Regular variation on Polish spaces, continuous maps and compound maxima

Inaugural dissertation
of the Faculty of Science,
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presented by

Nikolina Milinčević

from Croatia

Supervisors of the doctoral thesis:
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Contents

Preface

In this work, we focus on developing the theory of regular variation on general Polish spaces, i.e. complete separable metric spaces.

We first revisit some familiar results regarding the theory of extremes and regular variation in $\mathbb{R}^d$, $d \geq 1$. These results are mainly retrieved from [2], [17], [48] and [54]. The reader can find them in Chapter 1 together with some basic theoretical background that is crucial in our approach (Section 1.3). Motivated by [54] and the authors' approach to polar decomposition on star-shaped spaces using the modulus, we extend our analysis to a more general setting. Specifically, we consider spaces that are not necessarily star-shaped and equip them with a general scaling, modulus, and bornologies generated by the modulus. It is important to note that these bornologies, as families of bounded sets, may not necessarily be the families of sets whose closure does not intersect the origin. Moreover, similar to [40], where the authors remove a fixed closed cone from the underlying space, we often remove a cone.

In Chapter 2, we introduce the notion of vague convergence on a Polish space with respect to the bornology on a given space (Section 2.1). Using vague convergence, we define regular variation and show some basic properties that are naturally extended from simpler spaces and also hold in our setting (Section 2.2). We consider reductions to a subcone (Section 2.3), where the so-called *hidden regular variation phenomenon* may occur (see [46], and [40] for the case of a star-shaped space). With the introduction of general scaling and modulus, the theory becomes more widely applicable. In this context, we present some representative examples to illustrate its broader scope.

In Chapter 3, we extensively analyse continuous transformations of a regularly varying random element and show that under certain conditions on the space and the transformation, the mapped variant is also regularly varying (Section 3.2). One such transformation is *the polar decomposition* (Section 3.3). We also explore the connection of regular variation with point process convergence (Section 3.5) and consider set-valued maps as well (Section 3.6). Finally, we investigate an equivalence relation on a Polish space and find conditions under which the whole class of equivalence is regularly varying as an element from the quotient space and as a closed subset of the underlying space. Moreover, we examine what happens if we continuously select an element from a class. All of these topics are covered in Section 3.8. Throughout the sections, we provide an extensive and detailed list of examples.

In Chapter 4, we examine an independent and identically distributed sequence of random variables from a certain maximum domain of attraction and investigate the compound maxima of such a sequence, where the number of observed variables can depend on the observations themselves (see [34] and [59] for the independent case). By using results on point process convergence (we refer to [8]), we show that as long as the number of observations considered in the compound maxima is a stopping time with respect to the natural filtration of the sequence of observation and the expected number of observations is finite, then the compound maxima belongs to the same maximum domain of attraction (Section 4.2). We then apply these results to some marked renewal cluster processes (Section 4.3).

For a better understanding of the content, we recommend that the reader consider some preliminary literature on extreme value analysis, regular variation and cluster models, such as [17], [38], [42], and [48]. The results presented in this paper originate from two articles written jointly with Bojan Basrak, Ilya Molchanov, and Petra Žugec, one of which has been published (see [4] and Chapter 4), whereas the second paper is still in preparation (Chapters 1–3).

Chapter 1

Introduction

1.1 Preliminaries on maximum domains of attraction

Consider a sequence of independent and identically distributed (i.i.d.) random variables $(X_i)_{i\in\mathbb{N}}$ on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with a distribution function F. Let $(\bigvee_{i=1}^n X_i)_{n\in\mathbb{N}}$ denote a sequence of partial maxima, defined as

$$\left(\bigvee_{i=1}^n X_i\right)(\omega) = \max(X_1(\omega), X_2(\omega), \ldots, X_n(\omega)), \quad \omega \in \Omega.$$

Let x_F denote the right endpoint of the distribution function F, defined as $x_F = \sup\{x : F(x) < 1\}$. Note that x_F may take the value ∞. To establish convergence in distribution of a sequence of partial maxima, it is straightforward that

$$\mathbf{P}\left\{\bigvee_{i=1}^n X_i \leq x\right\} = \mathbf{P}\left\{X \leq x\right\}^n \to \begin{cases} 1, & x \geq x_F, \\ 0, & \text{otherwise.} \end{cases}$$

Due to this simple convergence, it is of greater interest to centre and scale the sequence of partial maxima and inquire whether, after appropriate normalisation, the sequence of partial maxima converges in distribution to some non-degenerate distribution.

Definition 1.1.1. Random variable X (or its distribution function F) belongs to the maximum domain of attraction of distribution G ($X \in \mathrm{MDA}(G)$) if there exists a sequence of positive constants $(a_n)_{n\in\mathbb{N}}$ and a sequence of real constants $(b_n)_{n\in\mathbb{N}}$ such that

$$\mathbf{P}\left\{\frac{\bigvee_{i=1}^n X_i - b_n}{a_n} \leq x\right\} \to G(x) \quad \text{as } n \to \infty \tag{1.1}$$

for all $x \in \mathbb{R}$ continuity points of G, where X_i, $i = 1, 2, \ldots$, are independent copies of X.

Sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are called *normalising sequences*. By saying that X belongs to $\mathrm{MDA}(G)$ with normalising sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, we assume that (1.1) holds with the first sequence being the denominator and the second sequence being the centring sequence. In 1928, Fisher and Tippet in [24] recognised that G belongs to the class of distributions called *extreme value distributions* (see also [27] and [16]). This is a family of distributions whose distribution function is of the form $G_\gamma(ax+b)$, where $a>0$, $b\in\mathbb{R}$, and $\gamma\in\mathbb{R}$, Here, $\{G_\gamma,\gamma\in\mathbb{R}\}$ is a parametric family of distribution functions of the following type:

$$G_\gamma(x)=\exp\{-(1+\gamma x)^{-1/\gamma}\},\quad 1+\gamma x>0.$$

For $\gamma=0$, G_γ is interpreted as $\exp\{-e^{-x}\}$. In other words, after proper centring and scaling, G belongs to one of the three types of distributions: Fréchet (Φ_α), Weibull (Ψ_α) and Gumbel (Λ). The distribution functions of these distributions (see [48, Proposition 0.3]) are given by

$$\begin{aligned}
\Phi_\alpha(x)&=\begin{cases}0, & x<0,\\ \exp\{-x^{-\alpha}\}, & x\geq 0,\end{cases}\quad \text{for some } \alpha>0,\\
\Psi_\alpha(x)&=\begin{cases}\exp\{-(-x)^{\alpha}\}, & x<0,\\ 1, & x\geq 0,\end{cases}\quad \text{for some } \alpha>0,\\
\Lambda(x)&=\exp\{-e^{-x}\},\quad x\in\mathbb{R}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\Phi_\alpha(x)&=G_\gamma((x-1)/\gamma),\qquad \text{for } \alpha=1/\gamma>0,\\
\Psi_\alpha(x)&=G_\gamma(-(1+x)/\gamma),\qquad \text{for } \alpha=-1/\gamma>0,\\
\Lambda(x)&=G_0(x).
\end{aligned}$$

The family of extreme value distributions coincides with the family of *max-stable* distributions, which are distributions that belong to their own maximum domain of attraction. There exists a functional relation between these distribution. The following equivalences hold (see [41, Remark 1.4.14])

$$X\sim\Phi_\alpha \text{ (Fréchet)} \quad\Longleftrightarrow\quad -X^{-1}\sim\Psi_\alpha \text{ (Weibull)} \quad\Longleftrightarrow\quad \log X^\alpha\sim\Lambda \text{ (Gumbel)}.$$

Example 1.1.2. Examples of well-known distributions that belong to the maximum domain of attraction of Fréchet distribution include Pareto, Cauchy, t, and F distributions. The maximum domain of attraction of Weibull distribution, which consists only of distributions with a finite right endpoint, includes Beta and uniform distributions. Exponential, Gamma, normal, and log-normal distributions are all examples of distributions that belong to the maximum domain of attraction of Gumbel distribution.

For characterisations of maximum domain of attraction of Fréchet, Weibull and Gumbel distribution, we refer to [17] and [48]. We state some of their main results in the following theorem, which characterises each of the maximum domains of attraction.

Theorem 1.1.3. *[17, Theorem 1.2.1] The distribution function F is in the maximum domain of attraction of the extreme value distribution G_γ if and only if* *(i) for $\gamma > 0$: $x_F = \infty$ and*

$$\frac{1 - F(tx)}{1 - F(t)} \to x^{-\frac{1}{\gamma}} \qquad \textit{as } t \to \infty \tag{1.2}$$

for all $x > 0$;

(ii) for $\gamma < 0$: $x_F < \infty$ and

$$\frac{1 - F(x_F - tx)}{1 - F(x_F - t)} \to x^{-\frac{1}{\gamma}} \qquad \textit{as } t \to 0 \tag{1.3}$$

for all $x > 0$;

(iii) for $\gamma = 0$: x_F can be finite or infinite and

$$\frac{1 - F(t + xf(t))}{1 - F(t)} \to e^{-x} \qquad \textit{as } t \to x_F \tag{1.4}$$

for all real x, where f is a suitable positive function. If (1.4) holds for some f, then $\int_t^{x_F}(1 - F(s))ds < \infty$ for $t < x_F$ and (1.4) holds with

$$f(t) = \frac{\int_t^{x_F}(1 - F(s))ds}{1 - F(t)}.$$

The authors of [17] state the explicit form of sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ (see also [41]) which we mention below. The condition (1.2) is closely related to the definition of a regularly varying function. A positive measurable function $g : (a, \infty) \to (0, \infty)$, for some $a \in \mathbb{R}$, is said to be regularly varying at infinity with index $\alpha \in \mathbb{R}$ if

$$\frac{g(tx)}{g(t)} \to x^{\alpha} \qquad \text{as } t \to \infty$$

for all $x > 0$. The family of all functions regularly varying at infinity with index α is denoted by RV_α. A function is said to be regularly varying at zero by letting $t \downarrow 0$ instead of $t \to \infty$. A function is said to be regularly varying at an arbitrary point $x_0 \in \mathbb{R}$ if $g(x_0 - x^{-1})$ is a regularly varying function at infinity. If we denote the tail distribution function $1 - F$ by $\bar{F}$, then a random variable X with distribution function F is said to be regularly varying with index $\alpha > 0$ if $\bar{F}$ is a regularly varying function at infinity with index $-\alpha$. We say that a random variable X is regularly varying at $x_0 \in \mathbb{R}$ if $\bar{F}$ is regularly varying at x_0. When we use the term "regular variation," we usually refer to regular variation at infinity. If any other case arises, we will explicitly mention it. Some authors use the term "regularly varying random variable" for variables that have regularly varying tail distribution functions in a balanced manner (see [2]), so that $|X|$ is regularly varying at infinity and that there exists $p \in [0, 1]$ such that $\mathbf{P}\{X > tx\} / \mathbf{P}\{|X| > t\} \to px^{-\alpha}$ as $t \to \infty$, and $\mathbf{P}\{X < -tx\} / \mathbf{P}\{|X| > t\} \to (1 - p)x^{-\alpha}$ as $t \to \infty$ for every $x > 0$.

Thus, the first statement of Theorem 1.1.3 says that Fréchet domain of attraction coincides with the family of regularly varying random variables. Normalising sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ from (1.1) are given by

$$a_n = F^{\leftarrow}(1-1/n) = (1/\bar{F})^{\leftarrow}(n), \quad b_n = 0,$$

where $F^{\leftarrow}$ is a quantile function $F^{\leftarrow}(a) = \inf\{x : F(x) \geq a\}$. The second statement of Theorem 1.1.3 states that a random variable X with right endpoint x_F belongs to the Weibull domain of attraction if and only if $x_F - X$ has a distribution function that is regularly varying at zero. Equivalently, $(x_F - X)^{-1}$ is regularly varying with index $-1/\gamma$, which is further equivalent to X being regularly varying at x_F. These statements follow from (1.3) and:

$$x^{-1/\gamma} = \lim_{t\downarrow 0} \frac{1-F(x_F - tx)}{1-F(x_F - t)} = \lim_{t\downarrow 0} \frac{\mathbf{P}\{x_F - X < tx\}}{\mathbf{P}\{x_F - X < t\}} = \lim_{t\to\infty} \frac{\mathbf{P}\{(x_F - X)^{-1} > ty\}}{\mathbf{P}\{(x_F - X)^{-1} > t\}} =$$
$$= \lim_{t\to\infty} \frac{\mathbf{P}\{X > x_F - (ty)^{-1}\}}{\mathbf{P}\{X > x_F - t^{-1}\}} = y^{1/\gamma},$$

where $y = 1/x > 0$ and $\gamma < 0$. Since the MDA of the Weibull distribution consists only of distributions F such that $x_F < \infty$ (as well as some of the distributions from the Gumbel domain of attraction), they are considered to be "light-tailed" and hence are often not considered when dealing with maximal value since $\bar{F}(L) = 0$ for $L \geq x_F$. Therefore, distributions whose right endpoint is finite are often omitted in the analysis of the maximum. The normalising sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ from (1.1) in the case of the Weibull domain of attraction are given by

$$a_n = x_F - F^{\leftarrow}(1-1/n), \quad b_n = x_F.$$

As already mentioned, the Gumbel domain of attraction consists of distributions whose right endpoint is finite and distributions whose right endpoint is infinite. For the function f from (1.4), sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ that satisfy (1.1) are given by

$$a_n = f(F^{\leftarrow}(1-1/n)), \quad b_n = F^{\leftarrow}(1-1/n).$$

1.2 Preliminaries on regular variation in $\mathbb{R}^d$, $d \geq 1$, and star-shaped metric spaces

For any real function f, the set $\{x : f(x) \neq 0\}$ is called the *support of* f, and throughout, it will be denoted by $\operatorname{supp} f$. The condition (1.1) is equivalent to the following: for every $x \in \operatorname{supp} G$,

$$n\mathbf{P}\{X > a_n x + b_n\} \to -\log G(x) \quad \text{as } n \to \infty, \tag{1.5}$$

see [48, page 68]. Let $\mathbb{X} = \operatorname{supp} G$. Note that in the Fréchet case, $\mathbb{X}$ is equal to $\mathbb{R}_+$ where $\mathbb{R}_+ = [0,\infty)$, while in the Weibull and Gumbel cases, $\mathbb{X} = \mathbb{R}$. In the first case, we say that a set $B_1 \subseteq \mathbb{X}$ is *bounded* if it is a subset of (a, ∞) for some $a > 0$, which is often said to be *bounded away from zero*. To be precise, this terminology refers to all sets that are contained in the complement of a ball centred at the origin. In other words, the closure of sets does not intersect the origin. Note that in this case, the origin is the single point 0, and a ball is considered with respect to the usual metric in $\mathbb{R}_+$, $||a|| = a$. One may wonder if different metrics and different origins give different notions of boundedness away from zero. This is indeed the case, and more details can be found in Section 1.3. In the second case, a set $B_2 \subseteq \mathbb{X}$ is *bounded* if it is a subset of (a, ∞) for some $a \in \mathbb{R}$. Therefore, the notion of bounded sets in $\mathbb{X}$ is closely connected to the choice of extreme value distribution G.

We denote by $\mathcal{S}(\mathbb{X})$ the family of all bounded subsets of $\mathbb{X}$, by $\mathcal{B}(\mathbb{X})$ the Borel σ-algebra on $\mathbb{X}$, and by $C_{\mathcal{S}}$ the collection of all bounded continuous functions $f : \mathbb{X} \to \mathbb{R}$ with bounded support, as previously defined. A measure $\eta : \mathcal{B}(\mathbb{X}) \to [0, \infty]$ is said to be *boundedly finite* if $\eta(B) < \infty$ for every measurable and bounded B. The spaces of boundedly finite measures and boundedly finite point measures on $\mathbb{X}$ are denoted by $\mathcal{M}(\mathbb{X})$ and $\mathcal{M}_p(\mathbb{X})$, respectively.

A sequence of measures $(\eta_n)_{n \in \mathbb{N}}$, $\eta_n \in \mathcal{M}(\mathbb{X})$ (or $\eta_n \in \mathcal{M}_p(\mathbb{X})$), converges *vaguely* to $\eta \in \mathcal{M}(\mathbb{X})$ (or $\eta \in \mathcal{M}_p(\mathbb{X})$) if $\eta_n(f) = \int_{\mathbb{X}} f d\eta_n \to \int_{\mathbb{X}} f d\eta = \eta(f)$, for every $f \in C_{\mathcal{S}}$. Vague convergence of this type will be denoted by $\eta_n \xrightarrow{v} \eta$ as $n \to \infty$.

Note that a Borel σ-algebra on $\mathcal{M}(\mathbb{X})$ generated by the vague topology coincides with the smallest σ-algebra on $\mathcal{M}(\mathbb{X})$ making the maps $\eta \mapsto \eta(f)$, from $\mathcal{M}(\mathbb{X})$ to $\mathbb{R}$, measurable for all $f \in C_{\mathcal{S}}$. Both $\mathcal{M}(\mathbb{X})$ and $\mathcal{M}_p(\mathbb{X})$ with their respective vague topologies are Polish (see [48, Chapter 3]).

Using a variant of the Portmanteau theorem for vague convergence [48, Proposition 3.12], one can conclude that the vague convergence of measures η_n towards η in $\mathcal{M}(\mathbb{X})$ can be characterised using the following condition

$$\eta_n(B) \to \eta(B), \quad B \in \mathcal{S}(\mathbb{X}) \cap \mathcal{B}(\mathbb{X}) \text{ such that } \eta(\partial B) = 0,$$

where ∂B denotes the boundary of the set B. It suffices to observe sets (x, ∞), for $x > 0$ or $x \in \mathbb{R}$, depending on $\mathbb{X}$ as before, see [48]. Thus, the condition (1.5) is equivalent to the condition

$$n\mathbf{P}\left\{\frac{X - b_n}{a_n} \in \cdot\,\right\} \xrightarrow{v} \mu_G(\,\cdot\,) \quad \text{as } n \to \infty, \tag{1.6}$$

where μ_G denotes the measure induced by the function $-\log G(x)$, that is, $\mu_G(x, \infty) = -\log G(x)$. Note that $\mu_G(\{x\}) = 0$. Moreover, [17, Theorem 1.1.2]) shows that instead of parametrising with $n \in \mathbb{N}$, one can use $t > 0$ and functions $a(t) = a_{\lfloor t \rfloor}$, $b(t) = b_{\lfloor t \rfloor}$, so that the analogous convergence holds as $t \to \infty$.

The definition of a regularly varying random variable can also be introduced using vague convergence. A random variable X is said to be regularly varying with index $\alpha > 0$ if there exist $g \in \mathrm{RV}_\alpha$ and a nontrivial measure $\mu \in \mathcal{M}(\mathbb{X})$ such that

$$g(t)\mathbf{P}\left\{t^{-1}X \in \cdot\,\right\} \xrightarrow{v} \mu(\,\cdot\,) \quad \text{as } t \to \infty. \tag{1.7}$$

It is not difficult to see that these two definitions are equivalent. First, assume that X is regularly varying in the sense that its tail distribution function is regularly varying. Then, it suffices to set $g(t) = \mathbf{P}\{X > t\}^{-1}$, which is in RV_α. In that case $\mu(x,\infty) = x^{-\alpha}$. Now assume that X is regularly varying in the sense of (1.7) and consider the ratio of $g(t)\mathbf{P}\{X > xt\}$ and $g(t)\mathbf{P}\{X > t\}$ for some $x \in \mathbb{R}$. This ratio equals the ratio of the tail distribution functions and converges towards $\mu(x,\infty)/\mu(1,\infty)$. Since the ratio of $g(xt)\mathbf{P}\{X > xt\}$ and the same term with $x = 1$ converges to 1 and g is regularly varying, these two facts imply that the tail distribution of X is regularly varying. Note that under (1.7) μ is α-homogeneous in the sense that

$$\mu(tA) = t^{-\alpha}\mu(A), \quad A \in \mathcal{B}(\mathbb{X}),\ t > 0.$$

Definition of regular variation in more general spaces again uses vague convergence of measures: for example in $\mathbb{R}^d$, see [47], in the space of non-empty compact subsets of a separable Banach space with origin excluded, see [44] and in star-shaped metric spaces, see [54]. We follow the notation as before for the set $\mathbb{X} = (\mathbb{R} \cup \{\pm\infty\})^d$ where set $A \subseteq \mathbb{X}$ is considered to be bounded if it is bounded away from zero or, in other words, its closure does not intersect the origin. Vague convergence $\nu_n \to \nu$ is then defined in the same way, as convergence of $\nu_n(f)$ to $\nu(f)$ for every bounded continuous real function f whose support is bounded. We say that a random vector X is regularly varying in $\mathbb{X}$ (see [2]) with a tail measure $\mu \in \mathcal{M}(\mathbb{X} \setminus \{\mathbf{0}\})$ if there exists a measurable and bounded set C such that $\mu(\partial C) = 0$, $tC \in \mathcal{S}(\mathbb{X}) \cap \mathcal{B}(\mathbb{X})$, $t \in T$, where T is dense in $(0,\infty)$, and the following holds:

$$\frac{\mathbf{P}\{X \in t\,\cdot\,\}}{\mathbf{P}\{X \in tC\}} \xrightarrow{v} \mu(\,\cdot\,) \quad \text{as } t \to \infty.$$

Note that this can easily be transformed into relation of the type (1.7). It is often practical to analyse joint distribution

$$\left(\frac{X}{\|X\|}, \|X\|\right)$$

for an arbitrary norm $\|\cdot\|$ in $\mathbb{X}$. The mapping $x \mapsto (x/\|x\|, \|x\|)$ from $\mathbb{X}\setminus\{\mathbf{0}\}$ to $\mathbb{S}^{d-1}\times(0,\infty)$, where $\mathbb{S}^{d-1} = \{x \in \mathbb{X} : \|x\| = 1\}$, is called *the polar decomposition* of $x \in \mathbb{X} \setminus \{\mathbf{0}\}$. This mapping is a continuous bijection, and the inverse mapping is also continuous. The polar decomposition is one of the ways to characterise regular variation. A random vector X is regularly varying in $\mathbb{X}$ if and only if there exists a random vector $\theta \in \mathbb{S}^{d-1}$ such that for some $\alpha \geq 0$ we have

$$\frac{\mathbf{P}\{\|X\| > tu, X/\|X\| \in \cdot\,\}}{\mathbf{P}\{\|X\| > t\}} \xrightarrow{v} u^{-\alpha}\,\mathbf{P}\{\theta \in \cdot\,\} \quad \text{as } t \to \infty$$

for all $u > 0$, where the vague convergence is considered with respect to the Borel σ-algebra on $\mathbb{S}^{d-1}$. Note that for $u = 1$, the term on the right-hand side equals to the distribution of $X/\|X\|$ conditioned on $\|X\| > t$. The measure $\mathbf{P}\{\theta \in \cdot\}$ is called *the spectral measure* of X. Moreover, if X is a regularly varying random vector, then $\|X\|$ is a regularly varying random variable. This holds because $\mathbf{P}\left\{\|X\| > ut, X/\|X\| \in \mathbb{S}^{d-1}\right\}/\mathbf{P}\{\|X\| > t\}$ converges to $u^{-\alpha}$ as $t \to \infty$ for every $u > 0$.

Now let $\mathbb{X}$ be a general complete and separable metric space with origin $\mathbf{0}$, and completely metrisable with metric d. For such a space, the authors of [54] consider a more general polar decomposition which uses the so-called *modulus* instead of a norm. In the following, we briefly present the results from [54]. To start with, we define scaling (called in [54] the *scalar multiplication*) as a map $[0,\infty)\times\mathbb{X}\to\mathbb{X}$ that maps (t,x) to $T_t x$ continuously in the product topology, and has the following properties:

(i) $T_t T_s x = T_{ts} x$ for all $t,s\geq 0$, $x\in\mathbb{X}$,

(ii) $T_1 x = x$ for all $x\in\mathbb{X}$.

If $\mathbb{X}$ is such that for every $x\in\mathbb{X}\setminus\{\mathbf{0}\}$ and $t,s\geq 0$, $t<s$ we have $\mathsf{d}(\mathbf{0},T_t x)<\mathsf{d}(\mathbf{0},T_s x)$, then $\mathbb{X}$ is said to be *star-shaped.* For the rest of this section, we assume that $\mathbb{X}$ is star-shaped. We say that a function $f:\mathbb{X}\to[0,\infty)$ is homogeneous if $f(T_t x)=tf(x)$ for all $t\geq 0$ and $x\in\mathbb{X}$. *The modulus* $\tau:\mathbb{X}\to[0,\infty)$ is defined as a continuous homogeneous function with the property that for every $\varepsilon>0$, $\inf\{\tau(x):\mathsf{d}(x,\mathbf{0})>\varepsilon\}>0$. With these notions of scaling and modulus, polar decomposition is now given by

$$\rho(x)=(T_{\tau(x)^{-1}}x,\ \tau(x)),$$

as a mapping from $\mathbb{X}\setminus\{\mathbf{0}\}$ to $\{x:\tau(x)=1\}\times(0,\infty)$. The family of bounded sets $\mathcal{S}(\mathbb{X})$ in this case is a class of all sets bounded away from zero in the sense that $B\in\mathcal{S}(\mathbb{X})$ if there exists $r>0$ such that $B\cap\{x\in\mathbb{X}:\mathsf{d}(x,\mathbf{0})<r\}=\varnothing$. In this case, we say that a sequence of boundedly finite measures $(\eta_n)_{n\in\mathbb{N}}\subseteq\mathcal{M}(\mathbb{X})$ converges vaguely to $\nu\in\mathcal{M}(\mathbb{X})$, denoted by $\nu_n\xrightarrow{v}\nu$, if $\nu_n(f)\to\nu(f)$ for all continuous bounded real functions f with $\operatorname{supp} f\in\mathcal{S}(\mathbb{X})$. On the other hand, for a sequence $(\eta_n)_{\in\mathbb{N}}$ of finite measures on $\mathbb{X}$, we define *weak convergence* to a finite measure η and denote it by $\eta_n\xrightarrow{w}\eta$ if $\eta_n(f)\to\eta(f)$ for all continuous bounded real functions f. The weak convergence arises if the bornology $\mathcal{S}(\mathbb{X})$ is trivial, that is, it contains all subsets of $\mathbb{X}$. Similarly to the previous definitions, a random element X in $\mathbb{X}$ is said to be regularly varying if there exists $g\in\mathrm{RV}_\alpha$ and a nontrivial boundedly finite measure μ such that

$$g(t)\mathbf{P}\left\{T_{t^{-1}}X\in\cdot\,\right\}\xrightarrow{v}\mu(\,\cdot\,)\qquad\text{as } t\to\infty.$$

Let θ_α denote the measure on $(0,\infty)$ with density $\alpha y^{-\alpha-1}dy$. We recall a result on polar decomposition from [54, Proposition 3.1]:

Proposition 1.2.1. *Let X be a random element in $\mathbb{X}$ and let $\alpha>0$. Assume that a modulus $\tau:\mathbb{X}\to[0,\infty)$ exists. The following properties are equivalent:*

(i) X is regularly varying with index $\alpha>0$.

(ii) The function $t\mapsto\mathbf{P}\{\tau(X)>t\}$ is in $\mathrm{RV}_{-\alpha}$ and there exists a probability measure σ on $\{x:\tau(x)=1\}$ such that

$$\mathbf{P}\left\{T_{\tau(X)^{-1}}X\in\cdot\mid\tau(X)>t\right\}\xrightarrow{w}\sigma(\,\cdot\,)\qquad\text{as } t\to\infty.$$

(iii) There exists a probability measure σ on $\{x:\tau(x)=1\}$ such that for all $u>0$

$$\mathbf{P}\left\{\left(t^{-1}\tau(X),T_{\tau(X)^{-1}}X\right)\in(u,\infty)\times\cdot\mid\tau(X)>t\right\}\xrightarrow{w}(\theta_\alpha\otimes\sigma)((u,\infty)\times\,\cdot\,)\qquad\text{as } t\to\infty.$$

In that case, we have

$$\frac{\mathbf{P}\{T_{t^{-1}}X \in \cdot\}}{\mathbf{P}\{\tau(X) > t\}} \xrightarrow{v} \mu(\,\cdot\,) \qquad \textit{as } t \to \infty,$$

where μ is determined by

$$\mu \circ \rho^{-1}(dr, d\phi) = \alpha r^{-\alpha-1} dr \sigma(d\phi), \quad (r, \phi) \in (0, \infty) \times \{x : \tau(x) = 1\}.$$

In a similar way, regular variation can be defined on some subsets of $\mathbb{R}^d$ or $\mathbb{R}^\infty$. In such cases, sets are considered bounded if they are bounded away from some closed set, such as the origin $\mathbf{0}$ or some or all of the axes. However, one may encounter the phenomenon of *hidden regular variation* when observing different subsets A and B, where $B \subseteq A$. Hidden regular variation is discussed in detail in [40], [46], and [50]. Briefly, if a random element X is regularly varying in A, then it is not necessarily regularly varying in B. If it is regularly varying in B, then the index of regular variation may differ, but it cannot be smaller, meaning the tail cannot be heavier. This phenomenon is often referred to as hidden regular variation and is observed in more detail in several places throughout, including Section 2.3 and Example 2.4.8.

1.3 Scaling, bornology and modulus

In this section we introduce general scaling, a family of bounded sets (the bornology), and a homogeneous positive Borel function (the modulus). The theory presented in this section is a result of a joint work with Bojan Basrak and Ilya Molchanov.

Let $\mathbb{X}$ be a topological space equipped with its Borel σ-algebra $\mathcal{B}(\mathbb{X})$. Assume that $\mathbb{X}$ is equipped with a group action by the multiplicative group $(0, \infty)$, that is, with a family of $\mathcal{B}(\mathbb{X}) \otimes \mathcal{B}((0, \infty))/\mathcal{B}(\mathbb{X})$-measurable maps $(x, t) \mapsto T_t x$ from $\mathbb{X} \times (0, \infty)$ to $\mathbb{X}$ which satisfy the following conditions:

(i) for all $x \in \mathbb{X}$ and $t, s > 0$, $T_t(T_s x) = T_{ts} x$,

(ii) for all $x \in \mathbb{X}$, $T_1 x = x$.

The action of T_t is called *scaling* by t, and the space $\mathbb{X}$ equipped with a scaling is called a *measurable cone*, see [13]. If T_t is a scaling, then T_{t^β} also defines a scaling for all $\beta \neq 0$. In particular, $T_{t^{-1}}$ is called the inverted scaling, as $T_{t^{-1}} = T_t^{-1}$.

For $I \subseteq (0, \infty)$, $x \in \mathbb{X}$, and $A \subseteq \mathbb{X}$, we denote $T_I x = \{T_t x : t \in I\}$, $T_t A = \{T_t x : x \in A\}$ and $T_I A = \{T_t x : t \in I, x \in A\}$.

Definition 1.3.1. A *cone* in $\mathbb{X}$ is a set $B \subseteq \mathbb{X}$ such that $T_t B = B$ for all $t > 0$. A *semicone* $C \subseteq \mathbb{X}$ is a set such that $T_t C \subseteq C$ for all $t \geq 1$, equivalently, $T_{[1,\infty)} C = C$.

We additionally impose the continuity property on the scaling:

(iii) $(x,t) \mapsto T_t x$ is continuous with respect to the product topology on $\mathbb{X} \times (0,\infty)$.

The closure, interior and the boundary of $B \subseteq \mathbb{X}$, we denote by $\mathrm{cl}_{\mathbb{X}} B$, $\mathrm{Int}_{\mathbb{X}} B$ and $\partial_{\mathbb{X}} B$.

Lemma 1.3.2. *If the scaling is continuous, then it is an open and closed map, and* $\mathrm{cl}_{\mathbb{X}}(T_t B) = T_t \,\mathrm{cl}_{\mathbb{X}} B$, $\mathrm{Int}_{\mathbb{X}}(T_t B) = T_t \,\mathrm{Int}_{\mathbb{X}} B$ *for all* $B \subseteq \mathbb{X}$ *and* $t > 0$.

Proof. If A is open in $\mathbb{X}$, then $T_t A = T_{t^{-1}}^{-1} A$ is also open, as the inverse image of A under continuous map $T_{t^{-1}}$. The same argument shows that $T_t A$ is closed if A is closed.

The set $T_t \,\mathrm{Int}_{\mathbb{X}} B$ is an open subset of $T_t B$, so $T_t \,\mathrm{Int}_{\mathbb{X}} B \subseteq \mathrm{Int}_{\mathbb{X}}(T_t B)$. In the other direction, if $x \in \mathrm{Int}_{\mathbb{X}} T_t B$, then an open neighbourhood U of x is a subset of $T_t B$. Then $T_{t^{-1}} U \subseteq B$ is an open neighbourhood of $T_{t^{-1}} x \in B$, so that $T_{t^{-1}} x \in \mathrm{Int}_{\mathbb{X}} B$ and $x \in T_t \,\mathrm{Int}_{\mathbb{X}} B$. Thus, $T_t \,\mathrm{Int}_{\mathbb{X}} B = \mathrm{Int}_{\mathbb{X}}(T_t B)$.

For the closure, $T_t \,\mathrm{cl}_{\mathbb{X}} B$ is a closed set, which contains $T_t B$. Then $T_t \,\mathrm{cl}_{\mathbb{X}} B \supseteq \mathrm{cl}_{\mathbb{X}}(T_t B)$. Assume that $x \in T_t \,\mathrm{cl}_{\mathbb{X}} B$. Consider an open neighbourhood of U of x. Then U intersects $T_t \,\mathrm{cl}_{\mathbb{X}} B$, so that $T_{t^{-1}} U$ intersects B, and U intersects $T_t B$, meaning that $x \in \mathrm{cl}_{\mathbb{X}}(T_t B)$. Therefore, $T_t \,\mathrm{cl}_{\mathbb{X}} B = \mathrm{cl}_{\mathbb{X}}(T_t B)$. □

The following is our standard assumption in the sequel

(A) $\mathbb{X}$ is a Polish space equipped with a continuous scaling.

If there exists an element $x \in \mathbb{X}$ that is invariant under T_t (i.e., $T_t x = x$) for at least one $t > 0$, where $t \neq 1$, then x is called scaling invariant or a zero, and the set of these elements is denoted by $\mathbf{0}$. In many cases, an element is either invariant under T_t for all t or not scaling invariant. Remark 3.6.3 describes the situation when there are elements that are only invariant for some $t > 0$.

Lemma 1.3.3. *Assume that (A) holds. Then* $\mathbf{0}$ *is an* F_σ *set (that is, a countable union of closed sets). If* $T_t x = x$ *for at least one* $t \neq 1$ *implies that* $T_t x = x$ *for all* $t > 0$*, then* $\mathbf{0}$ *is a closed set.*

Proof. For $\varepsilon \in (0, 1/2)$, let A_ε be the set of $x \in \mathbf{0}$ such that $T_t x = x$ for $t \in [\varepsilon, 1-\varepsilon]$. Assume that $x_n \in A_\varepsilon$, $n \geq 1$, and $x_n \to x$. Then $T_{t_n} x_n = x_n$ for $t_n \in [\varepsilon, 1-\varepsilon]$. By passing to subsequences, assume that $t_n \to t \in [\varepsilon, 1-\varepsilon]$. By continuity of scaling, we have that $T_{t_n} x_n \to T_t x$, so that $T_t x = x$ and $x \in A_\varepsilon$. Finally, note that $\mathbf{0}$ is the union of $A_{1/n}$ for $n \geq 2$. For the second statement, let (x_n) be a sequence of zero elements that converges to some $x \in \mathbb{X}$. Then for all $t > 0$, we have $x_n = T_t x_n \to T_t x$, so that $T_t x = x$ for all $t > 0$. □

If d is a complete metric on $\mathbb{X}$, $\mathbf{0} = \{0\}$ is a singleton and if

$$\mathsf{d}(0, T_t x) < \mathsf{d}(0, T_s x)$$

for all $t < s$ and $x \neq 0$, then $\mathbb{X}$ with the imposed scaling is a *star-shaped* metric space, see [54].

An *ideal* $\mathcal{S}$ is a family of subsets of $\mathbb{X}$ closed under finite unions and such that, if $B \in \mathcal{S}$, the family $\mathcal{S}$ contains all subsets of B, see [25]. The ideal $\mathcal{S}$ is also called a *boundedness*, see [31, Section V.5]. Note that an ideal may contain also non-measurable sets as well. If $\mathcal{S}$ contains a countable subfamily $\{B_n, n \in \mathbb{N}\}$ cofinal in $\mathcal{S}$ (meaning that each element of $\mathcal{S}$ is a subset of a certain member of this subfamily), then $\mathcal{S}$ is said to have a *countable base* or a *countable open base* if the sets B_n are open. Without loss of generality, we can assume that $B_n \subseteq B_{n+1}$ for all n.

An ideal $\mathcal{S}$ is called *scaling consistent* if $B \in \mathcal{S}$ implies $T_t B \in \mathcal{S}$ for all $t > 0$. If $\mathcal{S}$ is scaling consistent, then the union of its sets $\mathbb{U}(\mathcal{S}) = \cup\{B : B \in \mathcal{S}\}$ is a cone in $\mathbb{X}$. Further on, we write $\operatorname{cl} B$, $\operatorname{Int} B$ and ∂B for the closure, interior and the boundary of $B \in \mathcal{S}$, with respect to $\mathbb{U}(\mathcal{S})$ and the corresponding subspace topology. An ideal is *topologically consistent* if $B \in \mathcal{S}$ implies $\operatorname{cl} B \in \mathcal{S}$.

If an ideal $\mathcal{S}$ consists of subsets of $\mathbb{X}$ and covers $\mathbb{X}$, it is called a *bornology* on $\mathbb{X}$, see [30]. The sets from $\mathcal{S}$ are said to be *bounded*, and so a bornology is also called a boundedness, see [38] and [6]. Each ideal is a bornology on its union $\mathbb{U}(\mathcal{S})$. In the following we often deal with bornologies on the space $\mathbb{X}'$, which consist of all but zero elements of $\mathbb{X}$, that is,

$$\mathbb{X}' = \mathbb{X} \setminus \mathbf{0}.$$

Remark 1.3.4. A *countable base* of a bornology on $\mathbb{X}'$ is called a *localising sequence*, see, e.g., [36, Sec. 1.1]. If a topologically consistent bornology contains a countable open base $(B_n)_{n\in\mathbb{N}}$, then without loss of generality we can assume that the closure of B_n is a subset of B_{n+1}. In this case, the authors of [6] say that $\mathcal{S}$ admits a *properly localising sequence*, and the space $\mathbb{X}'$ is *localised*. It is easy to see that the localisation condition (the existence of such a properly localising sequence) implies that the bornology is topologically consistent since for each bounded B, there exists $n \in \mathbb{N}$ such that $B \subseteq B_n \subseteq \operatorname{cl} B_n \subseteq B_{n+1}$, which means that $\operatorname{cl} B$ is contained in B_{n+1} and thus is bounded.

While many subsequent results hold for all scaling and topologically consistent ideals with a countable base, we often impose the following stronger condition on an ideal $\mathcal{S}$.

(B) There exists a semicone $C \subseteq \mathbb{X}$ such that the sequence $(T_{n^{-1}}C)_{n\in\mathbb{N}}$ is a base of $\mathcal{S}$, $\cap_{t>1} T_t C = \varnothing$, and

$$\operatorname{cl}(T_t C) \subseteq \operatorname{Int}(T_s C), \quad 0 < s < t. \tag{1.8}$$

Then we say that the ideal $\mathcal{S}$ is generated by C.

Condition (B) implies that $\mathcal{S}$ does not contain any point $x \in \mathbf{0}$, so that all members of $\mathcal{S}$ are subsets of $\mathbb{X}'$. Indeed, otherwise, $x \in T_{n^{-1}}C$ for some n, and $T_t x = x$ for some $t > 0$, $t \neq 1$. If $t \geq 1$, find $k \in \mathbb{N}$ large enough such that $t^k n^{-1} \geq 1$. Then $x = T_{t^k} x \in T_{t^k n^{-1}} C \subseteq C$. If $t < 1$, find $l \in \mathbb{N}$ large enough so that $t^{-l} n^{-1} > 1$. Then $x = T_{t^{-l}} x \in T_{t^{-l} n^{-1}} C \subseteq C$, so that $x \in C$. Hence, $x \in \cap_{t>1} T_t C$, which is a contradiction.

It is easily seen that condition (1.8) is equivalent to the fact that the boundaries of $T_t C$ and $T_s C$ are disjoint for $t \neq s$. First assume that (1.8) holds. Then $\partial(T_t C) \subseteq \operatorname{cl}(T_t C) \subseteq \operatorname{Int}(T_s C)$, $0 < s < t$, so $\partial(T_t C)$ and $\partial(T_s C)$ are disjoint. Now assume that the boundaries of T_t and T_s are disjoint for $t \neq s$. Since $T_s C$ is also a semicone for each $s > 0$, then for $t > s$ we have $T_t C = T_{\frac{t}{s}} T_s C \subseteq T_s C$ which implies that $\operatorname{cl} T_t \subseteq \operatorname{cl} T_s C$. Disjoint boundaries imply that $\operatorname{cl}(T_t C) \subseteq \operatorname{Int}(T_s C)$.

Lemma 1.3.5. *If (B) holds and the scaling on a topological space $\mathbb{X}$ is continuous, then the semicone C in* (1.8) *can be chosen to be open or closed in* $\mathbb{U}(\mathcal{S})$.

