathbf{R}}^{\forall\exists} \alpha_1 \vee \cdots \vee \alpha_n \iff \models_{\mathbf{R}}^{\forall\exists} \sum_{j=1}^{n} \lambda_j \alpha_j \text{ for some } \lambda_1, \ldots, \lambda_n \in \mathbb{N} \text{ that are not all } 0.$$

*Proof.* Let $\beta_1, \ldots, \beta_m$ denote all predicates $P_i(c)$ occurring in $\alpha_1, \ldots, \alpha_n$. We can assume without loss of generality that each $\alpha_j$ is of the form $\sum_{i=1}^{m} m_{ij}\beta_j$, where $M = (m_{ij}) \in \mathbb{Z}^{m \times n}$. It can now be checked that $\alpha_1 \vee \cdots \vee \alpha_n$ is not $\mathbf{R}$-valid if and only if there exists $y \in \mathbb{R}^m$ such that $y^T M < \mathbf{0}$, by, for each $i = 1, \ldots, m$, identifying the coordinates $y_i$ of $y$ with $\|\beta_i\|_{\mathfrak{M},v}^{\mathbf{R}}$ for an $\mathbf{R}$-structure $\mathfrak{M}$ and $\mathfrak{M}$-evaluation $v$ in which $\alpha_1 \vee \cdots \vee \alpha_n$ fails. Hence, by the duality principle mentioned above, the $\mathbf{R}$-validity of $\alpha_1 \vee \cdots \vee \alpha_n$ is equivalent to $Mx = \mathbf{0}$ for some $x \in \mathbb{N}^n \setminus \{\mathbf{0}\}$. The latter condition is in turn equivalent to the existence of $x \in \mathbb{N}^n \setminus \{\mathbf{0}\}$ such that $x^T M^T y = 0$ for all $y \in \mathbb{R}^m$. By construction of $M$, this happens if and only if there exists $x \in \mathbb{N}^n \setminus \{\mathbf{0}\}$ such that $\sum_{j=1}^{n} x_j \|\alpha_j\|_{\mathfrak{M},v}^{\mathbf{R}} = 0$ for all $\mathbf{R}$-structures $\mathfrak{M}$ and $\mathfrak{M}$-evaluations $v$, which finally corresponds to the existence of $x_1, \ldots, x_n \in \mathbb{N}$ not all zero such that $\sum_{j=1}^{n} x_j \alpha_j$ is $\mathbf{R}$-valid. $\square$

We now have all the tools necessary to prove the completeness theorem for $\mathcal{S}5(\mathcal{A}_{\mathsf{m}})$.

**Theorem 4.12.** *For all $\varphi \in \mathrm{Fm}_{\square\lozenge}(\mathcal{L}_{\mathsf{A}}^{\mathsf{m}})$,*

$$\vdash_{\mathcal{S}5(\mathcal{A}_{\mathsf{m}})} \varphi \iff \models_{\mathsf{S5}(\mathbf{R})^{\mathsf{C}}} \varphi.$$

*Proof.* The left-to-right direction follows from Lemma 4.7. For the converse, suppose that $\varphi$ is $\mathsf{S5}(\mathbf{R})^{\mathsf{C}}$-valid. By Lemma 4.9, there exists formulas $\varphi_0, \ldots, \varphi_n, \psi_1, \ldots, \psi_m \in \mathrm{Fm}(\mathcal{L}_{\mathsf{A}}^{\mathsf{m}})$ such that $\vdash_{\mathcal{S}5(\mathcal{A}_{\mathsf{m}})} \varphi \equiv \psi$, where

$$\psi = \varphi_0 + \sum_{i=1}^{n} \square\varphi_i + \sum_{j=1}^{m} \lozenge\psi_j.$$

To show that $\vdash_{\mathcal{S}5(\mathcal{A}_{\mathsf{m}})} \varphi$, it now suffices to show that $\vdash_{\mathcal{S}5(\mathcal{A}_{\mathsf{m}})} \psi$. As $\varphi$ is $\mathsf{S5}(\mathbf{R})^{\mathsf{C}}$-valid, it follows that under the first-order translation,

$$\models_{\mathbf{R}}^{\forall\exists} \varphi_0^{\circ}(x) + \sum_{i=1}^{n} (\forall x)\varphi_i^{\circ}(x) + \sum_{j=1}^{m} (\exists x)\psi_j^{\circ}(x).$$

Let us now leave the one-variable setting, and rename each variable in $\psi^{\circ}$ with a fresh variable, giving

$$\models_{\mathbf{R}}^{\forall\exists} \varphi_0^{\circ}(x_0) + \sum_{i=1}^{n}(\forall x_i)\varphi_i^{\circ}(x_i) + \sum_{j=1}^{m}(\exists x_{n+j})\psi_j^{\circ}(x_{n+j}).$$

We then use the quantifier shifts mentioned in Section 4.1 to bring the formula in prenex form, and quantify over $x_0$, giving

$$\models_{\mathbf{R}}^{\forall\exists} (\forall x_0)\ldots(\forall x_n)(\exists x_{n+1})\ldots(\exists x_{n+m})(\sum_{i=0}^{n}\varphi_i^{\circ}(x_i) + \sum_{j=1}^{m}\psi_j^{\circ}(x_{n+j})).$$

It follows that for object constants $\bar{c} = c_0,\ldots,c_n$,

$$\models_{\mathbf{R}}^{\forall\exists} (\exists x_{n+1})\ldots(\exists x_{n+m})(\sum_{i=0}^{n}\varphi_i^{\circ}(c_i) + \sum_{j=1}^{m}\psi_j^{\circ}(x_{n+j})).$$

We can now apply the Herbrand Theorem (Theorem 4.1) to deduce that

$$\models_{\mathbf{R}}^{\forall\exists} \bigvee\{\sum_{i=0}^{n}\varphi_i^{\circ}(c_i) + \sum_{j=1}^{m}\psi_j^{\circ}(d_j) \mid \{d_1,\ldots,d_m\} \subseteq \{c_0,\ldots,c_n\}\}.$$

The duality principle from Lemma 4.11 now gives us $\lambda_{\bar{d}} \in \mathbb{N}$ for each $\bar{d} \subseteq \bar{c}$ that are not all zero such that

$$\models_{\mathbf{R}}^{\forall\exists} \sum_{\bar{d}\subseteq\bar{c}}\lambda_{\bar{d}}\left(\sum_{i=0}^{n}\varphi_i^{\circ}(c_i) + \sum_{j=1}^{m}\psi_j^{\circ}(d_j)\right).$$

Rewriting this, with $\mu := \sum_{\bar{d}\subseteq\bar{c}}\lambda_{\bar{d}}$, we obtain

$$\models_{\mathbf{R}}^{\forall\exists} \sum_{i=0}^{n}\mu\,\varphi_i^{\circ}(c_i) + \sum_{\bar{d}\subseteq\bar{c}}\lambda_{\bar{d}}\sum_{j=1}^{m}\psi_j^{\circ}(d_j).$$

Regrouping the second part of this formula gives

$$\models_{\mathbf{R}}^{\forall\exists} \sum_{i=0}^{n}\mu\,\varphi_i^{\circ}(c_i) + \sum_{i=0}^{n}\sum_{j=1}^{m}\lambda_{ij}\,\psi_j^{\circ}(c_j),$$

where $\lambda_{ij} \in \mathbb{N}$ for $i = 0,\ldots,n$, $j = 1,\ldots,m$, and $\sum_{i=0}^{n}\lambda_{ij} = \mu$ for each $j = 1,\ldots,m$. Another reformulation then gives

$$\models_{\mathbf{R}}^{\forall\exists} \sum_{i=0}^{n}\left(\mu\,\varphi_i^{\circ}(c_i) + \sum_{j=1}^{m}\lambda_{ij}\,\psi_j^{\circ}(c_i)\right).$$

