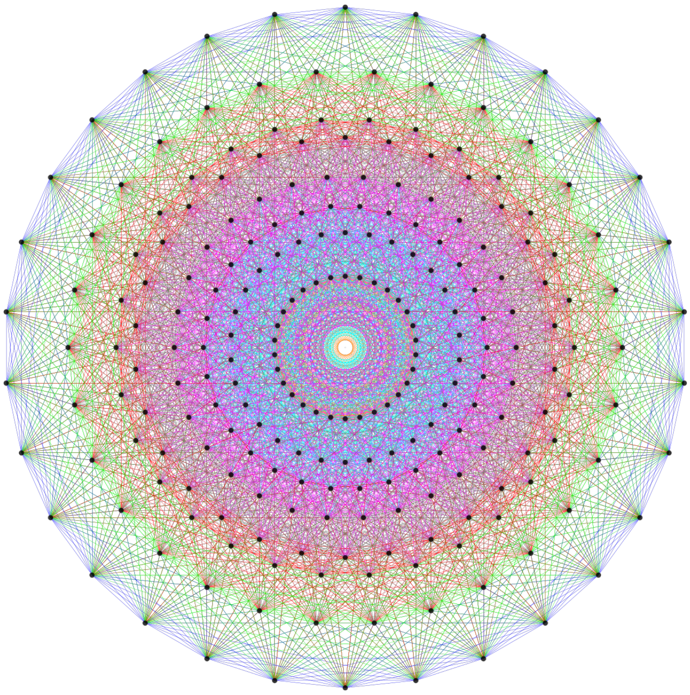
Holomorphic Factorization of Mappings into $\mathrm{Sp}_{2n}(\mathbb{C})$



Inauguraldissertation
der Philosophisch-naturwissenschaftlichen Fakultät
der Universität Bern

vorgelegt von

Josua Schott

von Meinisberg BE

Leiter der Arbeit:

Prof. Dr. Frank Kutzschebauch
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Der Dekan
Prof. Dr. Marco Herwegh

Contents

0 Introduction

It is well known that over any field, in particular over the field of complex numbers, a matrix in the Special Linear Group $\mathrm{SL}_n(\mathbb{C})$ is a product of elementary matrices. The proof is usually an application of a Gauss or a Gauss-Jordan process. The same question for $\mathrm{SL}_n(R)$ over a commutative ring R is much more difficult and has been studied a lot. For the ring $R = \mathbb{C}[z]$ of polynomials in one variable, it is true since R is an Euclidean ring. For $n = 2$ and the ring $R = \mathbb{C}[z, w]$, Cohn [5] found the following counterexample: the matrix

$$\begin{pmatrix} 1 - zw & z^2 \\ -w^2 & 1 + zw \end{pmatrix} \in \mathrm{SL}_2(R)$$

does not decompose as a finite product of unipotent matrices. In the 1970s, Suslin [22] gave a positive answer in the case $\mathrm{SL}_n(\mathbb{C}[z_1, ..., z_m])$, for $n \geq 3$ and $m \geq 1$. For the ring R of complex-valued continuous functions on a normal topological space, Vaserstein [23] showed, that a matrix in $\mathrm{SL}_n(R)$ decomposes into a product of elementary matrices if and only it is null-homotopic. In Gromov's seminal paper on the Oka principle, the starting point of modern Oka theory, he asks this question for the ring of complex-valued holomorphic functions (he calls it the *Vaserstein problem*, see [11, p. 886]). Ivarsson and Kutzschebauch [14] were able to give a positive answer to this problem in full generality.

The same question for the symplectic group $\mathrm{Sp}_{2n}(R)$ hasn't been studied to the same degree. Again, there is a positive answer for the ring $\mathbb{C}[z]$, since this is an Euclidean ring. For $n \geq 2$, Kopeiko [20] proved it for the polynomial ring $R = k[z_1, ..., z_m]$; and Grunewald, Mennicke and Vaserstein [12] for the ring $R = \mathbb{Z}[z_1, ..., z_m]$. In [16], Ivarsson, Kutzschebauch and Løw prove it for every commutative ring R with identity and bass stable rank $sr(R) = 1$. Furthermore, they show that, for the ring $R = C(X)$ of complex-valued continuous functions on a normal topological space, a matrix in $\mathrm{Sp}_{2n}(C(X))$ can be decomposed into a product of elementary matrices if and only it is null-homotopic; we call this the *Continuous Vaserstein problem for symplectic matrices*. The same authors [17] show a similar result in the case of $\mathrm{Sp}_4(\mathcal{O}(X))$, where $\mathcal{O}(X)$ denotes the ring of holomorphic functions on a reduced Stein space X. In this thesis, we see a full solution for $\mathrm{Sp}_{2n}(\mathcal{O}(X))$ for $n \geq 1$, the *Holomorphic Vaserstein problem for symplectic matrices*.

We call a $(2n \times 2n)$-matrix M *symplectic* if it satisfies $M^T \Omega M = \Omega$ with respect to the skew-symmetric matrix

$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

where I_n denotes the $n \times n$-identity matrix and 0 the $n \times n$-zero matrix. For symmetric matrices $A \in \mathbb{C}^{n \times n}$, i.e. $A^T = A$, matrices of the form

$$\begin{pmatrix} I_n & A \\ 0 & I_n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} I_n & 0 \\ A & I_n \end{pmatrix}$$

are symplectic and we call them *elementary symplectic matrices*. For simplicity reasons, let's identify the space of symmetric matrices $\mathrm{Sym}_n(\mathbb{C}) = \{A \in \mathbb{C}^{n \times n} : A^T = A\}$ with $\mathbb{C}^{\frac{n(n+1)}{2}}$.

Theorem 0.0.1 (Main theorem)**.** *There exists a natural number $K = K(n, d)$ such that given any finite dimensional reduced Stein space X of dimension d and any null-homotopic holomorphic mapping $f : X \to \mathrm{Sp}_{2n}(\mathbb{C})$ there exist a holomorphic mapping*

$$G = (G_1, ..., G_K) : X \to (\mathbb{C}^{\frac{n(n+1)}{2}})^K$$

such that

$$f(x) = \begin{pmatrix} I_n & 0 \\ G_1(x) & I_n \end{pmatrix} \begin{pmatrix} I_n & G_2(x) \\ 0 & I_n \end{pmatrix} \cdots \begin{pmatrix} I_n & 0 \\ G_{K-1}(x) & I_n \end{pmatrix} \begin{pmatrix} I_n & G_K(x) \\ 0 & I_n \end{pmatrix}, \quad x \in X.$$

It is immediate that any product of elementary symplectic matrices is connected by a path to the constant identity matrix I_{2n}, by multiplying the off-diagonal entries by $t \in [0, 1]$, i.e.

$$\begin{pmatrix} I_n & tA \\ 0 & I_n \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} I_n & 0 \\ tA & I_n \end{pmatrix}.$$

Therefore the requirement of null-homotopy of the map f is neccessary. Also, in general, we cannot expect the mapping to be null-homotopic, as the following example shows.

Example 0.0.2. *For $X = \mathrm{Sp}_2(\mathbb{C})$ the identity map*

$$\mathrm{Sp}_2(\mathbb{C}) \to \mathrm{Sp}_2(\mathbb{C})$$

is not null-homotopic, since $\mathrm{Sp}_2(\mathbb{C})$ is not contractible.

0.1 Strategy of proof

There are basically two main ingredients for the proof. In a first step, one proves a continuous version of the theorem (we also call this the *Continuous Vaserstein problem for symplectic matrices* - see Theorem 2.1.1). Once this is shown, we want to apply an Oka principle, which allows us, very roughly speaking, to deform the continuous solution into a holomorphic one. But one step at a time. Let's introduce the *elementary symplectic matrix mapping* $M_k : \mathbb{C}^{\frac{n(n+1)}{2}} \to \mathrm{Sp}_{2n}(\mathbb{C})$ by

$$M_k(Z) = \begin{cases} \begin{pmatrix} I_n & Z \\ 0 & I_n \end{pmatrix} & \text{if } k = 2l \\ \begin{pmatrix} I_n & 0 \\ Z & I_n \end{pmatrix} & \text{if } k = 2l + 1 \end{cases}$$

and then define the mapping $\Psi_K : (\mathbb{C}^{\frac{n(n+1)}{2}})^K \to \mathrm{Sp}_{2n}(\mathbb{C})$ by

$$\Psi_K(Z_1, ..., Z_K) = M_1(Z_1)M_2(Z_2) \cdots M_K(Z_K).$$

As already said, Ivarsson, Kutzschebauch and Løw [16] solved the Continuous Vaserstein problem for symplectic matrices. In fact, there is a natural number $K = K(n, d)$ such that given any finite dimensional normal topological space Y of dimension d and any null-homotopic continuous mapping $f : Y \to \mathrm{Sp}_{2n}(\mathbb{C})$ there exists a continuous mapping $F : Y \to (\mathbb{C}^{\frac{n(n+1)}{2}})^K$ such that the diagram

$$\begin{tikzcd} & (\mathbb{C}^{\frac{n(n+1)}{2}})^K \arrow[d, "\Psi_K"] \\ Y \arrow[ur, "F"] \arrow[r, "f"'] & \mathrm{Sp}_{2n}(\mathbb{C}) \end{tikzcd}$$

commutes. Unfortunately, the mapping Ψ_K is not a submersion and, in addition, the fibers are difficult to analyse. We therefore consider $\Phi_K := \pi_{2n} \circ \Psi_K$, where π_{2n} denotes the projection of a $(2n \times 2n)$-matrix to its last row. We obtain the commutative diagram

$$\begin{tikzcd} & (\mathbb{C}^{\frac{n(n+1)}{2}})^K \arrow[d, "\Phi_K=\pi_{2n}\circ\Psi_K"] \\ Y \arrow[ur, "F"] \arrow[r, "\pi_{2n}\circ f"'] & \mathbb{C}^{2n} \setminus \{0\}. \end{tikzcd}$$

The mapping Φ_K is surjective for $K \geq 3$ (see Theorem 3.1.2) and it is submersive outside some set of singularities $S_K \subset (\mathbb{C}^{\frac{n(n+1)}{2}})^K$ (see Theorem 3.1.1). We will find an open submanifold $E_K \subset (\mathbb{C}^{\frac{n(n+1)}{2}})^K \setminus S_K$ such that $\Phi_K|_{E_K} : E_K \to \mathbb{C}^{2n} \setminus \{0\}$ is a stratified elliptic submersion (see section 3). Further, the Continuous Vaserstein problem for symplectic matrices allows us, for a given any null-homotopic continuous mapping $f : Y \to \mathrm{Sp}_{2n}(\mathbb{C})$, to construct a continuous section $F : Y \to E_K$ such that

$$\begin{tikzcd} & E_K \arrow[d, "\Phi_K"] \\ Y \arrow[ur, "F"] \arrow[r, "\pi_{2n}\circ f"'] & \mathbb{C}^{2n} \setminus \{0\} \end{tikzcd}$$

commutes (see Theorem 2.2.3). Then, an application of the Oka principle allows us, roughly speaking, to homotopically deform F into a holomorphic mapping, such that the above diagram commutes.

0.2 Organisation of the thesis

The organisation of this thesis is as follows. The first chapter contains notations and definitions of elementary terms. For example, it includes a section on the symplectic group and some important results for the factorization of a symplectic matrix. In addition, we look at essential concepts of complex geometry such as Stein spaces, complete holomorphic vector fields and, of course, the Oka principle.

In the second chapter we prove the Main theorem under the assumption that the Oka principle can be applied. To do this, we will first state the continuous Vaserstein problem and see a sketch of the proof. Then we conclude the holomorphic Vaserstein problem by applying the Oka principle.

The third chapter is dedicated solely to the question of whether the Oka principle can actually be applied. We will clarify all the necessary details and construct stratified sprays. The climax is represented by the Spanning theorem. In its proof we carry out a nice induction, although technically demanding, and successively find a finite set of globally integrable vector fields which spans the tangent bundle of the fibers $\Phi_K^{-1}(x)$.

In the fourth chapter we see some interesting applications. The question of the number of factors is addressed. In general, this turns out to be extremely difficult. We state some known results from [15] for $\mathrm{SL}_n(\mathcal{O}(X))$. In addition, we can specify a result from the mentioned paper and find an optimal bound (see section 'Continuous vs. holomorphic factorization'). Then we show that the number of factors decreases monotonically as n increases. This can be done using Chevalley group theory (see section 'On the number of factors'). Last but not least, we deal with the question of the density property of the smooth fibers $\Phi_K^{-1}(x)$ in section 'Fibers with density property'.

An appendix follows in which some concrete calculations can be found and are intended to support the proof of the Spanning theorem.

0.3 Acknowledgements

First of all, I would like to thank my supervisor Prof. Dr. Frank Kutzschebauch for the opportunity to work on such an interesting problem and for the many exciting and supportive discussions. I am also very thankful to Dr. George Ionita for proofreading and his helpful feedback. Last but not least, I would like to thank my family, especially my wife Anna, who always supported me and believed in me.

1 Preliminaries

In the first section we introduce the symplectic group. We clarify our understanding of an *elementary symplectic matrix* and we will see some basic properties.

Then we introduce some basic notions of complex analytic geometry in the sections 1.2 - 1.6, as for instance *complex manifolds, Stein spaces* and *holomorphic vector fields*, following the book [8].

1.1 The symplectic group

Let R be a commutative ring with 1 and let R^* denote the group of units. For a positive integer $n \in \mathbb{N} = \{1, 2, 3, ...\}$, let R^n denote the n-dimensional R-module and $M_n(R)$ the ring of $n \times n$-matrices over the ring R. We let A^T denote the transpose of a matrix A. Matrices with $A^T = A$ (resp. $A^T = -A$) are called *symmetric* (resp. *skew-symmetric*).

Define the skew-symmetric $(2n \times 2n)$-matrix

$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

where I_n denotes the $n \times n$-identity matrix and 0 the $n \times n$-zero matrix.

Definition 1.1.1 (Symplectic matrix). *A matrix $M \in M_{2n}(R)$ over the ring R is **symplectic** if it satisfies the symplectic condition*

$$M^T \Omega M = \Omega.$$

The set of symplectic matrices (over R) is denoted by $\mathrm{Sp}_{2n}(R)$.

Remark 1.1.2. *The symplectic matrices $\mathrm{Sp}_{2n}(R)$ equipped with matrix multiplication forms an algebraic group.*

Taking the determinant of the defining equation yields $\det(M)^2 = 1$, hence

$$\mathrm{Sp}_{2n}(R) \subset \mathrm{GL}_{2n}(R),$$

where $\mathrm{GL}_{2n}(R)$ denotes the *general linear group*, the set of invertible $(2n \times 2n)$-matrices over R. Moreover, $\mathrm{Sp}_{2n}(R)$ equipped with matrix-multiplication and inversion is an algebraic group.

Sometimes it's useful to write a symplectic matrix $M \in \mathrm{Sp}_{2n}(R)$ in block notation

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1.1}$$

for some $n \times n$ matrices A, B, C and D in $M_n(R)$. The symplectic conditions are given by the equations

$$A^T C = C^T A \tag{SC.1}$$

$$B^T D = D^T B \tag{SC.2}$$

$$A^T D - C^T B = I_n. \tag{SC.3}$$

Here is an interesting alternative definition of a symplectic matrix.

Lemma 1.1.3. *A matrix $M \in M_{2n}(R)$, given in block notation (1.1), is symplectic if and only if its inverse is given by*

$$M^{-1} = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix}. \tag{1.2}$$

Proof. The symplectic conditions (SC.1) - (SC.3) are satisfied if and only if $M^{-1}M = I_{2n}$. □

Lemma 1.1.4. *The symplectic group $\mathrm{Sp}_{2n}(R)$ is closed under matrix transposition.*

Proof. Let M be a symplectic matrix. Then its inverse M^{-1} is symplectic too, hence

$$(M^{-1})^T \Omega M^{-1} = \Omega.$$

Next, we compute the inverse of this equation. Observe that $\Omega^{-1} = -\Omega$. We get

$$-\Omega = \left((M^{-1})^T \Omega M^{-1}\right)^{-1} = -\underbrace{(M^{-1})^{-1}}_{=M=(M^T)^T} \Omega \underbrace{((M^{-1})^T)^{-1}}_{=M^T} = -(M^T)^T \Omega M^T.$$

Therefore, the transpose M^T is symplectic. □

An ***elementary symplectic matrix*** is either of the form

$$\begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix} \tag{E.1}$$

where B is symmetric ($B^T = B$) or of the form

$$\begin{pmatrix} I_n & 0 \\ C & I_n \end{pmatrix} \tag{E.2}$$

where C is symmetric. Products of matrices of the first type (E.1) are additive in B. More precisely, for any pair of symmetric matrices B_1 and B_2, we have

$$\begin{pmatrix} I_n & B_1 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & B_2 \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} I_n & B_1 + B_2 \\ 0 & I_n \end{pmatrix}.$$

Analogously, products of matrices of the second type (E.2) are additive in C. Special cases are the matrices $E_{ij}(a)$ when B is the matrix with a in position ij and ji and otherwise zero. For $F_{ij}(a)$ the roles of B and C are changed. Clearly any elementary matrix of the first type is a product of matrices $E_{ij}(b_{ij})$ for $i \leq j$ and similarly for the second type.

We also introduce the *elementary symplectic matrices* $K_{ij}(a)$ defined by $B = C = 0$ and $A = I_n$ except in position ij where there is an a. Finally, $D = (A^T)^{-1}$. This equals I_n except in position ji where there is $-a$ if $i \neq j$ and a^{-1} if $i = j$ (this requires $a \in R^*$).

The following lemma shows, that the elementary symplectic matrices of the third type can be decomposed into a finite product of elementary symplectic matrices of the first two types.

Lemma 1.1.5 (Whitehead's lemma)**.** *We have*

$$K_{ii}(a) = E_{ii}(a-1)F_{ii}(1)E_{ii}(a^{-1}-1)F_{ii}(-a), \quad a \in R^*,$$

and if $i \neq j$, then

$$K_{ij}(a) = F_{jj}(-a)E_{ij}(1)F_{jj}(a)E_{ii}(a)E_{ij}(-1), \quad a \in R.$$

1.1.1 On the factorization of a symplectic matrix

Let $M \in \mathrm{Sp}_{2n-2}(R)$ be a symplectic matrix in block notation 1.1, that is,

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $A, B, C, D \in M_{n-1}(R)$ are $(n-1) \times (n-1)$-matrices with entries in the ring R. Then the mapping $\psi : \mathrm{Sp}_{2n-2}(R) \to M_{2n}(R)$ given by

$$\psi(M) = \begin{pmatrix} A & 0 & B & 0 \\ 0 & 1 & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{Sp}_{2n}(R)$$

defines a natural inclusion $\mathrm{Sp}_{2n-2}(R) \to \mathrm{Sp}_{2n}(R)$.

Moreover, let $\rho : M_{2n}(R) \to M_{2n-2}(R)$ be the projection obtained by removing columns and rows n and $2n$, i.e.

$$\rho \begin{pmatrix} a_{11} & \cdots & a_{1,2n} \\ \vdots & & \vdots \\ a_{2n,1} & \cdots & a_{2n,2n} \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1,n-1} & a_{1,n+1} & \cdots & a_{1,2n-1} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,n-1} & a_{n-1,n+1} & \cdots & a_{n-1,2n-1} \\ a_{n+1,1} & \cdots & a_{n+1,n-1} & a_{n+1,n+1} & \cdots & a_{n+1,2n-1} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{2n-1,1} & \cdots & a_{2n-1,n-1} & a_{2n-1,n+1} & \cdots & a_{2n-1,2n-1} \end{pmatrix}.$$

Observe that $\rho \circ \psi(M) = M$, that is, ρ projects $\psi(\mathrm{Sp}_{2n-2}(R))$ into $\mathrm{Sp}_{2n-2}(R)$. We can even relax the sufficient conditions on the matrix $M \in \mathrm{Sp}_{2n}(R)$ such that its projection $\rho(M) \in \mathrm{Sp}_{2n-2}(R)$ is symplectic.

Lemma 1.1.6. *Let $M \in \mathrm{Sp}_{2n}(R)$ be a symplectic matrix of the form*

$$M = \begin{pmatrix} \star & \star \\ 0 & 1 \end{pmatrix}.$$

Then the projection $\rho(M) \in \mathrm{Sp}_{2n-2}(R)$ is a symplectic matrix.

Proof. By Lemma 1.1.4, the matrix M^T is symplectic and of the form

$$\begin{pmatrix} \star & 0 \\ \star & 1 \end{pmatrix}.$$

Let $e_i \in R^{2n}$ denote the element with a 1 in entry i and zeros elsewhere. Then

$$e_n = \Omega e_{2n} = M\Omega \underbrace{M^T e_{2n}}_{=e_{2n}} = M \underbrace{\Omega e_{2n}}_{=e_n} = Me_n.$$

Hence the matrix M is even of the form

$$\begin{pmatrix} A & 0 & B & b_1 \\ a^T & 1 & b_2^T & b_3 \\ C & 0 & D & d \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $A, B, C, D \in M_{n-1}(R)$ and $a, b_1, b_2, d \in R^{n-1}$, as well as $b_3 \in R$. A simple calculation proves that

$$\rho(M) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

satisfies the symplectic conditions (SC.1)-(SC.3). $\square$

Lemma 1.1.7. *Let $M \in \mathrm{Sp}_{2n}(R)$ be a symplectic matrix of the form*

$$M = \begin{pmatrix} \star & \star \\ 0 & 1 \end{pmatrix}.$$

Then

$$M = \begin{pmatrix} I_{n-1} & 0 & 0 & b \\ a^T & 1 & b^T & \beta \\ 0 & 0 & I_{n-1} & -a \\ 0 & 0 & 0 & 1 \end{pmatrix} \psi(\rho(M))$$

for some $a, b \in R^{n-1}$ and $\beta \in R$.

Proof. By the previous lemma, the matrix M is of the form

$$\begin{pmatrix} A & 0 & B & b_1 \\ a^T & 1 & b_2^T & b_3 \\ C & 0 & D & d \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

for some $A, B, C, D \in M_{n-1}(R)$ and $a, b_1, b_2, d \in R^{n-1}$, as well as $b_3 \in R$. The projection $\rho(M)$ is symplectic and by Lemma 1.2, the inverse of $\psi(\rho(M))$ is given by

$$\begin{pmatrix} D^T & 0 & -B^T & 0 \\ 0 & 1 & 0 & 0 \\ -C^T & 0 & A^T & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore we obtain

$$\begin{aligned} M\psi(\rho(M))^{-1} &= \begin{pmatrix} A & 0 & B & b_1 \\ a^T & 1 & b_2^T & b_3 \\ C & 0 & D & d \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D^T & 0 & -B^T & 0 \\ 0 & 1 & 0 & 0 \\ -C^T & 0 & A^T & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} AD^T - BC^T & 0 & -AB^T + BA^T & b_1 \\ a^T D^T - b_2^T C^T & 1 & -a^T B^T + b_2^T A^T & b_3 \\ CD^T - DC^T & 0 & -CB^T + DA^T & d \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} I_{n-1} & 0 & 0 & b_1 \\ a^T D^T - b_2^T C^T & 1 & -a^T B^T + b_2^T A^T & b_3 \\ 0 & 0 & I_{n-1} & d \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

and for the last equation we simply applied the symplectic conditions (SC.1)-(SC.3). Another application of these conditions yields

$$a^T D^T - b_2^T C^T = -d^T \quad \text{and} \quad -a^T B^T + b_2^T A^T = b_1^T.$$

This proves the claim. □

Lemma 1.1.8 (2nd Whitehead lemma)**.** *Given $a, b \in R^{n-1}$ and $\beta \in R$, the matrix*

$$\begin{pmatrix} I_{n-1} & 0 & 0 & b \\ a^T & 1 & b^T & \beta \\ 0 & 0 & I_{n-1} & -a \\ 0 & 0 & 0 & 1 \end{pmatrix} \in M_{2n}(R)$$

can be written as a product of four elementary symplectic matrices.

Proof. Let $A, B, C, D \in M_n(R)$ be symmetric matrices. Then

$$\begin{pmatrix} I_n & 0 \\ A & I_n \end{pmatrix} \begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 \\ C & I_n \end{pmatrix} \begin{pmatrix} I_n & D \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} I_n + BC & B + (I_n + BC)D \\ A(I_n + BC) + C & AD + (I_n + AB)(I_n + CD) \end{pmatrix}$$

Choosing $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & a \\ a^T & 0 \end{pmatrix}$, we get

$$I_n + BC = \begin{pmatrix} I_{n-1} & 0 \\ a^T & 1 \end{pmatrix}.$$

This matrix is obviously regular and its inverse is obtained if we replace a^T by $-a^T$.

Then, we request $0 = A(I_n + BC) + C$, which is satisfied if and only if

$$A = -C(I_n + BC)^{-1} = \begin{pmatrix} 0 & -a \\ -a^T & 0 \end{pmatrix} \begin{pmatrix} I_{n-1} & 0 \\ -a^T & 1 \end{pmatrix} = \begin{pmatrix} aa^T & -a \\ -a^T & 0 \end{pmatrix}.$$

Finally, we want

$$\begin{pmatrix} 0 & b \\ b^T & \beta \end{pmatrix} = B + (I_n + BC)D$$

which is the case if and only if

$$D = \begin{pmatrix} I_{n-1} & 0 \\ -a^T & 1 \end{pmatrix} \begin{pmatrix} 0 & b \\ b^T & \beta - 1 \end{pmatrix} = \begin{pmatrix} 0 & b \\ b^T & \beta - 1 - a^T b \end{pmatrix}.$$

The last block of equations is automatically satisfied because of the symplectic conditions. This proves the claim. □

The following corollary is the foundation for induction in the factorization.

Corollary 1.1.9. *Let $M \in \mathrm{Sp}_{2n}(R)$ a symplectic matrix and suppose that there are finitely many elementary symplectic matrices $E_1, ..., E_k \in \mathrm{Sp}_{2n}(R)$ such that*

$$A := ME_1 \cdots E_k = \begin{pmatrix} \star & \star \\ 0 & 1 \end{pmatrix}.$$

Then M can be decomposed into a finite product if and only if $\rho(A)$ can be decomposed into a finite product.

1.2 Complex manifolds and holomorphic mappings

We let $\mathbb{R}$ and $\mathbb{C}$ denote the field of real and complex numbers, respectively. The model n-dimensional complex manifold is the Euclidean space $\mathbb{C}^n$. Let $z = (z_1, ..., z_n) \in \mathbb{C}^n$ denote the coordinates on $\mathbb{C}^n$. Write $z_j = x_j + iy_j$, where $x_j, y_j \in \mathbb{R}$ and $i = \sqrt{-1}$. Given a differentiable complex valued function $f : D \to \mathbb{C}$ on a domain $D \subset \mathbb{C}^n$, the differential $\mathrm{d}f$ splits as the sum of the $\mathbb{C}$-linear part ∂f and the $\mathbb{C}$-antilinear part $\bar{\partial} f$:

$$\mathrm{d}f = \partial f + \bar{\partial} f = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} \mathrm{d}z_j + \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j} \mathrm{d}\bar{z}_j.$$

Here $\mathrm{d}z_j = \mathrm{d}x_j + i\mathrm{d}y_j$, $\mathrm{d}\bar{z}_j = \mathrm{d}x_j - i\mathrm{d}y_j$, and

$$\frac{\partial f}{\partial z_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} - i\frac{\partial f}{\partial y_j}\right), \quad \frac{\partial f}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}\right).$$

The function f is *holomorphic* on D if $\mathrm{d}f = \partial f$ on D; that is, the differential df_z is $\mathbb{C}$-linear at every point $z \in D$. Equivalently, f is holomorphic if and only if $\bar{\partial} f = 0$, and this is equivalent to the n equations

$$\frac{\partial f}{\partial \bar{z}_j} = 0, \quad j = 1, ..., n.$$

A mapping $f = (f_1, f_2, ..., f_m) : D \to \mathbb{C}^m$ is holomorphic if each component function f_j is such. When $m = n$, f is biholomorphic onto its image $D' = f(D) \subset \mathbb{C}^n$ if it is bijective and its inverse $f^{-1} : D' \to D$ is holomorphic. An injective holomorphic map of a domain $D \subset \mathbb{C}^n$ to $\mathbb{C}^n$ is always biholomorphic onto its image [10, p.19].

A *topological manifold of dimension* n is a second countable Hausdorff topological space which is locally Euclidean, in the sense that each point has an open neighborhood homeomorphic to an open set in $\mathbb{R}^n$. Such a space is metrizable, countably compact and paracompact.

Assume now that X is a topological manifold of even dimension $2n$. A *complex atlas* on X is a collection $\mathcal{U} = \{(U_\alpha, \phi_\alpha)\}_{\alpha\in A}$, where $\{U_\alpha\}_{\alpha\in A}$ is an open cover of X and ϕ_α is aa homeomorphism of U_α onto an open subset U'_α in $\mathbb{R}^{2n} = \mathbb{C}^n$ such that for every pair of indices $\alpha, \beta \in A$ the *transition map*

$$\phi_{\alpha,\beta} = \phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_{\alpha,\beta}) \to \phi_\alpha(U_{\alpha,\beta})$$

is biholomorphic. Here $U_{\alpha,\beta} = U_\alpha \cap U_\beta$. An element (U_α, ϕ_α) of a complex atlas is called a *complex chart*, or a *local holomorphic coordinate system* on X. We also say that charts in a complex atlas are *holomorphically compatible*. For any three indices $\alpha, \beta, \gamma \in A$ we have

$$\phi_{\alpha,\alpha} = \mathrm{Id}, \quad \phi_{\alpha,\beta} = \phi_{\beta,\alpha}^{-1}, \quad \phi_{\alpha,\beta} \circ \phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$$

on the respective domains of these maps. Two complex atlases $\mathcal{U}, \mathcal{V}$ on a topological manifold X are said to be *homotopically compatible* if their union $\mathcal{U} \cup \mathcal{V}$ is also a complex atlas. This is an equivalence relation on the set of all complex atlases on X. Each equivalence class contains a unique *maximal complex atlas* - the union of all complex atlases in the given class.

A *complex manifold of complex dimension* n is a topological manifold X of real dimension $2n$ equipped with a complex atlas. Two complex atlases determine the same complex structure on X if and only if they are holomorphically compatible. A complex manifold of dimension $n = 1$ is called a *Riemann surface* and a *complex surface* is a complex manifold of dimension $n = 2$.

A function $f : X \to \mathbb{C}$ on a complex manifold is said to be holomorphic if for any chart (U, ϕ) from the maximal atlas on X the function $f \circ \phi^{-1} : \phi(U) \to \mathbb{C}$ is holomorphic on the open set $\phi(U) \subset \mathbb{C}^n$. We let $\mathcal{O}(X)$ denote the Fréchet algebra of all holomorphic functions on X with the compact-open topology.

Let X and Y be complex manifolds of dimensions n and m, respecively. A continuous map $f : X \to Y$ is said to be holomorphic if for any point $x \in X$ there are complex charts (U, ϕ) on X and (V, ψ) on Y such that $p \in U$, $f(U) \subset V$, and the map $\psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V) \subset \mathbb{C}^m$ is holomorphic on the open set $\phi(U) \subset \mathbb{C}^n$. Since the charts in a complex atlas are holomorphically compatible, the choice of charts is not important.

A map $f : X \to Y$ is *biholomorphic* if it is bijective and if both f and its inverse $f^{-1} : Y \to X$ are holomorphic. Note that a bijective holomorphic map between complex manifolds is actually biholomorphic.

A biholomorphic self-map $f : X \to X$ is called a *holomorphic automorphism* of X; the collection of all automorphisms is the *holomorphic automorphism group* $\mathrm{Aut}(X) = \mathrm{Aut}_{hol}(X)$.

Let X be a complex manifold of dimension n. A subset M of M is a *complex submanifold* of dimension $m \in \{0, ..., n\}$ (and codimension $d = n - m$) if every point $p \in M$ admits an open neighborhood $U \subset X$ and a holomorphic chart $\phi : U \to U' \subset \mathbb{C}^n$ such that $\phi(U \cap M) = U' \cap (\mathbb{C}^m \times \{0\}^{n-m})$.

1.3 Subvarieties and complex spaces

Let X be a complex manifold. We let $\mathcal{O}_x = \mathcal{O}_{X,x}$ denote the ring of germs of holomorphic functions at a point $x \in X$. A germ $[f]_x \in \mathcal{O}_x$ is represented by a holomorphic function in an open neighborhood of x; two such functions determine the same germ at x if and only if they agree in some neighborhood of x. The ring $\mathcal{O}_{X,x}$ is isomorphic to the ring $\mathcal{O}_{\mathbb{C}^n,0}$ via any holomorphic coordinate map sending x to 0. We can identify $\mathcal{O}_{\mathbb{C}^n,0}$ with the ring of convergent power series in n complex variables $(z_1, ..., z_n)$. This ring is Noetherian and a unique factorization domain. Its units are precisely the germs that do not vanish at 0. The disjoint union $\mathcal{O}_X = \cup_{x \in X} \mathcal{O}_{X,x}$ is equipped with the topology whose basis is given by sets $\{[f]_x : x \in U\}$, where $f : U \to \mathbb{C}$ is a holomorphic function on an open set $U \subset X$. This makes $\mathcal{O}_X$ into a sheaf of commutative rings, called the *sheaf of germs of holomorphic functions*.

A subset A of a complex manifold X is a *complex (analytic) subvariety* of X if for every point $p \in A$ there exists a neighborhood $U \subset X$ of p and functions $f_1, ..., f_d \in \mathcal{O}(U)$ such that

$$A \cap U = \{x \in U : f_1(x) = 0, ..., f_d(x) = 0\}.$$

If such A is topologically closed in X then A is a *closed complex subvariety* of X. Since the local ring $\mathcal{O}_x$ is Noetherian, a subset of X that is locally defined by infinitely many holomorphic equations is still a subvariety and can be locally defined by finitely many equations.

A point p in a subvariety A is a *regular* (or *smooth*) point if A is a complex submanifold near p; the set of all regular points is denoted A_{reg}. A point $p \in A \setminus A_{\mathrm{reg}} = A_{\mathrm{sing}}$ is a *singular point* of A.

A *reduced complex space* is a pair $(X, \mathcal{O}_X)$, where X is a paracompact Hausdorff topological space and $\mathcal{O}_X$ is a sheaf of rings of continuous functions on X (a subsheaf of the sheaf $\mathcal{C}_X$ of germs of continuous functions) such that for every point $x \in X$ there is a neighborhood $U \subset X$ and a homeomorphism $\phi : U \to A \subset \mathbb{C}^n$ onto a locally closed complex subvariety of $\mathbb{C}^n$ so that the homeomorphism $\phi^* : \mathcal{C}_A \to \mathcal{C}_X, f \mapsto f \circ \phi$, induces an isomorphism of $\mathcal{O}_A$ onto $\mathcal{O}_U = \mathcal{O}_X|_U$. Intuitively speaking, X is obtained by gluing pieces of subvarieties in Euclidean spaces using biholomorphic transition maps.

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be complex spaces. A continuous map $f : X \to Y$ is said to be holomorphic if for every $x \in X$ the composition $\mathcal{C}_{Y,f(x)} \ni g \mapsto g \circ f \in \mathcal{C}_{X,x}$ defines a homomorphism $f_x^* : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$.

Definition 1.3.1 (Submersion)**.** *Let Z and X be reduced complex spaces. A holomorphic map $\pi : Z \to X$ is a holomorphic submersion if for every point $z_0 \in Z$ there exist an open neighborhood $V \subset Z$ of z_0, an open neighborhood $U \subset X$ of $x_0 = \pi(z_0)$, an open set W in $\mathbb{C}^p$, and a biholomorphic map $\phi : V \to U \times W$ such that $pr_1 \circ \phi = \pi$. (Here $pr_1 : U \times W \to U$ is the projection on the first factor). Each such local chart ϕ will be called adapted to π.*

Definition 1.3.2 (Stratification)**.** *A stratification of a finite dimensional complex space X is a finite descending sequence*

$$X = X_0 \supset X_1 \supset \cdots \supset X_m = \emptyset$$

of closed complex subvarieties such that each connected component S (stratum) of a difference $X_k \setminus X_{k+1}$ is a complex manifold and $\bar{S} \setminus S \subset X_{k+1}$.

1.4 Tangent bundle and vector fields

We assume that the reader is familiar with the construction of the real tangent bundle TX of a smooth manifold X. A tangent vector $V_x \in T_xX$ is viewed as a derivation $\mathcal{C}_x^\infty \ni f \mapsto V_x(f) \in \mathbb{R}$ on the algebra of germs of smooth functions at x. Sections $X \to TX$ are called *vector fields* on X. The complexification $\mathbb{C}TX = TX \otimes \mathbb{C}$ of TX is the *complexified tangent bundle* of X; its sections are called *complex vector fields* on X.

Assume now that X is a complex manifold. There is a unique real linear endomorphism $J \in \mathrm{End}_{\mathbb{R}}TX$, called the *almost complex structure operator*, which is given in any local holomorphic coordinate system $z = (z_1, ..., z_n)$ on X by

$$J\frac{\partial}{\partial x_j} = \frac{\partial}{\partial y_j}, \quad J\frac{\partial}{\partial y_j} = -\frac{\partial}{\partial x_j}.$$

The operator J extends to $\mathbb{C}TX$ by $J(v\otimes\alpha) = J(v)\otimes\alpha$ for $v \in TX$ and $\alpha \in \mathbb{C}$. From $J^2 = -\mathrm{Id}$ we infer that the eigenvalues of J are $+i$ and $-i$. Hence we have a decomposition

$$\mathbb{C}TX = T^{1,0}X \oplus T^{0,1}X$$

into the $+i$ eigenspace $T^{1,0}X$ and the $-i$ eigenspace $T^{0,1}X$ of J. In holomorphic coordinates $z = (z_1, ..., z_n)$ on an open subset $U \subset X$ we have

$$T^{1,0}X|_U = \mathrm{Span}_{\mathbb{C}}\left\{\frac{\partial}{\partial z_1}, ..., \frac{\partial}{\partial z_n}\right\}, \quad T^{0,1}X|_U = \mathrm{Span}_{\mathbb{C}}\left\{\frac{\partial}{\partial \bar{z}_1}, ..., \frac{\partial}{\partial \bar{z}_n}\right\},$$

where

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

We have an $\mathbb{R}$-linear isomorphism $\Phi : TX \to T^{0,1}X$ given by

$$\Phi(V) = \frac{1}{2}(V - iJV).$$

In local coordinates the isomorphism Φ is given by

$$\Phi : \sum_{j=1}^{n}\left(a_j\frac{\partial}{\partial x_j} + b_j\frac{\partial}{\partial y_j}\right) \mapsto \sum_{j=1}^{n}(a_j + ib_j)\frac{\partial}{\partial z_j}.$$

Definition 1.4.1. *A real vector field V on X is said to be holomorphic if $\Phi(V) = \frac{1}{2}(V - iJV)$ is a holomorphic section of $T^{1,0}X$.*

Definition 1.4.2 ($\mathbb{C}$-complete vector field)**.** *Let X be a complex manifold. A holomorphic vector field V on X is called $\mathbb{C}$-complete if for each $x \in X$, the initial value problem*

$$\phi_0(x) = x, \quad \frac{d}{dt}\phi_t(x) = V(\phi_t(x))$$

can be solved in complex time $t \in \mathbb{C}$; the map $\phi_t : X \to X$ is called vector flow of V.