Proof. Note that the closure and interior of a semicone are semicones. If (B) holds, then the sequence $(T_{n^{-1}} \operatorname{Int} C)$ is localising, since $T_{n^{-1}} \operatorname{Int} C$ contains $T_{(n-1)^{-1}} C$, and

$$\operatorname{cl}(T_t \operatorname{Int} C) \subseteq \operatorname{cl}(T_t C) \subseteq \operatorname{Int}(T_s C) = T_s \operatorname{Int} C.$$

The latter equality holds by Lemma 1.3.2. The sequence $(T_{n^{-1}} \operatorname{cl} C)$ is also localising. By Lemma 1.3.2,

$$\operatorname{cl}(T_t \operatorname{cl} C) = T_t \operatorname{cl} C = \operatorname{cl}(T_t C) \subseteq \operatorname{cl}(\operatorname{Int}(T_s C)) = T_s(\operatorname{cl} \operatorname{Int} C) = T_s \operatorname{cl} C. \qquad \square$$

Thus, condition (B) implies that $\mathcal{S}$ admits a countable open base, is scaling and topologically consistent. If the scaling is not necessarily continuous, then the semicone C in (B) is required to be open, see [13, Item B4)].

Remark 1.3.6. In [38, Appendix B], the following conditions are imposed on the bornology:

(B1) For each $B \in \mathcal{S}$ and $t > 0$, we have $T_{[t,\infty)} B \in \mathcal{S}$.

(B2) There exists an open semicone $C \in \mathcal{S}$ such that $\cap_{t>1} T_t C = \varnothing$ and $\cup_{t \le 1} T_t C = \mathbb{X}'$. Furthermore, $\operatorname{cl}(T_s C) \subseteq T_t C$ for all $s > t$, and the sequence $(T_{n^{-1}} C)_{n \in \mathbb{N}}$ is localising.

(B3) Each measure μ on $\mathbb{X}'$, which takes finite values on $\mathcal{S}$, is uniquely determined by its values on bounded semicones.

The original formulation of (B2) also requires that, for each $x \in \mathbb{X}'$ and $t > 0$, there exists an $s_0 > 0$ such that $T_s x \notin T_t C$ for all $s \le s_0$. However, this assumption follows from other conditions, since $x \notin T_{[r,\infty)} C$ for some $r > 0$, so that $T_{(0,r^{-1}]} x \cap C = \varnothing$, and $T_{sr^{-1}t} x \notin T_t C$ for all $s \in (0,1)$. Lemma 1.3.9 below shows that there is also redundancy in (B1)–(B3).

A function $f : \mathbb{X}' \to \mathbb{R}$ is said to be *homogeneous* if

$$f(T_t x) = t f(x), \quad t > 0,\ x \in \mathbb{X}'.$$

Note that the domain of definition of f does not include scaling invariant elements for at least one $t > 0$, $t \neq 1$.

Definition 1.3.7. Let $\mathbb{U}$ be a cone in $\mathbb{X}'$. A *modulus* on $\mathbb{U}$ is a homogeneous Borel function $\tau : \mathbb{U} \to (0, \infty)$. For $A \subseteq \mathbb{U}$, denote

$$\hat{\tau}(A) = \inf\{\tau(x) : x \in A\}, \tag{1.9}$$

and let $\mathcal{S}_\tau$ be the family of sets $A \subseteq \mathbb{U}$ such that $\hat{\tau}(A) > 0$.

In the following we mostly assume that a modulus is defined on the whole $\mathbb{X}'$, so that $\mathbb{U} = \mathbb{X}'$. Note that $\mathcal{S}_\tau$ is the family of all sets A such that $A \subseteq \{x : \tau(x) \geq \varepsilon\}$ for some $\varepsilon > 0$. A modulus τ is said to be *compatible* with an ideal $\mathcal{S}$ if $\hat{\tau}(A) > 0$ for all $A \in \mathcal{S}$. In particular, τ is compatible with $\mathcal{S}_\tau$, which is said to be a bornology generated by τ. The set defined by

$$\mathbb{S} = \{x : \tau(x) = 1\} \tag{1.10}$$

is called a *sphere* or a transversal in $\mathbb{X}'$. It is possible to extend a modulus to a bijection as follows.

Definition 1.3.8. Let τ be a modulus on $\mathbb{X}'$. The *polar decomposition* is the map $\rho : \mathbb{X}' \to \mathbb{S} \times (0, \infty)$ given by

$$\rho(x) = (T_{\tau(x)^{-1}} x, \tau(x)).$$

The polar decomposition is a measurable bijection between $\mathbb{X}'$ and $\mathbb{S} \times (0, \infty)$ (see [22, Lemma 3]). If τ is continuous, then the polar decomposition is also continuous, with $\mathbb{S}$ equipped with the induced topology. The inverse map $(u, t) \mapsto T_t u$, $u \in \mathbb{S}$, $t > 0$, of the polar decomposition is continuous since the scaling is continuous.

The following result is formulated and proved for a general topological space $\mathbb{X}$.

Lemma 1.3.9. *Let $\mathbb{X}$ be a topological space with continuous scaling, and let $\mathcal{S}$ be an ideal on $\mathbb{X}'$. Then $\mathcal{S}$ satisfies (B) if and only if there exists a continuous modulus τ on a cone $\mathbb{U} \subseteq \mathbb{X}'$ such that $\mathcal{S} = \{B \cap \mathbb{U} : B \in \mathcal{S}_\tau\}$. Furthermore, if $\mathcal{S}$ is a bornology on $\mathbb{X}'$, then (B) is equivalent to (B2), which, in turn, implies (B1) and (B3).*

Proof. By changing the space $\mathbb{X}'$ to the union $\mathbb{U}$ of all sets from $\mathcal{S}$, it is always possible to assume that $\mathcal{S}$ is a bornology on $\mathbb{X}'$, which is done in the proof.

Sufficiency. It is easy to see that $\mathcal{S}_\tau$ is closed under finite unions and contains all subsets of any of its member sets. Condition (B) holds with $C = \{x : \tau(x) > 1\}$. Since τ is continuous, C is an open semicone. We show that $\mathrm{cl}(T_t C) = \{x : \tau(x) \geq t\}$. Indeed, $\{x : \tau(x) \geq t\}$ is a closed set which contains $T_t C$, and on the other hand, if $\tau(x) \geq t$, then for any sequence (t_n) converging to 1 from above, we have $T_{t_n} x \to x$ as $n \to \infty$, and $T_{t_n} x \in T_t C$ for all sufficiently large n, so that x belongs to the closure of $T_t C$. The sequence $(T_{n^{-1}} C)_{n \in \mathbb{N}}$ is localising, and $\mathrm{cl}(T_t C) = \{x : \tau(x) \geq t\}$ is a subset of $T_s C$ for all $s < t$.

Necessity. Assume that (B) holds and define the modulus by

$$\tau(x) = \sup\{t > 0 : x \in T_t C\}, \quad x \in \mathbb{X}'.$$

It is straightforward to see that the function defined in this way is homogeneous and strictly positive. For $t > 0$, the set $\{x : \tau(x) > t\} = T_t C$ is open. Thus, τ is lower semicontinuous. Next,

$$\{x : \tau(x) < t\} = \cup_{s<t} \{x : \tau(x) > s\}^c = (\cap_{s<t} T_s C)^c = (\cap_{s<t} \mathrm{cl}\, T_s C)^c.$$

The last equality follows from (B), since $T_s C \subseteq \mathrm{cl}\, T_s C \subseteq T_r C$ for $s < r < t$. Therefore, $\{x : \tau(x) < t\}$ is open as a complement of a closed set, so we conclude that τ is upper semicontinuous. Thus, τ is continuous on $\mathbb{X}'$.

Since $C = \{x : \tau(x) > 1\}$, the bornology $\mathcal{S}$ contains $\mathcal{S}_\tau$. Finally, since the sequence $(T_{n^{-1}}C)$ is localising, each $B \in \mathcal{S}$ is a subset of $\{x : \tau(x) > t\}$ for some $t > 0$. Thus, $\mathcal{S} = \mathcal{S}_\tau$.

Equivalence of (B) and (B1)–(B3). By Lemma 1.3.5, it is possible to assume that C is an open cone. Then, conditions (B) and (B2) become equivalent. The fact that $\cup_{t\leq 1} T_t C = \mathbb{X}'$ follows from the requirement that the sequence $T_{n^{-1}}C$, $n \geq 1$, is a base of $\mathcal{S}$. Condition (B1) follows from (B2), since each $B \in \mathcal{S}$ is a subset of $T_{n^{-1}}C$ for some n.

Consider the polar decomposition map. Since the family of sets $A \times (t, \infty)$ for all Borel $A \subseteq \mathbb{S}$ and $t > 0$ is measure determining in $\mathbb{S} \times (0, \infty)$, we confirm that the family of bounded semicones $T_{[t,\infty)}A = \{x \in \mathbb{X}' : T_{\tau(x)^{-1}}x \in A, \tau(x) \geq t\}$ for all Borel $A \subseteq \mathbb{S}$ and $t > 0$ is a measure determining class in $\mathbb{X}'$. Note that $T_{[t,\infty)}A$ is Borel due to measurability of the polar decomposition. Thus, (B3) also follows from (B2). □

The family of all metrically bounded sets in a metric space is called the Fréchet bornology. For a complete separable metric space $\mathbb{X}$, a bornology $\mathcal{S}$ on $\mathbb{X}$ can be realised as the Fréchet bornology with a metric that preserves the topology on $\mathbb{X}$ if and only if $\mathcal{S}$ has a countable open base, see [31, Corollary 5.12] and [6, Theorem 2.1].

The family of all subsets of a topological space $\mathbb{X}$ which do not intersect an open neighbourhood of the closure of $\mathbf{0}$ is said to be the *topological bornology* $\mathcal{S}_{\text{top}}$ on $\mathbb{X}'$. In a metric space $\mathbb{X}$, the topological bornology is generated by all closed sets which do not intersect the closure of $\mathbf{0}$. The topological bornology is topologically and scaling consistent; it has a countable open base if the closure of $\mathbf{0}$ equals the intersection of a countable family of open sets, which is the case in metric spaces. It is easy to see that $\mathcal{S}_{\text{top}}$ contains $\mathcal{S}_\tau$ for each continuous modulus τ.

The topological bornology was used in [54] for star-shaped metric spaces, where an additional condition $\hat{\tau}(B) > 0$ is imposed for B being the complement of any open ball centred at $\mathbf{0}$. This ensures that the topological bornology coincides with $\mathcal{S}_\tau$. Lemma 1.3.9 and continuity of the modulus guarantee that condition (B) holds in this case. However, in general, the topological bornology may fail to satisfy (B).

Remark 1.3.10. It is easy to see that all bornologies with a countable open base contain compact subsets of the underlying space. If the set $\mathbf{0}$ is rather large (see, e.g., Remark 3.6.3), then it may be convenient to work with ideals whose union is larger than $\mathbb{X}'$, including bounded sets containing all or some elements of zero. An example is provided by the *Hadamard bornology* on $\mathbb{X}$ which is generated by all relatively compact sets. The Hadamard bornology is topologically and scaling consistent and has a countable open base if $\mathbb{X}$ is a locally compact separable space. For instance, in $\mathbb{R}^d$ this bornology consists of all bounded sets and so contains zero, which is scaling invariant under the standard scaling. This example shows that the Hadamard bornology does not necessarily satisfy (B) and thus is not necessarily generated by a continuous modulus.

If $\mathbf{0}$ is empty, the topological bornology consists of all subsets of $\mathbb{X}$. If a bornology $\mathcal{S}$ on $\mathbb{X}'$ is topologically consistent, then each $B \in \mathcal{S}$ belongs to the topological bornology. This follows from the fact that $\mathcal{S}$ covers $\mathbb{X}'$ and thus sets from $\mathcal{S}$ do not intersect $\mathbf{0}$. If $\mathbf{0}$ has a countable base $\{U_n, n \geq 1\}$ of open neighbourhoods (which is always the case if $\mathbf{0}$ is a singleton and $\mathbb{X}$ is metric), then the sets U_n^c, $n \geq 1$, constitute a base of the topological bornology.

Consider the following condition:

(T) if $\tau(x_n) \to 0$ for a sequence $(x_n)_{n\in\mathbb{N}}$, then $x_n \in G$ for each open set $G \supseteq \mathbf{0}$ and all sufficiently large n.

Condition (T) holds if $\{x : \tau(x) \leq 1\}$ is a compact set. This is in turn the case if $\mathbf{0}$ is a compact set and the space $\mathbb{X}$ is locally compact.

Lemma 1.3.11. *(i) If (T) holds, then $\mathcal{S}_{top} \subseteq \mathcal{S}_\tau$.*

(ii) If $\mathcal{S}_\tau \subseteq \mathcal{S}_{top}$, then (T) holds.

(iii) If τ is upper semicontinuous and $\mathbf{0}$ is a compact set, then (T) is equivalent to $\mathcal{S}_\tau = \mathcal{S}_{top}$.

Proof. Let B have an empty intersection with G which is an open neighbourhood of $\mathbf{0}$. It suffices to show that $\hat{\tau}(B) > 0$. If this is not the case, then there exists $x_n \in B$, $n \geq 1$, such that $\tau(x_n) \to 0$. This contradicts (T).

Assume that (T) does not hold, so there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $\tau(x_n) \to 0$ and $x_n \notin G$ for an open neighbourhood G of $\mathbf{0}$. Then $B = \{x_n, n \geq 1\}$ belongs to the topological bornology, but $\hat{\tau}(B) = 0$ and $B \notin \mathcal{S}_\tau$, which is a contradiction. Therefore, (T) must hold.

If τ is upper semicontinuous, then each $B \in \mathcal{S}_\tau$ is a subset of $\{x : \tau(x) \geq \varepsilon\}$ for some $\varepsilon > 0$, that is, B is a subset of a closed set which does not intersect $\mathbf{0}$, hence its open neighbourhood due to compactness argument. If τ is upper semicontinuous, then $\{x : \tau(x) \geq t\}$ for $t > 0$ is a closed set which does not intersect $\mathbf{0}$, hence belongs to the topological bornology. □

1.4 Examples

It is worth noting that if τ_1 and τ_2 are two moduli on $\mathbb{X}'$, then $\min(a_1\tau_1, a_2\tau_2)$, $\max(a_1\tau_1, a_2\tau_2)$, and $a_1\tau_1 + a_2\tau_2$ are also moduli for arbitrary $a_1, a_2 > 0$.

Example 1.4.1. The mapping defined by $T_t(x_1, x_2) = (t^2 x_1, x_2)$, $(x_1, x_2) \in \mathbb{R}^2$, $t > 0$, is a scaling. The set of zeros in this case is given by $\{0\} \times \mathbb{R}$ and one possible choice for the sphere is the set $\{(1, x) : x \in \mathbb{R}\} \cup \{(-1, x) : x \in \mathbb{R}\}$. The modulus in this case is $\tau(x_1, x_2) = \sqrt{|x_1|}$.

Example 1.4.2. The mapping $T_t(x_1, x_2) = (t^{-1}x_1, t^{-1}x_2)$, $(x_1, x_2) \in \mathbb{R}^2$, $t > 0$, is another example of a scaling. The unique zero element is the origin $(0, 0)$. One possible choice for a sphere in this case is the same as in the previous example. The modulus induced by this sphere is given by $\tau(x_1, x_2) = 1/|x_1|$. However, the ideal induced by τ is not a bornology on $\mathbb{X}'$.

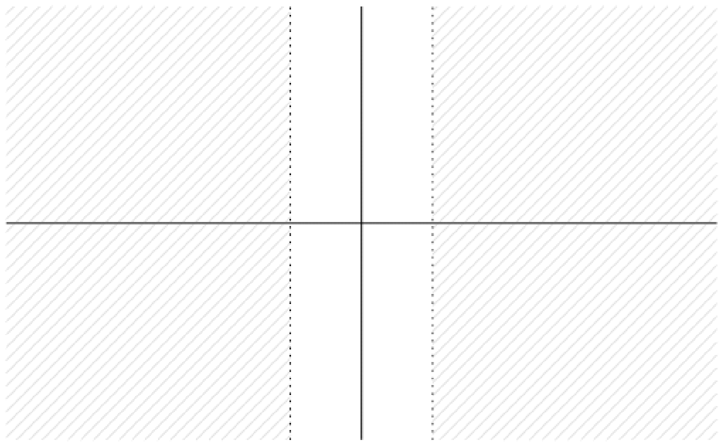


Figure 1.1: *A bounded set in bornology on* $(\mathbb{R} \setminus \{0\}) \times \mathbb{R}$ *induced by* τ *from Example 1.4.1*

Example 1.4.3. Let $\mathbb{X} = \mathbb{R}^d$ be the Euclidean space equipped with the *linear* scaling, which is defined by coordinatewise multiplication. More precisely, for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $t > 0$, we define $T_t x = (tx_1, \dots, tx_d)$. In this case, the origin is $\mathbf{0} = \{0\}$. The topological bornology on $\mathbb{R}^d$ with respect to this scaling consists of all sets whose closure does not contain the origin. This bornology satisfies (B), and therefore, it can be obtained as $\mathcal{S}_\tau$ for many choices of the modulus. For instance, we can let $\tau(x) = \|x\|_p$ for any ℓ_p-norm on $\mathbb{R}^d$ with $p \in (0, \infty]$. Note that $\tau_{\min}(x) = \min(|x_1|, \dots, |x_d|)$ is not a modulus on this space because it vanishes on the axes outside of the origin.

If we equip $\mathbb{R}^d$ with the inverted linear scaling, so that $T_t x = (t^{-1}x_1, \dots, t^{-1}x_d)$ for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $t > 0$, then a modulus is given by $\tau(x) = 1/\|x\|$, $x \neq 0$, where $\|\cdot\|$ denotes the Euclidean norm. The corresponding bornology on $\mathbb{R}^d \setminus \{0\}$ consists of bounded sets in $\mathbb{R}^d$ with the origin removed.

Example 1.4.4. Let $\mathbb{X} = [0, \infty]$ with the linear scaling. Then $\mathbf{0} = \{0, \infty\}$. All moduli on $\mathbb{X}' = (0, \infty)$ are necessarily continuous and correspond to the same bornology, which is generated by $[1, \infty)$. While it is possible to extend a continuous modulus to $\mathbb{R}_+$ by letting $\tau(0) = 0$, an extension is not possible on the whole $[0, \infty]$ unless τ is allowed to be infinite. In case of the inverted scaling given by $T_t x = t^{-1}x$, the bornology is generated by $(0, 1]$.

Example 1.4.5. Consider $\mathbb{X} = \mathbb{R}_+^2$ equipped with the linear scaling, and let $\mathbb{U} = (0, \infty)^2$ be a cone in $\mathbb{X}$. A valid choice of modulus on $\mathbb{U}$ is $\tau_{\min}(x_1, x_2) = \min(x_1, x_2)$, which is a special

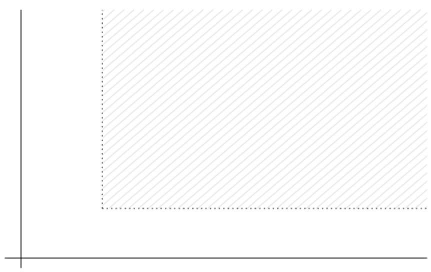


Figure 1.2: *A bounded set in bornology on* $(0,\infty)^2$ *induced by* τ_{min} *from Example 1.4.5*

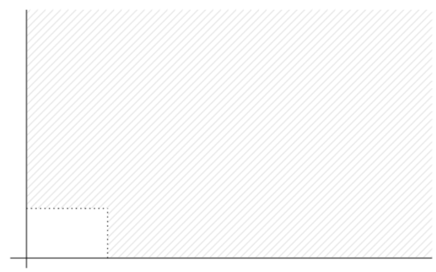


Figure 1.3: *A bounded set in bornology on* $\mathbb{R}_+^2 \setminus \{0\}$ *induced by* τ_{max} *from Example 1.4.5*

case of

$$\tau_\beta(x_1, x_2) = \min(x_1^\beta x_2^{1-\beta}, x_1^{1-\beta} x_2^\beta), \quad \beta \in [0, 1/2],$$

for $\beta = 0$. Another modulus $\tau_{\max}(x_1, x_2) = \max(x_1, x_2)$ is the special case of

$$\tau_\beta^*(x_1, x_2) = \max(x_1^\beta x_2^{1-\beta}, x_1^{1-\beta} x_2^\beta), \quad \beta \in [0, 1/2],$$

for $\beta = 0$. Other possible moduli on $\mathbb{U}$ are $\tau(x) = x_1$ or $\tau(x) = x_2$. These moduli induce different bornologies on $\mathbb{U}$. In particular, the bornology $\mathcal{S}_{\tau_\beta}$ induced by τ_β with $\beta > 0$ is strictly richer than $\mathcal{S}_{\tau_{\min}}$, and the bornology induced by $\tau_{\max}$ is strictly richer than $\mathcal{S}_{\tau_\beta}$ for $\beta > 0$. Note also that the moduli $\tau_{\max}$ and $\tau(x) = \|x\|_1 = x_1 + x_2$ (as well as any other modulus given by a norm on $\mathbb{R}_+^2$) generate the same bornology. On the space $\mathbb{X}' = \mathbb{R}_+^2 \setminus \{0\}$, the function τ_β^* is also a modulus, while τ_β vanishes on the axes and thus is not a modulus. Note that $\mathcal{S}_{\tau_\beta}$ is an ideal in $\mathbb{R}_+^2 \setminus \{0\}$ which satisfies condition (B) on its union set being $(0,\infty)^2$.

Example 1.4.6. Let $\mathbb{X} = \mathbb{R}$ with the linear scaling. The semicone $C = (-\infty, 1]$ generates a bornology on $\mathbb{X}$. However, this bornology does not satisfy (B), since $\cap_{t>1} T_t C = (-\infty, 1]$ and so is not empty. A bornology on $\mathbb{X}' = \mathbb{X} \setminus \{0\}$ satisfying (B) is generated by the semicone $(-\infty, -1] \cup [1, \infty)$ and the corresponding modulus is $\tau(x) = |x|$.

Example 1.4.7. Let $\mathbb{X} = (-\infty, 0]$, and define the scaling by $T_t x = t^{-1} x$. The modulus $\tau(x) = 1/|x|$ generates a bornology on $\mathbb{X}'$ with the corresponding semicone $C = [-1, 0)$. By translating $\mathbb{X}$ by $c \in \mathbb{R}$, we obtain the space $(-\infty, c]$ with the scaling $T_t x = t^{-1}(x - c) + c$, which is important for characterising the maximum domain of attraction of Weibull laws in the theory of extreme values. The same scaling can also be used on $\mathbb{X} = \mathbb{R}_+$, which is important for studying minimum values.

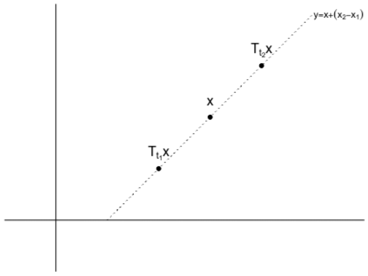


Figure 1.4: *An example of scaling of the vector $x = (x_1, x_2) \in \mathbb{R}_+^2$ under the scaling T_t from Example 1.4.9, where $t = t_1 < 1$ and $t = t_2 > 1$.*

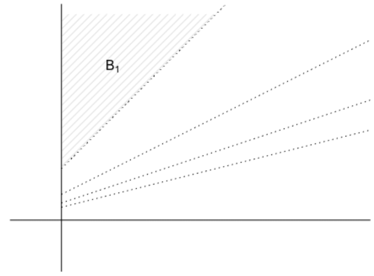


Figure 1.5: *The sketch of the family (B_n) that forms an open base for the bornology $\mathcal{S}$ from Example 1.4.10.*

Example 1.4.8. Motivated by extreme value theory, let $\mathbb{X} = \mathbb{X}' = \mathbb{R}$ with the scaling $T_t x = x + \kappa \log t$ for some $\kappa \neq 0$. A possible modulus on $\mathbb{X}'$ could be taken as $\tau(x) = e^{x/\kappa}$ which generates a bornology on $\mathbb{X}'$ with the corresponding semicone $C = [0, \infty)$ if $\kappa > 0$ and by $C = (-\infty, 0]$ if $\kappa < 0$.

Example 1.4.9. Define a scaling on $\mathbb{X} = \mathbb{R}_+^2$ by letting

$$T_t(x_1, x_2) = \big(x_1 + (t-1)\min(x_1, x_2), x_2 + (t-1)\min(x_1, x_2)\big).$$

Then $\mathbf{0}$ is the set $\{(x_1, x_2) \in \mathbb{R}_+^2 : \min(x_1, x_2) = 0\}$, which is the union of two semiaxes. A continuous modulus is $\tau_{\min}(x_1, x_2) = \min(x_1, x_2)$. Note that the functions $(x, y) \mapsto x + y$ and $(x, y) \mapsto \sqrt{xy}$ are not homogeneous under this scaling.

Example 1.4.10. Let $\mathbb{X} = (0, \infty)^2$ with the linear scaling. Define a bornology $\mathcal{S}$ by specifying its open base as

$$B_n = \big\{(x_1, x_2) \in \mathbb{X} : x_2 > n^{-1}(1 + x_1)\big\}, \quad n \geq 1.$$

This bornology does not satisfy condition (B). Indeed, if C is a bounded semicone from (B), then we should have $C \subseteq B_n$ for some $n \geq 1$. This is not possible, since C contains points (a_m, b_m), $m \geq 1$, with $b_m \downarrow 0$ as $m \to \infty$, while b_m is not larger than $n^{-1}(1 + a_m)$ for fixed n and sufficiently large m. However, if the scaling on $\mathbb{X}$ is defined by $T_t x = (x_1, t x_2)$, then $\mathcal{S}$ satisfies condition (B).

Example 1.4.11 (Uplift to counting measures). A scaling on $\mathbb{X}$ can be extended to act on measures on $\mathbb{X}$. Let $\mathcal{M}_p(\mathbb{X}', \mathcal{S})$ denote the set of counting measures m on $\mathbb{X}'$, which are finite on a bornology $\mathcal{S} = \mathcal{S}_\tau$ generated by a modulus τ. A metric on $\mathcal{M}_p(\mathbb{X}', \mathcal{S})$ can be constructed as

$$\mathsf{d}(m_1, m_2) = \sum_{n=1}^{\infty} 2^{-n} \mathsf{d}(m_1|_{B_n}, m_2|_{B_n}),$$

where $\{B_n, n \geq 1\}$ is a base of $\mathcal{S}$ and $\mathsf{d}(m_1|_{B_n}, m_2|_{B_n})$ is the distance between finite counting measures obtained as restrictions of m_1 and m_2 onto B_n, as shown in [53]. A natural continuous scaling on $\mathcal{M}_p(\mathbb{X}', \mathcal{S})$ is given by $T_t m(A) = m(T_{t^{-1}}A)$, that is, any atom x of m is moved to $T_t x$. A modulus can be defined as $\bar{\tau}(m) = \max\{\tau(x) : m(\{x\}) \geq 1\}$. Note that the maximum is well-defined, since only a finite number of atoms of m have modulus larger than any given positive number.

Example 1.4.12. Let $\mathbb{Y}$ be the family of integrable probability measures m on $\mathbb{R}_+$. Each $m \in \mathbb{Y}$ corresponds to a random variable ξ, and $T_t m$ is the distribution of the random variable $t\xi$. Hence, $\mathbf{0}$ is the Dirac measure at 0. The metric on $\mathbb{Y}$ is chosen to be the 1-Wasserstein distance, which is the integral of the absolute difference of the quantile functions. A continuous modulus is given by the expectation. For integrable probability measure on the whole line, a modulus is provided by the expectation of the absolute value.

Example 1.4.13. The family $\mathcal{S}$ of all finite subsets of $\mathbb{X}$ forms a bornology on $\mathbb{X}$. While this bornology is topologically and scaling consistent, it does not have a countable base if $\mathbb{X}$ is uncountable. Moreover, this bornology does not satisfy condition (B) and hence cannot be generated by a continuous modulus.

Example 1.4.14. Let $\mathbb{X} = \mathbb{R}_+^\infty$ with the scaling applied coordinatewisely and the usual metric d_∞ determining coordinatewise convergence, see, e.g., [40],

$$d_\infty(x, y) = \sum_{i=1}^{\infty} \frac{\|x_i - y_i\| \wedge 1}{2^i}, \quad x = (x_1, x_2, \dots),\ y = (y_1, y_2, \dots) \in \mathbb{X}.$$

This space does not admit a continuous modulus. To show this, we prove that property (B) does not hold (Lemma 1.3.9). Assume that a bornology on $\mathbb{X}'$ (which is $\mathbb{X}$ excluding the sequence consisting of all zeros) is generated by a closed semicone C. By $\mathrm{pr}_j C$ we denote the projection of C onto the jth coordinate. Let $y_j = \inf \mathrm{pr}_j C$, $j \geq 1$. Note that if $y_j = 0$ for some $j \in \mathbb{N}$, then $\inf \mathrm{pr}_j T_t C = 0$ for all $t > 1$. Moreover, $\inf \mathrm{pr}_j \cap_{t>1} T_t C = 0$. Since sets $T_t C$ cover $\mathbb{X}'$, then $(0, \infty) \subseteq \mathrm{pr}_j T_t C$, for all $t \geq 1$ and $(0, \infty) \subseteq \mathrm{pr}_j \cap_{t>1} T_t C$. Thus the intersection is not empty which is a contradiction so $y_j > 0$ for all $j \in \mathbb{N}$. Then C is a subset of the set D which consists of all sequences (x_n) such that $x_n \geq y_n$ for all $n \in \mathbb{N}$. We claim that $\cup_{t>0} T_t C$ is a strict subset of $\mathbb{X}'$, so that $(T_{n^{-1}}C)_{n\in\mathbb{N}}$ is not a base of $\mathcal{S}$. For this, consider a sequence $x = (x_n)$ of strictly positive numbers such that $x_n / y_n \to 0$. If $x \in T_t C$ for some $t > 0$, then $x_n \geq t y_n$ for all n, which is not possible, hence, $x \notin \cup_{t>0} T_t C$. Thus, the space $\mathbb{R}_+^\infty$ is not covered by the setting used in [54] and [38]. However, it should be noted that $\mathbb{X}'$ admits a bornology which is topologically and scaling consistent and has a countable open base. This bornology is generated by the sets $\{x \in \mathbb{X} : x_i > t\}$, $i \in \mathbb{N}$, $t > 0$.

The space $\mathbb{X} = (0, \infty)^\infty$ with the same scaling admits continuous moduli. For instance, a simple example of a continuous modulus is $\tau(x) = x_1$, which is just the first component of x. More generally, one can choose a modulus on a finite-dimensional subspace of $\mathbb{X}$. The authors of [40] considered a subset of $\mathbb{R}_+^\infty$ consisting of sequences (x_n) with at most j non-zero components. This space admits a continuous modulus given by $\sum x_n$, which then becomes a finite sum.

Chapter 2

Regular variation in Polish spaces

In this chapter we present general definition of vague convergence of boundedly finite measures, and regular variation in Polish spaces. We give characterisations of vague convergence and regularly varying measures, and consider reduction of the space to a subcone. We follow notation introduced in Chapter 1.

2.1 Vague convergence in general Polish spaces

Recall that $\mathcal{B}(\mathbb{X})$ denotes the Borel σ-algebra on a topological space $\mathbb{X}$. Fix an ideal $\mathcal{S}$ on $\mathbb{X}$ (see Section 1.3). A Borel measure ν on $\mathbb{X}$ is said to be *boundedly finite* if $\nu(B) < \infty$ for all $B \in \mathcal{S} \cap \mathcal{B}(\mathbb{X})$. The family of such measures is denoted by $\mathcal{M}(\mathbb{X}, \mathcal{S})$, or by $\mathcal{M}(\mathbb{U}, \mathcal{S})$ for measures supported by any cone $\mathbb{U} \subseteq \mathbb{X}$. We assume that $\mathbb{X}$ is equipped with a scaling T_t defined in Section 1.3.

Let $\mathsf{C}_{\mathcal{S}}$ be the collection of bounded continuous functions $f : \mathbb{X} \to \mathbb{R}$ such that the support of f (i.e., the set $\{x : f(x) \neq 0\}$) belongs to $\mathcal{S}$. We adopt the following definition from [38, Definition B.1.16], where it is formulated, assuming that $\mathcal{S}$ is a bornology on $\mathbb{X}'$, and $\mathbb{X}'$ is localised. The latter property (in case of a bornology) is equivalent to the conditions on $\mathcal{S}$ imposed below, see Remark 1.3.4.

Definition 2.1.1. Let $\mathcal{S}$ be a topologically consistent ideal with a countable open base on a Polish space $\mathbb{X}$. A sequence of Borel measures $\mu_n \in \mathcal{M}(\mathbb{X}, \mathcal{S})$, $n \in \mathbb{N}$, is said to *vaguely converge* to $\mu \in \mathcal{M}(\mathbb{X}, \mathcal{S})$ (with respect to $\mathcal{S}$; notation $\mu_n \xrightarrow{\mathcal{S}} \mu$) if

$$\int f d\mu_n \to \int f d\mu \quad \text{as } n \to \infty \tag{2.1}$$

for all $f \in \mathsf{C}_{\mathcal{S}}$.

In the same way, we can define vague convergence for families of measures (μ_t) parametrised by $t > 0$. However, as indicated by the notation above, the definition of vague convergence depends on the choice of the ideal $\mathcal{S}$ (see Example 2.4.8). It is important to note that the vague limit is unique only if it is restricted to the union of all sets from $\mathcal{S}$, $\mathbb{U}(\mathcal{S})$. For instance, it is possible to let μ vanish outside of $\mathbb{U}(\mathcal{S})$. If $\mathcal{S}$ is trivial, that is, $\mathcal{S}$ contains $\mathbb{X}$ and thus all subsets of $\mathbb{X}$, then all involved measures are finite and we obtain the definition of weak convergence denoted by $\mu_n \xrightarrow{w} \mu$.

A Borel set B is said to be a μ-continuity set of a Borel measure μ if $\mu(\partial B) = 0$.

Lemma 2.1.2. *Let $\mathcal{S}$ be a topologically consistent ideal with a countable open base on a Polish space $\mathbb{X}$. For measures $\mu, \mu_n \in \mathcal{M}(\mathbb{X}, \mathcal{S})$, $n \geq 1$, the following statements are equivalent:*

(i) $\mu_n \xrightarrow{\mathcal{S}} \mu$.

(ii) There exists a countable open base $(D_m)_{m\in\mathbb{N}}$ of $\mathcal{S}$ such that $\mu_n|_{D_m} \xrightarrow{w} \mu|_{D_m}$ as $n \to \infty$ for all m, where $\mu|_{D_m}$ is the restriction of μ to D_m.

(iii) We have $\mu_n(B) \to \mu(B)$ as $n \to \infty$ for all μ-continuity sets B from $\mathcal{S}$.

If $\mathcal{S}$ satisfies (B), and is generated by a semicone C and a modulus τ, then it is possible to let $D_m = T_{s_m}C$ for a suitable sequence $s_m \to \infty$ and the above conditions are equivalent to:

(iv) $\mu_n(T_{[t,\infty)}A) \to \mu(T_{[t,\infty)}A)$ for all closed $A \subseteq \{x : \tau(x) = 1\}$ and all $t > 0$ such that $\mu(\partial T_{[t,\infty)}A) = 0$.

Remark 2.1.3. Note that the assumption above guarantees that any two disjoint closed subsets A and B of $\mathbb{X}$ can be separated by a continuous function. In other words, there exists a continuous function $f : \mathbb{X} \to [0, 1]$ such that $f(x) = 0$ for every $x \in A$ and $f(x) = 1$ for every $x \in B$. This result can be derived using several results from [21]. First, since $\mathbb{X}$ is a separable metric space, Theorem 4.1.16 implies that $\mathbb{X}$ is second–countable. Moreover, by Theorem 4.2.9, every second–countable metrisable space is regular. We can use Theorem 1.5.16 to conclude that $\mathbb{X}$ is normal. Finally, Urysohn's lemma [21, Theorem 1.5.11] gives the final statement. Note that completeness of $\mathbb{X}$ is not necessary for this conclusion.

Proof. (i) $\Rightarrow$ (ii) Let (B_m) be an open base of $\mathcal{S}$. Since $\mathcal{S}$ is topologically consistent, we can assume that $\operatorname{cl} B_m \subseteq B_{m+1}$ for all $m \in \mathbb{N}$, which implies that the closed sets $\operatorname{cl} B_m$ and B^c_{m+1} are disjoint. Using Remark 2.1.3 we can separate $\operatorname{cl} B_m$ and B^c_{m+1} by a continuous function g that takes the value 0 on $\operatorname{cl} B_m$ and 1 on B^c_{m+1}. Since $\mu(B_{m+1}) < \infty$, we have $\mu(\partial D_m) = 0$, where $D_m = \{x : g(x) < t\} \subseteq B_{m+1}$ for some $t \in (0, 1)$. The collection of sets (D_m) forms a countable open base that consists of μ-continuity sets. Suppose h is a continuous bounded function on D_m. Then, $h\mathbf{1}_{D_m}$ is a bounded function on $\mathbb{X}$ with bounded support, and is discontinuous on a set of μ-measure zero. Note that bornology consists of metrically bounded sets according to [6]. Therefore, by Kallenberg [36, Lemma 4.1], we can conclude that under the assumption (i), (2.1) holds for f that is bounded, measurable, has bounded support, and is μ-a.e. continuous. We can now apply this statement to the function $h\mathbf{1}_{D_m}$ to conclude that (ii) holds.
(ii) $\Rightarrow$ (i) If $f \in \mathsf{C}_{\mathcal{S}}$, then the support of f is a subset of D_m for some $m \in \mathbb{N}$. Hence

$$\int f d\mu_n = \int f\mathbf{1}_{D_m} d\mu_n = \int f d\mu_n|_{D_m} \to \int f d\mu|_{D_m} = \int f d\mu \quad \text{as } n \to \infty.$$

(ii) $\Leftrightarrow$ (iii) This equivalence is well-known for the weak convergence, see, e.g., [12] and can be easily amended to the vague convergence by taking restrictions of measures onto a base of $\mathcal{S}$.
(iii) $\Rightarrow$ (iv) $\Rightarrow$ (ii) It is immediate that (iii) implies (iv). Denote by $\mathcal{A}_\mu$ the family of sets $T_{[t,\infty)}A$ for all closed $A \subseteq \mathbb{S}$ and $t > 0$ and such that $\mu(\partial T_{[t,\infty)}A) = 0$. Condition (iv) implies that $\mu_n(B) \to \mu(B)$ as $n \to \infty$ for all sets B from the family $\tilde{\mathcal{A}}_\mu$ of sets-differences of sets from $\mathcal{A}_\mu$. Because the polar decomposition map $\rho(x) = (T_{\tau(x)^{-1}}x, \tau(x))$ is not only continuous, but also continuously invertible, for any $x \in \mathbb{X}$ and $\varepsilon > 0$, the ball $B_\varepsilon(x)$ contains a set $B \in \tilde{\mathcal{A}}_\mu$, such that $x \in \operatorname{Int} B \subseteq B \subseteq B_\varepsilon(x)$.

To show that, let $x \in \mathbb{X}$ and $\varepsilon > 0$ be arbitrary. Since ρ is continuously invertible, $\rho(U)$ in open in $\mathbb{S} \times (0, \infty)$ for every U open in $\mathbb{X}'$. Thus, $\rho(B_\varepsilon(x))$ is open in $\mathbb{S} \times (0, \infty)$. Since $\{\operatorname{Int}(\rho(B)) : B \in \tilde{\mathcal{A}}_\mu\} = \{\operatorname{Int}(S \times (u_1, u_2)) : S \subseteq \mathbb{S},\ 0 < u_1 < u_2 \leq \infty\}$ is a base for the topology on $\mathbb{S} \times (0, \infty)$, then $\rho(B_\varepsilon(x))$ can be written as a union of such sets. Moreover, $\rho(x) \in \rho(B_\varepsilon(x))$, so we can find B_x such that $\operatorname{Int}(\rho(B_x))$ is an element of the base and $\rho(x) \in \operatorname{Int}(\rho(B_x)) \subseteq \rho(B_x) \subseteq \rho(B_\varepsilon(x))$. Applying the bijective and continuous ρ^{-1} to both sides yields

$$x \in \rho^{-1}(\operatorname{Int}(\rho(B_x))) = \operatorname{Int} \rho^{-1}\rho(B_x) = \operatorname{Int} B_x \subseteq B_x \subseteq B_\varepsilon(x).$$

Furthermore, the family $\tilde{\mathcal{A}}_\mu$ is closed under the formation of finite intersections. It follows from Theorem 2.2 in [12] that $\mu_n|_{D_m} \xrightarrow{w} \mu|_{D_m}$ as $n \to \infty$ for all m such that $D_m = T_{s_m}C$ is a μ-continuity set and $s_m \to 0$. Thus, (ii) holds. $\square$

As shown in the proof of Lemma 2.1.2, we emphasise once again that $\mu_n \xrightarrow{\mathcal{S}} \mu$ if and only if $\int f d\mu_n \to \int f d\mu$ for every bounded μ-a.e. continuous function f with bounded support. It is possible to construct a metric on $\mathcal{M}(\mathbb{X}', \mathcal{S})$ which generates the vague topology; see the proof of Proposition 3.1 in [6], cf. Proposition A2.6.II(i) in [15].