Since the object constants in each member of the summand are distinct, it follows that for each $i = 0,\ldots,n$,

$$\models_{\mathbf{R}}^{\forall\exists} \mu\,\varphi_i^{\circ}(c_i) + \sum_{j=1}^{m}\lambda_{ij}\,\psi_j^{\circ}(c_i),$$

and hence

$$\models_{\mathbf{R}}^{\forall\exists} \mu\,\varphi_i^{\circ}(x) + \sum_{j=1}^{m} \lambda_{ij}\,\psi_j^{\circ}(x).$$

Moving back to the modal setting, we obtain that for all $i = 0, \ldots, n$,

$$\models_{\mathsf{S5(R)^C}} \mu\,\varphi_i + \sum_{j=1}^{m} \lambda_{ij}\,\psi_j.$$

Since these formulas do not contain any modalities, Proposition 4.5 tells us that for each $i = 0, \ldots, n$,

$$\vdash_{\mathcal{S}5(\mathcal{A}_{\mathsf{m}})} \mu\,\varphi_i + \sum_{j=1}^{m} \lambda_{ij}\,\psi_j.$$

We can now apply (nec) and use formulas (vi) and (vii) from Lemma 4.8 to obtain for all $i = 1, \ldots, n$,

$$\vdash_{\mathcal{S}5(\mathcal{A}_{\mathsf{m}})} \mu\,\varphi_0 + \sum_{j=1}^{m} \lambda_{0j}\,\Diamond\psi_j \quad \text{and} \quad \vdash_{\mathcal{S}5(\mathcal{A}_{\mathsf{m}})} \mu\,\Box\varphi_i + \sum_{j=1}^{m} \lambda_{ij}\,\Diamond\psi_j,$$

where we also used (T) in the case $i = 0$. Combining these results using the axiom schema (+1) gives

$$\vdash_{\mathcal{S}5(\mathcal{A}_{\mathsf{m}})} \mu\,\varphi_0 + \sum_{i=1}^{n} \mu\,\Box\varphi_i + \sum_{i=0}^{n}\sum_{j=1}^{m} \lambda_{ij}\,\Diamond\psi_j.$$

We recall that $\mu = \sum_{i=0}^{n} \lambda_{ij}$ for each $j = 1, \ldots, m$. Using this fact and some rewriting gives

$$\vdash_{\mathcal{S}5(\mathcal{A}_{\mathsf{m}})} \mu\,\varphi_0 + \mu\sum_{i=1}^{n} \Box\varphi_i + \mu\sum_{j=1}^{m} \Diamond\psi_j.$$

An application of $(\mathrm{con}_\mu)$ then finally shows that $\vdash_{\mathcal{S}5(\mathcal{A}_{\mathsf{m}})} \psi$, as required. $\qquad\square$

Let us finally note that, as for first-order classical logic, the monadic fragment of multiplicative first-order Abelian logic coincides (up to equivalence of sentences) with its one-variable fragment. Indeed, consider any sentence $\alpha \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{A}}^{\mathsf{m}})$. We apply the $\mathbf{R}$-valid quantifier shifts $(\forall x)(\alpha_1 + \alpha_2) \leftrightarrow ((\forall x)\alpha_1 + \alpha_2)$ and $(\exists x)(\alpha_1 + \alpha_2) \leftrightarrow ((\exists x)\alpha_1 + \alpha_2)$ (where $x$ is not free in $\alpha_2$) repeatedly, pushing quantifiers inwards. We then obtain a sentence $\beta \in \mathrm{Fm}_{\forall\exists}(\mathcal{L}_{\mathsf{A}}^{\mathsf{m}})$ such that $\models_{\mathbf{R}}^{\forall\exists} \alpha \leftrightarrow \beta$, and no subformula $(\forall x)\beta'$ or $(\exists x)\beta'$ contains a free variable different to $x$. We can hence rename all variables to a single variable to obtain a sentence $\chi \in \mathrm{Fm}_1(\mathcal{L}_{\mathsf{A}}^{\mathsf{m}})$ such that $\models_{\mathbf{R}}^{\forall\exists} \alpha \leftrightarrow \chi$. Since $\mathsf{S5(R)^C}$ is decidable by Corollary 4.3, multiplicative first-order Abelian logic provides a first interesting example (as far as we know) of a first-order infinite-valued logic with a decidable monadic fragment.

## 4.3  The Full One-Variable Fragment

We will now consider the one-variable fragment of first-order Abelian logic, i.e., the logic $\mathsf{S5(R)^C}$, over the full language, including the lattice operations. The goal of this

section is to prove completeness of a proof system for $\mathsf{S5}(\mathbf{R})^{\mathsf{C}}$. More precisely, we show completeness of $\mathcal{S}5(\mathcal{A})$, which denotes the proof system extending $\mathcal{HA}$ with the modal axiom and rule schema from Figure 4.1 together with the axiom schema

$$(\wedge\Box) \quad (\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$$
$$(\wedge\Diamond) \quad (\Diamond\varphi \wedge \Diamond\psi) \to \Diamond(\varphi \wedge \Diamond\psi).$$

Completeness will be established through algebraic methods, following a method similar to that of Section 2.4. In particular, we strengthen the generation result of the variety of monadic abelian $\ell$-groups from Corollary 2.45, and then use a finite embedding result of abelian $\ell$-groups into $\mathbf{R}$ to obtain the desired completeness.

To express these results, we first require the necessary terminology from the theory of abelian $\ell$-groups. For the details, we refer to, e.g., [1]. An abelian $\ell$-group $\mathbf{A}$ is called an *abelian o-group* if the lattice order of $\mathbf{A}$ is linear. A non-empty subset $B \subseteq A$ that is closed under the operations of $\mathbf{A}$ forms an $\ell$-*subgroup* $\mathbf{B}$ of $\mathbf{A}$, where $\mathbf{B}$ is called an $\ell$-*ideal* of $\mathbf{A}$ if it is also convex. For an abelian $\ell$-group, its $\ell$-ideals play the same role as the convex $f$-free subuniverses do for an $\mathsf{FL}_e$-algebra; that is, the lattice of congruences of $\mathbf{A}$ is isomorphic to the lattice of its $\ell$-ideals, where both lattices are ordered by inclusion.[2] Using this isomorphism, we can consider the quotient $\mathbf{A}/\mathbf{K}$ of $\mathbf{A}$ by an $\ell$-ideal $\mathbf{K}$. We can represent this quotient by the right cosets of $\mathbf{K}$ in $\mathbf{A}$. Indeed, the set of right cosets of $\mathbf{K}$ in $\mathbf{A}$ forms an abelian $\ell$-group $\mathbf{A}/\mathbf{K}$ with lattice order $K + a \leq K + b :\Leftrightarrow a \leq b + c$ for some $c \in K$. An $\ell$-ideal $\mathbf{K}$ of $\mathbf{A}$ is called *prime* if $\mathbf{A}/\mathbf{K}$ is linearly ordered. For an element $a \in A$ and $\ell$-ideal $\mathbf{K}$ of $\mathbf{A}$, we write $\mathbf{K}(a)$ to denote the smallest $\ell$-ideal of $\mathbf{A}$ containing $K \cup \{a\}$. Some useful properties of $\ell$-ideals are summarized in the following lemma. We define $|b| := b \vee -b$ for any $b \in A$.

**Lemma 4.13** (cf. [1, Proposition 1.2.3, Theorem 1.2.10]). *Let $\mathbf{A}$ be an abelian $\ell$-group, $\mathbf{K}$ an $\ell$-ideal of $\mathbf{A}$, and $a, b \in A$. Then:*

(1) $K(a) = \{b \in A \mid |b| \leq |k| + n\,|a|$ *for some* $k \in K, n \in \mathbb{N}\}$;

(2) *if* $a, b \geq 0$, *then* $K(a) \cap K(b) = K(a \wedge b)$;

(3) $\mathbf{K}$ *is prime if and only if for each* $a, b \in A$ *such that* $a \wedge b = 0$, $a \in K$ *or* $b \in K$.