1.5 Stein spaces and Stein manifolds

Definition 1.5.1 (Stein manifold)**.** *Suppose X is a complex manifold of complex dimension n and let $\mathcal{O}(X)$ denote the ring of holomorphic functions on X. We call X a **Stein manifold** if the following conditions hold:*

(i) X is holomorphically convex, i.e. for every compact subset $K \subset X$, the holomorphically convex hull

$$\hat{K} = \left\{ x \in X : |f(x)| \leq \sup_{w \in K} |f(w)| \ \forall f \in \mathcal{O}(X) \right\},$$

is also a compact subset of X.

(ii) X is holomorphically separable, i.e. if $x \neq y$ are two points in X, then there exists $f \in \mathcal{O}(X)$ such that $f(x) \neq f(y)$.

(iii) For every point $p \in X$ there exist functions $f_1, ..., f_n \in \mathcal{O}(X)$, whose differentials $\mathrm{d}f_j$ are $\mathbb{C}$-linearly independent at p.

Remark 1.5.2. *Property (iii) is redundant in the sense that it follows from properties (i) and (ii) because of Cartan's Theorem A.*

Examples:

- $\mathbb{C}^n$ is Stein.
- Every closed complex submanifold of a Stein manifold is Stein.
- The Cartesian product $X \times Y$ of two Stein manifolds X, Y is Stein.
- Stein manifolds are non-compact: holomorphic functions on a compact manifold are constant by the maximum principle and hence they don't separate points.

Definition 1.5.3 (Stein space)**.** *A second countable complex space X is a **Stein space** if it satisfies properties (i), (ii) in Definition 1.5.1 and also*

(iii') Every local ring $\mathcal{O}_{X,x}$ is generated by functions in $\mathcal{O}(X)$.

Condition (iii') means that there is a holomorphic map $X \to \mathbb{C}^N$ which embeds a neighborhood of x as a local complex subvariety of $\mathbb{C}^N$.

1.6 Elliptic Complex-geometry and Oka principle

The Oka principle is a powerful tool. Roughly speaking, it allows us to homotopically deform a continuous mapping into a holomorphic one in certain situations. We want to make this more precise in this section and therefore introduce some definitions and terminologies.

Let's start with the notion of a *stratified elliptic submersion* $h : Z \to X$ from a complex space Z onto a complex space X, following [11] and [7].

Let $h\colon Z \to X$ be a holomorphic submersion of a complex manifold Z onto a complex manifold X. For any $x \in X$ the fiber over x of this submersion will be denoted by Z_x. At each point $z \in Z$ the tangent space T_zZ contains *the vertical tangent space* $VT_zZ = \ker Dh$. For holomorphic vector bundles $p\colon E \to Z$ we let 0_z denote the zero element in the fiber E_z.

Definition 1.6.1. *Let $h\colon Z \to X$ be a holomorphic submersion of a complex manifold Z onto a complex manifold X. A spray on Z associated with h is a triple (E, p, s), where $p\colon E \to Z$ is a holomorphic vector bundle and $s\colon E \to Z$ is a holomorphic map such that for each $z \in Z$ we have*

(i) $s(E_z) \subset Z_{h(z)}$,

(ii) $s(0_z) = z$, *and*

(iii) the derivative $Ds(0_z)\colon T_{0_z}E \to T_zZ$ maps the subspace $E_z \subset T_{0_z}E$ surjectively onto the vertical tangent space VT_zZ.

Remark 1.6.2. *We will also say that the submersion admits a spray. A spray associated with a holomorphic submersion is sometimes called a (fiber) dominating spray.*

One way of constructing dominating sprays, as pointed out by Gromov, is to find finitely many $\mathbb{C}$-complete vector fields that are tangent to the fibers and span the tangent space of the fibers at all points in Z. One can then use the flows φ_j^t of these vector fields V_j to define $s\colon Z \times \mathbb{C}^N \to Z$ via $s(z, t_1, \ldots, t_N) = \varphi_1^{t_1} \circ \cdots \circ \varphi_N^{t_N}(z)$ which gives a spray.

Definition 1.6.3. *We say that a submersion $h\colon Z \to X$ is stratified elliptic if there is a descending chain of closed complex subspaces $X = X_m \supset \cdots \supset X_0$ such that each stratum $Y_k = X_k \setminus X_{k-1}$ is regular and the restricted submersion $h\colon Z|_{Y_k} \to Y_k$ admits a spray over a small neighborhood of any point $x \in Y_k$.*

Remark 1.6.4. *We say that the submersion admits stratified sprays and that the stratification $X = X_m \supset \cdots \supset X_0$ is associated with the stratified spray.*

Let's consider the following diagram

$$\begin{tikzcd} P_0 \times X \arrow[r, "F"] \arrow[d, "\mathrm{incl}"'] & E \arrow[d, "\pi"] \\ P \times X \arrow[r, "f"] \arrow[ur, "F"] & B \end{tikzcd}$$

Here $\pi : E \to B$ is a holomorphic submersion of a complex space E onto a complex space B, X is a Stein space, $P_0 \subset P$ are compact Hausdorff spaces (the parameter spaces), and $f : P \times X \to B$ is an X-holomorphic map, meaning that $f(p, \cdot) : X \to B$ is holomorphic on X for every fixed $p \in P$. A map $F : P \times X \to E$ such that $\pi \circ F = f$ is said to be a *lifting* of f; such F is X-holomorphic on P_0 if $F(p, \cdot)$ is holomorphic for every $p \in P_0$.

Definition 1.6.5. *A holomorphic map $\pi : E \to B$ between reduced complex spaces enjoys the Parametric Oka Property (POP) if for any collection $(X, X', K, P, P_0, f, F_0)$ where X is a reduced Stein space, X' is a closed complex subvariety of X, $P_0 \subset P$ are compact Hausdorff spaces, $f : P \times X \to B$ is an X-holomorphic map, and $F_0 : P \times X \to E$ is a continuous map such that $\pi \circ F = f$, the map $F_0(p, \cdot)$ is holomorphic on X for all $p \in P_0$ and is holomorphic on $K \cup X'$ for all $p \in P$, there exists a homotopy $F_t : P \times X \to E$ such that the following hold for all $t \in [0, 1]$:*

(i) $\pi \circ F_t = f$,

(ii) $F_t = F_0$ *on* $(P_0 \times X) \cup (P \times X')$,

(iii) F_t is X-holomorphic on K and uniformly close to F_0 on $P \times K$, and

(iv) the map $F_1 : P \times X \to E$ is X-holomorphic.

The following version of the Oka principle has first been shown for Euclidean compact parameter spaces P by Forstneric. A proof can be found in [8, Corollary 6.14.4 (i)]. Kusakabe has shown (see [21, Corollary 5.7]) that it holds even more generally for compact Hausdorff parameter spaces.

Theorem 1.6.6 (Oka principle)**.** *Every stratified elliptic submersion enjoys POP.*

1.6.1 Construction of a dominating spray

In the previous section we saw one way to construct a dominating spray associated to a submersion (see paragraph before Definition 1.6.3). Namely, we define some finite composition of flow maps of complete fiber-preserving vector fields. It turns out that in practice it can be difficult to find enough $\mathbb{C}$-complete fiber-preserving vector fields that span the tangent space of the fibers at all points. In the following section we develop a strategy to solve these difficulties.

Let M be a Stein manifold and let $\mathcal{VC}_{hol}(M)$ denote the set of $\mathbb{C}$-complete holomorphic vector fields on M. For a vector field $V \in \mathcal{VC}_{hol}(M)$ its corresponding flow α_t^V, $t \in \mathbb{C}$, is a one-parameter subgroup of $\mathrm{Aut}_{hol}(M)$, the holomorphic automorphism group on M. For a set $A \subset \mathcal{VC}_{hol}(M)$ of complete holomorphic vector fields on M we define

$$S_A := \bigcup_{V \in A} \{\alpha_t^V : t \in \mathbb{C}\} \subset \mathrm{Aut}_{hol}(M).$$

Let $G_A := \langle S_A \rangle$ denote the subgroup generated by S_A. Furthermore, we let $\alpha^*\theta$ denote the pull-back of a vector field θ by an automorphism α.

Definition 1.6.7. *For a set $A \subset \mathcal{VC}_{hol}(M)$ of complete holomorphic vector fields on M, define*

$$\Gamma(A) := \{\alpha^* X : \alpha \in G_A, X \in A\}$$

the collection of complete holomorphic vector fields generated by A, and for an open set $Y \subset M$ define

$$C_A(Y) := \{\alpha(y) : \alpha \in G_A, y \in Y\}$$

the G_A-closure of Y.

Some basic properties follow directly from the definition. Let $Y \subset M$ be an open set. Then $C_A(Y)$ is the smallest set containing Y, which is invariant under G_A. Moreover, $C_A(Y)$ is open in M, hence, for a fixed collection $A \subset \mathcal{VC}_{hol}(M)$, C_A can be interpreted as a map $C_A : \mathcal{T}_M \to \mathcal{T}_M$, where $\mathcal{T}_M$ denotes the natural topology on M. In particular, C_A satisfies the conditions of a topological closure operator.

The following lemma describes another basic property.

Lemma 1.6.8. *Let $A, B \subset \mathcal{VC}_{hol}(M)$ be finite collections of complete holomorphic vector fields on M with $A \subset B \subset \Gamma(A)$. Then $C_A(X) = C_B(X)$ for all open subsets $X \subset M$.*

Proof. We're going to prove that $G_A = G_B$. To do this, it suffices to show that $G_B \subset G_A$, since the reverse inclusion trivially holds by assumption $A \subset B$.

At first, we consider $V \in B$. There is $\beta \in G_A$ and $W \in A$ with $V = \beta^* W$, since $B \subset \Gamma(A)$. The flow of V satisfies

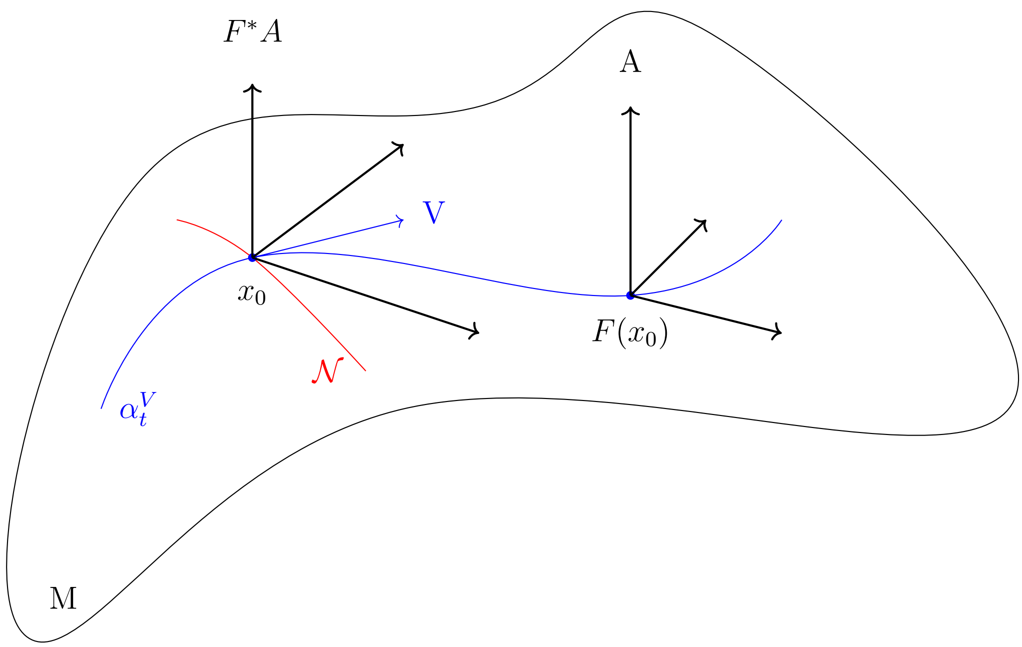
$$\alpha_t^V = \alpha_t^{\beta^* W} = \beta \circ \alpha_t^W \circ \beta^{-1} \in G_A,$$

since $\beta, \alpha_t^W \in G_A$.

In a next step, let $\beta \in G_B$ be any automorphism. By definition of G_B, there are vector fields $V_{i_1}, ..., V_{i_m} \in B$ and times $t_1, ..., t_m \in \mathbb{C}$ with $\beta = \alpha_{t_1}^{V_{i_1}} \circ \cdots \circ \alpha_{t_m}^{V_{i_m}}$. From the previous step we know that each $\alpha_{t_j}^{V_{i_j}} \in G_A$ is an automorphism in G_A and hence so is β, i.e. $\beta \in G_A$ and this proves the claim. $\square$

Before clarifying the technical details, let us use a simple example to explain the basic idea of suitably extending a given collection of vector fields so that the new collection spans the tangent space of a manifold over a larger set of points.

Consider a Stein manifold M and suppose that there is finite collection $A \subset \mathcal{VC}_{hol}(M)$ which spans the tangent space T_xM for all points $x \in M \setminus \mathcal{N}$ outside some analytic subset $\mathcal{N} \subset M$. Let $x_0 \in \mathcal{N}$ be some point and assume that there is a vector field $V \in \mathcal{VC}_{hol}(M)$ whose flow α_t^V through x_0 leaves the set $\mathcal{N}$. For some fixed time $t \in \mathbb{C}$ the map $F := \alpha_t^V$ is an automorphism of M with $F(x_0) \in M \setminus \mathcal{N}$. Hence the collection A spans the tangent space $T_{F(x_0)}M$. Now we pull the vector fields in A back with the automorphism F, add them to the collection A and obtain a larger collection which spans the tangent space $T_{x_0}M$ in $x_0 \in \mathcal{N}$.



So much for the idea. Now we have to make sure that this idea is applicable in a finite process. The following result represents the main step for this.

Lemma 1.6.9. *Let M be a Stein manifold, $X \subset M$ an open subset and $A \subset \mathcal{VC}_{hol}(M)$ a finite set of $\mathbb{C}$-complete holomorphic vector fields on M which spans the tangent bundle TX. Then there is a finite subset $A \subset B \subset \Gamma(A)$ which spans the tangent bundle $TC_A(X)$.*

Proof. For each field $V \in A$ let $\alpha_t^V, t \in \mathbb{C}$, be the corresponding vector flow. Let N_0 be the set of points $x \in C_A(X)$ where the fields of A don't span the tangent space T_xM. This is an analytic subset $N_0 \subset C_A(X) \setminus X$. Further, we define

$$N_k := \{x \in N_{k-1} : \alpha_t^V(x) \in N_{k-1}, \forall V \in A, \forall t \in \mathbb{C}\}, \quad k \geq 1.$$

Let $k \geq 0$ be arbitrary but fixed. Then N_k has at most countably many connected components. Let $A_i^k, i \in I_k$, denote those connected components of N_k which aren't entirely contained in N_{k+1} and let $a_k := \max_{i \in I_k} \dim A_i^k$ be the maximal dimension of them. Choose a point $x_i^k \in A_i^k \cap N_k \setminus N_{k+1}, i \in I_k$, of each such component. By definition of the sets N_k and N_{k+1}, there is a field $V \in A$ for each point x in the sequence $\{x_i^k\}_{i \in I}$, such that $\alpha_t^V(x) \notin N_k$ for some $t \in \mathbb{C}$. For $V \in A$ define

$$u_V^k := \{x \in \{x_i^k\}_{i \in I_k} : \alpha_t^V(x) \notin N_k, \text{ for some } t \in \mathbb{C}\}.$$

Then this yields
$$\{x_i^k\}_{i\in I_k} = \bigcup_{V\in A} u_V^k.$$
For each point $x \in u_V^k$ the set $\{t \in \mathbb{C} : \alpha_t^V(x) \in N_k\}$ is discrete and hence
$$\bigcup_{x\in u_V^k} \{t \in \mathbb{C} : \alpha_t^V(x) \in N_k\}$$
is a meagre set in $\mathbb{C}$. This implies the existence of a time $t_V^k \in \mathbb{C}$ such that $\alpha_{t_V^k}^V(x) \notin N_k$ for all $x \in u_V^k$. Define
$$\tilde{N}_{k+1} := \{x \in N_k : \alpha_{t_V^k}^V(x) \in N_k, \forall V \in A\}.$$
Clearly,
$$N_{k+1} \subset \tilde{N}_{k+1} \subset N_k$$
holds true. The set $\tilde{N}_{k+1}$ has at most countably many connected components. Let $\tilde{a}_{k+1}$ denote the maximal dimension of them. By construction, we have $a_{k+1} \le \tilde{a}_{k+1} < a_k$. Since M is finite dimensional, this implies that there is $L \in \mathbb{N}$ such that $N_k = \emptyset$ for all $k > L$.

Let B_k be the set of pullbacks $(\alpha_{t_V^k}^V)^*(W)$ for $V, W \in A$ and set
$$B := \bigcup_{k\ge 0}^{L} A \cup B_k.$$
Again by construction, the collection $B \subset \Gamma(A)$ is a finite set of $\mathbb{C}$-complete holomorphic vector fields on M that spans the tangent bundle $TC_A(X)$. Moreover, $C_A(X) = C_B(X)$ by the previous lemma, which implies that $C_A(X)$ is invariant under the flows of B. This finishes the proof. □

In a next step, we want to adapt this argument to our setting so that we can apply it to every fiber simultaneously, so to speak. Moreover, we are ready to define a dominating spray.

Lemma 1.6.10. *Let M be a Stein manifold, $\pi : M \to Y$ a holomorphic mapping and $X \subset M$ a (connected) open subset such that the restriction $\pi|_X : X \to Y$ is a surjective submersion with connected fibers.*

Suppose that there is a finite set $A \subset \mathcal{VC}_{hol}(M)$ of $\mathbb{C}$-complete fiber-preserving holomorphic vector fields on M which spans the tangent bundle $T(M_y \cap X)$ of each fiber $M_y := \pi^{-1}(y)$. Then there is a finite set $B \subset \Gamma(A)$ of $\mathbb{C}$-complete fiber-preserving holomorphic vector fields which spans the tangent bundle $T(C_A(M_y \cap X))$ of each fiber M_y. In particular, the surjective submersion $\pi|_{C_A(X)}$ admits a spray.

Proof. We can proceed similarly as in the previous lemma to obtain a finite collection $B \subset \Gamma(A)$ which spans the tangent bundle $T(C_A(M_y \cap X))$ of each fiber M_y. This follows from the assumption that all fields in A are fiber-preserving.

Moreover, the map $\pi|_X : X \to Y$ is a surjective submersion, hence $\pi|_{C_A(X)} : C_A(X) \to Y$ is also a surjective submersion. Write $B = \{W_1, ..., W_L\}$. Then the map $s : C_A(X) \times \mathbb{C}^L \to C_A(X)$ given by
$$s(z, t_1, ..., t_L) = \alpha_{t_1}^{W_1} \circ \cdots \circ \alpha_{t_L}^{W_L}(z)$$
is a dominating spray associated to $\pi|_{C_A(X)}$, since $C_A(X)$ is invariant with respect to the flows of $W_1, ..., W_L$ by Lemma 1.6.8. □

This corollary shows that we can relax the assumptions of the previous lemma and we will apply it in this form later.

Corollary 1.6.11. *Let M be a Stein manifold, $\pi : M \to Y$ a holomorphic map and $X \subset M$ a connected open set such that the restriction $\pi|_X$ is a surjective submersion with connected fibers. Furthermore, we are given an open subset $W \subset X$ and a finite collection $A \subset \mathcal{VC}_{hol}(M)$ of complete fiber-preserving holomorphic vector fields on M which spans the tangent bundle $T(M_y \cap W)$ of each fiber M_y.*

Suppose that there is a finite collection $B \subset \mathcal{VC}_{hol}(M)$ of complete fiber-preserving holomorphic vector fields on M such that $X \subset C_B(W)$ and $A \subset B$. Then $\pi|_{C_B(W)}$ admits a spray.

Proof. Since B contains A by assumption, B spans the tangent bundle $T(M_y \cap W)$ of each fiber, hence there is a finite collection $\tilde{B} \subset \Gamma(B)$ which spans the tangent bundle $T(M_y \cap C_B(W))$ of each fiber. Since $X \subset C_B(W)$ by assumption, $\tilde{B}$ spans the tangent bundle $T(M_y \cap X)$ for each fiber. Now, we apply Lemma 1.6.10 to finish the proof. □

Remark 1.6.12. *Given a finite collection $A \subset \mathcal{VC}_{hol}(M)$ and two open sets $X, W \subset M$ with $W \subset X$, $X \subset C_A(W)$ is true if and only if $X \setminus W \subset C_A(W)$. This follows from the fact that C_A is extensive, i.e. $X \subset C_A(X)$ for all open sets $X \subset M$.*

This lemma will help us decide if we have $X \subset C_A(X)$ for a suitable finite collection A.

Lemma 1.6.13. *Let M be a Stein manifold and $N \subset M$ an analytic subvariety given by*

$$N := \{x \in M : f(x) = 0\}$$

for some holomorphic mapping $f : M \to \mathbb{C}$. Assume that there are complete holomorphic vector fields $V_1, ..., V_k$ on M (and we let $\alpha_t^1, ..., \alpha_t^k$ denote the respective flows) such that

$$V_{i_n} \circ \cdots \circ V_{i_1}(f(x)) \neq 0, \forall x \in N,$$

for some finite sequence $\{i_1, ..., i_n\} \subset \{1, ..., k\}$. Then there is a composition of the flows $\alpha_t^1, ..., \alpha_t^k$ which leaves the subvariety N. More precisely,

$$\{\alpha_{t_n}^{i_n} \circ \cdots \circ \alpha_{t_1}^{i_1}(x) : t_1, ..., t_n \in \mathbb{C}\} \not\subset N, \quad \forall x \in N.$$

Proof. Define the subvariety $N_1 := \{x \in N : V_{i_1}(f(x)) = 0\}$. The orbit of $\alpha_t^{i_1}$ through points of $N \setminus N_1$ is leaving N. Next, define the subvariety $N_2 := \{x \in N_1 : V_{i_2}(V_{i_1}(f(x))) = 0\}$. Then the orbit of $\alpha_t^{i_2}$ through points of $N_1 \setminus N_2$ is leaving N_1. We proceed inductively and set $N_l := \{x \in N_{l-1} : V_{i_l} \circ \cdots \circ V_{i_1}(f(x)) = 0\}$. Then the orbit of $\alpha_t^{i_l}$ through points of $N_{l-1} \setminus N_l$ is leaving N_{l-1}. This is true for all $1 \leq l \leq n-1$, which implies that an invariant set with respect to the fields $V_1, ..., V_k$ has to be in the set N_n. By assumumption, we have $V_{i_n} \circ \cdots \circ V_{i_1}(f(x)) \neq 0$. Hence the set N_n is empty and there is no invariant set in N with respect to the fields $V_1, ..., V_k$. This proves the claim. □

2 Holomorphic Vaserstein problem for symplectic matrices

We are interested in decomposing a symplectic matrix into a finite product of elementary symplectic matrices. In the simplest case the entries are complex numbers - we call this situation the *factorization over* $\mathbb{C}$ - for which several proofs are known. The standard strategy in linear algebra is to use a Gauss-Jordan process. Another strategy is provided by Jin, Lin and Xiao in [18]. They have shown that every symplectic matrix over $\mathbb{C}$ can be written by at most *five* elementary symplectic matrices. The third strategy comes from K-theory; in a first step, the last row is solved correctly and then the factorization can be completed by induction (for more information, see section 1.1.1). The latter will prove most useful for tackling questions like the following: What if the entries are not just complex numbers, but depend on parameters in some way, such as algebraically, continuously, or holomorphically? More precisely, what if the entries are polynomials, continuous or holomorphic functions? We call the corresponding factorization problems *polynomial, continuous* and *holomorphic factorization*, respectively.

It is known that the polynomial factorization for 2-by-2 symplectic matrices is in general not true (see Cohn [5]): assume that the entries are polynomials $\mathbb{C}[z, w]$ in two variables. Then the matrix

$$\begin{pmatrix} 1 - zw & -z^2 \\ w^2 & 1 + zw \end{pmatrix}$$

cannot be decomposed into a finite product of elementary matrices.

However, if we consider sufficiently large matrices, at least 4-by-4 matrices, then polynomial factorization is possible: Kopeiko [20] proved it for polynomials $k[z_1, ..., z_m]$ in several variables and coefficients in a field k. Grunewald, Mennicke and Vaserstein [12] were even able to prove it for polynomials $\mathbb{Z}[z_1, ..., z_m]$ with integer coefficients.

We are primarily interested in *holomorphic factorization*, that is, we require the entries to be holomorphic functions on some suitable space X. For the proof we orientate ourselves to the holomorphic factorization problem for the special linear group $\mathrm{SL}_m(\mathcal{O}(X))$ (Gromov [11] called it the *Vaserstein problem*). In 2012, Ivarsson and Kutzschebauch [14] solved this problem by applying an *Oka principle*, which roughly states that holomorphic factorization is solvable if continuous factorization is; and the latter was shown by Vaserstein [23] as early as 1988.

In the first section we will discuss continuous factorization. This was solved by Ivarsson, Kutzschebauch and Løw [16] in 2020.

In section 2, we will consider important details for the application of the Oka principle and solve the last row of the matrix $M \in \mathrm{Sp}_{2n}(\mathcal{O}(X))$ correctly. In other words, we will prove the existence of elementary symplectic matrices $E_1, ..., E_k \in \mathrm{Sp}_{2n}(\mathcal{O}(X))$ such that

$$ME_1 \cdots E_k = \begin{pmatrix} \star & \star \\ 0 & 1 \end{pmatrix}.$$

In the third section, we will prove holomorphic factorization by induction on n.

2.1 Continuous Vaserstein problem for symplectic matrices

Let's start with the so-called *Continuous Vaserstein problem* for symplectic matrices. This has been proved by Ivarsson, Kutzschebauch and Løw ([16, Theorem 1.3]) and is one of the key ingredients.

Theorem 2.1.1 (Continuous Vaserstein problem for symplectic matrices)**.** *There exists a natural number $K(n,d)$ such that given any finite dimensional normal topological space X of (covering) dimension d and any null-homotopic continuous mapping $M\colon X \to \mathrm{Sp}_{2n}(\mathbb{C})$ there exist K continuous mappings*

$$G_1, \ldots, G_K \colon X \to \mathbb{C}^{n(n+1)/2}$$

such that

$$M(x) = M_1(G_1(x)) \ldots M_K(G_K(x)).$$

Sketch of proof. Let $P_t : X \to \mathrm{Sp}_{2n}(\mathbb{C})$ denote the null-homotopy, i.e. $P_1 = M$ and $P_0 = I$. By a Gram-Schmidt process for symplectic matrices (see [16, Lemma 6.1]), there are elementary symplectic matrices $F_1, ..., F_L$ such that

$$V_t = F_1(P_t)F_2(P_t) \cdots F_L(P_t)P_t$$

is null-homotopic with values in the compact symplectic group $\mathrm{Sp}(n)$ and such that

$$V_1 = F_1(M)F_2(M) \cdots F_L(M)M.$$

By a result of Calder and Siegel (see [4]), there is a uniform null-homotopy $M_t : X \to \mathrm{Sp}(n)$ with

$$M_1 = F_1(M) \cdots F_L(M)M.$$

For any integer $k \geq 1$, we can write

$$M_1 = (M_1 M_{\frac{k-1}{k}}^{-1})(M_{\frac{k-1}{k}} M_{\frac{k-2}{k}}^{-1}) \cdots (M_{\frac{2}{k}} M_{\frac{1}{k}}^{-1})M_{\frac{1}{k}}.$$

Thus M_1 can be seen as a product of k matrices $N_i \in \mathrm{Sp}(n)$ such that

$$M_1(x) = N_1(x) \cdots N_k(x), \quad x \in X.$$

Moreover, the matrices $N_i(x)$ are near the identity for k large enough. By a Gauss-Jordan process (see [16, Lemma 4.1]), we find N elementary symplectic matrices $E_1, ..., E_N$ such that

$$N_i(x) = E_1(N_i(x)) \cdots E_N(N_i(x))$$

for all $x \in X$ and for all $i = 1, ..., k$. This implies, that

$$M(x) = F_L(M(x))^{-1} \cdots F_1(M(x))^{-1} \prod_{i=1}^{k} \prod_{j=1}^{N} E_j(N_i(x))$$

is a product of elementary symplectic matrices depending continuously on $x \in X$.

Theorem 1.3 in [16] does not give a uniform bound on the number of factors depending on n and d. Suppose such a bound would not exist, i.e., for all natural numbers i there are normal topological spaces X_i of dimension d and null-homotopic continuous maps $f_i\colon X_i \to \mathrm{Sp}_{2n}(\mathbb{C})$ such that f_i does not factor over a product of less than i elementary symplectic matrices. Set $X = \cup_{i=1}^{\infty} X_i$ the disjoint union of the spaces X_i and $F\colon X \to \mathrm{Sp}_{2n}(\mathbb{C})$ the map that is equal to f_i on X_i. By Theorem 1.3. in [16] F factors over a finite number of elementary symplectic matrices. Consequently all f_i factor over the same number of elementary symplectic matrices which contradicts the assumption on f_i. □

2.2 Application of the Oka principle

Recall the mapping $\Psi_K : (\mathbb{C}^{\frac{n(n+1)}{2}})^K \to \mathrm{Sp}_{2n}(\mathbb{C})$ given by $\Psi_K(Z_1, ..., Z_K) = M_1(Z_1) \cdots M_K(Z_K)$, where $M_k : \mathbb{C}^{\frac{n(n+1)}{2}} \to \mathrm{Sp}_{2n}(\mathbb{C})$ is defined by

$$M_k(Z) = \begin{cases} \begin{pmatrix} I_n & Z \\ 0 & I_n \end{pmatrix} & \text{if } k = 2l \\ \begin{pmatrix} I_n & 0 \\ Z & I_n \end{pmatrix} & \text{if } k = 2l+1. \end{cases}$$

Further, recall the map $\Phi_K = \pi_{2n} \circ \Psi_K : (\mathbb{C}^{\frac{n(n+1)}{2}})^K \to \mathbb{C}^{2n} \setminus \{0\}$, where $\pi_{2n} : \mathbb{C}^{2n \times 2n} \to \mathbb{C}^{2n}$ denotes the projection of a $2n \times 2n$-matrix to its last row.

Define the set

$$\mathcal{W}_K := \left\{ (Z_1, ..., Z_K) \in (\mathbb{C}^{\frac{n(n+1)}{2}})^K : Z_{2i-1} e_n \neq 0 \text{ for some } 1 \leq i \leq \lceil \tfrac{K-1}{2} \rceil \right\}$$

and let $S_K \subset (\mathbb{C}^{\frac{n(n+1)}{2}})^K$ denote the set of points, where the mapping Φ_K is not submersive.

The following is another cornerstone in the proof of the main theorem and we will prove it in the next chapter.

Theorem 2.2.1. *For $K \geq 3$, there exists an open submanifold $E_K \subset (\mathbb{C}^{\frac{n(n+1)}{2}})^K$ satisfying*

(i) $\mathcal{W}_K \subset E_K \subset (\mathbb{C}^{\frac{n(n+1)}{2}})^K \setminus S_K$

(ii) the restriction $\Phi_K|_{E_K} : E_K \to \mathbb{C}^{2n} \setminus \{0\}$ is a stratified elliptic submersion.

The Oka principle tells us, that $\Phi_K|_{E_K}$ has the Parametric Oka Property, i.e. we get the following

Corollary 2.2.2 (Application of the Oka principle)**.** *Let P be a compact Hausdorff space, X a finite dimensional reduced Stein space and $f : P \times X \to \mathrm{Sp}_{2n}(\mathbb{C})$ a null-homotopic, continuous X-holomorphic mapping. Assume there is a natural number K and a continuous map $F : P \times X \to E_K$ such that*

$$\begin{tikzcd} & E_K \arrow[d, "\Phi_K"] \\ P \times X \arrow[ur, "F"] \arrow[r, "\pi_{2n} \circ f"'] & \mathbb{C}^{2n} \setminus \{0\} \end{tikzcd}$$

is commutative. Then there exists a continuous homotopy $F_t : P \times X \to E_K$ with $F_0 = F$, $\pi_{2n} \circ f = \Phi_K \circ F_t$ and such that $F_1 : P \times X \to E_K$ is X-holomorphic.

In a next step, we prove the existence of a natural number K and a continuous lifting $F : P \times X \to E_K$ such that the diagram in the corollary commutes. This is where continuous factorization comes into play.

Theorem 2.2.3 (Existence of a continuous lifting)**.** *There exists a natural number $L(n, d)$ such that given any compact Hausdorff space P, any finite dimensional reduced Stein space X, such that $P \times X$ has covering dimension d, and any null-homotopic, continuous X-holomorphic mapping $f : P \times X \to \mathrm{Sp}_{2n}(\mathbb{C})$, there exists a continuous lifting $F : P \times X \to E_L$ of $\pi_{2n} \circ f$. In particular, there exists a continuous homotopy $F_t : P \times X \to E_L$ of liftings of $\pi_{2n} \circ f$, such that $F_0 = F$ and F_1 is X-holomorphic.*

Proof. The Continuous Vaserstein problem for symplectic matrices provides us with a natural number $K = K(n, d)$ such that given any normal topological space Y of dimension d and any null-homotopic continuous mapping $f : Y \to \mathrm{Sp}_{2n}(\mathbb{C})$ there exists a continuous lifting $G = (G_1, ..., G_K) : Y \to (\mathbb{C}^{\frac{n(n+1)}{2}})^K$ with $f = \Psi_K \circ G$. For $L = K + 2$, we define the mapping $F : Y \to (\mathbb{C}^{\frac{n(n+1)}{2}})^L$ by

$$F = (I_n, 0, G_1 - I_n, G_2, ..., G_K)$$

where I_n denotes the $n \times n$-identity matrix and 0 the $n \times n$-zero matrix. Observe that

$$\begin{pmatrix} I_n & 0 \\ G_1 & I_n \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ I_n & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 \\ G_1 - I_n & I_n \end{pmatrix}$$

and therefore F is a lifting of f, more precisely, $f = \Psi_L \circ F$. The image of F is contained in $\mathcal{W}_L$, since $I_n e_n \neq 0$. By Theorem 2.2.1, we have $\mathcal{W}_L \subset E_L$ which means that F is a mapping $F : Y \to E_L$ and we obtain a commuting diagram

$$\begin{tikzcd} & E_L \arrow[d, "\Psi_L"] \arrow[dr, "\Phi_L"] & \\ Y \arrow[ur, "F"] \arrow[r, "f"] & \mathrm{Sp}_{2n}(\mathbb{C}) \arrow[r, "\pi_{2n}"] & \mathbb{C}^{2n} \setminus \{0\}. \end{tikzcd}$$

Choose $Y = P \times X$, where P is compact Hausdorff space and X a reduced Stein space such that Y has dimension d. An application of Corollary 2.2.2 completes the proof. $\square$

2.3 Holomorphic factorization - Proof of the Main theorem

The proof is by induction on n. Note that the theorem is true for $n = 1$, since $\mathrm{Sp}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})$ (see [14]), that is, the base case is fine.

For the induction step, we first observe that, according to Theorem 2.2.3, there exists a natural number $L(n, d)$ such that given any compact Hausdorff space P, any finite dimensional Stein space X, such that $P \times X$ has covering dimension d, and any null-homotopic X-holomorphic mapping $f : P \times X \to \mathrm{Sp}_{2n}(\mathbb{C})$, there exists a continuous homotopy $F_t : P \times X \to E_L$ of liftings of $\pi_{2n} \circ f$ (i.e. $\pi_{2n} \circ f = \Phi_L \circ F_t$ for all $0 \leq t \leq 1$), such that F_1 is X-holomorphic. In particular, this implies that $\Phi_L \circ F_t$ doesn't depend on t, hence we get

$$\Psi_L(F_t(p, x))f(p, x)^{-1} = \begin{pmatrix} A_{1,t}(p, x) & a_{2,t}(p, x) & B_{1,t}(p, x) & b_{2,t}(p, x) \\ a_{3,t}(p, x) & a_{4,t}(p, x) & b_{3,t}(p, x) & b_{4,t}(p, x) \\ C_{1,t}(p, x) & c_{2,t}(p, x) & D_{1,t}(p, x) & d_{1,t}(p, x) \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $A_{1,t}(p, x), B_{1,t}(p, x), C_{1,t}(p, x)$ and $D_{1,t}(p, x)$ are $(n - 1) \times (n - 1)$ matrices, and the remaining mappings are vectors of appropriate dimension. Since $\Psi_L(F_t(p, x))f(p, x)^{-1}$ is a symplectic matrix for all $0 \leq t \leq 1$, Lemma 1.1.6 implies $a_{2,t}(p, x) \equiv 0, a_{4,t}(p, x) \equiv 1$ and $c_{2,t}(p, x) \equiv 0$, so that

$$\Psi_L(F_t(p, x))f(p, x)^{-1} = \begin{pmatrix} A_{1,t}(p, x) & 0 & B_{1,t}(p, x) & b_{2,t}(p, x) \\ a_{3,t}(p, x) & 1 & b_{3,t}(p, x) & b_{4,t}(p, x) \\ C_{1,t}(p, x) & 0 & D_{1,t}(p, x) & d_{1,t}(p, x) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and in addition

$$\tilde{f}_t(p, x) = \begin{pmatrix} A_{1,t}(p, x) & B_{1,t}(p, x) \\ C_{1,t}(p, x) & D_{1,t}(p, x) \end{pmatrix}$$

is symplectic. The fact that $\Psi_L(F_0(p,x)) = f(p,x)$ implies $\tilde{f}_0 = I_{2n-2}$ and thus the X-holomorphic map $\tilde{f} := \tilde{f}_1 : P \times X \to \mathrm{Sp}_{2n-2}(\mathbb{C})$ is null-homotopic. Let ψ be the standard inclusion of Sp_{2n-2} into Sp_{2n} given by

$$\psi : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A & 0 & B & 0 \\ 0 & 1 & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By the induction hypothesis,

$$\psi(\tilde{f}(p,x)^{-1}) = \begin{pmatrix} D_{1,1}(p,x)^T & 0 & -B_{1,1}(p,x)^T & 0 \\ 0 & 1 & 0 & 0 \\ -C_{1,1}(p,x)^T & 0 & A_{1,1}(p,x)^T & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is a finite product of holomorphic elementary symplectic matrices. Then the matrix $M(p,x) := \Psi_L(F_1(p,x))f(p,x)^{-1}\psi(\tilde{f}(p,x)^{-1})$ is given by

$$M(p,x) = \begin{pmatrix} I_{n-1} & 0 & 0 & b_{2,1}(p,x) \\ -d_{2,1}(p,x)^T & 1 & b_{2,1}(p,x)^T & b_{4,1}(p,x) \\ 0 & 0 & I_{n-1} & d_{2,1}(p,x) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

according to Lemma 1.1.7. An application of the 2nd Whitehead lemma implies that this matrix is a product of four elementary symplectic matrices of $\mathrm{Sp}_{2n}(\mathcal{O}(X))$. In summary, this proves that $f(p,x)$ is indeed a finite product of elementary symplectic matrices.