2.2 Regularly varying measures

Assume condition (A) throughout, that is, let $\mathbb{X}$ be a Polish space equipped with a continuous scaling T_t. The scaling T_t on $\mathbb{X}$ transforms a measure μ on $\mathcal{B}(\mathbb{X})$ to its pushforward measure defined as

$$(T_t\mu)(B) = \mu(T_{t^{-1}}B), \quad B \in \mathcal{B}(\mathbb{X}),\ t > 0.$$

Let $\mathcal{S}$ be a scaling consistent ideal on $\mathbb{X}$. A measure $\mu \in \mathcal{M}(\mathbb{X}, \mathcal{S})$ is said to be *α-homogeneous* if

$$(T_t\mu)(B) = t^\alpha \mu(B), \quad B \in \mathcal{B}(\mathbb{X}) \cap \mathcal{S}, t > 0.$$

If μ is α-homogeneous and $\mathcal{S}$ contains a set B which has a nonempty intersection with $\mathbf{0}$, then $\mu(B) = \mu(B \setminus \mathbf{0})$, since otherwise we would have $\mu(B) = \infty$. Suppose that condition (B) holds. Then, for any $B \in \mathcal{S}$ and $0 < t \leq s < \infty$, we have $T_{[t,s]}B \in \mathcal{S}$. Since μ is finite on $\mathcal{S}$, we then have that $\mu(T_{[t,s]}B) < \infty$. Letting $s > 1$, we obtain

$$\mu(T_{[1,\infty)}B) \leq \sum_{i=0}^{\infty} \mu(T_{[s^i,s^{i+1}]}B) = \sum_{i=0}^{\infty} s^{-i\alpha}\mu(T_{[1,s]}B) < \infty$$

for all $B \in \mathcal{S}$. Under (B), since $\mathcal{S}$ is generated by a continuous modulus τ, each α-homogeneous measure, which is finite on a semicone C, satisfies $\mu(\{x \in C : \tau(x) = t\}) = 0$ for all $t > 0$. This is because otherwise, we would have $\mu(\{x \in C : \tau(x) = st\}) = s^{-\alpha}\mu(\{x \in C : \tau(x) = t\})$ strictly positive for all $s > 0$, which is impossible if t is chosen such that $\mu(\{x \in C : \tau(x) = t\}) > 0$. This is because the sets $\{x : \tau(x) = t\}$ are disjoint for all $t > 0$.

Definition 2.2.1. Let $\mathcal{S}$ be a scaling and topologically consistent ideal on $\mathbb{X}$ which admits a countable open base with union $\mathbb{U}$. A measure $\nu \in \mathcal{M}(\mathbb{X}, \mathcal{S})$ is said to be *regularly varying* on $\mathcal{S}$ if there exists a $B \in \mathcal{S}$ such that $\nu(T_t B) \to 0$ and

$$\frac{1}{\nu(T_t B)} T_{t^{-1}}\nu \xrightarrow{\mathcal{S}} \mu \quad \text{as } t \to \infty \tag{2.2}$$

for a nontrivial measure $\mu \in \mathcal{M}(\mathbb{U}, \mathcal{S})$. In this case, we write $\nu \in \mathrm{RV}(\mathbb{X}, \mathcal{S}, \mu)$ and call μ the *tail measure* of ν.

By definition, the regular variation property applies to the restriction of μ onto $\mathbb{U}(\mathcal{S})$, which is equal to $\mathbb{X}'$ if $\mathcal{S}$ is a bornology on $\mathbb{X}'$. It is important to note that the tail measure μ is assumed to be supported by $\mathbb{U}(\mathcal{S})$, and in particular, $\mu \in \mathcal{M}(\mathbb{X}', \mathcal{S})$ if $\mathcal{S}$ is a bornology on $\mathbb{X}'$. A measure ν on $\mathbb{X}$ is regularly varying if and only if its restriction on $\mathbb{U}(\mathcal{S})$ is regularly varying (it is worth noting that $\mathbb{U}(\mathcal{S})$ is a Borel set, since $\mathcal{S}$ has a countable open base). If ν assigns positive mass to $\mathbf{0}$ and $\mathbf{0}$ has a nonempty intersection with $\mathbb{U}(\mathcal{S})$, then (2.2) fails. Furthermore, if (B) holds, the regular variation property is not affected by the restriction of ν to the set $\{x : \tau(x) \le c\}$ for any constant $c > 0$.

Let ν be the probability distribution of a random element ξ in $\mathbb{X}'$ or a (nontrivial) subprobability measure obtained as the restriction to a cone $\mathbb{U} \subseteq \mathbb{X}$ of the distribution of a random element distributed in $\mathbb{X}$. If ν satisfies the regular variation property defined in Definition 2.2.1, then ξ is said to be regularly varying, and we write $\xi \in \mathrm{RV}(\mathbb{X}, \mathcal{S}, \mu)$. Equivalently,

$$\frac{\mathbf{P}\{\xi \in T_t \,\cdot\}}{\mathbf{P}\{\xi \in T_t B\}} \xrightarrow{\mathcal{S}} \mu(\cdot) \quad \text{as } t \to \infty,$$

or

$$g(t)\mathbf{E}\left[f(T_{t^{-1}}\xi)\right] \to \int f d\mu \quad \text{as } t \to \infty \tag{2.3}$$

for a $g \in \mathrm{RV}_\alpha$, $\alpha > 0$, and all $f \in \mathsf{C}_{\mathcal{S}}$. Recall that RV_α denotes the family of functions that are regularly varying at infinity with index $\alpha \in \mathbb{R}$. That is, RV_α is the family of all positive measurable functions $g : (a, \infty) \to (0, \infty)$ for some $a \in \mathbb{R}$ with the property that

$$\frac{g(tx)}{g(t)} \to x^\alpha \quad \text{as } t \to \infty$$

for all $x > 0$.

Lemma 2.2.2. *If $\mathcal{S}$ satisfies (B) and $\nu \in \mathrm{RV}(\mathbb{X}, \mathcal{S}, \mu)$, then* (2.2) *holds with $B = C$, where C is a semicone generating $\mathcal{S}$.*

Proof. Lemma 1.3.9 guarantees the existence of a continuous modulus τ such that $\mathcal{S} = \mathcal{S}_\tau$. Let $C = \{x : \tau(x) > 1\}$. The set $C = T_{[1,\infty)}C$ is in $\mathcal{S}_\tau$ and so has a finite measure. Since the boundaries $\partial T_t C$ are disjoint, all but countable number of them have zero measure. Moreover, since $T_{n^{-1}}C$, $n \in \mathbb{N}$, form a base for $\mathcal{S}_\tau$ and μ is nontrivial on $\mathcal{S}_\tau$, we can find $s > 0$ such that $\mu(T_s C) > 0$.

Fixing s, we observe that $\tau_1(x) = \tau(x)/s$ is a continuous modulus that generates the same bornology, such that $T_s C = \{x : \tau_1(x) > 1\}$. Therefore, without loss of generality, we assume that C is a μ-continuity set with $\mu(C) > 0$.

Since $\mu(\partial C) = 0$, we have that $\nu(T_t C)/\nu(T_t B) \to \mu(C) \in (0, \infty)$ as $t \to \infty$. Thus,

$$\frac{\nu(T_t\,\cdot\,)}{\nu(T_t C)} = \frac{\nu(T_t\,\cdot\,)}{\nu(T_t B)} \frac{\nu(T_t B)}{\nu(T_t C)} \xrightarrow{\mathcal{S}} \frac{\mu(\,\cdot\,)}{\mu(C)} \quad \text{as } t \to \infty. \qquad \square$$

The regular variation property is often formulated by assuming the existence of an increasing sequence $(a_n)_{n\in\mathbb{N}}$ of positive constants such that $n(T_{a_n^{-1}}\nu)$ vaguely converges in $\mathcal{S}$ to μ as $n \to \infty$. A variant of this definition goes back to [32], who also require that the sequence (a_n) is regularly varying; another variant can be found in [38, Definition B.2.1], where the sequence (a_n) is assumed to be monotonic and converging to infinity. Below we show that this definition is equivalent to ours, see Theorem 2.2.3(iii).

Theorem 2.2.3. *Assume (A), and let $\mathcal{S}$ be an ideal on $\mathbb{X}$ that satisfies (B). The following statements are equivalent:*

(i) Measure ν is regularly varying, that is, $\nu \in \mathrm{RV}(\mathbb{X}, \mathcal{S}, \mu)$.

(ii) There exist an $\alpha > 0$ and a function $g \in \mathrm{RV}_\alpha$, such that $g(t)T_{t^{-1}}\nu \xrightarrow{\mathcal{S}} \mu_g$, for some nontrivial measure $\mu_g \in \mathcal{M}(\mathbb{X}, \mathcal{S})$.

(iii) There exists a nondecreasing sequence $(a_n)_{n\in\mathbb{N}}$ of positive constants such that $nT_{a_n^{-1}}\nu \xrightarrow{\mathcal{S}} \mu_a$ for some nontrivial measure $\mu_a \in \mathcal{M}(\mathbb{X}, \mathcal{S})$.

If either of these condition holds, then μ is α-homogeneous with $\alpha > 0$, the measures μ and μ_g agree up to a multiplicative constant, and it is possible to choose g such that $\mu = \mu_g$. Moreover, the sequence (a_n) is regularly varying and it is possible to choose (a_n) such that $\mu = \mu_a$.

Proof. The implication (i) $\Rightarrow$ (ii) is immediate by letting $g(t) = 1/\nu(T_t C)$. Define $h(u) = \lim_{t\to\infty} g(ut)/g(t)$. Then for all except for countably many $u, v > 0$ we have $0 < h(uv) < \infty$ and

$$h(uv) = \lim_{t\to\infty} \frac{g(uvt)}{g(t)} = \lim_{t\to\infty} \frac{g(ut)}{g(t)} \lim_{t\to\infty} \frac{g(vt)}{g(t)} = h(u)h(v),$$

which means that $h(u) = u^\alpha$, that is, g is regularly varying (see [55, Theorem 1.3]). For the converse, (ii) implies that $g(t)\nu(T_t D) \to \mu_g(D) \in (0, \infty)$ for some $D \in \mathcal{S}$. Thus,

$$\frac{T_{t^{-1}}\nu}{\nu(T_t D)} = \frac{g(t)T_{t^{-1}}\nu}{g(t)T_{t^{-1}}\nu(D)} \xrightarrow{\mathcal{S}} \frac{\mu_g}{\mu_g(D)} \quad \text{as } t \to \infty.$$

Condition (ii) implies that μ_g is α-homogeneous with α being the index of regular variation of g. Indeed, if $B \in \mathcal{S}$ and $\mu(\partial B) = 0$, then $g(t)\nu(T_t B) \to \mu(B)$ as $t \to \infty$, and for all $s > 0$ we have

$$(T_s\mu)(B) = \mu(T_{s^{-1}}B) = \lim_{t\to\infty} g(t)T_{t^{-1}}\nu(T_{s^{-1}}B) = \lim_{t\to\infty} \frac{g(t)}{g(s^{-1}t)} g(s^{-1}t)T_{st^{-1}}\nu(B) \to s^\alpha \mu(B)$$

in view of the regular variation property of g. Consider a $B \in \mathcal{S}$ such that $\mu(T_{[1,\infty)}B) < \infty$. Since $\mu(T_t T_{[1,\infty)}B) = t^{-\alpha}\mu(T_{[1,\infty)}B)$ is decreasing in $t > 0$, we necessarily have $\alpha > 0$. This establishes the equivalence between conditions (i) and (ii).

The equivalence of (ii) and (iii) is essentially the content of Theorem B.2.2 in [38]. However, we provide an independent and complete proof for the benefit of the reader, filling in some details that were omitted, and not relying on the Prokhorov theorem.

Assuming that (ii) holds, let $a_n = \inf\{t : g(t) \geq n\}$. Since g is regularly varying with a positive index α, the number a_n is well-defined for each n. By (ii), we have

$$g(a_n)(T_{a_n^{-1}}\nu) \xrightarrow{\mathcal{S}} \mu_g \,.$$

The regular variation property of g implies that $g(a_n) \sim n$ as $n \to \infty$, see Section 0.4.1 in Resnick [48], which yields (iii) with $\mu_a = \mu_g$. Note that condition (B) is not used for this implication.

What is left to show is that (iii) implies (ii). We actually show that (iii) implies (i) which is then equivalent to the required. Let us first show that under (iii), $\lim_n a_n = \infty$. Assume that $\sup_n a_n = s < \infty$. Choose a constant $t > 0$ such that $\nu(T_{st}C) > 0$. This is always possible since the union of $T_t C$ over $t > 0$ equals $\mathbb{X}'$. Furthermore, note that for the same t, the set $T_t C$ is a μ-continuity set. Then

$$n(T_{a_n^{-1}}\nu)(T_t C) = n\nu(T_{ta_n}C) \to \mu(T_t C) \in (0,\infty) \quad \text{as } n \to \infty \,.$$

However, $n\nu(T_{ta_n}C) \geq n\nu(T_{st}C) \to \infty$ as $n \to \infty$, since we assumed $\nu(T_{st}C) > 0$. Thus we obtain a contradiction. This shows that $\lim_n a_n = \infty$.

Take $u > 0$ and $A \subseteq \mathbb{S}$ such that $\mu_a(\partial T_{[u,\infty)}A) = 0$. Since a_n is monotonically increasing to ∞, any $t > 0$ belongs to a unique interval $[a_{n(t)}, a_{n(t)+1})$. Thus,

$$\frac{\nu(T_t T_{[u,\infty)}A)}{\nu(T_t C)} \leq \frac{n(t)+1}{n(t)} \frac{n(t)\nu(T_{a_{n(t)}}T_{[u,\infty)}A)}{(n(t)+1)\nu(T_{a_{n(t)+1}}C)} \,.$$

The right-hand side converges to $\mu_a(T_{[u,\infty)}A)/\mu_a(C)$ as $t \to \infty$. Applying a similar bound from below, it follows that the left-hand side above has the same limit as $t \to \infty$. By Lemma 2.1.2, $\nu_t \xrightarrow{\mathcal{S}} \mu$ as $t \to \infty$, that is, (i) follows from (iii). $\square$

The regular variation property can often be easily verified by checking the conditions of Proposition 3.3.3 proved in Section 3.3. As shown in Theorem 2.2.3, the tail measure μ is necessarily homogeneous, and its homogeneity index α is referred to as the *tail index* of ν. Each α-homogeneous measure on $(0,\infty)$ is proportional to the measure $\theta_\alpha(dt) = t^{-\alpha-1}dt$. The following result from [22] specifies a representation of general homogeneous measures.

Proposition 2.2.4. *Assume that $\mathbb{X}$ is a topological space with a measurable scaling such that $\mathbf{0} = \varnothing$, and let $\mu \in \mathcal{M}(\mathbb{X}, \mathcal{S})$ be an α-homogeneous Borel measure with $\alpha > 0$. Assume that $\mathcal{S}$ is a bornology on $\mathbb{X}$ which admits a countable base (B_n), such that each B_n is a semicone and $\cup_{n\in\mathbb{N}} T_{(0,\infty)} B_n = \mathbb{X}$. Then μ is the pushforward of $\sigma \otimes \theta_\alpha$ by the map $(u,t) \mapsto T_t u$, where σ is a probability measure on $\mathbb{X}$. Equivalently,*

$$\mu(B) = \alpha \int_0^\infty \sigma(T_t B) t^{\alpha-1}\, dt, \quad B \in \mathcal{S} \cap \mathcal{B}(\mathbb{X}). \tag{2.4}$$

The measure σ is called the *spectral measure* of μ. The condition on $\mathcal{S}$ imposed in Proposition 2.2.4 holds if $\mathcal{S}$ satisfies (B) and then σ can be chosen to be a finite measure on $\mathbb{S} = \{x \in \mathbb{X} : \tau(x) = 1\}$. In this form, (2.4) also appears in [38], under additional assumptions that $\mathbf{0}$ is a singleton and the scaling is continuous. If σ is supported by $\mathbb{S}$, then, for each Borel $A \subseteq \mathbb{S}$,

$$\sigma(A) = \mu(T_{(1,\infty)} A).$$

Note that the spectral measure is not necessarily unique unless its support $\mathbb{S}$ is fixed as above. It is always possible to normalise σ to obtain a probability measure supported by $T_u \mathbb{S} = \{x \in \mathbb{X} : \tau(x) = u\}$ for some $u > 0$.

The following result concerns regular variation with respect to smaller and larger ideals than the original one.

Proposition 2.2.5. *Assume (A), and equip $\mathbb{X}$ with a scaling and topologically consistent ideal $\mathcal{S}$ which admits a countable open base. Let $\nu \in \mathrm{RV}(\mathbb{X}, \mathcal{S}, \mu)$ and let B be the set from (2.2).*

(i) Let $\mathcal{S}_1 \subseteq \mathcal{S}$ be a scaling consistent ideal with a countable base. If μ is nontrivial on $\mathcal{S}_1$, then $\nu \in \mathrm{RV}(\mathbb{X}, \mathcal{S}_1, \mu)$.

(ii) For each $A \in \mathcal{B}(\mathbb{X})$ with $\mu(\partial A) = 0$, the ratio $\nu(T_t A)/\nu(T_t B)$ has a (possibly infinite) limit as $t \to \infty$.

Proof. (i) By condition, there is a $D \in \mathcal{S}_1$ such that

$$\frac{\nu(T_t D)}{\nu(T_t B)} \to \mu(D) \in (0, \infty).$$

Thus, (2.2) holds on $\mathcal{S}_1$ with B replaced by D, similarly as in the proof of Lemma 2.2.2.
(ii) Since $\nu(T_t B) \to 0$, we have

$$\frac{\nu(T_t A)}{\nu(T_t B)} \geq \frac{\nu(T_t(A \cap \mathbf{0}))}{\nu(T_t B)} \to \infty \quad \text{as } t \to \infty$$

if $\nu(A \cap \mathbf{0}) > 0$. Assume that $A \subseteq \mathbb{X}'$ and let (B_n) be a countable open base of $\mathcal{S}$ with $\mu(\partial B_n) = 0$ for all n (its existence is shown in the proof of Lemma 2.1.2). Then

$$\frac{\nu(T_t(A \cap B_n))}{\nu(T_t B)} \to \mu(A \cap B_n).$$

By the monotone convergence, we have also

$$\frac{\nu(T_t A)}{\nu(T_t C)} \to \mu(A) = \lim_{n\to\infty} \mu(A \cap B_n).$$

□

2.3 Reduction to a subcone

It is common in the theory of extreme values to consider not only the basic space $\mathbb{X}$ but also its strict subcones. This phenomenon is well-studied in $\mathbb{R}^d$ and is called *hidden regular variation*, see, for instance, Section 9.4 of [50] and Example 2.4.8 below.

Recall that for an ideal $\mathcal{S}$, we denote $\mathbb{U}(\mathcal{S}) = \cup\{B : B \in \mathcal{S}\}$. Let $\mathcal{S}$ be an ideal on $\mathbb{X}$, and let $\mathcal{S}_1$ be another ideal such that $\mathcal{S}_1 \subseteq \mathcal{S}$. Without loss of generality, we assume that $\mathbb{U}(\mathcal{S}) = \mathbb{X}$ and $\mathbb{U}(\mathcal{S}_1)$ is a strict subcone in $\mathbb{X}$, denoted by $\mathbb{X}_1$, so that $\mathcal{S}$ and $\mathcal{S}_1$ are bornologies on $\mathbb{X}$ and $\mathbb{X}_1$, respectively. We equip $\mathbb{X}_1$ with the topology induced by $\mathbb{X}$ or any stronger topology. By assumption, $\mathcal{S}_1$ is the subset of the induced bornology $\mathcal{S} \cap \mathbb{X}_1 = \{B \cap \mathbb{X}_1 : B \in \mathcal{S}\}$. We assume that all bornologies satisfy condition (B), and denote the semicones generating $\mathcal{S}$ and $\mathcal{S}_1$ by C and C_1, respectively. Note that $C_1 \in \mathcal{S}$ and that $C \cap \mathbb{X}_1$ generates $\mathcal{S} \cap \mathbb{X}_1$.

If ν is a measure on $\mathcal{B}(\mathbb{X})$, then $\nu|_{\mathbb{X}_1}$ denotes the restriction of ν to $\mathbb{X}_1$. It is worth noting that this restriction may be regularly varying even if ν is not regularly varying, see Example 2.4.1.

The following result provides a condition that guarantees the preservation of the regular variation property when passing to a subcone.

Proposition 2.3.1. *Assume (A) and let $\mathbb{X}_1$ be a G_δ subset of $\mathbb{X}$ that is a cone. Endow $\mathbb{X}$ and its subcone $\mathbb{X}_1$ with bornologies $\mathcal{S}$ and $\mathcal{S}_1$ generated by semicones C and C_1, respectively. Let $\nu \in \mathrm{RV}(\mathbb{X}, \mathcal{S}, \mu)$ with the tail index α.*

(i) If μ is nontrivial on the bornology $\mathcal{S} \cap \mathbb{X}_1$ (equivalently, if $\mu(C \cap \mathbb{X}_1) \in (0, \infty)$), then $\nu|_{\mathbb{X}_1} \in \mathrm{RV}(\mathbb{X}_1, \mathcal{S} \cap \mathbb{X}_1, \mu_{\mathbb{X}_1})$ with tail index α.

(ii) If $\nu|_{\mathbb{X}_1} \in \mathrm{RV}(\mathbb{X}_1, \mathcal{S}_1, \mu_1)$ with tail index α_1, then either $\mu(C_1) \in (0, \infty)$ and then $\mu_1 = (\mu(C_1))^{-1}\mu$, $\alpha_1 = \alpha$ and $\nu|_{\mathbb{X}_1} \in \mathrm{RV}(\mathbb{X}_1, \mathcal{S} \cap \mathbb{X}_1, \mu_1)$ or $\mu(C_1) = 0$ and then $\alpha_1 \geq \alpha$.

Proof. Theorem 4.3.23 from [21] implies that $\mathbb{X}_1$ is a Polish space.

(i) follows from Proposition 2.2.5.

(ii) We have that

$$\frac{\nu(T_t\,\cdot)}{\nu(T_t C)} \xrightarrow{s} \mu(\,\cdot\,) \quad \text{as } t \to \infty. \tag{2.5}$$

In particular,

$$\frac{\nu(T_{ts} C_1)}{\nu(T_t C)} \to \mu(T_s C_1) = s^{-\alpha}\mu(C_1) \quad \text{as } t \to \infty$$

for all $s > 0$, hence also for $s = 1$. Assume first that $\mu(C_1) \in (0, \infty)$. Since (2.5) holds in $\mathcal{S}_1$ as well, we have

$$\frac{\nu(T_t\,\cdot\,)}{\nu(T_tC_1)}\frac{\nu(T_tC_1)}{\nu(T_tC)} \xrightarrow{\mathcal{S}_1} \mu(\,\cdot\,) \quad \text{as } t \to \infty.$$

Thus,

$$\frac{\nu|_{\mathbb{X}_1}(T_t\,\cdot\,)}{\nu|_{\mathbb{X}_1}(T_tC_1)} \xrightarrow{\mathcal{S}_1} \frac{1}{\mu(C_1)}\mu(\,\cdot\,) \quad \text{as } t \to \infty.$$

If $\mu(C_1) = 0$, then

$$\frac{\nu(T_{ts}C_1)}{\nu(T_tC)} \to 0 \quad \text{as } t \to \infty,$$

meaning that the tail index of $\nu|_{\mathbb{X}_1}$ is not smaller than the tail index of ν. □

2.4 Examples

Standard examples of regularly varying vectors in Euclidean space with linear scaling are well-known. In this section, we present various examples and demonstrate the application of the results shown in previous sections. In many of our examples, we use Proposition 3.3.3 from Chapter 3 to pass to polar coordinates and simplify calculations.

Example 2.4.1. Let $\mathbb{X} = \mathbb{R}_+^2$ equipped with the scaling $T_t(x_1, x_2) = (tx_1, x_2)$, $(x_1, x_2) \in \mathbb{X}$ and consider the subcone $\mathbb{X}_1 = \mathbb{R}_+ \times \{1\}$. We define the modulus τ by setting $\tau(x_1, x_2) = x_1$. Therefore, the bornology $\mathcal{S}_\tau$ on $\mathbb{X}$ is induced by $C = (1, \infty) \times \mathbb{R}_+$. We consider a random vector $\xi = (\varepsilon\xi_1 + (1 - \varepsilon)\xi_2, \varepsilon)$ in $\mathbb{X}$, where ε is a Bernoulli distributed random variable with parameter $1/2$, ξ_1 is a Pareto-1 distributed random variable that is independent of ε, and ξ_2 is a random variable that is independent of both ξ_1 and ε, with distribution function $F(t) = 1 - (\log t)^{-1}$. Note that ξ_2 is distributed as e^η for a Pareto-1 distributed random variable η. We denote the probability measure of ξ with ν. For $t > 0$, we have

$$\nu(T_tC) = \mathbf{P}\{\tau(\xi) > t\} = \frac{1}{2}\left(\mathbf{P}\{\xi_1\varepsilon > t \mid \varepsilon = 1\} + \mathbf{P}\{\xi_2 > t \mid \varepsilon = 0\}\right) = \frac{1}{2}\left(t^{-1} + (\log t)^{-1}\right).$$

Equip $\mathbb{X}_1$ with the induced bornology. Then, for $t > 0$, we have

$$\nu|_{\mathbb{X}_1}(T_tC) = \mathbf{P}\{\xi \in (t, \infty) \times \{1\}\} = \frac{1}{2}\mathbf{P}\{\xi_1 > t\} = \frac{1}{2}t^{-1}.$$

Thus, even though the measure ν is not regularly varying (see Proposition 3.3.3), its restriction $\nu|_{\mathbb{X}_1}$ is regularly varying.

Example 2.4.2. Let $\mathbb{X} = \mathbb{R}$ with the linear scaling and the modulus $\tau(x) = |x|$. A measure ν on $\mathbb{R}$ is regularly varying if its pushforward by the map $x \mapsto |x|$ is regularly varying on $\mathbb{R}_+$. In particular, it is possible that ν restricted on the right and left half-lines is regularly varying with different indices α. Then the tail measure μ may be supported by a half-line. For example, consider a random variable $X = \varepsilon Y_\alpha + (\varepsilon - 1)Y_\beta$, where ε is a Bernoulli distributed with parameter $1/2$ independent of both Y_α and Y_β which are mutually independent Pareto distributed with indices α and β respectively. In this case, $\tau(X)$ is regularly varying with index $\alpha \wedge \beta$. The random variable X is regularly varying with the same index, which means that its probability distribution is a regularly varying measure.

Example 2.4.3. Let $\mathbb{X} = \mathbb{R}_+^2$ with the scaling $T_t(x_1, x_2) = (t^{-1}x_1, t^{-1}x_2)$, $(x_1, x_2) \in \mathbb{X}$. Then $\mathbb{X}' = \mathbb{R}_+^2 \setminus \{(0,0)\}$. Define modulus with $\tau(x_1, x_2) = \max(x_1, x_2)^{-1}$. Note, this modulus generates a bornology on $\mathbb{X}'$. Let $V = (V_1, V_2)$ be a random element in $\mathbb{X}$ where V_1 and V_2 are independent uniform random variables on $[0, 1]$. It is easily seen that $\tau(V)$ is regularly varying random variable:

$$\mathbf{P}\left\{\tau(V) > t\right\} = \mathbf{P}\left\{\max(V_1, V_2) < t^{-1}\right\} = t^{-2}, \quad t \geq 1.$$

We use Proposition 3.3.3 to conclude that V is a regularly varying random vector in $\mathbb{X}$ with index 2. To show this, consider $A = \{1\} \times [0, a] \subseteq \mathbb{S}$ for some $0 < a \leq 1$, let $u > 0$ and $t > 0$ such that $ut \geq 1$. We have

$$\begin{aligned}
\mathbf{P}\left\{\rho(V) \in A \times (ut, \infty)\right\} = \mathbf{P}\left\{V_1 > V_2, \frac{V_2}{V_1} \leq a, V_1^{-1} > ut\right\} = \\
&= \int_0^1 \mathbf{P}\left\{V_1 > s, V_1 \geq s/a, V_1 < (ut)^{-1}\right\} ds = \\
&= \int_0^{a(ut)^{-1}} \mathbf{P}\left\{s/a \leq V_1 < (ut)^{-1}\right\} ds = \\
&= \frac{1}{2} a (ut)^{-2}.
\end{aligned}$$

The conclusion follows after dividing this term with $\mathbf{P}\left\{\tau(V) > t\right\}$ and taking the limit as t approaches infinity, using the symmetry of coordinates.

Example 2.4.4. Let $\mathbb{X} = (0, \infty)^2$ and define the scaling $T_t x = (t^{-1}x_1, t^{-1}x_2)$, $x = (x_1, x_2) \in \mathbb{X}$. Then $\mathbf{0}$ is empty. It is easily seen that $\mathbb{X}$ is a G_δ subset of a Polish space, since it is an open subset of $\mathbb{R}^2$. Define the modulus by setting $\tau(x_1, x_2) = \min(x_1, x_2)^{-1}$. For the same V with independent uniform components V_1, V_2 as in Example 2.4.3, we have

$$\mathbf{P}\left\{\tau(V) > t\right\} = \mathbf{P}\left\{\min(V_1, V_2) < t^{-1}\right\} = 1 - (1 - t^{-1})^2, \quad t \geq 1,$$

so $\tau(V)$ is a regularly varying function with index 1. We can again use polar decomposition to conclude that V is a regularly varying random vector: let $A = \{1\} \times [a, \infty) \subseteq \mathbb{S}$ for some

$a \geq 1$ and let $u > 0$ and $t > 0$ such that $ut \geq 1$. We have

$$\begin{aligned}\mathbf{P}\left\{\rho(V) \in A \times (ut, \infty)\right\} &= \mathbf{P}\left\{V_1 < V_2,\, \frac{V_2}{V_1} \geq a,\, V_1^{-1} > ut\right\} = \\ &= \int_0^1 \mathbf{P}\left\{V_1 < s, V_1 \leq s/a, V_1 < (ut)^{-1}\right\} ds = \\ &= \int_0^{a(ut)^{-1}} \mathbf{P}\left\{V_1 \leq s/a\right\} ds + \int_{a(ut)^{-1}}^1 \mathbf{P}\left\{V_1 < (ut)^{-1}\right\} ds = \\ &= (ut)^{-1} - \frac{1}{2} a (ut)^{-2},\end{aligned}$$

so dividing by $\mathbf{P}\left\{\tau(V) > t\right\}$ and letting $t \to \infty$ gives us $u^{-1}/2$, which does not depend on a. Therefore, we can let $a \to \infty$ to conclude that V is not regularly varying in the bornology generated by this modulus.

Example 2.4.5. Let $\mathbb{X} = \mathsf{C}([0,1])$ be the space of continuous functions $x : [0,1] \to \mathbb{R}$ equipped with the uniform metric and the linear scaling applied to functions' values. Consider the bornology generated by the modulus $\tau(x) = \|x\|_\infty$ given by the uniform norm of the function x. By Proposition 3.3.3, a random continuous function ξ is regularly varying if and only if the conditional distribution of $\xi/\|\xi\|$ given that $\|\xi\| > t$ weakly converges to a probability measure on $\mathbb{S} = \{x \in \mathbb{X} : \tau(x) = 1\}$. This is the case if the conditional finite-dimensional distributions of the normalised function converge and the sequence of the conditional distributions of normalised functions is tight in the space of continuous functions. Regular variation on the space $\mathsf{C}([0,1])$ has been studied in [18] and [26].

Example 2.4.6. Let $\mathbb{X} = \mathbb{R}^d$ with the scaling $T_t x = (t^{\alpha_1} x_1, \dots, t^{\alpha_d} x_d)$ with $\alpha_1, \dots, \alpha_d \in \mathbb{R}$. To ease the notation, assume that $d = 3$ and $\alpha_1 = 1, \alpha_2 = -1, \alpha_3 = 0$. Then $\mathbf{0} = \{0\} \times \{0\} \times \mathbb{R}$. A modulus can be defined as

$$\tau(x) = |x_1| + |x_2|^{-1}, \quad x = (x_1, x_2, x_3).$$

The corresponding bornology is generated by the semicone $C = [-1,1]^c \times [-1,1] \times \mathbb{R}$. A random vector $\xi = (\xi_1, \xi_2, \xi_3)$ is regularly varying if the distribution of $(t^{-1}\xi_1, t\xi_2, \xi_3)$ normalised by $\mathbf{P}\left\{|\xi_1| \geq t, |\xi_2| \leq t\right\}$ vaguely converges.

Example 2.4.7. Let $\mathbb{X} = \mathbb{R}_+^\infty$ with the linear scaling. This space does not admit a continuous modulus, as shown in Example 1.4.14. Instead, consider the bornology generated by sets of the form $\{x \in \mathbb{X} : x_i > t\}$, $i \in \mathbb{N}$, $t > 0$. A random element ξ in $\mathbb{X}$ is regularly varying if the distribution of $T_{t^{-1}}\xi$ normalised by $\mathbf{P}\left\{T_{t^{-1}}\xi \in B\right\}$ vaguely converges for a suitable choice of $B \in \mathcal{S}$. This convergence is verified by checking it for finite-dimensional distributions, as stated in [35, Proposition 2.2]. For instance, if the components of $\xi = (\xi_n)$ are i.i.d. regularly varying random variables, then ξ is regularly varying and its tail measure is supported by the set of sequences with exactly one nonzero component, see [50, page 192]. In what follows, we provide more details showing that any finite-dimensional distribution of a sequence with Pareto-distributed components is regularly varying with induced bornology.

Let ξ_n be i.i.d. Pareto distributed with parameter $\alpha > 0$. Then, it suffices to analyse finite-dimensional distributions of the form $(\xi_1, \xi_2, \dots, \xi_m)$, $m \in \mathbb{N}$. For each finite-dimensional distribution $(\xi_1, \xi_2, \dots, \xi_m)$, $m \geq 2$, note that the bornology induced on $\mathbb{R}_+^m$ corresponds to the bornology generated by $\tau_{\max}$ as given in Example 1.4.5. Furthermore, observe that $\tau_{\max}(\xi_1, \xi_2, \dots, \xi_m)$ is regularly varying with index α:

$$\frac{\mathbf{P}\left\{\bigvee_{i=1}^m \xi_i > ut\right\}}{\mathbf{P}\left\{\bigvee_{i=1}^m \xi_i > t\right\}} = \frac{1-(1-(ut)^{-\alpha})^m}{1-(1-t^{-\alpha})^m} \to u^{-\alpha} \quad \text{as } t \to \infty$$

for all $u > 0$, using L'Hospital rule, for example (starting from t large enough so that $ut > 1$). By Lemma 1.3.9, let $C = \{x \in \mathbb{R}^m : \tau_{\max}(x) > 1\}$. Then $(T_{n^{-1}}C)_{n \in \mathbb{N}}$ is a base for the bornology. We now follow [50, page 192]. First define

$$a_n = \left(\frac{1}{1-F}\right)^{\leftarrow}(n) = F^{\leftarrow}\left(1-\frac{1}{n}\right), \quad \text{where } F(x) = 1 - x^{-\alpha},\ x \geq 1.$$

Since $F^{\leftarrow}(x) = (1-x)^{-1/\alpha}$, we get the explicit form of the sequence (a_n):

$$a_n = n^{1/\alpha}, \quad n \in \mathbb{N}.$$

For this choice of a_n, it can be easily shown that $n\mathbf{P}\{a_n^{-1}\xi_1 > c\} \to c^{-\alpha}$, $c > 0$. Let $c_1, c_2, \dots, c_m \in (0, \infty) = \operatorname{supp} F$, and denote by $[\mathbf{0}, \mathbf{c}]$ the set $[0, c_1] \times [0, c_2] \times \dots \times [0, c_m]$ (we use [50, Lemma 6.1]). We have

$$\begin{aligned}
n\mathbf{P}\left\{T_{a_n^{-1}}(\xi_1, \xi_2, \dots, \xi_m) \in [\mathbf{0}, \mathbf{c}]^c\right\} &= n\mathbf{P}\left\{\bigcup_{i=1}^m \left(\frac{\xi_i}{a_n} > c_i\right)\right\} = \\
&= \sum_{i=1}^m n\mathbf{P}\left\{\frac{\xi_i}{a_n} > c_i\right\} - \sum_{1 \leq i < j \leq m} n\mathbf{P}\left\{\frac{\xi_i}{a_n} > c_i, \frac{\xi_j}{a_n} > c_j\right\} + \dots \\
&\quad \dots + (-1)^{m-1} n\mathbf{P}\left\{\bigcap_{i=1}^m \left(\frac{\xi_i}{a_n} > c_i\right)\right\} \\
&\to \sum_{i=1}^m c_i^{-\alpha},
\end{aligned}$$

where we use the regularly varying property of each coordinate (equation (1.6) and the paragraph underneath). All summands except for the first one converge to 0. If there exists $j \in \{1, 2, \dots, m\}$ such that $c_j \neq 0$ and $c_i = 0$ for $i \neq j$, we have the marginal distribution:

$$n\mathbf{P}\left\{T_{a_n^{-1}}(\xi_1, \xi_2, \dots, \xi_m) \in [\mathbf{0}, \mathbf{c}]^c\right\} = n\mathbf{P}\left\{\frac{\xi_j}{a_n} > c_j\right\} \to c_j^{-\alpha}.$$

To conclude, the tail measure μ_a is given by $\mu_a([\mathbf{0}, \mathbf{c}]^c) = \sum_{i=1}^m c_i^{-\alpha}$, $\mathbf{c} = (c_1, c_2, \dots, c_m) \in \mathbb{R}_+^m \setminus \{\mathbf{0}\}$, where we set $0^{-\alpha} = 0$.

As we will see from the following example, it is possible for a measure ν to belong to both $\mathrm{RV}(\mathbb{X}, \mathcal{S}, \mu)$ and $\mathrm{RV}(\mathbb{X}_1, \mathcal{S}_1, \mu_1)$, where μ and μ_1 are two measures on $\mathbb{X}$ and $\mathbb{X}_1$, respectively, but the tail index α of μ is strictly smaller than the tail index α_1 of μ_1.

Example 2.4.8. Consider the space $\mathbb{X} = \mathbb{R}_+^2$ and its subcone $\mathbb{X}_1 = (0,\infty)^2$ with the linear scaling and the moduli introduced in Example 1.4.5. Let $\xi = (\xi_1, \xi_2)$ be a random vector in $\mathbb{R}_+^2$, where ξ_1, ξ_2 are independent and both Pareto-1 distributed, that is, $\mathbf{P}\{\xi_i > t\} = t^{-1}$ for $t \geq 1$, $i = 1, 2$. It is well-known that ξ is regularly varying of index 1 on $\mathbb{X}$ with the bornology generated by $\tau_{\max}$ (see Example 2.4.7). The spectral measure is supported by the axes, it vanishes on $\mathbb{X}_1$, and so Proposition 2.3.1(i) does not apply. We use the characterisation from Proposition 3.3.3 to easily prove this. For example if we denote by $\rho_{\max}$ the polar decomposition map that uses $\tau_{\max}$, and if we take $A = [0, a] \times \{1\} \subseteq \mathbb{S} = \{x : \tau_{\max} = 1\}$, $0 < a \leq 1$, $u > 0$ and t sufficiently large so that $ut > 1$, we have

$$\begin{aligned}
\mathbf{P}\{\rho_{\max}(\xi) \in A \times (ut, \infty)\} &= \mathbf{P}\{\xi_2 \geq \xi_1,\ 0 \leq \xi_1/\xi_2 \leq a,\ \xi_2 > ut\} = \\
&= \int_1^{aut} \mathbf{P}\{\xi_2 > ut\} s^{-2} ds + \int_{aut}^{\infty} \mathbf{P}\{\xi_2 \geq s/a\} s^{-2} ds = \\
&= (ut)^{-1} - \frac{1}{a}(ut)^{-2},
\end{aligned}$$

which, divided by $\mathbf{P}\{\tau_{\max}(\xi) > t\}$, on the limit as $t \to \infty$, gives $\frac{1}{2}u^{-1}$. Since this final term does not depend on a, we can allow $a \to 0$ to see that the tail measure is supported by the y-axis. Symmetry and equality in distribution of the coordinates of ξ result in analogous conclusion for other choices of A.