Note that for any abelian $\ell$-group $\mathbf{A}$ and non-empty set $W$, the algebra $\mathbf{A}^W$ whose universe consists of all functions $f \colon W \to A$ with operations defined pointwise is again an abelian $\ell$-group. For our purposes, we are particularly interested in the case where $\mathbf{A}$ is an abelian $o$-group. We generalize the notion of a bounded function to this setting: a function $f \colon W \to A$ is called *bounded* if there exists $a \in A$ such that $0 \leq a$ and $|f(w)| \leq a$ for all $w \in W$. The set of bounded functions from $W$ to $A$ forms an $\ell$-subgroup $\mathbf{B}(W, \mathbf{A})$ of $\mathbf{A}^W$.

Now let us recall the definition of a monadic abelian $\ell$-group from Example 2.8.

**Definition 4.14.** A *monadic abelian $\ell$-group* is an algebra $\langle A, \wedge, \vee, +, -, 0, \Box \rangle$, also written $\langle \mathbf{A}; \Box \rangle$, such that $\mathbf{A} = \langle A, +, -, 0 \rangle$ is an abelian $\ell$-group with defined operator

---

[2]In fact, using the term equivalence given in Example 1.7, the universe of an $\ell$-ideal of an abelian $\ell$-group $\mathbf{A}$ is a convex 0-free subuniverse of the term-equivalent $\mathsf{FL}_e$-algebra, and vice versa.

$\Diamond a := -\Box -a$ satisfying for all $a, b \in A$,

| | | | |
|---|---|---|---|
| (M1) | $\Box(a + b) \leq \Box a + \Diamond b$ | (M4) | $\Box(a \wedge b) = \Box a \wedge \Box b$ |
| (M2) | $\Box a \leq a$ | (M5) | $\Box(a \vee \Box b) = \Box a \vee \Box b$ |
| (M3) | $\Diamond a = \Box \Diamond a$ | (M6) | $\Box(a + a) = \Box a + \Box a.$ |

**Remark 4.15.** In [53], Cimadamore and Díaz Varela introduced a monadic version of abelian $\ell$-groups with a distinguished element, playing the role of a strong unit, proving a categorical equivalence with monadic MV-algebras. Their notion of monadic abelian $\ell$-groups coincides with our notion as defined in this thesis, if we discard the distinguished element. To state this formally, let us recall the definition as found in [53]. A *CD-monadic $\ell$-group* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, +, -, 0, u, \Diamond \rangle$ such that $\langle A, \wedge, \vee, +, -, 0 \rangle$ is an abelian $\ell$-group, $u > 0$ is a fixed element of $A$, and $\Diamond \colon A \to A$ is a unary operation satisfying for all $a, b, c \in A$ with $c \geq 0$,

| | | | |
|---|---|---|---|
| (G1) | $a \leq \Diamond a$ | (G6) | $\Diamond(\Diamond a + \Diamond b) = \Diamond a + \Diamond b$ |
| (G2) | $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$ | (G7) | $\Diamond(c \wedge u) = \Diamond c \wedge u$ |
| (G3) | $\Diamond 0 = 0$ | (G8) | $\Diamond(c + c) = \Diamond c + \Diamond c$ |
| (G4) | $\Diamond u = u$ | (G9) | $\Diamond(a \wedge 0) = \Diamond a \wedge 0.$ |
| (G5) | $\Diamond(-\Diamond a) = -\Diamond a$ | | |

It is not hard to show that for any monadic abelian $\ell$-group $\langle \mathbf{A}; \Box \rangle$, $\langle A, \wedge, \vee, +, -, 0, 0, \Diamond \rangle$ is a CD-monadic $\ell$-group. Conversely, for a CD-monadic $\ell$-group $\mathbf{A}$, we leave it to the reader to verify that $\langle A, \wedge, \vee, +, -, 0, \Box \rangle$, where $\Box a := -\Diamond -a$ for all $a \in A$, satisfies conditions (M1)–(M4). Proofs for conditions (M5) and (M6) can be found in [127, Proposition 2.1]. For completeness, we recount them here. We require two additional useful properties for CD-monadic $\ell$-groups proved in [53]: for all $a, b \in A$,

$$(G13) \; \Diamond(a - \Diamond b) = \Diamond a - \Diamond b$$
$$(G13^+) \; \Diamond(a + \Diamond b) = \Diamond a + \Diamond b.$$

To prove that (M5) holds, note that $a \wedge \Diamond b = ((a - \Diamond b) \wedge 0) + \Diamond b$. It follows by (G9), (G13), and (G13$^+$), that

$$
\begin{aligned}
\Diamond(a \wedge \Diamond b) = \Diamond(((a - \Diamond b) \wedge 0) + \Diamond b) &= \Diamond((a - \Diamond b) \wedge 0) + \Diamond b \\
&= (\Diamond(a - \Diamond b) \wedge 0) + \Diamond b \\
&= ((\Diamond a - \Diamond b) \wedge 0) + \Diamond b \\
&= \Diamond a \wedge \Diamond b.
\end{aligned}
$$

For (M6), observe that $-a \leq -a \vee 0 \leq \Diamond(-a \vee 0)$, so $0 \leq a + \Diamond(-a \vee 0)$. Using (G6) and (G13$^+$), we obtain

$$
\begin{aligned}
\Diamond(2(a + \Diamond(-a \vee 0))) = \Diamond(2a + 2\Diamond(-a \vee 0)) &= \Diamond(2a + \Diamond 2\Diamond(-a \vee 0)) \\
&= \Diamond 2a + \Diamond 2\Diamond(-a \vee 0) \\
&= \Diamond 2a + 2\Diamond(-a \vee 0).
\end{aligned}
$$

Again using (G13$^+$), we obtain

$$2\Diamond(a + \Diamond(-a \vee 0)) = 2(\Diamond a + \Diamond(-a \vee 0)) = 2\Diamond a + 2\Diamond(-a \vee 0).$$

Finally, condition (G8) implies that $\lozenge(2(a + \lozenge(-a \vee 0))) = 2\lozenge(a + \lozenge(-a \vee 0))$. It follows that $\lozenge 2a + 2\lozenge(-a \vee 0) = 2\lozenge a + 2\lozenge(-a \vee 0)$, and so $\lozenge 2a = 2\lozenge a$ as required.

A standard Lindenbaum-Tarski argument, which we will not recount here, can be used to prove that $\mathcal{S}5(\mathcal{A})$ is complete with respect to the variety $\mathcal{M}\ell\mathcal{G}$ of monadic abelian $\ell$-groups.

**Lemma 4.16.** *For all $\varphi \in \mathrm{Fm}_{\square\lozenge}(\mathcal{L}_{\mathsf{A}})$,*

$$\vdash_{\mathcal{S}5(\mathcal{A})} \varphi \quad \Longleftrightarrow \quad \mathcal{M}\ell\mathcal{G} \models \bar{0} \leq \varphi.$$

Similar to the terminology adopted for monadic $\mathsf{FL}_e$-algebras in Chapter 2, we say that a monadic abelian $\ell$-group $\langle \mathbf{A}; \square \rangle$ is *functional* if the term-equivalent monadic $\mathsf{FL}_e$-algebra is functional. That is, $\mathbf{A}$ is the subalgebra of $\mathbf{B}^W$ for some abelian $\ell$-group $\mathbf{B}$ and non-empty set $W$, and for each $f \in A$, $w \in W$, $\bigwedge\{f(w) \mid w \in W\}$ exists in $A$ and

$$\square f(w) = \bigwedge\{f(w) \mid w \in W\}.$$

Note that then, each $f \in A$ is bounded and we can assume $\mathbf{A}$ to be a subalgebra of $\mathbf{B}(W, \mathbf{B})$. We are particularly interested in the case when $\mathbf{B} = \mathbf{R}$, in which case we call $\mathbf{A}$ *standard*. If $\square f(w) = \min\{f(w) \mid w \in W\}$ for each $f \in A$, $w \in W$, we call $\mathbf{A}$ *witnessed*. We say that $\langle \mathbf{A}; \square \rangle$ is *chain-monadic* if $\square \mathbf{A}$ is an abelian $o$-group.