So far, the number of factors depends on the mapping $f : P \times X \to \mathrm{Sp}_{2n}(\mathbb{C})$. Assume there is no uniform bound $K(n,d)$, that is, for each natural number i, there is a compact Hausdorff space P_i and a reduced Stein space X_i, such that $Y_i = P_i \times X_i$ has dimension d, and a null-homotopic X-holomorphic mapping $f_i : Y_i \to \mathrm{Sp}_{2n}(\mathbb{C})$ which does not factor into less than i elementary matrix factors. Set $Y = \bigcup_i Y_i$ and let $F : Y \to \mathrm{Sp}_{2n}(\mathbb{C})$ be the null-homotopic mapping, which equals f_i on Y_i. We just proved the existence of a constant K which bounds the number of elementary factors in which F decomposes. But then K is an upper bound for each f_i which contradicts the assumption. Hence there is a uniform bound $K(n,d)$ of factors and this complectes the proof. Actually, we've shown a generalized version of the Main theorem.

Theorem 2.3.1 (Generalized version of main theorem). *There is a natural number $K = K(n,d)$ such that given any compact Hausdorff space P, any finite dimensional reduced Stein space X, such that $P \times X$ has covering dimension d, and any null-homotopic X-holomorphic mapping $f : P \times X \to \mathrm{Sp}_{2n}(\mathbb{C})$ there exist K X-holomorphic mappings*

$$G_1, ..., G_K : P \times X \to \mathbb{C}^{\frac{n(n+1)}{2}}$$

with

$$f(p,x) = M_1(G_1(p,x))M_2(G_2(p,x)) \cdots M_K(G_K(p,x)).$$

3 Stratified ellipticity of the mapping Φ_K

Recall that we identified $\mathrm{Sym}_n(\mathbb{C}) = \{Z \in \mathbb{C}^{n\times n} : Z^T = Z\}$ with $\mathbb{C}^{\frac{n(n+1)}{2}}$ for simplicity. Furthermore, the *elementary symplectic matrix mapping* $M_K : \mathbb{C}^{\frac{n(n+1)}{2}} \to \mathrm{Sp}_{2n}(\mathbb{C})$ we defined by

$$M_K(Z) = \begin{cases} \begin{pmatrix} I_n & Z \\ 0 & I_n \end{pmatrix} & \text{if } K = 2k+1 \\ \begin{pmatrix} I_n & 0 \\ Z & I_n \end{pmatrix} & \text{if } K = 2k. \end{cases}$$

Let's write $\vec{Z}_K := (Z_1, ..., Z_K) \in (\mathbb{C}^{\frac{n(n+1)}{2}})^K$. Then the mapping $\Phi_K : (\mathbb{C}^{\frac{n(n+1)}{2}})^K \to \mathbb{C}^{2n} \setminus \{0\}$ is defined by

$$\Phi_K(\vec{Z}_K) := M_K(Z_K) \cdots M_1(Z_1) e_{2n}.$$

Remark 3.0.1. *The attentive reader may have noticed that we now define Φ_K in a transposed way compared to the previous sections. This is purely for aesthetic reasons.*

The following recursive formula can be derived immediately from the definition.

Corollary 3.0.2 (Recursive formula of Φ_K)**.** *For $K \geq 1$, the mapping $\Phi_K : (\mathbb{C}^{\frac{n(n+1)}{2}})^K \to \mathbb{C}^{2n} \setminus \{0\}$ satisfies*

$$\Phi_K(\vec{Z}_K) = M_K(Z_K)\Phi_{K-1}(\vec{Z}_{K-1}), \tag{3.1}$$

with the convention $\Phi_0 := e_{2n}$.

The main goal of this section is to prove

Theorem 3.0.3. *For $K \geq 3$, there is an open submanifold E_K of $(\mathbb{C}^{\frac{n(n+1)}{2}})^K$ such that*

$$\Phi_K|_{E_K} : E_K \to \mathbb{C}^{2n} \setminus \{0\}$$

is a stratified elliptic submersion.

In subsection 3.1 we will classify the points for which Φ_K is not submersive. We will also show that Φ_K is surjective.

The subsequent subsections are then devoted to the task of finding stratified sprays. In subsection 3.2, we stratify $\mathbb{C}^{2n} \setminus \{0\}$ suitably. In 3.3, we find formulas of holomorphic vector fields which are fiber-preserving for Φ_K. Unfortunately, some of those fields aren't $\mathbb{C}$-complete. We therefore classify some complete vector fields in subsection 3.4. In subsection 3.5, we analyze the fibers of Φ_K from a topological point of view. In subsection 3.6 we lay the mathematical basis for the construction of the sprays. And finally we carry out all the necessary calculations in 3.7.

3.1 Notations and basic properties

Let's start with some notations. Let E_{ij} be the $n \times n$ matrix having a 1 at entry (i,j) and is zero elsewhere. Then $\tilde{E}_{ij} = \frac{1}{1+\delta_{ij}}(E_{ij} + E_{ji})$ is an *elementary symmetric matrix*.

In the following we will make the identification of $\mathrm{Sym}_n(\mathbb{C})$ and $\mathbb{C}^{\frac{n(n+1)}{2}}$ more precise. The set $\{\tilde{E}_{ij} : 1 \leq i \leq j \leq n\}$ forms a basis of $\mathrm{Sym}_n(\mathbb{C})$. The sets $I := \{i \in \mathbb{N} : 1 \leq i \leq \frac{n(n+1)}{2}\}$ and $J := \{(i,j) \in \mathbb{N}^2 : 1 \leq i \leq j \leq n\}$ have the same order. Hence there is a bijection $\alpha : I \to J$, which induces an isomorphism $S : \mathbb{C}^{\frac{n(n+1)}{2}} \to \mathrm{Sym}_n(\mathbb{C})$ by defining $S(e_i) = \tilde{E}_{\alpha(i)}$ for all $1 \leq i \leq \frac{n(n+1)}{2}$. By an abuse of notation, $Z \in \mathbb{C}^{\frac{n(n+1)}{2}}$ denotes both, the vector and the corresponding symmetric matrix $S(Z)$, depending on the corresponding context, of course.

In this very first subsection, we will compute the set of points where Φ_K is *not* submersive. Also we'll give a proof for the surjectivity of Φ_K. But let's spell that out in more detail first.

We let S_K denote the set of points in $(\mathbb{C}^{\frac{n(n+1)}{2}})^K$ where Φ_K is not submersive. We also define the open set

$$\mathcal{W}_K := \left\{ \vec{Z}_K \in (\mathbb{C}^{\frac{n(n+1)}{2}})^K : Z_{2i-1}e_n \neq 0 \text{ for some } 1 \leq i \leq \lceil \tfrac{K-1}{2} \rceil \right\}$$

and $\mathcal{W}_K^c$ denotes its complement in $(\mathbb{C}^{\frac{n(n+1)}{2}})^K$.

Theorem 3.1.1 (Singularity set of Φ_K)**.** *For $K \geq 2$, the set S_K is given by*

$$S_K = \left\{ \vec{Z}_K \in \mathcal{W}_K^c : \mathrm{rank}(W_K(\vec{Z}_K)) < n \right\}$$

where $W_K(\vec{Z}_K)$ is the augmented matrix $(Z_2|Z_4|\cdots|Z_{2k})$ for $k = \lfloor \frac{K-1}{2} \rfloor$.

Proof. The proof of this theorem is subject of subsection "Singularity set of Φ_K". □

Theorem 3.1.2. *For $K \geq 3$, the mapping $\Phi_K|_{\mathcal{W}_K} : \mathcal{W}_K \to \mathbb{C}^{2n} \setminus \{0\}$ is surjective.*

Proof. The proof is given in subsection "Surjectivity of Φ_K ". □

A direct consequence of these two statements is

Corollary 3.1.3. *For $K \geq 3$ and for any open submanifold E in $(\mathbb{C}^{\frac{n(n+1)}{2}})^K$ with*

$$\mathcal{W}_K \subset E \subset (\mathbb{C}^{\frac{n(n+1)}{2}})^K \setminus S_K,$$

the mapping $\Phi_K|_E : E \to \mathbb{C}^{2n} \setminus \{0\}$ is a surjective submersion.

3.1.1 Singularity set of Φ_K

In order to compute the singularity set S_K, we need to know the Jacobian of Φ_K, denoted by $J\Phi_K$. For the computations, we need some auxiliary tools. For a fixed $1 \leq i \leq n$, let's define the mapping $F_i : \mathbb{C}^n \to \mathbb{C}^{n \times n}$ by

$$F_i(v) := \left[\tilde{E}_{i1}v \cdots \tilde{E}_{in}v \right] = \begin{pmatrix} v_i & & & & \\ & \ddots & & & \\ v_1 & \cdots & v_i & \cdots & v_n \\ & & & \ddots & \\ & & & & v_i \end{pmatrix}.$$

Furthermore, we define $F : \mathbb{C}^n \to \mathbb{C}^{n \times \frac{n(n+1)}{2}}$ by

$$F(v) := \left[\tilde{E}_{\alpha(1)} v \cdots \tilde{E}_{\alpha(\frac{n(n+1)}{2})} v\right].$$

Observe that the matrix $F_i(v)$ is a submatrix of $F(v)$, for every $1 \leq i \leq n$. This implies that $F(v)$ is surjective if and only if $v \neq 0$. And if $F(v)$ is not surjective, then we even have $F(v) = 0$.

Lemma 3.1.4. *For $u, v \in \mathbb{C}^n$ such that $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{C}^{2n} \setminus \{0\}$, the Jacobian of the mapping $\mathbb{C}^{\frac{n(n+1)}{2}} \to \mathbb{C}^{2n} \setminus \{0\}, Z \mapsto M_K(Z) \begin{pmatrix} u \\ v \end{pmatrix}$ is given by*

$$A_K(u,v) := \begin{cases} \begin{pmatrix} F(v) \\ 0 \end{pmatrix} & \text{if } K = 2k+1 \\ \begin{pmatrix} 0 \\ F(u) \end{pmatrix} & \text{if } K = 2k. \end{cases}$$

In particular, $M_K(Z) A_K(u,v) = A_K(u,v)$ for all $Z \in \mathbb{C}^{\frac{n(n+1)}{2}}$.

Proof. Note that each symmetric matrix $Z \in \mathbb{C}^{\frac{n(n+1)}{2}}$ can be written as a sum $Z = \sum_{1 \leq i \leq j \leq n} z_{ij} \tilde{E}_{ij}$. Hence $\frac{\partial}{\partial z_{ij}} Z = \tilde{E}_{ij}$. Further, note

$$\frac{\partial}{\partial z_{ij}} M_{2k+1}(Z) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & \tilde{E}_{ij} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \tilde{E}_{ij} v \\ 0 \end{pmatrix}$$

and

$$\frac{\partial}{\partial z_{ij}} M_{2k}(Z) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \tilde{E}_{ij} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{E}_{ij} u \end{pmatrix},$$

respectively. From here, the claim follows by definition of the mapping F. □

We are now ready to compute the Jacobian of Φ_K. By the recursive formula (3.1), the product rule and the previous lemma, we obtain

Corollary 3.1.5 (Jacobian of Φ_K)**.** *The Jacobian $J\Phi_1$ is given by $A_1(e_{2n})$. For $K \geq 2$, the Jacobian $J\Phi_K$ of Φ_K is given by*

$$J\Phi_K(\vec{Z}_K) = \left(M_K(Z_K) J\Phi_{K-1}(\vec{Z}_{K-1}) \;\;|\;\; A_K(\Phi_{K-1}(\vec{Z}_{K-1}))\right).$$

If the Jacobian $J\Phi_{K-1}$ is surjective, then so is $J\Phi_K$, since $M_K(Z_K)$ is a regular matrix. Or, equivalently, if $\vec{Z}_K = (\vec{Z}_{K-1}, Z_K) \in S_K$ is a singular point for Φ_K, then $\vec{Z}_{K-1} \in S_{K-1}$ is a singular point for Φ_{K-1}. This observation suggests that we will compute S_K recursively.

Lemma 3.1.6. *Let $\vec{Z}_K \in (\mathbb{C}^{\frac{n(n+1)}{2}})^K$ and assume that there is $1 \leq k \leq \lceil \frac{K-1}{2} \rceil$ such that $Z_{2i-1} e_n = 0$ for all $1 \leq i \leq k$. Then $\Phi_j(\vec{Z}_j) = e_{2n}$ for all $1 \leq j \leq 2k$.*

Proof. We prove this by induction on j. For the base step, observe that

$$\Phi_1(\vec{Z}_1) = M_1(Z_1) e_{2n} = \begin{pmatrix} Z_1 e_n \\ e_n \end{pmatrix}$$

by definition. Since we assume $Z_1 e_n = 0$, $\Phi_1(\vec{Z}_1) = e_{2n}$ follows immediately.

For the induction step, let $1 < j \le 2k$. By the induction hyptothesis, we have $\Phi_{j-1}(\vec{Z}_{j-1}) = e_{2n}$. Then we obtain

$$\Phi_j(\vec{Z}_j) = M_j(Z_j)\Phi_{j-1}(\vec{Z}_{j-1}) = M_j(Z_j)e_{2n}$$

by the recursive formula (3.1). If j is even, then we're done, by definition of M_j. Let's assume that $j = 2l-1$ for some integer l. Observe that $l \le k$ is satisfied, hence $Z_j e_n = 0$ by assumption. This implies

$$\Phi_j(\vec{Z}_j) = \begin{pmatrix} I_n & Z_j \\ 0 & I_n \end{pmatrix} \begin{pmatrix} 0 \\ e_n \end{pmatrix} = \begin{pmatrix} Z_j e_n \\ e_n \end{pmatrix} = e_{2n}.$$

This completes the proof. $\square$

Lemma 3.1.7. *Let $K \ge 2$ be a natural number. Then the restriction $\Phi_K|_{\mathcal{W}_K}$ is a submersion.*

Proof. Consider $\vec{Z}_K \in \mathcal{W}_K$. There is a smallest index $1 \le k \le \lceil \frac{K-1}{2} \rceil$ such that $Z_{2k-1}e_n \neq 0$ and $Z_{2i-1}e_n = 0$ for all $1 \le i < k$. Setting $L := 2k-1$, the Jacobian $J\Phi_L$ is of the form

$$\begin{pmatrix} * & F(P_s^{L-1}) \\ * & 0 \end{pmatrix}$$

and

$$J\Phi_{L+1} = \begin{pmatrix} M_{L+1}(Z_{L+1})J\Phi_L & A_{L+1}(\Phi_L) \end{pmatrix} = \begin{pmatrix} * & F(P_s^{L-1}) & 0 \\ * & 0 & F(P_f^L) \end{pmatrix}.$$

By definition of F, the Jacobian $J\Phi_{L+1}$ has full rank, if $P_s^{L-1} \neq 0$ and $P_f^L \neq 0$. By Lemma 3.1.6, we have $\Phi_{L-1} = e_{2n}$. Hence

$$P_s^{L-1} = \begin{pmatrix} 0 & I_n \end{pmatrix} \Phi_{L-1} = e_n \neq 0.$$

Furthermore,

$$P_f^L = \begin{pmatrix} I_n & 0 \end{pmatrix} \Phi_L = \begin{pmatrix} I_n & 0 \end{pmatrix} M_L(Z_L)\Phi_{L-1} = \begin{pmatrix} I_n & Z_L \end{pmatrix} e_{2n} = Z_L e_n \neq 0.$$

This showes that the Jacobian $J\Phi_{L+1}$ has full rank. Note that $L+1 \le K$ by construction. By the recursive formula of the Jacobian and regularity of $M_i(Z_i)$, $1 \le i \le K$, we conclude that $J\Phi_K$ has full rank, too. $\square$

Let's write $C_K := \begin{pmatrix} I_n & 0 \end{pmatrix} J\Phi_K$ and $D_K := \begin{pmatrix} 0 & I_n \end{pmatrix} J\Phi_K$.

Lemma 3.1.8. *For a point $\vec{Z}_{2k+2} = (\vec{Z}_{2k+1}, Z_{2k+2}) \in \mathcal{W}^c_{2k+2}$ the following statements are equivalent.*

(i) *The Jacobian $J\Phi_{2k+2}(\vec{Z}_{2k+2})$ is surjective in $\vec{Z}_{2k+2}$.*

(ii) *The Jacobian $J\Phi_{2k+1}(\vec{Z}_{2k+1})$ is surjective in $\vec{Z}_{2k+1}$.*

(iii) $\operatorname{rank}(D_{2k+1}) = \operatorname{rank}(D_{2k}) = n$.

Proof. Note that $\mathcal{W}^c_K \subset \mathcal{W}^c_{K-1} \times \mathbb{C}^{\frac{n(n+1)}{2}}$ by definition. Moreover,

$$A_{2k+1}(e_{2n}) = \begin{pmatrix} F(e_n) \\ 0 \end{pmatrix} \quad \text{and} \quad A_{2k+2}(e_{2n}) = 0,$$

for $\vec{Z}_{2k+2} \in \mathcal{W}^c_{2k+2}$, by Lemma 3.1.6 and by definition of F. We conclude

$$J\Phi_{2k+2}(\vec{Z}_{2k+2}) = M_{2k+2}(Z_{2k+2}) \begin{pmatrix} J\Phi_{2k+1}(\vec{Z}_{2k+1}) & 0 \end{pmatrix}$$

and this shows equivalence of (i) and (ii), since $M_{2k+2}(Z_{2k+2})$ is a regular matrix.

To show equivalence of (ii) and (iii) observe that

$$J\Phi_{2k+1}(\vec{Z}_{2k+1}) = \begin{pmatrix} C_{2k} + Z_{2k+1}D_{2k} & F(e_n) \\ D_{2k} & 0 \end{pmatrix}.$$

Since $F(e_n)$ is surjective by definition of F, the claim follows immediately. □

In order to prove Theorem 3.1.1, it remains to show the next lemma.

Lemma 3.1.9. *The following equation is satisfied for all $\vec{Z}_{2k+2} \in \mathcal{W}^c_{2k+2}$:*

$$\operatorname{im}(D_{2k+2}) = \operatorname{im}(D_{2k}|Z_{2k+2}). \tag{$\star$}$$

In particular, the singularity set of Φ_K, $K \geq 2$, is given by

$$S_K := \{\vec{Z}_K \in \mathcal{W}^c_K : \operatorname{rank}(W_K(\vec{Z}_K)) < n\},$$

where W_K is an augmented matrix $W_K(\vec{Z}_K) := (Z_2|Z_4|\cdots|Z_{2k})$, for $k = \lfloor \frac{K-1}{2} \rfloor$.

Proof. By the previous lemma, it is enough to show $(\star)$. We have

$$J\Phi_{2k+2}(\vec{Z}_{2k+2}) = \begin{pmatrix} C_{2k+1} & 0 \\ D_{2k+1} + Z_{2k+2}C_{2k+1} & 0 \end{pmatrix} = \begin{pmatrix} C_{2k} + Z_{2k+1}D_{2k} & F(e_n) & 0 \\ D_{2k} + Z_{2k+2}(C_{2k} + Z_{2k+1}D_{2k}) & Z_{2k+2}F(e_n) & 0 \end{pmatrix}.$$

We get $\operatorname{im}(Z_{2k+2}) = \operatorname{im}(Z_{2k+2}F(e_n))$, since $F(e_n)$ is surjective. Thus $\operatorname{im}(D_{2k+2}) = \operatorname{im}(D_{2k}|Z_{2k+2})$. □

3.1.2 Surjectivity of Φ_K

The proof of surjectivity is based on the following lemma.

Lemma 3.1.10. *For $a \in \mathbb{C}^n \setminus \{0\}$ fixed, the mapping $\varphi_a : \operatorname{Sym}_n(\mathbb{C}) \to \mathbb{C}^n, Z \mapsto Za$ is surjective.*

Proof. Let $a \neq 0$ being fixed. Then the linear mapping $F(a) : \mathbb{C}^{\frac{n(n+1)}{2}} \to \mathbb{C}^n$ is surjective, by definition of F. Therefore it is enough to show that the following diagram

$$\begin{tikzcd} & \mathbb{C}^{\frac{n(n+1)}{2}} \arrow[d, "F(a)"] \\ \operatorname{Sym}_n(\mathbb{C}) \arrow[ur, "S^{-1}"] \arrow[r, "\varphi_a"'] & \mathbb{C}^n. \end{tikzcd}$$

is commutative. This is the case if and only if $F(a)v = S(v)a$ for all $v \in \mathbb{C}^{\frac{n(n+1)}{2}}$. Write $v = \sum_{i=1}^{\frac{n(n+1)}{2}} v_i e_i$. By definition of F and S, we get

$$F(a)v = \sum_{i=1}^{\frac{n(n+1)}{2}} v_i F(a)e_i = \sum_{i=1}^{\frac{n(n+1)}{2}} v_i \tilde{E}_{\alpha(i)}a = \left(\sum_{i=1}^{\frac{n(n+1)}{2}} v_i \tilde{E}_{\alpha(i)}\right) a = S(v)a.$$

This completes the proof. □

We are now ready for the

Proof of Theorem 3.1.2. In a first step, let $K = 3$ and consider $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{C}^{2n} \setminus \{0\}$. Observe that $M_K(-Z)$ is the inverse of $M_K(Z)$ for every K and every Z. Then $\Phi_3(\vec{Z}_3) = \begin{pmatrix} a \\ b \end{pmatrix}$ if and only if

$$M_2(Z_2)M_1(Z_1)e_{2n} = M_3(-Z_3)\begin{pmatrix} a \\ b \end{pmatrix}.$$

The left-hand-side is given by

$$\begin{pmatrix} I_n & 0 \\ Z_2 & I_n \end{pmatrix}\begin{pmatrix} Z_1 e_n \\ e_n \end{pmatrix} = \begin{pmatrix} Z_1 e_n \\ (I_n + Z_2 Z_1)e_n \end{pmatrix}$$

and the right-hand-side by

$$\begin{pmatrix} I_n & -Z_3 \\ 0 & I_n \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a - Z_3 b \\ b \end{pmatrix}.$$

The symmetric matrix Z_3 can be chosen such that $a - Z_3 b \neq 0$. To see this, observe that it is automatically satisfied if $b = 0$, since a and b aren't simultaneously zero. On the other hand, if $b \neq 0$, then an application of Lemma 3.1.10 provides the existence of such Z_3.

The vector e_n is obviously non-zero, hence another application of Lemma 3.1.10 yields the existence of a symmetric matrix Z_1 such that

$$Z_1 e_n = a - Z_3 b.$$

We have $Z_1 e_n \neq 0$ by construction, which enables a third application of Lemma 3.1.10 and proves the existence of a symmetric matrix Z_2 such that $(I_n + Z_2 Z_1)e_n = b$. Thus we've found $\vec{Z}_3 \in (\mathbb{C}^{\frac{n(n+1)}{2}})^3$ such that

$$\Phi_3(\vec{Z}_3) = \begin{pmatrix} a \\ b \end{pmatrix}.$$

Moreover, since $Z_1 e_n \neq 0$, we even have $\vec{Z}_3 \in \mathcal{W}_3$, which completes the proof for $K = 3$.

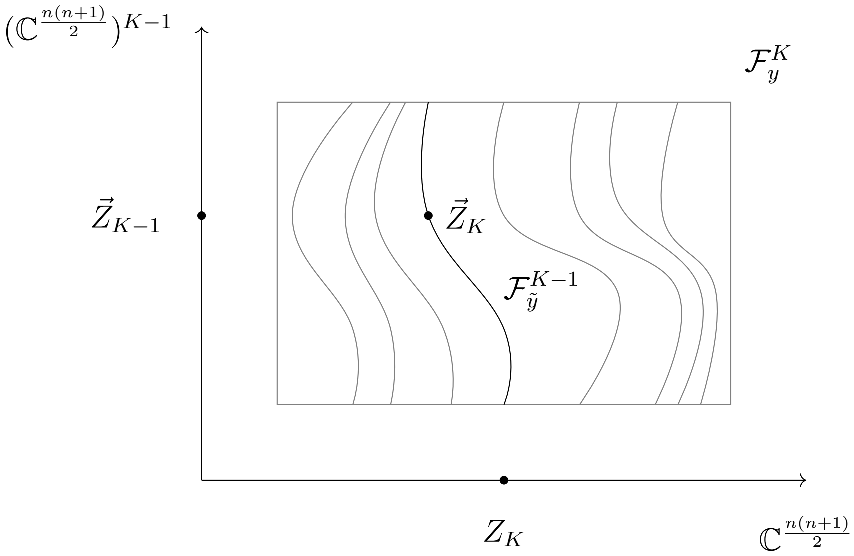
For $K > 3$ and $x \in \mathbb{C}^{2n} \setminus \{0\}$ we find $\vec{Z}_3 \in \mathcal{W}_3$ such that $\Phi_3(\vec{Z}_3) = x$. Now, we set $Z_K := (\vec{Z}_3, 0, ..., 0) \in (\mathbb{C}^{\frac{n(n+1)}{2}})^K$. Then we have $\Phi_K(\vec{Z}_K) = \Phi_3(\vec{Z}_3) = x$ and, moreover, $\vec{Z}_K \in \mathcal{W}_K$ by definition. This completes the proof. □

3.2 Stratification of $\mathbb{C}^{2n} \setminus \{0\}$

Consider a fixed point $y := (a, b) \in \mathbb{C}^{2n} \setminus \{0\}$ and let $\mathcal{F}_y^K := \mathcal{F}_{a,b}^K := \Phi_K^{-1}(y)$ denote the fiber of Φ_K over y. By the recursive formula (3.1) of Φ_K we can write the K-fiber $\mathcal{F}_y^K$ as a union of $(K-1)$-fibers

$$\mathcal{F}_y^K = \bigcup_{Z \in \mathbb{C}^{\frac{n(n+1)}{2}}} \mathcal{F}_{M_K(Z)y}^{K-1}.$$

Equivalently, a given a point $\vec{Z}_K = (\vec{Z}_{K-1}, Z_K)$ is contained in the fiber $\mathcal{F}_y^K$ if and only if $\vec{Z}_{K-1} \in \mathcal{F}_{\tilde{y}}^{K-1}$ for $\tilde{y} = M_K(-Z_K)y$. We have the following picture in mind



For an "appropriate" stratification of $\mathbb{C}^{2n} \setminus \{0\}$, we will use the projection $\pi_K : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ given by

$$\pi_K(u, v) := \begin{cases} u & \text{if } K = 2k \\ v & \text{if } K = 2k + 1. \end{cases}$$

The following stratification turns out to be a natural one. Let

$$Y_{ng}^K = \{y \in \mathbb{C}^{2n} \setminus \{0\} : \pi_K(y) = 0\}$$

denote the *non-generic* stratum and its complement $Y_g^K = (\mathbb{C}^{2n} \setminus \{0\}) \setminus Y_{ng}^K$ the *generic* stratum.

Lemma 3.2.1. *For a point $y \in Y_{ng}^K$ in the non-generic stratum, the corresponding non-generic fiber $\mathcal{F}_y^K$ satisfies*

$$\mathcal{F}_y^K = \mathcal{F}_y^{K-1} \times \mathbb{C}^{\frac{n(n+1)}{2}},$$

where $\mathcal{F}_y^{K-1}$ is a generic $(K-1)$-fiber.

Proof. We carry out the proof only for $K = 2k+1$, since it applies equally to $K = 2k$ for reasons of symmetry. Let $y = (a, b) \in Y_{ng}^K$, that is, $\pi_K(y) = 0$. Observe that $y = (\pi_{K-1}(y), \pi_K(y))$. On the one hand, this means $b = \pi_K(y) = 0$ and on the other $a = \pi_{K-1}(y) \neq 0$, since $y \neq 0$ by definition. This implies that $\mathcal{F}_y^{K-1}$ is a generic $(K-1)$-fiber.

The non-generic K-fiber $\mathcal{F}_y^K$ is given by the defining equations $\Phi_K(\vec{Z}_K) = y$. By the recursive formula (3.1) of Φ_K this system of equations is equivalent to $\Phi_{K-1}(\vec{Z}_{K-1}) = M_K(-Z_K)y$. But we have

$$M_K(-Z_K)y = \begin{pmatrix} I_n & -Z_K \\ 0 & I_n \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} = y,$$

which means that the defining equations are independent of the matrix $Z_K \in \mathbb{C}^{\frac{n(n+1)}{2}}$. In fact, we obtain

$$\mathcal{F}_y^K = \mathcal{F}_y^{K-1} \times \mathbb{C}^{\frac{n(n+1)}{2}}.$$

□

Informally, the next statement tells us that for fibers in the generic stratum we can reduce the number of defining equations from $2n$ to n.

First we introduce the following convention. Let $\pi : \mathbb{C}^k \to \mathbb{C}^l$ be the standard projection $(z_1, ..., z_l, ..., z_k) \mapsto (z_1, ..., z_l)$. For a continuous mapping $f : \mathbb{C}^l \to \mathbb{C}^m$ its pullback $\pi^* f$ is a mapping $\pi^* f : \mathbb{C}^k \to \mathbb{C}^m$ and by an abuse of notation, we just write f instead of $\pi^* f$.

Also define $\tilde{P}^K := \pi_{K+1} \circ \Phi_K$. Then, $\tilde{P}_j^K$ and $\pi_K(y)_j$ denote the j-th component of $\tilde{P}^K$ and $\pi_K(y)$, respectively.

Lemma 3.2.2. *Set $X := (\mathbb{C}^{\frac{n(n+1)}{2}})^{K-1} \times \mathbb{C}^{\frac{n(n-1)}{2}}$ and define the variety*

$$\mathcal{G}_{\pi_K(y)} := \{\vec{Z} \in X : \tilde{P}^{K-1}(\vec{Z}) = \pi_K(y)\}.$$

Then there are n meromorphic mappings $\psi_j^K : X \to X \times \mathbb{C}^n$, $1 \leq j \leq n$, such that each generic fiber $\mathcal{F}_y^K$ is biholomorphic to $\mathcal{G}_{\pi_K(y)}$ via ψ_j^K for some $1 \leq j \leq n$.

Proof. Let $y \in Y_g^K$ be a point in the generic stratum, i.e. $\pi_K(y)_j \neq 0$ for some $1 \leq j \leq n$. From the definition of π_K and $\tilde{P}^K$ on the one hand and the recursive formula (3.1) on the other hand, it follows that the fiber $\mathcal{F}_y^K$ is given by

$$\mathcal{F}_y^K = \{\vec{Z}_K \in (\mathbb{C}^{\frac{n(n+1)}{2}})^K : \tilde{P}^K(\vec{Z}_K) = \pi_{K+1}(y), \tilde{P}^{K-1}(\vec{Z}_K) = \pi_K(y)\}$$

and furthermore

$$\tilde{P}^K = \tilde{P}^{K-2} + Z_K \tilde{P}^{K-1}. \tag{3.2}$$

Hence $\tilde{P}_j^K \neq 0$ is satisfied on the fiber $\mathcal{F}_y^K$ and the latter equation can be rearranged to

$$Z_K e_j = \frac{1}{\tilde{P}_j^{K-1}} \left(\tilde{P}^K - \tilde{P}^{K-2} - \sum_{k=1, k\neq j}^{n} \tilde{P}_k^{K-1} Z_K e_k \right).$$

We obtain

$$f_{ij} := z_{K,ij} = \frac{1}{\tilde{P}_j^{K-1}} \left(\tilde{P}_i^K - \tilde{P}_i^{K-2} - \sum_{k=1, k\neq j}^{n} \tilde{P}_k^{K-1} z_{K,ik} \right) \quad 1 \leq i \leq n, i \neq j, \tag{3.3}$$

$$f_{jj} := z_{K,jj} = \frac{1}{\tilde{P}_j^{K-1}} \left(\tilde{P}_j^K - \tilde{P}_j^{K-2} - \frac{1}{\tilde{P}_j^{K-1}} \sum_{k=1, k\neq j}^{n} b_k \left(\tilde{P}_k^K - \tilde{P}_k^{K-2} - \sum_{l=1, l\neq j}^{n} \tilde{P}_l^{K-1} z_{K,kl} \right) \right). \tag{3.4}$$

Set $f_j := (f_{1j}, ..., f_{nj}) : X \to \mathbb{C}^n$ and $\psi_j : X \to X \times \mathbb{C}^n$, $\Psi_j(x) = (x, f_j(x))$. By construction, the variety $\mathcal{G}_{\pi_K(y)}$ is mapped biholomorphically onto $\mathcal{F}_y^K$ by ψ_j. □

3.2.1 On the singularities of the fibers

In this short section we will classify the fibers; we distinguish between smooth and singular fibers. In fact, most of the fibers $\mathcal{F}_y^K$ are completely contained in $\mathcal{W}_K$ and therefore smooth, by Lemma 3.1.7.

Lemma 3.2.3. *A fiber $\mathcal{F}_y^K$ contains singularities if and only if $\pi_1(y) = e_n$.*

Proof. We start with the case $K = 2k$. Suppose there is a singularity $\vec{Z}_K \in \mathcal{F}_y^K \cap S_K$. Then $Z_{2i-1}e_n = 0$ for all $1 \leq i \leq k$, by Lemma 3.1.9. Lemma 3.1.6 implies $\Phi_K(\vec{Z}_K) = e_{2n}$ and thus $\pi_1(y) = \pi_1(\Phi_K(\vec{Z}_K)) = \pi_1(e_{2n}) = e_n$.

Now let $K = 2k + 1$ and suppose again there is a singularity $\vec{Z}_K \in \mathcal{F}_y^K \cap S_K$. Again, we have $Z_{2i-1}e_n = 0$ for all $1 \leq i \leq k$, by Lemma 3.1.9. From the even case, we know that $\Phi_{K-1}(\vec{Z}_{K-1}) = e_{2n}$. The recursive formula (3.1) of Φ_K implies

$$y = \Phi_K(\vec{Z}_K) = M_K(Z_K)\Phi_{K-1}(\vec{Z}_{K-1}) = \begin{pmatrix} I_n & Z_K \\ 0 & I_n \end{pmatrix} \begin{pmatrix} 0 \\ e_n \end{pmatrix} = \begin{pmatrix} Z_K e_n \\ e_n \end{pmatrix},$$

and therefore $\pi_1(y) = e_n$. This completes the proof of necessary condition.

For the proof of the sufficient condition, we consider a fiber $\mathcal{F}_y^K$ with $\pi_1(y) = e_n$ and set $k := \lceil \frac{K-1}{2} \rceil$. Let $\vec{Z}_K \in \mathcal{F}_y^K$ and recall that $\vec{Z}_K = (Z_1, Z_2, ..., Z_K) \in (\mathbb{C}^{\frac{n(n+1)}{2}})^K$. The matrices $Z_2, Z_4, ..., Z_{2k} \in \mathbb{C}^{\frac{n(n+1)}{2}}$ can take any value, since $Z_1 e_n = Z_3 e_n = ... = Z_{2k-1} e_n = 0$ by assumption and therefore $\Phi_i(\vec{Z}_i) = e_{2n}$ for all $1 \leq i \leq 2k$ by Lemma 3.1.6. Hence we set $Z_2 = Z_4 = ... = Z_{2k} = 0$ and then $\vec{Z}_K \in \mathcal{F}_y^K \cap S_K$ is a singularity by Lemma 3.1.9. $\square$

3.3 Holomorphic vector fields tangential to the fibers

We will construct stratified sprays using $\mathbb{C}$-complete holomorphic vector fields. The main goal of this subsection is therefore to find enough vector fields that are holomorphic on $(\mathbb{C}^{\frac{n(n+1)}{2}})^K$, complete and, in particular, tangential to the fibers $\mathcal{F}_y^K$.

3.3.1 Fiber-preserving vector fields

Let X be some Stein manifold and $f = (f_1, ..., f_n) : X \to \mathbb{C}^n$ a holomorphic mapping. A holomorphic vector field $V : X \to TX, x \mapsto (x, V_x)$ is *fiber-preserving* for f, if it is tangential to the fibers of f. This is the case if and only if V is in the kernel of the tangent map df, that is, $df_x(V_x) = 0$. This is equivalent to say that the Lie derivative $\mathcal{L}_{V_x}(f_i) = V_x(f_i) = (df_i)_x(V_x) = 0$ vanishes for all $1 \leq i \leq n$.

Lemma 3.3.1. *For $N > n$, let $P \in \mathbb{C}[z_1, ..., z_N]^n$ be a polynomial mapping $P : \mathbb{C}^N \to \mathbb{C}^n$ and let $x = (z_{\alpha_0}, ..., z_{\alpha_n})$, with $1 \leq \alpha_0 < ... < \alpha_n \leq N$. Let's write $\frac{\partial}{\partial z} P := (\frac{\partial}{\partial z} P_1, ..., \frac{\partial}{\partial z} P_n)^T$. Then*

$$D_x(P) := \det \begin{pmatrix} \frac{\partial}{\partial z_{\alpha_0}} & \cdots & \frac{\partial}{\partial z_{\alpha_n}} \\ \frac{\partial}{\partial z_{\alpha_0}} P & \cdots & \frac{\partial}{\partial z_{\alpha_n}} P \end{pmatrix}$$

is a holomorphic vector field on $\mathbb{C}^N$ which is fiber-preserving for P.

Proof. The Lie derivative $\mathcal{L}_{D_x(P)}(P_i) = D_x(P)(P_i) = 0$ vanishes for each $1 \leq i \leq n$, since the first and the $(i+1)$-th row of $D_x(P)(P_i)$ are the same. $\square$

We now introduce a few more notations. For a fixed natural number K, the mapping $\tilde{P}^K$ (defined before Lemma 3.2.2) is a polynomial mapping in $\mathbb{C}[z_1, ..., z_{N_K}]^n$ with $N_K := K\frac{n(n+1)}{2}$. Since we'll only be interested in $K > 1$, the constraint $N_K > n$ is given.

There are $\binom{N_K}{n+1}$ possibilities to choose $(n+1)$ of the N_K variables. Let $\mathcal{T}_K$ denote the set of all such possible choices. Recall that we see $\mathbb{C}^{N_K}$ as a product of K copies of $\mathbb{C}^{\frac{n(n+1)}{2}}$ and we write $\vec{Z}_K = (Z_1, ..., Z_K) \in (\mathbb{C}^{\frac{n(n+1)}{2}})^K$. With this convention, the set $\mathcal{T}_K$ can be given by

$$\mathcal{T}_K := \{x = (z_{i_0, j_0 k_0}, ..., z_{i_n, j_n k_n}) : 1 \leq i_0 \leq ... \leq i_n \leq K, 1 \leq j_r \leq k_r \leq n, 0 \leq r \leq n\}.$$

The vector fields $\partial_x^K := D_x(\tilde{P}^K), x \in \mathcal{T}_K$, are fiber-preserving for $\tilde{P}^K$, by Lemma 3.3.1; or, equivalently, they're tangential to the variety $\mathcal{G}_{\pi_{K+1}(y)}$ by construction (c.f. Lemma 3.2.2).

The following lemma collects some interesting examples of fiber-preserving vector fields for Φ_K. In fact, they'll play a cruical role in the construction of a dominating spray (see Theorem 3.6.2)

Lemma 3.3.2. *Let $1 \leq j^* \leq n$, $x \in \mathcal{T}_{K-1}$ and $u := (\partial_x^{K-1}(\tilde{P}_1^{K-2}), ..., \partial_x^{K-1}(\tilde{P}_n^{K-2}))^T$. Then the vector field*

$$\varphi_{x,j^*}^K = (\tilde{P}_{j^*}^{K-1})^2 \partial_x^{K-1} - \tilde{P}_{j^*}^{K-1} \sum_{\substack{i=1 \\ i \neq j^*}}^n u_i \frac{\partial}{\partial z_{K,j^*i}} + \left(\sum_{\substack{i=1 \\ i \neq j^*}}^n \tilde{P}_i^{K-1} u_i - \tilde{P}_{j^*}^{K-1} u_{j^*} \right) \frac{\partial}{\partial z_{K,j^*j^*}}$$

is holomorphic on $(\mathbb{C}^{\frac{n(n+1)}{2}})^K$ and fiber-preserving for Φ_K. Moreover, φ_{x,j^*}^K is complete if and only if ∂_x^{K-1} is complete.