For other moduli and all sufficiently large t,

$$\begin{aligned}
\mathbf{P}\{\tau_{\min}(\xi) > t\} &= t^{-2}, \\
\mathbf{P}\{\tau_\beta(\xi) > t\} &= \frac{1}{1-2\beta} t^{-2} - \frac{2\beta}{1-2\beta} t^{-1/\beta}, \quad \beta \in (0, 1/2), \\
\mathbf{P}\left\{\tau_{1/2}(\xi) > t\right\} &= t^{-2}(1 + 2\log t), \\
\mathbf{P}\left\{\tau^*_\beta(\xi) > t\right\} &= \frac{2-2\beta}{1-2\beta} t^{-1/(1-\beta)} - \frac{1}{1-2\beta} t^{-2}, \quad \beta \in (0, 1/2).
\end{aligned}$$

Note that the main asymptotical term for $\tau_{1/2}$ has also been derived in [34, Lemma 4.1(4)]. These moduli of ξ are regularly varying with index 1 or 2 or $1/(1-\beta)$ in the last case. These calculations are briefly below. For the first modulus $\tau_{\min}(\xi) = \min(\xi_1, \xi_2)$, it is straightforward that $\mathbf{P}\{\tau_{\min}(\xi) > t\} = \mathbf{P}\{\xi_1 > t\}^2 = t^{-2}$ is regularly varying with index 2 since ξ_1 and

ξ_2 are i.i.d. Let $t > 0$ be large enough. For τ_β we have

$$\begin{aligned}
\mathbf{P}\left\{\tau_\beta(\xi_1,\xi_2) > t\right\} &= \mathbf{P}\left\{\xi_1 < \xi_2,\, \xi_1^{1-\beta}\xi_2^{\beta} > t\right\} + \mathbf{P}\left\{\xi_1 \geq \xi_2,\, \xi_1^{\beta}\xi_2^{1-\beta} > t\right\} = \\
&= \mathbf{P}\left\{\xi_1 < \xi_2, \xi_2 > (t\xi_1^{-1+\beta})^{\frac{1}{\beta}}\right\} + \mathbf{P}\left\{\xi_1 \geq \xi_2,\, \xi_1 > (t\xi_2^{-1+\beta})^{\frac{1}{\beta}}\right\} = \\
&= 2\int_1^\infty \mathbf{P}\left\{s < \xi_2,\, \xi_2 > t^{\frac{1}{\beta}} s^{-\frac{1}{\beta}+1}\right\} s^{-2} ds = \\
&= 2t^{-\frac{1}{\beta}} \int_1^t s^{\frac{1}{\beta}-3} ds + 2\int_t^\infty s^{-3} ds = \\
&= \frac{1}{1-2\beta} t^{-2} - \frac{2\beta}{1-2\beta} t^{-\frac{1}{\beta}}.
\end{aligned}$$

Since $\beta < \frac{1}{2}$, we have $1/\beta > 2$, so the second power term decays to zero much faster than the first one. This implies that $\tau_\beta(\xi)$ is regularly varying with index 2. For $\tau_{1/2}$ we have

$$\begin{aligned}
\mathbf{P}\left\{\sqrt{\xi_1\xi_2} > t\right\} &= \mathbf{P}\left\{\xi_1 > t^2\xi_2^{-1}\right\} = \int_1^\infty \mathbf{P}\left\{\xi_1 > t^2 s^{-1}\right\} s^{-2} ds = \\
&= \int_1^{t^2} \mathbf{P}\left\{\xi_1 > t^2 s^{-1}\right\} s^{-2} ds + \int_{t^2}^\infty 1 \cdot s^{-2} ds = \\
&= t^{-2}\left(1 + 2\log t\right),
\end{aligned}$$

which is regularly varying with index 2. For τ_β^* we have

$$\begin{aligned}
\mathbf{P}\left\{\tau_\beta^*(\xi) > t\right\} &= \mathbf{P}\left\{\xi_1 < \xi_2,\; \xi_1^{\beta}\xi_2^{1-\beta} > t\right\} + \mathbf{P}\left\{\xi_1 \geq \xi_2,\; \xi_1^{1-\beta}\xi_2^{\beta} > t\right\} = \\
&= 2\int_1^\infty \mathbf{P}\left\{\xi_2 > s, \xi_2 > (ts^{-\beta})^{\frac{1}{1-\beta}}\right\} s^{-2} ds = \\
&= 2\left(t^{-\frac{1}{1-\beta}} \int_1^t s^{\frac{\beta}{1-\beta}-2} + \int_t^\infty s^{-3} ds\right) = \\
&= \frac{2-2\beta}{1-2\beta} t^{-\frac{1}{1-\beta}} - \frac{1}{1-2\beta} t^{-2}.
\end{aligned}$$

Since $\beta < 1/2$ we have $1/(1-\beta) < 2$ so the second term decays to zero faster. Thus, the index of regular variation for $\tau_\beta^*(\xi)$ is $1/(1-\beta)$.

To establish the regular variation property of ξ, we will use the characterisation from Proposition 3.3.3. Let τ be a continuous modulus on $\mathbb{X}_1$ and let $\mathbb{S}_1$ be the unit Euclidean circle in $(0,\infty)^2$. Since τ is continuous and $\tau(x) > 0$ for all $x \in \mathbb{X}_1$, it follows that $\tau(u) \geq \varepsilon > 0$ for all $u \in A$, where A is any compact subset of $\mathbb{S}_1$. Suppose that A is a compact subset of $\mathbb{S}_1$ and $A' = \{T_{\tau(u)^{-1}} u : u \in A\}$ is a subset of $\{x : \tau(x) = 1\}$. Passing to polar coordinates yields that the probability that the polar decomposition $\rho(\xi)$ is in $A' \times (t,\infty)$ equals

$$\mathbf{P}\left\{(T_{\tau(\xi)^{-1}}\xi, \tau(\xi)) \in A' \times (t,\infty)\right\} = \int_A du \int_{t\tau(u)} (su_1)^{-2}(su_2)^{-2} s ds = t^{-2}\int_A u_1^{-2}u_2^{-2}\tau(u)^{-2} du$$

for all sufficiently large t, since then $t\tau(u)u \in [1,\infty)^2$, which is the support of ξ. Note that the integral over A is taken with respect to the one-dimensional Hausdorff measure on $\mathbb{S}_1$. This integral is finite since A is compact and $\tau(u)$ is bounded away from zero. Thus, the spectral measure on $\mathbb{X}_1$ exists if and only if $\tau(\xi)$ has a power tail of order 2. This is not the case for the modulus $\tau_{\max}$, but holds for the modulus $\tau_{\min}(x) = \min(x_1, x_2)$. Therefore, ξ is regularly varying with index 2 in $(0,\infty)^2$ with the modulus $\tau_{\min}$. This is well-known, for instance, one can take $g(t) = t^2$ in Theorem 2.2.3(ii). In this case, ξ is said to exhibit the effect of hidden regular variation, see [49].

It is worth noting that the same result holds for the modulus τ_β with $\beta \in (0, 1/2)$, but it fails for $\tau_{1/2}$ due to the presence of the logarithmic term. Furthermore, $\tau^*_\beta(\xi)$ exhibits regular variation of order $1/(1-\beta) < 2$, which implies that ξ is not regularly varying on $\mathbb{X}_1$ with this modulus.

A similar construction that leads to the hidden regular variation phenomenon can be applied to the space $\mathbb{R}^2$ with the modulus $\tau(x_1, x_2) = \min(|x_1|, |x_2|)$.

Example 2.4.9. We adapt the setting of Example 2.4.7. Hidden regular variation on $\mathbb{R}_+^\infty$ has been studied in [40]. Let $\mathbb{X}_1 = (0,\infty)^m \times \mathbb{R}_+^\infty$, which is a subset of $\mathbb{X} = \mathbb{R}_+^\infty$. Then $\tau(x) = \min(x_1, \dots, x_m)$, $x = (x_1, x_2, \dots, x_m, \dots) \in \mathbb{R}_+^\infty$, is a modulus on $\mathbb{X}_1$. If we have a sequence of i.i.d. random variables $\xi = (\xi_n)$ in $\mathbb{R}_+$ with Pareto tail, then ξ is regularly varying in $\mathbb{X}_1$, and its tail measure is supported by $(0,\infty)^m \times \{0\}^\infty$.

Example 2.4.10. Let $\mathbb{X}$ be the family of continuous functions $x : [0,1] \to \mathbb{R}_+$ with the topology generated by the uniform metric and the scaling $(T_t x)(u) = t^{-1}x(u)$, $u \in [0,1]$, for $x \in \mathbb{X}$. Let $\mathbb{X}_1$ be a subcone of $\mathbb{X}$, which consists of all functions $x : [0,1] \to (0,\infty)$ with the induced topology. We consider moduli on $\mathbb{X}'$ and $\mathbb{X}'_1$ given by

$$\begin{aligned}
\tau_{\inf}(x) &= \inf\big\{x(u)^{-1} : u \in [0,1]\big\}, \quad x \in \mathbb{X}', \\
\tau_{\sup}(x) &= \sup\big\{x(u)^{-1} : u \in [0,1]\big\}, \quad x \in \mathbb{X}'_1
\end{aligned}$$

and the corresponding bornologies, where $0^{-1} = \infty$. The hidden regular variation phenomenon arises by considering a random continuous function $\xi(u) = V_1 u + V_2(1-u)$, $u \in [0,1]$, where V_1 and V_2 are two independent random variables uniformly distributed on $[0,1]$. Note that $\tau_{\inf}(\xi) = (\max(V_1, V_2))^{-1}$ is regularly varying on $\mathbb{R}_+$ of index 2 since

$$\mathbf{P}\{\tau_{\inf}(\xi) > t\} = \mathbf{P}\big\{\max(V_1, V_2) < t^{-1}\big\} = t^{-2},$$

and $\tau_{\sup}(\xi) = (\min(V_1, V_2))^{-1}$ is regularly varying on $\mathbb{R}_+$ of index 1:

$$\mathbf{P}\{\tau_{\sup}(\xi) > t\} = \mathbf{P}\big\{\min(V_1, V_2) < t^{-1}\big\} = 1 - (1 - t^{-1})^2.$$

The random function ξ is regularly varying in the bornology generated by $\tau_{\inf}$, which can be seen by applying the continuous mapping argument from Theorem 3.2.1. To see this, let $\psi : \mathbb{R}_+^2 \setminus \{(0,0)\} \to \mathbb{X}_1$ be defined as $\psi(x_1, x_2) = x_1 u + x_2(1-u)$ so that $\psi(V_1, V_2) = \xi(u)$ for (V_1, V_2) from Example 2.4.3. The tail measure of ξ here is supported by functions $au + b(1-u)$, where $a, b > 0$, and its index of regular variation is 2. The random function ξ is also regularly

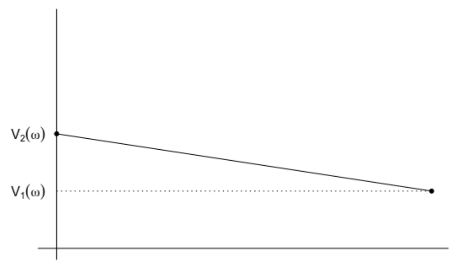


Figure 2.1: *One realisation of a random function ξ from Example 2.4.10 where $V_1(\omega) < V_2(\omega)$ for an elementary event ω.*

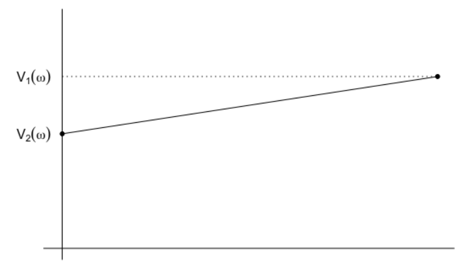


Figure 2.2: *One realisation of a random function ξ from Example 2.4.10 where $V_1(\omega) \geq V_2(\omega)$ for an elementary event ω.*

varying in the bornology generated by τ_{sup}, which can be seen by using Example 2.4.4 and applying Theorem 3.2.1. However, its tail measure is supported by functions au and $a(1-u)$, $u \in [0,1]$, where $a > 0$. These functions are not strictly positive on $[0,1]$ and therefore do not belong to $\mathbb{X}_1$. Thus, the tail measure μ is trivial on $\mathbb{X}_1$, and so ξ is not regularly varying in $\mathbb{X}_1$ with the bornology induced by τ_{sup}.

Example 2.4.11. Let $\mathbb{X}$ be the family of positive upper semicontinuous functions $x : [0,1] \to \mathbb{R}_+$ equipped with the hypo-topology, see [10], [44] and [45], and the linear scaling. Let $\mathbb{X}_1$ be the family of continuous functions $x : [0,1] \to (0,\infty)$, which is a subcone in $\mathbb{X}$. The topology on $\mathbb{X}_1$ generated by the uniform metric is stronger than the topology induced from $\mathbb{X}$. Equip $\mathbb{X} = \mathbb{X}'$ with the bornology generated by the modulus $\sup x$. Consider a random function given by

$$\xi(u) = \begin{cases} V^2(1/V - |u-U|), & |u-U| \leq 1/V, \\ 0, & \text{otherwise}, \end{cases} \qquad u \in [0,1],$$

where U is a continuous random variable on $[0,1]$ and V is a Pareto-1 random variable independent of U. Then ξ is regularly varying with the tail measure μ supported by functions $x(u) = c\mathbf{1}_{\{a\}}(u)$ for $c > 0$ and $a \in [0,1]$, and the tail index is 1. Since $\mu(\mathbb{X}_1) = 0$, ξ is not regularly varying on $\mathbb{X}_1$ with the induced bornology.

Another example involving spaces of sets is given later in Example 3.7.6.

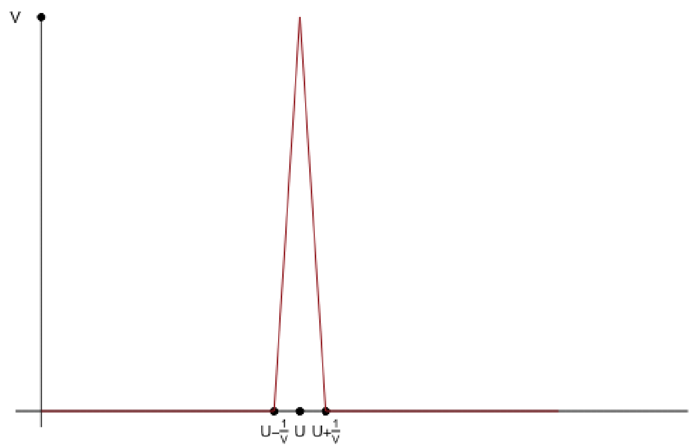


Figure 2.3: *One realisation of a random function ξ from Example 2.4.11*

Chapter 3

Continuous mapping of regular variation and regular variation of random closed sets

In the following chapter, we will discuss continuous transformations of regularly varying measures. To preserve the regularly varying property, the mapping must satisfy additional conditions as given in Theorem 3.2.1. Previous results of this type have been observed, for example, in [32, Theorem 2.5] where the space is assumed to be star-shaped. We apply our result to polar decomposition map to get a result analogous to [54, Proposition 3.1]. Furthermore, we will observe the inverse mapping ψ^{-1} of a continuous transformation ψ as a set-valued map and derive similar results for mappings of this type. Additionally, we will discuss quotient maps and selection maps as specific types of continuous transformations.

3.1 Transformations of scaling and bornology

Consider Polish spaces $\mathbb{X}$ and $\mathbb{Y}$ equipped with continuous scalings $T^{\mathbb{X}}$ and $T^{\mathbb{Y}}$, and zeros $\mathbf{0}_{\mathbb{X}}$ and $\mathbf{0}_{\mathbb{Y}}$, respectively (see Section 1.3).

Definition 3.1.1. A Borel map $\psi : \mathbb{X} \to \mathbb{Y}$ is said to be a *morphism* of order $\gamma > 0$ if for all $x \in \mathbb{X}$ and $t > 0$,

$$\psi(T_t^{\mathbb{X}} x) = T_{t^\gamma}^{\mathbb{Y}} \psi(x). \tag{3.1}$$

If $\gamma = 1$ we omit mentioning the order.

Lemma 3.1.2. *If* (3.1) *holds, then* $\psi(\mathbf{0}_{\mathbb{X}}) \subseteq \mathbf{0}_{\mathbb{Y}}$ *and* $\psi^{-1}(T_{t^\gamma}^{\mathbb{Y}} y) = T_t^{\mathbb{X}} \psi^{-1}(y)$ *for all* $y \in \mathbb{Y}$ *and* $t > 0$.

Proof. For $y \in \mathbb{Y}$ and $t > 0$,

$$\begin{aligned} T_t^{\mathbb{X}} \psi^{-1}(y) &= \{T_t^{\mathbb{X}} x : \psi(x) = y\} = \{x : \psi(T_{t^{-1}}^{\mathbb{X}} x) = y\} \\ &= \{x : T_{t^{-\gamma}}^{\mathbb{Y}} \psi(x) = y\} = \{x : \psi(x) = T_{t^\gamma}^{\mathbb{Y}} y\} = \psi^{-1}(T_{t^\gamma}^{\mathbb{Y}} y). \end{aligned}$$

If $T_t^{\mathbb{X}}x = x$ for some $t > 0$, $t \neq 1$, then

$$y = \psi(x) = \psi(T_t^{\mathbb{X}}x) = T_{t^\gamma}^{\mathbb{Y}}\psi(x) = T_{t^\gamma}^{\mathbb{Y}}y,$$

which implies that y is invariant under $T_t^{\mathbb{Y}}$ and so $\psi(\mathbf{0}_{\mathbb{X}}) \subseteq \mathbf{0}_{\mathbb{Y}}$. □

Example 3.1.3. Let the scaling on $\mathbb{Y}$ be the identity, so that $\mathbb{Y} = \mathbf{0}_{\mathbb{Y}}$. Equation (3.1) yields that $\psi(T_t^{\mathbb{X}}x) = \psi(x)$ for all x and $t > 0$. In this case, $\psi(\mathbf{0}_{\mathbb{X}})$ may be a strict subset of $\mathbf{0}_{\mathbb{Y}}$.

Lemma 3.1.2 shows that the inverse map ψ^{-1} is a morphism between $\mathbb{Y}$ and the family of subsets of $\mathbb{X}$, and that $\psi^{-1}\mathbb{Y}' \subseteq \mathbb{X}'$. If $y \in \mathbf{0}_{\mathbb{Y}}$, then

$$\psi^{-1}(y) = \psi^{-1}(T_t^{\mathbb{Y}}y) = T_{t^{1/\gamma}}^{\mathbb{X}}\psi^{-1}(y).$$

Therefore, $\psi^{-1}(y)$ may contain elements from $\mathbb{X}'$ and so $\psi^{-1}(\mathbf{0}_{\mathbb{Y}})$ may be strictly larger than $\mathbf{0}_{\mathbb{X}}$. Furthermore, $y = \psi(x) \in \mathbf{0}_{\mathbb{Y}}$ if and only if $\psi(T_t^{\mathbb{X}}x) = \psi(x)$ for some $t > 0$, $t \neq 1$.

Definition 3.1.4. Let $\mathbb{X}$ and $\mathbb{Y}$ be two spaces with ideals $\mathcal{S}(\mathbb{X})$ and $\mathcal{S}(\mathbb{Y})$, respectively. A function $\psi : \mathbb{X} \to \mathbb{Y}$ is said to be a *bornologically consistent* if $\psi^{-1}(B) \in \mathcal{S}(\mathbb{X})$ for all $B \in \mathcal{S}(\mathbb{Y})$.

Example 3.1.5. Let $\mathbb{X}$ and $\mathbb{Y}$ be star-shaped metric spaces with continuous moduli, so that $\mathbf{0}_X = \{0_{\mathbb{X}}\}$ and $\mathbf{0}_{\mathbb{Y}} = \{0_{\mathbb{Y}}\}$ are singletons, and the corresponding ideals are topological bornologies. Recall, this means that they are generated by all closed sets which do not intersect the closure of $\mathbf{0}$. Then each continuous morphism $\psi : \mathbb{X} \to \mathbb{Y}$ is bornologically consistent. To see this, let B be a closed set in $\mathbb{Y}$ that does not contain $\mathbf{0}_{\mathbb{Y}}$, which is equivalent to $B \in \mathcal{S}(\mathbb{Y})$. If $\psi^{-1}(B)$ contains $0_{\mathbb{X}}$, then B contains $\psi(0_{\mathbb{X}}) = 0_{\mathbb{Y}}$, see Lemma 3.1.2.

Remark 3.1.6. If the ideals $\mathcal{S}(\mathbb{X})$ and $\mathcal{S}(\mathbb{Y})$ are generated by semicones $C_{\mathbb{X}}$ and $C_{\mathbb{Y}}$, respectively, and satisfy condition (B), then a morphism ψ is bornologically consistent if and only if $\psi^{-1}(C_{\mathbb{Y}}) \subseteq T_t^{\mathbb{X}}C_{\mathbb{X}}$ for some $0 < t < 1$. To prove this, assume that ψ is bornologically consistent. Then, for $B = C_{\mathbb{Y}}$, we have $\psi^{-1}(C_{\mathbb{Y}}) \in \mathcal{S}(\mathbb{X})$, which implies that there exists $n \in \mathbb{N}$ such that $\psi^{-1}(C_{\mathbb{Y}}) \subseteq T_{n^{-1}}^{\mathbb{X}}C_{\mathbb{X}}$. For the converse, let $B \in \mathcal{S}(\mathbb{Y})$ be arbitrary. Then there exists $n \in \mathbb{N}$ such that $B \subseteq T_{n^{-1}}^{\mathbb{Y}}C_{\mathbb{Y}}$. Using the fact that ψ is a morphism, we get $\psi^{-1}(B) \subseteq T_{n^{-1/\gamma}}^{\mathbb{X}}\psi^{-1}(C_{\mathbb{Y}}) \subseteq T_t^{\mathbb{X}}C_{\mathbb{X}} \in \mathcal{S}(\mathbb{X})$ for some $t > 0$. Recall that $\mathcal{S}(\mathbb{X})$ contains all subsets of each set from $\mathcal{S}(\mathbb{X})$.

Example 3.1.7. Consider the map $\psi(x_1, x_2) = x_1 + x_2$ from $\mathbb{X} = (0, \infty)^2$ to $\mathbb{Y} = (0, \infty)$. Let $\mathcal{S}$ be the bornology on $(0, \infty)$ generated by the modulus $\tau(y) = y$, $y > 0$ (see Definition 1.3.7). The map ψ is bornologically consistent if the bornology on $\mathbb{X}$ contains $\psi^{-1}\{y \in \mathbb{Y} : \tau(y) > t\} = \{(x_1, x_2) \in \mathbb{X} : x_1 + x_2 > t\}$ for all $t > 0$. This is the case if $\mathcal{S}$ is generated by the modulus $\tau(x_1, x_2) = x_1 + x_2$ or by the modulus $\tau_{\max}(x_1, x_2) = \max(x_1, x_2)$, as well as any other modulus induced by a norm. However, if we equip $\mathbb{X}$ with the modulus $\tau_{\min}(x_1, x_2) = \min(x_1, x_2)$, then ψ is not bornologically consistent. Indeed, $\psi^{-1}(y) = \{(x_1, x_2) : x_1 + x_2 = y\}$, $y \in \mathbb{Y}$, does not belong to $\mathcal{S}_{\tau_{\min}}$ as well as to any other bornology which is compatible with the hidden regular variation property.

Proposition 3.1.8. *Assume that $\mathbb{X}$ and $\mathbb{Y}$ are topological spaces with continuous scalings and that the bornology $\mathcal{S}(\mathbb{Y}')$ is generated by a continuous modulus $\tau_{\mathbb{Y}}$. If a continuous map $\psi : \mathbb{X} \to \mathbb{Y}$ is a morphism, then $\psi^{-1}\mathcal{S}(\mathbb{Y}') = \{\psi^{-1}(B) : B \in \mathcal{S}(\mathbb{Y}')\}$ is a bornology on $\mathbb{X}'$, which satisfies (B).*

Proof. Let $C = \{y \in \mathbb{Y} : \tau_{\mathbb{Y}}(y) > 1\}$ and let ψ be a morphism of order $\gamma > 0$. Then $C' = \psi^{-1}(C)$ is also an open set, which is a semicone in $\mathbb{X}$, since, for all $t \geq 1$,

$$T_t^{\mathbb{X}} C' = \bigcup_{y \in C} T_t^{\mathbb{X}} \psi^{-1}(y) = \bigcup_{y \in C} \psi^{-1}(T_{t^\gamma}^{\mathbb{Y}} y) = \psi^{-1}(T_{t^\gamma}^{\mathbb{Y}} C) \subseteq \psi^{-1}(C) = C'.$$

The family $\{T_{n^{-1}} C', n \geq 1\}$ is an open base of $\psi^{-1}\mathcal{S}(\mathbb{Y}')$, and

$$\begin{aligned}\sup\{t : x \in T_t^{\mathbb{X}} C'\} &= \sup\{t : x \in T_t^{\mathbb{X}} \psi^{-1}(C)\} = \sup\{t : x \in \psi^{-1}(T_{t^\gamma}^{\mathbb{Y}} C)\} \\ &= \sup\{t : \psi(x) \in T_{t^\gamma}^{\mathbb{Y}} C\}\end{aligned}$$

is a continuous modulus on $\mathbb{X}'$ (follows from the proof of Lemma 1.3.9) where for $\gamma = 1$ we get $\tau_{\mathbb{Y}}(\psi(x))$. □

Alternatively to Proposition 3.1.8, it is possible to fix a bornology $\mathcal{S} = \mathcal{S}(\mathbb{X}')$ on $\mathbb{X}'$ and identify the largest bornology on $\mathbb{Y}'$ that ensures that ψ is bornologically consistent. This bornology, denoted by $\psi\mathcal{S}$, is the *pushforward* of $\mathcal{S}$ under ψ. It consists of all $B \subseteq \mathbb{Y}'$ such that $\psi^{-1}(B) \in \mathcal{S}$. Note that $\psi\mathcal{S}$ is a bornology that does not require ψ to be a morphism.

Proposition 3.1.9. *Let $\mathbb{X}$ and $\mathbb{Y}$ be topological spaces equipped with continuous scalings. Assume that $\psi : \mathbb{X} \to \mathbb{Y}$ is a continuous closed and open morphism of order* 1*. Let $\mathcal{S}$ be a bornology on $\mathbb{X}'$ that is generated by a continuous modulus τ such that $\mathbf{0}_{\mathbb{X}} \cup \{x \in \mathbb{X}' : \tau(x) \leq 1\}$ is compact in $\mathbb{X}$. Then $\psi\mathcal{S}$ is a bornology on $\mathbb{Y}'$ that satisfies (B).*

Proof. We denote

$$\tau_{\mathbb{Y}}(y) = \hat{\tau}(\psi^{-1}(y)), \quad y \in \mathbb{Y}', \tag{3.2}$$

where $\hat{\tau}$ is defined at (1.9) and we write $\psi^{-1}(y)$ for $\psi^{-1}(\{y\})$. By (3.1), $\tau_{\mathbb{Y}}$ is homogeneous. In order to show that $\tau_{\mathbb{Y}}$ is a continuous modulus, we need to show that $\tau_{\mathbb{Y}}$ is continuous and non-vanishing on $\mathbb{Y}'$. First, we have

$$\{y : \tau_{\mathbb{Y}}(y) \geq t\} = \Big\{y : \psi^{-1}(y) \cap \{x : \tau(x) < t\} = \varnothing\Big\} = \psi\big(\{x : \tau(x) < t\}\big)^c.$$

Since ψ is an open map, the set on the right-hand side is closed, hence $\tau_{\mathbb{Y}}$ is upper semicontinuous. Furthermore, since ψ is a closed map,

$$\{y : \tau_{\mathbb{Y}}(y) \leq t\} = \bigcap_{\varepsilon > 0} \Big\{y : \psi^{-1}(y) \cap \{x : \tau(x) \leq t + \varepsilon\} \neq \varnothing\Big\} = \bigcap_{\varepsilon > 0} \psi\big(\{x : \tau(x) \leq t + \varepsilon\}\big)$$

is also closed, so $\tau_{\mathbb{Y}}$ is lower semicontinuous and hence continuous.

If $\tau_{\mathbb{Y}}(y) = 0$, then for all $n \geq 1$, there exists $x_n \in \psi^{-1}(y)$ such that $\tau(x_n) \leq 1/n$. Since $\mathbf{0}_{\mathbb{X}} \cup \{x : \tau(x) \leq 1\}$ is compact, there exists a subsequence (n_k) such that $x_{n_k} \to x \in \mathbb{X}$. By continuity of τ, we have $\tau(x_{n_k}) \to \tau(x)$, so it cannot be that $x \in \mathbb{X}'$ since for each $z \in \mathbb{X}'$, we have $\tau(z) > 0$. Thus, $x \in \mathbf{0}_{\mathbb{X}}$, so $y = \psi(x) \in \mathbf{0}_{\mathbb{Y}}$ by continuity of ψ and Lemma 3.1.2.

It remains to show that the pushforward $\psi\mathcal{S}$ is generated by the modulus $\tau_{\mathbb{Y}}$. Indeed, $B \in \psi\mathcal{S}$ if $\hat{\tau}(\psi^{-1}(B)) = \hat{\tau}_{\mathbb{Y}}(B) > 0$. Thus, $\psi\mathcal{S}$ consists of sets $B \subseteq \mathbb{Y}$ such that $\hat{\tau}_{\mathbb{Y}}(B) > 0$. □

3.2 Continuous images of regularly varying measures

The following result establishes that the regular variation property is preserved under certain continuous bornologically consistent morphisms. A measure $\mu \in \mathcal{M}(\mathbb{X}, \mathcal{S})$ is said to be *nontrivial* on $\mathcal{S}_1 \subseteq \mathcal{S}$ if $\mu(B) > 0$ for some $B \in \mathcal{S}_1$. We follow the notation introduced in Section 3.1.

Theorem 3.2.1. *Let $\mathbb{X}$ and $\mathbb{Y}$ be Polish spaces equipped with continuous scalings $T^{\mathbb{X}}$ and $T^{\mathbb{Y}}$ and ideals $\mathcal{S}(\mathbb{X})$ and $\mathcal{S}(\mathbb{Y})$, which are scaling and topologically consistent and have countable open bases. Assume that $\psi : \mathbb{X} \to \mathbb{Y}$ is a continuous bornologically consistent morphism of order γ. If $\nu \in \mathrm{RV}(\mathbb{X}, \mathcal{S}(\mathbb{X}), \mu)$ with tail index α and the pushforward $\psi\mu$ is nontrivial on $\mathcal{S}(\mathbb{Y})$, then $\psi\nu \in \mathrm{RV}(\mathbb{Y}, \mathcal{S}(\mathbb{Y}), \psi\mu)$ with the tail index α/γ.*

Proof. Since $\mu \in \mathcal{S}(\mathbb{X}, \mathcal{S}(\mathbb{X}))$, we have that for every $A \in \mathcal{S}(\mathbb{Y})$, $(\psi\mu)(A) = \mu(\psi^{-1}(A)) < \infty$ which follows from bornological consistency. Therefore, $\psi\mu \in \mathcal{M}(\mathbb{Y}, \mathcal{S}(\mathbb{Y}))$. Note that for $f \in \mathsf{C}_{\mathcal{S}_{\mathbb{Y}}}$, the function $f \circ \psi$ is continuous, bounded, and such that the set $\{x \in \mathbb{X} : (f \circ \psi)(x) \neq 0\} = \mathbb{X} \setminus \psi^{-1}(f^{-1}(0)) = \psi^{-1}(\mathbb{Y} \setminus f^{-1}(0))$ belongs to $\mathcal{S}(\mathbb{X})$. Consider a random element $\xi \in \mathrm{RV}(\mathbb{X}, \mathcal{S}(\mathbb{X}), \mu)$ and apply Theorem 2.2.3(ii) to conclude that for each $f \in \mathsf{C}_{\mathcal{S}(\mathbb{Y})}$ we have $f \circ \psi \in \mathsf{C}_{\mathcal{S}(\mathbb{X})}$ and

$$\lim_{t\to\infty} g(t^{1/\gamma})\mathbf{E}f\left(T^{\mathbb{Y}}_{t^{-1}}\psi(\xi)\right) = \lim_{s\to\infty} g(s)\mathbf{E}f\left(\psi(T^{\mathbb{X}}_{s^{-1}}\xi)\right) = \int_{\mathbb{X}} (f\circ\psi)d\mu_g = \int_{\mathbb{Y}} fd(\psi\mu_g)\,.$$

Finally, note that $g(t^{1/\gamma})$ is regularly varying of order α/γ. Thus, $\psi(\xi) \in \mathrm{RV}(\mathbb{Y}, \mathcal{S}(\mathbb{Y}), \psi\mu)$, provided $\psi\mu$ is nontrivial on $\mathcal{S}(\mathbb{Y})$. □

If μ admits a spectral measure σ, then $\psi\sigma$ is a spectral measure of $\psi\mu$: for Borel B in $\mathcal{S}(\mathbb{Y})$,

$$\psi\mu(B) = \alpha\int_0^\infty \sigma(T^{\mathbb{X}}_t\psi^{-1}(B))t^{\alpha-1}dt = \alpha\int_0^\infty \sigma(\psi^{-1}(T^{\mathbb{Y}}_t B))t^{\alpha-1}dt,$$

so that $\psi\sigma$ is the spectral measure of $\psi\mu$.

Remark 3.2.2. If ψ is not continuous, then the statement of Theorem 3.2.1 holds if the μ-measure of the set of discontinuity points of ψ is zero. By (3.1), if ψ is not continuous at $x \in \mathbb{X}'$, then it is also not continuous on all points in $T_{(0,\infty)}x$. Therefore, we can consider discontinuity points on the set $\{x \in \mathbb{X}' : \tau(x) = 1\}$ and require that the spectral measure of this set vanishes. This version of the continuous mapping theorem appears in [32, Theorem 2.5] and [38, Theorem B.1.21].

Remark 3.2.3. Assume that the conditions of Theorem 3.2.1 hold and that the map $\psi : \mathbb{X} \to \mathbb{Y}$ is a bijective continuous morphism. Assume that the ideals $\mathcal{S}(\mathbb{X})$ and $\mathcal{S}(\mathbb{Y})$ on $\mathbb{X}$ and $\mathbb{Y}$ are bornologies on $\mathbb{X}'$ and $\mathbb{Y}'$ generated by continuous moduli $\tau_{\mathbb{X}}$ and $\tau_{\mathbb{Y}}$ such that $\tau_{\mathbb{Y}}(\psi(x)) = \tau_{\mathbb{X}}(x)$. Then $\psi(\mathbf{0}_{\mathbb{X}}) = \mathbf{0}_{\mathbb{Y}}$, and the image $\psi(\xi)$ of each regularly varying random element ξ in $\mathbb{X}$ is regularly varying in $\mathbb{Y}$. In this case, the spaces $(\mathbb{X}, \mathcal{S}(\mathbb{X}))$ and $(\mathbb{Y}, \mathcal{S}(\mathbb{Y}))$ with the corresponding scalings are said to be *indistinguishable*.

3.3 Modulus and polar decomposition maps

Assume that the bornology $\mathcal{S}$ on $\mathbb{X}$ satisfies (B), equivalently, $\mathcal{S} = \mathcal{S}_\tau$ for a continuous modulus τ (see Lemma 1.3.9). The modulus is a continuous morphism from $(\mathbb{X}, \mathcal{S})$ to $(\mathbb{R}_+, \mathcal{S}_1)$, where $\mathcal{S}_1$ is the standard bornology generated by the semicone $[1, \infty)$. For a measure ν on $\mathbb{X}$, the pushforward of ν under τ is denoted by $\tau\nu$. Theorem 3.2.1 implies the following result.

Corollary 3.3.1. *If $\nu \in \mathrm{RV}(\mathbb{X}, \mathcal{S}, \mu)$, then $\tau\nu \in \mathrm{RV}(\mathbb{R}_+, \mathcal{S}_1, \tau\mu)$. In particular, if a random element ξ is $\mathrm{RV}(\mathbb{X}, \mathcal{S}, \mu)$ of index α, then $\tau(\xi)$ is regular varying random variable with index α, and for all $u > 0$,*

$$g(t)\mathbf{P}\left\{\tau(\xi) > ut\right\} \to \mu(\{x : \tau(x) > u\}) = cu^{-\alpha} \quad \textit{as } t \to \infty,$$

where $c = \mu(\{x : \tau(x) > 1\})$.

The following result deals with the regular variation properties of the polar decomposition map between $\mathbb{X}'$ and $\mathbb{Y} = \mathbb{S} \times (0, \infty)$, see Definition 1.3.8. The scaling on $\mathbb{Y}$ acts only on the second component, that is, $T_s^{\mathbb{Y}}(u, t) = (u, ts)$. The space $\mathbb{Y}$ does not have scaling invariant elements. The bornology $\mathcal{S}(\mathbb{Y})$ on $\mathbb{Y}$ is generated by sets of the form $\mathbb{S} \times [t, \infty)$ for $t > 0$, and the modulus on $\mathbb{Y}$ is given by $\tau_{\mathbb{Y}}(u, t) = t$. Recall that θ_α is a measure on $(0, \infty)$ with density $\alpha y^{-\alpha-1} dy$.

Remark 3.3.2. Note that the polar decomposition is a bijective continuous morphism that is also bornologically consistent. Being a morphism comes from the following calculation:

$$\rho(T_s x) = \big(T_{\tau((T_s x)^{-1})} T_s x, \tau(T_s x)\big) = \big(T_{s^{-1}\tau(x)^{-1}} T_s x, s\tau(x)\big) = (T_{\tau(x)^{-1}} x, s\tau(x)) = T_s^{\mathbb{Y}} \rho(x),$$

whereas bornological consistency follows from:

$$\rho^{-1}(A \times (t, \infty)) = \{x \in \mathbb{X}' : \tau(x) > t,\, T_{\tau(x)^{-1}} x \in A\} \subseteq \{x : \tau(x) > t\},$$

for any $A \subseteq \mathbb{S}$ and $t > 0$. The set on the right is bounded (belongs to $\mathcal{S}_\tau$) since there exists $n \in \mathbb{N}$ such that it is contained in $T_{n^{-1}} C = \{x : \tau(x) > n^{-1}\}$.

Proposition 3.3.3. *Assume that $\mathbb{X}$ is a Polish space equipped with a continuous scaling T_t, and let $\mathbb{X}'$ be equipped with a bornology $\mathcal{S}$ generated by a continuous modulus τ. Let ξ be a random element in $\mathbb{X}'$, and let σ be a finite measure on $\mathbb{S}$ defined at* (1.10). *Then the following are equivalent:*

(i) $\xi \in \mathrm{RV}(\mathbb{X}, \mathcal{S}, \mu)$*, where* μ *is given at (2.4).*

(ii) The random variable $\tau(\xi)$ *is regularly varying with index* α*, and*

$$\mathbf{P}\left\{T_{\tau(\xi)^{-1}}\xi \in \cdot \mid \tau(\xi) > t\right\} \xrightarrow{w} \sigma(\cdot)/\sigma(\mathbb{S}) \quad \text{as } t \to \infty, \tag{3.3}$$

where the convergence is the weak convergence of probability measures on $\mathbb{S}$*.*

(iii) We have

$$\frac{\mathbf{P}\left\{(T_{\tau(\xi)^{-1}}\xi, t^{-1}\tau(\xi)) \in A \times (u, \infty), \tau(\xi) > t\right\}}{\mathbf{P}\left\{\tau(\xi) > t\right\}} \to (\sigma \otimes \theta_\alpha)(A \times (u, \infty)\,) \tag{3.4}$$

as $t \to \infty$ *for all Borel* $A \subseteq \mathbb{S}$ *such that* $\sigma(\partial A) = 0$*, and all* $u > 0$*. In that case,*

$$\frac{\mathbf{P}\left\{T_{t^{-1}}\xi \in \cdot\,\right\}}{\mathbf{P}\left\{\tau(\xi) > t\right\}} \xrightarrow{s} \mu(\,\cdot\,) \quad \text{as } t \to \infty, \tag{3.5}$$

where μ *is determined by* $\rho\mu = \sigma \otimes \theta_\alpha$*.*

Proof. By Theorem 3.2.1, $\xi \in \mathrm{RV}(\mathbb{X}, \mathcal{S}, \mu)$ if and only if the polar decomposition $\eta = \rho(\xi) = (T_{\tau(\xi)^{-1}}\xi, \tau(\xi))$ is regularly varying in $(\mathbb{Y}, \mathcal{S}(\mathbb{Y}))$. Let σ be a spectral measure of ξ, which is supported by $\{x : \tau(x) = 1\} = \mathbb{S}$ such that (2.4) holds. Then η has the tail measure $\rho\mu = \sigma \otimes \theta_\alpha$. This measure is finite on $\mathcal{S}(\mathbb{Y})$, so that Proposition 2.2.4 yields for a Borel $A \subseteq \mathbb{S}$ and $s > 0$

$$\rho\mu(A \times (s, \infty)) = \alpha \int_0^\infty \sigma(T_{st}T_{(1,\infty)}A)t^{\alpha-1}dt = \alpha s^{-\alpha} \int_0^1 \sigma(T_{(r,\infty)}A)r^{\alpha-1}dr = s^{-\alpha}\sigma(A).$$

Thus, the tail measure of η is $\sigma \otimes \theta_\alpha$.

Since the sets $A \times (s, \infty)$ for Borel $A \subseteq \mathbb{S}$ such that $\sigma(\partial A) = 0$, and $s > 0$ are Borel, bounded in $\mathbb{Y}$, and convergence-determining in $\mathbb{Y}$, the regular variation property of η is equivalent to

$$g(t)\mathbf{P}\left\{\eta \in A \times (ts, \infty)\right\} \to \rho\mu(A \times (s, \infty)) = s^{-\alpha}\sigma(A) \quad \text{as} \quad t \to \infty$$

for all Borel σ-continuity sets A and $s > 0$. The regular variation property of g implies that it suffices to let $s = 1$. Note that

$$\mathbf{P}\left\{\eta \in A \times (t, \infty)\right\} = \mathbf{P}\left\{\xi \in T_{(t,\infty)}A\right\} = \mathbf{P}\left\{T_{\tau(\xi)^{-1}}\xi \in A, \tau(\xi) > t\right\}.$$

(i) $\Leftrightarrow$ (ii): First, since ξ is regularly varying which is equivalent to η being regularly varying, we can use arguments from above to obtain

$$\mathbf{P}\left\{T_{\tau(\xi)^{-1}}\xi \in \,\cdot \mid \tau(\xi) > t\right\} = \frac{g(t)\mathbf{P}\left\{T_{\tau(\xi)^{-1}}\xi \in \,\cdot\,,\ \tau(\xi) > t\right\}}{g(t)\mathbf{P}\left\{T_{\tau(\xi)^{-1}}\xi \in \mathbb{S},\ \tau(\xi) > t\right\}} \xrightarrow{w} \frac{\sigma(\cdot)}{\sigma(\mathbb{S})}$$

as $t \to \infty$. For the converse, we set $g(t) = \mathbf{P}\{\tau(\xi) > t\}^{-1}$ and let $A \subseteq \mathbb{S}$ be a Borel σ-continuity set, $s > 0$. Then, we have

$$g(t)\mathbf{P}\left\{T_{\tau(\xi)^{-1}}\xi \in A,\ \tau(\xi) > ts\right\} = \frac{g(t)}{g(ts)}\, g(ts)\, \mathbf{P}\left\{T_{\tau(\xi)^{-1}}\xi \in A,\ \tau(\xi) > ts\right\} \to cs^{-\alpha}\sigma(\cdot)$$

as $t \to \infty$, where $c = \sigma(\mathbb{S})^{-1}$. Properties of σ discussed above guarantee that σ is a probability measure and thus $c = 1$.