The remainder of this section is dedicated to proving that $\mathcal{M}\ell\mathcal{G}$ is generated by its witnessed standard members. Recall that we have already obtained a weaker generation result for the variety $\mathcal{M}\ell\mathcal{G}$ in Corollary 2.45. We first strengthen this generation result by showing that $\mathcal{M}\ell\mathcal{G}$ is generated by all witnessed functional chain-monadic abelian $\ell$-groups, and then we use a finite embedding theorem of abelian $o$-groups into the reals to obtain the desired result. We will follow largely the same strategy as outlined in Section 2.4, using some additional properties of monadic abelian $\ell$-groups.

Before we get to this stronger generation result, let us first recount some of the results from Chapter 2 using the terminology of monadic abelian $\ell$-groups. Recall from Example 2.8 that monadic abelian $\ell$-groups are term-equivalent to monadic $\mathsf{FL}_e$-algebras $\langle \mathbf{A}; \square, \lozenge \rangle$ such that $\mathbf{A}$ is term-equivalent to an abelian $\ell$-group and the identities $\square(x \vee \square y) \approx \square x \vee \square y)$ and $\lozenge(x \cdot x) \approx \lozenge x \cdot \lozenge x$ are satisfied. The following result is then an immediate corollary of Theorem 2.22.

**Corollary 4.17.** *There exists a one-to-one correspondence between*

(1) *monadic abelian $\ell$-groups $\langle \mathbf{A}; \square \rangle$;*

(2) *pairs $\langle \mathbf{A}, \mathbf{A}_0 \rangle$ of abelian $\ell$-groups such that $\mathbf{A}_0$ is an $\mathcal{M}\ell\mathcal{G}$-relatively complete $\ell$-subgroup of $\mathbf{A}$,*

*witnessed by the maps $\langle \mathbf{A}; \square \rangle \mapsto \langle \mathbf{A}, \square\mathbf{A} \rangle$ and $\langle \mathbf{A}, \mathbf{A}_0 \rangle \mapsto \langle \mathbf{A}; \square_0 \rangle$.*

Recall that an abelian $\ell$-group $\mathbf{A}_0$ is an $\mathcal{M}\ell\mathcal{G}$-relatively complete $\ell$-subgroup of an abelian $\ell$-group $\mathbf{A}$ if $\mathbf{A}_0$ is a relatively complete $\ell$-subgroup of $\mathbf{A}$ and for all $a, b \in A$,

$$\square_0(a \vee \square_0 b) = \square_0 a \vee \square_0 b \text{ and } \square_0(a + a) = \square_0 a + \square_0 a.$$

We can also give a more concrete characterization, as done in the following lemma. Note that this result resembles the definition of m-relative completeness for MV-algebras (see Example 2.23).

**Lemma 4.18.** *Let* **A** *be an abelian $\ell$-group and* **B** *a relatively complete $\ell$-subgroup of* **A**. *Then* **B** *is $\mathcal{M\ell G}$-relatively complete if and only if for all $a \in A$, $c_1, c_2 \in A_0$, the following conditions hold:*

(1) *if $c_1 \leq c_2 \vee a$, there exists $c_3 \in A_0$ such that $c_3 \leq a$ and $c_1 \leq c_2 \vee c_3$;*

(2) *if $c_1 \leq a + a$, there exists $c_3 \in A_0$ such that $c_3 \leq a$ and $c_1 \leq c_3 + c_3$.*

*Proof.* We show first that (1) is equivalent to $\square_0(a \vee \square_0 b) = \square_0 a \vee \square_0 b$ for $a, b \in A$. For one direction, suppose that (1) holds. Consider any $a, b \in A$. Then $\square_0 a \vee \square_0 b \leq a \vee \square_0 b$ and so $\square_0 a \vee \square_0 b = \square_0(\square_0 a \vee \square_0 b) \leq \square_0(a \vee \square_0 b)$. To prove that $\square_0(a \vee \square_0 b) \leq \square_0 a \vee \square_0 b$, consider any $c_1 \in A_0$ such that $c_1 \leq a \vee \square_0 b$. By (1), we obtain $c_3 \in A_0$ such that $c_1 \leq c_3 \vee \square_0 b$ and $c_3 \leq a$. It then follows that $c_3 \leq \square_0 a$. Hence, $c_1 \leq c_3 \vee \square_0 b \leq \square_0 a \vee \square_0 b$. Since this holds for any $c_1 \in A_0$ such that $c_1 \leq a \vee \square_0 b$, we obtain $\square_0(a \vee \square_0 b) \leq \square_0 a \vee \square_0 b$ by definition of $\square_0$. For the converse direction, consider $c_1, c_2 \in A_0$ and $a \in A$ such that $c_1 \leq c_2 \vee a$. We let $c_3 = \square_0 a$. Obviously, $c_3 \leq a$. Moreover,

$$c_2 \vee c_3 = \square_0 c_2 \vee \square_0 a = \square_0(\square_0 c_2 \vee a) = \square_0(c_2 \vee a).$$

Since $c_1 \in A_0$ and $c_1 \leq c_2 \vee a$, we obtain $c_1 \leq \square_0(c_2 \vee a) = c_2 \vee c_3$ as required.

Secondly, we show that (2) is equivalent to $\square_0(a + a) = \square_0 a + \square_0 a$ for all $a \in A$. For one direction, suppose that (2) holds and let $a, b \in A$. Note that $\square_0 a + \square_0 a \leq a + a$, and hence $\square_0 a + \square_0 a = \square_0(\square_0 a + \square_0 a) \leq \square_0(a + a)$. For the converse inequality, consider $c_1 \in A_0$ such that $c_1 \leq a + a$. By (2), there exists $c_3 \in A_0$ such that $c_1 \leq c_3 + c_3$ and $c_3 \leq a$. It follows that $c_3 \leq \square_0 a$ by definition of $\square_0$, so $c_1 \leq c_3 + c_3 \leq \square_0 a + \square_0 a$. Hence,

$$\square_0(a + a) = \bigvee \{ d \in A_0 \mid d \leq a + a \} \leq \square_0 a + \square_0 a.$$

Conversely, suppose that $\square_0 a + \square_0 a = \square_0(a + a)$ for all $a \in A$, and consider $c_1 \in A_0$ and $a \in A$ such that $c_1 \leq a + a$. Then $c_1 = \square c_1 \leq \square_0(a + a) = \square_0 a + \square_0 a$, so it suffices to take $c_3 = \square_0 a$. $\qquad\square$

For a monadic abelian $\ell$-group $\langle \mathbf{A}; \square \rangle$, we say that **K** is a *monadic $\ell$-ideal* of $\langle \mathbf{A}; \square \rangle$ if **K** is an $\ell$-ideal of **A** and $a \in K$ implies $\square a \in K$. As a consequence of Theorem 2.28, the lattice of congruences of $\langle \mathbf{A}; \square \rangle$ is isomorphic to the lattice of monadic $\ell$-ideals of $\langle \mathbf{A}; \square \rangle$, where both lattices are ordered by inclusion.[3] We can again represent the quotient $\langle \mathbf{A}; \square \rangle / \mathbf{K}$ of $\langle \mathbf{A}; \square \rangle$ by a monadic $\ell$-ideal **K** using the right cosets; that is, we define $\langle \mathbf{A}; \square \rangle / \mathbf{K} := \langle \mathbf{A}/\mathbf{K}; \square_K \rangle$ where $\square_K(K + a) := K + \square a$ for all $a \in A$. The following result is then an immediate consequence of Theorem 2.32.