For $1 \leq i \leq j \leq n, i \neq j^*, j \neq j^*$, the vector field

$$\gamma_{ij,j^*}^K = (\tilde{P}_{j^*}^{K-1})^2 \frac{\partial}{\partial z_{K,ij}} + \frac{1}{1+\delta_{ij}} \left(2\tilde{P}_i^{K-1} \tilde{P}_j^{K-1} \frac{\partial}{\partial z_{K,j*j*}} - \tilde{P}_i^{K-1} \tilde{P}_{j^*}^{K-1} \frac{\partial}{\partial z_{K,j^*j}} - \tilde{P}_j^{K-1} \tilde{P}_{j^*}^{K-1} \frac{\partial}{\partial z_{K,j^*i}} \right)$$

is complete, holomorphic on $(\mathbb{C}^{\frac{n(n+1)}{2}})^K$ and fiber-preserving for Φ_K.

Proof. Observe that $\tilde{P}^{K-1} \equiv 0$ on non-generic fibers $\mathcal{F}_y^K$. Hence the above fields are trivial and there is nothing to show. We therefore consider $y \in Y_g^K$ in the generic stratum, i.e. $\pi_K(y) \neq 0$. Without loss of generality assume $\pi_K(y)_1 \neq 0$. According to Lemma 3.2.2, the mapping ψ_1^K is defined by $\Psi_1(x) = (x, f_1(x))$ for some meromorphic map $f_1 = (f_{11}, ..., f_{n1})$ and it maps $\mathcal{G}_{\pi_K(y)}$ biholomorphically onto the generic fiber $\mathcal{F}_y^K$.

Consider a vector field V tangential to $\mathcal{G}_{\pi_K(y)}$. Then the push-forward $W := (\Psi_1)_*(V)$ is given by

$$W = V + \sum_{i=1}^n V(f_{i1}) \frac{\partial}{\partial z_{K,i1}}.$$

On the one hand W is tangential to the fiber $\mathcal{F}_y^K$ and on the other hand it is complete if and only if V is complete.

Let's write $W(\tilde{P}^K) := (W(\tilde{P}_1^K), ..., W(\tilde{P}_n^K))^T$ and $W(Z_K) := \sum_{1 \leq i \leq j \leq n} W(z_{K,ij}) \tilde{E}_{ij}$. By the recursive formula (3.2) and since $W(\tilde{P}^{K-1}) = 0$, we get

$$0 = W(\tilde{P}^K) = W(\tilde{P}^{K-2}) + W(Z_K)\tilde{P}^{K-1} = u + W(Z_K)\tilde{P}^{K-1}.$$

In the special case, where $V = \partial_x^{K-1}$ for some $x \in \mathcal{T}_{K-1}$, we have

$$W(Z_K)\tilde{P}^{K-1} = \left(\sum_{i=1}^n V(f_{i1}) \tilde{E}_{i1} \right) \tilde{P}^{K-1} = \begin{pmatrix} \sum_{i=1}^n V(f_{i1}) \tilde{P}_i^{K-1} \\ \tilde{P}_1^{K-1} V(f_{21}) \\ \vdots \\ \tilde{P}_1^{K-1} V(f_{n1}) \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} \tilde{P}_1^{K-1} & \cdots & \tilde{P}_n^{K-1} \\ & \ddots & \\ & & \tilde{P}_1^{K-1} \end{pmatrix}}_{=:A} \underbrace{\begin{pmatrix} V(f_{11}) \\ \vdots \\ V(f_{n1}) \end{pmatrix}}_{=:b},$$

where A is a regular matrix, since $\tilde{P}_1^{K-1} \neq 0$. Therefore we obtain $b = -A^{-1}u$ with

$$A^{-1} = \frac{1}{(\tilde{P}_1^{K-1})^2} \begin{pmatrix} \tilde{P}_1^{K-1} & -\tilde{P}_2^{K-1} & -\tilde{P}_3^{K-1} & \cdots & -\tilde{P}_n^{K-1} \\ & \tilde{P}_1^{K-1} & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \tilde{P}_1^{K-1} \end{pmatrix}.$$

The vector field $\varphi_{x,1}^K := (\tilde{P}_1^{K-1})^2 W$ is holomorphic on $(\mathbb{C}^{\frac{n(n+1)}{2}})^K$ and fiber-preserving for Φ_K. Moreover, it is complete if and only if W is complete, since $\tilde{P}^{K-1}$ is in the kernel of W. This proves the first part of the lemma.

In the special case, where $V = \frac{\partial}{\partial z_{K,ij}}, 1 < i \le j \le n$, we have $u = W(\tilde{P}^{K-2}) = 0$ and

$$W(Z_K)\tilde{P}^{K-1} = \tilde{E}_{ij}\tilde{P}^{K-1} + \left(\sum_{i=1}^n V(f_{i1})\tilde{E}_{i1}\right)\tilde{P}^{K-1} = \tilde{E}_{ij}\tilde{P}^{K-1} + Ab.$$

Therefore

$$\begin{aligned} b = -A^{-1}\tilde{E}_{ij}\tilde{P}^{K-1} &= -\frac{1}{1+\delta_{ij}}A^{-1}(\tilde{P}_i^{K-1}e_j + \tilde{P}_j^{K-1}e_i) \\ &= \frac{1}{1+\delta_{ij}}\frac{1}{(\tilde{P}_1^{K-1})^2}\left(2\tilde{P}_i^{K-1}\tilde{P}_j^{K-1}e_1 - \tilde{P}_1^{K-1}\tilde{P}_i^{K-1}e_j - \tilde{P}_1^{K-1}\tilde{P}_j^{K-1}e_i\right). \end{aligned}$$

As before, we multiply W by $(\tilde{P}_1^{K-1})^2$ to obtain a fiber-preserving vector field $\gamma_{ij,1}^K := (\tilde{P}_1^{K-1})^2 W$ which is holomorphic on $(\mathbb{C}^{\frac{n(n+1)}{2}})^K$. Note that $\frac{\partial}{\partial z_{K,ij}}$ is complete on $\mathcal{G}_{\pi_K(y)}$ by definition, hence $\gamma_{ij,1}^K$ is complete. This proves the second part of the lemma. □

3.4 Complete holomorphic vector fields tangent to the fibers

In the previous subsection, we've constructed vector fields tangent to the fibers. Unfortunately, some of those fields aren't complete (see Example 3.4.5 below). In addition, it is quite laborious to decide whether a given field is complete. The goal of this subsection is to build machinery that will make this decision easier. Furthermore, we will list the most important examples of complete fields.

For $K > L$, let $f_{K,L} : (\mathbb{C}^{\frac{n(n+1)}{2}})^K \to (\mathbb{C}^{\frac{n(n+1)}{2}})^L, (Z_1, ..., Z_K) \mapsto (Z_1, ..., Z_L)$ be the standard projection. Given a vector field V on $(\mathbb{C}^{\frac{n(n+1)}{2}})^L$ tangential to the fibers $\mathcal{F}_y^L$, its pullback $f_{K,L}^* V$ is a vector field on $(\mathbb{C}^{\frac{n(n+1)}{2}})^K$ tangential to the fibers $\mathcal{F}_{y'}^K$, by the recursive formula (3.1). By an abuse of notation, we just write V instead of $f_{K,L}^* V$. Furthermore, set

$$\mathcal{T}_K^C := \{x \in \mathcal{T}_K : \partial_x^K \text{ is a complete vector field}\}.$$

Definition 3.4.1. *For $K \ge 3$, define the collection*

$$\mathcal{V}_K := \bigcup_{L=3}^K \left(\bigcup_{j=1}^n \{\varphi_{x,j}^L : x \in \mathcal{T}_{L-1}^C\} \cup \{\gamma_{rs,j}^L : 1 \le r, s \le n\}\right)$$

*of **principal vector fields** for Φ_K.*

Remark 3.4.2. *We can consider $\mathcal{V}_{K-1}$ as a subset of $\mathcal{V}_K$ using the convention introduces just before the definition.*

We are now working on a machinery that should make it easier for us to decide whether a tuple x corresponds to a $\mathbb{C}$-complete vector field ∂_x^K. Let $x = (x_1, ..., x_m) \in \mathbb{C}^m$ and $P : \mathbb{C}^m \to \mathbb{C}^{m-1}$ be a polynomial mapping. We define an equivalence relation on $\mathbb{C}^l$ in the following way

$$u, v \in \mathbb{C}^l, u \sim_x v \quad :\Leftrightarrow \quad u_i - v_i \in \bigcap_{k=1}^m \ker\left(\frac{\partial}{\partial x_k}\right), \text{ for all } 1 \le i \le l.$$

A vector $v \in \mathbb{C}^l$ is called *constant in x* if $v \sim_x 0$.

Lemma 3.4.3. *Let $x = (x_1, ..., x_m) \in \mathbb{C}^m$ and $P : \mathbb{C}^m \to \mathbb{C}^{m-1}, x \mapsto P(x)$ be a polynomial mapping. Assume there exists $v \in \mathbb{C}^{m-1}$, $v \sim_x 0$, and $\lambda_{ij} \in \mathbb{C}$, $\lambda_{ij} \sim_x 0$, for all $1 \le i, j \le m$, such that*

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}P(x) = \lambda_{ij} v. \tag{3.5}$$

Then $V = D_x(P)$ is $\mathbb{C}$-complete.

Proof. We consider the vector field $V = D_x(P) = \sum_{j=1}^m V_j \frac{\partial}{\partial x_j}$, where V_j is given by

$$V_j = \det\left(\frac{\partial P}{\partial x_1}, ..., \frac{\partial P}{\partial x_{j-1}}, \frac{\partial P}{\partial x_{j+1}}, ..., \frac{\partial P}{\partial x_m}\right).$$

For $1 \le j \le m$, define $f_j(x) = \sum_{k=1}^m \lambda_{kj} x_k$. Then $\frac{\partial}{\partial x_j} P(x) \sim_x f_j(x) v$ by construction, this means

$$\frac{\partial}{\partial x_j} P(x) = c_j + f_j(x) v,$$

for some $c_j \in \mathbb{C}, c_j \sim_x 0$. Obviously, $f_k(x)v$ and $f_l(x)v$ are linearly dependent, therefore

$$\begin{aligned} V_j &= \det\left(c_1 + f_1 v, ..., c_{j-1} + f_{j-1} v, c_{j+1} + f_{j+1} v, ..., c_m + f_m v\right) \\ &= \underbrace{\det(c_1, ..., c_{j-1}, c_{j+1}, ..., c_m)}_{=:\alpha_{j0}} + f_1 \underbrace{\det(v, c_2, ..., c_{j-1}, c_{j+1}, ..., c_m)}_{=:\alpha_{j1}} + \\ &+ \cdots + f_n \underbrace{\det(c_1, ..., c_{j-1}, c_{j+1}, ..., c_{m-1}, v)}_{=:\alpha_{jm}}. \end{aligned}$$

With the convention $\alpha_{jj} := 0$, we obtain

$$V_j = \alpha_{j0} + \sum_{k=1}^m \alpha_{jk} f_k = \alpha_{j0} + \sum_{k=1}^m \alpha_{jk} \sum_{l=1}^m \lambda_{lk} x_l = \alpha_{j0} + \sum_{l=1}^m x_l \underbrace{\sum_{k=1}^m \lambda_{lk} \alpha_{jk}}_{=:a_{jl}}.$$

Set $b := (\alpha_{10}, ..., \alpha_{m0})^T$. Then we just proved that $V(x) = Ax + b$, where $A = (a_{jl})_{1 \le j,l \le m}$ is a $m \times m$-matrix with $a_{jl} \sim_x 0$.

Let γ be a flow curve, i.e. a holomorphic map $\gamma : \mathbb{C} \to \mathbb{C}^m$ with $\frac{d}{dt}\gamma(t) = V(\gamma(t))$. This leads to the system

$$\frac{d}{dt}\gamma(t) = A\gamma(t) + b,$$

which implies that γ exists for all time $t \in \mathbb{C}$. □

The mapping $\tilde{P}^K : (\mathbb{C}^{\frac{n(n+1)}{2}})^K \to \mathbb{C}^n$ does not a priori fit into the setting of the previous lemma, but this problem can be solved with a simple trick. By fixing all but $(n+1)$ of the $\frac{n(n+1)}{2} K$ variables, we may interpret $\tilde{P}^K$ as a polynomial mapping $\mathbb{C}^{n+1} \to \mathbb{C}^n$. More precisely, each $(n+1)$-tupel $x \in \mathcal{T}_K$ corresponds to a natural inclusion map $i_x : \mathbb{C}^{n+1} \to (\mathbb{C}^{\frac{n(n+1)}{2}})^K$. Then $\tilde{P}^K \circ i_x$ is a polynomial mapping $\mathbb{C}^{n+1} \to \mathbb{C}^n$.

Proposition 3.4.4. ***(List of complete vector fields)***

(Type 1) *For $1 \le m \le n$, $x = (z_{k-1,mm}, z_{k,11}, ..., z_{k,nn}) \in \mathcal{T}_K^C$, $\quad K \ge k$.*

(Type 2) *For $l \ne m$, $x = (z_{k-1,mm}, z_{k,l1}, ..., z_{k,ln}) \in \mathcal{T}_K^C$, $\quad K \ge k$.*

(Type 3) *For $(n+1)$ distinct pairs of indices $(i_0, j_0), ..., (i_n, j_n)$, $x = (z_{k,i_0j_0}, ..., z_{k,i_nj_n}) \in \mathcal{T}_K^C$, $K \ge k$.*

*(**Type 4**) For $1 \le i^* \le n$, $x = (z_{1,ni^*}, z_{2,11}, ..., z_{2,nn}) \in \mathcal{T}_K^C, K \ge 2$.*

*(**Type 5**) Let $k < l$, $1 \le j^* \le n$ and let $(i_1, j_1), ..., (i_n, j_n)$ be n distinct pairs of indices. Then*

$$x = (z_{k,i_1j_1}, ..., z_{k,i_nj_n}, z_{l,j^*j^*}) \in \mathcal{T}_K^C, K \ge l.$$

*(**Type 6**) For $1 \le r \le n$ consider the partition $\{1, ..., n\} = \{i_1, ..., i_r\}\dot\cup\{j_1, ..., j_{n-r}\}$. Let $i^* \in \{i_1, ..., i_r\}$ and $j^*, j' \in \{j_1, ..., j_{n-r}\}$. Then*

$$x = (z_{k-1,j^*j_1}, ..., z_{k-1,j^*j_{n-r}}, z_{k,i^*i_1}, ..., z_{k,i^*i_r}, z_{k,i^*j'}) \in \mathcal{T}_K^C, K \ge k.$$

*(**Type 7**) For $0 \le r \le n$ consider the partition $\{1, ..., n\} = \{i_1, ..., i_r\}\dot\cup\{j_1, ..., j_{n-r}\}$. Let $i^* \in \{i_1, ..., i_r\}$ and $j^*, j' \in \{j_1, ..., j_{n-r}\}$. Then*

$$x = (z_{k,j^*j_1}, ..., z_{k,j^*j_{n-r}}, z_{k+1,i^*i_1}, ..., z_{k+1,i^*i_r}, z_{k+2,j'j'}) \in \mathcal{T}_K^C, K \ge k+2.$$

*(**Type 8**) For $1 \le i \le n, i \ne j$, $x = (z_{1,in}, z_{2,j1}, ..., z_{2,jn}) \in \mathcal{T}_K^C, K \ge 2$.*

Proof. For the proof of (*Type 1*), we consider the mapping $P : \mathbb{C}^{n+1} \to \mathbb{C}^n$ given by

$$x := (z_{k-1,mm}, z_{k,11}, ..., z_{k,nn}) \mapsto \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} I_n & Z_k \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 \\ Z_{k-1} & I_n \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix},$$

where A and B are arbitrary $n \times n$-matrices, both constant in x; whereas c and d are arbitrary vectors in $\mathbb{C}^n$ both constant in x. Observe that

$$\begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} I_n & Z_k \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 \\ Z_{k-1} & I_n \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} B & A \end{pmatrix} \begin{pmatrix} I_n & 0 \\ Z_k & I_n \end{pmatrix} \begin{pmatrix} I_n & Z_{k-1} \\ 0 & I_n \end{pmatrix} \begin{pmatrix} d \\ c \end{pmatrix}.$$

Thanks to this symmetry condition, we don't need to make a case distinction between even and odd K. In fact, it is enough to prove that $D_x(P)$ is a $\mathbb{C}$-complete vector field. At first, note that

$$\frac{\partial^2}{\partial z_{k-1,mm}^2} P(x) \equiv 0, \quad \frac{\partial}{\partial z_{k,ii}} \frac{\partial}{\partial z_{k,jj}} P(x) \equiv 0, \quad 1 \le i, j \le n.$$

Hence most of the λ's in Lemma 3.4.3 can be chosen to be zero. It remains to consider $\frac{\partial}{\partial z_{k,ii}} \frac{\partial}{\partial z_{k-1,mm}} P(x)$, $1 \le i \le n$. We get

$$\begin{aligned} \frac{\partial}{\partial z_{k,ii}} \frac{\partial}{\partial z_{k-1,mm}} P(x) &= \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} 0 & \tilde{E}_{ii} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \tilde{E}_{mm} & 0 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} \\ &= \begin{pmatrix} 0 & A\tilde{E}_{ii} \end{pmatrix} \begin{pmatrix} 0 \\ \tilde{E}_{mm} c \end{pmatrix} \\ &= A\tilde{E}_{ii}\tilde{E}_{mm}c = c_m \delta_{mi} A e_m, \end{aligned}$$

for all $1 \le i \le n$. Since $v := Ae_m$ is independent of i, the conditions of Lemma 3.4.3 are satisfied. Therefore $D_x(P)$ is a complete field and we conclude that vector fields ∂_x^K, $K \ge k$, of (*Type 1*) are complete.

For the proof of (*Type 2*) we choose P as before. Again we have $\frac{\partial^2}{\partial z_{k-1,mm}^2} P \equiv 0$ and $\frac{\partial}{\partial z_{k,li}} \frac{\partial}{\partial z_{k,lj}} P \equiv 0$, $1 \le i, j \le n$. Further, we compute

$$\frac{\partial}{\partial z_{k,li}} \frac{\partial}{\partial z_{k-1,mm}} P(x) = \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} 0 & \tilde{E}_{li} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \tilde{E}_{mm} & 0 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = A\tilde{E}_{li}\tilde{E}_{mm}c = c_m A\tilde{E}_{li} e_m.$$

Observe that $\tilde{E}_{li}e_m = \delta_{im}e_l$, since we assume $l \neq m$. Therefore we obtain

$$\frac{\partial}{\partial z_{k,li}} \frac{\partial}{\partial z_{k-1,mm}} P(x) = \delta_{im} c_m A e_l$$

for all $1 \leq i \leq n$. This proves that the conditions of Lemma 3.4.3 are satisfied and hence the field $D_x(P)$ is complete. We conclude that the fields ∂_x^K, $K \geq k$, of (*Type 2*) are complete.

For the proof of (*Type 3*), we set $P(x) = \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} I_n & Z_k \\ 0 & I_n \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix}$. Observe that $\frac{\partial}{\partial z_{k,i_r j_r}} P(x) \sim_x 0$ for all $r = 0, ..., n$. Hence the conditions of Lemma 3.4.3 are trivially satisfied and we conclude that the vector fields ∂_x^K of (*Type 3*) are complete.

For the proof of (*Type 4*) we set $P(x) = \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} I_n & 0 \\ Z_2 & I_n \end{pmatrix} \begin{pmatrix} I_n & Z_1 \\ 0 & I_n \end{pmatrix} e_{2n}$. Note that $\frac{\partial^2}{\partial z_{1,ni^*}^2} P \equiv 0$ and $\frac{\partial}{\partial z_{2,ii}} \frac{\partial}{\partial z_{2,jj}} P \equiv 0$, $1 \leq i, j \leq n$. Furthermore, we compute

$$\frac{\partial}{\partial z_{2,jj}} \frac{\partial}{\partial z_{1,ni^*}} P(x) = \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \tilde{E}_{jj} & 0 \end{pmatrix} \begin{pmatrix} 0 & \tilde{E}_{ni^*} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ e_n \end{pmatrix} = B \tilde{E}_{jj} \underbrace{\tilde{E}_{ni^*} e_n}_{=e_{i^*}} = \delta_{ji^*} B e_{i^*}.$$

Hence we can apply Lemma 3.4.3 and conclude that vector fields ∂_x^K, $K \geq 2$, of (*Type 4*) are complete.

For the proof of (*Type 5*) we first consider

$$P(x) = \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} I_n & 0 \\ Z_l & I_n \end{pmatrix} \begin{pmatrix} U & V \\ W & X \end{pmatrix} \begin{pmatrix} I_n & Z_k \\ 0 & I_n \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix},$$

where U, V, W and X are arbitrary $n \times n$-matrices constant in x. As in the previous cases, we have $\frac{\partial^2}{\partial z_{l,j^*j^*}^2} P \equiv 0$ and $\frac{\partial}{\partial z_{k,i_r j_r}} \frac{\partial}{\partial z_{k,i_s j_s}} P \equiv 0$, $1 \leq r, s \leq n$. Further, let's compute

$$\begin{aligned} \frac{\partial}{\partial z_{l,j^*j^*}} \frac{\partial}{\partial z_{k,ij}} P(x) &= \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \tilde{E}_{j^*j^*} & 0 \end{pmatrix} \begin{pmatrix} U & V \\ W & X \end{pmatrix} \begin{pmatrix} 0 & \tilde{E}_{ij} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} \\ &= B \tilde{E}_{j^*j^*} U \tilde{E}_{ij} d = \underbrace{(e_{j^*}^T U \tilde{E}_{ij} d)}_{\sim_x 0} B e_{j^*}. \end{aligned}$$

If we replace $\begin{pmatrix} I_n & 0 \\ Z_l & I_n \end{pmatrix}$ by $\begin{pmatrix} I_n & Z_l \\ 0 & I_n \end{pmatrix}$ in P, we obtain

$$\frac{\partial}{\partial z_{l,j^*j^*}} \frac{\partial}{\partial z_{k,ij}} P(x) = A \tilde{E}_{j^*j^*} W \tilde{E}_{ij} d = (e_{j^*}^T W \tilde{E}_{ij} d) A e_{j^*}.$$

In both cases, Lemma 3.4.3 implies that $D_x(P)$ is complete and in conclusion, the vector fields ∂_x^K, $K \geq l$, of (*Type 5*) are complete.

For the proof of (*Type 6*), we set P as for (*Type 1*). Observe that $\tilde{E}_{i^*i} \tilde{E}_{j^*j} = 0$ for $i^*, i \in \{i_1, ..., i_r\}$ and $j^*, j \in \{j_1, ..., j_{n-r}\}$. Then we get

$$\begin{aligned} \frac{\partial}{\partial z_{k-1,j^*j}} \frac{\partial}{\partial z_{k,i^*i}} P(x) &= \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} 0 & \tilde{E}_{i^*i} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \tilde{E}_{j^*j} & 0 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} \\ &= A \tilde{E}_{i^*i} \tilde{E}_{j^*j} c = 0, \end{aligned}$$

for all $i \in \{i_1, ..., i_r\}$ and $j \in \{j_1, ..., j_{n-r}\}$. Furthermore,

$$\frac{\partial}{\partial z_{k-1,j^*j}} \frac{\partial}{\partial z_{k,i^*j'}} P(x) = A \tilde{E}_{i^*j'} \tilde{E}_{j^*j} c \sim_x 0,$$

and hence the vector fields ∂_x^K, $K \geq k$, of (*Type 6*) are complete, by Lemma 3.4.3.

For the proof of (*Type 7*) we set $P(x) = \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} I_n & Z_{k+2} \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 \\ Z_{k+1} & I_n \end{pmatrix} \begin{pmatrix} I_n & Z_k \\ 0 & I_n \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix}$. By the same argument as in (*Type 6*) we obtain $\frac{\partial}{\partial z_{k+2,j'j'}} \frac{\partial}{\partial z_{k+1,i^*i}} P(x) = 0$ for all $i \in \{i_1, ..., i_r\}$ and $\frac{\partial}{\partial z_{k+1,i^*i}} \frac{\partial}{\partial z_{k,j^*j}} P(x) = 0$ for all $i \in \{i_1, ..., i_r\}$, $j \in \{j_1, ..., j_{n-r}\}$. It remains to compute

$$\begin{aligned} \frac{\partial}{\partial z_{k+2,j'j'}} \frac{\partial}{\partial z_{k,j^*j}} P(x) &= \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} 0 & \tilde{E}_{j'j'} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I_n & 0 \\ Z_{k+1} & I_n \end{pmatrix} \begin{pmatrix} 0 & \tilde{E}_{j^*j} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} \\ &= A\tilde{E}_{j'j'} \begin{pmatrix} Z_{k+1} & I_n \end{pmatrix} \begin{pmatrix} \tilde{E}_{j^*j} d \\ 0 \end{pmatrix} \\ &= A\tilde{E}_{j'j'}(Z_{k+1}\tilde{E}_{j^*j}d) = (e_{j'}^T Z_{k+1}\tilde{E}_{j^*j}d) A e_{j'}. \end{aligned}$$

Observe that $e_{j'}^T Z_{k+1}$ is constant in x, since we assume $j' \in \{j_1, ..., j_{n-r}\}$. Hence the conditions of Lemma 3.4.3 are satisfied and we conclude that the fields ∂_x^K, $K \geq k+2$, of (*Type 7*) are complete.

For the proof of (*Type 8*), let $P(x) = \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} I_n & 0 \\ Z_2 & I_n \end{pmatrix} \begin{pmatrix} Z_1 e_n \\ e_n \end{pmatrix}$. For $1 \leq r \leq n$, we compute

$$\frac{\partial}{\partial z_{1,in}} \frac{\partial}{\partial z_{2,jr}} P(x) = \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \tilde{E}_{jr} & 0 \end{pmatrix} \begin{pmatrix} e_i \\ 0 \end{pmatrix} = \delta_{ir} B e_j.$$

Hence the conditions of Lemma 3.4.3 are met and the fields ∂_x^K, $K \geq 2$, of (*Type 8*) are complete. □

In the following, we find an example of an incomplete vector field ∂_x^2. As a direct consequence of this, we don't know how large the collection $\mathcal{V}_K$ of principal vector fields is. We shall see later, however, that it is powerful enough to construct stratified sprays.

Example 3.4.5. *Consider the tupel $x = (z_{1,n1}, ..., z_{1,nn}, z_{2,n1})$ and the mapping $P(x) = e_n + Z_2 Z_1 e_n$. The Jacobian JP is given by $\begin{pmatrix} Z_2 & z_{1,nn}e_1 + z_{1,n1}e_n \end{pmatrix}$ and we get the vector field*

$$\partial_x^2 = \det \begin{pmatrix} \partial/\partial z_{1,n1} & \cdots & \partial/\partial z_{1,nn} & \partial/\partial z_{2,n1} \\ z_{2,11} & \cdots & z_{2,1n} & z_{1,nn} \\ z_{2,21} & \cdots & z_{2,2n} & 0 \\ \vdots & & \vdots & \vdots \\ z_{2,n1} & \cdots & z_{2,nn} & z_{1,n1} \end{pmatrix}.$$

Therefore $\partial_x^2(z_{2,n1}) = \pm \det(Z_2) = \pm(\alpha_1 z_{2,n1}^2 + \alpha_2 z_{2,n1} + \alpha_3)$ for some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C} \cap \ker(\partial_x^2)$. In fact, α_1 is the principal minor of order $n-2$ obtained by removing the first and last rows and columns from Z_2. Hence $\alpha_1 \not\equiv 0$ on $(\mathbb{C}^{\frac{n(n+1)}{2}})^2$, which means that the variable $z_{2,n1}$ occurs quadratically. We conclude that ∂_x^2 is incomplete.

3.5 Topological analysis of the fibers

So far we haven't learned anything about the fibers $\mathcal{F}_y^K$ from a topological perspective. In this subsection we will show that all fibers are connected for $K \geq 3$. In fact, all fibers are irreducible, except the singular fibers $\mathcal{F}_{a,e_n^T}^3$ and $\mathcal{F}_{0,e_n^T}^4$ which consist of two irreducible components, with the smooth part breaking down in two connected components.

Lemma 3.5.1. *The fibers $\mathcal{F}_{a,b}^3$ are connected. The singular fibers, i.e. $\mathcal{F}_{a,e_n^T}^3$ consists of two irreducible components.*

Proof. We start with the non-generic fibers, i.e. we assume $b = 0$. In this case, we have $\mathcal{F}^3_{a,0} = \mathcal{F}^2_{a,0} \times \mathbb{C}^{\frac{n(n+1)}{2}}$ with $a \neq 0$. The defining equations of $\mathcal{F}^2_{a,0}$ are given by

$$\begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} I_n & Z_2 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} Z_1 e_n \\ e_n \end{pmatrix} = \begin{pmatrix} Z_1 e_n \\ (I_n + Z_2 Z_1) e_n \end{pmatrix},$$

and therefore $\mathcal{F}^2_{a,0} \cong \mathbb{C}^{\frac{n(n-1)}{2}} \times \{Z \in \mathbb{C}^{\frac{n(n+1)}{2}} : e_n + Za = 0\} \cong \mathbb{C}^{n(n-1)}$. In conclusion, the non-generic fiber $\mathcal{F}^3_{a,0}$ is biholomorphic to some $\mathbb{C}^N$ and hence connected.

We continue with the generic fibers $\mathcal{F}^3_{a,b}$, i.e. $b \neq 0$. In the first step, we consider the smooth fibers, that is, we assume $b \neq e_n$ in addition. Due to Lemma 3.2.2, the fiber $\mathcal{F}^3_{a,b}$ is biholomorphic to $\mathcal{G}_{\pi_3(a,b)} \times \mathbb{C}^{\frac{n(n-1)}{2}}$, where

$$\mathcal{G}_{\pi_3(a,b)} = \{\vec{Z}_2 \in (\mathbb{C}^{\frac{n(n+1)}{2}})^2 : b = P_s(\vec{Z}_2) = e_n + Z_2 Z_1 e_n\}.$$

Define $C_j := \{\vec{Z}_2 \in \mathcal{G}_{\pi_3(a,b)} : z_{1,nj} \neq 0\},\ 1 \leq j \leq n$. Similar as in Lemma 3.2.2, we can express the variables $z_{2,j1}, ..., z_{2,jn}$, which proves that C_j is biholomorphic to $\mathbb{C}^* \times \mathbb{C}^N$ for some natural number N and therefore connected. Since we assume $b \neq e_n$, $Z_1 e_n = 0$ is not possible in $\mathcal{G}_{\pi_3(a,b)}$, which means that $\mathcal{G}_{\pi_3(a,b)}$ is covered by $\cup_{j=1}^n C_j$. It remains to show, that the intersection $\cap_{j=1}^n C_j \neq \emptyset$ is non-empty. Choose a symmetric matrix Z_1^* with $Z_1^* e_n = (1, ..., 1)^T$. By Lemma 3.1.10, there is a symmetric matrix Z_2 with $b - e_n = Z_2(Z_1^* e_n)$. This shows that the intersection is indeed non-empty. In conclusion, $\mathcal{G}_{\pi_3(a,b)}$ and $\mathcal{F}^3_{a,b}$ are connected.

Finally, let's consider the singular fibers $\mathcal{F}^3_{a,e_n^T}$. By Lemma 3.2.2, such fibers are biholomorphic to $\mathcal{G}_{\pi_3(a,b)} \times \mathbb{C}^{\frac{n(n-1)}{2}}$ where

$$\mathcal{G}_{\pi_3(a,b)} = \{\vec{Z}_2 \in (\mathbb{C}^{\frac{n(n+1)}{2}})^2 : Z_2 Z_1 e_n = 0\}.$$

Observe that this variety has two irreducible components $A_1 = \{\vec{Z}_2 \in \mathcal{G}_{\pi_3(a,e_n^T)} : Z_1 e_n = 0\}$ and $A_2 = \{\vec{Z}_2 \in \mathcal{G}_{\pi_3(a,e_n^T)} : \det(Z_2) = 0\}$. This proves that singular fibers $\mathcal{F}^3_{a,e_n^T}$ have two irreducible components. Since the intersection of these components equals the singularity set S_3, $\mathcal{F}^3_{a,e_n^T}$ is connected. $\square$

Theorem 3.5.2. *The fibers $\mathcal{F}^K_y$ are connected for $K \geq 3$. Moreover, the smooth part of the singular fibers is connected for $K \geq 5$.*

Proof. We prove this theorem by induction on K. Note that we've shown the base case $K = 3$ in the previous lemma. Let $K \geq 4$ and assume that the $(K-1)$ fibers $\mathcal{F}^{K-1}_y$ are connected. Recall that

$$\mathcal{F}^K_y = \bigcup_{Z \in \mathbb{C}^{\frac{n(n+1)}{2}}} \mathcal{F}^{K-1}_{M_K(Z)y}. \tag{3.6}$$

We will now introduce the following auxiliary function. Let $\rho : \mathcal{F}^K_y \to \mathbb{C}^{\frac{n(n+1)}{2}}$ denote the restriction of the projection $(\vec{Z}_{K-1}, Z_K) \mapsto Z_K$ to the fiber $\mathcal{F}^K_y$. We'll show some useful facts.

(i) ρ is surjective: Observe that $\vec{Z}_K \in \mathcal{F}^K_y$ if and only if $\vec{Z}_{K-1} \in \mathcal{F}^{K-1}_{M_K(-Z_K)y}$, by (3.6). Hence the ρ-fibers are given by $\rho^{-1}(Z_K^*) = \mathcal{F}^{K-1}_{M_K(-Z_K^*)y}$. The $(K-1)$-fibers $\mathcal{F}^{K-1}_{M_K(Z)y}$ are non-empty for all $Z \in \mathbb{C}^{\frac{n(n+1)}{2}}$, by Theorem 3.1.2.

(ii) ρ is submersive in $\vec{Z}_K$ if $\vec{Z}_{K-1} \notin S_{K-1}$: Observe that $T_{\vec{Z}_K}(\mathcal{F}^K_y \cap S_K^c) = \ker J\Phi_K(\vec{Z}_K)$. By assumption, the Jacobian $J\Phi_{K-1}(\vec{Z}_{K-1})$ is surjective. Given $W_K \in \mathbb{C}^{\frac{n(n+1)}{2}}$, we find

$\vec{W}_{K-1} \in (\mathbb{C}^{\frac{n(n+1)}{2}})^{K-1}$ such that $J\Phi_{K-1}(\vec{Z}_{K-1})\vec{W}_{K-1} = -M_K(-Z_K)A_K(\Phi_{K-1}(\vec{Z}_{K-1}))W_K$. By the recursive formula of the Jacobian (see Corollary 3.1.5), we get

$$J\Phi_K(\vec{Z}_K)\begin{pmatrix}\vec{W}_{K-1}\\ W_K\end{pmatrix} = M_K(Z_K)J\Phi_{K-1}(\vec{Z}_{K-1})\vec{W}_{K-1} + A_K(\Phi_{K-1}(\vec{Z}_{K-1}))W_K = 0$$

and $d\rho_{\vec{Z}_K}(\vec{W}_{K-1}, W_K) = W_K$.

(iii) Each connected component $A \subset \mathcal{F}_y^K$ is ρ-saturated, that is, $A = \rho^{-1}(\rho(A))$: It is enough to show "$\supset$", by definition of the preimage. Each ρ-fiber is connected and therefore we have either $\rho^{-1}(b) \subset A$ or $\rho^{-1}(b) \cap A = \emptyset$.

(iv) $\rho(A)$ is open for each connected component $A \subset \mathcal{F}_y^K$: Given a point $b \in \rho(A)$ we find a regular point $a \in \rho^{-1}(b)$. To see this, notice that in every $(K-1)$ fiber we find points with $Z_1 e_n \neq 0$, by the previous lemma and by (3.6). Submersivity is a local property, hence there exists an open neighborhood $U \subset A$ of a in which ρ is submersive. Furthermore, U is mapped openly, that is, $\rho(U)$ is an open neighborhood of b in $\rho(A)$.

We can write $\mathcal{F}_y^K$ as a disjoint union of connected components $\dot{\bigcup}_{i\in I} A_i$. Then

$$\mathbb{C}^{\frac{n(n+1)}{2}} \underset{(i)}{=} \rho(\mathcal{F}_y^K) \underset{(iii)}{=} \dot{\bigcup}_{i\in I}\rho(A_i)$$

can be written as the disjoint union of open sets. Since $\mathbb{C}^{\frac{n(n+1)}{2}}$ is connected, we conclude that $\mathcal{F}_y^K$ has to be connected too.

It remains to show, that the smooth part of the fibers $\mathcal{F}_{a,b}^K$ is connected for $K \geq 5$. Let's start with the case $K = 2k+1$. Then the singular fibers are $\mathcal{F}_{a,e_n^T}^K$, by Lemma 3.2.3. If we now also note (3.6), then the singularities are all in the subfiber $\mathcal{F}_{0,e_n^T}^{2k}$. Since $\mathcal{F}_{0,e_n^T}^{2k} \subset \mathcal{F}_{a,e_n^T}^{2k+1}$ has codimension n, the complement $U := \mathcal{F}_{a,e_n^T}^{2k+1} \setminus \mathcal{F}_{0,e_n^T}^{2k}$ is connected. If we consider a smooth point $p \in \mathcal{F}_{0,e_n^T}^{2k}$, then each open neighborhood of p intersects with U. Hence p has to be in the same connected component as U. This proves that the smooth part $\mathcal{F}_{a,e_n^T}^K \setminus \operatorname{Sing}(\mathcal{F}_{a,e_n^T}^K)$ is connected.

In the case $K = 2k$, observe that $\mathcal{F}_{0,e_n^T}^{2k}$ is the only singular fiber. Furthermore,

$$\mathcal{F}_{0,e_n^T}^{2k} \setminus \operatorname{Sing}(\mathcal{F}_{0,e_n^T}^{2k}) = \left(\mathcal{F}_{0,e_n^T}^{2k-1} \setminus \operatorname{Sing}(\mathcal{F}_{0,e_n^T}^{2k-1})\right) \times \mathbb{C}^{\frac{n(n+1)}{2}},$$

and this is connected by the induction hypothesis. This completes the proof of the theorem. $\square$

Theorem 3.5.3. *The fibers of the submersion $\Phi_K : \mathcal{W}_K \to \mathbb{C}^{2n} \setminus \{0\}$ are connected for $K \geq 3$.*

Proof. We want to show that the intersection $\mathcal{F}_y^K \cap \mathcal{W}_K$ is connected. Smooth fibers $\mathcal{F}_y^K$ are contained in $\mathcal{W}_K$, hence we only need to consider the case when $\mathcal{F}_y^K$ is a singular fiber. For $K = 3$ we can apply the strategy from Lemma 3.5.1 for smooth fibers and cover $\mathcal{F}_{a,e_n^T}^3 \cap \mathcal{W}_3$ by n intersection connected charts.