(i),(ii) $\Leftrightarrow$ (iii): From the regular variation of ξ and η, for each Borel σ-continuity set $A \subseteq \mathbb{S}$ and $s \geq 1$, we have

$$\frac{\mathbf{P}\left\{T_{\tau(\xi)^{-1}}\xi \in A,\ \tau(\xi) > st\right\}}{\mathbf{P}\{\tau(\xi) > t\}} \to s^{-\alpha}\sigma(A) \quad \text{as } t \to \infty,$$

whereas for $u < 0$ we have

$$\frac{\mathbf{P}\left\{T_{\tau(\xi)^{-1}}\xi \in A,\ \tau(\xi) > t\right\}}{\mathbf{P}\{\tau(\xi) > t\}} \to \sigma(A) \quad \text{as } t \to \infty.$$

Finally, (3.5) is equivalent to the regular variation of ξ, given that the normalising function can be chosen as $g(t) = \mathbf{P}\{\tau(\xi) > t\}^{-1}$. □

In part (iii) of Proposition 3.3.3, it suffices to consider A belonging to a convergence-determining class on $\mathbb{S}$, which may be smaller than the family of all Borel sets in $\mathbb{S}$.

Corollary 3.3.4. *Assume that $\mathbb{X}'$ is equipped with a bornology $\mathcal{S}$ generated by a continuous modulus τ. If η is a random element in $\mathbb{S}$ defined at* (1.10)*, and $\xi \in [0, \infty)$ is an independent of η regularly varying random variable with tail measure $c\theta_\alpha$, then $T_\xi \eta \in \mathrm{RV}(\mathbb{X}, \mathcal{S}, \mu)$, where μ is the pushforward of $\sigma \otimes c\theta_\alpha$ under the inverse of the polar decomposition map and σ is the distribution of η.*

3.4 Examples: continuous images of regularly varying elements

It is well-known that a random variable ξ belongs to the maximum domain of attraction (MDA) of the Fréchet distribution if and only if ξ is regularly varying with index $\alpha > 0$ on $\mathbb{X} = [0, \infty)$ in the linear scaling and bornology generated by the modulus $\tau(x) = x$, up to a multiplicative constant (see Theorem 1.1.3). Moreover, other MDA's can be also characterised by appropriate regular variation conditions, see Chapter 1 of [48]. In some cases, if we are allowed to choose the scaling, this observation can be straightforward, as illustrated by the following two examples. They also clarify the connection of this to continuous bijective maps.

Example 3.4.1. Let $\mathbb{X} = (0, \infty)$ with the linear scaling $T_t^{\mathbb{X}}$ and the bornology $\mathcal{S}(\mathbb{X})$ generated by a continuous modulus $\tau_{\mathbb{X}}(x) = x$ with $C = [1, \infty)$. Let $\psi(x) = -x^{-1} + c$ be a map from $\mathbb{X}$ to $\mathbb{Y} = (-\infty, c)$ for some fixed $c \in \mathbb{R}$. Equip $\mathbb{Y}$ with the scaling given by $T_t^{\mathbb{Y}} x = t^{-1}(x-c)+c$ and the bornology generated by the semicone $[c-1, c)$, see Example 1.4.7. This bornology corresponds to the continuous modulus $\tau_{\mathbb{Y}}(x) = |x-c|^{-1}$. The map ψ is a bornologically consistent morphism, and the spaces $\mathbb{X}$ and $\mathbb{Y}$ with the chosen bornologies are indistinguishable:

$$\psi(T_t^{\mathbb{X}} x) = -(tx)^{-1} + c, \quad T_t^{\mathbb{Y}} \psi(x) = t^{-1}(-x^{-1} + c - c) + c = \psi(T_t^{\mathbb{X}} x), \text{ for all } t > 0,\ x \in \mathbb{X},$$

and $\tau_{\mathbb{Y}}(\psi(x)) = |-x^{-1} - c + c|^{-1} = \tau_{\mathbb{X}}(x)$, for all $x \in \mathbb{X}$. By Theorem 3.2.1, a random variable ξ is regularly varying in $\mathbb{X}$ if and only if $\eta = -\xi^{-1} + c$ is regularly varying in $\mathbb{Y}$. The tail measure of η is given by the pushforward $\psi\mu$ of the tail measure μ of ξ. For example, if ξ is Pareto-1 distributed, then $-\xi^{-1}$ is regularly varying in $(-\infty, 0)$. The distribution of $-\xi^{-1}$ is uniform on $(-1, 0)$ which is in Weibull MDA (easy calculation shows that *(ii)* from Theorem 1.1.3 holds). More generally, assume that a random variable $\eta = \psi(\xi)$ with distribution function F has a finite right endpoint $c = \sup\{x : F(x) < 1\}$. By right continuity, we have $F(c) = 1$. If $\xi \in \mathrm{RV}(\mathbb{X}, \mathcal{S}(\mathbb{X}), \mu)$, then $\eta \in \mathrm{RV}(\mathbb{Y}, \mathcal{S}(\mathbb{Y}), \psi\mu)$ in this setting, and for any $u > 0$, we have

$$\frac{\mathbf{P}\left\{T_{t^{-1}}^{\mathbb{Y}} \eta \in (c-u, c)\right\}}{\mathbf{P}\left\{T_{t^{-1}}^{\mathbb{Y}} \eta \in (c-1, c)\right\}} = \frac{1 - F(c - u/t)}{1 - F(c - 1/t)} \to (\psi\mu)(c-u, c) = \mu(u^{-1}, \infty) = u^{\alpha}$$

as $t \to \infty$, where $\alpha > 0$ is the index of regular variation of ξ and η. For an i.i.d. sequence (η_i) with the distribution function F, define $M_n = \vee_{i=1}^n \eta_i$. Then there exists a corresponding sequence (a_n) such that

$$\mathbf{P}\left\{T_{a_n^{-1}}^{\mathbb{Y}} M_n \le c - u\right\} = \mathbf{P}\left\{a_n(M_n - c) \le -u\right\} = \left(1 - \mathbf{P}\left\{a_n(\eta_i - c) > -u\right\}\right)^n.$$

Take natural logarithm of this term and use the relation $\log(1-x) = -x$ for x small enough. Then, using the regularly varying property, we obtain

$$-n\mathbf{P}\left\{a_n^{-1}(\eta_i - c) > -u\right\} \to -u^{\alpha} \text{ as } n \to \infty,$$

so the term from above converges:

$$\mathbf{P}\left\{T_{a_n^{-1}}^{\mathbb{Y}} M_n \le c - u\right\} \to e^{-u^{\alpha}} \text{ as } n \to \infty. \tag{3.6}$$

As expected, the regular variation of η in this setting corresponds to the fact that the tail distribution function $\overline{F}$ is regularly varying with index $\alpha > 0$ at c, which is a well-known necessary and sufficient condition for the convergence of M_n towards the limiting distribution of the type given in (3.6).

Example 3.4.2. Consider $\mathbb{X} = (0, \infty)$ equipped with the linear scaling and bornology generated by $\tau_{\mathbb{X}}(x) = x$, that is, by a semicone $[1, \infty)$, and let $\psi(x) = \kappa \log x$ be a map from $\mathbb{X}$ to $\mathbb{Y} = \mathbb{R}$ for fixed $\kappa \in \mathbb{R}$, $\kappa \neq 0$. Equip $\mathbb{Y}$ with the scaling and the bornology from Example 1.4.8:

$$T_t^{\mathbb{Y}} y = y + \kappa \log t,\ t > 0, \quad \tau_{\mathbb{Y}}(y) = e^{y/\kappa},\ y \in \mathbb{Y}.$$

Then ψ is a bornologically consistent morphism, and the spaces $\mathbb{X}$ and $\mathbb{Y}$ are indistinguishable:

$$\psi(T_t^{\mathbb{X}}x) = \psi(tx) = \kappa\log(tx) = \kappa\log x + \kappa\log t = T_t^{\mathbb{Y}}\psi(x), \quad \text{for all } t > 0,\ x \in \mathbb{X},$$

and $\tau_{\mathbb{Y}}(\psi(x)) = e^{\kappa\log x/\kappa} = \tau_{\mathbb{X}}(x)$, $x \in \mathbb{X}$. A random variable ξ is regularly varying in $\mathbb{X}$ if and only if $\eta = \log\xi$ is regularly varying in $\mathbb{Y}$, see Theorem 3.2.1. Then η lies in the maximum domain of attraction for the Gumbel distribution.

To show that, let $\xi \in \mathrm{RV}(\mathbb{X}, \mathcal{S}(\mathbb{X}), \mu)$ and let $\eta = \log\xi$. Denote by F the distribution function of η. Then $\kappa\eta \in \mathrm{RV}(\mathbb{Y}, \mathcal{S}(\mathbb{Y}), \psi\mu)$. For $u > 0$ we have

$$\frac{\mathbf{P}\left\{T^{\mathbb{Y}}_{t^{-1}}\eta \in (u,\infty)\right\}}{\mathbf{P}\left\{T^{\mathbb{Y}}_{t^{-1}}\eta \in (0,\infty)\right\}} = \frac{1 - F(u + \kappa\log t)}{1 - F(\kappa\log t)} \to (\psi\mu)(\kappa u, \infty) = \mu(e^u, \infty) = e^{-\alpha u}$$

as $t \to \infty$, where α is the index of regular variation of ξ and η. Similarly to Example 3.4.1, for an i.i.d. sequence (η_i) with the distribution function F and $M_n = \vee_{i=1}^n \eta_i$ there exists an appropriate sequence (a_n) such that for any $u > 0$

$$\mathbf{P}\left\{T^{\mathbb{Y}}_{a_n^{-1}} M_n \le u\right\} = \mathbf{P}\left\{M_n - \kappa\log a_n \le u\right\} \to e^{-e^{-\alpha u}} \quad \text{as } n \to \infty,$$

which confirms that ξ belongs to the MDA of the Gumbel distribution. However, it should be noted that in general, the Gumbel distribution with distribution function $G(x) = e^{-e^{-(x-\mu)/\beta}}$, $x \in \mathbb{R}$ ($u \in \mathbb{R}$, $\beta > 0$), arises from a two-parametric (affine) scaling of random variables, and thus it does not fit our current framework.

Example 3.4.3. Let ψ be a linear map from $\mathbb{X} = \mathbb{R}^d$ to $\mathbb{Y} = \mathbb{R}^m$ determined by a matrix A of rang $m \le d$. Assume that $\mathbb{X}$ and $\mathbb{Y}$ are equipped with the linear scaling and bornologies generated by the moduli given as the Euclidean norms. Then ψ is a bornologically consistent continuous morphism. The condition of Theorem 3.2.1 holds if $A\mu$ is nontrivial on $\mathbb{Y}'$, see also [3]. This is not the case if the tail measure μ is supported by the (nontrivial) kernel of A, so that $\psi\mu$ vanishes on $\mathbb{Y}'$.

Example 3.4.4. Let $\mathbb{X} = \mathsf{C}([0,1])$ be the space of continuous functions $x : [0,1] \to \mathbb{R}$ with the uniform metric, and let $\mathbb{Y} = \mathbb{R}$. Define the scalings on $\mathbb{X}$ and $\mathbb{Y}$ as usual multiplications by scalars and equip the spaces with bornologies generated by the moduli given as the corresponding norms. Let $\psi(x) = \int_0^1 x(u)du$ be a map from $\mathbb{X}$ to $\mathbb{Y}$. Then

$$\psi^{-1}(\{y : |y| \ge \varepsilon\}) = \{x \in \mathbb{X} : |\psi(x)| \ge \varepsilon\} \subseteq \{x \in \mathbb{X} : \|x\|_\infty \ge \varepsilon\}$$

for each $\varepsilon > 0$, so that ψ is a continuous bornologically consistent morphism. Note that $\psi(x)$ may vanish on a function x that is not identically zero. By Theorem 3.2.1, if ξ is a regularly varying random function in $\mathsf{C}([0,1])$, then $\psi(\xi)$ is a regularly varying random variable, unless $\psi(\xi) = 0$ almost surely. For instance, the latter is the case if $\xi(u) = V\sin(2\pi(u+U))$, where V has a Pareto tail and U is uniform on $[0,1]$. We show this by direct calculation:

$$\psi(\xi) = \int_0^1 \xi(u)du = V\int_U^{U+1} \sin(2\pi t)dt = \frac{-V}{2\pi}(\cos(2\pi + 2\pi U) - \cos(2\pi U)) = 0.$$

Consider also a map $\tilde{\psi}(x) = \sup x - \inf x$. Since

$$\tilde{\psi}^{-1}(\{y : |y| \geq \varepsilon\}) = \{x \in \mathbb{X} : |\tilde{\psi}(x)| \geq \varepsilon\} \subseteq \{x \in \mathbb{X} : \|x\|_\infty \geq \varepsilon/2\},$$

for all $\varepsilon > 0$, the map $\tilde{\psi}$ is a bornologically consistent morphism. The set $\tilde{\psi}^{-1}(\mathbf{0}_\mathbb{Y})$ consists of constant functions. Thus, if ξ is regularly varying with the tail measure not entirely supported by constant functions, then $\tilde{\psi}(\xi)$ is a regularly varying random variable. For the function $\xi(u) = V \sin(2\pi(u + U))$, we have $\tilde{\psi}(\xi) = 2V$, which is regularly varying.

Example 3.4.5. Let ξ be regularly varying in $\mathbb{X} = \mathbb{R}_+^d$ with the linear scaling and the bornology $\mathcal{S}_\tau$ generated by a continuous modulus τ, see Example 1.4.5 for $d = 2$. Let $\psi(x) = x_1 \cdots x_d$ for $x = (x_1, \dots, x_d) \in \mathbb{X}$. This map becomes a continuous morphism of order $\gamma = d$ between $\mathbb{X}$ and $\mathbb{Y} = \mathbb{R}_+$ with the linear scaling. If the bornology $\mathcal{S}_\tau$ contains the set $\{x : x_1 \cdots x_d \geq 1\}$, then ψ is also bornologically consistent (see Remark 3.1.6). Let $\xi \in \mathrm{RV}(\mathbb{X}, \mathcal{S}_\tau, \mu)$. If $\psi\mu$ is nontrivial, then $\psi(\xi)$ is regularly varying in $\mathbb{Y}$. Assume that ξ is composed of i.i.d. random variables. If $\tau(x) = \max(x_1, \dots, x_d)$, then $\psi\mu$ is supported by $\{0\}$ and the above argument does not apply.

Note that if τ is chosen to be $\min(x_1, \dots, x_d)$, then ψ is not bornologically consistent, and thus it is not possible to use the hidden regular variation property to deduce the regular variation of the product of two i.i.d. random variables, see [34, Lemma 4.1]. It should be noted that the regular variation of the product $\xi_1 \cdots \xi_d$ does not imply that ξ is regularly varying in the bornology on $(0, \infty)^d$ generated by the modulus $(x_1 \cdots x_d)^{1/d}$. This is demonstrated in Example 2.4.8 for the case $d = 2$.

Example 3.4.6. Let ξ be a regularly varying random vector in $\mathbb{X} = \mathbb{R}^d$ with the linear scaling, and let the bornology $\mathcal{S}$ on $\mathbb{X}'$ be generated by a norm. Let $\psi : \mathbb{R}^d \mapsto \mathbb{R}_+$ be a continuous homogeneous function. Then $x \mapsto \psi(x)$ is a continuous morphism from $\mathbb{X}$ to $\mathbb{Y} = \mathbb{R}_+$, which is also bornologically consistent if $\{x : \psi(x) \geq 1\} \in \mathcal{S}$. If $\xi \in \mathrm{RV}(\mathbb{R}^d, \mathcal{S}, \mu)$ and $\psi\mu$ is nontrivial on $(0, \infty)$, then $\psi(\xi)$ is regularly varying in $\mathbb{R}_+$ (Theorem 3.2.1).

This argument can be generalised for functions that are homogeneous of different orders in some of their arguments. For this, we adapt the setting of [20]. We equip $\mathbb{X} = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ with the linear scaling. Let $\psi : \mathbb{X} \to \mathbb{R}^{d_3}$ be a continuous function that is homogeneous of positive orders β_1 and β_2 in each of its two arguments. Then $(x, y) \mapsto \psi(x, y)$ becomes a morphism of order $\gamma = \beta_1 + \beta_2$ from $\mathbb{X}$ to $\mathbb{R}^{d_3}$ equipped with the linear scaling. Let $\mathcal{S}_i$ be ideals on $\mathbb{R}^{d_i}$, $i = 1, 2, 3$, that satisfy (B) with the corresponding cones C_i. Denote by $\mathbb{U}_i$ the unions of sets from the corresponding bornologies, by τ_i the modulus on $\mathbb{U}_i$, and by $\mathbb{S}_i$ the corresponding sphere. Assume that the range of ψ is a subset of $\mathbb{U}_3$.

Let $\mathcal{S}$ be an ideal on $\mathbb{X}$. The morphism ψ is bornologically consistent if $\{(x, y) \in \mathbb{X} : \tau_3(\psi(x, y)) \geq 1\} \in \mathcal{S}$. This set can be written as

$$\{(x, y) \in \mathbb{X} : \tau_1(x)^{\beta_1} \tau_2(y)^{\beta_2} \tau_3(\psi(T_{\tau_1(x)^{-1}} x, T_{\tau_2(y)^{-1}} y)) \geq 1\},$$

where $T_{\tau_1(x)^{-1}} x \in \mathbb{S}_1$, $T_{\tau_2(y)^{-1}} y \in \mathbb{S}_2$. Assume that

$$\inf\{\tau_3(\psi(u, v)) : u \in \mathbb{S}_1, v \in \mathbb{S}_2\} > 0.$$

Then ψ is bornologically consistent if

$$\{(x,y) \in \mathbb{X} : \tau_1(x)^{\beta_1}\tau_2(y)^{\beta_2} \geq 1\} \in \mathcal{S}.$$

This is the case if $\mathcal{S}$ contains the bornology on $\mathbb{X}$ generated by the modulus

$$\tau(x,y) = \tau_1(x)^{\beta_1/(\beta_1+\beta_2)}\tau_2(y)^{\beta_2/(\beta_1+\beta_2)}. \tag{3.7}$$

Let $\xi_i \in \mathrm{RV}(\mathbb{R}^{d_1}, \mathcal{S}_i, \mu_i)$, $i = 1, 2$, be independent random vectors that are regularly varying of orders α_1 and α_2, respectively. If $(\xi_1, \xi_2) \in \mathrm{RV}(\mathbb{X}, \mathcal{S}, \mu)$, where $\mathcal{S}$ is generated by the modulus τ such that (3.7) holds, then $\psi(\xi_1, \xi_2) \in \mathrm{RV}(\mathbb{R}^{d_3}, \mathcal{S}_3, \psi\mu)$, provided $\psi\mu$ is nontrivial on $\mathcal{S}_3$.

Example 3.4.7. Let $\mathbb{X} = \mathbb{R}^{d+1}$ and let $\mathbb{Y} = \mathsf{C}([0,1])$ be the space of continuous functions on $[0,1]$. We equip $\mathbb{X}$ and $\mathbb{Y}$ with the linear scalings and the moduli given by the corresponding norms. Define a map $\psi : \mathbb{X} \to \mathbb{Y}$ by letting $\psi(x)$ be the polynomial $\psi(x)(u) = x_1 + x_2 u + \cdots + x_{d+1}u^d$, $u \in [0,1]$. This map is a continuous and bornologically consistent morphism. To see that it is bornologically consistent, observe that

$$\psi^{-1}(\{y : \|y\| \geq \varepsilon\}) = \{x \in \mathbb{X} : \|\psi(x)\| \geq \varepsilon\} \subseteq \{x \in \mathbb{X} : \|x\| \geq \varepsilon/(d+1)\}$$

for each $\varepsilon > 0$, where $\|x\| = \max(x_1, \dots, x_{d+1})$ and $\|\psi(x)\|$ is the sup-norm induced by the uniform metric. In this case, $\psi^{-1}(\mathbf{0}_\mathbb{Y}) = \mathbf{0}_\mathbb{X}$, so ψ preserves the regular variation property.

Example 3.4.8. Let $\mathbb{X}$ be the space of continuously differentiable functions on $[0,1]$, and let $\mathbb{Y}$ be the space of continuous functions. The scaling on both spaces are given by multiplying the function's values with a scaling parameter. Equip $\mathbb{X}$ with the bornology generated by the modulus $\tau(x) = \|x\| = \|x\|_\infty + \|x'\|_\infty$, which is the sum of the uniform norm of the function and its derivative. Similarly, equip $\mathbb{Y}$ with the bornology generated by the modulus which is the uniform norm. Then the map $\psi(x)$, associating a function $x \in \mathbb{X}$ with its derivative x', is a continuous bornologically consistent morphism (note that the norm of $\psi(x) = x'$ in $\mathbb{Y}$ is smaller than or equal to the norm of x in $\mathbb{X}$). If ν is regularly varying on $\mathbb{X}$ and its tail measure is not supported only by constant functions, then $\psi\nu$ is regularly varying on $\mathbb{Y}$. This example can be generalised to higher-order differential operators.

Example 3.4.9. Consider $\mathbb{X} = \mathbb{R}_+^2$ with the scaling defined in Example 1.4.9:

$$T_t(x,y) = (x + (t-1)\min(x,y), y + (t-1)\min(x,y)).$$

Choose $\tau(x,y) = \min(x,y)$ as a modulus and let $\eta = (\eta_1, \eta_2)$ be any random vector with values in $\mathbb{S} = \{(x,y) : \min(x,y) = 1\}$. If ξ is a regularly varying random variable with index α, then $T_\xi \eta$ is regularly varying in $\mathbb{X}$ with the same index, where

$$T_\xi \eta = \big(\eta_1 + (\xi-1)\min(\eta_1,\eta_2), \eta_2 + (\xi-1)\min(\eta_1,\eta_2)\big) = (\eta_1,\eta_2) + (\xi-1, \xi-1),$$

since $\min(\eta_1, \eta_2) = 1$. Therefore, $\mathbf{P}\{\tau(T_\xi\eta) > t\} = \mathbf{P}\{\xi > t\}$, so $\tau(T_\xi\eta)$ is a regularly varying random variable with the same index as ξ. Thus, the regularly varying property of

$T_\xi \eta$ follows from the regular variation of ξ, since $\rho(T_\xi \eta) = ((\eta_1, \eta_2), \xi)$, so for Borel $A \subseteq \mathbb{S}$ and $u, t > 0$,

$$\frac{\mathbf{P}\{\rho(T_\xi \eta) \in A \times (ut, \infty)\}}{\mathbf{P}\{\tau(T_\xi \eta) > t\}} = \mathbf{P}\{(\eta_1, \eta_2) \in A\} \frac{\mathbf{P}\{\xi > ut\}}{\mathbf{P}\{\xi > t\}} \to \mathbf{P}\{\eta \in A\} u^{-\alpha} \quad \text{as } t \to \infty.$$

If (η_1, η_2) has a light tail, then $T_\xi \eta$ is regularly varying in the classical sense with the spectral measure concentrated on the diagonal.

Note that a pair (X, Y) of independent Pareto distributed random variables is not regularly varying in the new scaling. Indeed, let X and Y be two independent Pareto distributed random variables with index 1. Then $\tau(X, Y)$ is a regularly varying random variable with index 2, and $\mathbf{P}\{\tau(X, Y) > t\} = t^{-2}$. Let $\rho(X, Y)$ be the polar decomposition of (X, Y). Let $b > a \geq 0$, and let $A = S \cap \{(x, y) : x + a < y < x + b\}$. We have, for t large enough,

$$\begin{aligned} \mathbf{P}\{\rho(X, Y) \in A \times (ut, \infty)\} &= \mathbf{P}\{X + a < Y < X + b,\, \min\{X, Y\} > ut\} = \\ &= \mathbf{P}\{X + a < Y < X + b,\, X > ut\} = \\ &= \mathbf{P}\{Y > X + a,\, X > ut\} - \mathbf{P}\{Y > X + b,\, X > ut\}. \end{aligned}$$

Consider an arbitrary $c \geq 0$ and calculate,

$$\mathbf{P}\{Y > X + c,\, X > ut\} = \int_{ut}^{\infty} \mathbf{P}\{Y > s + c\} s^{-2} ds = \int_{ut}^{\infty} \frac{ds}{(s + c)s^2}.$$

We can now bound this term as follows:

$$\frac{1}{2}(ut + c)^{-2} = \int_{ut}^{\infty} \frac{ds}{(s + c)^3} \leq \int_{ut}^{\infty} \frac{ds}{(s + c)s^2} \leq \int_{ut}^{\infty} \frac{ds}{s^{-3}} = \frac{1}{2}(ut)^{-2}.$$

After applying these bounds to the calculation above, we have

$$\begin{aligned} \mathbf{P}\{\rho(X, Y) \in A \times (ut, \infty)\} &= \mathbf{P}\{Y > X + a,\, X > ut\} - \mathbf{P}\{Y > X + b,\, X > ut\} \\ &\leq \frac{1}{2}((ut)^{-2} - (ut + b)^{-2}). \end{aligned}$$

Dividing the latter term by $\mathbf{P}\{\min(X, Y) > t\}$ and taking the limit as $t \to \infty$, we have

$$\lim_{t \to \infty} \frac{\mathbf{P}\{\rho(X, Y) \in A \times (ut, \infty)\}}{\mathbf{P}\{\min\{X, Y\} > t\}} \leq \lim_{t \to \infty} \frac{1}{2} \left(u^{-2} - \left(u + \frac{b}{t} \right)^{-2} \right) = 0.$$

This implies that the random element (X, Y) is not regularly varying in $\mathbb{X}$.

One example of a regularly varying element in this space is a random vector $(X, X + l)$, where X is Pareto-1 distributed and $l \geq 0$ is a constant.

3.5 Regular variation and point processes

We follow the setting of Example 1.4.11 with $\mathbb{X} = \mathbb{R}_+$. Let $\mathcal{S}(\mathbb{X})$ be an ideal generated by the modulus $\tau(x) = x$ (see Definition 1.3.7 and Lemma 1.3.9). Recall that $\mathcal{M}_p(\mathbb{X}, \mathcal{S}(\mathbb{X}))$ denotes the set of counting measures m on $\mathbb{X}$. A random element in $\mathcal{M}_p(\mathbb{X}, \mathcal{S}(\mathbb{X}))$ is called a *point process*.

Consider a sequence of i.i.d. random elements (ξ_i) on $\mathbb{X}$ and a sequence of positive constants (a_n). We construct the sequence of point processes

$$N_n = \sum_{\{i=1,\dots,n:\xi_i \in \mathbb{X}'\}} \delta_{T_{a_n^{-1}}\xi_i}$$

on the state space $\mathbb{X}'$. We ignore the values of ξ_i which belong to $\mathbf{0}$. We typically assume that ξ_i is regularly varying, and in that case, the sequence (a_n) is given in Theorem 2.2.3(iii).

Endow $\mathcal{M}_p(\mathbb{X}, \mathcal{S}(\mathbb{X}))$ with the corresponding vague topology. Recall that a Laplace functional of a point process $\eta \in \mathcal{M}_p(\mathbb{X}, \mathcal{S}(\mathbb{X}))$ is a map Ψ_η (see [48]) that maps a non-negative measurable function to $[0,1]$ in the following way:

$$\Psi_\eta(f) = \mathbf{E}[e^{-\eta(f)}].$$

By [15, Proposition 11.1.VIII], weak convergence in $\mathcal{M}_p(\mathbb{X}, \mathcal{S}(\mathbb{X}))$ is equivalent to convergence of corresponding Laplace functionals for $f \in C_{\mathcal{S}(\mathbb{X})}$.

A Poisson point process with mean measure μ or a Poisson random measure will be denoted by $\mathrm{PRM}(\mu)$. The following theorem is classical in locally compact spaces, but the same argument applies in our setting as well, cf. Proposition 3.21 in Resnick [48].

Theorem 3.5.1. *Assume that $\mathbb{X}$-valued random elements (ξ_n) are i.i.d. Their common distribution is* $\mathrm{RV}(\mathbb{X}, \mathcal{S}(\mathbb{X}), \mu)$ *if and only if N_n converges in distribution to a* $\mathrm{PRM}(\mu)$ *on $\mathbb{X}'$.*

Proof. Let Ψ_{N_n} be the Laplace functional of N_n. For $f \in C_{\mathcal{S}(\mathbb{X})}$,

$$\begin{aligned}\Psi_{N_n}(f) = \mathbf{E}\left[\exp\left\{-\sum_{i=1}^{n} f(T_{a_n^{-1}}\xi_i)\right\}\right] &= \left(\mathbf{E}\left[\exp\{-f(T_{a_n^{-1}}\xi_i)\}\right]\right)^n = \\ &= \left(1 - \frac{\int_{\mathbb{X}}(1-e^{-f(x)})n\mathbf{P}\left\{T_{a_n^{-1}}\xi_i \in dx\right\}}{n}\right)^n,\end{aligned}$$

which converges towards

$$\exp\left\{-\int_{\mathbb{X}}(1-e^{-f(x)})\mu(dx)\right\} = \Psi_{\mathrm{PRM}(\mu)}(f)$$

if and only if $\xi \in \mathrm{RV}(\mathbb{X}, \mathcal{S}(\mathbb{X}), \mu)$, due to the pointwise convergence of composition of sequences of functions. $\square$

An immediate corollary of the theorem above is that

$$\mathbf{P}\left\{T_{a_n^{-1}}\xi_i \notin B, \text{ for any } i = 1,\dots,n\right\} = \mathbf{P}\left\{N_n(B) = 0\right\} \to e^{-\mu(B)} \quad \text{as } n \to \infty \tag{3.8}$$

whenever $\mu(\partial B) = 0$ for $B \in \mathcal{S}(\mathbb{X})$. The importance of this simple relation is widely recognised in extreme value theory. This result can be used in Examples 3.4.1 and 3.4.2 to simplify the calculations.

The following result concerns the regular variation property on the product space $\mathbb{X}^\infty$, which consists of sequences of points from $\mathbb{X}$ that is endowed with the bornology induced by continuous modulus τ. The scaling on $\mathbb{X}$ is applied coordinatewisely to $\mathbb{X}^\infty$, and a continuous modulus is defined as

$$\bar{\tau}(x) = \sup\{\tau(x_i) : i \in \mathbb{N}\}, \quad x = (x_1, x_2, \dots) \in \mathbb{X}^\infty.$$

Recall (Example 2.4.7) that this modulus does not induce a bornology on $\mathbb{X}^\infty \setminus \{\mathbf{0}\}$, but only a bornology $\mathcal{S}$ on $\mathbb{U} = \cup_{t>0} T_t C$, where $C = \{x \in \mathbb{X} : \bar{\tau}(x) \geq 1\}$. We assume that $\mathbb{R}^n$ is equipped with the linear scaling and the bornology generated by the modulus τ_n given by $\tau_n(x_1, \dots, x_n) = \sup\{\tau(x_i) : i = 1, \dots, n\}$.

Proposition 3.5.2. *If a random element $\xi = (\xi_1, \xi_2, \dots)$ in $\mathbb{X}^\infty$ is regularly varying on $\mathcal{S}$, then its finite-dimensional projection $(\xi_1, \dots, \xi_n)$ is regularly varying in $\mathbb{X}^n$ for all $n \geq 1$.*

Proof. Assume that ξ is regularly varying with tail measure μ and fix $n \in \mathbb{N}$. Let h be continuous bounded function $h : \mathbb{X}^n \to \mathbb{R}$ with support bounded in $\mathbb{X}^n$. This means that $\operatorname{supp} h = \{(x_1, \dots, x_n) : h(x_1, \dots, x_n) > 0\}$ is bounded in $\mathbb{R}^n$ which is equipped with the modulus τ_n. Note, $\hat{\tau}(\operatorname{supp} h) > t$, for some $t > 0$. Then the function $f : \mathbb{X}^\infty \to \mathbb{R}$ defined by

$$f(x_1, x_2, \dots) = h(x_1, x_2, \dots, x_n)$$

is continuous bounded with support bounded in $\mathbb{X}^\infty$, since the support of f is the set $\{(x_1, x_2, \dots) : h(x_1, \dots, x_n) > 0\}$. Hence,

$$g(t)\mathbf{E}h(T_{t^{-1}}(\xi_1, \dots, \xi_n)) = g(t)\mathbf{E}f(T_{t^{-1}}\xi) \to \int f d\mu = \int h d\mu_{(n)} \quad \text{as } t \to \infty,$$

where $\mu_{(n)}(A) = \mu(A \times \mathbb{X}^\infty)$, for a measurable and bounded set A in $\mathbb{X}^n$.

□

Consider now the point process

$$N = \sum_{i=1}^{\Lambda} \delta_{\xi_i},$$

where $(\xi_i)_{i\geq 1}$ are i.i.d. in $(0, \infty)$, and Λ is a random variable with values in $\mathbb{N}$ that has a finite expectation and is independent of all ξ_i. Recall that the modulus $\bar{\tau}$ on $\mathcal{M}_p(\mathbb{X}, \mathcal{S}(\mathbb{X}))$ is defined as the supremum of the moduli of each point from its support, so that $\bar{\tau}(N) = \vee_{i=1}^{\Lambda} \xi_i$.

The bornology on $\mathcal{M}_p(\mathbb{X}, \mathcal{S}(\mathbb{X}))$ induced by $\bar{\tau}$ is denoted by $\mathcal{S}$. Assume that ξ_1 is regularly varying in $\mathbb{R}_+$ with the linear scaling and the bornology generated by $[1, \infty)$.

Consider first the case when $\Lambda = n$ is deterministic. It is easy to see that the random vector $(\xi_1, \ldots, \xi_n)$ is regularly varying in $\mathbb{R}_+^n$ with bornology induced by the modulus $\max(x_1, \ldots, x_n)$ and its tail measure μ_n is the sum of the measures $c\theta_\alpha(dx_i)$ supported by each of the coordinate half-lines in $\mathbb{R}_+^n$, where $c\theta_\alpha$ is the tail measure of ξ_1, see Example 2.4.7 and [50, page 192]. This follows from the fact that the map $\psi_n(x_1, \ldots, x_n) = \sum_{i=1}^n \delta_{x_i}$ is a continuous bornologically consistent morphism between $\mathbb{R}_+^n$ and $\mathcal{M}_p(\mathbb{X}, \mathcal{S}(\mathbb{X}))$. Moreover, the spaces $\mathbb{R}_+^n$ and $\mathcal{M}_p(\mathbb{X}, \mathcal{S}(\mathbb{X}))$ are indistinguishable. Thus, N is regularly varying in $\mathcal{M}_p(\mathbb{X}, \mathcal{S}(\mathbb{X}))$ with the tail measure $\psi_n \mu_n$. The normalising function g can be chosen to be the tail distribution function of ξ_1, that is, $g(t) = \mathbf{P}\{\xi_1 > t\}$.

Now consider a possibly random Λ. It is well-known (see [34, Lemma 5.1]) that $\bar{\tau}(N) = \vee_{i=1}^{\Lambda} \xi_i$ is also regularly varying on $\mathbb{R}_+$ with the tail measure $(\mathbf{E}[\Lambda])c\theta_\alpha$. For each $f \in \mathsf{C}_{\mathcal{S}}$ on $\mathcal{M}_p((0, \infty), \mathcal{S}(\mathbb{X}))$ we have to check (2.3) as

$$\frac{1}{\mathbf{P}\{\bar{\tau}(N) > t\}} \mathbf{E} f(T_{t^{-1}} N) = \sum_{n=1}^{\infty} \frac{\mathbf{P}\{\xi_1 > t\}}{\mathbf{P}\{\bar{\tau}(N) > t\}} \frac{1}{\mathbf{P}\{\xi_1 > t\}} \mathbf{E} f\Big(T_{t^{-1}} \sum_{i=1}^{n} \delta_{\xi_i}\Big) \mathbf{P}\{\Lambda = n\}. \quad (3.9)$$

The regular variation of $\psi_n(\xi_1, \ldots, \xi_n)$ implies that

$$\frac{1}{\mathbf{P}\{\xi_1 > t\}} \mathbf{E}\left[f\Big(T_{t^{-1}} \sum_{i=1}^{n} \delta_{\xi_i}\Big)\right] \to \int f d(\psi_n \mu_n) \quad \text{as } t \to \infty.$$

By [34, Lemma 5.1],

$$\frac{\mathbf{P}\{\xi_1 > t\}}{\mathbf{P}\{\bar{\tau}(N) > t\}} \to \frac{1}{\mathbf{E}[\Lambda]} \quad \text{as } t \to \infty.$$

Since $f \in C_{\mathcal{S}}$, there exists a constant $a > 0$ such that $\{\mu : f(\mu) > 0\} \subseteq \{\mu : \bar{\tau}(\mu) \geq a\}$, which means that f vanishes on $\{\mu : \bar{\tau}(\mu) < a\}$. Hence, $f(T_{t^{-1}} N) = 0$ if $\vee_{i=1}^n \xi_i < at$. If f is bounded by $c > 0$, then each term in the sum on the right-hand side of (3.9) is bounded by

$$\frac{c\mathbf{P}\{\vee_{i=1}^n \xi_i > at\}}{\mathbf{P}\{\vee_{i=1}^{\Lambda} \xi_i > t\}} \mathbf{P}\{\Lambda = n\} \leq \frac{cn\mathbf{P}\{\xi_1 > at\}\mathbf{P}\{\Lambda = n\}}{\sum_{k=1}^{\infty} \mathbf{P}\{\xi_1 > t\}\mathbf{P}\{\Lambda = k\}} = c\frac{n\mathbf{P}\{\xi_1 > at\}}{\mathbf{P}\{\xi_1 > t\}} \mathbf{P}\{\Lambda = n\}.$$

The ratio is uniformly bounded for sufficiently large t. By the dominated convergence theorem,

$$\frac{1}{\mathbf{P}\{\tau(N) > t\}} \mathbf{E}[f(T_{t^{-1}} N)] \to \frac{1}{\mathbf{E}[\Lambda]} \sum_{n=1}^{\infty} \mathbf{P}\{\Lambda = n\} \int f d(\psi_n \mu_n) = \int f d\mu,$$

where

$$\mu = \frac{1}{\mathbf{E}[\Lambda]} \sum_{n=1}^{\infty} \mathbf{P}\{\Lambda = n\} \psi_n \mu_n$$

becomes the tail measure of N.

Point processes of this type, and more specifically, the maximum of points from such a process, are further analysed in Chapter 4. Additionally, we also examine marked renewal cluster models, where each cluster has a representation similar to the marked version of the processes observed in this section.

3.6 Regular variation of random closed sets and their continuous maps

When dealing with regular variation of random closed sets, we assume that the carrier space $\mathbb{X}$ is a locally compact separable metric space equipped with a continuous scaling T_t. The family $\mathcal{F} = \mathcal{F}(\mathbb{X})$ of closed sets in $\mathbb{X}$ is endowed with the Fell topology, see [11, 44]. The Fell topology is generated by the subbase consisting of the following families:

$$\{F \in \mathcal{F} : F \cap G \neq \varnothing\}, \text{ for all open sets } G,$$
$$\{F \in \mathcal{F} : F \cap K = \varnothing\}, \text{ for all compact sets } K,$$

which means that finite intersections of these families form a base for the Fell topology. The Fell topology on $\mathcal{F}(\mathbb{X})$ is a topology in which a sequence (F_n) converges to F if, for each open set G such that $F \cap G \neq \varnothing$ and each compact set K such that $K \cap F = \varnothing$, we have $F_n \cap G \neq \varnothing$ and $F_n \cap K = \varnothing$ for all sufficiently large n. The Fell topology makes $\mathcal{F}$ a Polish space, see [44, Appendix C]. The Fell topology on $\mathcal{F} \setminus \{\varnothing\}$ is equivalent to pointwise convergence of distance functions (see [11, Theorem 2.5.4]) and can be metrised by the Hausdorff–Busemann metric

$$\mathsf{d}_{\mathcal{F}}(F_1, F_2) = \sup_{x \in \mathbb{R}^d} e^{-\mathsf{d}(x_0, x)} |\mathsf{d}(x, F_1) - \mathsf{d}(x, F_2)|, \tag{3.10}$$

where x_0 is any fixed point in $\mathbb{X}$, d is the metric on $\mathbb{X}$, and $\mathsf{d}(x, F) = \inf\{\mathsf{d}(x, y) : y \in F\}$ is the distance function of F. The scaling operation on $\mathbb{X}$ is naturally extended to act on $\mathcal{F}$ and will also be denoted by T_t.