**Corollary 4.19.** *Let $\langle \mathbf{A}; \square \rangle$ be a monadic abelian $\ell$-group. Then the lattice of monadic $\ell$-ideals of $\langle \mathbf{A}; \square \rangle$ (ordered by inclusion) and the lattice of $\ell$-ideals of $\square \mathbf{A}$ (ordered by inclusion) are isomorphic, witnessed by the maps $K \mapsto K \cap \square A$ and $K \mapsto K^{\square \lozenge} := \{ a \in A \mid \square a \in K \text{ and } \lozenge a \in K \}$.*

As instances of Theorem 2.41 and Lemma 2.42, we obtain the following results.

**Corollary 4.20.** *Each monadic abelian $\ell$-group is isomorphic to a subdirect product of chain-monadic abelian $\ell$-groups.*

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[3]Under the term equivalence from Example 2.8, the universe of a monadic $\ell$-ideal of a monadic abelian $\ell$-group is an $f$-free subuniverse of the corresponding monadic $\mathsf{FL}_e$-algebra, and vice versa.

**Corollary 4.21.** *Let $\langle \mathbf{A}; \square \rangle$ be a chain-monadic abelian $\ell$-group and $a \in A \setminus 0$. Then there exists a prime $\ell$-ideal $\mathbf{P}$ of $\mathbf{A}$ such that $a \notin P$ and $P \cap \square A = \{0\}$.*

The setting of monadic abelian $\ell$-groups allows for an additional prime generation result that was not considered in Section 2.4. This result, shown in the following lemma, will prove instrumental in obtaining the desired completeness result. Note that a similar result was obtained for monadic MV-algebras in [45].

**Lemma 4.22.** *Let $\langle \mathbf{A}; \square \rangle$ be a chain-monadic abelian $\ell$-group and $a \in A$. Then there exists a prime $\ell$-ideal $\mathbf{P}$ of $\mathbf{A}$ such that $P + a = P + \square a$ and $P \cap \square A = \{0\}$.*

*Proof.* Let $\langle \mathbf{A}; \square \rangle$ be a chain-monadic abelian $\ell$-group and $a \in A$. We apply Zorn's Lemma to the set $\mathcal{D}$ of all $\ell$-ideals $\mathbf{K}$ of $\mathbf{A}$ such that $K \cap \square A = \{0\}$ and $a - \square a \in K$, ordered by inclusion. First, we verify that $\mathcal{D}$ is non-empty. We show that the $\ell$-ideal $\mathbf{K}(a - \square a)$ of $\mathbf{A}$ generated by the element $a - \square a$ is in $\mathcal{D}$. Consider any $b \in K(a - \square a) \cap \square A$. Then by Lemma 4.13(1) there exists some $n \in \mathbb{N}$ such that $|b| \leq n\,|a - \square a|$. In particular, since $0 \leq |a - \square a|$, we obtain $|b| \leq 2^n\,|a - \square a|$. Then,

$$
\begin{aligned}
|b| = \square\,|b| &\leq \square(2^n\,|a - \square a|) && \text{since } |b| \in \square A \\
&= 2^n \square\,|a - \square a| && \text{using (M6)} \\
&= 2^n \square (a - \square a) && \text{using (M2)} \\
&= 2^n (\square a - \square a) && \text{using (M1), (M2), and (M3)} \\
&= 0.
\end{aligned}
$$

So $b = 0$ and $\mathcal{D} \neq \emptyset$. Moreover, it is easy to see that $\mathcal{D}$ is closed under taking unions of chains, so Zorn's Lemma yields a maximal element $\mathbf{P} \in \mathcal{D}$.

Suppose for a contradiction that $\mathbf{P}$ is not prime. Then Lemma 4.13(3) implies that there exist $b, c \in A$ with $b \wedge c = 0$ but $b, c \notin P$. By the maximality of $\mathbf{P}$, there exist $r \in (P(b) \cap \square G) \setminus \{0\}$ and $s \in (P(c) \cap \square G) \setminus \{0\}$. Since $\square \mathbf{A}$ is linearly ordered, we can assume without loss of generality that $|r| \leq |s|$. Convexity of $P(c)$ then implies that also $r \in P(c) \cap \square A$. Hence, using Lemma 4.13(2), $r \in P(b) \cap P(c) = P(b \wedge c) = P(0) = P$. But $P \cap \square A = \{0\}$, so $r = 0$, a contradiction. That is, $\mathbf{P}$ is prime. Finally, note that since $a - \square a \in P$, also $P + a = P + \square a$. $\qquad\square$

We can now prove the strengthened analogue of Theorem 2.43 for $\mathcal{M}\ell\mathcal{G}$. Its proof is analogous to that of Theorem 2.43, aside from an application of Lemma 4.22 at the end that shows that the obtained monadic abelian $\ell$-group is in fact witnessed. Recall that the class of abelian $o$-groups has the amalgamation property [128], and hence by Lemma 2.36, it has the generalized amalgamation property.

**Theorem 4.23.** *Any chain-monadic abelian $\ell$-group $\langle \mathbf{A}; \square \rangle$ is isomorphic to a witnessed functional chain-monadic abelian $\ell$-group.*

*Proof.* Let $\langle \mathbf{A}; \square \rangle$ be a chain-monadic abelian $\ell$-group, and let $\{\mathbf{P}_i\}_{i \in I}$ be the family of all prime $\ell$-ideals $\mathbf{P}$ of $\mathbf{A}$ such that $P \cap \square A = \{0\}$. It follows from Corollary 4.21 that $\bigcap\{P_i \mid i \in I\} = \{0\}$ and hence that $\sigma \colon \mathbf{A} \to \prod_{i \in I} \mathbf{A}/\mathbf{P}_i$; $a \mapsto (a + P_i)_{i \in I}$ is an embedding between abelian $\ell$-groups. Moreover, for each $i \in I$, since $P_i \cap \square A = \{0\}$, the map $\pi_i \circ \sigma|_{\square A}$ is an $\ell$-embedding, where $\pi_i$ is the $i$-th projection map.

For the abelian $o$-group $\square\mathbf{A}$, family of abelian $o$-groups $\{\mathbf{A}/\mathbf{P}_i\}_{i\in I}$ and family of $\ell$-embeddings $\{\pi_i \circ \sigma|_{\square A}\colon \square\mathbf{A} \to \mathbf{A}/\mathbf{P}_i\}_{i\in I}$, the generalized amalgamation property provides an amalgam $\mathbf{B}$ with $\ell$-embeddings $\gamma_i\colon \mathbf{A}/\mathbf{P}_i \to \mathbf{B}$ for each $i \in I$. Defining $\gamma := \prod_{i\in I}\gamma_i\colon \prod_{i\in I}A/P_i \to B^I$ yields an $\ell$-embedding $\rho := \gamma \circ \sigma\colon \mathbf{A} \to \mathbf{B}^I$. Observe now that for all $r \in \square A$ and $i,j \in I$,

$$\rho(r)(i) = \gamma_i(\sigma(r)(i)) = \gamma_i(\pi_i(\sigma(r))) = \gamma_j(\pi_j(\sigma(r))) = \gamma_j(\sigma(r)(j)) = \rho(r)(j).$$

That is, $\rho(r)$ is a constant function. Moreover, for each $a \in A$, there exists, by Lemma 4.22, an $i \in I$ such that $P_i + a = P_i + \square a$ and hence $\rho(\square a)(i) = \rho(a)(i)$. So for any $a \in A$ and $i \in I$, we obtain $\rho(\square a)(i) = \min\{\rho(a)(j) \mid j \in I\}$. $\qquad\square$

To prove the promised completeness result for $\mathcal{S}5(\mathcal{A})$, we make use of the following folklore result from the theory of abelian $\ell$-groups.

**Lemma 4.24** (cf. [51])**.** *Let $\mathbf{A}$ be an abelian $o$-group. For each finite subset $S$ of $A$, there exists a function $h\colon S \to \mathbb{R}$ satisfying for all $a,b,c \in S$,*

(i) $a \le b$ *if and only if* $h(a) \le h(b)$;

(ii) *if* $0 \in S$, *then* $h(0) = 0$;

(iii) $a + b = c$ *if and only if* $h(a) + h(b) = h(c)$;

(iv) $b = -a$ *if and only if* $h(b) = -h(a)$.