Next, assume the claim to be true for $K - 1 = 2k - 1$. For

$$\mathcal{R} := \{\vec{Z}_{2k-1} \in (\mathbb{C}^{\frac{n(n+1)}{2}})^{2k-1} : Z_1 e_n = ... = Z_{2k-3}e_n = 0, Z_{2k-1}e_n \neq 0\},$$

we can rewrite

$$\mathcal{W}_{2k} = \mathcal{W}_{2k-1} \times \mathbb{C}^{\frac{n(n+1)}{2}} \dot{\cup} \mathcal{R} \times \mathbb{C}^{\frac{n(n+1)}{2}}.$$

Observe that $\mathcal{F}_{0,e_n^T}^{2k-1} \cap \mathcal{R} = \emptyset$, by definition. Therefore, the non-generic singular fiber $\mathcal{F}_{0,e_n^T}^{2k}$ satisfies

$$\mathcal{F}^{2k} \cap \mathcal{W}_{2k} = \left(\mathcal{F}_{0,e_n^T}^{2k-1} \cap \mathcal{W}_{2k-1}\right) \times \mathbb{C}^{\frac{n(n+1)}{2}},$$

which is connected by the induction hypothesis.

Finally, we assume the claim to be true for $K-1=2k$. Let $\rho : \mathcal{F}^K_{a,e_n^T} \cap \mathcal{W}_K \to \mathbb{C}^{\frac{n(n+1)}{2}}$ be the restriction of the projection $(\vec{Z}_{K-1}, Z_K) \mapsto Z_K$ to $\mathcal{F}^K_{a,e_n^T} \cap \mathcal{W}_K$. The fiber $\rho^{-1}(Z_K)$ is given by $\mathcal{F}^{K-1}_{M_K(-Z_K)y} \cap \mathcal{W}_{K-1}$. From here we can argue as in the proof of Theorem 3.5.2. □

Lemma 3.5.4. *Each generic fiber $\mathcal{F}^3_y$ containes points $\vec{Z}_3$ with $Q^1_f(\vec{Z}_3) = z_{1,n1} \cdots z_{1,nn} \neq 0$. Moreover, each generic fiber $\mathcal{F}^K_y$ containes points $\vec{Z}_K$ with*

$$Q^{K-2}_f(\vec{Z}_K) Q^{K-2}_s(\vec{Z}_K) = P^{K-2}_1(\vec{Z}_{K-2}) \cdots P^{K-2}_n(\vec{Z}_{K-2}) P^{K-2}_{n+1}(\vec{Z}_{K-2}) \cdots P^{K-2}_{2n}(\vec{Z}_{K-2}) \neq 0.$$

Proof. Let's start with the case $K=3$. We consider a generic fiber $\mathcal{F}^3_{a,b}$, $b \neq 0$. Recall that $\vec{Z}_3 \in \mathcal{F}^3_{a,b}$ if and only if $\vec{Z}_2 \in \mathcal{F}^2_{a-bZ_3,b}$ by the recursive formula (3.1). Since we assume $b \neq 0$, there is a symmetric matrix $Z_3 \in \mathbb{C}^{\frac{n(n+1)}{2}}$ with $a - Z_3 b = (1,...,1)^T$, by Lemma 3.1.10. The fiber $\mathcal{F}^2_{(1,...,1),b} \neq \emptyset$ is non-empty by Theorem 3.1.2. Furthermore, $P^2_f \equiv P^1_f$, by the recursive formula (3.1). This implies $Q^1_f \equiv 1$ on the fiber $\mathcal{F}^2_{(1,...,1),b} \subset \mathcal{F}^3_{a,b}$ and we're done with the case $K=3$.

We now consider the case $K \geq 4$. The fiber $\mathcal{F}^{K-2}_{(1,...,1),(1,...,1)} \neq \emptyset$ is non-empty, by Theorem 3.1.2, and clearly $Q^{K-2}_f Q^{K-2}_s \equiv 1$ on the whole fiber. Therefore it is enough to show, that this fiber sits inside each generic K-fiber, that is, $\mathcal{F}^{K-2}_{(1,...,1),(1,...,1)} \subset \mathcal{F}^K_y$. We prove this claim only for $K = 2k+1$, since the proof is symmetric for $K = 2k$. In this case, a point $y = (a,b) \in Y^K_g$ in the generic stratum satisfies $b \neq 0$. Let's write $v := (1,...,1)^T$. Then the fiber $\mathcal{F}^{K-2}_{v,v}$ sits inside $\mathcal{F}^K_{a,b}$ if and only if we find symmetric matrices $Z_{K-1}, Z_K \in \mathbb{C}^{\frac{n(n+1)}{2}}$ with

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} I_n & Z_K \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 \\ Z_{K-1} & I_n \end{pmatrix} \begin{pmatrix} v \\ v \end{pmatrix},$$

or equivalently,

$$\begin{pmatrix} a - Z_K b \\ b \end{pmatrix} = \begin{pmatrix} v \\ Z_{K-1} v + v \end{pmatrix}.$$

We can split up this system of equations into two independent systems $a - Z_K b = v$ and $b = Z_{K-1} v + v$. An application of Lemma 3.1.10 to both systems yields the existence of such matrices Z_{K-1} and Z_K, since $b \neq 0$ and $v \neq 0$. This completes the proof. □

3.6 Construction of stratified sprays

The most convenient way to construct a dominating spray associated to a submersion is to define some finite composition of flow maps of complete fiber-preserving vector fields. The collection of complete vector fields in Proposition 3.4.4 does not span the tangent space of the fibers $\mathcal{F}^K_y$ in every point. This is the reason why these are a priori not sufficient to define a dominating spray. However, we can enlarge this collection until it spans the tangent space in a *sufficiently large* set of points. In the following section we discuss the meaning of 'sufficiently large' and the mathematical details for the enlargement.

3.6.1 Construction of a spray over the generic stratum

We begin this subsection with a definition. Recall the collection $\mathcal{V}_K$ of $\mathbb{C}$-complete fiber-preserving holomorphic vector fields from Definition 3.4.1. These collections are defined for $K \geq 3$. According to Lemma 3.2.2, there are n meromorphic mappings $\psi_1, ..., \psi_n$ such that each generic fiber $\mathcal{F}_y^2$ is biholomorphic to $\mathcal{G}_{\pi_2(y)} \cong \mathbb{C}^{n(n-1)}$ via ψ_j for some $1 \leq j \leq n$. Let $\frac{\partial}{\partial x_i}$, $1 \leq i \leq n(n-1)$ denote the standard vector fields on $\mathbb{C}^{n(n-1)}$. Then

$$\mathcal{V}_2 = \bigcup_{j=1}^{n} \left\{ (\psi_j)_* \left(\frac{\partial}{\partial x_i} \right) : 1 \leq i \leq n(n-1) \right\}$$

is the collection obtained by pushing forward the standard vector fields via $\psi_1, ..., \psi_n$.

Definition 3.6.1. *For $K \geq 2$, we define $\mathcal{Q}_K := \Gamma(\mathcal{V}_K)$ the collection of $\mathbb{C}$-complete $\mathcal{F}_y^K$-fiber-preserving holomorphic vector fields generated by $\mathcal{V}_K$.*

Recall the set $\mathcal{W}_K$ which is given by

$$\mathcal{W}_K := \left\{ \vec{Z}_K \in (\mathbb{C}^{\frac{n(n+1)}{2}})^K : Z_{2i-1}e_n \neq 0, \text{ for some } 1 \leq i \leq \lceil \tfrac{K-1}{2} \rceil \right\},$$

where $\lceil x \rceil$ is the *ceiling function*, which maps x to the least integer greater than or equal to x. The set $\mathcal{W}_K$ is open and connected.

The following result, from now on we will call it *Spanning theorem*, is cruical for the construction of a spray over the generic stratum.

Theorem 3.6.2 (Spanning theorem)**.** *Let $K \geq 2$. Then there is a finite set $A_K \subset \mathcal{Q}_K$ of $\mathbb{C}$-complete fiber-preserving holomorphic vector fields on $(\mathbb{C}^{\frac{n(n+1)}{2}})^K$ spanning the tangent bundle $T(\mathcal{F}_y^K \cap \mathcal{W}_K)$ of every generic fiber $\mathcal{F}_y^K$. In particular, $A_K \subset A_{K+1}$, when considering A_K as a subset of $\mathcal{Q}_{K+1}$.*

Let $K \geq 3$. Then the mapping $\Phi_K|_{\mathcal{W}_K} : \mathcal{W}_K \to \mathbb{C}^{2n} \setminus \{0\}$ is a surjective submersion with connected fibers $\mathcal{F}_y^K$ (see Theorem 3.1.2 and Theorem 3.5.3). By the Spanning theorem, the conditions of Lemma 1.6.10 are satisfied. Hence the submersion

$$\Phi_K : C_{A_K}(\mathcal{W}_K)|_{Y_g^K} \to Y_g^K$$

over the generic stratum $Y_g^K \subset \mathbb{C}^{2n} \setminus \{0\}$ admits a spray.

Remark 3.6.3 (Application of the Spanning theorem)**.** *Recall that, by Lemma 3.2.3, a fiber $\mathcal{F}_y^K$ contains singularities if and only if $y = (y_1, ..., y_n, 0, ..., 0, 1)$ for $y_1, ..., y_n \in \mathbb{C}$. In particular, the intersection of singular fibers and $\mathcal{W}_K^c$, the complement of $\mathcal{W}_K$, is non-empty. We do not know, whether the collection A_K from the Spanning theorem can be supplemented by finitely many fields in $\mathcal{Q}_K$ so that A_K spans the tangent space $T_{\vec{Z}_K}\mathcal{F}_y^K$ for smooth points $\vec{Z}_K$ in $\mathcal{W}_K^c$. To be precise, it was proved for $n = 2$ in [17], but for $n > 2$ it is an open question. In contrast, the collection A_K spans the tangent bundle $T\mathcal{F}_y^K$ of each smooth generic fiber $\mathcal{F}_y^K$, since smooth fibers $\mathcal{F}_y^K$ are completely contained in $\mathcal{W}_K$.*

3.6.2 Construction of a spray over the non-generic stratum

In this subsection, we show that the submersion

$$\Phi_K : C_{A_K}(\mathcal{W}_K)|_{Y_{ng}^K} \to Y_{ng}^K$$

over the non-generic stratum $Y_{ng}^K \subset \mathbb{C}^{2n} \setminus \{0\}$ admits a spray.

We need the following result.

Lemma 3.6.4. *Let $K \geq 3$ and let $A \subset \mathcal{Q}_K$ be a finite collection of complete holomorphic fiber preserving vector fields. Then*

$$C_A(\mathcal{W}_K \cap \mathcal{F}_y^K) = C_A(\mathcal{W}_{K-1} \cap \mathcal{F}_y^{K-1}) \times \mathbb{C}^{\frac{n(n+1)}{2}}$$

for each non-generic fiber $\mathcal{F}_y^K$.

Proof. In a first step, we prove that

$$\mathcal{W}_K \cap \mathcal{F}_y^K = (\mathcal{W}_{K-1} \cap \mathcal{F}_y^{K-1}) \times \mathbb{C}^{\frac{n(n+1)}{2}} \tag{3.7}$$

for each non-generic fiber $\mathcal{F}_y^K$ and $K \geq 3$. From the definition of the set $\mathcal{W}_K$ we directly conclude $\mathcal{W}_{2k+1} = \mathcal{W}_{2k} \times \mathbb{C}^{\frac{n(n+1)}{2}}$. Since each non-generic fiber satisfies $\mathcal{F}_y^K = \mathcal{F}_y^{K-1} \times \mathbb{C}^{\frac{n(n+1)}{2}}$ by Lemma 3.2.1, equation (3.7) follows for $K = 2k+1$.

Consider now $K = 2k$ even and a non-generic fiber $\mathcal{F}_y^K$, that is, $y = (0, b)^T$ for some non-zero $b \in \mathbb{C}^n$. Further, observe that $\mathcal{W}_{2k} = \mathcal{W}_{2k-1} \times \mathbb{C}^{\frac{n(n+1)}{2}} \cup \mathcal{R}$, where

$$\mathcal{R} := \{\vec{Z}_{2k} \in (\mathbb{C}^{\frac{n(n+1)}{2}})^{2k} : \pi_n(Z_{2i-1}) = 0, 1 \leq i \leq k-1, \pi_n(Z_{2k-1}) \neq 0\}.$$

It suffices to show that $\mathcal{F}_y^K \cap \mathcal{R} = \emptyset$, in order to prove (3.7). Assume for contradiction there is $\vec{Z}_{2k} \in \mathcal{F}_y^K \cap \mathcal{R}$. Lemma 3.1.6 implies $\Phi_{2k-2}(Z_{2k-2}) = e_{2n}$ and $\Phi_{2k-1}(\vec{Z}_{2k-1}) = \begin{pmatrix} 0 & b \end{pmatrix}^T$ follows by Lemma 3.2.1. According to the recursive formula (3.1), we obtain

$$\begin{pmatrix} 0 \\ b \end{pmatrix} = \begin{pmatrix} I_n & Z_{2k-1} \\ 0 & I_n \end{pmatrix} e_{2n} = \begin{pmatrix} \pi_n(Z_{2k-1}) \\ e_n \end{pmatrix}$$

contradicting assumption $\pi_n(Z_{2k-1}) \neq 0$. This proves equation (3.7).

In a second step, we show that, over the non-generic stratum, none of the vector fields $V \in \mathcal{Q}_K$ flows in a new direction. It suffices to prove the claim for the generating set $\mathcal{V}_K$. Note that there is nothing to show for vector fields in $\mathcal{V}_{K-1}$. And fields in $\mathcal{V}_K \setminus \mathcal{V}_{K-1}$ vanish over the non-generic stratum, by Lemma 3.3.2. This proves the claim. In particular, the vector flow α_t^V of $V \in \mathcal{Q}_K$ fixes the new directions, i.e. $\alpha_t^V(\vec{Z}_{K-1}, Z_K) = (f(\vec{Z}_{K-1}, Z_K), Z_K)$ for some suitable holomorphic function f.

In the last step, we apply equation (3.7) and step two. We get

$$C_A(\mathcal{W}_K \cap \mathcal{F}_y^K) = C_A(\mathcal{W}_{K-1} \cap \mathcal{F}_y^{K-1} \times \mathbb{C}^{\frac{n(n+1)}{2}}) = C_A(\mathcal{W}_{K-1} \cap \mathcal{F}_y^{K-1}) \times \mathbb{C}^{\frac{n(n+1)}{2}}.$$

This finishes the proof. $\square$

Lemma 3.6.5. *Let $K \geq 3$ and $A_K \subset \mathcal{Q}_K$ be the finite collection provided by the Spanning theoremsuch that the tangent bundle $T(\mathcal{F}_y^K \cap \mathcal{W}_K)$ of every generic fiber $\mathcal{F}_y^K$ is spanned by A_K. Then the restricted submersion*

$$\Phi_K : C_{A_K}(\mathcal{W}_K)|_{Y_{ng}^K} \to Y_{ng}^K$$

over the non-generic stratum $Y_{ng}^K \subset \mathbb{C}^{2n} \setminus \{0\}$ admits a spray.

Proof. Let A_{K-1} and A_K be the collections from the Spanning theorem and let $\mathcal{F}_y^K$ be a non-generic fiber. We have $\mathcal{F}_y^K = \mathcal{F}_y^{K-1} \times \mathbb{C}^{\frac{n(n+1)}{2}}$ by Lemma 3.2.1, where $\mathcal{F}_y^{K-1}$ is a generic $(K-1)$-fiber. According to Lemma 3.2.1, the vector fields from $A_K \setminus A_{K-1}$ vanish over $\mathcal{F}_y^K$ and we get

$$C_{A_K}(\mathcal{W}_K \cap \mathcal{F}_y^K) = C_{A_{K-1}}(\mathcal{W}_{K-1} \cap \mathcal{F}_y^{K-1}) \times \mathbb{C}^{\frac{n(n+1)}{2}}.$$

Collection A_{K-1} spans the tangent bundle $T(\mathcal{W}_{K-1} \cap \mathcal{F}_y^{K-1})$ for every generic fiber $\mathcal{F}_y^{K-1}$, by the Spanning theorem. By Theorem 1.6.10, there is a finite collection $B \subset \Gamma(A_{K-1})$ spanning the tangent bundle $T(C_{A_{K-1}}(\mathcal{W}_{K-1} \cap \mathcal{F}_y^{K-1}))$ for every generic fiber $\mathcal{F}_y^{K-1}$. We add the vector fields

$$\left(\frac{\partial}{\partial z_{K,ij}}\right)_{1 \leq i \leq j \leq n}$$

which span $\mathbb{C}^{\frac{n(n+1)}{2}}$ to the collection B. This new collection, let's call it $\tilde{B}$, spans the tangent bundle $T(C_{A_K}(\mathcal{W}_K \cap \mathcal{F}_y^K))$ for every non-generic fiber $\mathcal{F}_y^K$. Similar as in Theorem 1.6.10, we can use the vector flows α_t^V, $V \in \tilde{B}$, to construct a dominating spray associated to the submersion $\Phi_K : C_{A_K}(\mathcal{W}_K)|_{Y_{ng}^K} \to Y_{ng}^K$. $\square$

3.7 The Spanning theorem

In this subsection we prove the Spanning theorem, which we will do by induction on the number of factors K. As it turns out, for various reasons, it requires several base steps before we get to the actual induction step. It makes sense to explain the proof strategy continuously. So let's first introduce or recall some notations and then we start with the first base step, $K = 2$.

We write $\Phi_K = (P_1^K, ..., P_{2n}^K)^T$ as well as $P_f^K = (P_1^K, ..., P_n^K)^T$ and $P_s^K = (P_{n+1}^K, ..., P_{2n}^K)^T$. Similarly, we write $Q_f^K = P_1^K \cdots P_n^K$ and $Q_s^K = P_{n+1}^K \cdots P_{2n}^K$. For $y \in \mathbb{C}^{2n} \setminus \{0\}$ we sometimes write $y = (a, b)$ with $a = (a_1, ..., a_n)$ and $b = (b_1, ..., b_n)$.

Lemma 3.7.1 (Spanning theorem for $K = 2$). *Let $\mathcal{F}_y^2$ be a generic fiber, i.e. $\pi_2(y)_{j^*} = P_{j^*}^1 \neq 0$ for some $1 \leq j^* \leq n$. Then the collection*

$$A_2 = \left\{(P_{j^*}^1)^2 \frac{\partial}{\partial z_{1,ij}} : 1 \leq i \leq j < n\right\} \cup \left\{\gamma_{ij,j^*}^2 : 1 \leq i \leq j \leq n, i \neq j^*, j \neq j^*\right\}$$

consist of complete holomorphic vector fields which are fiber-preserving for Φ_2. Moreover, A_2 spans the tangent bundle $T\mathcal{F}_y^2$.

Proof. According to Lemma 3.2.2, we are able to express the variables $z_{2,1j^*}, ..., z_{2,nj^*}$. This gives us a meromorphic mapping ψ_{j^*} which maps $\mathcal{G}_{\pi_2(y)} \cong \mathbb{C}^{n(n-1)}$ biholomorphic to $\mathcal{F}_y^2$. In particular, the vector fields $\frac{\partial}{\partial z_{1,ij}}, 1 \leq i \leq j < n$ and $\frac{\partial}{\partial z_{2,ij}}, 1 \leq i \leq j \leq n, i \neq j^*, j \neq j^*$ are complete holomorphic and tangential to $\mathcal{G}_{\pi_2(y)}$. Moreover, they span the tangent bundle $T\mathcal{G}_{\pi_2(y)}$. The collection A_2 is obtained by computing the push-forwards with respect to the mapping ψ_{j^*}. $\square$

3.7.1 Preparation and explanation of the induction step

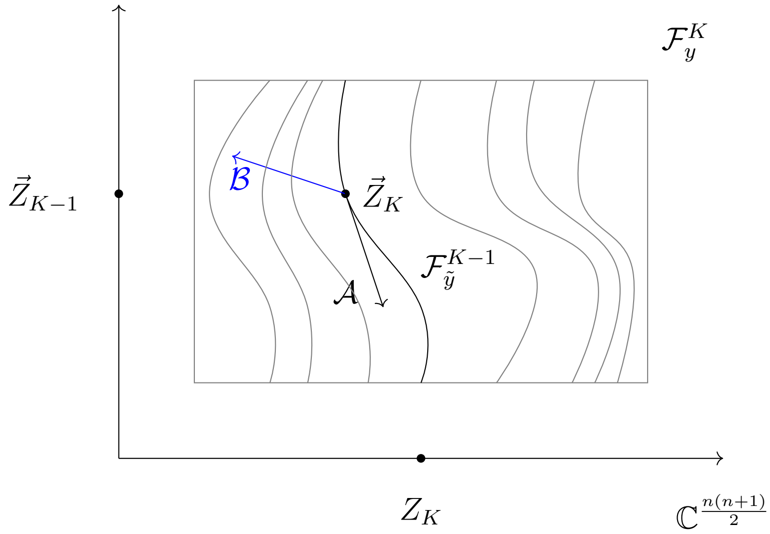
Recall that each K-fiber $\mathcal{F}_y^K$ can be written as a union of $(K-1)$-fibers, that is,

$$\mathcal{F}_y^K = \bigcup_{Z \in \mathbb{C}^{\frac{n(n+1)}{2}}} \mathcal{F}_{M_K(Z)y}^{K-1}.$$

Equivalently, $\vec{Z}_K = (\vec{Z}_{K-1}, Z_K) \in \mathcal{F}_y^K$ if and only if $\vec{Z}_{K-1} \in \mathcal{F}_{\tilde{y}}^{K-1}$ with $\tilde{y} = M_K(-Z_K)y$. The directions

$$\left(\frac{\partial}{\partial z_{K,ij}}\right)_{1 \leq i \leq j \leq n}$$

are somehow transversal to the 'fibration' and we use to speak of *the new directions*. The basic idea of the induction is the following. Assume there is a finite collection $\mathcal{A} \subset \mathcal{Q}_{K-1}$ of complete fiber-preserving vector fields which spans the tangent space $T_{\vec{Z}_K}\mathcal{F}^{K-1}_{\tilde{y}}$. Then we are looking for a finite collection $\mathcal{B} \subset \mathcal{Q}_K$ which spans the new directions in $\vec{Z}_K$. Since the new directions are complementary, the union $\mathcal{A} \cup \mathcal{B}$ spans the tangent space $T_{\vec{Z}_K}\mathcal{F}^{K}_{y}$. The following picture, while mathematically inaccurate, can illustrate the idea fairly well.



The following statement is very useful for the induction step.

Lemma 3.7.2. *For $K = 3$, there are finitely many vector fields from $\mathcal{V}_3$ spanning the new directions in a generic fiber $\mathcal{F}^3_y$ in points with*

$$Q^1_f \neq 0.$$

For $K \geq 4$, there are finitely many vector fields from $\mathcal{V}_K$ spanning the new directions in a generic fiber $\mathcal{F}^K_y$ in points with

$$Q^{K-2}_f \neq 0 \quad \textit{and} \quad Q^{K-2}_s \neq 0.$$

Before we prove this lemma, let's apply it. We therefore define the sets

$$\mathcal{U}_3 := \{\vec{Z}_3 \in \mathcal{W}_3 : Q^1_f(\vec{Z}_3) \neq 0\}$$

and for $K \geq 4$,

$$\mathcal{U}_K := \{\vec{Z}_K \in \mathcal{W}_K : Q^{K-2}_f(\vec{Z}_K) Q^{K-2}_s(\vec{Z}_K) \neq 0\}.$$

What now follows is one argument of the induction step.

Lemma 3.7.3. *Let $K \geq 3$ and suppose that the Spanning theorem is true for $K-1$. Then there is a finite collection $A \subset \mathcal{Q}_K$ spanning the tangent bundle $T(\mathcal{F}^K_y \cap \mathcal{U}_K)$ for every generic fiber $\mathcal{F}^K_y$.*

Proof. In a first step, assume $K = 3$ and consider a point $\vec{Z}_3 \in \mathcal{U}_3$ such that $y := \Phi_3(\vec{Z}_3) \in Y^3_g$ is in the generic stratum, i.e. $b = P^3_s(\vec{Z}_3) \neq 0$. By definition of the set $\mathcal{U}_3$, we have $Q^1_f(\vec{Z}_3) \neq 0$. According to Lemma 3.7.2, there is a finite collection $A \subset \mathcal{Q}_3$ which spans the new directions in points with $Q^1_f \neq 0$. Hence $\vec{Z}_3$ is a point, where the new directions are spanned.

By the recursive formula (3.1), we have $\Phi_2(\vec{Z}_2) = M_3(-Z_3)y$. Moreover,

$$Z_1 e_n = \pi_2 \circ \Phi_2(\vec{Z}_2) = \pi_2 \circ M_3(-Z_3)y = a - Z_3 b$$

implies that $\vec{Z}_2 \in \mathcal{F}^2_{M_3(-Z_3)y}$ is contained a generic 2-fiber, since $\vec{Z}_3 = (\vec{Z}_2, Z_3) \in \mathcal{U}_3$. Therefore the tangent space $T_{\vec{Z}_3}\mathcal{F}^2_{M_3(-Z_3)y} \subset T_{\vec{Z}_3}\mathcal{F}^3_y$ is spanned by $A_2 \subset \mathcal{Q}_2$ according to Lemma 3.7.1. Since the new directions are complementary, the collection $A \cup A_2$ spans the tangent space $T_{\vec{Z}_3}\mathcal{F}^3_y$. This is true for every generic fiber $\mathcal{F}^3_y$, hence there is a finite collection in $\mathcal{Q}_3$ which spans the tangent bundle $T(\mathcal{F}^3_y \cap \mathcal{U}_3)$ for every generic fiber.

In the case $K \geq 4$ we'll argue similarly. Let $\vec{Z}_K \in \mathcal{U}_K$ be a point such that $y := \Phi_K(\vec{Z}_K) \in Y^K_g$ is in the generic stratum, i.e. $\pi_K(y) \neq 0$. Write $\vec{Z}_K = (\vec{Z}_{K-1}, Z_K)$ and $\vec{Z}_{K-1} = (\vec{Z}_{K-2}, Z_{K-1})$. Note that

$$\Phi_{K-1}(\vec{Z}_{K-1}) = M_K(-Z_K)y \quad \text{and} \quad \Phi_{K-1}(\vec{Z}_{K-1}) = M_{K-1}(Z_{K-1})\Phi_{K-2}(\vec{Z}_{K-2})$$

by the recursive formula (3.1). By defintion of π_K and M_K, we have $\pi_K \circ M_K = \pi_K$, hence $\pi_{K-1} \circ \Phi_{K-1} = \pi_{K-1} \circ \Phi_{K-2}$. Moreover, by definition of $\mathcal{U}_K$, we have $Q_f^{K-2}Q_s^{K-2}(\vec{Z}_K) \neq 0$, which implies $\pi_{K-1} \circ \Phi_{K-1}(\vec{Z}_{K-1}) \neq 0$, that is, $\vec{Z}_{K-1}$ is contained in the generic fiber $\mathcal{F}^{K-1}_{M_K(-Z_K)y}$. In addition, we have $\vec{Z}_{K-1} \in \mathcal{W}_{K-1}$, since otherwise this would contradict $Q_f^{K-2}Q_s^{K-2} \neq 0$ by Lemma 3.1.6. Therefore the tangent space $T_{\vec{Z}_{K-1}}\mathcal{F}^{K-1}_{M_K(-Z_K)y}$ is spanned by the finite collection in $\mathcal{Q}_{K-1}$ provided from the Spanning theorem for $K-1$. Similarly as in the previous case, $\vec{Z}_K$ is a point where the new directions are spanned, according to Lemma 3.7.2. Again, the new directions are complementary and we've shown that there is a finite collection in $\mathcal{Q}_K$ spanning the tangent space $T_{\vec{Z}_K}\mathcal{F}^K_y$ for each generic fiber. And this proves the lemma. $\square$

Remark 3.7.4. *An application of Corollary 1.6.11 and the former lemma leads us to the following observation. To complete the induction step, it suffices to find a finite collection $A \subset \mathcal{Q}_K$ satisfying*

$$\mathcal{F}^K_y \cap \mathcal{W}_K \subset C_A(\mathcal{F}^K_y \cap \mathcal{U}_K), \quad \textit{for each generic fiber } \mathcal{F}^K_y. \tag{3.8}$$

The idea of the proof is to stratify $\mathcal{W}_K$ suitably, i.e. find a finite descending chain of subspaces

$$\mathcal{W}_K =: X_N \supset ... \supset X_0 = \emptyset,$$

where the spaces $X_0, ..., X_{N-1}$ are closed. In a first step, we'll find a finite collection $A_N \subset \mathcal{Q}_K$ such that the stratum $S_N := X_N \setminus X_{N-1}$ satisfies

$$\mathcal{F}^K_y \cap S_N \subset C_{A_N}(\mathcal{F}^K_y \cap \mathcal{U}_K)$$

for each generic fiber $\mathcal{F}^K_y$. Then, we find a finite collection $A_k \subset \mathcal{Q}_K$, for each stratum $S_k := X_k \setminus X_{k-1}$, $1 \leq k \leq N-1$, such that

$$\mathcal{F}^K_y \cap S_k \subset C_{A_k}(\mathcal{F}^K_y \cap S_{k+1})$$

for each generic fiber $\mathcal{F}^K_y$. Define the finite collection $A := \bigcup_{k=1}^N A_k \subset \mathcal{Q}_K$. Then we get

$$\mathcal{F}^K_y \cap S_1 \subset C_A(\mathcal{F}^K_y \cap S_2) \subset \cdots \subset C_A(\mathcal{F}^K_y \cap S_N) \subset C_A(\mathcal{F}^K_y \cap \mathcal{U}_K)$$

for each generic fiber $\mathcal{F}^K_y$. As it turns out, the stratification of $\mathcal{W}_K$ in the cases $K = 3$, $K = 4$ and $K \geq 5$ differ in some elementary ways.

Proof of Lemma 3.7.2. Let $\mathcal{F}^K_y$ be a generic fiber, that is, $\tilde{P}^K_{j^*} \neq 0$ for some $1 \leq j^* \leq n$. Recall that this fiber is biholomorphic to $\mathcal{G}_{\pi_K(y)} \times \mathbb{C}^{\frac{n(n-1)}{2}}$ with the biholomorphism obtained by expressing the n variables $z_{K,j^*1}, ..., z_{K,j^*n}$ in terms of the remaining variables $(z_{K,ij})_{1\leq i \leq j \leq n; i,j \neq j^*}$.

Clearly, the fields $\left(\frac{\partial}{\partial z_{K,ij}}\right)_{1\leq i\leq j\leq n; i,j\neq j^*}$ are tangential to $\mathcal{G}_{\pi_K(y)} \times \mathbb{C}^{\frac{n(n-1)}{2}}$ and moreover, they span $\{0\} \times \mathbb{C}^{\frac{n(n-1)}{2}}$. Hence the corresponding lifts span the new directions $\left(\frac{\partial}{\partial z_{K,ij}}\right)_{1\leq i\leq j\leq n; i,j\neq j^*}$ in $\mathcal{F}_y^K$. It remains to find n vector fields $\partial_{x_1}^{K-1}, ..., \partial_{x_n}^{K-1}$ tangent to $\mathcal{G}_{\pi_K(y)} \times \{0\}$ such that the projection of the corresponding lifts $\varphi_{x_1,j^*}^K, ..., \varphi_{x_n,j^*}^K$ to the new directions $\left(\frac{\partial}{\partial z_{K,j^*j}}\right)_{1\leq j\leq n}$ are linearly independent. Let $\alpha_{j^*,1}, ..., \alpha_{j^*,n}$ denote the component vectors of such projections in the frame $\left(\frac{\partial}{\partial z_{K,j^*j}}\right)_{1\leq j\leq n}$. We need to show that the matrix $A := (\alpha_{j^*,1}|\cdots|\alpha_{j^*,n})$ is regular. Set $u_m := (\partial_{x_m}^{K-1}(\tilde{P}_1^{K-2}), ..., \partial_{x_m}^{K-1}(\tilde{P}_n^{K-2}))^T$ and recall from the proof of Lemma 3.3.2, that there is a regular matrix $B := F_{j^*}(\tilde{P}^K)^{-1}$ with $\alpha_{j^*,m} = Bu_m$. Therefore A is regular if and only if $U := (u_1|\cdots|u_n)$ is regular.

In a first step, we assume $K \geq 4$ and consider the tupels of (*Type 1*)

$$x_m := (z_{K-2,mm}, z_{K-1,11}, ..., z_{K-1,nn}), 1 \leq m \leq n.$$

Without loss of generality, let $K = 2k+1$. The entries of U are given by $u_{ij} := \partial_{x_i}^{2k}(P_j^{2k})$. Let's compute the following derivatives

$$\frac{\partial}{\partial z_{K-2,ii}} P_s^{2k} = \begin{pmatrix} Z_{K-1} & I_n \end{pmatrix} \begin{pmatrix} 0 & \tilde{E}_{ii} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P_f^{2k-2} \\ P_s^{2k-2} \end{pmatrix} = P_{n+i}^{2k-2} Z_{K-1} e_i,$$

and

$$\frac{\partial}{\partial z_{K-1,mm}} P_s^{2k} = \begin{pmatrix} \tilde{E}_{mm} & 0 \end{pmatrix} \begin{pmatrix} P_f^{2k-1} \\ P_s^{2k-1} \end{pmatrix} = P_m^{2k-1} e_m.$$

Furthermore, we have $\frac{\partial}{\partial z_{K-1,mm}} P_f^{2k} \equiv 0$ and

$$\frac{\partial}{\partial z_{K-2,ii}} P_f^{2k} = \tilde{E}_{ii} P_s^{2k-1} = P_{n+i}^{2k-1} e_i.$$

We obtain

$$u_{ij} = \partial_{x_i}^{2k}(P_j^{2k}) = \det \begin{pmatrix} \delta_{ij} P_{n+i}^{2k-1} & 0 & \cdots & 0 \\ P_{n+i}^{2k-1} Z_{K-1} e_i & P_1^{2k-1} e_1 & \cdots & P_n^{2k-1} e_n \end{pmatrix} = \delta_{ij} P_{n+i}^{2k-1} Q_f^{2k-1},$$

and hence the matrix

$$U = \begin{pmatrix} P_{n+1}^{2k-1} Q_f^{2k-1} & & \\ & \ddots & \\ & & P_{2n}^{2k-1} Q_f^{2k-1} \end{pmatrix}$$

is regular, provided $Q_f^{2k-1} Q_s^{2k-1} \neq 0$.

In a second step, let $K = 3$ and consider the tupels of (*Type 4*)

$$x_j := (z_{1,nj}, z_{2,11}, ..., z_{2,nn}), 1 \leq j \leq n.$$

We obtain derivatives $\frac{\partial}{\partial z_{1,nj}} P_s^2 = Z_2 e_j$, $\frac{\partial}{\partial z_{2,ii}} P_s^2 = z_{1,ni} e_i$ and $\frac{\partial}{\partial z_{1,nj}} P_f^2 = e_j$, $\frac{\partial}{\partial z_{2,ii}} P_f^2 \equiv 0$, for $1 \leq i, j \leq n$. Hence we get

$$u_{ij} = \det \begin{pmatrix} \delta_{ij} & 0 & \cdots & 0 \\ Z_2 e_j & z_{1,n1} e_1 & \cdots & z_{1,nn} e_n \end{pmatrix} = \delta_{ij} Q_f^1$$

and $U = Q_f^1 I_n$. This completes the proof. □

3.7.2 Spanning theorem for $K = 3$

We want to find a finite collection $A \subset \mathcal{Q}_3$ such that (3.8) is fulfilled for $K = 3$.

Let $1 \le i, j \le n$, $i \ne j$, and consider the tupels of (*Type 8*)

$$x_{i,j} := (z_{1,ni}, z_{2,j1}, ..., z_{2,jn}).$$

The corresponding vector fields $\partial^2_{x_{i,j}}$ are of the form (see (A.1) for more details)

$$\partial^2_{x_{i,j}} = \det \begin{bmatrix} \partial/\partial z_{1,ni} & \partial/\partial z_{2,j1} & \cdots & \partial/\partial z_{2,jj} & \cdots & \partial/\partial z_{2,jn} \\ z_{2,1i} & z_{1,nj} & & 0 & & 0 \\ \vdots & & \ddots & & & \\ z_{2,ji} & z_{1,n1} & \cdots & z_{1,nj} & \cdots & z_{1,nn} \\ \vdots & & & & \ddots & \\ z_{2,ni} & 0 & & 0 & & z_{1,nj} \end{bmatrix}$$

and they satisfy

$$\partial^2_{x_{i,j}}(z_{1,kn}) = \delta_{ik}(z_{1,nj})^n,$$

for $1 \le i, j, k \le n$.

By definition, points in $\mathcal{N} := \mathcal{W}_3 \setminus \mathcal{U}_3$ satisfy $Q^1_f = z_{1,1n} \cdots z_{1,nn} = 0$ and $P^1_f = Z_1 e_n \ne 0$. Hence there are indices $1 \le i_1 < ... < i_k \le n$ and $1 \le j_1 < ... < j_{n-k} \le n$ satisfying

(i) $\{1, ..., n\} = \{i_1, ..., i_k\} \dot\cup \{j_1, ..., j_{n-k}\}$,

(ii) $z_{1,in} = 0$, for all $i \in \{i_1, ..., i_k\}$,

(iii) $z_{1,jn} \ne 0$, for all $j \in \{j_1, ..., j_{n-k}\}$.

Fix an index $j \in \{j_1, ..., j_{n-k}\}$. Then observe

$$\partial^2_{x_{i_1,j}} \circ \cdots \circ \partial^2_{x_{i_k,j}}(z_{1,i_1n} \cdots z_{1,i_kn}) = (z_{1,jn})^n \partial^2_{x_{i_1,j}} \circ \cdots \circ \partial^2_{x_{i_{k-1},j}}(z_{1,i_1n} \cdots z_{1,i_{k-1}n}),$$

which inductively implies

$$\partial^2_{x_{i_1,j}} \circ \cdots \circ \partial^2_{x_{i_k,j}}(z_{1,i_1n} \cdots z_{1,i_kn}) = (z_{1,jn})^{kn} \ne 0.$$

Let $\vec{Z}_3 \in \mathcal{N}$ be a point over the generic stratum, that is, $\Phi_3(\vec{Z}_3) \in Y^3_g$. According to Lemma 1.6.13, we find finitely many of the above fields $\partial^2_{x_{i,j}}$ such that a suitable finite composition of the respective flows moves $\vec{Z}_3$ away from $\mathcal{N}$. More precisely, we have a finite collection $A \subset \mathcal{Q}_3$ and an automorphism $\alpha \in G_A$ such that $\alpha(\vec{Z}_3) \in C_A(\mathcal{U}_3)$. That's exactly what we need to get (3.8).

3.7.3 Spanning theorem for $K = 4$

As in the previous step, we want to find a suitable finite collection A such that (3.8) is satisfied. In order to do this, we'll do a *divide and conquer* with the set $\mathcal{N}_4 := \mathcal{W}_4 \setminus \mathcal{U}_4$.

Lemma 3.7.5. *Define the set*

$$\mathcal{N}_1 := \{\vec{Z}_4 \in \mathcal{W}_4 : Q^2_f(\vec{Z}_4) Q^2_s(\vec{Z}_4) = 0, P^2_s(\vec{Z}_4) \ne 0\}.$$

Then there is a finite collection $A \subset \mathcal{Q}_4$ such that

$$\mathcal{F}^4_y \cap \mathcal{N}_1 \subset C_A(\mathcal{F}^4_y \cap \mathcal{U}_4)$$

for each generic fiber $\mathcal{F}^4_y$.