Lemma 3.6.1. *The scaling on $\mathcal{F}(\mathbb{X})$ is continuous in the Fell topology.*

Proof. The scaling on a Polish space is continuous if it is separately continuous, that is, the map $x \mapsto T_t x$ is continuous on $\mathbb{X}$ for all $t \in (0, \infty)$ and the map $t \mapsto T_t x$ is continuous on $(0, \infty)$ for all $x \in \mathbb{X}$, see [37, Theorem 9.14]. Let $F_n \to F$ as $n \to \infty$. Assume that $T_t F \cap G \neq \varnothing$ for an open set G. Then $F \cap T_{t^{-1}} G \neq \varnothing$, so $F_n \cap T_{t^{-1}} G \neq \varnothing$ for sufficiently large n, taking into account that $T_{t^{-1}} G$ is open. If $T_t F \cap L = \varnothing$ for a compact set L, then F does not intersect the set $T_{t^{-1}} L$, which is compact as a continuous image of a compact set. Thus, $T_t F_n \cap L = \varnothing$ for all sufficiently large n.

Assume that $t_n \to 1$ as $n \to \infty$ with the aim to show that $T_{t_n} F$ converges to $T_1 F = F$ for each $F \in \mathcal{F}$. Let $F \in \mathcal{F}$, and let G be an open set such that $F \cap G \neq \varnothing$. Then for any $x \in F \cap G$, we have $T_{t_n} x \in T_{t_n} F \cap G$ for all sufficiently large n, by continuity of the scaling on $\mathbb{X}$. Therefore, $T_{t_n} F \cap G \neq \varnothing$ for all sufficiently large n. Now assume that $F \cap L = \varnothing$ for a compact set L. If $T_{t_n} F$ does not miss L for all sufficiently large n, then $T_{t_{n_k}} x_{n_k} \in L$

for a subsequence (n_k) and $x_{n_k} \in F$, $k \geq 1$. By compactness of L and passing to a further subsequence, assume that $x_{n_k} \to x$. Since F is closed, $x \in (F \cap L)$, which is a contradiction, and so $T_{t_n}F$ converges to F in the Fell topology. $\square$

Example 3.6.2. Let $\mathbb{X} = \mathbb{R}^d$ with the linear scaling and the Euclidean distance d, so that $\mathsf{d}(T_t x, T_t F) = t\mathsf{d}(x, F)$. If $x_0 = 0$ in (3.10), then

$$\mathsf{d}_{\mathcal{F}}(T_t F_1, T_t F_2) = \sup_{x \in \mathbb{R}^d} t e^{-t\|x\|} \big|\mathsf{d}(x, F_1) - \mathsf{d}(x, F_2)\big|, \quad t > 0.$$

If C is a closed cone, then

$$\mathsf{d}_{\mathcal{F}}(T_t F, C) = \sup_{x \in \mathbb{R}^d} t e^{-t\|x\|} \big|\mathsf{d}(x, F) - \mathsf{d}(x, C)\big|.$$

Remark 3.6.3. The family $\mathbf{0}_{\mathcal{F}}$ of scaling invariant elements in $\mathcal{F}$ is larger than the family of subsets of $\mathbf{0}$ in $\mathbb{X}$. For example, all cones are scaling invariant under T_t for all $t > 0$, as well as the empty set and $\mathbb{X}$ itself. In the space of closed sets, one can easily identify members which are scaling invariant under T_t only for some $t \neq 1$. Let $x \notin \mathbf{0}$ and $t \neq 1$ be fixed. Define $F = \operatorname{cl}\{T_{t^k}x : k \in \mathbb{Z}\}$, where $\mathbb{Z}$ is the set of all integers. Then F is scaling invariant under T_t.

Remark 3.6.4. The vague convergence on the family of nonempty closed sets is usually defined by working with the values of measures on the families $\{F \in \mathcal{F} : F \cap K \neq \varnothing\}$ for all nonempty compact sets K in $\mathbb{X}$, see [44, page 130]. This choice is motivated by the fact that these families are compact in the Fell topology on $\mathcal{F} \setminus \{\varnothing\}$. This family of sets is a topologically and scaling consistent ideal which covers $\mathcal{F} \setminus \{\varnothing\}$. This ideal is also the Hadamard bornology on the family of nonempty closed sets and is denoted by $\mathcal{S}_H$. Note that this ideal contains all nonempty scaling invariant closed sets, and thus does not satisfy property (B) and is not generated by a continuous modulus. To see why, assume that property (B) holds and denote the semicone from (B) with C. If $B \in C$ and $T_t B = B$ for some $t > 1$, then $B \in T_{t^k}C$ for every $k \geq 1$. Now $k \to \infty$ leads to contradiction.

Denote by $\mathcal{F}' = \mathcal{F} \setminus \mathbf{0}_{\mathcal{F}}$ the family of all nonempty closed sets excluding those which are scaling invariant for at least one $t > 0$, $t \neq 1$. While a bornology on $\mathcal{F}'$ can be constructed as in Remark 3.6.4, it does not seem feasible to define a continuous modulus on the whole $\mathcal{F}'$. However, it is possible to define moduli on some subfamilies of $\mathcal{F}'$, some of which are given in Propositions 3.6.5 and 3.6.7. We often impose the following condition:

(C) The space $\mathbb{X}'$ is equipped with a continuous modulus τ and $\mathbf{0} \cup \{x \in \mathbb{X}' : \tau(x) \leq 1\}$ is a relatively compact set in $\mathbb{X}$.

Condition (C) implies that, if $x_n \to x$ and $\tau(x_n) \leq a$ for some $a > 0$ and all n, then either $\tau(x) \leq a$ or $x \in \mathbf{0}$. It is satisfied in $\mathbb{R}^d$ with the linear scaling and the topological bornology.

Proposition 3.6.5. *Assume condition (C). Then $\hat{\tau}$ defined in* (1.9) *is a continuous modulus on the family $\mathcal{F}^{\mathbf{0}}(\mathbb{X})$ of nonempty closed sets which do not intersect $\mathbf{0}$.*

Remark 3.6.6. Note that the family $\mathcal{F}^{\mathbf{0}}(\mathbb{X})$ does not contain any scaling invariant element. Indeed, if $T_t F = F$ for some $F \in \mathcal{F}^{\mathbf{0}}(\mathbb{X})$ and $t \neq 1$, then we can assume that $t < 1$. Hence, for any $x \in F$ we have $T_{t^n} x \in F$, $n \geq 1$. By (C), this sequence has a convergent subsequence which converges to a point from $\mathbf{0}$, so that F has nonempty intersection with $\mathbf{0}$.

Proof. We show that the function

$$\hat{\tau}(A) = \inf\{\tau(x) : x \in A\}, \quad A \in \mathcal{F}^0(\mathbb{X})$$

is a continuous modulus on $\mathcal{F}^{\mathbf{0}}(\mathbb{X})$. This function is evidently homogeneous. First, note that $\hat{\tau}(F) > a$ if and only if F does not intersect $\{x : \tau(x) \leq a\}$. Indeed, if F intersects this set, then $\hat{\tau}(F) \leq a$. On the other hand, if F does not intersect this set and $\hat{\tau}(F) \leq a$, then there is a sequence $x_n \in F$ with $\tau(x_n) \leq a + 1/n$. By (C), (x_n) admits a subsequence that converges to $x \in F$. By continuity of τ, we have either $x \in \mathbf{0}$ or $\tau(x) \leq a$, which is impossible.

Assume that $F_n \to F$ in the Fell topology induced on $\mathcal{F}^{\mathbf{0}}(\mathbb{X})$. If $\hat{\tau}(F) < a$ for some $a > 0$, then there exists an $x \in F$ with $\tau(x) < a$. The Fell convergence implies that there exists a sequence $x_n \in F_n$ such that $x_n \to x$, so that $\tau(x_n) < a$ and hence $\hat{\tau}(F_n) < a$ for all sufficiently large n. Assume now that $\hat{\tau}(F) > a > 0$. Then $F \cap \{x : \tau(x) \leq a\} = \varnothing$. If $F_n \cap \{x : \tau(x) \leq a\} \neq \varnothing$ for infinitely many n, then there exists a subsequence $x_{n_k} \in F_{n_k}$ such that $\tau(x_{n_k}) \leq a$ for all k. By (C), (x_{n_k}) has a further subsequence which converges to x with $\tau(x) \leq a$. By the Fell convergence, $x \in F$, which is a contradiction. Therefore, $F_n \cap \{x : \tau(x) \leq a\} = \varnothing$, equivalently, $\hat{\tau}(F_n) > a$ for all sufficiently large n.

Finally, if $\hat{\tau}(F) = 0$, then there exists a sequence $x_n \in F$ with $\tau(x_n) \to 0$. By (C), there is a subsequence x_{n_k} which converges to $x \in F$. By continuity of the modulus, $x \notin \mathbf{0}$ is not possible, so that $F \cap \mathbf{0} \neq \varnothing$, a contradiction, since $F \in \mathcal{F}^{\mathbf{0}}(\mathbb{X})$. $\square$

The bornology generated by $\hat{\tau}$ consists of all families of sets whose union belongs to the bornology $\mathcal{S}_\tau$ generated on $\mathbb{X}'$ by the modulus τ. As a result, any scaling consistent ideal on $\mathbb{X}$ transforms into a scaling consistent ideal on $\mathcal{F}$. In the following proposition, we analyse another subfamily of $\mathcal{F}(\mathbb{X})$ along with the corresponding modulus.

Proposition 3.6.7. *Assume condition (C). Then $\bar{\tau}$ defined by*

$$\bar{\tau}(F) = \sup\{\tau(x) : x \in F\} \tag{3.11}$$

is a continuous modulus on the family $\mathcal{F}_b(\mathbb{X})$ of τ - bounded star-shaped sets in $\mathbb{X}$.

Remark 3.6.8. The family $\mathcal{F}_b(\mathbb{X})$ consists of all sets $F \in \mathcal{F}(\mathbb{X})$ such that $T_t F \subseteq F$ for all $t \in (0, 1]$ and $\bar{\tau}(F) < \infty$. Note that the set of zeros, $\mathbf{0}_{\mathcal{F}}$, is the family of closed subsets of $\mathbf{0}$.

Proof. The function $\bar{\tau}$ is strictly positive and homogeneous on $\mathcal{F}_b(\mathbb{X}) \setminus \mathbf{0}_{\mathcal{F}}$. We will now show that it is also a continuous modulus. Assume that $F_n \to F$ in the Fell topology. If $\bar{\tau}(F) > t$ for some $t > 0$, then there exists $x \in F$ such that $\tau(x) > t$. By the Fell convergence, there exists a sequence $x_n \in F_n$ such that $x_n \to x$. By continuity of τ, we have $\tau(x_n) > t$

for all sufficiently large n. Hence, $\bar{\tau}(F_n) > t$ for all sufficiently large n. Now assume that $\bar{\tau}(F) < t$, but $\bar{\tau}(F_n) \geq t$ for infinitely many n. Passing to a subsequence, we can assume that $\tau(x_n) \geq t_n$ with $x_n \in F_n$ and $t_n \uparrow t$. Then

$$y_n = T_{\tau(x_n)^{-1}} x_n \in \{x \in \mathbb{X}' : \tau(x) \leq 1\}, \quad n \geq 1.$$

By passing to a convergent subsequence, assume that $y_n \to y$ as $n \to \infty$. Then $T_{t_n} y_n \in F_n$ by the star-shapedness assumption, since $x_n \in F_n$ and $t_n \tau(x_n)^{-1} \leq 1$. Hence, $T_{t_n} y_n \to T_t y \in F$ with $\tau(T_t y) = t$, which is a contradiction. $\square$

The corresponding bornology is generated by the semicone

$$\mathcal{C} = \{F \in \mathcal{F}_b(\mathbb{X}) : F \cap C \neq \varnothing\}.$$

Note that if we drop the star-shapedness assumption, $\bar{\tau}$ is no longer continuous. For example, let $F_n = F \cup \{x_n\}$, where $F \in \mathcal{F}_b(\mathbb{X})$ and $\tau(x_n) \to \infty$ (so that x_n eventually escapes any compact set). Then $F_n \to F$ in the Fell topology, but $\bar{\tau}(F_n) \geq \tau(x_n)$ does not converge to $\bar{\tau}(F)$.

A random element in $\mathcal{F}$ with the Borel σ-algebra generated by the Fell topology is said to be a *random closed set*. The concept of regular variation for random closed sets is introduced in [44, Definition 4.2.4], implicitly referring to the bornology from Remark 3.6.4. Clearly, if a random closed set is scaling invariant with a positive probability, then it is not regularly varying on such a bornology, since the tail measure acquires infinite values. We restrict our attention to random closed sets taking values from a subfamily $\mathcal{D}$ of $\mathcal{F}(\mathbb{X}) \setminus \{\varnothing\}$. Assume that $\mathcal{D}$ is closed under scaling, that is, $\mathcal{D}$ is a cone in $\mathcal{F}$. We equip $\mathcal{D}$ with an ideal $\mathcal{S}$ that is topologically and scaling consistent and has a countable open base. We then define a regularly varying random set X by specializing the general definition to the space $\mathcal{D}$ with the ideal $\mathcal{S}$.

By Proposition 2.2.5, if X is a random closed set regularly varying in $\mathcal{F}$ with the Hadamard bornology $\mathcal{S}_H$ and X almost surely takes values from $\mathcal{D}$ and $\mathcal{S}$ is a subideal of $\mathcal{S}_H$, then $X \in \mathrm{RV}(\mathcal{D}, \mathcal{S}, \mu)$ if the tail measure μ is nontrivial on $\mathcal{S}$.

In the following, we consider set-valued maps for which Theorem 3.2.1 can be applied.

Let ψ be a continuous map from $\mathbb{X}$ to a Hausdorff space $\mathbb{Y}$. Then $\Psi(x) = \psi^{-1}(\psi(x))$ becomes a set-valued map from $\mathbb{X}$ to $\mathcal{F}(\mathbb{X})$. Equivalently,

$$\Psi(x) = \{z \in \mathbb{X} : \psi(z) = \psi(x)\}. \tag{3.12}$$

Continuity of ψ and closedness of singletons in $\mathbb{Y}$ ensure that $\Psi(x)$ is a closed set. Examples of such set-valued inverses arise in the context of quotient spaces in Section 3.8.

The map Ψ is a morphism between $\mathbb{X}$ and $\mathcal{F}(\mathbb{X})$ if (3.1) holds, that is,

$$\Psi(T_t x) = T_t \Psi(x), \quad t > 0, x \in \mathbb{X}. \tag{3.13}$$

This holds if $\psi(z) = \psi(x)$ for any $x, z \in \mathbb{X}$ implies $\psi(T_t z) = \psi(T_t x)$ for all $t > 0$. To see this, assume that $\psi(z) = \psi(x)$ implies that $\psi(T_t z) = \psi(T_t x)$ for each $t > 0$. Then

$$\begin{aligned}\Psi(T_t x) &= \{z \in \mathbb{X} : \psi(z) = \psi(T_t x)\} = \{z \in \mathbb{X} : \psi(T_{t^{-1}} z) = \psi(x)\} \\ &= \{T_t z \in \mathbb{X} : \psi(z) = \psi(x)\} = T_t \Psi(x).\end{aligned}$$

Lemma 3.6.9. *If ψ is a continuous open map, then Ψ is continuous in the Fell topology.*

Proof. Let $x_n \to x$ as $n \to \infty$. If $\Psi(x) \cap G \neq \varnothing$ for an open set G, then $\psi(x) \in \psi(G)$. Since $\psi(x_n) \to \psi(x)$ and $\psi(G)$ is open, we have that $\psi(x_n) \in \psi(G)$, so that $\Psi(x_n) \cap G \neq \varnothing$. If $\Psi(x) \cap K = \varnothing$ for a compact set K, then $\psi(x) \notin \psi(K)$. Since $\psi(K)$ is compact, being a continuous image of a compact set, hence, closed, we have that $\psi(x_n) \notin \psi(K)$ for all sufficiently large n. $\square$

Lemma 3.6.10. *Assume that* (3.13) *holds. If $\mathbf{0} \cap \Psi(x) \neq \varnothing$ for some $x \in \mathbb{X}$, then $T_t\Psi(x) = \Psi(x)$ for some $t \neq 1$.*

Proof. Assume that $y \in \Psi(x) \cap \mathbf{0}$ and $T_t y = y$ for some $t \neq 1$. Then $\psi(x) = \psi(y)$, and

$$\begin{aligned} T_t\Psi(x) &= \{T_t z : \psi(z) = \psi(y)\} = \{z : \psi(T_t^{-1}z) = \psi(T_t^{-1}y)\} = \\ &= \{z : \psi(z) = \psi(y)\} = \Psi(x). \end{aligned}$$

$\square$

Therefore, each set from the family $\mathcal{D} = \{\Psi(x) : x \in \mathbb{X}\}$ either belongs to $\mathbf{0}_{\mathcal{F}}$ or is a closed set that has an empty intersection with $\mathbf{0}$, that is, $\mathcal{D}' = \mathcal{D} \setminus \mathbf{0}_{\mathcal{F}} \subseteq \mathcal{F}^{\mathbf{0}}(\mathbb{X})$. In view of this, it is possible to use the modulus on the family $\mathcal{F}^{\mathbf{0}}(\mathbb{X})$ of nonempty closed sets that do not intersect $\mathbf{0}$, which is further analysed in Proposition 3.6.5. We define

$$\hat{\tau}(\Psi(x)) = \inf\{\tau(y) : y \in \mathbb{X},\ \psi(y) = \psi(x)\}.$$

This function is clearly homogeneous and does not vanish if $\Psi(x)$ is not a cone, that is, if $\psi(x) \notin \psi(\mathbf{0})$. Since $\hat{\tau}$ is continuous on $\mathcal{D}$ (see Proposition 3.6.5), $\hat{\tau}(\Psi(x))$ is continuous as a function of x if Ψ is continuous. Therefore, $\hat{\tau}$ is a continuous modulus if Ψ is continuous.

Theorem 3.6.11. *Assume that the assumption (A) holds. Let $\xi \in \mathrm{RV}(\mathbb{X}, \mathcal{S}, \mu)$ be regularly varying with bornology $\mathcal{S}$ satisfying (B). Assume that Ψ defined at* (3.12) *is continuous in the Fell topology and satisfies* (3.13). *If the tail measure μ attaches positive mass to the set $\{x \in \mathbb{X} : \Psi(x) \cap \mathbf{0} = \varnothing\}$, then $\Psi(\xi)$ is regularly varying in $\mathcal{F}^{\mathbf{0}}(\mathbb{X})$ with the bornology generated by $\hat{\tau}$.*

Proof. The map Ψ is a bornologically consistent morphism since

$$\big\{x : \hat{\tau}(\Psi(x)) \geq t\big\} = \big\{x : \tau(y) \geq t, \psi(y) = \psi(x)\big\} \subseteq \big\{x : \tau(x) \geq t\big\}.$$

The result follows from Theorem 3.2.1. $\square$

Remark 3.6.12. It is possible to generalise (3.12) as follows. Let $\Phi : \mathbb{X} \mapsto \mathcal{F}(\mathbb{Y})$ be a continuous function from $\mathbb{X}$ to the family $\mathcal{F}(\mathbb{Y})$ of closed sets (equipped with the Fell topology) in a locally compact Hausdorff separable space $\mathbb{Y}$. Define

$$\Psi(x) = \{y \in \mathbb{X} : \Phi(y) \cap \Phi(x) \neq \varnothing\}. \tag{3.14}$$

Note that we recover (3.12) if Φ is singleton-valued. Another possible definition is

$$\Psi(x) = \Phi^{-}(\Phi(x)),$$

where

$$\Phi^{-}(F) = \{y \in X : \Phi(y) \cap F \neq \varnothing\}$$

is the inverse set-valued function to Φ. Assuming that $\Phi(x) \cap \Phi(y) \neq \varnothing$ implies $\Phi(T_t x) \cap \Phi(T_t y) \neq \varnothing$ for all $t > 0$, property (3.13) and the conclusion of Lemma 3.6.10 hold. If Ψ is also continuous, then Theorem 3.6.11 applies.

3.7 Examples: random closed sets

An important subfamily of $\mathcal{F}(\mathbb{X})$ is the family $\mathcal{K}(\mathbb{X})$ of nonempty compact sets in $\mathbb{X}$ with the scaling uplifted from $\mathbb{X}$. The zeroes in $\mathcal{K}(\mathbb{X})$ are compact subsets of $\mathbf{0}$ in $\mathbb{X}$. The family of nonempty compact sets is metrised by the Hausdorff metric d_H. A sequence (K_n) converges to $K \in \mathcal{K}(\mathbb{X})$ if, whenever $K \cap G \neq \varnothing$ for an open G and $K \cap F = \varnothing$ for a closed F, these conditions are fulfilled for K_n with all sufficiently large n. This convergence is stronger than one induced on $\mathcal{K}(\mathbb{X}) \subseteq \mathcal{F}$ by the Fell topology on $\mathcal{F}$, for instance, $K \cup \{x_n\}$ does not converge to K in the Hausdorff metric if x_n escapes to infinity. To illustrate this, consider K to be the unit ball in $\mathbb{R}^d$ induced by some metric, and define the sequence $(x_n) = ((n, 0, \ldots, 0))$. Then K does not intersect $F = [1, \infty) \times \mathbb{R}^d$ but there exists $n_0 \in \mathbb{N}$ such that $K_n = K \cup \{x_n\}$ intersects F for all $n \geq n_0$.

Example 3.7.1. Let $\mathcal{K}(\mathbb{R}^d)$ be the family of nonempty compact sets in $\mathbb{R}^d$ metrised by the Hausdorff metric, which makes it a Polish space, and equipped with the linear scaling. Then $\mathbf{0}$ is only the set $\{0\}$, and

$$\bar{\tau}(K) = \sup\{\|x\| : x \in K\} \tag{3.15}$$

is a continuous modulus on $\mathcal{K}(\mathbb{R}^d)$. Note that the space $\mathcal{K}(\mathbb{R}^d)$ with this modulus and the Hausdorff metric is star-shaped.

Let $\psi(K)$ be the Lebesgue measure (the volume) of $K \in \mathcal{D}$, so that $\psi : \mathcal{D} \to \mathbb{R}_+$. If $\mathbb{R}_+$ is equipped with the linear scaling and the bornology generated by $[1, \infty)$, then ψ becomes a bornologically consistent morphism of order d. Indeed, it is not possible that $\bar{\tau}(K_n) \to 0$ for a sequence of compact sets K_n with $\psi(K_n) \geq \varepsilon > 0$. If the tail measure of a random compact set X is not only supported by sets of volume zero, then the volume of X is a regularly varying random variable. If X is also convex, similar examples can be constructed by considering other intrinsic volumes, see [51].

Example 3.7.2. Let $\mathcal{D}$ be the family of singletons. Then a modulus on $\mathbb{X}'$ becomes a modulus on $\mathcal{D} \setminus \{\{0\}\}$, so that a regularly varying random closed set in $\mathcal{D}$ is a regularly varying random element in $\mathbb{X}$.

One can take as $\mathcal{D}$ families of closed sets from Propositions 3.6.5 or 3.6.7. In $\mathbb{X} = \mathbb{R}^d$ we also often consider the family of nonempty compact convex sets $\mathcal{K}_c(\mathbb{R}^d)$, which is a closed subset of the family $\mathcal{K}(\mathbb{R}^d)$ from Example 3.7.1, and thus, Polish. The trace of the Fell topology on $\mathcal{K}_c(\mathbb{R}^d)$ coincides with the Hausdorff metric topology, see [52, Theorem 12.3.4]. In the following we concentrate on the case when $\mathcal{D}$ is the family of compact convex subsets of $\mathbb{X}$.

Example 3.7.3. Let $\mathcal{D} = \mathcal{K}_c(\mathbb{R}^d)$ be the family of compact convex sets in $\mathbb{R}^d$ metrised by the Hausdorff metric and equipped with the linear scaling $T_tK = \{tx : x \in K\}$ for $K \in \mathcal{D}$ and $t > 0$. The unique zero element in $\mathcal{D}$ is the set $\{0\}$. Equip $\mathcal{D}' = \mathcal{D} \setminus \{\{0\}\}$ with the bornology generated by the modulus $\bar{\tau}$ from (3.15). Let $\psi(K) = \mathsf{s}(K)$ be the *Steiner point* of $K \in \mathcal{D}$ defined by

$$\mathsf{s}(K) = \frac{1}{\kappa_d} \int_{\mathbb{S}^{d-1}} h_K(u)\, u\, \mathcal{H}^{d-1}(du) \tag{3.16}$$

where

$$h_K(u) = \sup\{\langle u, a\rangle : a \in K\}, \quad u \in \mathbb{S}^{d-1}, \tag{3.17}$$

is the *support function* of K, κ_d is the volume of the unit Euclidean ball in $\mathbb{R}^d$, and $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure on the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$, see [51, Eq. (1.31)]. By [51, Eq. (1.34)], $\mathsf{s}(K)$ always belongs to K. Furthermore, $\bar{\tau}$ from (3.11) can be expressed as

$$\bar{\tau}(K) = \sup\{h_K(u) : u \in \mathbb{S}^{d-1}\}. \tag{3.18}$$

Then $\psi : \mathcal{D} \to \mathbb{R}^d$ is a continuous morphism, where the target space $\mathbb{R}^d$ is equipped with the linear scaling and the bornology generated by $\tau(y) = \|y\|$. Furthermore, ψ is bornologically consistent, which follows from the star-shapedness of the spaces. If X is regularly varying in $\mathcal{D}$, then $\psi(X)$ is regularly varying in $\mathbb{R}^d$ if the tail measure of X attaches positive mass to the family of sets whose Steiner point is not the origin.

If X is an almost surely nonempty random closed set, then a random element ξ in $\mathbb{X}$ is called a *selection* of X if $\xi \in X$ a.s. In some cases it is possible to prove that a regularly varying random closed set admits a regularly varying selection. The discussion in Example 3.7.3 shows that this is the case if X is a random compact convex set and its tail measure is nontrivial on the family of sets with Steiner point not being the origin.

Consider further continuous maps of random compact convex sets.

Example 3.7.4. Consider $\mathcal{D} = \mathcal{K}_c(\mathbb{R}^d)$ from Example 3.7.3. Define a map ψ from $\mathcal{D}$ to the space $\mathbb{Y} = \mathsf{C}(\mathbb{S}^{d-1})$ of continuous functions on the unit sphere $\mathbb{S}^{d-1}$ by associating each compact convex set K with its support function h_K, see (3.17). The space $\mathbb{Y}$ is equipped with the uniform metric and the scaling being the usual multiplication of a function with scalar. Equip $\mathbb{Y}$ with the modulus

$$\tau_{\mathbb{Y}}(h) = \|h\|_\infty = \sup\{|h(u)| : \|u\| = 1\}$$

given by the uniform norm. Recall that

$$||h_K - h_L||_\infty = \mathsf{d}_H(K, L), \quad K, L \in \mathcal{D},$$

where d_H denotes the Hausdorff metric on $\mathcal{D}$. Then ψ is an isometry, which satisfies conditions of Theorem 3.2.1. See also Remark 3.2.3. In this case, the space of compact convex sets $\mathcal{D}$ and its image in the space $\mathbb{Y}$ are indistinguishable. A random compact convex set X is regular varying in $\mathcal{D}$ if and only if its support function h_X is regularly varying in the family of continuous functions on the unit sphere. In this way, we recover the observation made in [43].

Example 3.7.5. In the setting of Example 3.7.3, consider a map from $\mathcal{D}$ to $\mathcal{D}$ that maps $K \in \mathcal{D}$ to the convex body $L = \Pi K$ such that $h_L(u)$ equals the $(d-1)$-dimensional volume of the projection of K onto the hyperplane orthogonal to u. The set ΠK is called the projection body of K, see [51, Section 10.9]. The map from K to ΠK is a continuous bornologically consistent morphism that preserves the regular variation property. To see that it is a morphism, note that $h_{T_t K} = t \cdot h_K$ for all $K \in \mathcal{D}$ and $t > 0$, where the multiplication on the right is the usual multiplication of function values. Bornological consistency follows from observing that the set $\{K : \bar{\tau}(\Pi K) \geq \varepsilon\} = \{K : \sup\{h_{\Pi K}(u) : u \in \mathbb{S}^{d-1}\} \geq \varepsilon\}$ is bounded in $\mathcal{D}$ for every $\varepsilon > 0$. This is a consequence of $h_{\Pi K}$ being the $(d-1)$-dimensional volume of projection of K.

The following example concerns passing to a subcone in the family of compact convex sets.

Example 3.7.6. Let $\mathcal{D} = \mathcal{K}_c(\mathbb{R}^d)$ with the scaling and bornology generated by the modulus $\bar{\tau}$, see Example 3.7.3. For a regularly varying random vector $\xi = (\xi_1, \dots, \xi_d)$ in $\mathbb{R}^d_+$, let

$$X = \psi(\xi) = \Big\{ u \in \mathbb{R}^d : \sum_{i=1}^d \xi_i^{-2} u_i^2 \leq 1 \Big\}$$

be the ellipsoid with semi-axes given by the components of ξ where $u/0 = 0$ if $u = 0$. If ξ is regularly varying in $\mathbb{R}^d_+$ with the modulus $\tau_{\max}$, then X is regularly varying in $\mathcal{D}$ by Theorem 3.2.1. Note that ψ is a bornologically consistent morphism, which can be seen from

$$\begin{aligned}\psi^{-1}\{K \in \mathcal{D} : \bar{\tau}(K) \geq \varepsilon\} &= \{\xi \in \mathbb{R}^d_+ : \sup\{h_{\psi(\xi)}(u) : u \in \mathbb{S}^{d-1}\} \geq \varepsilon\} \\ &= \{\xi \in \mathbb{R}^d_+ : \tau_{\max}(\xi) \geq \varepsilon\},\end{aligned}$$

where we also note that (3.1) holds. Assuming now that the components of ξ are i.i.d., the tail measure of ξ is supported by the axes. Consequently, the tail measure of X is supported by the family of segments $[-ue_i, ue_i]$ passing through the origin, where $u > 0$ and $e_1, \dots, e_d$ are standard basis vectors. If $\mathcal{D}_1$ is a subcone of $\mathcal{D}$ that consists of sets containing the origin in their interiors, then X is not regularly varying on $\mathcal{D}_1$. However, if $\mathcal{D}_1$ is equipped with the modulus $\tau(K) = \sup\{r \geq 0 : B_r(0) \subseteq K\}$, that is, the radius of the largest inscribed ball in $K \in \mathcal{D}_1$, then $\tau(X) = \min(\xi_1, \dots, \xi_d) = \tau_{\min}(\xi)$. Moreover, X becomes regularly varying if ξ is regularly varying in $(0, \infty)^d$ with the bornology generated by $\tau_{\min}$, that is, if ξ exhibits the hidden regular variation phenomenon.

Furthermore, it is possible to equip $\mathcal{D}_1$ with the modulus $\tau(K) = V_d(K)^{1/d}$, where V_d denotes the d-dimensional volume. Then X is regularly varying in $\mathcal{D}_1$ with this modulus if and only if the product $\xi_1 \cdots \xi_d$ is regularly varying. This condition is satisfied if and only if ξ is regularly varying on $(0, \infty)^d$ with the bornology generated by the modulus $\tau(x) = (x_1 \cdots x_d)^{1/d}$. This can be seen by using the formula for volume of an ellipsoid in $\mathbb{R}^d$ (see [39]):

$$V_d(\psi(\xi)) = \frac{\pi^{d/2}}{\Gamma(d/2 + 1)} \prod_{i=1}^d \xi_i,$$

where Γ denotes the Gamma function $\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx$, $t > 0$.

Several examples below deal with set-valued maps of random vectors.

Example 3.7.7. Recall the space $\mathcal{D}$ from Example 3.7.3. Consider i.i.d. copies $\eta_1, \dots, \eta_m$ of a regularly varying random vector $\eta \in \mathrm{RV}(\mathbb{R}^d, \mathcal{S}, \mu)$ with the topological bornology $\mathcal{S}$ on $\mathbb{R}^d$. Let X be the convex hull of $\{\eta_1, \dots, \eta_m\}$. By Theorem 3.2.1, X is regularly varying and its tail measure is the pushforward of $\mu^{\otimes m}$ under the map $(x_1, \dots, x_m) \mapsto \mathrm{conv}\{x_1, \dots, x_m\}$.

Example 3.7.8. Let $\psi : \mathbb{X} \to \mathbb{Y}$ be a continuous morphism between a locally compact Polish space $\mathbb{X}$ and a Polish space $\mathbb{Y}$ equipped with a continuous scaling. Assume that ψ is an open map. Then it is easy to show that the inverse map $\psi^{-1} : \mathbb{Y} \to \mathcal{F}(\mathbb{X})$ is a continuous morphism. Continuity follows from

$$(\psi^{-1})^{-1}\{F \in \mathcal{F}(\mathbb{X}) : F \cap G \neq \varnothing\} = \{y \in \mathbb{Y} : \psi^{-1}(y) \cap G \neq \varnothing\} = \psi(G),$$

which is open in $\mathbb{Y}$ if G is open in $\mathbb{X}$ by openness of ψ, and

$$(\psi^{-1})^{-1}\{F \in \mathcal{F}(\mathbb{X}) : F \cap K = \varnothing\} = \{y \in \mathbb{Y} : \psi^{-1}(y) \cap K = \varnothing\} = \psi(K)^c,$$

which is open for compact K. Endow $\mathbb{Y}$ with a bornology $\mathcal{S}$ that contains all compact subsets of $\mathbb{Y}$, and equip $\mathcal{F}(\mathbb{X})$ with Hadamard bornology. Then ψ^{-1} is bornologically consistent, as $\{y \in \mathbb{Y} : \psi^{-1}(y) \cap K \neq \varnothing\} = \psi(K)$ is a compact set in $\mathbb{Y}$. Let $\xi \in \mathrm{RV}(\mathbb{Y}, \mathcal{S}, \mu)$ be a regularly varying random element in $\mathbb{Y}$. Then $X = \psi^{-1}(\xi)$ is a random closed set in $\mathbb{X}$, which is regularly varying in the Hadamard bornology.

3.8 Regular variation on quotient spaces

Let $\mathbb{X}$ be a Polish space, and let $\sim$ be an equivalence relation on $\mathbb{X}$. Denote by $[x] = \{z \in \mathbb{X} : z \sim x\}$ the subset of $\mathbb{X}$ which consists of all elements equivalent to x. For a set $B \subseteq \mathbb{X}$, let $[B]$ denote the saturation of B, which is the set of all $y \in \mathbb{X}$ such that $y \sim x$ for some $x \in B$. Equivalently, $[B]$ is the union of all $[x]$ for $x \in B$.

The family of equivalence classes is called the *quotient space* (or the factor space). The quotient space is denoted by $\tilde{\mathbb{X}}$ and its generic element by $\tilde{x}$. The *quotient map* $q : \mathbb{X} \to \tilde{\mathbb{X}}$ associates with each $x \in \mathbb{X}$ the corresponding equivalence class. It should be noted that q is surjective and its inverse $q^{-1}\tilde{x} = [x]$ is a set-valued map. Furthermore, $qq^{-1}\tilde{x} = \tilde{x}$ but $q^{-1}qx = [x]$. The space $\tilde{\mathbb{X}}$ is endowed with the finest topology under which q is continuous; this topology is called the quotient (or factor) topology. The Borel σ-algebra induced by the quotient topology is denoted by $\mathcal{B}(\tilde{\mathbb{X}})$. Since $\mathbb{X}$ is sequential, its topological properties can be described in terms of sequences. It is known that the quotient space of a sequential space is also sequential, as seen in [21, Ex. 2.4.G].

Remark 3.8.1. The map q being a quotient map is equivalent to the following property: $q^{-1}U$ is open in $\mathbb{X}$ if and only if U is open in $\tilde{\mathbb{X}}$, see [21, Proposition 2.4.3]. This implies that $\tilde{x}_n \to \tilde{x}$ in $\tilde{\mathbb{X}}$ if and only if $q^{-1}\tilde{x} \cap G \neq \varnothing$ for an open G in $\mathbb{X}$ implies that $q^{-1}\tilde{x}_n \cap G \neq \varnothing$ for all sufficiently large n.

The topological space $\tilde{\mathbb{X}}$ is Hausdorff if the quotient map is open (equivalently, $[G]$ is open for each open G in $\mathbb{X}$) and the set $\{(x,y) : x \sim y\}$ is closed in $\mathbb{X} \times \mathbb{X}$, see [14, Proposition I.8.3.8]. In this case, each equivalence class $[x]$ is closed, being the inverse image of a closed singleton under continuous q.

Definition 3.8.2. A $\mathcal{B}(\tilde{\mathbb{X}})/\mathcal{B}(\mathbb{X})$-measurable map $\tilde{q} : \tilde{\mathbb{X}} \to \mathbb{X}$ is called a *selection map* if $\tilde{q}qx \in [x]$ for every $x \in \mathbb{X}$.

Note that $\tilde{q}$ is necessarily injective, since it selects an element from different (and disjoint) classes of equivalences. Moreover,

$$q\tilde{q}\tilde{x} = \tilde{x}, \tag{3.19}$$

but the reverse composition $\tilde{q}q$ is not necessarily identity since $\tilde{q}qx = \tilde{q}qy$ if $x \sim y$.

Lemma 3.8.3. *If q is an open map and $\{(x,y) : x \sim y\}$ is closed in $\mathbb{X} \times \mathbb{X}$, then a selection map always exists.*

Proof. A selection map is indeed a selection of the set-valued map $\tilde{x} \mapsto q^{-1}\tilde{x}$. This set-valued map has closed values. The set-valued map q^{-1} is Effros measurable (see [44, Definition 1.3.1]), since

$$\{\tilde{x} : q^{-1}\tilde{x} \cap G \neq \varnothing\} = qG.$$

The set qG is open and so Borel in $\tilde{\mathbb{X}}$. The existence of a measurable selection follows from the Kuratowski–Ryll-Nardzewski theorem, see [44, Theorem 1.4.1]. For this, it is essential that $\mathbb{X}$ is Polish. □

In the following we usually require the existence of a *continuous* selection map.

Lemma 3.8.4. *The selection map $\tilde{q}$ is continuous if and only if the saturation $[G \cap Q]$ is open for all open sets G in $\mathbb{X}$, where $Q = \{\tilde{q}\tilde{x} : \tilde{x} \in \tilde{\mathbb{X}}\}$ is the image of $\tilde{q}$.*

Proof. For open G in $\mathbb{X}$, we have $\tilde{q}^{-1}G = \tilde{q}^{-1}(G \cap Q)$. The right-hand side is open in $\tilde{\mathbb{X}}$ if and only if $q^{-1}\tilde{q}^{-1}(G \cap Q) = [G \cap Q]$ is open in $\mathbb{X}$. □

While the quotient space of a metrisable and even Polish space is not necessarily metrisable (see [1]), the existence of a continuous selection map ensures the metrisability of $\tilde{\mathbb{X}}$ as shown below.

Theorem 3.8.5. *Let $\mathbb{X}$ be a separable metric space completely metrisable by a metric d. If there exists a continuous selection map $\tilde{q}$, then $\tilde{\mathbb{X}}$ with its quotient topology is a separable metric space completely metrisable by*

$$\tilde{\mathsf{d}}(\tilde{x}, \tilde{y}) = \mathsf{d}(\tilde{q}\tilde{x}, \tilde{q}\tilde{y}). \tag{3.20}$$

Proof. Let $\tilde{x}, \tilde{y}$ and $\tilde{z}$ be arbitrary elements from $\tilde{\mathbb{X}}$. First, $\tilde{\mathsf{d}}(\tilde{x}, \tilde{y}) = \mathsf{d}(\tilde{q}\tilde{x}, \tilde{q}\tilde{x}) \geq 0$. Moreover, if $\tilde{\mathsf{d}}(\tilde{x}, \tilde{x}) = 0$, then $\tilde{q}\tilde{x} = \tilde{q}\tilde{y}$, which implies $\tilde{x} = \tilde{y}$ in view of (3.19). Secondly, $\tilde{\mathsf{d}}(\tilde{x}, \tilde{y})$ is obviously symmetric and the triangle inequality follows from

$$\tilde{\mathsf{d}}(\tilde{x}, \tilde{y}) \leq \mathsf{d}(\tilde{q}\tilde{x}, \tilde{q}\tilde{z}) + \mathsf{d}(\tilde{q}\tilde{z}, \tilde{q}\tilde{y}) = \tilde{\mathsf{d}}(\tilde{x}, \tilde{z}) + \tilde{\mathsf{d}}(\tilde{z}, \tilde{y}).$$

Thus, $\tilde{\mathsf{d}}$ is a metric on $\tilde{\mathbb{X}}$.

We now prove that the topology induced by $\tilde{\mathsf{d}}$ coincides with the quotient topology on $\tilde{\mathbb{X}}$. Since every metric space is first countable and hence sequential, it suffices to check convergence of sequences. Assume that $\tilde{\mathsf{d}}(\tilde{x}_n, \tilde{x}) \to 0$ as $n \to \infty$. Then $\mathsf{d}(\tilde{q}\tilde{x}_n, \tilde{q}\tilde{x}) \to 0$, so that $\tilde{q}\tilde{x}_n \to \tilde{q}\tilde{x}$ in $\mathbb{X}$. Since q is continuous in the quotient topology, and using (3.19), we conclude that $\tilde{x}_n \to \tilde{x}$ in the quotient topology on $\tilde{\mathbb{X}}$.