**Theorem 4.25.** *For any $\varphi \in \mathrm{Fm}_{\square\Diamond}(\mathcal{L}_\mathbf{A})$,*

$$\vdash_{\mathcal{S}5(\mathcal{A})} \varphi \quad \Longleftrightarrow \quad \models_{\mathsf{S5}(\mathbf{R})^\mathsf{C}} \varphi.$$

*Proof.* For the left-to-right direction, it is easily checked that the axioms of $\mathcal{S}5(\mathcal{A})$ are $\mathsf{S5}(\mathbf{R})^\mathsf{C}$-valid and its rules preserve $\mathsf{S5}(\mathbf{R})^\mathsf{C}$-validity. For the converse direction, suppose that $\nvdash_{\mathcal{S}5(\mathcal{A})} \varphi$. By Lemma 4.16 and Corollary 4.20, there exist a chain-monadic abelian $\ell$-group $\langle\mathbf{A};\square\rangle$ and a valuation $e\colon \mathrm{Fm}_{\square\Diamond}(\mathcal{L}_\mathbf{A}) \to A$, preserving all connectives in $\mathcal{L}_\mathbf{A}$ as well as $\square$ and $\Diamond$, such that $0 \not\le e(\varphi)$. By Theorem 4.23, we may assume that $\mathbf{A}$ is a witnessed $\ell$-subgroup of $\mathbf{B}(W,\mathbf{B})$ for some non-empty set $W$ and abelian $o$-group $\mathbf{B}$. Hence there exists $w_0 \in W$ such that $e(\varphi)(w_0) < 0$. Let $\Sigma$ be the set of subformulas of $\varphi$. For each $\square\psi, \Diamond\psi \in \Sigma$, we choose $w_{\square\psi} \in W$ and $w_{\Diamond\psi} \in W$ such that

$$e(\square\psi)(w_{\square\psi}) = e(\psi)(w_{\square\psi}) \text{ and } e(\Diamond\psi)(w_{\Diamond\psi}) = e(\psi)(w_{\Diamond\psi}),$$

respectively. These exist since $\mathbf{A}$ is witnessed. Let $W' := \{w_{\square\psi} \in W \mid \square\psi \in \Sigma\} \cup \{w_{\Diamond\psi} \mid \Diamond\psi \in \Sigma\} \cup \{w_0\}$ and define

$$S := \{e(\psi)(w) \mid w \in W', \psi \in \Sigma\} \cup \{-e(\psi)(w) \mid w \in W', \psi \in \Sigma\} \cup \{0\}.$$

Since both $W'$ and $\Sigma$ are finite, so is $S$. Using Lemma 4.24, we obtain a function $h\colon S \to \mathbb{R}$ satisfying the properties (i)–(iv). We consider the standard monadic abelian $\ell$-group $\langle\mathbf{B}(W',\mathbf{R});\square\rangle$ and any valuation $e'\colon \{p_i\}_{i\in\mathbb{N}} \to \mathbb{R}$ such that for each propositional variable $p_i \in \Sigma$ and $w \in W'$,

$$e'(p_i)(w) = h(e(p_i)(w)).$$

When extending $e'$ to a map $e' \colon \mathrm{Fm}_{\square \lozenge}(\mathcal{L}_{\mathsf{A}}) \to \mathbb{R}$ preserving all connectives, a simple induction on formula length shows that $e'(\psi)(w) = h(e(\psi)(w))$ for all $\psi \in \Sigma$ and $w \in W'$, and in particular,

$$e'(\varphi)(w_0) = h(e(\varphi)(w_0)) < h(0) = 0.$$

Finally, consider the $\mathsf{S5}(\mathbf{R})^{\mathsf{C}}$-model $\langle W', V \rangle$ where $V(p_i, w) \coloneqq e'(p_i)(w)$ for each $w \in W'$ and $i \in \mathbb{N}$, and observe that $\bar{V}(\varphi, w_0) = e'(\varphi)(w_0) < 0$. Hence $\not\models_{\mathsf{S5}(\mathbf{R})^{\mathsf{C}}} \varphi$. $\qquad \square$

As an immediate corollary, we obtain that the variety $\mathcal{M}\ell\mathcal{G}$ is generated by the witnessed standard monadic abelian $\ell$-groups.

**Corollary 4.26.** *The variety $\mathcal{M}\ell\mathcal{G}$ is generated by the witnessed standard monadic abelian $\ell$-groups. In particular, it is generated by the standard monadic abelian $\ell$-group $\langle \mathbf{B}(\mathbb{N}, \mathbf{R}); \square \rangle$.*

# Conclusions and Open Problems

In this chapter, we present a short summary of the thesis and its major contributions. We conclude by sketching a number of interesting questions that have arisen while writing this thesis, and further open problems.

## 5.1 Summary of the Thesis

This thesis focused on the intersection between the study of one-variable fragments of first-order logics and the study of many-valued modal logics. Its contents are hence of interest to researchers in either field. We have considered three (classes of) first-order logics in particular.

In Chapter 3, we matched the one-variable fragments of first-order intermediate logics $\mathsf{IKL}(\mathbf{K})$ defined over some linearly ordered frame $\mathbf{K}$ to the many-valued modal logics $\mathsf{S5}(\mathbf{A})$ for some Gödel set $A$ (Theorems 3.3 and 3.12). This extends a related correspondence obtained by Beckmann and Preining in [13] that matched first-order intermediate logics $\mathsf{CDIKL}(\mathbf{K})$ to first-order Gödel logics with truth values in some Gödel set $A$. In particular, we matched the one-variable fragment $\mathsf{IKL}_1$ of the first-order intermediate logic $\mathsf{IKL}$, defined over all linearly ordered intuitionistic Kripke frames, to the modal Gödel logic $\mathsf{S5}(\mathbf{G})$. Although $\mathsf{IKL}$ was axiomatized by Corsi in [56], an axiomatization of its one-variable fragment was lacking until now (Theorem 3.7). Additionally, we provided an interpretation from each modal Gödel logic $\mathsf{S5}(\mathbf{A})$ into its crisp counterpart $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$. Using an alternative "relativized" semantics, we obtained decidability and co-NP-completeness for a large class of the logics $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ (Corollary 3.23). Using the interpretation and the mentioned correspondences, this also gives decidability and complexity for a large class of logics $\mathsf{S5}(\mathbf{A})$, and of one-variable fragments $\mathsf{IKL}_1(\mathbf{K})$. In fact, this work provides a basis to answer the question of decidability for all logics $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$, $\mathsf{S5}(\mathbf{A})$, and $\mathsf{IKL}_1(\mathbf{K})$. As mentioned in Remark 3.26, Caicedo has in an unpublished manuscript employed the methods from Sections 3.4 and 3.5 to obtain decidability for all logics $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$.

In Chapter 4, we investigated the one-variable fragment of first-order Abelian logic or, equivalently, the modal Abelian logic $\mathsf{S5}(\mathbf{R})^{\mathsf{C}}$. First-order Abelian logic had (to our knowledge) not yet been studied in the literature, and is an interesting first-order logic in its own right: it has a semantics based on well-known mathematical structures (both the

real numbers and abelian $\ell$-groups), and its language is rich enough to interpret other logics such as first-order Łukasiewicz logic, as shown in Theorem 1.15. We proved completeness of the proof system $\mathcal{S}5(\mathcal{A})$ with respect to $\mathsf{S5}(\mathbf{R})^{\mathsf{C}}$-validity, as well as completeness of $\mathcal{S}5(\mathcal{A}_{\mathsf{m}})$ with respect to the multiplicative fragment of $\mathsf{S5}(\mathbf{R})^{\mathsf{C}}$, that is, the fragment consisting of formulas that do not contain the lattice connectives. The novelty of these results lies in particular in the two distinct methods of proving completeness. The proof of completeness for the multiplicative fragment of $\mathsf{S5}(\mathbf{R})^{\mathsf{C}}$ (Theorem 4.12) is syntactic in nature, and is based on a partial Herbrand theorem, a normal form theorem, and a linear programming result known as Gordan's Theorem. The proof of completeness for the full logic $\mathsf{S5}(\mathbf{R})^{\mathsf{C}}$ (Theorem 4.25) is algebraic in nature, taking ideas from [45]. We defined the variety of monadic abelian $\ell$-groups, to which we applied methods established in Chapter 2, as well as a finite embedding theorem for abelian $o$-groups. Translating $\mathcal{S}5(\mathcal{A})$ into first-order logic also gives a proof system for the one-variable fragment of first-order Abelian logic. Decidability of $\mathsf{S5}(\mathbf{R})^{\mathsf{C}}$ follows rather easily from the partial Herbrand theorem (Corollary 4.3).