Proof. In a first step, recall that $P_f^2 = P_f^1$ by the recursive formula (3.1). Consider a point $\vec{Z}_4$ in the set

$$X_1 = \{\vec{Z}_4 \in \mathcal{N}_1 : Q_f^1 \neq 0\}.$$

By definition of this set, we find indices $1 \leq i^*, i \leq n$ with $P_{n+i^*}^2(\vec{Z}_4) \neq 0$ and $P_{n+i}^2(\vec{Z}_4) = 0$. Pick the vector field $\partial_{x_i}^3$ corresponding to the tupel $x_i = (z_{2,ii}, z_{3,i^*1}, ..., z_{3,i^*n})$ of (*Type 2*). It is of the form (cf. (A.2))

$$\pm(P_{n+i^*}^2)^n \frac{\partial}{\partial z_{2,ii}} + \sum_{l=1}^{n} \alpha_l \frac{\partial}{\partial z_{3,i^*l}}$$

for some suitable holomorphic functions $\alpha_1, ..., \alpha_n$. We show that the function P_{n+i}^2 is not contained in the kernel of ∂_x^3. Note that P_{n+i}^2 does not depend on the variables $z_{3,rs}$, $1 \leq r \leq s \leq n$. Therefore

$$\begin{aligned}
\partial_x^3(P_{n+i}^2) &= \pm(P_{n+i^*}^2)^n \frac{\partial}{\partial z_{2,ii}}(P_{n+i}^2) \\
&= \pm(P_{n+i^*}^2)^n \frac{\partial}{\partial z_{2,ii}}(\delta_{in} + e_i^T Z_2 Z_1 e_n) \\
&= \pm(P_{n+i^*}^2)^n \underbrace{e_i^T \tilde{E}_{ii}}_{=e_i^T} Z_1 e_n \\
&= \pm z_{1,in}(P_{n+i^*}^2)^n \neq 0
\end{aligned}$$

and this applies for every $1 \leq i \leq n$ with $P_{n+i}^2(\vec{Z}_4) = 0$. Hence there is a finite collection $A \subset \mathcal{Q}_4$ such that

$$\mathcal{F}_y^4 \cap X_1 \subset C_A(\mathcal{F}_y^4 \cap \mathcal{U}_4)$$

for each generic fiber $\mathcal{F}_y^4$.

In a second step, let's consider a point $\vec{Z}_4 = (\vec{Z}_3, Z_4)$ in the set

$$X_2 = \{\vec{Z}_4 : Q_f^1 = 0, P_f^1 \neq 0, P_s^2 \neq 0\} \subset \mathcal{N}_1.$$

Observe that the projection of $\vec{Z}_4$ to the first component $\vec{Z}_3$ is contained in a generic 3-fiber $\mathcal{F}_{\tilde{y}}^3$ and in $\mathcal{W}_3$. From the case $K = 3$, we know the existence of a finite collection $B \subset \mathcal{Q}_3$ such that $\vec{Z}_3 \in C_B(\mathcal{F}_{\tilde{y}}^3 \cap \mathcal{U}_3)$. Put the corresponding pullbacks into the collection $A \subset \mathcal{Q}_4$. Then we've found a finite collection A such that

$$\mathcal{F}_y^4 \cap X_2 \subset C_A(\mathcal{F}_y^4 \cap X_1) \subset C_A(\mathcal{F}_y^4 \cap \mathcal{U}_4)$$

for each generic fiber $\mathcal{F}_y^4$.

In a third step, let's consider a point $\vec{Z}_4$ in the set

$$X_3 = \{\vec{Z}_4 : P_f^1 = 0\} \subset \mathcal{N}_1,$$

that is, $Z_1 e_n = 0$. The vector field ∂_x^3 corresponding to the tupel $x = (z_{1,1n}, ..., z_{1,nn}, z_{3,nn})$ of (*Type 7*), satisfies

$$\partial_x^3(z_{1,nn}) = \det(e_1 + Z_3 Z_2 e_1, ..., e_{n-1} + Z_3 Z_2 e_{n-1}, e_n)$$

in $\vec{Z}_4$. Hence there is a finite collection $A \subset \mathcal{Q}_4$ such that

$$\mathcal{F}_y^4 \cap \underbrace{\{\vec{Z}_4 : P_f^1(\vec{Z}_4) = 0, \partial_x^3(z_{1,nn}) \neq 0\}}_{=:\tilde{N}} \subset C_A(\mathcal{F}_y^4 \cap \{\vec{Z}_4 : P_f^1 \neq 0, P_s^2 \neq 0\})$$

for each generic fiber $\mathcal{F}_y^4$.

Next, we consider the vector fields $\partial_{x_i}^3$ and $\partial_{y_i}^3$, $i = 1, ..., n-1$, corresponding to the tupels $x_i = (z_{2,ii}, z_{3,1n}, ..., z_{3,nn})$ and $y_i = (z_{3,ii}, z_{3,1n}, ..., z_{nn})$ of (*Type 2*) and (*Type 3*), respectively. By Lemma A.1.1 and Lemma A.1.2, they are of the form $\partial_{x_i}^3 = \frac{\partial}{\partial z_{2,ii}}$ and $\partial_{y_i}^3 = \frac{\partial}{\partial z_{3,ii}}$ on $\mathcal{N}^4(P_f^2)$. The formula

$$\frac{d}{dt}\det(A_1(t), ..., A_n(t)) = \sum_{l=1}^{n} \det(A_1(t), ..., A_l'(t), ..., A_n(t)) \tag{3.9}$$

is sort of a product rule, where $A_1(t), ..., A_n(t)$ denote columns of a $n \times n$-matrix $A(t)$ depending on t. In our case, we consider a matrix A, where the first $(n-1)$ columns are given by

$$A_i = e_i + \sum_{l=1}^{n} z_{2,li} Z_3 e_l, \quad 1 \leq i \leq n-1,$$

and the n-th column $A_n = e_n$. Observe that

$$\frac{\partial}{\partial z_{2,ii}} A_j = \frac{\partial}{\partial z_{2,ii}} (e_j + \sum_{l=1}^{n} z_{2,lj} Z_3 e_l) = \delta_{ij} Z_3 e_i.$$

Hence

$$\frac{\partial}{\partial z_{2,11}} \cdots \frac{\partial}{\partial z_{2,(n-1)(n-1)}} \det(A) = \det(Z_3 e_1, ..., Z_3 e_{n-1}, e_n).$$

Furthermore, we obtain

$$\frac{\partial}{\partial z_{3,11}} \cdots \frac{\partial}{\partial z_{3,(n-1)(n-1)}} \det(Z_3 e_1, ..., Z_3 e_{n-1}, e_n) = \det(e_1, ..., e_n) = 1.$$

By Lemma 1.6.13 and the previous step, we conclude the existence of a finite collection $A \subset \mathcal{Q}_4$ such that

$$\mathcal{F}_y^4 \cap \{\vec{Z}_4 \in \mathcal{N}_1 : P_f^1(\vec{Z}_4) = 0, \partial_x^3(z_{1,nn}) = 0\} \subset C_A(\mathcal{F}_y^4 \cap \tilde{N})$$

and

$$\mathcal{F}_y^4 \cap \tilde{N} \subset C_A(\mathcal{F}_y^4 \cap \{\vec{Z}_4 : Q_f^1 \neq 0, P_s^2 \neq 0\})$$

for each generic fiber $\mathcal{F}_y^4$. In particular, this implies

$$\mathcal{F}_y^4 \cap X_3 \subset C_A(\mathcal{F}_y^4 \cap X_2) \subset C_A(\mathcal{F}_y^4 \cap \mathcal{U}_4)$$

for each generic fiber $\mathcal{F}_y^4$.

For the final step, observe that $\mathcal{N}_1 = X_1 \cup X_2 \cup X_3$. We put all of the involved vector fields above together and obtain a finite collection $A \subset \mathcal{Q}_4$ satisfying

$$\mathcal{F}_y^4 \cap \mathcal{N}_1 \subset C_A(\mathcal{F}_y^4 \cap \mathcal{U}_4)$$

for each generic fiber $\mathcal{F}_y^4$. This finishes the proof. □

Lemma 3.7.6. *Define the set*

$$\mathcal{N}_2 := \{\vec{Z}_4 : P_s^2(\vec{Z}_4) = 0\}.$$

Then there is a finite collection $A \subset \mathcal{Q}_4$ such that

$$\mathcal{F}_y^4 \cap \mathcal{N}_2 \subset C_A(\mathcal{F}_y^4 \cap \mathcal{U}_4)$$

for each generic fiber $\mathcal{F}_y^4$.

In order to prove this lemma, the symplectic nature of the elementary matrices $M_i(Z)$ comes into play. The following result, also called *complementary bases theorem*, is proved by Dopico and Johnson [6].

Theorem 3.7.7 (Complementary bases theorem)**.** *Let*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{C})$$

be a symplectic $2n \times 2n$-matrix in block notation (1.1) and let $k := \mathrm{rank}(B)$, i.e. there are k indices $j_1, ..., j_k$ such that the vectors $Be_{j_1}, ..., Be_{j_k}$ form a basis of the image $\mathrm{Im}(B)$. Let $i_1, ..., i_{n-k}$ denote the complementary indices in $\{1, ..., n\}$, that is, $\{1, ..., n\} = \{i_1, ..., i_{n-k}\} \dot\cup \{j_1, ..., j_k\}$.

Then the $n \times n$ matrix

$$X = \begin{pmatrix} Ae_{i_1} & \cdots & Ae_{i_{n-k}} & Be_{j_1} & \cdots & Be_{j_k} \end{pmatrix}$$

is regular.

This theorem proves an important property of symmetric matrices.

Corollary 3.7.8 (Application of the Complementary bases theorem)**.** *Let $Z \in \mathbb{C}^{\frac{n(n+1)}{2}}$ be a symmetric matrix of rank $k := \mathrm{rank}(Z)$ with $1 \leq k \leq n$. Then Z has a non-vanishing principal minor of order k. More precisely, there is a regular $k \times k$-matrix $(Z)_{i_1,...,i_{n-k}}$, obtained by removing columns and rows $i_1, ..., i_{n-k}$ from Z for some suitable indices $1 \leq i_1 < ... < i_{n-k} \leq n$.*

Proof. Consider the elementary symplectic matrix

$$\begin{pmatrix} I_n & Z \\ 0 & I_n \end{pmatrix}.$$

By assumption, there are indices $1 \leq j_1 < ... < j_k \leq n$ such that the vectors $Ze_{j_1}, ..., Ze_{j_k}$ span the k-dimensional image of Z. Let $i_1, ..., i_{n-k}$ denote the complementary indices in $\{1, ..., n\}$, that is, $\{i_1, ..., i_{n-k}\} \dot\cup \{j_1, ..., j_k\}$. Then the Complementary bases theorem implies that the matrix

$$X = \begin{pmatrix} e_{i_1} & \cdots & e_{i_{n-k}} & Ze_{j_1} & \cdots & Ze_{j_k} \end{pmatrix}$$

is regular. An application of some Gauss-elimination process yields the desired result. □

Proof of Lemma 3.7.6. Let's stratify $\mathcal{N}_2$ in the following way. For $0 \leq k \leq n$, define

$$\mathcal{A}_k := \{\vec{Z}_4 \in \mathcal{N}_2 : \mathrm{rank}(Z_3) \leq k\}$$

and as a convention let $\mathcal{A}_{-1} := \emptyset$. Then each stratum

$$\mathcal{B}_k := \mathcal{A}_k \setminus \mathcal{A}_{k-1}, \quad 0 \leq k \leq n,$$

consists of those points $\vec{Z}_4 \in \mathcal{N}_2$ with $\mathrm{rank}(Z_3) = k$.

Now we want to proceed as in Remark 3.7.4. We start by assuming A to be the finite collection of Lemma 3.7.5, and then successively add matching fields to A.

For each point in $\mathcal{N}_2$, there is an index $1 \leq i* \leq n$ with $z_{1,i^*n} \neq 0$, since $P_f^1 = Z_1 e_n$ and $P_s^2 = 0$ don't vanish simultaneously by the recursive formula (3.1). So this is especially true for points in the stratum $\mathcal{B}_n$. The tupel $x = (z_{1,nj}, z_{2,i^*1}, ..., z_{2,i^*n})$, $j \neq i^*$, is of (*Type 8*) and by Lemma A.1.3, the corresponding vector field ∂_x^3 satisfies

$$\partial_x^3(z_{1,nj}) = (z_{1,ni^*})^n \det(Z_3) \neq 0.$$

Moreover, observe that $\partial_x^3(z_{3,ij}) = 0$ and $\partial_x^3(P_i^3) = 0$ for all $1 \le i \le j \le n$, by construction. The recursive formula (3.1) implies

$$P_i^3 = z_{1,ni} + \sum_{l=1}^{n} z_{3,il} P_{n+l}^2.$$

Hence

$$0 = \partial_x^3(P_i^3) = \partial_x^3(z_{1,ni}) + \sum_{l=1}^{n} z_{3,il} \partial_x^3(P_{n+l}^2), \quad 1 \le i \le n.$$

Suppose that $\partial_x^3(P_{n+l}^2) = 0$, for all $1 \le l \le n$. Then we obtain a contradiction

$$0 = \partial_x^3(P_j^3) = \partial_x^3(z_{1,nj}) \neq 0.$$

Therefore, the flow of ∂_x^3 through points of $\mathcal{B}_n$ leaves $\mathcal{N}_2$. We add ∂_x^3 (actually its n push-forwards with respect to the biholomorphisms from Lemma 3.2.2) to the collection A and by Lemma 3.7.5, we get

$$\mathcal{F}_y^4 \cap \mathcal{B}_n \subset C_A(\mathcal{F}_y^4 \cap \mathcal{N}_1) \subset C_A(\mathcal{F}_y^4 \cap \mathcal{U}_4).$$

In a next step, we assume $1 \le k \le n-1$ and we divide the stratum $\mathcal{B}_k$ into two more strata. Define

$$\mathcal{C}_k := \{\vec{Z}_4 \in \mathcal{B}_k : Z_3 e_i \neq 0 \Rightarrow z_{1,ni} = 0, \forall 1 \le i \le n\}$$

and consider a point $\vec{Z}_4 \in \mathcal{B}_k \setminus \mathcal{C}_k$. By definition, there is an index $1 \le j^* \le n$ such that $Z_3 e_{j^*} \neq 0$ and $z_{1,j^*n} \neq 0$. Furthermore, again by definition, we assume that the matrix Z_3 has rank k. By the complementary bases theorem, we can choose complementary indices $\{i_1, ..., i_{n-k}\} \dot\cup \{j_1, ..., j_k\} = \{1, ..., n\}$ such that $j^* \in \{j_1, ..., j_k\}$ and such that the vectors $Z_3 e_{j_1}, ..., Z_3 e_{j_k}$ form a basis of the image $\mathrm{Im}(Z_3)$. Choose $i \in \{i_1, ..., i_{n-k}\}$, then the tupel $x = (z_{1,i_1 n}, ..., z_{1,i_{n-k}n}, z_{2,j^* j_1}, ..., z_{2,j^* j_k}, z_{3,ii})$ is of (*Type 7*). Since $\frac{\partial}{\partial z_{3,ii}} P_f^3 = \tilde{E}_{ii} P_s^2 = 0$ on $\mathcal{N}_2$, the vector field ∂_x^3 is given by $\pm \det(B) \frac{\partial}{\partial z_{3,ii}}$ on $\mathcal{N}_2$ for some matrix B. Using the notation from Corollary 3.7.8, Lemma A.1.4 yields

$$\det(B) = (z_{1,j^* n})^k \det((Z_3)_{i_1, ..., i_{n-k}}) \neq 0.$$

Next consider the $(k+1) \times (k+1)$ submatrix of Z_3

$$Z := \begin{pmatrix} z_{3,ii} & z_{3,ij_1} & \cdots & z_{3,ij_k} \\ z_{3,j_1 i} & z_{3,j_1 j_1} & \cdots & z_{3,j_1 j_k} \\ \vdots & \vdots & \ddots & \vdots \\ z_{3,j_k i} & z_{3,j_k j_1} & \cdots & z_{3,j_k j_k} \end{pmatrix},$$

for $i \in \{i_1, ..., i_{n-k}\}$. Its determinant $\det(Z)$, written as a function in $z_{3,ii}$, is given by

$$\det(Z) = \det((Z_3)_{i_1, ..., i_{n-k}}) z_{3,ii} + \alpha,$$

where $\alpha \in \mathbb{C}$ is constant in $z_{3,ii}$. We have $\det(Z) = 0$ on $\mathcal{B}_k$, since the rank of Z_3 is k on this stratum. Apply the vector field ∂_x^3 to the equation $\det(Z) = 0$. This gives us

$$\partial_x^3(\det(Z)) = \det(B) \frac{\partial}{\partial z_{3,ii}} \det(Z) = \det(B) \det((Z_3)_{i_1, ..., i_{n-k}}) \neq 0.$$

Hence the flow of ∂_x^3 through the given point $\vec{Z}_4 \in \mathcal{B}_k \setminus \mathcal{C}_k$ leaves the set $\mathcal{A}_k$ and intersects the stratum $\mathcal{B}_{k+1}$. Hence there exists a finite collection $A \subset \mathcal{Q}_4$ of complete fiber-preserving holomorphic vector fields such that

$$\mathcal{F}_y^4 \cap \mathcal{B}_k \setminus \mathcal{C}_k \subset C_A(\mathcal{F}_y^4 \cap \mathcal{B}_{k+1})$$

for each generic fiber $\mathcal{F}_y^4$.

Now, consider a point $\vec{Z}_4 \in \mathcal{C}_k$. Choose again a complementary set of indices $\{i_1, ..., i_{n-k}\} \dot\cup \{j_1, ..., j_k\} = \{1, ..., n\}$ such that the vectors $Z_3 e_{j_1}, ..., Z_3 e_{j_k}$ form a basis of the image $\mathrm{Im}(Z_3)$. By definition of $\mathcal{C}_k$, we have $z_{1,j_1 n} = ... = z_{1,j_k n} = 0$. Recall that by definition of $\mathcal{N}_2$, there is an index $1 \leq i^* \leq n$ with $z_{1,i^* n} \neq 0$. In particular, $Z_3 e_{i^*} = 0$, again by definition of $\mathcal{C}_k$. The vector fields γ^2_{jj,i^*}, $j \in \{j_1, ..., j_k\}$, from Lemma 3.3.1 are given by

$$\gamma^2_{jj,i^*} = (z_{1,i^* n})^2 \frac{\partial}{\partial z_{2,jj}}, \quad j \in \{j_1, ..., j_k\},$$

in the given point $\vec{Z}_4 \in \mathcal{C}_k$. Consider the tupel $x = (z_{1,n1}, ..., z_{1,nn}, z_{3,i^* i^*})$ of (*Type 5*). Its corresponding vector field ∂_x^3 is of the form $\pm \det(I_n + Z_3 Z_2) \frac{\partial}{\partial z_{3,i^* i^*}}$ in $\vec{Z}_4$. Apply a suitable composition to the equation $z_{3,i^* i^*} = 0$, namely

$$\begin{aligned} \gamma^2_{j_k j_k, i^*} \circ \cdots \circ \gamma^2_{j_1 j_1, i^*} \circ \partial_x^3 (z_{3,i^* i^*}) &= \pm \gamma^2_{j_k j_k, i^*} \circ \cdots \circ \gamma^2_{j_1 j_1, i^*} (\det(I_n + Z_3 Z_2)) \\ &= \pm (z_{1,i^* n})^{2k} \frac{\partial}{\partial z_{2,j_k j_k}} \circ \cdots \circ \frac{\partial}{\partial z_{2,j_1 j_1}} (\det(I_n + Z_3 Z_2)). \end{aligned}$$

Note that $\frac{\partial}{\partial z_{2,jj}} (I_n + Z_3 Z_2) = Z_3 \tilde{E}_{jj}$ which means that the j-th column $(I_n + Z_3 Z_2) e_j$ is the only column in $I_n + Z_3 Z_2$ depending on the variable $z_{2,jj}$. In particular, the product formula for determinants (3.9) and a suitable rearrangement of the columns yields

$$\frac{\partial}{\partial z_{2,j_k j_k}} \circ \cdots \circ \frac{\partial}{\partial z_{2,j_1 j_1}} (\det(I_n + Z_3 Z_2)) = \pm \det((I_n + Z_3 Z_2) e_{i_1}, ..., (I_n + Z_3 Z_2) e_{i_{n-k}}, Z_3 e_{j_1}, ..., Z_3 e_{j_k}).$$

Another application of the complementary bases theorem implies

$$\gamma^2_{j_k j_k, i^*} \circ \cdots \circ \gamma^2_{j_1 j_1, i^*} \circ \partial_x^3 (z_{3,i^* i^*}) \neq 0.$$

By Lemma 1.6.13, we have found a suitable finite collection $A \subset \mathcal{Q}_4$ such that

$$\mathcal{F}_y^4 \cap \mathcal{C}_k \subset C_A(\mathcal{F}_y^4 \cap \mathcal{B}_k \setminus \mathcal{C}_k)$$

for all generic fibers $\mathcal{F}_y^4$.

In summary, we have found a finite collection $A \subset \mathcal{Q}_4$ satisfying

$$\mathcal{F}_y^4 \cap \mathcal{B}_k \subset C_A(\mathcal{F}_y^4 \cap \mathcal{B}_{k+1}),$$

for each $1 \leq k \leq n-1$ and each generic fiber $\mathcal{F}_y^4$.

In a last step, consider a point $\vec{Z}_4 \in \mathcal{B}_0$. Then the field ∂_x^3, corresponding to the tupel $x = (z_{1,1n}, ..., z_{1,nn}, z_{3,nn})$ of (*Type 7*), is given by

$$\partial_x^3 = \pm \det(I_n + Z_3 Z_2) \frac{\partial}{\partial z_{3,nn}} = \pm \frac{\partial}{\partial z_{3,nn}}.$$

Therefore, there is a finite $A \subset \mathcal{Q}_4$ with $\mathcal{F}_y^4 \cap \mathcal{B}_0 \subset C_A(\mathcal{F}_y^4 \cap \mathcal{B}_1)$ for each generic fiber.

It remains to put everything together and apply the properties of the closure operator C_A. More precisely, we have found a finite $A \subset \mathcal{Q}_4$ such that

$$C_A(\mathcal{F}_y^4 \cap \mathcal{B}_k) \subset C_A(\mathcal{F}_y^4 \cap \mathcal{B}_{k+1})$$

for each $0 \leq k \leq n-1$ and each generic fiber. And since we already know

$$\mathcal{F}_y^4 \cap \mathcal{B}_n \subset C_A(\mathcal{F}_y^4 \cap \mathcal{U}_4)$$

for each generic fiber, the proof is complete. $\square$

3.7.4 Spanning theorem for $K \geq 5$

The strategy for $K \geq 5$ is very similar to that for $K = 4$. Let's write the function $Q_f^{K-2} Q_s^{K-2}$ in terms of the notation $\tilde{P}^K$ (see Lemma 3.2.2). Recall

$$\tilde{P}^K = \begin{cases} P_f^K & \text{if } K = 2k+1 \\ P_s^K & \text{if } K = 2k \end{cases}$$

and the recursive formula (3.2)

$$\tilde{P}^K = \tilde{P}^{K-2} + Z_K \tilde{P}^{K-1}.$$

Then we get,

$$Q_f^{K-2} Q_s^{K-2} = \tilde{P}_1^{K-2} \cdots \tilde{P}_n^{K-2} \tilde{P}_1^{K-3} \cdots \tilde{P}_n^{K-3}$$

for all $K \geq 5$.

Lemma 3.7.9. *Let $K \geq 5$ and define*

$$\mathcal{N}_1 := \{\vec{Z}_K \in \mathcal{W}_K : \tilde{P}_1^{K-2} \cdots \tilde{P}_n^{K-2} \tilde{P}_1^{K-3} \cdots \tilde{P}_n^{K-3} = 0, \tilde{P}^{K-2} \neq 0\}.$$

Then there is a finite collection $A \subset \mathcal{Q}_K$ such that

$$\mathcal{F}_y^K \cap \mathcal{N}_1 \subset C_A(\mathcal{F}_y^K \cap \mathcal{U}_K)$$

for each generic fibers $\mathcal{F}_y^K$.

Proof. In a first step, consider a point $\vec{Z}_K$ in the set

$$X_1 = \{\vec{Z}_K \in \mathcal{N}_1 : \tilde{P}_1^{K-3} \cdots \tilde{P}_n^{K-3} \neq 0\}.$$

By definition of this set, we find indices $1 \leq i, i^* \leq n$ with $\tilde{P}_i^{K-2}(\vec{Z}_K) = 0$ and $\tilde{P}_{i^*}^{K-2}(\vec{Z}_K) \neq 0$. Pick the vector field $\partial_{x_i}^{K-1}$ corresponding to the tupel $x_i = (z_{K-2,ii}, z_{K-1,i^*1}, ..., z_{K-1,i^*n})$ of (*Type 2*). It is of the form

$$\pm(\tilde{P}_{i^*}^{K-2})^n \frac{\partial}{\partial z_{K-2,ii}} + \sum_{l=1}^{n} \alpha_l \frac{\partial}{\partial z_{K-1,i^*l}}$$

for some suitable holomorphic functions $\alpha_1, ..., \alpha_n$. By the recursive formula (3.2), we get

$$\tilde{P}_i^{K-2} = \tilde{P}_i^{K-4} + e_i^T Z_{K-2} \tilde{P}^{K-3}$$

and therefore

$$\partial_{x_i}^{K-1}(\tilde{P}_i^{K-2}) = \pm(\tilde{P}_{i^*}^{K-2})^n \tilde{P}_i^{K-3} \neq 0.$$

This applies for every $1 \leq i \leq n$ with $\tilde{P}_i^{K-2} = 0$. Hence there is a finite collection $A \subset \mathcal{Q}_K$ such that

$$\mathcal{F}_y^K \cap X_1 \subset C_A(\mathcal{F}_y^K \cap \mathcal{U}_K)$$

for each generic fiber $\mathcal{F}_y^K$.

In a second step, consider a point $\vec{Z}_K = (\vec{Z}_{K-1}, Z_K)$ in the set

$$X_2 = \{\vec{Z}_K \in \mathcal{N}_1 : \tilde{P}_1^{K-3} \cdots \tilde{P}_n^{K-3} = 0, \tilde{P}^{K-3} \neq 0\}.$$

Observe that the projection of $\vec{Z}_K$ to the first component $\vec{Z}_{K-1}$ is contained in a generic $(K-1)$-fiber $\mathcal{F}_{\tilde{y}}^{K-1}$ and in $\mathcal{W}_{K-1}$. By the induction hypothesis, there is a finite collection $B \subset \mathcal{Q}_{K-1}$ such that

$$\vec{Z}_{K-1} \in C_B(\mathcal{F}_{\tilde{y}}^{K-1} \cap \{\vec{Z}_{K-1} : \tilde{P}_1^{K-3} \cdots \tilde{P}_n^{K-3} \neq 0\}).$$

Put the corresponding pullbacks into the collection A. This gives us a finite collection $A \subset \mathcal{Q}_K$ such that

$$\mathcal{F}_y^K \cap X_2 \subset C_A(\mathcal{F}_y^K \cap X_1) \subset C_A(\mathcal{F}_y^K \cap \mathcal{U}_K)$$

for each generic fiber $\mathcal{F}_y^K$.

In a third step, consider a point $\vec{Z}_K = (\vec{Z}_{K-1}, Z_K)$ in the set

$$X_3 = \{\vec{Z}_K \in \mathcal{N}_1 : \tilde{P}^{K-3} = 0\}.$$

Define the subset

$$\tilde{X}_3 = \{(\vec{Z}_{K-1}, Z_K) \in X_3 : \vec{Z}_{K-1} \in \mathcal{W}_{K-1}\}.$$

By Lemma 3.7.6, we can apply the induction hypothesis, to obtain a finite collection $A \subset \mathcal{Q}_K$ such that

$$\mathcal{F}_y^K \cap \tilde{X}_3 \subset C_A(\mathcal{F}_y^K \cap X_2)$$

for each generic fiber $\mathcal{F}_y^K$. Observe that $\mathcal{W}_{2k+1} = \mathcal{W}_{2k} \times \mathbb{C}^{\frac{n(n+1)}{2}}$, hence the two sets X_3 and $\tilde{X}_3$ coincide for $K = 2k+1$. Now, let us assume $K = 2k$ and consider a point $\vec{Z}_K \in X_3 \setminus \tilde{X}_3$. This implies $Z_1 e_n = Z_3 e_n = ... = Z_{2k-3} e_n = 0$. Pick the vector field ∂_x^{K-1} corresponding to the tupel $x = (z_{K-3,1n}, ..., z_{K-3,nn}, z_{K-1,nn})$ of (*Type 5*). It satisfies

$$\partial_x^{K-1}(z_{K-3,nn}) = \det(e_1 + Z_{K-1}Z_{K-2}e_1, ..., e_{n-1} + Z_{K-1}Z_{K-2}e_{n-1}, e_n)$$

in $\vec{Z}_K$ (cf. (A.10)). Also consider the fields $\partial_{x_i}^{K-1}$ and $\partial_{y_i}^{K-1}$, $i = 1, ..., n-1$, corresponding to the tupels $x_i = (z_{K-2,ii}, z_{K-1,1n}, ..., z_{K-1,nn})$ and $y_i = (z_{K-1,ii}, z_{K-1,1n}, ..., z_{K-1,nn})$ of (*Type 2*) and (*Type 3*), respectively. By Lemma A.1.5 and Lemma A.1.6, they are of the form $\partial_{x_i}^{K-1} = \frac{\partial}{\partial z_{K-2,ii}}$ and $\partial_{y_i}^{K-1} = \frac{\partial}{\partial z_{K-1,ii}}$ on $X_3 \setminus \tilde{X}_3$. With the very same reasoning as in the third step of Lemma 3.7.5, we obtain a finite collection $A \subset \mathcal{Q}_K$ such that

$$\mathcal{F}_y^K \cap X_3 \setminus \tilde{X}_3 \subset C_A(\mathcal{F}_y^K \cap X_2)$$

for each generic fiber $\mathcal{F}_y^K$. In summary, we obtain a finite collection $A \subset \mathcal{Q}_K$ such that

$$\mathcal{F}_y^K \cap X_3 \subset C_A(\mathcal{F}_y^K \cap X_2) \subset C_A(\mathcal{F}_y^K \cap \mathcal{U}_K)$$

for each generic fiber $\mathcal{F}_y^K$ and for each $K \geq 5$.

In a last step, observe that $\mathcal{N}_1 = X_1 \cup X_2 \cup X_3$. We put all the involved vector fields above together and obtain a finite collection $A \subset \mathcal{Q}_K$ satisfying

$$\mathcal{F}_y^K \cap \mathcal{N}_1 \subset C_A(\mathcal{F}_y^K \cap \mathcal{U}_K)$$

for each generic fiber $\mathcal{F}_y^K$. This finishes the proof. □

Lemma 3.7.10. *Let $K \geq 5$ and define*

$$\mathcal{N}_2 := \{\vec{Z}_K \in \mathcal{W}_K : \tilde{P}^{K-2} = 0\}.$$

Then there is a finite collection $A \subset \mathcal{Q}_K$ such that

$$\mathcal{F}_y^K \cap \mathcal{N}_2 \subset C_A(\mathcal{F}_y^K \cap \mathcal{U}_K)$$

for each generic fibers $\mathcal{F}_y^K$.

Proof. The proof of this lemma follows more or less the strategy of Lemma 3.7.6. We stratify $\mathcal{N}_2$ in the following way. For $0 \leq k \leq n$, define

$$\mathcal{A}_k = \{\vec{Z}_K \in \mathcal{N}_2 : \operatorname{rank}(Z_{K-1}) \leq k\}$$

and as a convention let $\mathcal{A}_{-1} := \emptyset$. Then each stratum

$$\mathcal{B}_k := \mathcal{A}_k \setminus \mathcal{A}_{k-1}, \quad 0 \leq k \leq n,$$

consists of those points $\vec{Z}_K \in \mathcal{N}_2$ with $\operatorname{rank}(Z_{K-1}) = k$.

In a first step, we prove the following claim: for $\vec{Z}_K = (\vec{Z}_{K-2}, Z_{K-1}, Z_K) \in \mathcal{N}_2$, the projection to the first component $\vec{Z}_{K-2}$ is contained in a generic $(K-2)$-fiber and in $\mathcal{W}_{K-2}$. The first part of the claim follows directly from the fact that $\tilde{P}^{K-3}$ and $\tilde{P}^{K-2}$ don't vanish simultaneously, by the recursive formula (3.1). For the second part of the claim, recall that $\vec{Z}_K \in \mathcal{W}_K$ by definition of $\mathcal{N}_2$. Assume that $\vec{Z}_{K-2} \notin \mathcal{W}_{K-2}$ and observe

$$\lceil \tfrac{K-3}{2} \rceil = \begin{cases} k-1 & \text{if } K = 2k+1 \\ k-1 & \text{if } K = 2k. \end{cases}$$

Then $Z_1 e_n = Z_3 e_n = ... = Z_{2k-3} e_n = 0$ and by Lemma 3.1.6, we conclude $\Phi_{2k-2} = e_{2n}$. In the case $K = 2k$, this contradicts $0 = \tilde{P}^{K-2} = P_s^{2k-2} = e_n$. And in the case $K = 2k+1$, this leads to

$$0 = \tilde{P}^{K-2} = P_f^{2k-1} = Z_{2k-1} e_n,$$

which contradicts $\vec{Z}_K \in \mathcal{W}_K$. This proves the claim. We are now able to apply the induction hypothesis (the Spanning theorem for $K-2$) to $\vec{Z}_{K-2}$ and assume without loss of generality that $\tilde{P}_1^{K-4} \cdots \tilde{P}_n^{K-4} \neq 0$.

For the second step, observe that for each point in $\mathcal{N}_2$, there is an index $1 \leq j \leq n$ with $\tilde{P}_j^{K-3} \neq 0$, since $\tilde{P}^{K-3}$ and $\tilde{P}^{K-2}$ don't vanish simultaneously. So this is especially true for points $\vec{Z}_K$ in the stratum $\mathcal{B}_n$. Consider the tupel $x = (z_{K-3,ii}, z_{K-2,j1}, ..., z_{K-2,jn})$, $i \neq j$, of (*Type 2*). The corresponding vector field ∂_x^{K-1} is of the form

$$\alpha \frac{\partial}{\partial z_{K-3,ii}} + \sum_{r=1}^{n} \beta_r \frac{\partial}{\partial z_{K-2,jr}}$$

for some suitable holomorphic functions $\alpha, \beta_1, ..., \beta_n$. By construction we have $\partial_x^{K-1}(\tilde{P}^{K-1}) = 0$. Furthermore, $\tilde{P}^{K-1} = \tilde{P}^{K-3} + Z_{K-1}\tilde{P}^{K-2}$ by the recursive formula (3.2). Hence

$$0 = \partial_x^{K-1}(\tilde{P}^{K-1}) = \partial_x^{K-1}(\tilde{P}^{K-3}) + Z_{K-1}\partial_x^{K-1}(\tilde{P}^{K-2}),$$

which implies that if $\tilde{P}^{K-2}$ is in the kernel of ∂_x^{K-1}, then $\tilde{P}^{K-3}$ is in that kernel too.

Observe that $\tilde{P}^{K-4}$ and $\tilde{P}^{K-5}$ are in the kernel of ∂_x^{K-1}, since they don't depend on the matrices Z_{K-3} and Z_{K-2}. Therefore

$$\partial_x^{K-1}(\tilde{P}^{K-3}) = \partial_x^{K-1}(Z_{K-3})\tilde{P}^{K-4} = \alpha \tilde{E}_{ii}\tilde{P}^{K-4} = \alpha \tilde{P}_i^{K-4},$$

and by the previous paragraph, we may assume $\tilde{P}_i^{K-4} \neq 0$. It remains to compute α. First, let's compute the derivatives

$$\frac{\partial}{\partial z_{K-2,jr}} \tilde{P}^{K-1} = Z_{K-1} \frac{\partial}{\partial z_{K-2,jr}} \tilde{P}^{K-2} = Z_{K-1}\tilde{E}_{jr}\tilde{P}^{K-3}, \quad 1 \leq r \leq n.$$

This gives us

$$\begin{aligned}
\alpha &= \det(Z_{K-1}\tilde{E}_{j1}\tilde{P}^{K-3}, ..., Z_{K-1}\tilde{E}_{jn}\tilde{P}^{K-3}) \\
&= \det(Z_{K-1})\det(\tilde{E}_{j1}\tilde{P}^{K-3}, ..., \tilde{E}_{jn}\tilde{P}^{K-3}) \\
&= \det(Z_{K-1})\det\begin{pmatrix} \tilde{P}_j^{K-3} & & & & \\ & \ddots & & & \\ \tilde{P}_1^{K-3} & \cdots & \tilde{P}_j^{K-3} & \cdots & \tilde{P}_n^{K-3} \\ & & & \ddots & \\ & & & & \tilde{P}_j^{K-3} \end{pmatrix} \\
&= (\tilde{P}_j^{K-3})^n \det(Z_{K-1}) \neq 0.
\end{aligned}$$

Hence $\partial_x^{K-1}(\tilde{P}^{K-3}) \neq 0$ and therefore also $\partial_x^{K-1}(\tilde{P}^{K-2}) \neq 0$. We conclude that there is a finite $A \subset \mathcal{Q}_K$ such that

$$\mathcal{F}_y^K \cap \mathcal{B}_n \subset C_A(\mathcal{F}_y^K \cap \mathcal{N}_1) \subset C_A(\mathcal{F}_y^K \cap \mathcal{U}_K)$$

for each generic fiber $\mathcal{F}_y^K$.