Now assume $\tilde{x}_n \to \tilde{x}$ in the quotient topology. Since $\tilde{q}$ is continuous, $\tilde{q}\tilde{x}_n \to \tilde{q}\tilde{x}$ as $n \to \infty$ in $\mathbb{X}$. Hence $\tilde{\mathsf{d}}(\tilde{x}_n, \tilde{y}) = \mathsf{d}(\tilde{q}\tilde{x}_n, \tilde{q}\tilde{x}) \to 0$ as $n \to \infty$. This shows that the topology induced by $\tilde{\mathsf{d}}$ coincides with the quotient topology on $\tilde{\mathbb{X}}$.

Since $\mathbb{X}$ is separable, there exists a countable set D that is dense in $\mathbb{X}$. Define $qD = \{qx : x \in D\}$, and let $\tilde{U}$ be an arbitrary open subset of $\tilde{\mathbb{X}}$. Since q is continuous, $q^{-1}\tilde{U}$ is open in $\mathbb{X}$, so there exists some $x \in D \cap q^{-1}\tilde{U}$. Therefore, $qx \in qD \cap \tilde{U}$, which implies that the intersection of $\tilde{U}$ and qD is not empty, meaning that qD is a countable dense set in $\tilde{\mathbb{X}}$.

The completeness property of $\tilde{\mathbb{X}}$ follows from the completeness of $\mathbb{X}$. Take a Cauchy sequence $(\tilde{x}_n)$ in $\tilde{\mathbb{X}}$. Then the sequence $(\tilde{q}\tilde{x}_n)$ is a Cauchy sequence in $\mathbb{X}$, hence it converges to some $x \in \mathbb{X}$. The continuity of q yields that $\tilde{x}_n = q\tilde{q}\tilde{x}_n \to qx \in \tilde{\mathbb{X}}$ as $n \to \infty$. □

From now on assume that there exists a continuous selection map $\tilde{q}$. Assume that the scaling T_t on $\mathbb{X}$ and the selection map $\tilde{q}$ satisfy the following condition:

(S) $T_t\tilde{q}qx = \tilde{q}qT_tx$ for every $x \in \mathbb{X}$ and $t > 0$.

This condition implies that $\tilde{q}q : \mathbb{X} \to \mathbb{X}$ is a morphism.

Lemma 3.8.6. *Condition (S) implies the following:*

(S′) if $x \sim y$, then $T_tx \sim T_ty$ for all $t > 0$.

Proof. For $t > 0$,

$$\tilde{q}qT_tx = T_t\tilde{q}qx = T_t\tilde{q}qy = \tilde{q}qT_ty.$$

From injectivity of $\tilde{q}$ we conclude that $qT_tx = qT_ty$ and that $T_tx \sim T_ty$. □

If (S) holds, we equip $\tilde{\mathbb{X}}$ with the scaling given by

$$\tilde{T}_t\tilde{x} = qT_t\tilde{q}\tilde{x}. \tag{3.21}$$

For $s, t > 0$,

$$\tilde{T}_s\tilde{T}_t\tilde{x} = \tilde{T}_sqT_t\tilde{q}\tilde{x} = qT_s\tilde{q}qT_t\tilde{q}\tilde{x} = qT_sT_t\tilde{q}\tilde{x} = \tilde{T}_{st}\tilde{x}.$$

Furthermore,

$$\tilde{T}_1\tilde{x} = qT_1\tilde{q}\tilde{x} = q\tilde{q}\tilde{x} = \tilde{x},$$

so (3.21) indeed defines a scaling, which is continuous as it is a composition of continuous functions. Elements that are scaling invariant in $\tilde{\mathbb{X}}$ for some $t > 0$, $t \neq 1$ build the set of zero elements denoted by $\tilde{\mathbf{0}}$. The family of all nonzero elements from $\tilde{\mathbb{X}}$ is denoted by $\tilde{\mathbb{X}}'$.

Lemma 3.8.7. *Assume that (S) holds. Then the set of zero elements $\tilde{\mathbf{0}}$ in $\tilde{\mathbb{X}}$ consists of qx for $x \in \mathbf{0}$, i.e. $\tilde{\mathbf{0}} = q\mathbf{0}$.*

Proof. By Lemma 3.1.2, $q\mathbf{0} \subseteq \tilde{\mathbf{0}}$. Let $\tilde{x} = qx$ for $x \in \mathbb{X}$. Assume that $\tilde{x}$ is scaling invariant for some $t \neq 1$. Then

$$qx = \tilde{x} = \tilde{T}_t\tilde{x} = qT_t\tilde{q}qx = q\tilde{q}qT_tx = qT_tx.$$

By applying $\tilde{q}$, we have $\tilde{q}qT_tx = \tilde{q}qx$. By (S), the left-hand side equals $T_t\tilde{q}qx$, so now

$$T_t\tilde{q}qx = \tilde{q}qx.$$

Thus, $\tilde{q}qx$ is scaling invariant under T_t for some $t \neq 1$, that is, $\tilde{q}qx \in \mathbf{0}$. Finally, note that $\tilde{x} = q\tilde{q}qx$ so $\tilde{x}$ is the image of some $y \in \mathbf{0}$ under q. This gives $\tilde{\mathbf{0}} \subseteq q\mathbf{0}$. □

Below we introduce a natural bornology on $\tilde{\mathbb{X}}'$. Let $\tilde{\mathcal{S}}$ be the family of sets $\tilde{B} \subset \tilde{\mathbb{X}}'$ such that

$$q^{-1}\tilde{B} = \{x \in \mathbb{X} : qx \in \tilde{B}\} \in \mathcal{S},$$

where $\mathcal{S}$ is the bornology on $\mathbb{X}'$ generated by a continuous modulus τ. The bornology $\tilde{\mathcal{S}}$ is the pushforward of $\mathcal{S}$ under q, see Proposition 3.1.9. The following result relies on a special choice of the selection map that relies on selecting an element from $q^{-1}\tilde{x}$ with the smallest modulus which realises the infimum in $\hat{\tau}(q^{-1}\tilde{x})$. In other words, $\tilde{q}qx$ is a point $y \sim x$ with the smallest modulus.

Lemma 3.8.8. *Assume that $\mathbb{X}'$ is equipped with a bornology generated by a continuous modulus τ and that there exists a continuous selection map $\tilde{q}$ such that (S) holds and*

$$\hat{\tau}(q^{-1}\tilde{x}) = \tau(\tilde{q}\tilde{x}), \quad \tilde{x} \in \tilde{\mathbb{X}}'. \tag{3.22}$$

Then

$$\tilde{\tau}(\tilde{x}) = \tau(\tilde{q}\tilde{x}) \tag{3.23}$$

is a continuous modulus on $\tilde{\mathbb{X}}'$, which generates the bornology $\tilde{\mathcal{S}}$.

Proof. Note that $\tilde{q}\tilde{x} \in \mathbb{X}'$ if $\tilde{x} \in \tilde{\mathbb{X}}'$. Indeed, if $\tilde{q}\tilde{x} \in \mathbf{0}$, then applying q to the both sides we have $\tilde{x} \in \tilde{\mathbf{0}}$. By construction, $\tilde{\tau}$ generates $\tilde{\mathcal{S}}$. The map $\tilde{\tau}$ is homogeneous, since

$$\tilde{\tau}(\tilde{T}_t\tilde{x}) = \tau(\tilde{q}qT_t\tilde{q}\tilde{x}) = \tau(T_t\tilde{q}q\tilde{q}\tilde{x}) = t\tau(\tilde{q}\tilde{x}).$$

Furthermore, $\tilde{\tau}$ is continuous since it is a composition of continuous maps. It is not possible that $\tilde{\tau}(\tilde{x}) = 0$, for $\tilde{x} \in \tilde{\mathbb{X}}'$ since then $\tau(\tilde{q}\tilde{x}) = 0$ for $\tilde{q}\tilde{x} \in \mathbb{X}'$ which is a contradiction. □

Remark 3.8.9. If $\mathbb{X}'$ is equipped with a modulus that takes constant values on equivalence classes, then (3.22) automatically holds.

Lemma 3.8.10. *Assume that $\mathbf{0}$ is a singleton and that the selection map $\tilde{q}$ satisfies* (3.22). *If the bornology on $\mathbb{X}'$ is generated by complements to open balls (in metric d) centred at $\mathbf{0}$, then the bornology $\tilde{\mathcal{S}}$ on $\tilde{\mathbb{X}}$ is generated by complements to open balls centred at $\tilde{\mathbf{0}}$ in the metric $\tilde{\mathsf{d}}$.*

Proof. By Lemma 3.8.7, $\tilde{\mathbf{0}}$ is a singleton in $\tilde{\mathbb{X}}$. We need to show that the complement to the ball $\tilde{B}_r(\tilde{\mathbf{0}})$ in $\tilde{\mathbb{X}}$ of radius $r > 0$ belongs to $\tilde{\mathcal{S}}$. Then

$$q^{-1}\tilde{B}_r(\tilde{\mathbf{0}})^c = \{y : qy \in \tilde{B}_r(\tilde{\mathbf{0}})^c\} = \{y : \mathsf{d}(\mathbf{0}, \tilde{q}qy) \geq r\} = \{y : \tilde{q}qy \in B_r(\mathbf{0})^c\}.$$

By Lemma 3.8.8, the set on the right-hand side belongs to $\tilde{\mathcal{S}}$. Now we show that if $q^{-1}A \in \mathcal{S}$, then A is a subset of $\tilde{B}_r(\tilde{\mathbf{0}})^c$ for some $r > 0$. By Lemma 3.8.8,

$$0 < \inf\{\tau(x) : qx \in A\} = \inf\{\tau(\tilde{q}\tilde{x}) : \tilde{x} \in A\} = \hat{\tau}(\tilde{q}A).$$

Thus, $\tilde{q}A \subseteq B_r(\mathbf{0})^c$ for some $r > 0$, that is, $\mathsf{d}(x, \mathbf{0}) \geq r$ for all $x \in \tilde{q}A$. By (3.20), $\tilde{\mathsf{d}}(\tilde{x}, \tilde{\mathbf{0}}) \geq r$ for all $\tilde{x} \in A$. □

If ξ is a random element in $\mathbb{X}$, then $q\xi$ is a random element in $\tilde{\mathbb{X}}$. If $[\xi]$ contains at least one deterministic element, then $q\xi$ is also deterministic in $\tilde{\mathbb{X}}$. With the equivalence relation, bornology, and scaling defined on $\tilde{\mathbb{X}}$, we can consider regular variation on the quotient space $\tilde{\mathbb{X}}$.

Theorem 3.8.11. *Assume that $\mathbb{X}$ is a Polish space equipped with a continuous scaling and bornology $\mathcal{S}$ generated by a continuous modulus τ. Let $\tilde{\mathbb{X}}$ be a quotient space of $\mathbb{X}$ with a continuous selection map $\tilde{q}$ satisfying (S) and* (3.22)*. Let $\tilde{\mathcal{S}}$ be the bornology on $\tilde{\mathbb{X}}'$ generated by $\tilde{\tau}$.*

(i) If $\xi \in \mathrm{RV}(\mathbb{X}, \mathcal{S}, \mu)$ and its tail measure μ is not concentrated only on $[\mathbf{0}]$*, then $q\xi \in \mathrm{RV}(\tilde{\mathbb{X}}, \tilde{\mathcal{S}}, q\mu)$.*

(ii) If $\tilde{\xi} \in \mathrm{RV}(\tilde{\mathbb{X}}, \tilde{\mathcal{S}}, \tilde{\mu})$, then $\tilde{q}\tilde{\xi} \in \mathrm{RV}(\mathbb{X}, \mathcal{S}_\tau, \tilde{q}\tilde{\mu})$.

Proof. (i) Assume that ξ is regularly varying in $\mathbb{X}$ with a tail measure μ that is not concentrated only on $[\mathbf{0}]$. The quotient map $q : \mathbb{X} \to \tilde{\mathbb{X}}$ is continuous and since

$$\tilde{T}_t qx = qT_t\tilde{q}qx = q\tilde{q}qT_t x = qT_t x$$

for all $x \in \mathbb{X}$, if follows that (3.1) holds for q, meaning that q is a continuous morphism. By Lemma 3.8.8, it is bornologically consistent.

Let $q\mu$ be the pushforward of μ. For each $t > 0$,

$$(q\mu)(\{\tilde{x} : \tilde{\tau}(\tilde{x}) \geq t\}) = \mu(q^{-1}\{\tilde{x} : \tau(\tilde{q}\tilde{x}) \geq t\}) = \mu([T_t C]).$$

where $C = \{x : \tau(x) \geq 1\}$. Since $[T_t C] = T_t[C]$ increases to $[\mathbf{0}]^c$ as $t \downarrow 0$, we have that $\mu([T_t C]) > 0$ for some t, so that $q\mu$ is nontrivial on $\tilde{\mathcal{S}}$. By Theorem 3.2.1, $q\xi$ is regularly varying in $\tilde{\mathbb{X}}$ with the tail measure $q\mu$.

(ii) Assume that $\tilde{\xi}$ is regularly varying in $\tilde{\mathbb{X}}$. By (S) and (3.21), we have

$$\tilde{q}\tilde{T}_t\tilde{x} = \tilde{q}qT_t\tilde{q}\tilde{x} = T_t\tilde{q}q\tilde{q}\tilde{x} = T_t\tilde{q}\tilde{x},$$

which implies that $\tilde{q}$ satisfies (3.1). Denoting by Q the image of $\tilde{q}$, we have

$$\tilde{q}^{-1}(\{x : \tau(x) \geq t\}) = \tilde{q}^{-1}(\{x \in Q : \tau(x) \geq t\}) = \tilde{q}^{-1}(\{\tilde{q}\tilde{x} : \tau(\tilde{q}\tilde{x}) \geq t\}) = \{\tilde{x} : \tau(\tilde{q}\tilde{x}) \geq t\}$$

for all $t > 0$, so that $\tilde{q}$ is continuous and bornologically consistent morphism. Finally, $\tilde{q}\tilde{\mu}$ is not supported entirely by $\mathbf{0}$, since

$$\begin{aligned}\tilde{q}\tilde{\mu}(\{x : \tau(x) \geq t\}) &= \tilde{\mu}(\{\tilde{q}^{-1}x : x \in Q, \tau(x) \geq t\}) \\ &= \tilde{\mu}(\{\tilde{q}^{-1}\tilde{q}\tilde{x} : \tau(\tilde{q}\tilde{x}) \geq t\}) \\ &= \tilde{\mu}(\{\tilde{x} : \tilde{\tau}(\tilde{x}) \geq t\}, \quad t > 0.\end{aligned}$$

By Theorem 3.2.1, $\tilde{q}\tilde{\xi}$ is regularly varying in $\mathbb{X}$. □

Now consider the regular variation property of the equivalence classes as random closed sets in $\mathbb{X}$. For this, we do not need to equip $\tilde{\mathbb{X}}$ with a scaling, so it is no longer necessary to use a selection map. The map $x \mapsto [x] = \Psi(x)$, which associates to $x \in \mathbb{X}$ its equivalence class falls under the scope of Theorem 3.6.11. Namely, $\Psi(x) = q^{-1}qx$ becomes the map $\Psi(x)$ from (3.12) if ψ is the factor map q. Note that $\Psi(x)$ is a closed set, since q is continuous. If, additionally to continuity, we assume that q is an open map, then Lemma 3.6.9 yields that $\Psi(x)$ is continuous as a map from $\mathbb{X}$ to $\mathcal{F}(\mathbb{X})$ with the Fell topology if the space $\mathbb{X}$ is locally compact.

If ξ is regularly varying, $[\xi]$ can contain more than one regularly varying selection, but does not contain deterministic elements, whereas if ξ is regularly varying, $q\xi$ could turn out to be deterministic.

Note that we do not assume the existence of a continuous selection map. Instead, assume that condition (S′) holds, so that Ψ satisfies (3.13). By Lemma 3.6.10, $\Psi(x) = [x]$ is either in $\mathbf{0}_{\mathcal{F}}$ or has empty intersection with $\mathbf{0}$. In the latter case $[x]$ belongs to the family $\mathcal{F}^{\mathbf{0}}(\mathbb{X})$. The next result follows from Theorem 3.6.11.

Corollary 3.8.12. *Assume that $\mathbb{X}$ is a locally compact Polish space equipped with a continuous scaling and a bornology $\mathcal{S}$ which satisfies (B). Let $\xi \in \mathrm{RV}(\mathbb{X}, \mathcal{S}_\tau, \mu)$. If condition (S′) holds and μ is nontrivial on the complement of $[\mathbf{0}]$, then $[\xi]$ is a regularly varying random closed set in $\mathcal{F}^{\mathbf{0}}(\mathbb{X})$ with the bornology generated by the modulus $\hat{\tau}$.*

Under conditions of Theorem 3.8.11, if $\tilde{\xi}$ is regularly varying in $\tilde{\mathbb{X}}$, then $\tilde{q}\tilde{\xi}$ is regularly varying in $\mathbb{X}$, so that $q^{-1}\tilde{\xi} = [\tilde{q}\tilde{\xi}]$ is a regularly varying random closed set in $\mathcal{F}^{\mathbf{0}}(\mathbb{X})$.

Remark 3.8.13. In many examples, the equivalence relation is generated by a continuous function $\psi : \mathbb{X} \mapsto \mathbb{Y}$, where $\mathbb{Y}$ is a Hausdorff space. Then the image $\psi(\mathbb{X})$ can be considered as the quotient space $\tilde{\mathbb{X}}$ with the quotient map $q = \psi$ and the corresponding quotient topology.

3.9 Examples: quotient maps

Example 3.9.1. Let $\mathfrak{G}$ be a topological group that acts continuously on a Polish space $\mathbb{X}$ such that $x_n \to x$ and $y_n = g_n x_n \to y$ imply that g_n converges. For instance, this is the case if

$\mathfrak{G}$ is compact. Assume that $x \sim y$ if $y = gx$ for some $g \in \mathfrak{G}$, so that $[x] = \{gx : g \in \mathfrak{G}\}$ is the orbit of $x \in \mathbb{X}$. The quotient map q is open if $q^{-1}qG = [G]$ is open in $\mathbb{X}$ for all open $G \subseteq \mathbb{X}$, meaning that the saturation of each open set is open. This is indeed the case, since $[G] = \cup_{g\in\mathfrak{G}} gG$ and gG is open due to the continuity of g, and the fact that gx coincides with the inverse image of x under the mapping $x \mapsto g^{-1}x$. Furthermore, $\{(x, y) : x \sim y\}$ is closed in $\mathbb{X}^2$, since $x_n \to x$ and $y_n \to y$ for $x_n \sim y_n$ imply that $y_n = g_n x_n$, so that $x \sim y$ by the imposed condition on the group. By Lemma 3.8.3, a measurable selection map exists. If there exists a continuous selection map that satisfies (3.22), then Theorem 3.8.11 is applicable.

Example 3.9.2. Let $\mathbb{X}$ be a Polish space equipped with continuous scaling and assume that $\mathbf{0}_\mathbb{X}$ is empty. Consider the bornology generated by a continuous modulus τ. For $x, y \in \mathbb{X}$, let $x \sim y$ if $\tau(x) = \tau(y)$. The quotient space can be identified with $\mathbb{R}_+$. A selection map can be defined by fixing any $u \in \mathbb{X}'$ and letting $\tilde{q}\tilde{x} = T_{\tilde{x}}u$ for $\tilde{x} > 0$ and $\tilde{q}0 = 0_\mathbb{X}$. The image of $\tilde{q}$ is the set $Q = T_{(0,\infty)}u$. It is easy to see that $[G \cap Q]$ is open for each open G in $\mathbb{X}$. Indeed, if $x \in [G \cap Q]$, then $x \sim y$ with $y \in G \cap Q$. Hence, $T_{(1-\varepsilon,1+\varepsilon)}y \subseteq G \cap Q$ and so the open set $\{z \in \mathbb{X} : \tau(z)/\tau(y) \in (1-\varepsilon, 1+\varepsilon)\}$ is an open neighbourhood of x, by continuity of τ, which is also a subset of $[G \cap Q]$. Thus, $\tilde{q}$ is continuous by Lemma 3.8.4. Furthermore, it is clear that (3.22) holds in this case, since all $y \in q^{-1}\tilde{x}$ share the same modulus. If ξ is regularly varying in $\mathbb{X}$, then $q\xi = \tau(\xi)$ is regularly varying in $\tilde{\mathbb{X}}$. Conversely, if $\tilde{\xi}$ is regularly varying in $\tilde{\mathbb{X}}$, then $\tilde{q}\tilde{\xi}$ is regularly varying in $\mathbb{X}$ and $q^{-1}\tilde{\xi} = [\tilde{q}\tilde{\xi}]$ is a regularly varying random closed set.

Example 3.9.3. Consider the equivalence relation on a Polish $\mathbb{X}'$ equipped with a continuous modulus τ by letting $x \sim y$ if $T_{\tau(x)^{-1}}x = T_{\tau(y)^{-1}}y$ for $x, y \in \mathbb{X}'$. The latter is the case if $\rho(x) = (u, t)$ and $\rho(y) = (v, s)$ are polar decompositions of x and y with $u = v$. The quotient space is identified as $\mathbb{S} = \{x : \tau(x) = 1\}$ with $qx = T_{\tau(x)^{-1}}x$. A continuous selection map is the identity $\tilde{q}\tilde{x} = \tilde{x}$ for $\tilde{x} \in \mathbb{S}$. In this case, the condition (S) does not hold. Moreover, all equivalence classes $[x]$ are cones and so are invariant under scaling. Therefore, it is not possible to define regularly varying elements on this quotient space.

Example 3.9.4. Let $\mathbb{X} = \mathbb{R}^2$ with the Euclidean metric d and the bornology generated by the modulus given by the norm. Consider the linear scaling

$$T_t(x_1, x_2) = (tx_1, tx_2), \quad x = (x_1, x_2) \in \mathbb{X},\ t > 0.$$

Then $\mathbf{0} = \{(0, 0)\}$. For $x = (x_1, x_2)$ and $y = (y_1, y_2)$ from $\mathbb{X}$, let $x \sim y$ if $x_1 = y_1$. Then

$$[(x_1, x_2)] = \{(x_1, t) : t \in \mathbb{R}\}.$$

The quotient space is identified as $\tilde{\mathbb{X}} = \mathbb{R}$ with the quotient map given by $q(x_1, x_2) = x_1$. Define a selection map $\tilde{q}$ by

$$\tilde{q}\tilde{x} = (\tilde{x}, 0), \quad \tilde{x} \in \mathbb{R}.$$

This map is continuous by Lemma 3.8.4, and both (S) and (3.22) hold. For the latter, it is essential that the second component of the selection map is set to zero. By Theorem 3.8.5,

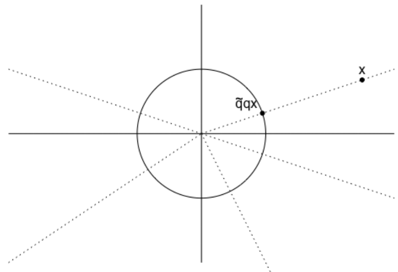


Figure 3.1: *Classes of equivalence from Example 3.9.3 assuming $\mathbb{X} = \mathbb{R}^2$ with the usual scaling and modulus induced by the Euclidean metric.*

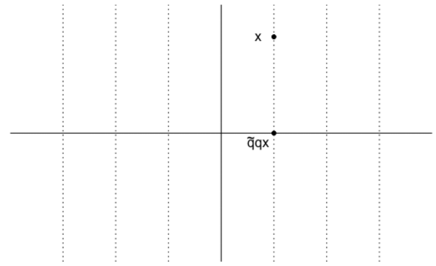


Figure 3.2: *Classes of equivalence from Example 3.9.4 which are parallel vertical lines and the selection is given as the intersection with x-axis.*

$\tilde{\mathbb{X}}$ is a Polish space with the metric $\tilde{\mathsf{d}}(\tilde{x}, \tilde{y}) = |\tilde{x} - \tilde{y}|$. The scaling $\tilde{T}_t$ on $\tilde{\mathbb{X}}$ becomes $\tilde{T}_t\tilde{x} = q(t\tilde{x}, 0) = t\tilde{x}$ for $\tilde{x} \in \mathbb{R}$, and $\tilde{\mathbf{0}} = \{0\}$. By Lemma 3.8.8, the bornology $\tilde{\mathcal{S}}$ on $\tilde{\mathbb{X}}$ is generated by the modulus $\tilde{\tau}(\tilde{x}) = |\tilde{x}|$. Theorem 3.8.11 implies that if a random element $\tilde{\xi}$ is regularly varying in $\tilde{\mathbb{X}}$, then $\tilde{q}\tilde{\xi} = (\tilde{\xi}, 0)$ is regularly varying in $\mathbb{X}$. For the reverse implication, let $\xi = (\xi_1, \xi_2)$ be regularly varying in $\mathbb{X}$. By Theorem 3.8.11, $q\xi = \xi_1$ is regularly varying in $\tilde{\mathbb{X}} = \mathbb{R}$ if the spectral measure of ξ has a mass outside $q^{-1}0 = \{(0, x_2) : x_2 \in \mathbb{R}\}$. For example, the random vector $\xi = (\xi_1, \xi_2)$ is regularly varying in $\mathbb{X}$ if ξ_1 has a light tail (e.g., is normal) and ξ_2 has a heavy tail (e.g., is Pareto). However, in this case, $q\xi = \xi_1$ is not regularly varying in $\tilde{\mathbb{X}}$.

Now consider a random vector $\xi = (\xi_1, \xi_2)$, where ξ_1 and ξ_2 are independent random variables such that ξ_1 is regularly varying and $\xi_2 = e^{\eta}$ for Pareto-1 random variable η, so that ξ_2 has a super-heavy tail. Then the vector ξ is not regularly varying but $\xi \sim \eta = (\xi_1, 1)$ is regularly varying. Thus, the equivalence class $[\xi]$ admits both regularly varying and not regularly varying selections.

Example 3.9.5. Let $\mathbb{X} = \mathbb{R}^3$ with the Euclidean metric d. Define the scaling by $T_t x = (tx_1, x_2, x_3)$ for $x = (x_1, x_2, x_3) \in \mathbb{X}$. Then the zero $\mathbf{0} = \{0\} \times \mathbb{R}^2$ is not a singleton. Define the modulus $\tau(x) = |x_1|$ and the corresponding bornology $\mathcal{S}$. Let $x \sim y$ if $x_1 = y_1$ for $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ from $\mathbb{X}$. The quotient space is then $\mathbb{R}$ and a continuous selection map given by $\tilde{q}\tilde{x} = (\tilde{x}, 0, 0)$ satisfies (S) and (3.22). The scaling on $\tilde{\mathbb{X}}$ is defined by $\tilde{T}_t\tilde{x} = t\tilde{x}$. Then the image $q\xi$ of a regularly varying element ξ in $\mathbb{X}$ is regularly varying in $\tilde{\mathbb{X}}$ if the tail measure of ξ is not exclusively supported by $[\mathbf{0}]$.

Now redefine equivalence so that $x \sim y$ if $x_2 = y_2$ for $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ from $\mathbb{X}$. Then again we can identify $\tilde{\mathbb{X}} = \mathbb{R}$. However, now $\tilde{\mathbf{0}} = q\mathbf{0} = \{(0, 0, z) : z \in \mathbb{R}\}$, so $\tilde{\mathbb{X}}'$ consists of a single point $\tilde{\mathbf{0}}$ and we cannot talk about regular variation on $\tilde{\mathbb{X}}$. For any continuous selection map $\tilde{q}\tilde{x} = (0, \tilde{x}, y)$, $y \in \mathbb{R}$, that satisfies condition (S), the condition (3.22) is not satisfied since $q^{-1}\tilde{x} = \mathbb{R} \times \{\tilde{x}\} \times \mathbb{R}$ so $\hat{\tau}(q^{-1}\tilde{x}) = -\infty$.

Example 3.9.6. Let $\mathbb{X} = \mathsf{C}[0,1]$ be the family of all continuous functions $x : [0,1] \to \mathbb{R}$ with the linear scaling $T_t x = tx$, being the usual multiplication of function's values by $t > 0$, so that $\mathbf{0} = \{0\}$ is the function identically equal 0. Endow $\mathbb{X}$ with the bornology $\mathcal{S}$ generated by the modulus $\tau(x) = \sup\{|x(u)| : u \in [0,1]\} = \|x\|_\infty$.

For $x, y \in \mathbb{X}$, let $x \sim y$ if $x - y$ is a constant function. Then $[x] = \{x + c : c \in \mathbb{R}\}$. If $\tilde{x} = qx$, let

$$\tilde{q}\tilde{x} = x - \frac{1}{2}\big(\inf_{u\in[0,1]} x(u) + \sup_{u\in[0,1]} x(u)\big).$$

Note that the choice of $x \in q^{-1}\tilde{x}$ does not influence $\tilde{q}\tilde{x}$. The scaling T_t and the selection map $\tilde{q}$ satisfy (S), so we define scaling $\tilde{T}_t$ on $\tilde{\mathbb{X}}$ by (3.21). The zero element $\tilde{\mathbf{0}}$ in $\tilde{\mathbb{X}}$ is the equivalence class of a constant function on $[0,1]$. The choice of the selection map ensures that

$$\hat{\tau}([x]) = \inf\{\tau(x + a) : a \in \mathbb{R}\},$$

so that (3.22) holds and

$$\tilde{\tau}(\tilde{x}) = \tau(\tilde{q}\tilde{x})$$

is a modulus on $\tilde{\mathbb{X}}'$, which generates the bornology $\tilde{\mathcal{S}}$, see Lemma 3.8.8. Theorem 3.8.11 states that if ξ is regularly varying in $C[0,1]$ with tail measure μ and $q\mu$ is not supported only on $[\mathbf{0}]$ (that is, on constant functions), then $q\xi$ is regularly varying in $\tilde{\mathbb{X}}$. A trivial example is provided by the function $\xi(u) = \eta$, where η is a regularly varying random variable. This function is regularly varying, but $q\xi$ is not regularly varying since $\xi \sim 0_{\mathbb{X}}$.

A more complicated equivalence relation arises by letting $x \sim y$ if $x - y \in P_k$, where P_k is the family of polynomials of degree at most k for a fixed $k \in \mathbb{N}$. Then

$$\hat{\tau}([x]) = \inf\{\tau(x + p) : p \in P_k\}.$$

It is easy to see that the infimum is attained at a unique $-p = p_x$ (which is actually the best approximating polynomial of order at most k, see, e.g., [33]), so that the selection map becomes $\tilde{q}qx = x - p_x$.

Example 3.9.7. Let $\mathbb{X}$ be the space of continuous cumulative distribution functions (c.d.f.) of random variables from $L^\infty(\mathbb{R})$, that is, the space of continuous increasing functions $x : \mathbb{R} \to [0,1]$ such that $x(-u) = 0$ and $x(u) = 1$ for all sufficiently large u. Endow $\mathbb{X}$ with the uniform metric (which is the Kolmogorov distance between random variables), the scaling $(T_t x)(u) = x(t^{-1}u)$, and the modulus $\tau(x) = \operatorname{esssup}|\xi|$, where ξ is a random variable with c.d.f. x. If ξ is non-negative, then the modulus represents the right endpoint of x, which is finite due to conditions on $\mathbb{X}$. The only scaling invariant element is the c.d.f. of the random variable which a.s. equals zero, but since it is not continuous, there are no zero elements in $\mathbb{X}$. For $x, y \in \mathbb{X}$, let $x \sim y$ if $x(u + c) = y(u)$ for all $u \in \mathbb{R}$ and some $c \in \mathbb{R}$, meaning that the corresponding random variables differ by an additive deterministic constant. A continuous selection map is given by

$$\tilde{q}qx(u) = x\left(u + \frac{\mathrm{essinf}(\xi) + \mathrm{esssup}(\xi)}{2}\right), \quad x \in \mathbb{X}.$$

This selection map selects an element y from the class such that $|\mathrm{essinf}(\eta)| = |\mathrm{esssup}(\eta)|$, where η has the c.d.f. y. Moreover, for $t > 0, x \in \mathbb{X}$, we have

$$\begin{aligned}
\tilde{q}qT_t x(u) &= (T_t x)\left(u + \frac{\mathrm{essinf}(T_t\xi) + \mathrm{esssup}(T_t\xi)}{2}\right) \\
&= x\left(t^{-1}u + \frac{\mathrm{essinf}(\xi) + \mathrm{esssup}(\xi)}{2}\right) \\
&= T_t\tilde{q}qx(u).
\end{aligned}$$

Hence, $\tilde{q}$ satisfies (S). By the definition of the selection map, it satisfies (3.22). By Theorem 3.8.11, if x is regularly varying in $\mathbb{X}$, then $\tilde{q}qx$ is regularly varying in $\mathbb{X}$ as well, and qx is regularly varying in the quotient space $\tilde{\mathbb{X}}$.

An example of a regularly varying element in $\mathbb{X}$ is the function $x(u) = \frac{u-1}{\xi-1}\mathbf{1}_{(1,\xi)}(u) + \mathbf{1}_{(\xi,\infty)}(u)$, where ξ is a Pareto-1 distributed random variable. In other words, x is the c.d.f. of the uniform random variable on the interval $(1,\xi)$. To see that x is regularly varying, note first that $\tau(x) = \xi$ is a regularly varying random variable. Let $A = \mathbb{S} \cap \{x \in \mathbb{X} : \min(|\mathrm{esssup}(\xi)|, |\mathrm{essinf}(\xi)|) \le c\}$, where $\mathbb{S}$ is the transversal corresponding to τ, and $c > 0$. Then, if we denote by ρ the polar decomposition, we have

$$\mathbf{P}\{\rho(x) \in A \times (ut,\infty)\} = \mathbf{P}\{\xi > ut, 1/\xi \le c\} = \mathbf{P}\{\xi > ut\} = (ut)^{-1},$$

for all $u > 0$ and t large enough. Thus, x is regularly varying by Proposition 3.3.3. By Theorem 3.8.11, the function $\tilde{q}x(u) = \frac{u+\xi/2}{\xi}\mathbf{1}_{(-\xi/2,\xi/2)}(u) + \mathbf{1}_{(\xi/2,\infty)}(u)$, which is the c.d.f. of the uniform random variable on $(-\xi/2, \xi/2)$, is also regularly varying in $\mathbb{X}$.

Example 3.9.8. Let $\mathbb{X} = \mathcal{M}_p(\mathbb{R}^d, \mathcal{S})$ be the space of finite counting measures. Here, $\mathbb{R}^d$ is equipped with linear scaling, and the bornology $\mathcal{S}$ is generated by the norm. The space of counting measures is equipped with the scaling given in Example 1.4.11. Assume that two counting measures ν and ν' are equivalent if there exists $z \in \mathbb{R}^d$ such that $\theta_z\nu = \nu'$, where $\theta_z(\nu)(B) = \nu(B - z)$ for all Borel B. In other words, ν and ν' are equivalent if they are identical up to a translation. This quotient space $\tilde{\mathbb{X}}$ of counting measures is useful when considering convergence of clusters of extremes for time series (see [7]) and marked point processes (see [5]). In these works, measures were translated to move a specified point (called an anchor) to the origin. However, since this anchoring map is not necessarily continuous, we work with an alternative map that associates any $\tilde{\nu} = q\nu$ with the measure $\tilde{q}\tilde{\nu} = \theta_z\nu$ for $z = \bar{\nu} = \nu(\mathbb{R}^d)^{-1}\int x\nu(dx)$, which is the centre of mass for ν. This map is continuous and satisfies condition (S). In order to ensure the validity of (3.22), we endow $\mathbb{X}$ with the modulus given by

$$\tau(\nu) = \|\bar{\nu}\| + \Big(\int \|x - \bar{\nu}\|^2\nu(dx)\Big)^{1/2}.$$

Figure 3.3: *An example of $\xi, \eta \in \mathbb{X}$ from Example 3.9.7 where $\xi \sim \eta$.*

Over all $\nu' \in [\nu]$, the first summand on the right-hand side is minimal for $\tilde{q}q\nu$, while the second summand is translation invariant. Note that $[\mathbf{0}]$ is the family of counting measures with support being a singleton. If ν is a regularly varying random counting measure and its tail measure is not entirely supported by $[\mathbf{0}]$, then $q\nu$ is regularly varying in $\tilde{\mathbb{X}}$ with the bornology generated by the corresponding modulus $\tilde{\tau}$.

Example 3.9.9. Let $\mathbb{X} = \mathcal{K}_c(\mathbb{R}^d)$ be the family of convex compact sets in $\mathbb{R}^d$ equipped with the Hausdorff metric, linear scaling T_t and bornology $\mathcal{S}$ defined in Example 3.7.3. For $K, L \in \mathbb{X}$, let $K \sim L$ if $K + a = L$ for some $a \in \mathbb{R}^n$, meaning that a translation of K coincides with L. Note that $[\mathbf{0}]$ is the family of all singletons. Let $\tilde{\mathbb{X}}$ be the corresponding quotient space, so that $\tilde{\mathbf{0}}$ is the equivalence class of a singleton in $\mathbb{R}^d$. To fulfil (3.22), the selection map $\tilde{q}$ should be such that

$$\tau(\tilde{q}qK) = \inf\{\tau(K + a) : a \in \mathbb{R}^d\}.$$

For instance, the Steiner point defined at (3.16) is continuous and satisfies (S), but does not deliver the infimum in the above equation. Instead, we define $\tilde{q}qK = K - c(K)$, where $c(K)$ is the centre of the smallest Euclidean ball circumscribed around K. This ball is unique by [51, Lemma 3.1.5]. Then (3.22) holds, and a bornology on the quotient space can be defined using the modulus $\tilde{\tau}(qK)$, which is the radius of the smallest circumscribed ball around K.

By Theorem 3.8.11, if a random set X is regularly varying in $\mathcal{K}_c(\mathbb{R}^d)$ with the tail measure not exclusively supported by singletons, then the equivalence class is a regularly varying element in $\tilde{\mathbb{X}}$. If $\tilde{X}$ is regularly varying in $\tilde{\mathbb{X}}$, then $\tilde{q}\tilde{X}$ is regularly varying in $\mathcal{K}_c(\mathbb{R}^d)$.

For instance, let $X = B_\eta(\xi)$ be the closed ball in $\mathbb{R}^d$ of radius η centred at ξ. The regular variation property of $\tilde{X} = qX$ depends only on the distribution of η if η does not have a strictly lighter tail than ξ. Namely, $\tilde{X}$ is regularly varying if and only if η is a regularly varying random variable, which is equivalent to $\tilde{q}\tilde{X} = B_\eta(0)$ being regularly varying. If the tail of ξ is heavier than that of η, then X is regularly varying. However, the pushforward of the spectral measure is concentrated on singletons, so that qX is not regularly varying in the quotient space.

Chapter 4

On compound maxima and point processes

When collecting data in various fields, such as non-life insurance, hydrology, climatology (see [42], [57], and [58]), it is common to observe clustering, where several observations come in the same observed time interval or geographical area. This type of behaviour can often be seen in the plot of data, where clusters of data are more apparent. In this section, we focus on the limit behaviour of the maximum of one cluster that has randomly many observations from the maximum domain of attraction of some extreme value distribution. This topic has been studied in [34], where the main assumption is independence between the observations and the number of observations. Our goal is to examine the scenario where this assumption does not hold. Similar problem has already been treated in [59] in the special case of independence. In additional to generalising results therein, we also provide different mathematical arguments for an analogous results. The results presented in this chapter are based on a paper [4], which was co-authored with B. Basrak and P. Žugec.

In Section 3.5, we discussed the relationship between regular variation and point processes. We examined a point process first assuming the number of observations to be deterministic and then allowing it to be random but independent of the observations. In this chapter, we continue our analysis of this type by considering more general case where the number of observations is a stopping time with respect to the natural filtration induced by the observations. Our main focus in this chapter is on studying the maximum of observations in such a point process. We will also apply the obtained results to marked renewal cluster processes, as each cluster can be represented as a marked version of the point process discussed in Section 3.5.

4.1 Previous results and setup

In the upcoming section, we will examine the connection between vague convergence in Polish spaces, point processes and order statistics. Although connection between point processes and extremes, or point processes and order statistics is very well-known (see [8], [38, Chapter 7], [48, Chapter 3]), we present some main ideas and well known results necessary in some of our proofs. We use the terminology and notation introduced in previous chapters.

As discussed in Section 3.4, concept of regular variation, with different choices of scaling and bornology, plays an important role in characterising distributions in all of the maximum domains of attraction.

The Poisson point process, also known as a Poisson random measure, is perhaps the most commonly used random measure, and it will be denoted by $\mathrm{PRM}(\lambda)$ throughout this text. Here, λ is the mean measure of the process defined on the underlying space, and it is assumed to be boundedly finite (see Section 2.1). We denote Lebesgue measure on a subset E of $\mathbb{R}^d$ by Leb and Dirac measures will be denoted by δ_x, for $x \in E$. We often work with a Poisson random measure and other processes from the space $\mathcal{M}_p(\mathbb{R}_+ \times E)$, which contains all point processes on $\mathbb{R}_+ \times E$. Here, $\mathbb{R}_+$ is the domain for the scaled time component and $E = \operatorname{supp} G$ for some extreme value distribution G. It is worth noting that a set is considered bounded in $\mathbb{R}_+ \times E$ if it is contained in $[0, M] \times B$, for some $M > 0$ and some bounded set $B \subseteq E$. One example of such a process is given by

$$N_n = \sum_{i \in \mathbb{N}} \delta_{\left(\frac{i}{n}, \frac{X_i - b_n}{a_n}\right)}, \tag{4.1}$$

where X_i are random variables and a_n and b_n are sequences of real numbers. Some of the previous results describing the relation between maximum domain of attraction and Poisson random measure, as well as the previously defined point process N_n are given in the following theorems. The first is slightly reformulated (see [48, Proposition 3.21]) to fit in within our context. Recall that for an extreme value distribution G, the measure μ_G is given by $\mu_G(x, \infty) = -\log G(x)$, for $x \in E$.