In Chapter 2, we investigated one-variable fragments in a much more general setting. The contents of that chapter sprouted from the study of various algebras that emerge when researching one-variable fragments, including monadic Boolean algebras, monadic Heyting algebras, monadic MV-algebras, (crisp) monadic Gödel algebras, and monadic abelian $\ell$-groups. We conducted an initial investigation into the algebraic semantics of one-variable fragments in the rather general setting of substructural first-order logics that admit the rule of exchange. We defined the variety of monadic $\mathsf{FL}_e$-algebras, intended as the algebraic semantics for the one-variable fragment of the first-order substructural logic $\mathsf{QFL}_e$. Equivalently, these monadic $\mathsf{FL}_e$-algebras can be viewed as the intended semantics for the many-valued modal logic defined over all $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-models, where $\mathbf{A}$ ranges over all $\mathsf{FL}_e$-algebras. We showed that the algebraic semantics of the one-variable fragment of $\mathsf{QFL}_e$ necessarily consists of monadic $\mathsf{FL}_e$-algebras (Corollary 2.12), but sufficiency still remains an interesting open problem.

We also generalized to this setting a number of interesting properties that many monadic algebras from the literature share. Firstly, we characterized the modalities of a monadic $\mathsf{FL}_e$-algebra in terms of a relatively complete subalgebra of the $\mathsf{FL}_e$-reduct, which leads to a characterization of monadic $\mathsf{FL}_e$-algebras in terms of pairs of $\mathsf{FL}_e$-algebras (Theorems 2.17 and 2.22). Secondly, we gave an alternative characterization of the congruences of monadic $\mathsf{FL}_e$-algebras in terms of $f$-free subuniverses (Theorem 2.28). We also proved that such congruences are in fact completely determined by the relatively complete subalgebra (Theorem 2.32). Finally, we have investigated the issue of functional representation for particular varieties of monadic $\mathsf{FL}_e$-algebras. Inspired by methods employed in [24, 45], we used the amalgamation property and semilinearity to prove that some varieties of monadic $\mathsf{FL}_e$-algebras are generated by particular relatively functional members (Corollary 2.44). These results have already lead to a completeness proof of $\mathcal{S}5(\mathcal{A})$ with respect to $\mathsf{S5}(\mathbf{R})^{\mathsf{C}}$-models (Theorem 4.25). We hope they can lead to completeness results for other modal logics defined over particular $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-models.

## 5.2   Open Problems and Future Work

To conclude, let us present some of the questions that have arisen while working on this thesis, together with some ideas on how to tackle them.

**Projection Operator $\Delta$**   In [4], Baaz proposed a projection operator $\Delta$ to add to the language of Gödel logic, defined for all $x \in [0, 1]$ as

$$\Delta x = \begin{cases} 1 & x = 1; \\ 0 & x < 1. \end{cases}$$

Adding this operator greatly increases the expressive power of Gödel logic, as it allows one to recover classical reasoning inside Gödel logic. It does however render the resulting logic more complicated. Propositional Gödel logic $\mathsf{G}$ extended with this operator has been axiomatized, and for the first-order extensions there exists a characterization similar to that of Baaz et al. from [11]; for a survey of these results, see [131]. Decidability of validity and satisfiability problems for these first-order logics and some of their fragments have been considered in, e.g., [6, 7].

Modal Gödel logics extended with operations such as $\Delta$ have been studied by Caicedo et al. in [38] in the context of "order-based" modal logics. In that paper, it is shown that the modal Gödel logic $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$ extended with $\Delta$ is decidable.[1] We expect that the methods outlined in Sections 3.4 and 3.5 can be used to show that the same is true for all modal Gödel logics $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ and $\mathsf{S5}(\mathbf{A})$ extended with $\Delta$. In fact, we deem it likely that the results of these sections can be replicated for the more general order-based modal logics $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$, where $\mathbf{A}$ is an "order-based algebra" as defined in [38], rather than a Gödel algebra.

**Real-Valued Modal Logics**   Although first-order Abelian logic had previously not been considered in the literature, modal Abelian logics have been studied in [61]. In that paper, Diaconescu et al. considered a minimal modal Abelian logic $\mathsf{K}(\mathbf{R})$ defined over real-valued Kripke models with a classical two-valued accessibility relation. They prove decidability for $\mathsf{K}(\mathbf{R})$ and provide a proof system that is complete with respect to the multiplicative fragment of $\mathsf{K}(\mathbf{R})$. In [140], Schnüriger obtains the same results for those real-valued Kripke models whose accessibility relation is reflexive; in Chapter 4, we have studied those models whose accessibility relation is an equivalence relation. This leaves a large class of real-valued Kripke models — and their corresponding modal Abelian logics — for further study, for example those with a transitive or symmetric accessibility relation. Of such modal Abelian logics, we can study aspects such as decidability, complexity, and axiomatizability. Similar questions can be asked about real-valued Kripke models with a many-valued accessibility relation and their corresponding modal Abelian logics.

**"Relativized" Semantics**   A theme that is found throughout the thesis is that of a "relativized" semantics. It appears in Section 3.4, where we define the notion of an $\mathsf{ruS5}(\mathbf{A})^{\mathsf{C}}$-model to establish a finite model property. Rather than interpreting the value of a modal formula $\Box\varphi$ or $\Diamond\varphi$ as an infimum or supremum of the relevant truth values of

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[1]Due to the flawed proof pointed out in footnote 2 on page 71, it cannot be concluded from [38] that all logics $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ extended with $\Delta$ are decidable.

$\varphi$, respectively, we approximate the infima and suprema using a designated set of values $T$. A similar phenomenon occurs in Chapter 2, where we consider relatively complete subalgebras. Indeed, for a relatively complete subalgebra $\mathbf{A}_0$ of an $\mathsf{FL}_e$-algebra $\mathbf{A}$, we defined modalities $\Box_0$ and $\Diamond_0$ such that for each $a \in A$,

$$\Box_0 a = \max\{c \in A_0 \mid c \leq a\} \text{ and } \Diamond_0 a = \min\{c \in A_0 \mid a \leq c\}.$$

That is, we approximate the value of $a$ inside $A_0$, either from below or above. The functional representation proved in Theorem 2.43 has a similar flavor; indeed, the relatively functional monadic $\mathsf{FL}_e$-algebras are exactly those algebras where the modal values $\Box f$ and $\Diamond f$ are approximations of $\bigwedge_{w \in W} f(w)$ and $\bigvee_{w \in W} f(v)$, respectively.