In a third step, let's assume $1 \leq k \leq n-1$ and we divide the stratum $\mathcal{B}_k$ into two more strata. Define

$$\mathcal{C}_k := \{\vec{Z}_K \in \mathcal{B}_k : Z_{K-1}e_i \neq 0 \Rightarrow \tilde{P}_i^{K-3} = 0, \forall 1 \leq i \leq n\}$$

and consider a point $\vec{Z}_K \in \mathcal{B}_k \setminus \mathcal{C}_k$. By definition, there is an index $1 \leq j^* \leq n$ such that $Z_{K-1}e_{j^*} \neq 0$ and $\tilde{P}_{j^*}^{K-3} \neq 0$. Moreover, also by definition of the stratum $\mathcal{B}_k$, we assume that the matrix Z_{K-1} has rank k. By the complementary bases theorem, we can choose complementary indices $\{i_1, ..., i_{n-k}\}\dot{\cup}\{j_1, ..., j_k\} = \{1, ..., n\}$ such that $j^* \in \{j_1, ..., j_k\}$ and such that the vectors $Z_{K-1}e_{j_1}, ..., Z_{K-1}e_{j_k}$ form a basis of the image $\mathrm{Im}(Z_{K-1})$. Choose $i, i^* \in \{i_1, ..., i_{n-k}\}$, then the tupel $x = (z_{K-3,i^*i_1}, ..., z_{K-3,i^*i_{n-k}}, z_{K-2,j^*j_1}, ..., z_{K-2,j^*j_k}, z_{K-1,ii})$ is of (*Type 7*). Since $\frac{\partial}{\partial z_{K-1,ii}}\tilde{P}^{K-1} = \tilde{E}_{ii}\tilde{P}^{K-2} = 0$ on $\mathcal{N}_2$, the vector field ∂_x^{K-1} is given $\pm\det(B)\frac{\partial}{\partial z_{K-1,ii}}$ on $\mathcal{N}_2$, where

$$B = \begin{pmatrix} \frac{\partial}{\partial z_{K-3,i^*i_1}}\tilde{P}^{K-1} & \cdots & \frac{\partial}{\partial z_{K-3,i^*i_{n-k}}}\tilde{P}^{K-1} & \frac{\partial}{\partial z_{K-2,j^*j_1}}\tilde{P}^{K-1} & \cdots & \frac{\partial}{\partial z_{K-2,j^*j_k}}\tilde{P}^{K-1} \end{pmatrix}$$

By Lemma A.1.8,

$$\det(B) = (\tilde{P}_{i^*}^{K-4})^{n-k}(\tilde{P}_j^{K-3})^k \det((Z_{K-1})_{i_1,...,i_{n-k}}) \neq 0$$

in the given point $\vec{Z}_K$. The same argument as in Lemma 3.7.6 implies that the flow of ∂_x^{K-1} through $\vec{Z}_K$ leaves the set $\mathcal{A}_k$ and intersects the stratum $\mathcal{B}_{k+1}$. Hence there exists a finite collection $A \subset \mathcal{Q}_K$ such that

$$\mathcal{F}_y^K \cap \mathcal{B}_k \setminus \mathcal{C}_k \subset C_A(\mathcal{F}_y^K \cap \mathcal{B}_{k+1})$$

for each generic fiber $\mathcal{F}_y^K$.

Next, consider a point $\vec{Z}_K \in \mathcal{C}_k$. Choose again a complementary set of indices $\{i_1, ..., i_{n-k}\}\dot{\cup}\{j_1, ..., j_k\} = \{1, ..., n\}$ such that the vectors $Z_{K-1}e_{j_1}, ..., Z_{K-1}e_{j_k}$ form a basis of the image $\mathrm{Im}(Z_{K-1})$. By definition of $\mathcal{C}_k$, $\tilde{P}_{j_1}^{K-3} = ... = \tilde{P}_{j_k}^{K-3} = 0$. Recall that there is an index $1 \leq i^* \leq n$ with $\tilde{P}_{i^*}^{K-3} \neq 0$, since $\tilde{P}^{K-3}$ and $\tilde{P}^{K-2}$ don't vanish simultaneously. The vector fields γ_{jj,i^*}^{K-2}, $j \in \{j_1, ..., j_k\}$, from Lemma 3.3.1, are given by

$$\gamma_{jj,i^*}^{K-2} = (\tilde{P}_{i^*}^{K-3})^2 \frac{\partial}{\partial z_{K-2,jj}}, \quad j \in \{j_1, ..., j_k\},$$

in $\vec{Z}_K$. Now consider the tupel $x = (z_{K-3,n1}, ..., z_{K-3,nn}, z_{K-1,i^*i^*})$ of (*Type 5*). Its corresponding vector field ∂_x^{K-1} is of the form $\pm(\tilde{P}_n^{K-4})^n \det(I_n + Z_{K-1}Z_{K-2})\frac{\partial}{\partial z_{K-1,i^*i^*}}$ in $\vec{Z}_K$. As in Lemma 3.7.6, we have

$$\gamma_{j_kj_k,i^*}^{K-2} \circ \cdots \circ \gamma_{j_1j_1,i^*}^{K-2} \circ \partial_x^{K-1}(z_{K-1,i^*i^*}) \neq 0.$$

This implies the existence of a finite collection $A \subset \mathcal{Q}_K$ such that

$$\mathcal{F}_y^K \cap \mathcal{C}_k \subset C_A(\mathcal{F}_y^K \cap \mathcal{B}_k \setminus \mathcal{C}_k)$$

for each generic fiber $\mathcal{F}_y^K$. In summary, this leads to

$$\mathcal{F}_y^K \cap \mathcal{B}_k \subset C_A(\mathcal{F}_y^K \cap \mathcal{B}_{k+1})$$

for each generic fiber $\mathcal{F}_y^K$ and $1 \leq k \leq n-1$.

It remains to consider a point $\vec{Z}_K$ in the stratum $\mathcal{B}_0$, that is, we assume $Z_{K-1} = 0$. Choose again the vector field ∂_x^{K-1} corresponding to the tupel $x = (z_{K-3,n1}, ..., z_{K-3,nn}, z_{K-1,i^*i^*})$ of (*Type 5*). Then

$$\partial_x^{K-1}(z_{K-1,i^*i^*}) = \pm(\tilde{P}_n^{K-4})^n \underbrace{\det(I_n + Z_{K-1}Z_{K-2})}_{=1} \neq 0,$$

and therefore there is a finite collection $A \subset \mathcal{Q}_K$ such that

$$\mathcal{F}_y^K \cap \mathcal{B}_0 \subset C_A(\mathcal{F}_y^K \cap \mathcal{B}_1)$$

for each generic fiber $\mathcal{F}_y^K$.

For the final step, we only need to put everything together. We have

$$C_A(\mathcal{F}_y^K \cap \mathcal{B}_0) \subset C_A(\mathcal{F}_y^K \cap \mathcal{B}_1) \subset \cdots \subset C_A(\mathcal{F}_y^K \cap \mathcal{B}_n) \subset C_A(\mathcal{F}_y^K \cap \mathcal{U}_K)$$

for each generic fiber $\mathcal{F}_y^K$, and since $\mathcal{N}_2 = \bigcup_{k=0}^n \mathcal{B}_k$, we obtain

$$\mathcal{F}_y^K \cap \mathcal{N}_2 \subset C_A(\mathcal{F}_y^K \cap \mathcal{U}_K)$$

for each generic fiber $\mathcal{F}_y^K$. This finishes the proof. $\square$

4 Applications

In the following we present some applications of the Main theorem as we can find them in [13]. In addition, we devote a section to the density property. This is not directly an application of the Main theorem, but from its proof.

4.1 Preliminaries

In this section we introduce some notions which are important for this chapter. Let R be a commutative ring with 1. An element $(a_1, ..., a_n) \in R^n$ is called *unimodular* if there exist $b_1, ..., b_n \in R$ such that

$$a_1 b_1 + \cdots + a_n b_n = 1.$$

Let $U_n(R)$ denote the set of unimodular elements in R^n. A unimodular element $(a_1, ..., a_{n+1}) \in U_{n+1}(R)$ is called *stable*, if there exist $r_1, ..., r_n \in R$ such that

$$(a_1 + r_1 a_{n+1}, ..., a_n + r_n a_{n+1}) \in U_n(R)$$

is unimodular. Let $\mathrm{sr}(R)$ denote the least natural number such that each unimodular element in $U_{\mathrm{sr}(R)}(R)$ is stable. This number is called *Bass stable rank* of R. For the ring $\mathcal{O}(X)$ we have

$$\mathrm{sr}(\mathcal{O}(X)) = \lfloor \tfrac{\dim X}{2} \rfloor + 1 \tag{4.1}$$

as for instance (see [2]).

4.2 On the number of factors

The previous choice of elementary symplectic matrices turns out to be suboptimal if we want to estimate the number of factors. We will consider *unitriangular* symplectic matrices instead. Here, a unitriangular matrix is an upper (resp. lower) triangular matrix having only ones on the diagonal. However, this does not give us an advantage a priori. We need an alternative definition of a symplectic matrix. Essentially, we just want to do a suitable basis transformation such that we also have unitriangular elementary matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \tag{$\star$}$$

where A, B are unitriangular matrices. Let L be the $n \times n$-matrix with ones on the skew-diagonal and zeros elsewhere, i.e.

$$L = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

Then we define the basis transformation matrix

$$B := \begin{pmatrix} I_n & 0 \\ 0 & L \end{pmatrix}$$

and

$$\tilde{\Omega} = B\Omega B = \begin{pmatrix} 0 & L \\ -L & 0 \end{pmatrix}.$$

Then a matrix M is symplectic with respect to Ω if and only if BMB is symplectic with respect to $\tilde{\Omega}$. The elementary Ω-symplectic matrices of (E.1) and (E.2) are still unitriangular, since $L^2 = I_n$. The Ω-symplectic matrix in $(\star)$ reads as

$$\begin{pmatrix} A & 0 \\ 0 & LDL \end{pmatrix}$$

in the new basis and in particular, it is unitriangular, if A is unitriangular. So the choice of unitriangular matrices, together with the change of bases, actually extends the set of elementary symplectic matrices.

We will use the Tavgen reduction to find an estimate for the number of factors. For this we make a short disgression in the setting of elementary Chevalley groups. Let Φ be a reduced irreducible root system of rank $l \geq 2$ and let R be a commutative ring with 1. We choose an order on Φ and a system of fundamental roots $\Pi = \{\alpha_1, ..., \alpha_l\}$. Each root $\alpha \in \Phi$ is an integral sum of the fundamental roots

$$\alpha = \sum_{i=1}^{l} k_i(\alpha)\alpha_i,$$

where the integer coefficients $k_i(\alpha)$ are either all non-negative or all non-positive. For $r = 1, l$, we define the following subsets of Φ

$$\Delta_r = \{\alpha \in \Phi : k_r(\alpha) = 0\}, \quad \Sigma_r = \{\alpha \in \Phi : k_r(\alpha) > 0\}, \quad \Sigma_r^- = \{\alpha \in \Phi : k_r(\alpha) < 0\}.$$

Δ_r is itself a root system of rank $l-1$. On the level of Dynkin diagram, we obtain Δ_r from Φ by taking away the first $(r = 1)$ or the last $(r = l)$ fundamental root. The elementary Chevalley group $E(\Phi, R)$ of type Φ over R is generated by root subgroups $x_\alpha, \alpha \in \Phi$

$$E(\Phi, R) = \{x_\alpha(r) : \alpha \in \Phi, r \in R\}.$$

The positive unipotent subgroup $U(\Phi, R)$ is generated by the root subgroups of positive roots

$$U(\Phi, R) = \{x_\alpha(r) : \alpha \in \Phi^+, r \in R\}.$$

Similarly, $U^-(\Phi, R) = \{x_\alpha(r) \mid \alpha \in \Phi^-, r \in R\}$. The following theorem was originally proved by Oleg Tavgen and adapted in [25] , where the number of factors is even. For our estimates, we need the same result allowing odd number of factors. We remark that the shape of the starting factor, upper or lower, is also immaterial. The above mentioned article, as well as [24], are recommended for more information.

Theorem 4.2.1 (Tavgen's reduction)**.** *Let Φ be a reduced irreducible root system of rank $l \geq 2$ and let R be a commutative ring with 1. Suppose that for subsystems $\Delta = \Delta_1, \Delta_l$ of rank $l-1$ the elementary Chevalley group $E(\Delta, R)$ admits the unitriangular factorization with L factors*

$$E(\Delta, R) = U^-(\Delta, R)U(\Delta, R) \cdots U^{\pm}(\Delta, R).$$

Then the elementary Chevalley group $E(\Phi, R)$ admits the unitriangular factorization with the same number of factors

$$E(\Phi, R) = U^-(\Phi, R)U(\Phi, R) \cdots U^{\pm}(\Phi, R).$$

Proof. We take

$$Y = U^-(\Phi, R)U(\Phi, R) \cdots U^{\pm}(\Phi, R).$$

Y is a nonempty subset of $E(\Phi, R)$, in particular it contains 1. Since the group $E(\Phi, R)$ is generated by the following root elements $X = \{x_\alpha(r) \mid \alpha \in \pm\Pi, r \in R\} \subset E(\Phi, R)$. Notice that the generating set X is symmetric, i.e. $X^{-1} = X$. We claim that $x_\alpha(r)Y \subset Y$ for $\alpha \in \pm\Pi$: Since $l \geq 2$, α lies in at least one of the subsystems Δ_1, Δ_l. Suppose that α belongs to $\Delta = \Delta_r$, then we consider the Levi decomposition

$$U(\Phi, R) = U(\Delta, R) \ltimes E(\Sigma, R), \qquad U^-(\Phi, R) = U^-(\Delta, R) \ltimes E(\Sigma^-, R),$$

where $\Sigma = \Sigma_r$ and $E(\Sigma, R) = \langle x_\alpha(r) \mid \alpha \in \Sigma, r \in R\rangle$. Since $U^{\pm}(\Delta, R)$ normalizes $E^{\pm}(\Sigma, R)$, we can rewrite Y as

$$\begin{aligned} Y &= U^-(\Phi, R)U(\Phi, R) \cdots U^{\pm}(\Phi, R) \\ &= U^-(\Delta, R)E(\Sigma^-, R)\, U(\Delta, R)E(\Sigma, R) \,\cdots\, U^{\pm}(\Delta, R)E(\Sigma^{\pm}, R) \\ &= \big(U^-(\Delta, R)U(\Delta, R) \cdots U^{\pm}(\Delta, R)\big)\, E(\Sigma^-, R)E(\Sigma, R) \cdots E(\Sigma^{\pm}, R) \\ &= E(\Delta, R)\, E(\Sigma^-, R)E(\Sigma, R) \cdots E(\Sigma^{\pm}, R), \end{aligned}$$

where the last step follows from the assumption. For $\alpha \in \Delta$, $x_\alpha(r)$ is an element in $E(\Delta, R)$, hence $x_\alpha(r)Y \subset Y$. This proves $XY \subset Y$. But this implies $Y = E(\Phi, R)$: choose any $g \in E(\Phi, R)$ and choose some $y \in Y$. Since X is a generating set of $E(\Phi, R)$ and X is symmetric, there are $x_1, ..., x_k \in X$ such that $gy^{-1} = x_1 \cdots x_k$. In particular, this proves

$$g = gy^{-1}y = x_1 \cdots \underbrace{x_{k-1}\, \underbrace{x_k y}_{\in Y}}_{\in Y} \in Y.$$

This finishes the proof. $\square$

Corollary 4.2.2. *Let X be a reduced Stein space of dimension d and let $n \geq 2$. Then there exists an upper bound $t = t(n, d)$ for the number of unitriangular factors of any null-homotopic holomorphic mapping $f : X \to \mathrm{Sp}_{2n}(\mathbb{C})$. In particular, the bound $t(n, d)$ smaller than or equal to $t(1, d)$ the corresponding bound for $\mathrm{SL}_2(\mathcal{O}(X))$, i.e.*

$$t(n, d) \leq t(1, d).$$

Proof. The main theorem guarantees the existence of a natural number $t(n, d)$ such that any holomorphic, null-homotopic mapping $f : X \to \mathrm{Sp}_{2n}(\mathbb{C})$ can be written as a product of t unitriangular matrices. The group $\mathrm{Ep}_{2n}(\mathcal{O}(X))$ generated by unipotent symplectic matrices coincides with the elementary Chevalley group $E(C_n, \mathcal{O}(X))$. According to the main theorem in [14], there exists a natural number $t(1, d)$ such that any holomorphic, null-homotopic mapping $f : X \to \mathrm{SL}_2(\mathbb{C})$ can be written by $t(1, d)$ unitriangular matrices. By Tavgen's reduction, we conclude $t(n, d) \leq t(1, d)$. $\square$

Question 4.2.3. *On the one hand, we wish to find a (sharp) upper bound for $t(1, d)$ and on the other, there exist natural numbers m and N such that*

$$t(n, d) = m$$

for all $n \geq N$. What numbers are m and N?

We have a full answer for Stein spaces of dimension one and a partial answer for Stein spaces of dimension 2. The following result was proved by Vavilov, Smolensky and Sury in [25]

Theorem 4.2.4. *Let Φ be a reduced irreducible root system and R be a commutative ring with stable rank $sr(R) = 1$. Then the elementary Chevalley group $E(\Phi, R)$ admits unitriangular factorization*

$$E(\Phi, R) = U(\Phi, R)U^{-}(\Phi, R)U(\Phi, R)U^{-}(\Phi, R)$$

of length 4.

The stable rank of the ring $\mathcal{O}(X)$ is given by

$$sr(\mathcal{O}(X)) = \lfloor \tfrac{\dim X}{2} \rfloor + 1$$

(see (4.1)). This leads to the following

Corollary 4.2.5. *Let X be a reduced Stein space of dimension $\dim X = 1$. Then every null-homotopic $f \in \mathrm{Sp}_{2n}(\mathcal{O}(X))_0$ can be written as a product of $t(n, 1) = 4$ unitriangular matrices.*

Ivarsson and Kutzschebauch proved the following for dimension 2 Stein spaces in [15]

Theorem 4.2.6. *Let X be a reduced Stein space of dimension $\dim X = 2$. Then every $f \in \mathrm{SL}_2(\mathcal{O}(X))$ can be written as a product of $t(1, 2) = 5$ unitriangular matrices.*

Corollary 4.2.7. *Let X be a reduced Stein space of dimension $\dim X = 2$. Then every $f \in \mathrm{Sp}_{2n}(\mathcal{O}(X))$ can be written as a product of $t(n, 2) \leq 5$ unitriangular matrices.*

4.3 Product of exponentials

The exponential of a $n \times n$-matrix A is given by the exponential series

$$\exp A = \sum_{k \geq 0} \frac{A^k}{k!}.$$

$\mathrm{Exp}_n(\mathcal{O}(X))$ denotes the subgroup of $\mathrm{GL}_n(\mathcal{O}(X))$ generated by exponentials and $e(n, \mathcal{O}(X))$ the minimal number such that any matrix in $\mathrm{Exp}_n(\mathcal{O}(X))$ factorizes as a product of $e(n, \mathcal{O}(X))$ exponentials. Let $t(n, \mathcal{O}(X))$ be the minimal number such that any element in the elementary Chevalley group $E(\Phi, \mathcal{O}(X)) \subset \mathrm{GL}_n(\mathcal{O}(X))$ factorizes as a product of $t(n, \mathcal{O}(X))$ unitriangular matrices. When no such number exists, set $t(n, \mathcal{O}(X)) = \infty$.

Observe that for a nilpotent matrix N,

$$\log(I_n + N) = \sum_{k \geq 1} \frac{(-1)^k}{k} N^k$$

is a finite sum. Thus every unipotent matrix A (i.e. $A - I_n$ is nilpotent) can be written as the exponential of $\log A$. Also under conjugation an exponential remains an exponential, since $BAB^{-1} = \exp(B \cdot A \cdot B^{-1})$ for any regular $n \times n$-matrix B.

An *alternating product* $U_1 U_2 \cdots U_k$ of unitriangular matrices is a product, where the odd factors $U_1, U_3, ...$, are lower unitriangular and the even $U_2, U_4, ...$ are upper unitriangular, or vice versa.

Lemma 4.3.1. *An alternating product of k unitriangular matrices $U_1 U_2 \cdots U_k$ can be written as a product of $\lfloor \frac{k}{2} \rfloor + 1$ exponentials.*

Proof. The proof is by induction on the number of factors. It is enough to prove the claim for products of odd length, for if we consider a product of even length, we simply add one exponential. More precisely, assume the claim is true for an odd number of factors and consider an alternating product $U_1U_2\cdots U_{2k-1}U_{2k}$. Then $U_1\cdots U_{2k-1}$ can be written as a product of $\lfloor\frac{2k-1}{2}\rfloor+1$ exponentials and we obtain

$$(\lfloor\tfrac{2k-1}{2}\rfloor+1)+1=\lfloor\tfrac{2k}{2}\rfloor+1,$$

which proves the claim for alternating products of even length.

The base case is fine, since the claim is trivially true for $k=1$. For the induction step, we consider an alternating product $U_1U_2\cdots U_{2k+1}$ and we want to write it as a product of

$$\lfloor\tfrac{2k+1}{2}\rfloor+1=k+1$$

exponentials. We apply the following trick presented in [3, Lemma 2.1]. Write

$$U_1U_2\cdots U_{2k-1}U_{2k}U_{2k+1}=U_1U_2\cdots U_{2k-2}\cdot\underbrace{U_{2k-1}U_{2k+1}}_{=:\tilde{U}_{2k-1}\text{ unitr.}}\cdot(U_{2k+1}^{-1}U_{2k}U_{2k+1}),$$

and note that the product $U_1U_2\cdots U_{2k-2}\tilde{U}_{2k-1}$ is alternating and of length $2k-1$. Hence it can be written as a product of k exponentials by the induction hypothesis. Since U_{2k} conjugated by U_{2k+1} is an exponential, this gives us $k+1$ exponential factors. This finishes the proof. □

An application of this lemma and the main theorem yields

Corollary 4.3.2 (Product of exponentials)**.** *Let X be a reduced Stein space of dimension d and let $f : X\to \mathrm{Sp}_{2n}(\mathbb{C})$ be a null-homotopic, holomorphic mapping with $n\geq 2$. Further, let $t(n,d)$ denote the number of unitriangular factors of f. Then there exist a natural number $e=e(n,d)$ and holomorphic mappings $A_1,...,A_e : X\to\mathfrak{sp}_{2n}(\mathbb{C})$ such that*

$$f(x)=\exp(A_1(x))\cdots\exp(A_e(x)).$$

Moreover,

$$e(n,d)\leq\lfloor\tfrac{t(n,d)}{2}\rfloor+1.$$

Another application of the above lemma, together with Corollary 4.2.5 and Corollary 4.2.7, yields

Corollary 4.3.3. *Let X be a Stein space of dimension* $\dim X\in\{1,2\}$*. Then*

$$e(n,\mathcal{O}(X))\leq 3.$$

Definition 4.3.4. *Let $e_{\mathrm{SL}}(n,\mathcal{O}(X))$ denote the minimal number such that any matrix in $\mathrm{SL}_n(\mathcal{O}(X))\cap\mathrm{Exp}_n(\mathcal{O}(X))$ factorizes as a product of $e_{\mathrm{SL}}(n,\mathcal{O}(X))$ exponentials, with holomorphic mappings $X\to\mathfrak{sl}_n(\mathbb{C})$ as exponents. Similarly, $e_{\mathrm{Sp}}(n,\mathcal{O}(X))$ is the minimal number such that any matrix in $\mathrm{Sp}_{2n}(\mathcal{O}(X))\cap\mathrm{Exp}_n(\mathcal{O}(X))$ factorizes as a product of $e_{\mathrm{Sp}}(n,\mathcal{O}(X))$ exponentials, with holomorphic mappings $X\to\mathfrak{sp}_{2n}(\mathbb{C})$ as exponents.*

Lemma 4.3.5. *Let n be a natural number. Then*

$$e(n,\mathcal{O}(X))\leq e_{\mathrm{SL}}(n,\mathcal{O}(X)).$$

Proof. Let $f \in \mathrm{GL}_n(\mathcal{O}(X))$ be null-homotopic. Then the composition with the determinant $\det \circ f : X \to \mathbb{C}^*$ is also null-homotopic. Thus there exists a holomorphic function $g : X \to \mathbb{C}$ such that $\exp \circ g = \det \circ f$. Observe that $\exp(-\frac{g}{n} I_n) \cdot f \in \mathrm{SL}_n(\mathcal{O}(X))$, since by Jacobi's formula

$$\det(\exp(-\frac{g}{n} I_n)) = \exp(\mathrm{trace}(-\frac{g}{n} I_n)) = \exp(-g).$$

Hence there exist $e = e_{\mathrm{SL}}(n, \mathcal{O}(X))$ holomorphic mappings $S_1, ..., S_e : X \to \mathfrak{sl}_n(\mathbb{C})$ such that

$$\exp(-\frac{g}{n} I_n) \cdot f = \exp(S_1) \cdots \exp(S_e).$$

This implies

$$\begin{aligned} f &= \exp(\frac{g}{n} I_n) \exp(S_1) \exp(S_2) \cdots \exp(S_e) \\ &= \exp(\frac{g}{n} I_n + S_1) \exp(S_2) \cdots \exp(S_e) \end{aligned}$$

which proves the claim. □

The proof of the following proposition is essentially the same as in [3, Theorem 1.1 (4)].

Proposition 4.3.6. *Let X be a Stein space with $\dim X > 0$ and let $n \geq 2$ be a natural number. Then $e(n, \mathcal{O}(X)) \geq 2$.*

Proof. Without loss of generality, we assume that X is irreducible. Then there exist two distinct points $x_1, x_2 \in X$ and a holomorphic function $h \in \mathcal{O}(X)$ such that $h(x_1) = 0, h(x_2) = 2\pi i$. Set $g = \exp h$ and let

$$T = \begin{pmatrix} g & 1 \\ 0 & 1 \end{pmatrix}.$$

Suppose for contradiction there is a logarithm, i.e. $T = \exp M$ for some $M \in M_n(\mathcal{O}(X))$. Then the matrix $S = \exp(\frac{1}{2}M) \in \mathrm{GL}_n(\mathcal{O}(X))$ satisfies $S^2 = T$. Let's write

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then we obtain

$$\begin{pmatrix} g & 1 \\ 0 & 1 \end{pmatrix} = T = S^2 = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix}.$$

Equations $b(a + d) = 1$ and $c(a + d) = 0$ imply $a + d \neq 0$ and $c = 0$. Moreover, $d^2 = 1$ and $d \in \mathcal{O}(X)$ implies $d \equiv 1$ or $d \equiv -1$. On the other hand, $a^2 = \exp h$ implies $a = \exp(\frac{h}{2})$ or $a = -\exp(\frac{h}{2})$. However, both points, -1 and 1, are in the image of $\exp(\frac{h}{2})$ by construction. This contradicts $a + d \neq 0$ for $d = \pm 1$. Therefore $e(2, \mathcal{O}(X)) \geq 2$.

For $n > 2$, fix $M \in \mathbb{C} \setminus \{0, 1\}$ and set

$$T_n = \begin{pmatrix} MI_{n-2} & 0 \\ 0 & T \end{pmatrix}.$$

Suppose that T_n had a logarithm, then there would exist

$$S_n = \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix} \in M_n(\mathcal{O}(X))$$

with the same block partition as T_n and such that $S_n^2 = T_n$. Then we have $S_n T_n = T_n S_n$, which implies that

$$L_2(T - MI_2) = 0 \quad \text{and} \quad (T - MI_2)L_3 = 0.$$

On $X' \setminus (\exp h)^{-1}(M)$, $T - MI_2$ is invertible, so $L_2 = L_3 = 0$. By the identity theorem, L_2 and L_3 vanish on X'. But this would imply that $L_4^2 = T$, a contradiction. Hence $e(n, \mathcal{O}(X)) \geq 2$ for all $n \geq 2$. □

4.4 Continuous vs. holomorphic factorization

Since the solution to the Gromov-Vasterstein problem involves the Oka principle, it is natural to compare the K-theoretic questions for the ring $\mathcal{O}(X)$ with the corresponding questions for the ring $\mathcal{C}(X)$ of continuous complex-valued functions on the Stein space X. Let $t(n,d,\mathcal{C},\mathcal{O})$ and $t(n,d,\mathcal{O})$ (see [15]) be the respective minimal numbers such that all null-homotopic holomorphic mappings $X \to \mathrm{SL}_n(\mathbb{C})$ from a d-dimensional Stein space X into the special linear group factorize as a product of $t(n,d,\mathcal{C},\mathcal{O})$ *continuous* and $t(n,d,\mathcal{O})$ *holomorphic* unipotent matrices (starting with a lower unitriangular one), respectively. Clearly,

$$t(n,d,\mathcal{C},\mathcal{O}) \leq t(n,d,\mathcal{O})$$

and by Tavgen's reduction theorem

$$t(n,d,\mathcal{O}) \leq t(n-1,d,\mathcal{O})$$

for all $n > 2$. We have a lower bound.

Lemma 4.4.1. *Let $n \geq 2$ be a natural number. Then*

$$4 \leq t(n,d,\mathcal{C},\mathcal{O}) \leq t(n,d,\mathcal{O}) \leq t(2,d,\mathcal{O}).$$

Proof. Consider the matrix

$$P = \begin{pmatrix} 2 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \frac{1}{2} \end{pmatrix} \in \mathrm{SL}_n(\mathcal{O}(X)).$$

This matrix cannot be written as a product of one or two unitriangular matrices, because of the 2 in the upper left corner. Moreover, it cannot be written as a product of three unitriangular matrices. Suppose for contradiction it would be possible with the unitriangular matrices X_1, X_2, X_3, where X_1, X_3 are lower unitriangular. Then X_3^{-1} is also lower unitriangular and we get

$$PX_3^{-1} = \begin{pmatrix} 2 & & \\ \star & \star & \\ \star & \star & \star \end{pmatrix}.$$

On the other hand,

$$X_1X_2 = \begin{pmatrix} 1 & \star & \star \\ \star & \star & \star \\ \star & \star & \star \end{pmatrix},$$

which is a contradiction. $\square$

Obviously we have

$$t(2,d,\mathcal{O}) = 4 \quad \Longrightarrow \quad t(n,d,\mathcal{C},\mathcal{O}) = t(n,d,\mathcal{O}) = 4, \quad \forall n \geq 2.$$

This is the case for instance if the dimension of the Stein space X is $\dim X = 1$ (see Corollary 4.2.5), or more generally, for a commutative ring R with 1 such that the bass stable rank is

$\mathrm{sr}(R) = 1$. However, we don't know, whether this is also a necessary condition. Consider for instance a unimodular row $(a, b) \in R^2$ for some commutative ring R with 1. Then there are $c, d \in R$ such that $ad - bc = 1$, i.e. the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(R).$$

This leads to the following question.

Question 4.4.2. *Suppose we can write each matrix in* $\mathrm{SL}_2(R)$ *as a product of four unitriangular matrices. Does this imply that the bass stable rank* $\mathrm{sr}(R) = 1$*?*

In [15] they prove the following.

Theorem 4.4.3. *Let X be a Stein space of dimension $d = 2$. Then*

$$4 \leq t(2, d, \mathcal{C}, \mathcal{O}) \leq t(2, d, \mathcal{O}) \leq 5.$$

Moreover, Ivarsson and Kutzschebauch show in the formentioned paper that Cohn's famous counterexample (see [5])

$$\begin{pmatrix} 1 - zw & -z^2 \\ w^2 & 1 + zw \end{pmatrix}$$

factorizes as a product of four continuous unitriangular matrices, but not less than five holomorphic unitriangular matrices. In other words, we have

$$t(2, 2, \mathcal{O}) = 5.$$

So the question remained, whether we have $t(2, 2, \mathcal{C}, \mathcal{O}) = 4$ or $t(2, 2, \mathcal{C}, \mathcal{O}) = 5$. We have now an answer.

Theorem 4.4.4. *Let X be a Stein space of dimension $d = 2$. Then*

$$t(2, 2, \mathcal{C}, \mathcal{O}) = t(2, 2, \mathcal{O}) = 5.$$

Proof. We construct an example of a matrix in $\mathrm{SL}_2(\mathcal{O}(X))$ which does not factorise as a product of four continuous unitriangular matrices. First we study what it means to be a product of four unitriangular matrices. Let $M \in \mathrm{SL}_2(\mathcal{O}(X))$ be given by

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, ad - bc = 1.$$

Assume there are four holomorphic functions $g_1, g_2, g_3, g_4 : X \to \mathbb{C}$ such that

$$M = \begin{pmatrix} 1 & \\ g_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & g_2 \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ g_3 & 1 \end{pmatrix} \begin{pmatrix} 1 & g_4 \\ & 1 \end{pmatrix}. \tag{4.2}$$

Bringing the first and the fourth matrix to the left-hand-side, we obtain

$$\begin{pmatrix} 1 + g_2 g_3 & g_2 \\ g_3 & 1 \end{pmatrix} = \begin{pmatrix} a & b - a g_4 \\ c - a g_1 & -g_4(c - a g_1) + d - b g_1 \end{pmatrix}$$

In case $a \neq 0$, the first three equations read

$$
\begin{aligned}
a &= 1 + g_2 g_3 \\
g_4 &= \frac{1}{a}(b - g_2) \\
g_1 &= \frac{1}{a}(c - g_3),
\end{aligned}
$$

and the fourth equation follows from the other three. If moreover $a \neq 1$, any choice of $g_3 : \{x \in X | a(x) \notin \{0,1\}\} \to \mathbb{C}^*$ gives a factorization in this part of X. The fiber of the fibration $f^*\Phi_4$ (see [15]) over $\{x \in X | a(x) \notin \{0,1\}\}$ is $\mathbb{C}^*$, where

$$
\Phi_4 : \mathbb{C}^4 \to \mathrm{SL}_2(\mathbb{C}), (z_1, z_2, z_3, z_4) \mapsto \begin{pmatrix} 1 & \\ z_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & z_2 \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ z_3 & 1 \end{pmatrix} \begin{pmatrix} 1 & z_4 \\ & 1 \end{pmatrix}.
$$

When $a = 0$, then

$$
1 + g_2 g_3 = 0, \quad g_2 = b, \quad g_3 = c, \quad 1 = -c g_4 + d - b g_1.
$$

Note that g_2 and g_3 are prescribed as b and c, respectively, and the fiber of $f^*\Phi_4$ here is $\mathbb{C}$. For $a = 1$, the fiber is the cross of axis.

Consider the following holomorphic mapping $f : \mathbb{C}^2 \to \mathrm{SL}_2(\mathbb{C})$

$$
f(z, w) = \begin{pmatrix} (zw-1)(zw-2) & (zw-1)z + (zw-2)z^2 \\ h_1(z,w) & h_2(z,w) \end{pmatrix},
$$

where the functions in the second row are chosen such that $f(z, w)$ has determinant 1. The existence of such polynomials follows from Hilbert's Nullstellensatz, or if one is looking for holomorphic functions from a standard application of Theorem B. For this obeserve that the functions in the first row have no common zeros.

Suppose that there were continuous $g_1, ..., g_4 : \mathbb{C}^2 \to \mathbb{C}$ such that f factorizes as in (4.2). Then $g_2(z, w) = -z^2$ on $\{(z, w) : zw = 1\}$ and $g_2(z, w) = z$ on $\{(z, w) : zw = 2\}$. Let η_1, η_2 denote the roots of $(x-1)(x-2) = 1$, and choose a continuous curve $\gamma : [0,1] \to \mathbb{C} \setminus \{\eta_1, \eta_2\}$ such that $\gamma(0) = 1$ and $\gamma(2)$. Then g_2 induces a family of continuous self-maps of $\mathbb{C}^*$

$$
F : [0,1] \times \mathbb{C}^* \to \mathbb{C}^*, (t, \theta) \mapsto g_2(\theta, \frac{1}{\theta}\gamma(t))
$$

connecting between $F(0, \theta) = -\theta^2$ and $F(1, \theta) = \theta$. But since these two self maps of $\mathbb{C}^*$ have different degrees, we find a contradiction. $\square$

Remark 4.4.5. *The example given in the proof above can be used to show that*

$$
t(2, d, \mathcal{O}) \geq 5
$$

for all Stein spaces X with $d = \dim X \geq 2$. Hence we have a positive answer of Question 4.4.2 in the special case $R = \mathcal{O}(X)$.

4.5 Fibers with density property

In the early 90s, Andersén and Lempert [1] established remarkable properties of the automorphism group $\mathrm{Aut}(\mathbb{C}^n), n \geq 2$. While $\mathrm{Aut}(\mathbb{C})$ is fairly easy to calculate, it is already hopeless for $\mathrm{Aut}(\mathbb{C}^2)$, let alone for manifolds. They prove that the subgroup of $\mathrm{Aut}(\mathbb{C})$ generated by so-called overshears is dense. A mapping $\mathbb{C}^n \to \mathbb{C}^n$ is an overshear, if it is of the form (up to permutations of the coordinates $z_1, ..., z_n$)

$$(z_1, ..., z_n) \mapsto (z_1, ..., z_{n-1}, f(z_1, ..., z_{n-1}) + h(z_1, ..., z_{n-1})z_n),$$

with f, h holomorphic functions on $\mathbb{C}^{n-1}$, $h \neq 0$. They also prove, that the subgroup generated by overshears is not all of $\mathrm{Aut}(\mathbb{C}^n)$. Since then, Andersén-Lempert theory has evolved and this is where the *density property* comes into play. A property that gives us an insight into the size of the automorphism group. Kaliman and Kutzschebauch [19] found a wonderful criteria for the density property of complex manifolds. Actually, it can be formulated in a even more general setting. We follow the article [9] which is dedicated to Lázló Lempert in honour of his 70th birthday.

Definition 4.5.1 (Density property)**.** *A complex manifold X has the density property if in the compact-open topology the Lie algebra $\mathrm{Lie}_{hol}(X)$ generated by completely integrable holomorphic vector fields on X is dense in the Lie algebra $\mathrm{VF}_{hol}(X)$ of all holomorphic vector fields on X.*

Definition 4.5.2. *A pair (θ_1, θ_2) of complete holomorphic vector fields on a Stein manifold X is a compatible pair if the following conditions hold:*

(i) the closure of the linear span of the product of the kernels $\ker\theta_1 \cdot \ker\theta_2$ contains a nontrivial ideal $I \subset \mathcal{O}(X)$, and

(ii) there is a function $h \in \mathcal{O}(X)$ with $h \in \ker\theta_2$ and $\theta_1(h) \in \ker\theta_1 \setminus \{0\}$.

Theorem 4.5.3. *Let X be a Stein manifold on which the group of holomorphic automorphisms $\mathrm{Aut}(X)$ acts transitively. If there are compatible pairs $(\theta_{1,k}, \theta_{2,k}), k = 1, ..., m$, such that there is a point $p \in X$ where the vectors $\theta_{2,k}(p)$ form a generating set for T_pX, then X has the density property.*

According to the Spanning theorem, the automorphism group acts transitively on smooth generic fibers. Moreover, every such fiber $\mathcal{F}_y^K$ is of the form $\mathbb{C}^m \times G$, since

$$\frac{\partial}{\partial z_{1,ij}} \Phi_K \equiv 0, \quad 1 \leq i \leq j < n.$$

We know the existence of a finite collection of complete holomorphic vector fields on $\mathcal{F}_y^K$ which span the tangent bundle $T\mathcal{F}_y^K$. Then it is rather trivial to find compatible pairs.

Corollary 4.5.4. *Every smooth generic fiber $\mathcal{F}_y^K$, $K \geq 2$, has the density property.*

Recall the notation $\tilde{P}^K$ (see Lemma 3.2.2):

$$\tilde{P}^K = \begin{cases} P_f^K & \text{if } K = 2k+1, \\ P_s^K & \text{if } K = 2k. \end{cases}$$

In the proof of the Spanning theorem, we actually prove that the automorphism group acts transitively on

$$\mathcal{G}_b^K := \{(z, W) \in \mathbb{C}^n \times (\mathbb{C}^{\frac{n(n+1)}{2}})^{K-1} : \tilde{P}^K(z, W) = b\}, b \in \mathbb{C}^n \setminus \{0, e_n\}.$$

We exclude $b = e_n$ such that we can omit a case distinction between K odd and even.

Theorem 4.5.5. *Let $K \geq 2$ and $n > 2$. Then the Stein manifold $\mathcal{G}_b^K$, $b \in \mathbb{C}^n \setminus \{0, e_n\}$, has the density property.*

Proof. Let's start with $K > 2$. According to Lemma 3.2.2, there is a meromorphic mapping ψ which maps $\mathcal{G}_b^K \times \mathbb{C}^m$ biholomorphic onto some smooth generic fiber $\mathcal{F}_y^{K+1}$ for some natural number m and some $y \in \mathbb{C}^{2n} \setminus \{0\}$. Since this fiber is smooth and generic, there exists a point $\vec{Z}_{K+1} \in \mathcal{F}_y^{K+1}$ with

$$Q_f^1 = z_{1,1n} \cdots z_{1,nn} \neq 0, \quad Q_f^k Q_s^k \neq 0, 2 \leq k \leq K-1.$$

Let (z_0, W_0) denote the corresponding copy in $\mathcal{G}_b^K$. From the Spanning theoremwe know the existence of a finite collection $A_{K+1} \subset \mathcal{Q}_{K+1}$ which spans the tangent bundle $T\mathcal{F}_y^{K+1}$ which is actually shown on the level of $\mathcal{G}_b^K$, that is, we construct a finite collection A of complete holomorphic vector fields on $\mathcal{G}_b^K$ which spans $T\mathcal{G}_b^K$. By construction of this set, each vector field in A was introduced for some $2 \leq k \leq K$ and thus is tangential to every $\mathcal{G}_b^L, b \in \mathbb{C}^n \setminus \{0\}, k \leq L \leq K$. Let $A_L \subset A$ denote the subset of those vector fields which are introduced for k with $k \leq L$. In particular every vector field $V \in A_L$, $L < K$, has $Z_K \in \mathbb{C}^{\frac{n(n+1)}{2}}$ in its kernel, i.e. $V(z_{K,ij}) = 0$ for all $1 \leq i \leq j \leq n$. Consider the complete vector field ∂_x^K corresponding to the (*Type 3*)-tupel

$$x = (z_{K,ij}, z_{K,11}, z_{K,22}, ..., z_{K,nn}), \quad i \neq j.$$

Then

$$\partial_x^K(z_{K,ij}) = \det \begin{pmatrix} \tilde{P}_1^{K-1} & & \\ & \ddots & \\ & & \tilde{P}_n^{K-1} \end{pmatrix} = \tilde{P}_1^{K-1} \cdots \tilde{P}_n^{K-1} \not\equiv 0,$$

since $\tilde{P}_1^{K-1} \cdots \tilde{P}_n^{K-1}(z_0, W_0) \neq 0$. Moreover, $\tilde{P}_1^{K-1} \cdots \tilde{P}_n^{K-1} \in \ker \partial_x^K$ and each variable $z_{l,rs}, 1 \leq l < K, 1 \leq r \leq s \leq n$ is in the kernel of ∂_x^K. This implies

$$\overline{\ker V \cdot \ker \partial_x^K} = \mathcal{O}(\mathcal{G}_b^K),$$

hence (V, ∂_x^K) is a compatible pair for each $V \in A_{K-1}$. Choose the n vector fields $\partial_{x_i}^K$ corresponding to the (*Type 1*)-tupels

$$x = (z_{K-1,ii}, z_{K,11}, ..., z_{K,nn}), 1 \leq i \leq n$$

from Lemma 3.7.2 which span the new expressed directions in the point (z_0, W_0). Hence $T_{(z_0,W_0)}\mathcal{G}_b^K$ is spanned by $A_{K-1} \cup \{\partial_{x_i}^K, i = 1, ..., n\}$. It remains to find compatible pairs for the fields $\partial_{x_i}^K$. Consider the complete vector fields ∂_y^L, $L \leq K-1$, corresponding to the (*Type 3*)-tupels

$$y = (z_{L,jj}, z_{L,1m}, z_{L,2m}, ..., z_{L,nm}), \quad j, m \neq i. \tag{4.3}$$

Such vector fields exist in the case where $n > 2$ and $K > 2$. Note that $(\partial_{x_i}^K, \partial_y^L)$ is a compatible pair. Therefore $\mathcal{G}_y^K$ has the density property by Theorem 4.5.3.

In a next step, let $K = 2$ and choose a point (z, W) in the manifold

$$\mathcal{G}_b^2 = \{(z, W) \in \mathbb{C}^n \times \mathbb{C}^{\frac{n(n+1)}{2}} : e_n + WZ = b\}$$

satisfying $z_1 \cdots z_n \neq 0$. Such a point exists by Lemma 3.5.4. Then the tangent space $T_{(z,W)}\mathcal{G}_b^2$ is spanned by complete vector fields ∂_x^2 corresponding to the tupels

$$\begin{aligned} x &= (z_{1,in}, z_{2,j1}, z_{2,j2}, ..., z_{2,jn}), && 1 \leq i \leq n, j \neq i, \\ x &= (z_{2,ij}, z_{2,11}, ..., z_{2,nn}), && i \neq j, \\ x &= (z_{1,1n}, ..., z_{1,nn}, z_{2,ii}), && 1 \leq i \leq n. \end{aligned}$$

of (*Type 8*), (*Type 3*) and (*Type 5*), respectively. For $n > 2$ we have

$$n + 1 < \binom{\frac{n(n+1)}{2}}{n+1}.$$

Therefore, given one of these vector fields ∂_x^2, we find distinct pairs of indices $(i_0, j_0), ..., (i_n, j_n)$ such that $(\partial_y^2, \partial_x^2)$ forms a compatible pair with the vector field ∂_y^2 corresponding to the (*Type 3*)-tupel $y = (z_{2,i_0j_0}, ..., z_{2,i_nj_n})$. Hence $\mathcal{G}_b^2$ has the density property by Theorem 4.5.3. $\square$

Corollary 4.5.6. *Let $n = 2$ and $K \geq 4$. Then the Stein manifold $\mathcal{G}_b^K$, $b \in \mathbb{C}^2 \setminus \{0, e_n\}$ has the density property.*

Proof. We can argue almost similarly as in the previous case. The only difference is that we need $L < K - 1$ instead of $L \leq K - 1$ for the choice of the fields in (4.3). $\square$

For $n = 2$ we have

$$\binom{\frac{n(n+1)}{2}}{n+1} = \binom{3}{3} = 1 < 2 = n,$$

which means that we are somehow 'running out of space' in this case. It turns out, that the arguments above no longer apply so easily. In particular, it is significantly more difficult to find a non-trivial ideal that satisfies property (i) in the definition of a compatible pair. This leads to the following

Question 4.5.7. *Let $n = 2$ and $b \in \mathbb{C}^2 \setminus \{0, e_2\}$. Do the Stein manifolds $\mathcal{G}_b^2$ and $\mathcal{G}_b^3$ have the density property?*

A Appendix

A.1 Calculation and properties of the vector fields ∂_x^K

In this section we calculate some concrete vector fields as a supplement and for a better understanding of the proof of the Spanning theorem. Consider the tupel $x = (x_0, ..., x_n)$. By definition, the vector field ∂_x^K is given by

$$\partial_x^K = \det \begin{pmatrix} \frac{\partial}{\partial x_0} & \cdots & \frac{\partial}{\partial x_n} \\ \frac{\partial}{\partial x_0}\tilde{P}^K & \cdots & \frac{\partial}{\partial x_n}\tilde{P}^K \end{pmatrix}.$$

We are primarily interested in whether there are some complete fields whose flows don't remain in a given subvariety. At first glance this may sound simple. However, the definition with the determinant suggests that the coefficients are not easy to understand in general, since they are polynomials in several variables. In addition, the subvariety might be given by some difficult equations. and some useful properties. Therefore, it makes sense to calculate some derivatives $\frac{\partial}{\partial x_i}\tilde{P}^K$ in advance. In order to make life easier, we start with $K = 2$.

A.1.1 Calculation of ∂_x^2

We are interested in the tupels of (*Type 8*)

$$x_{i,j} := (z_{1,ni}, z_{2,j1}, ..., z_{2,jn}), \quad 1 \le i, j \le n, i \ne j. \tag{A.1}$$

Compute

$$\frac{\partial}{\partial z_{1,ni}} P_s^2 = \frac{\partial}{\partial z_{1,ni}} \begin{pmatrix} Z_2 & I_n \end{pmatrix} \begin{pmatrix} Z_1 e_n \\ e_n \end{pmatrix} = \begin{pmatrix} Z_2 & I_n \end{pmatrix} \begin{pmatrix} e_i \\ 0 \end{pmatrix} = Z_2 e_i$$

and for $1 \le r \le n$,

$$\begin{aligned} \frac{\partial}{\partial z_{2,jr}} P_s^2 &= \frac{\partial}{\partial z_{2,jr}} \begin{pmatrix} Z_2 & I_n \end{pmatrix} \begin{pmatrix} Z_1 e_n \\ e_n \end{pmatrix} = \begin{pmatrix} \tilde{E}_{jr} & 0 \end{pmatrix} \begin{pmatrix} Z_1 e_n \\ e_n \end{pmatrix} = \tilde{E}_{jr} Z_1 e_n \\ &= \frac{1}{1+\delta_{jr}} \left(z_{1,nr} e_j + z_{1,nj} e_r \right). \end{aligned}$$

Hence the corresponding vector fields $\partial_{x_{i,j}}^2$ are of the form

$$\partial_{x_{i,j}}^2 = \det \begin{bmatrix} \partial/\partial z_{1,ni} & \partial/\partial z_{2,j1} & \cdots & \partial/\partial z_{2,jj} & \cdots & \partial/\partial z_{2,jn} \\ z_{2,1i} & z_{1,nj} & & 0 & & 0 \\ \vdots & & \ddots & & & \\ z_{2,ji} & z_{1,n1} & \cdots & z_{1,nj} & \cdots & z_{1,nn} \\ \vdots & & & & \ddots & \\ z_{2,ni} & 0 & & 0 & & z_{1,nj} \end{bmatrix}.$$

By construction, they satisfy

$$\partial^2_{x_{i,j}}(z_{1,kn}) = \delta_{ik}(z_{1,nj})^n,$$

for $1 \leq i, j, k \leq n$.

A.1.2 Calculation of ∂^3_x

Let $1 \leq i, j \leq n$ and compute the following derivatives.

$$\begin{aligned}
\frac{\partial}{\partial z_{1,ni}} P^3_f &= \frac{\partial}{\partial z_{1,ni}} \begin{pmatrix} I_n & Z_3 \end{pmatrix} \begin{pmatrix} I_n & 0 \\ Z_2 & I_n \end{pmatrix} \begin{pmatrix} Z_1 e_n \\ e_n \end{pmatrix} = \begin{pmatrix} I_n & Z_3 \end{pmatrix} \begin{pmatrix} I_n & 0 \\ Z_2 & I_n \end{pmatrix} \begin{pmatrix} e_i \\ 0 \end{pmatrix} \\
&= (I_n + Z_3 Z_2) e_i,
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial z_{2,ij}} P^3_f &= \frac{\partial}{\partial z_{2,ij}} \begin{pmatrix} I_n & Z_3 \end{pmatrix} \begin{pmatrix} I_n & 0 \\ Z_2 & I_n \end{pmatrix} \begin{pmatrix} Z_1 e_n \\ e_n \end{pmatrix} = \begin{pmatrix} I_n & Z_3 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \tilde{E}_{ij} & 0 \end{pmatrix} \begin{pmatrix} Z_1 e_n \\ e_n \end{pmatrix} \\
&= Z_3 \tilde{E}_{ij} Z_1 e_n = \frac{1}{1+\delta_{ij}} \left(z_{1,ni} Z_3 e_j + z_{1,nj} Z_3 e_i \right)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial z_{3,ij}} P^3_f &= \frac{\partial}{\partial z_{3,ij}} \begin{pmatrix} I_n & Z_3 \end{pmatrix} \begin{pmatrix} P^2_f \\ P^2_s \end{pmatrix} = \begin{pmatrix} 0 & \tilde{E}_{ij} \end{pmatrix} \begin{pmatrix} P^2_f \\ P^2_s \end{pmatrix} = \tilde{E}_{ij} P^2_s \\
&= \frac{1}{1+\delta_{ij}} \left(P^2_{n+i} e_j + P^2_{n+j} e_i \right).
\end{aligned}$$

At first, we are interested in the tupels

$$x_i = (z_{2,ii}, z_{3,i^*1}, ..., z_{3,i^*n}), 1 \leq i, i^* \leq n, i \neq i^*, \tag{A.2}$$

of (*Type 2*). It is of the form

$$\begin{aligned}
\partial^3_{x_i} &= \det \begin{pmatrix} \frac{\partial}{\partial z_{2,ii}} & \frac{\partial}{\partial z_{3,i^*1}} & \cdots & \frac{\partial}{\partial z_{3,i^*n}} \\ \star & \tilde{E}_{i^*1} P^2_s & \cdots & \tilde{E}_{i^*n} P^2_s \end{pmatrix} \\
&= \det \begin{pmatrix} P^2_{n+i^*} & & & & \\ & \ddots & & & \\ P^2_{n+1} & \cdots & P^2_{n+i^*} & \cdots & P^2_{2n} \\ & & & \ddots & \\ & & & & P^2_{n+i^*} \end{pmatrix} \frac{\partial}{\partial z_{2,ii}} + \sum_{k=1}^n \alpha_k \frac{\partial}{\partial z_{3,i^*k}} \\
&= (P^2_{n+i^*})^n \frac{\partial}{\partial z_{2,ii}} + \sum_{k=1}^n \alpha_k \frac{\partial}{\partial z_{3,i^*k}}
\end{aligned}$$

for some suitable holomorphic functions $\alpha_1, ..., \alpha_n$.

The vector field ∂^3_x corresponding to the (*Type 5*)-tupel

$$x = (z_{1,1n}, ..., z_{1,nn}, z_{3,nn}) \tag{A.3}$$

is given by

$$\partial^3_x = \det \begin{pmatrix} \frac{\partial}{\partial z_{1,1n}} & \cdots & \frac{\partial}{\partial z_{1,nn}} & \frac{\partial}{\partial z_{3,nn}} \\ (I_n + Z_3 Z_2) e_1 & \cdots & (I_n + Z_3 Z_2) e_n & P^2_{2n}(\vec{Z}_2) e_n \end{pmatrix},$$

having the notation $\Phi_2 = (P_1^2, ..., P_{2n}^2)$ in mind.

The vector fields $\partial^3_{x_i}$ corresponding to the (*Type 2*)-tupels

$$x_i = (z_{2,ii}, z_{3,1n}, ..., z_{3,nn}), \quad i = 1, ..., n-1, \tag{A.4}$$

are given by

$$\partial^3_{x_i} = \det \begin{pmatrix} \frac{\partial}{\partial z_{2,ii}} & \frac{\partial}{\partial z_{3,1n}} & \cdots & \frac{\partial}{\partial z_{3,n-1,n}} & \frac{\partial}{\partial z_{3,nn}} \\ z_{1,in} Z_3 e_i & P^2_{2n} e_1 + P^2_{n+1} e_n & \cdots & P^2_{2n} e_{n-1} + P^2_{2n-1} e_n & P^2_{2n} e_n \end{pmatrix}.$$

Lemma A.1.1. *The vector fields $\partial^3_{x_i}$, corresponding to the tupels in (A.4), are given by $\frac{\partial}{\partial z_{2,ii}}$ on the set of points with $Z_1 e_n = 0$.*

Proof. Let $\vec{Z}_3 \in (\mathbb{C}^{\frac{n(n+1)}{2}})^3$ be a point with $Z_1 e_n = 0$. Then the recursive formula (3.1) implies

$$P^2_s = e_n + Z_2 \underbrace{Z_1 e_n}_{=0} = e_n.$$

Applying this to the formula of the vector fields yields

$$\begin{aligned} (\partial^3_{x_i})_{\vec{Z}_3} &= \det \begin{pmatrix} \frac{\partial}{\partial z_{2,ii}} & \frac{\partial}{\partial z_{3,1n}} & \cdots & \frac{\partial}{\partial z_{3,n-1,n}} & \frac{\partial}{\partial z_{3,nn}} \\ z_{1,in} Z_3 e_i & P^2_{2n} e_1 + P^2_{n+1} e_n & \cdots & P^2_{2n} e_{n-1} + P^2_{2n-1} e_n & P^2_{2n} e_n \end{pmatrix} \\ &= \det \begin{pmatrix} \frac{\partial}{\partial z_{2,ii}} & \frac{\partial}{\partial z_{3,1n}} & \cdots & \frac{\partial}{\partial z_{3,n-1,n}} & \frac{\partial}{\partial z_{3,nn}} \\ 0 & e_1 & \cdots & e_{n-1} & e_n \end{pmatrix} \\ &= \det(I_n) \frac{\partial}{\partial z_{2,ii}} \\ &= \frac{\partial}{\partial z_{2,ii}} \end{aligned}$$

□

The vector fields $\partial^3_{y_i}$ corresponding to the (*Type 3*)-tupels

$$y_i = (z_{3,ii}, z_{3,1n}, ..., z_{3,nn}), \quad i = 1, ..., n-1, \tag{A.5}$$

are given by

$$\partial^3_{y_i} = \det \begin{pmatrix} \frac{\partial}{\partial z_{3,ii}} & \frac{\partial}{\partial z_{3,1n}} & \cdots & \frac{\partial}{\partial z_{3,n-1,n}} & \frac{\partial}{\partial z_{3,nn}} \\ P^2_{n+i} e_i & P^2_{2n} e_1 + P^2_{n+1} e_n & \cdots & P^2_{2n} e_{n-1} + P^2_{2n-1} e_n & P^2_{2n} e_n \end{pmatrix}.$$

Lemma A.1.2. *The vector fields $\partial^3_{y_i}$, corresponding to the tupels in (A.5), are given by $\frac{\partial}{\partial z_{3,ii}}$ on the set of points with $Z_1 e_n = 0$.*

Proof. The proof is analogous to that in the previous lemma. Recall that $P^2_s(\vec{Z}_3) = e_n$ for a point $\vec{Z}_3 \in (\mathbb{C}^{\frac{n(n+1)}{2}})^3$ with $Z_1 e_n = 0$. In particular, $P^2_{n+i} = 0$ for $i = 1, ..., n-1$. □

The vector field ∂^3_x corresponding to the (*Type 8*)-tupel

$$x = (z_{1,jn}, z_{2,i1}, ..., z_{2,in}), i \neq j, \tag{A.6}$$

is given by

$$\partial^3_x = \det \begin{pmatrix} \frac{\partial}{\partial z_{1,jn}} & \frac{\partial}{\partial z_{2,i1}} & \cdots & \frac{\partial}{\partial z_{2,in}} \\ (I_n + Z_3 Z_2) e_j & Z_3 \tilde{E}_{i1} Z_1 e_n & \cdots & Z_3 \tilde{E}_{in} Z_1 e_n \end{pmatrix}.$$

Lemma A.1.3. *The vector field ∂_x^3, corresponding to the tupel in (A.6), satisfies*

$$\partial_x^3(z_{1,jn}) = (z_{1,in})^n \det(Z_3).$$

Proof. Recall that the matrix (cf. section ...)

$$F_i(Z_1 e_n) := (\tilde{E}_{i1} Z_1 e_n, ..., \tilde{E}_{in} Z_1 e_n)$$

is given by

$$F_i(Z_1 e_n) = \begin{pmatrix} z_{1,in} & & & & \\ & \ddots & & & \\ z_{1,1n} & \cdots & z_{1,in} & \cdots & z_{1,nn} \\ & & & \ddots & \\ & & & & z_{1,in} \end{pmatrix}.$$

Hence

$$\det(F_i(Z_1 e_n)) = (z_{1,in})^n.$$

Now, observe that

$$\begin{aligned} \partial_x^3(z_{1,jn}) &= \det(Z_3 \tilde{E}_{i1} Z_1 e_n, ..., Z_3 \tilde{E}_{in} Z_1 e_n) \\ &= \det(Z_3) \det(\tilde{E}_{i1} Z_1 e_n, ..., \tilde{E}_{in} Z_1 e_n) \\ &= \det(Z_3) \det(F_i(Z_1 e_n)) \\ &= (z_{1,in})^n \det(Z_3) \end{aligned}$$

and this completes the proof. $\square$

The vector field ∂_x^3 corresponding to the (*Type 7*)-tupel

$$x = (z_{1,i_1 n}, ..., z_{1,i_{n-k} n}, z_{2,jj_1}, ..., z_{2,jj_k}, z_{3,ii}), \tag{A.7}$$

where $\{i_1, ..., i_{n-k}\} \dot\cup \{j_1, ..., j_k\} = \{1, ..., n\}, i \in \{i_1, ..., i_{n-k}\}, j \in \{j_1, ..., j_k\}$, is given by

$$\partial_x^3 = \det \begin{pmatrix} \frac{\partial}{\partial z_{1,i_1 n}} & \cdots & \frac{\partial}{\partial z_{1,i_{n-k} n}} & \frac{\partial}{\partial z_{2,jj_1}} & \cdots & \frac{\partial}{\partial z_{2,jj_k}} & \frac{\partial}{\partial z_{3,ii}} \\ (I_n + Z_3 Z_2) e_{i_1} & \cdots & (I_n + Z_3 Z_2) e_{i_{n-k}} & Z_3 \tilde{E}_{jj_1} Z_1 e_n & \cdots & Z_3 \tilde{E}_{jj_k} Z_1 e_n & P^2_{n+i} e_i \end{pmatrix}.$$

Lemma A.1.4. *Let ∂_x^3 be the vector field corresponding to the tupel in (A.7). Assume that $z_{1,jn} \neq 0$ and*

$$\operatorname{span}\{Z_3 e_{j_1}, ..., Z_3 e_{j_k}\} = \operatorname{Im}(Z_3) \tag{$\star$}$$

is satisfied. Then

$$\partial_x^3(z_{3,ii}) = \pm (z_{1,jn})^k \det((Z_3)_{i_1,...,i_{n-k}}) \neq 0$$

where $(Z_3)_{i_1,...,i_{n-k}}$ denotes the $k \times k$ matrix obtained by removing columns and rows $i_1, ..., i_{n-k}$ from Z_3.

Proof. We divide the proof into two parts. In the first part, we show that $\partial_x^3(z_{3,ii}) \neq 0$ under the given assumptions and in the second part, we explicitly compute $\partial_x^3(z_{3,ii})$. For simplicity reasons, let's substitute $A := Z_3$, $B := Z_2$ and $v := Z_1 e_n$.

In a first step, we prove

$$\operatorname{span}\{A \tilde{E}_{jj_1} v, ..., A \tilde{E}_{jj_k} v\} = \operatorname{Im}(A)$$

if $v_j \neq 0$. By assumption, $j \in \{j_1, ..., j_k\}$ and without loss of generality, let's assume $j = j_k$. This gives us $A\tilde{E}_{jj_k}v = A\tilde{E}_{jj}v = v_j Ae_j$ and moreover, $v_j Ae_j$ is non-zero, also by assumption. To prove the claim, we want to add suitable multiples of vector $v_j Ae_j$ to the other vectors

$$A\tilde{E}_{jj_1}v, ..., A\tilde{E}_{jj_{k-1}}v$$

which is a *span-preserving* operation. We get

$$\begin{aligned}\operatorname{span}\{A\tilde{E}_{jj_1}v, ..., A\tilde{E}_{jj_k}v\} &= \operatorname{span}\{v_j Ae_{j_1} + v_{j_1}Ae_j, ..., v_j Ae_{j_{k-1}} + v_{j_{k-1}}Ae_j, v_j Ae_j\} \\ &= \operatorname{span}\{v_j Ae_{j_1}, ..., v_j Ae_{j_{k-1}}, v_j Ae_j\},\end{aligned}$$

where the second equation is obtained by adding $-\frac{v_{j_l}}{v_j}v_j Ae_j$, $l = 1, ..., k-1$, to the l-th vector. Now, the claim follows by assumption $v_j \neq 0$ and by $(\star)$.

In a second step, consider the symplectic matrix

$$\begin{pmatrix} I_n + AB & A \\ B & I_n \end{pmatrix}.$$

By assumption $(\star)$ and the complementary bases theorem, the span

$$V := \operatorname{span}\{(I_n + AB)e_{i_1}, ..., (I_n + AB)e_{i_{n-k}}, Ae_{j_1}, ..., Ae_{j_k}\}$$

describes an n-dimensional vector space. An application of the first step implies

$$V = \operatorname{span}\{(I_n + AB)e_{i_1}, ..., (I_n + AB)e_{i_{n-k}}, A\tilde{E}_{jj_1}v, ..., A\tilde{E}_{jj_k}v\},$$

which shows that $\partial_x^3(z_{3,ii}) \neq 0$.

In the next step we compute $\partial_x^3(z_{3,ii})$ explicitly. From the first step we get

$$\begin{aligned}\partial_x^3(z_{3,ii}) &= \det((I_n + AB)e_{i_1}, ..., (I_n + AB)e_{i_{n-k}}, A\tilde{E}_{jj_1}v, ..., A\tilde{E}_{jj_k}v) \\ &= \det((I_n + AB)e_{i_1}, ..., (I_n + AB)e_{i_{n-k}}, v_j Ae_{j_1}, ..., v_j Ae_{j_k}) \\ &= v_j^k \det((I_n + AB)e_{i_1}, ..., (I_n + AB)e_{i_{n-k}}, Ae_{j_1}, ..., Ae_{j_k}),\end{aligned}$$

since adding multiples of columns to other columns has no effect on the determinant. By assumption $(\star)$, we have

$$ABe_l \in \operatorname{span}\{Ae_{j_1}, ..., Ae_{j_k}\}, l = i_1, ..., i_{n-k}.$$

This means that we may add suitable linear combinations of the last k columns to the first $n-k$ columns to get

$$\det((I_n + AB)e_{i_1}, ..., (I_n + AB)e_{i_{n-k}}, Ae_{j_1}, ..., Ae_{j_k}) = \det(e_{i_1}, ..., e_{i_{n-k}}, Ae_{j_1}, ..., Ae_{j_k}).$$

Recall that the indices $i_1, ..., i_{n-k}$ and $j_1, ..., j_k$ are complementary, i.e.

$$\{1, ..., n\} = \{i_1, ..., i_{n-k}\}\dot{\cup}\{j_1, ..., j_k\}.$$

Therefore we obtain

$$\det(e_{i_1}, ..., e_{i_{n-k}}, Ae_{j_1}, ..., Ae_{j_k}) = \det((A)_{i_1,...,i_{n-k}}).$$

Putting this together yields

$$\partial_x^3(z_{3,ii}) = v_j^k \det((A)_{i_1,...,i_{n-k}}) = (z_{1,jn})^k \det((Z_3)_{i_1,...,i_{n-k}})$$

and this completes the proof. □

The vector field ∂_x^3 corresponding to the tupel

$$x = (z_{1,1n}, ..., z_{1,nn}, z_{3,ii}), 1 \leq i \leq n, \tag{A.8}$$

is of the form

$$\det\begin{pmatrix} \frac{\partial}{\partial z_{1,1n}} & \cdots & \frac{\partial}{\partial z_{1,nn}} & \frac{\partial}{\partial z_{3,ii}} \\ (I_n + Z_3 Z_2)e_1 & \cdots & (I_n + Z_3 Z_2)e_n & P^2_{n+i}e_i \end{pmatrix}$$

and it satisfies $\partial_x^3(z_{3,ii}) = (-1)^{n+1}(z_{1,jn})^k \det(I_n + Z_3 Z_2)$.

A.1.3 Calculation of ∂_x^K for $K \geq 4$

We want to compute the vector field

$$\partial_x^K = D_x(\tilde{P}^K),$$

where $\tilde{P}^K$ is given by the recursive formula (3.2)

$$\begin{cases} \tilde{P}^K(\vec{Z}_K) = \tilde{P}^{K-2}(\vec{Z}_{K-2}) + Z_K\tilde{P}^{K-1}(\vec{Z}_{K-1}) \\ \tilde{P}^1(\vec{Z}_K) = Z_1 e_n, \quad \tilde{P}^0(\vec{Z}_K) = e_n. \end{cases}$$

with the convention $\vec{Z}_K = (\vec{Z}_{K-1}, Z_K) = (\vec{Z}_{K-2}, Z_{K-1}, Z_K) \in (\mathbb{C}^{\frac{n(n+1)}{2}})^K$. Compute the derivatives

$$\begin{aligned} \frac{\partial}{\partial z_{K,ij}}\tilde{P}^K &= \underbrace{\frac{\partial}{\partial Z_{K,ij}}(\tilde{P}^{K-2})}_{=0} + \frac{\partial}{\partial Z_{K,ij}}(Z_K\tilde{P}^{K-1}) \\ &= (\frac{\partial}{\partial z_{K,ij}}Z_K)\tilde{P}^{K-1} + Z_K(\underbrace{\frac{\partial}{\partial z_{K,ij}}\tilde{P}^{K-1}}_{=0}) \\ &= \tilde{E}_{ij}\tilde{P}^{K-1}, \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial z_{K-1,ij}}\tilde{P}^K &= \underbrace{\frac{\partial}{\partial z_{K-1,ij}}\tilde{P}^{K-2}}_{=0} + Z_K(\underbrace{\frac{\partial}{\partial z_{K-1,ij}}\tilde{P}^{K-1}}_{=\tilde{E}_{ij}\tilde{P}^{K-2}}) \\ &= Z_K\tilde{E}_{ij}\tilde{P}^{K-2} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial z_{K-2,ij}}\tilde{P}^K &= \underbrace{\frac{\partial}{\partial z_{K-2,ij}}\tilde{P}^{K-2}}_{=\tilde{E}_{ij}\tilde{P}^{K-3}} + Z_K(\underbrace{\frac{\partial}{\partial z_{K-2,ij}}\tilde{P}^{K-1}}_{Z_{K-1}\tilde{E}_{ij}\tilde{P}^{K-3}}) \\ &= (I_n + Z_K Z_{K-1})\tilde{E}_{ij}\tilde{P}^{K-3}. \end{aligned}$$

If we compare the derivatives of the previous subsection with the ones here, it is noticeable that they are pretty similar. It is thus no surprise that the tupels of interest look familiar.

The vector field ∂_x^{K-1} corresponding to the (*Type 2*)-tupel

$$x = (z_{K-2,ii}, z_{K-1,i^*1}, ..., z_{K-1,i^*n}) \tag{A.9}$$

is of the form

$$\pm(\tilde{P}_{i^*}^{K-2})^n \frac{\partial}{\partial z_{K-2,ii}} + \sum_{l=1}^{n} \alpha_l \frac{\partial}{\partial z_{K-1,i^*l}}$$

for some suitable holomorphic functions $\alpha_1, ..., \alpha_n$.

The vector field ∂_x^K corresponding to the (*Type 5*)-tupel

$$x = (z_{K-2,1n}, ..., z_{K-2,nn}, z_{K,nn}) \tag{A.10}$$

is given by

$$\partial_x^K = \det\begin{pmatrix} \frac{\partial}{\partial z_{K-2,1n}} & \cdots & \frac{\partial}{\partial z_{K-2,nn}} & \frac{\partial}{\partial z_{K,nn}} \\ (I_n + Z_K Z_{K-1})\tilde{E}_{1n}\tilde{P}^{K-3} & \cdots & (I_n + Z_K Z_{K-1})\tilde{E}_{nn}\tilde{P}^{K-3} & \tilde{E}_{nn}\tilde{P}^{K-1} \end{pmatrix}.$$

The vector fields $\partial^K_{x_i}$ corresponding to the (*Type 2*)-tupels

$$x_i = (z_{K-1,ii}, z_{K,1n}, ..., z_{K,nn}), \quad i = 1, ..., n-1, \tag{A.11}$$

are given by

$$\partial^K_{x_i} = \det \begin{pmatrix} \frac{\partial}{\partial z_{K-1,ii}} & \frac{\partial}{\partial z_{K,1n}} & \cdots & \frac{\partial}{\partial z_{K,nn}} \\ \tilde{P}^{K-2}_i Z_{K-1} e_i & \tilde{E}_{1n}\tilde{P}^{K-1} & \cdots & \tilde{E}_{nn}\tilde{P}^{K-1} \end{pmatrix}.$$

Lemma A.1.5. *The vector fields $\partial^K_{x_i}$,corresponding to the tupel in (A.11), are of the form $\frac{\partial}{\partial z_{K-1,ii}}$ on the set of points with $\tilde{P}^{K-2} = 0$ and $\tilde{P}^{K-1} = e_n$.*

Proof. The proof is similiar to the one of Lemma A.1.1. □

The vector fields $\partial^K_{y_i}$ corresponding to the (*Type 3*)-tupels

$$y_i = (z_{K,ii}, z_{K,1n}, ..., z_{K,nn}), \quad i = 1, ..., n-1, \tag{A.12}$$

are given by

$$\partial^K_{y_i} = \det \begin{pmatrix} \frac{\partial}{\partial z_{K,ii}} & \frac{\partial}{\partial z_{K,1n}} & \cdots & \frac{\partial}{\partial z_{K,nn}} \\ \tilde{P}^{K-1}_i e_i & \tilde{E}_{1n}\tilde{P}^{K-1} & \cdots & \tilde{E}_n n\tilde{P}^{K-1} \end{pmatrix}.$$

Again we have a similar result.

Lemma A.1.6. *The vector fields $\partial^K_{y_i}$, corresponding to the tupels in (A.12), are of the form $\frac{\partial}{\partial z_{K,ii}}$ on the set of points with $\tilde{P}^{K-2} = 0$ and $\tilde{P}^{K-1} = e_n$.*

The vector field ∂^K_x corresponding to the (*Type 2*)-tupel

$$x = (z_{K-2,ii}, z_{K-1,j1}, ..., z_{K-1,jn}), i \neq j, \tag{A.13}$$

is given by

$$\partial^K_x = \det \begin{pmatrix} \frac{\partial}{\partial z_{K-2,ii}} & \frac{\partial}{\partial z_{K-1,j1}} & \cdots & \frac{\partial}{\partial z_{K-1,jn}} \\ \tilde{P}^{K-3}_i (I_n + Z_K Z_{K-1}) e_i & Z_K \tilde{E}_{j1}\tilde{P}^{K-2} & \cdots & Z_K \tilde{E}_{jn}\tilde{P}^{K-2} \end{pmatrix}$$

The following result can be shown similar as Lemma A.1.3.

Lemma A.1.7. *The vector field ∂^K_x, corresponding to the tupel in (A.13), satisfies*

$$\partial^K_x (z_{K-2,ii}) = (\tilde{P}^{K-2}_j)^n \det(Z_K).$$

The vector field ∂^3_x corresponding to the (*Type 7*)-tupel

$$x = (z_{K-2,i_1 i^*}, ..., z_{K-2,i_{n-k} i^*}, z_{K-1,jj_1}, ..., z_{K-1,jj_k}, z_{K,ii}), \tag{A.14}$$

where $\{i_1, ..., i_{n-k}\} \dot\cup \{j_1, ..., j_k\} = \{1, ..., n\}, i, i^* \in \{i_1, ..., i_{n-k}\}, j \in \{j_1, ..., j_k\}$, is given by

$$\partial^K_x = \det \begin{pmatrix} \cdots & \frac{\partial}{\partial z_{K-2,i_l i^*}} & \cdots & \frac{\partial}{\partial z_{K-1,jj_m}} & \cdots & \frac{\partial}{\partial z_{K,ii}} \\ \cdots & (I_n + Z_K Z_{K-1})\tilde{E}_{i_l i^*}\tilde{P}^{K-3} & \cdots & Z_K \tilde{E}_{jj_m}\tilde{P}^{K-2} & \cdots & \tilde{E}_{ii}\tilde{P}^{K-1} \end{pmatrix}.$$

The following lemma is the analog of Lemma A.1.4.

Lemma A.1.8. *Let ∂_x^K be the vector field corresponding to the tupel in (A.14). Assume that $\tilde{P}_j^{K-2} \neq 0$, $\tilde{P}_{i^*}^{K-3} \neq 0$ and*

$$\operatorname{span}\{Z_K e_{j_1}, ..., Z_K e_{j_k}\} = \operatorname{Im}(Z_K)$$

is satisfied. Then

$$\partial_x^K(z_{K,ii}) = \pm(\tilde{P}_{i^*}^{K-3})^{n-k}(\tilde{P}_j^{K-2})^k \det((Z_K)_{i_1,...,i_{n-k}}).$$

Proof. For simplicity reasons, substitute $A := Z_K, B := Z_{K-1}, u := \tilde{P}^{K-3}$ and $v := \tilde{P}^{K-2}$. As in Lemma A.1.4, we can conclude that

$$\begin{aligned}
\partial_x^K(z_{K,ii}) &= \pm \det((I_n + AB)\tilde{E}_{i_1 i^*}u, ..., (I_n + AB)\tilde{E}_{i_{n-k} i^*}u, A\tilde{E}_{jj_1}v, ..., A\tilde{E}_{jj_k}) \\
&= \pm \det((I_n + AB)\tilde{E}_{i_1 i^*}u, ..., (I_n + AB)\tilde{E}_{i_{n-k} i^*}u, v_j A e_{j_1}, ..., v_j A e_{j_k}) \\
&= \pm(v_j)^k \det(\tilde{E}_{i_1 i^*}u + AB\tilde{E}_{i_1 i^*}u, ..., \tilde{E}_{i_{n-k} i^*}u + AB\tilde{E}_{i_{n-k} i^*}u, Ae_{j_1}, ..., Ae_{j_k}) \\
&= \pm(v_j)^k \det(\tilde{E}_{i_1 i^*}u, ..., \tilde{E}_{i_{n-k} i^*}u, Ae_{j_1}, ..., Ae_{j_k}).
\end{aligned}$$

Since we assume $i^* \in \{i_1, ..., i_{n-k}\}$ and $u_{i^*} \neq 0$, we can add a suitable multiple of column $u_{i^*}e_{i^*} = \tilde{E}_{i^* i^*}u$ to the remaining columns $\tilde{E}_{i_r i^*}u, r = 1, ..., n-k, i_r \neq i^*$, to conclude that

$$\begin{aligned}
\det(\tilde{E}_{i_1 i^*}u, ..., \tilde{E}_{i_{n-k} i^*}u, Ae_{j_1}, ..., Ae_{j_k}) &= \det(u_{i^*}e_{i_1}, ..., u_{i^*}e_{i_{n-k}}, Ae_{j_1}, ..., Ae_{j_k}) \\
&= (u_{i^*})^{n-k} \det(e_{i_1}, ..., e_{i_{n-k}}, Ae_{j_1}, ..., Ae_{j_k}) \\
&= (u_{i^*})^{n-k} \det((A)_{i_1,...,i_{n-k}}).
\end{aligned}$$

In summary, this yields

$$\partial_x^K(z_{K,ii}) = \pm(u_{i^*})^{n-k}(v_j)^k \det((A)_{i_1,...,i_{n-k}})$$

and this completes the proof. □

The vector field ∂_x^K corresponding to the (*Type 5*)-tupel

$$x = (z_{K-2,n1}, ..., z_{K-2,nn}, z_{K,ii}), 1 \leq i \leq n, \tag{A.15}$$

is given by

$$\partial_x^K = \det \begin{pmatrix} \frac{\partial}{\partial z_{K-2,1n}} & \cdots & \frac{\partial}{\partial z_{K-2,nn}} & \frac{\partial}{\partial z_{K,ii}} \\ (I_n + Z_K Z_{K-1})\tilde{E}_{1n}\tilde{P}^{K-3} & \cdots & (I_n + Z_K Z_{K-1})\tilde{E}_{nn}\tilde{P}^{K-3} & \tilde{E}_{ii}\tilde{P}^{K-1} \end{pmatrix}$$

and satisfies

$$\begin{aligned}
\partial_x^K(z_{K,ii}) &= \det(I_n + Z_K Z_{K-1}) \det(F_n(\tilde{P}^{K-3})) \\
&= (\tilde{P}_n^{K-3})^n \det(I_n + Z_K Z_{K-1}).
\end{aligned}$$

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<u>Erklärung</u>

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LeiterIn der Arbeit: Prof. Dr. Frank Kutzschebauch

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