Since the modal Gödel logics $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ can be equivalently characterized by $\mathsf{ruS5}(\mathbf{A})^{\mathsf{C}}$-models, the question arises whether the same thing can be said for modal Abelian logic $\mathsf{S5}(\mathbf{R})^{\mathsf{C}}$. This is not the case however. To make this precise, let $\mathbf{A}$ be any abelian $\ell$-group. We define a *relativized universal* $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-*model* ($\mathsf{ruS5}(\mathbf{A})^{\mathsf{C}}$-*model* for short) as a triple $\mathcal{M} = \langle W, V, T \rangle$ consisting of non-empty sets $W$ and $T$ such that $T$ is a relatively complete subuniverse of $\mathbf{A}$, and a map $V \colon \{p_i\}_{i \in \mathbb{N}} \times W \to \mathbb{R}$ such that for each $i \in \mathbb{N}$ the map $V_i \colon W \to \mathbb{R}; w \mapsto V(p_i, w)$, is bounded. The map $V$ is then extended inductively to $\bar{V} \colon \mathrm{Fm}_{\Box \Diamond}(\mathcal{L}_{\mathbf{A}}) \times W \to A$ as follows:

$$\bar{V}(\bar{0}, w) = 0$$
$$\bar{V}(-\varphi, w) = -\bar{V}(\varphi, w)$$
$$\bar{V}(\varphi + \psi, w) = \bar{V}(\varphi, w) + \bar{V}(\psi, w)$$
$$\bar{V}(\Box\varphi, w) = \bigvee\{r \in T \mid r \leq \bigwedge\{\bar{V}(\varphi, v) \mid v \in W\}\}$$
$$\bar{V}(\Diamond\varphi, w) = \bigwedge\{r \in T \mid r \geq \bigvee\{\bar{V}(\varphi, v) \mid v \in W\}\}.$$

We say that a formula $\varphi \in \mathrm{Fm}_{\Box \Diamond}(\mathcal{L}_{\mathbf{A}})$ is *valid* in $\mathcal{M}$ if $\bar{V}(\varphi, w) \geq 0$ for all $w \in W$, and $\mathsf{ruS5}(\mathbf{A})^{\mathsf{C}}$-*valid* if it is valid in all $\mathsf{ruS5}(\mathbf{A})^{\mathsf{C}}$-models.

We can now consider an $\mathsf{ruS5}(\mathbf{R})^{\mathsf{C}}$-model $\langle \{w_0\}, V, \mathbb{Z} \rangle$, where $V(p_1, w_0) = 0.75$. It then follows that $\bar{V}(\Box p_1, w_0) = \bar{V}(\Box p_1 + \Box p_1, w_0) = 0$, but $\bar{V}(\Box(p_1 + p_1), w_0) = 1$. We have hence found an $\mathsf{ruS5}(\mathbf{R})^{\mathsf{C}}$-model that falsifies the formula $\Box(p_1 + p_1) \to (\Box p_1 + \Box p_1)$. Since this formula is $\mathsf{S5}(\mathbf{R})^{\mathsf{C}}$-valid, $\mathsf{S5}(\mathbf{R})^{\mathsf{C}}$ cannot be equivalently characterized by $\mathsf{ruS5}(\mathbf{R})^{\mathsf{C}}$-models. Note that this formula plays an important role in proving completeness for $\mathsf{S5}(\mathbf{R})^{\mathsf{C}}$ in Section 4.3; indeed, its algebraic formulation $\Box(x + x) \approx \Box x + \Box x$ is instrumental in the proof of Lemma 4.22. We conjecture that it is precisely this formula that distinguishes $\mathsf{S5}(\mathbf{R})^{\mathsf{C}}$- and $\mathsf{ruS5}(\mathbf{R})^{\mathsf{C}}$-models, that is, that the axiom system $\mathcal{S}5(\mathcal{A})$ without the axiom schema (M) $\Box(\varphi + \varphi) \to (\Box\varphi + \Box\varphi)$ is complete with respect to $\mathsf{ruS5}(\mathbf{R})^{\mathsf{C}}$-validity. Note that Corollary 2.45 already implies a slightly weaker completeness result: it shows that a formula $\varphi \in \mathrm{Fm}_{\Box \Diamond}(\mathcal{L}_{\mathbf{A}})$ is derivable in the axiom system $\mathcal{S}5(\mathcal{A})$ without the axiom schema (M) if and only if $\varphi$ is $\mathsf{ruS5}(\mathbf{A})^{\mathsf{C}}$-valid for all abelian $o$-groups $\mathbf{A}$.

Investigating such a relativized semantics for other many-valued modal logics, particularly in the context of the functional representation result from Theorem 2.43, is left as future research.

**Constant Domain Axiom**  A formula with a recurring role in this thesis is the constant domain axiom (cd) $(\forall x)(\alpha \vee (\forall x)\beta) \to ((\forall x)\alpha \vee (\forall x)\beta)$. In intuitionistic Kripke

frames, it characterizes the presence of constant, rather than increasing, domains; in modal Gödel logics $\mathsf{S5}(\mathbf{G})$ and $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$, it characterizes the crispness of the accessibility relation; and in Theorem 2.43, it is used to ensure that the relatively complete subalgebra $\Box\mathbf{A}$ consists only of constant functions. This raises a number of intriguing questions. Does the constant domain axiom characterize crispness of the accessibility relation in other many-valued modal logics $\mathsf{S5}(\mathbf{A})$? Or perhaps in other more general many-valued modal logics? Moreover, can a functional representation result, more general than that of Theorem 2.43, be obtained without imposing the identity $\Box(x \vee \Box y) \approx \Box x \vee \Box y$ on monadic $\mathsf{FL}_e$-algebras?

**Non-commutative Monadic Residuated Lattices**   In Chapter 2 we have decided on the framework of $\mathsf{FL}_e$-algebras rather than that of non-commutative $\mathsf{FL}$-algebras. This is partly due to the fact that all algebras considered in this thesis fit into this framework, and partly due to the fact that non-commutative $\mathsf{FL}$-algebras are more involved. An obvious question to ask is then whether we can generalize the results of this chapter to the non-commutative setting. Recall that a *pointed residuated lattice* (or $\mathsf{FL}$-*algebra*) is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, f, e \rangle$ such that $\langle A, \cdot, e \rangle$ is a monoid, $\langle A, \wedge, \vee \rangle$ is a lattice, and $\cdot$ is left- and right-residuated with residuals $\backslash$ and $/$ respectively, that is, for all $a, b, c \in A$,

$$a \cdot b \leq c \iff b \leq a\backslash c \quad \text{and} \quad a \cdot b \leq c \iff a \leq c/b.$$

We propose the following definition of a monadic pointed residuated lattice.

**Definition 5.1.** A *monadic pointed residuated lattice* (or *monadic* $\mathsf{FL}$-*algebra*) is an algebra $\langle A, \wedge, \vee, \cdot, \backslash, /, f, e, \Box, \Diamond \rangle$ such that $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, f, e \rangle$ is a pointed residuated lattice and for all $a, b \in A$,

| | | | |
|---|---|---|---|
| (L1) | $\Box a \leq a$ | (L6) | $\Box(a\backslash\Box b) = \Diamond a\backslash\Box b$ |
| (L2) | $\Box\Diamond a = \Diamond a$ | (L7) | $\Box(\Box b/a) = \Box b/\Diamond a$ |
| (L3) | $\Box(a \wedge b) = \Box a \wedge \Box b$ | (L8) | $\Box(\Box a\backslash b) = \Box a\backslash\Box b$ |
| (L4) | $\Box f = f$ | (L9) | $\Box(b/\Box a) = \Box b/\Box a.$ |
| (L5) | $\Box e = e$ | | |

We conjecture that these monadic pointed residuated lattices play the same role for first-order substructural logics as monadic $\mathsf{FL}_e$-algebras play for those first-order substructural logics that admit exchange, and that we can obtain analogues of important results of Chapter 2, including analogues of Theorem 2.22, Theorem 2.28, and Theorem 2.32. Note that in [133], Rachůnek and Šalounová already studied a particular case of monadic pointed residuated lattices, namely those satisfying $\Box(x \vee \Box y) \approx \Box x \vee \Box y$ and $\Box(x \cdot x) \approx \Box x \cdot \Box x$ (see Remark 2.13).

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# Erklärung

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|---|---|
| **Name/Vorname:** | Tuyt, Olim Frits |
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| **Art des Abschlusses:** | Dissertation |
| **Titel der Arbeit:** | One-Variable Fragments of First-Order Many-Valued Logics |
| **LeiterIn der Arbeit:** | Prof. Dr. George Metcalfe |

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