Proposition 4.1.1. *Assume that $(X_i)_{i \in \mathbb{N}}$ are i.i.d. random elements in $(E, \mathcal{B}(E))$. Condition $X_1 \in \mathrm{MDA}(G)$ is equivalent to*

$$N_n \xrightarrow{d} \mathrm{PRM}(\mathrm{Leb} \times \mu_G) \quad \text{as } n \to \infty,$$

in $\mathcal{M}_p([0, \infty) \times E)$.

The fact that X_1 belongs to $\mathrm{MDA}(G)$ is equivalent to a simpler convergence:

$$\sum_{i=1}^{n} \delta_{\frac{X_i - b_n}{a_n}} \xrightarrow{d} \mathrm{PRM}(\mu_G) \quad \text{as } n \to \infty,$$

in $\mathcal{M}_p(E)$. Basrak and Špoljarić in [8] studied restrictions of point processes of a similar kind. Before we present their result (see [8, Lemma 1]) which we use in our further work, note that the restriction of a measure η on a set A, denoted by $\eta|_A$, is defined as $\eta|_A(B) = \eta(A \cap B)$. For this purpose, let E' denote an arbitrary measurable subset of $\mathbb{R}^d$.

Lemma 4.1.2. *Assume that N, $(N_t)_{t \geq 0}$ are point processes with values in $\mathcal{M}_p(\mathbb{R}_+ \times E')$. Assume further that Z, $(Z_t)_{t \geq 0}$ are $\mathbb{R}_+$-valued random variables. If $\mathbf{P}\{N(\{Z\} \times E') > 0\} = 0$ and $(N_t, Z_t) \xrightarrow{d} (N, Z)$, in the product topology as $t \to \infty$, then*

$$N_t|_{[0, Z_t] \times E'} \xrightarrow{d} N|_{[0, Z] \times E'} \quad \text{as } t \to \infty.$$

It is worth noting that the lemma mentioned above does not require the assumption of independence between the point processes N_t and the random variables Z_t. This lemma plays a significant role in our main results, which will also be discussed in the following section. For the sake of simplicity, we will use the notation X, A, etc. to refer to a generic member of an identically distributed sequence or array, such as (X_n) or $(A_{i,j})$.

We begin by analysing the limit of the maximum value within a single cluster. Consider an independent and identically distributed sequence of random variables $(X_i)_{i\in\mathbb{N}}$ drawn from $\mathrm{MDA}(G)$ for some extreme value distribution G, and let K be a positive random integer with finite expectation. Let us define

$$H = \bigvee_{i=1}^{K} X_i. \tag{4.2}$$

While the problem of determining the tail behaviour of sums involving a random number of observations has been studied extensively (see, e.g., [19], [56]), the current problem, concerning the maximal claim amount, is less common but of practical interest in the insurance industry. As such, we will often use insurance-related terminology to interpret our results.

When $K = k \in \mathbb{N}$ is a fixed integer, H belongs to the same maximum domain of attraction $\mathrm{MDA}(G)$ as X, which follows from the standard extreme value theory: let $X \in \mathrm{MDA}(G)$ with the normalising sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, then (1.1) holds. Since

$$\bigvee_{j=1}^{n} H_j = \bigvee_{i=1}^{nk} X_i$$

for i.i.d. copies H_j, $j = 1, 2, \ldots$, of H where H_j is interpreted as the maximum of the j-th cluster, we have that for all $x \in \mathbb{R}$,

$$\mathbf{P}\left\{ \frac{\bigvee_{j=1}^{n} H_j - b_{nk}}{a_{nk}} \le x \right\} \to G(x) \text{ as } n \to \infty.$$

Hence, H belongs to the same maximum domain of attraction $\mathrm{MDA}(G)$ as X, with normalising sequences $(a_{nk})_{n\in\mathbb{N}}$ and $(b_{nk})_{n\in\mathbb{N}}$.

The case of a random K that is independent of the sequence $(X_i)_{i\in\mathbb{N}}$ has already been studied in [34] and [59], where the authors show that H is in the same MDA as X. A similar problem in a multidimensional setting has been investigated in [29], where the authors assume independence as well but consider K that also belongs to some MDA. We extend these results to cases when K is not independent of the sequence $(X_i)_{i\in\mathbb{N}}$, but instead, it may be a stopping time with respect to the natural filtration of the sequence $(X_i)_{i\in\mathbb{N}}$. In such cases, H remains in the same MDA as X, as long as K has finite expectation. Furthermore, we allow K to be dependent on an additional source of randomness represented by W in the sequel.

4.2 Main results on compound maxima

Let $\mathbb{X}$ be a general Polish space. Assume that $((W_i, X_i))_{i\in\mathbb{N}}$ is a sequence of i.i.d. random elements in $\mathbb{X} \times \mathbb{R}$. For the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}} = (\sigma\{(W_i, X_i) : i \le n\})_{n\in\mathbb{N}}$, we further assume that K is a stopping time with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}}$. In this case, the distribution of H can be rather complicated, as we will see in the following.

Example 4.2.1. (a) Assume $(W_i)_{i\in\mathbb{N}}$ is independent of $(X_i)_{i\in\mathbb{N}}$ and integer-valued. When $K = W_1$, H has already been studied, for example, in [34] and [59], as mentioned above, since this is the aforementioned independent case of X and K.

(b) Assume $((W_i, X_i))_{i\in\mathbb{N}}$ is i.i.d. as before (note that some mutual dependence between W_i and X_i is allowed) and $\mathbf{P}\{X > W\} > 0$. Let $K = \inf\{k \in \mathbb{N} : X_k > W_k\}$, and see that for $k \in \mathbb{N}$,

$$\begin{aligned}\mathbf{P}\{K = k\} &= \mathbf{P}\{X_k > W_k,\, X_{k-1} \leq W_{k-1},\, X_{k-2} \leq W_{k-2}, \ldots, X_1 \leq W_1\} = \\ &= \mathbf{P}\{X > W\}\,(1 - \mathbf{P}\{X > W\})^{k-1}.\end{aligned}$$

Thus, K has geometric distribution and finite expectation and we will show that this implies that H is in the same MDA as X.

(c) Assume $(W_i)_{i\in\mathbb{N}}$ and $(X_i)_{i\in\mathbb{N}}$ are two independent i.i.d. sequences. Let $K = \inf\{k \in \mathbb{N} : X_k > W_1\}$. Clearly $H = X_K > W_1$. Therefore, H has a tail at least as heavy as W.

In order to analyse the extremal behaviour of H and prove that H is in the same MDA as X, we consider i.i.d. copies H_i of H and the maximum of n such copies, given by

$$\bigvee_{i=1}^{n} H_i. \tag{4.3}$$

Let $((W_{i,j}, X_{i,j}))_{j\in\mathbb{N}}$, for $i \in \mathbb{N}$, be i.i.d. copies of the sequence $((W_i, X_i))_{i\in\mathbb{N}}$ and define σ-algebras $\mathcal{F}_{i,j} = \sigma\{(W_{i,l}, X_{i,l}) : l \leq j\}$. For $i \in \mathbb{N}$ let K_i be a stopping time with respect to the filtration $(\mathcal{F}_{i,j})_{j\in\mathbb{N}}$, such that that for each $k \in \mathbb{N}$,

$$\{K_i = k\} \in \mathcal{F}_{i,k},$$

and such that $(K_i, (W_{i,j}, X_{i,j})_{j\in\mathbb{N}})_{i\in\mathbb{N}}$ is an i.i.d. sequence with the same distribution as $(K, (W_i, X_i)_{i\in\mathbb{N}})$. It is clear that K_i, $i = 1, 2, \ldots$, are i.i.d. We assume that K has finite expectation, i.e.

$$\mathbb{E}[K] < \infty, \tag{4.4}$$

which implies that $\mathbb{E}[K_i] < \infty$, for every $i \in \mathbb{N}$. Here, we interpret $(X_{i,1}, X_{i,2}, \ldots, X_{iK_i})$, for fixed $i \in \mathbb{N}$, as one of the i.i.d. clusters. Note that every variable from the i-th cluster is independent of each variable from the j-th cluster for $i \neq j$. The aforementioned i.i.d. copies H_i of H are defined as

$$H_i = \bigvee_{j=1}^{K_i} X_{i,j}, \quad i \in \mathbb{N}. \tag{4.5}$$

Clearly,

$$\bigvee_{i=1}^{n} H_i = \bigvee_{i=1}^{n} \bigvee_{j=1}^{K_i} X_{i,j}.$$

Our first step is to prove the following lemma:

Lemma 4.2.2. *Let $K_1, K_2, \ldots,$ be i.i.d. copies of K as described. Then*

$$\bigvee_{i=1}^{n} H_i \stackrel{d}{=} \bigvee_{i=1}^{K_1+K_2+\cdots+K_n} X_i.$$

This equality in distribution implies that even though we need to consider the maximum of each cluster separately, we can instead concatenate all the observations into a single i.i.d. sequence. To see why this is true, we align the observations from all clusters in a sequence: $X_{1,1}, X_{1,2}, \ldots, X_{1,K_1}, X_{2,1}, X_{2,2}, \ldots, X_{2,K_2}, \ldots$. We show that this aligned sequence is also i.i.d., by using two different approaches: one based on the connection between the aligned sequence and the original sequence, and the other based on the strong Markov property.

Proof. First approach: To start, rename the sequence $X_{1,1}, X_{1,2}, \ldots, X_{1,K_1}, X_{2,1}, X_{2,2}, \ldots, X_{2,K_2}, \ldots$ into a sequence $Y_1, Y_2, \ldots, Y_{K_1}, Y_{K_1+1}, Y_{K_1+2}, \ldots, Y_{K_1+K_2}, \ldots$. Note that now the maximum of H_i has a representation:

$$\bigvee_{i=1}^{n} H_i = \bigvee_{i=1}^{K_1+K_2+\cdots+K_n} Y_i. \tag{4.6}$$

Our goal is to show that $(Y_i)_{i\in\mathbb{N}}$ is an i.i.d. sequence. Once we manage to show that, the statement of the lemma easily follows. To do that, define auxiliary sequences of random variables $(P_n)_{n\in\mathbb{N}}$ and $(R_n)_{n\in\mathbb{N}}$ with values in $\mathbb{N}$ by

$$R_n = \inf\left\{k : \sum_{i=1}^{k} K_i \geq n\right\}, \tag{4.7}$$

$$P_n = n - \sum_{i=1}^{R_n-1} K_i, \quad n \in \mathbb{N}. \tag{4.8}$$

They give us the information about the "row" number, that is, the cluster number, or the first index i, and the position number within that cluster, that is, the second index j. In other words, on an event $\{R_n = i, P_n = j\}$, $i, j \in \mathbb{N}$, the variable Y_n corresponds to $X_{i,j}$, and thus, $Y_n = X_{R_n,P_n}$. For an arbitrary natural number $n \geq 2$, and Borel measurable sets

$A \subseteq \mathbb{R}$, $B \subseteq \mathbb{R}^{n-1}$, calculate

$$\begin{aligned}
&\mathbf{P}\{(Y_1, Y_2, \ldots, Y_{n-1}) \in B,\, Y_n \in A\} = \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n-i+1} \mathbf{P}\{(Y_1, Y_2, \ldots, Y_{n-1}) \in B,\, Y_n \in A,\, R_n = i,\, P_n = j\} = \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n-i+1} \mathbf{P}\{(Y_1, Y_2, \ldots, Y_{n-1}) \in B,\, X_{i,j} \in A,\, R_n = i,\, P_n = j\}.
\end{aligned}$$

In this calculation, we used the fact that since Y_n is in the i-th cluster, then there are at least $i-1$ random variables in the previous clusters. Therefore, in the i-th cluster, there are at most $n-i$ random variables positioned before Y_n. We also note that $X_{i,j}$ is independent of Y_k, for $k < n$. Moreover, the set $\{R_n = i,\, P_n = j\}$ can be rewritten as a set

$$\begin{aligned}
\left\{\sum_{l=1}^{i-1} K_l < n,\, \sum_{l=1}^{i} K_l \geq n,\, \sum_{l=1}^{i-1} K_l = n-j\right\} &= \left\{\sum_{l=1}^{i-1} K_l = n-j,\, K_i \geq j\right\} = \\
&= \left\{\sum_{l=1}^{i-1} K_l = n-j\right\} \cap \{K_i \leq j-1\}^c
\end{aligned}$$

The variables $K_1, K_2, \ldots, K_{i-1}$ come from the first $i-1$ clusters and are independent of $X_{i,j}$, since $X_{i,j}$ is a variable from i-th cluster. On the other hand, although the random variable K_i is not independent of observations from i-th cluster, they are independent of all "future" observations. Specifically, $\{K_i \leq j-1\}^c \in \mathcal{F}_{i,j-1}$ and this same event is independent of $\mathcal{F}_{i,j}$. With this said, we can conclude that $\{X_{i,j} \in A\}$ is independent of $\{R_n = i,\, P_n = j\}$. This gives

$$\begin{aligned}
&\mathbf{P}\{(Y_1, Y_2, \ldots, Y_{n-1}) \in B,\, Y_n \in A\} = \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n-i+1} \mathbf{P}\{X \in A\} \mathbf{P}\{(Y_1, Y_2, \ldots, Y_{n-1}) \in B,\, R_n = i,\, P_n = j\} = \\
&= \mathbf{P}\{X \in A\}\, \mathbf{P}\{(Y_1, Y_2, \ldots, Y_{n-1}) \in B\} = \\
&= \mathbf{P}\{Y_n \in A\}\, \mathbf{P}\{(Y_1, Y_2, \ldots, Y_{n-1}) \in B\}
\end{aligned}$$

Setting $B = \mathbb{R}^{n-1}$, we get the last equality $Y_n \overset{d}{=} X$. This implies that $(Y_i)_{i \in \mathbb{N}}$ is an i.i.d. sequence with the same distribution as the original sequence $(X_i)_{i \in \mathbb{N}}$. Therefore, in (4.6), on the right-hand side we have a maximum of i.i.d. random variables with the same distribution as X, so the statement of the lemma holds. $\square$

Second approach: Since the sequence $((W_i, X_i))_{i \in \mathbb{N}}$ is i.i.d., apply the strong Markov property to see that after a stopping time $K_1' = K$, the sequence $((W_{K_1'+i},\, X_{K_1'+i}))_{i \in \mathbb{N}}$ has the same distribution as the original sequence. Therefore, it has its own stopping time K_2' distributed as K_1' and such that $((W_{K_1'+K_2'+i}, X_{K_1'+K_2'+i}))_{i\in\mathbb{N}}$ again has the same distribution. By applying this argument iteratively, we break the original sequence into i.i.d. clusters

$$((W_{T(i-1)+1}, X_{T(i-1)+1}),\ (W_{T(i-1)+2}, X_{T(i-1)+2}),\ \ldots,\ (W_{T(i)}, X_{T(i)}))_{i\in\mathbb{N}},$$
$$\text{where} \quad T(0)=0, \quad T(i)=\sum_{j=1}^{i} K_j' \stackrel{d}{=} \sum_{j=1}^{i} K_j, \quad i\in\mathbb{N}.$$

Clearly,

$$H_i' = \bigvee_{j=T(i-1)+1}^{T(i)} X_j\,, \qquad i\in\mathbb{N},$$

are i.i.d. with the same distribution as the original compound maximum H. Since H_i from (4.5) are also i.i.d. with the same distribution as H, conclude that $H_i' \stackrel{d}{=} H_i$ and

$$\bigvee_{i=1}^{n} H_i \stackrel{d}{=} \bigvee_{i=1}^{n} H_i' \stackrel{d}{=} \bigvee_{i=1}^{K_1+K_2+\cdots+K_n} X_i.$$

□

We show one additional auxiliary result before stating our main theorem in this section.

Lemma 4.2.3. *Assume that* $\xi = \mathbb{E}[K] < \infty$. *Then*

$$\sum_{i=1}^{n}\sum_{j=1}^{K_i} \delta_{\frac{X_{i,j}-b_{\lfloor n\xi\rfloor}}{a_{\lfloor n\xi\rfloor}}} \xrightarrow{d} \mathrm{PRM}(\mu_G) \qquad \text{as } n\to\infty.$$

Proof. As in the proof of Lemma 4.2.2, we denote the random walk induced by the i.i.d. sequence $(K_i)_{i\in\mathbb{N}}$ by $T(n)$. First note that by Lemma 4.2.2,

$$\sum_{i=1}^{n}\sum_{j=1}^{K_i} \delta_{\frac{X_{i,j}-b_{\lfloor n\xi\rfloor}}{a_{\lfloor n\xi\rfloor}}} \stackrel{d}{=} \sum_{i=1}^{T(n)} \delta_{\frac{X_i-b_{\lfloor n\xi\rfloor}}{a_{\lfloor n\xi\rfloor}}}.$$

To use Lemma 4.1.2, let $Z=1$, $(Z_n)_{n\in\mathbb{N}} = (T(n)/(n\xi))_{n\in\mathbb{N}}$ be $\mathbb{R}_+$-valued random variables, denote $N = \mathrm{PRM}(\mathrm{Leb}\times\mu_G)$ and define point processes $(N_n')_{n\in\mathbb{N}}$,

$$N_n' = \sum_{i\in\mathbb{N}} \delta_{\left(\frac{i}{n\xi},\, \frac{X_i-b_{\lfloor n\xi\rfloor}}{a_{\lfloor n\xi\rfloor}}\right)},$$

with values in the space $[0,\infty)\times E$, where E depends on G as before. By the weak law of large numbers and by Proposition 4.1.1, since $X\in\mathrm{MDA}(G)$, we have

$$Z_n \xrightarrow{p} Z = 1 \quad \text{and} \quad N_n' \xrightarrow{d} N \qquad \text{as } n\to\infty.$$

Hence, by the standard Slutsky argument (see [12, Theorem 4.4])

$$(N_n',\, Z_n) \xrightarrow{d} (N,\, Z) \quad \text{as } n \to \infty.$$

Note that $\mathbf{P}\{N(\{Z\} \times E) > 0\} = 0$, so by Lemma 4.1.2,

$$N_n'\Big|_{[0,Z_n]\times E} \xrightarrow{d} N\Big|_{[0,Z]\times E}.$$

Conclude that

$$N_n'\Big|_{\left[0,\frac{T(n)}{n\xi}\right]\times E}([0,\infty)\times\cdot\,) = \sum_{i=1}^{T(n)} \delta_{\frac{X_i - b_{\lfloor n\xi\rfloor}}{a_{\lfloor n\xi\rfloor}}}(\,\cdot\,) \xrightarrow{d} N\Big|_{[0,1]\times E}([0,\infty)\times\cdot\,) \quad \text{as } n \to \infty,$$

where the point process on the right is a $\mathrm{PRM}(\mu_G)$, see [8, Theorem 2] for details. □

We present our main theorem which guarantees that the maximum of a cluster stays in the same MDA as observations, under our given assumptions. Moreover, we present a very short and straightforward proof for which we use Lemma 4.2.3.

Theorem 4.2.4. *Let $(W_i, X_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random elements in $\mathbb{X}\times\mathbb{R}$, where $\mathbb{X}$ is a general Polish space. Assume that K is a stopping time with respect to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}} = (\sigma\{(W_i, X_i) : i \le n\})_{n\in\mathbb{N}}$ and has a finite mean, $\mathbb{E}[K] = \xi < \infty$. If X belongs to $\mathrm{MDA}(G)$, then the same holds for $H = \bigvee_{i=1}^{K} X_i$.*

Proof. For (H_i) i.i.d. copies of H, using Lemma 4.2.3 and the notation therein, as $n \to \infty$ we have

$$\begin{aligned}\mathbf{P}\left\{\frac{\bigvee_{i=1}^{n} H_i - b_{\lfloor n\xi\rfloor}}{a_{\lfloor n\xi\rfloor}} \le x\right\} &= \mathbf{P}\left\{\sum_{i=1}^{n}\sum_{j=1}^{K_i} \delta_{\frac{X_{i,j} - b_{\lfloor n\xi\rfloor}}{a_{\lfloor n\xi\rfloor}}}(x,\infty) = 0\right\} \\ &\to \mathbf{P}\{\mathrm{PRM}(\mu_G)(x,\infty) = 0\} = G(x).\end{aligned}$$

□

Example 4.2.5 (Example 4.2.1 continued). Provided $\mathbb{E}[W] < \infty$, we recover known results for example (a). Since $\mathbb{E}[K] < \infty$, in case (b) H belongs to the same MDA as X. As we have seen, case (c) is more involved, but the theorem implies that if W_1 has a heavier tail index than X, then $\mathbb{E}[K] = \infty$ and $H \notin \mathrm{MDA}(G)$. On the other hand, for bounded or lighter tailed W, we can still have $H \in \mathrm{MDA}(G)$. For a simulation of H and corresponding QQ plots, see Figure 4.1.

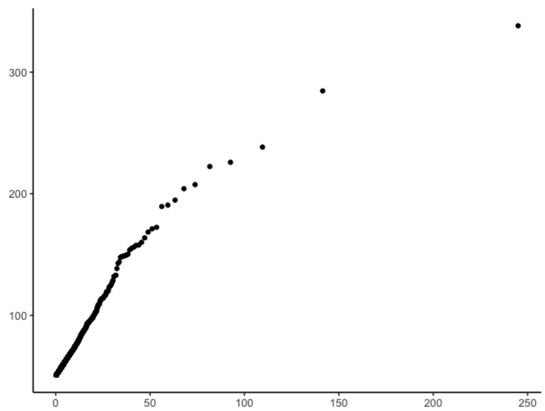
Note, the proof of this theorem provides normalising sequences for H:

$$(c_n)_{n\in\mathbb{N}} = (a_{\lfloor n\xi\rfloor})_{n\in\mathbb{N}}, \quad (d_n)_{n\in\mathbb{N}} = (b_{\lfloor n\xi\rfloor})_{n\in\mathbb{N}}, \tag{4.9}$$

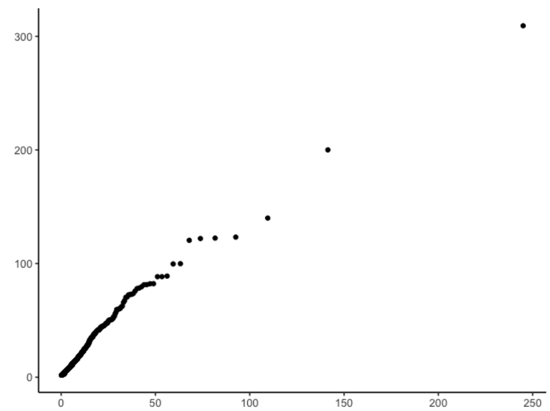
for which the following convergence holds:

$$n\mathbf{P}\{H > c_n x + d_n\} \to -\log G(x) \quad \text{as } n \to \infty, \tag{4.10}$$

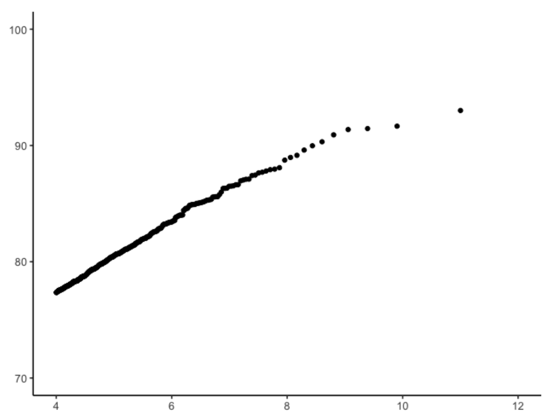
for every $x \in \operatorname{supp} G$. Intuitively, H behaves as $\bigvee_{i=1}^{\lfloor n\xi\rfloor} X_i$ where we consider deterministic fixed number of elements from the i.i.d. sequence $(X_i)_{i\in\mathbb{N}}$.



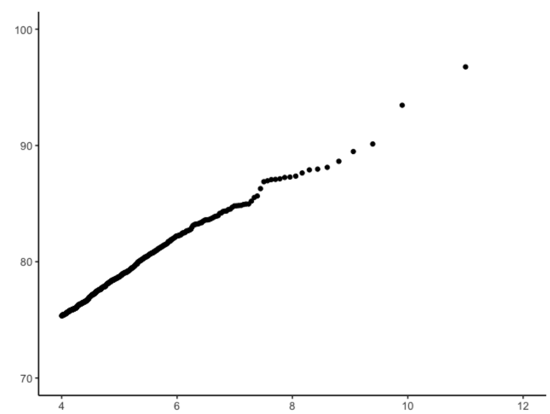
(a) *Fréchet* MDA*: independent case. The number K of observations in a cluster is independent of X_i, $K = 1 + N$, $N \sim$* Poi(5).



(b) *Fréchet* MDA*: dependent case. The number K of observations in a cluster is $K = \inf\{i : X_i \geq 2\}$.*



(c) *Gumbel* MDA*: independent case. The number K of observations in a cluster is independent of X_i, $K = 1 + N$, $N \sim$* Poi(5).



(d) *Gumbel* MDA*: dependent case. The number K of observations in a cluster is $K = \inf\{i : X_i \geq 55\}$.*

Figure 4.1: *QQ plots with theoretical quantiles of the marginal distribution of the sequence X_i on x-axis and empirical quantiles of the marginal distribution of simulated maximas $H = \vee_{i=1}^{K} X_i$ on y-axis, where X_i are i.i.d. In (a) – (b): observations X_i come from Pareto(2) distribution. In (c) – (d): observations X_i come from* $\mathrm{N}(50, 10^2)$ *distribution.*

4.3 Application to marked renewal cluster processes

Even though clusters can appear in various fields, we primarily focus on their application in non-life insurance, and therefore adopt the terminology related to insurance claims as our main example (observations are referred to as claims and so on). We apply our results in analysis of maximal claim size of some cluster processes. Cluster processes are commonly used in non-life insurance, where, for example after a natural phenomenon affects some geographical area (for example earthquake or flood), there occur a lot of demands for claims in a short time period. These clusters are often triggered by the first claim in the cluster, although the point processes we deal with could be represented so that the clusters are triggered by an event outside of the point process itself. Models with triggered clusters of subsequent payments with possible delay are referred to as incurred but not reported (IBNR) claims models, see [42] for more information.

Let $(Y_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. non-negative random variables and let $(A_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random elements in a general Polish space $\mathbb{X}$ with distribution Q, independent of the sequence $(Y_i)_{i\in\mathbb{N}}$. Assume that

$$\mathbb{E}[Y] = \frac{1}{\nu} < \infty.$$

We define a random walk $(\Gamma_i)_{i\in\mathbb{N}}$ generated by the sequence $(Y_i)_{i\in\mathbb{N}}$:

$$\Gamma_0 = 0, \quad \Gamma_i = Y_1 + Y_2 + \cdots + Y_i, \quad i \in \mathbb{N}.$$

We then define an independently marked renewal process N^0 on $\mathbb{R}_+ \times \mathbb{X}$ by

$$N^0 = \sum_{i\in\mathbb{N}} \delta_{\Gamma_i, A_i}.$$

The sequence $(\Gamma_i)_{i\in\mathbb{N}}$ represents arrival times (with the sequence $(Y_i)_{i\in\mathbb{N}}$ representing inter-arrival times), while $(A_i)_{i\in\mathbb{N}}$ is a sequence of marks that are often referred to as claims. In general, marks provide much more information than just the claim size. They can include all relevant information about the claim, such as its type, size, severity, location, and more. Recall that a subset of $\mathbb{R}_+$ is considered bounded if it is contained in $[0, M]$ for some $M > 0$. We equip $\mathbb{X}$ with an arbitrary family of bounded sets $\mathcal{S}(\mathbb{X})$. Subsets of $\mathbb{R}_+ \times \mathbb{X}$ are considered bounded if they are contained in $[0, M] \times B$ for some $M > 0$ and $B \in \mathcal{S}(\mathbb{X})$. Let C be one such set such that $C \subseteq [0, M_1] \times B_1$, where $M_1 > 0$, $B_1 \in \mathcal{S}(\mathbb{X})$. Then we have

$$N^0(C) = \sum_{i\in\mathbb{N}} \delta_{\Gamma_i, A_i}(C) \le \sum_{i\in\mathbb{N}} \delta_{\Gamma_i}[0, M_1] < \infty \quad \text{a.s.},$$

which follows from the fact that $(\Gamma_i)_{i\in\mathbb{N}}$ is a non-decreasing sequence.

The process N^0 represents the process of all first claims from each cluster, and we say N^0 is the *parent process*. We attach to each mark A_i, $i \in \mathbb{N}$, at time Γ_i, a complete cluster of points denoted by G_i as another point process from the space $\mathcal{M}_p(\mathbb{R}_+ \times \mathbb{X})$. These processes

$(G_i)_{i\in\mathbb{N}}$ are called the *descendant processes* and they represent clusters without the first claim. We assume that the clusters G_i, $i \in \mathbb{N}$, are mutually independent. Formally, we assume that there exists a probability kernel K from $\mathbb{X}$ to $\mathcal{M}_p(\mathbb{R}_+ \times \mathbb{X})$ such that, conditionally on the parent process N^0, the point processes G_i, $i \in \mathbb{N}$, are independent, almost surely finite and they have distribution equal to $K(A_i, \cdot)$. We emphasise that this permits dependence between G_i and A_i for $i \in \mathbb{N}$.

We write G_i as

$$G_i = \sum_{j=1}^{K_i} \delta_{T_{i,j}, A_{i,j}},$$

where K_i is a non-negative integer-valued random variable (called also a $\mathbb{Z}_+$-valued random variable), $(T_{i,j})_{j\in\mathbb{N}}$ is a sequence of non-negative random variables, and $(A_{i,j})_{j\in\mathbb{N}}$ is a sequence of marks representing claims in a cluster. Note that if we take into account the original point arriving at time Γ_i, the cluster size is $K_i + 1$. Additionally, each mark $A_{i,j}$ in a cluster G_i arrives at time $\Gamma_i + T_{i,j}$, as will be observed in our cluster process later on.

Throughout, we assume that processes G_i are independently marked with the same mark distribution Q as before, independent of A_i. This means that all the marks $A_{i,j}$ are i.i.d. Note that K_i may depend on A_i. We assume throughout that

$$\mathbb{E}[K_i] < \infty,$$

which ensures that there are almost surely finitely many claims in each cluster. Finally, to describe the size and other characteristics of all observations (claims) together with their arrival times, we use a marked point process N as a random element in $\mathcal{M}_p(\mathbb{R}_+ \times \mathbb{X})$ given by

$$N = \sum_{i=1}^{\infty} \sum_{j=0}^{K_i} \delta_{\Gamma_i + T_{i,j}, A_{i,j}}, \tag{4.11}$$

where $T_{i,0} = 0$ and $A_{i,0} = A_i$. In this representation, the claims arriving at time Γ_i and corresponding to the index $j = 0$ are called ancestral or immigrant claims, while the claims arriving at times $\Gamma_i + T_{i,j}$, $j \in \mathbb{N}$, are referred to as progeny or offspring. Note that N is a.s. boundedly finite because $\Gamma_i \to \infty$ as $i \to \infty$ and K_i is a.s. finite for every i. These properties allow us to rearrange claims chronologically to get the following representation of the process N:

$$N = \sum_{k=1}^{\infty} \delta_{\tau_k, A^k}, \tag{4.12}$$

where $\tau_k \le \tau_{k+1}$ for all $k \in \mathbb{N}$ and A^k are i.i.d. marks which are in general not independent of the arrival times $(\tau_k)_{k\in\mathbb{N}}$. This representation, however, ignores the information regarding cluster structure of the process N. Note also that eventual ties turn out to be irrelevant asymptotically. We often use both representations of N; the first one from (4.11) when we use the cluster structure, and the second one from (4.12) when we need to analyse claims that arrive in some time interval, $[0, t]$ say, for $t > 0$.

In the special case when inter-arrival times are exponential with parameter ν, the renewal counting process which generates the arrival times in the parent process is a homogeneous Poisson process. Associated marked renewal cluster model is then called marked Poisson cluster process, see [15], cf. [9].

Recall that marks $A_{i,j}$ can contain more information about claims than just the size of the claim. Numerical observations, i.e. the size of the claims, are produced by an application of a measurable function on these marks, say $f : \mathbb{X} \to \mathbb{R}_+$. The maximum of all claims in cluster G_i is denoted by H_i and equals

$$H_i = \bigvee_{j=0}^{K_i} X_{i,j}, \tag{4.13}$$

where $X_{i,j} = f(A_{i,j})$ are i.i.d. random variables for all i and j. We interpret H_i as the maximal claim size coming from the ith immigrant and its progeny. Similarly, set $X^k = f(A^k)$ and by $M(t)$ denote the maximal claim size in the period $[0, t]$, $t > 0$. Note,

$$M(t) = \sup \left\{ X^k : \tau_k \leq t \right\}.$$

We can fit this model in the context of Theorem 4.2.4 by letting $W_k = A^k$, for $k \in \mathbb{N}$. Let us introduce the first passage time process $(\tau(t))_{t \geq 0}$ defined by

$$\tau(t) = \inf \left\{ n : \Gamma_n > t \right\}.$$

In other words, $\tau(t)$ is the renewal counting process generated by the sequence $(Y_n)_{n \in \mathbb{N}}$ from the beginning, and we interpret it as the ordinal number of the first cluster that comes after time t. According to the strong law for counting processes (see [28, Chapter 2, Theorem 5.1]), for every $c \geq 0$,

$$\frac{\tau(tc)}{\nu t} \xrightarrow{a.s.} c \quad \text{as } t \to \infty.$$

Denote by

$$M^\tau(t) = \bigvee_{i=1}^{\tau(t)} H_i,$$

the maximal claim size from the first $\tau(t)$ clusters. We can define the leftover effect at time t denoted by ε_t defined by

$$\varepsilon_t = \max\{X_{i,j} : 0 \leq \Gamma_i \leq t,\, t < \Gamma_i + T_{i,j}\}, \quad t \geq 0.$$

It represents the maximum of all claims that arrived after t but belong to the cluster whose at least one claim arrived before t. The number of members in the set above we denote by J_t:

$$J_t = \#\{(i,j) : 0 \leq \Gamma_i \leq t,\, t < \Gamma_i + T_{i,j}\}. \tag{4.14}$$

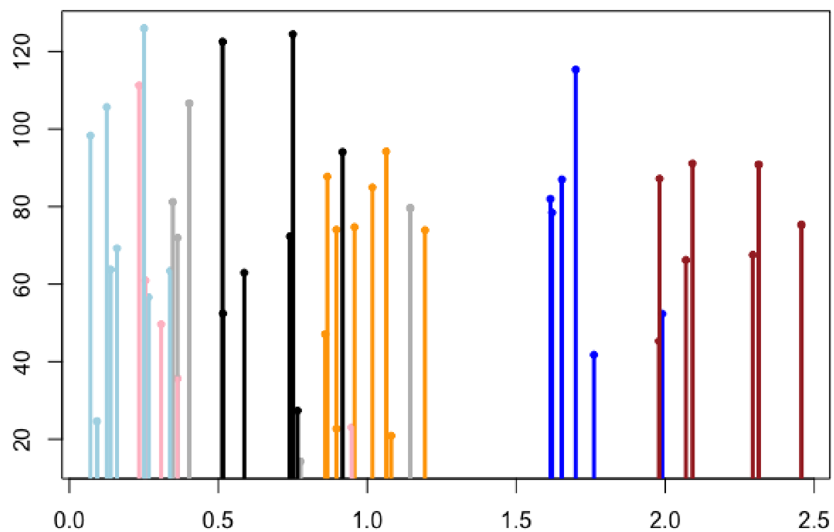


Figure 4.2: *Simulation of the mixed binomial cluster model represented in Example 4.4.1 where* $\Gamma_i \sim \mathrm{Exp}(1)$ *is the arrival time of the first observation in cluster* $i \in \{1, 2, \ldots, 7\}$, $K_i \sim \mathrm{Poi}(8)$ *is the size of cluster* i, $V_{i,j} \sim \mathrm{Exp}(5)$ *are arrival times of observations in cluster* i, *and* $X_{i,j} \sim \mathrm{N}(70, 30^2)$ *are observations in cluster* i *where those that are in the same cluster are denoted with the same colour.*

Figure 4.2 shows an example of cluster point processes analysed in this section. This illustration can aid in understanding that for $t \geq 0$,

$$M^\tau(t) = M(t) \bigvee H_{\tau(t)} \bigvee \varepsilon_t. \tag{4.15}$$

Our objective is to investigate the limiting behaviour of the maximal claim size up to time t, given by $M(t)$, and and in the following, we provide sufficient conditions under which $M(t)$ converges in distribution to a nontrivial limit after appropriate centring and normalisation. A similar problem in the independent case (where K is independent of A) has been the subject of study in [59].

Proposition 4.3.1. *Assume that* H *belongs to* $\mathrm{MDA}(G)$ *so that* (4.10) *holds and that the error term in* (4.14) *satisfies:*

$$J_t = o_P(t).$$

Then

$$\frac{M(t) - d_{\lfloor \nu t \rfloor}}{c_{\lfloor \nu t \rfloor}} \xrightarrow{d} G \qquad \textit{as } t \to \infty. \tag{4.16}$$

Proof. Using the equation (4.15), we have

$$\frac{M^\tau(t)-d_{\lfloor\nu t\rfloor}}{c_{\lfloor\nu t\rfloor}} = \frac{M(t)-d_{\lfloor\nu t\rfloor}}{c_{\lfloor\nu t\rfloor}} \bigvee \frac{H_{\tau(t)}-d_{\lfloor\nu t\rfloor}}{c_{\lfloor\nu t\rfloor}} \bigvee \frac{\varepsilon_t-d_{\lfloor\nu t\rfloor}}{c_{\lfloor\nu t\rfloor}}.$$

Since for $x \in E$,

$$\begin{aligned} 0 &\le \mathbf{P}\left\{\frac{M^\tau(t)-d_{\lfloor\nu t\rfloor}}{c_{\lfloor\nu t\rfloor}} > x\right\} - \mathbf{P}\left\{\frac{M(t)-d_{\lfloor\nu t\rfloor}}{c_{\lfloor\nu t\rfloor}} > x\right\} \\ &\le \mathbf{P}\left\{\frac{H_{\tau(t)}-d_{\lfloor\nu t\rfloor}}{c_{\lfloor\nu t\rfloor}} > x\right\} + \mathbf{P}\left\{\frac{\varepsilon_t-d_{\lfloor\nu t\rfloor}}{c_{\lfloor\nu t\rfloor}} > x\right\}, \end{aligned}$$

it suffices to show that

$$\frac{M^\tau(t)-d_{\lfloor\nu t\rfloor}}{c_{\lfloor\nu t\rfloor}} \xrightarrow{d} G \quad \text{as } t\to\infty, \tag{4.17}$$

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{H_{\tau(t)}-d_{\lfloor\nu t\rfloor}}{c_{\lfloor\nu t\rfloor}} > x\right\} = 0, \quad \text{and} \quad \lim_{t\to\infty} \mathbf{P}\left\{\frac{\varepsilon_t-d_{\lfloor\nu t\rfloor}}{c_{\lfloor\nu t\rfloor}} > x\right\} = 0. \tag{4.18}$$

Recall that H_i represents the maximum of all claims in the i-th cluster (all claims due to the arrival of an immigrant claim at time Γ_i) and by (4.13), it equals

$$H_i = \bigvee_{i=0}^{K_i} X_{i,j}\,.$$

Note that $(H_i)_{i\in\mathbb{N}}$ is an i.i.d. sequence because the ancestral mark in every cluster comes from an independently marked renewal point process. As in the proofs of Lemma 4.2.3 and Theorem 4.2.4,

$$\begin{aligned} \mathbf{P}\left\{\frac{M^\tau(t)-d_{\lfloor\nu t\rfloor}}{c_{\lfloor\nu t\rfloor}} \le x\right\} &= \mathbf{P}\left\{\sum_{i=1}^{\tau(t)} \delta_{\frac{H_i-d_{\lfloor\nu t\rfloor}}{c_{\lfloor\nu t\rfloor}}}(x,\infty) = 0\right\} \\ &\to \mathbf{P}\left\{\mathrm{PRM}(\mu_G)(x,\infty)=0\right\} = G(x) \end{aligned}$$

as $t\to\infty$, which shows (4.17). To prove (4.18), we begin by noting that $\{\tau(t)=k\} = \{\Gamma_1 \le t,\ldots,\Gamma_{k-1}\le t,\, \Gamma_k > t\} = \{\sum_{i=1}^{k-1} Y_i \le t, \sum_{i=1}^{k} Y_i > t\} \in \sigma(Y_1,\ldots Y_k)$. Moreover, for any Borel subset B of $\mathbb{R}$ we have $\{H_k \in B\}$ is independent of $\sigma(Y_1,\ldots Y_k)$ for every k, once again because the marks are independent of arrival times. This implies that $H_{\tau(t)} \stackrel{d}{=} H_1 \in \mathrm{MDA}(G)$ so the first part of (4.18) follows easily from (4.10). For the second part of (4.18), we note that the leftover effect ε_t admits the following representation

$$\varepsilon_t \stackrel{d}{=} \bigvee_{i=1}^{J_t} X_i,$$

for $(X_i)_{i\in\mathbb{N}}$ i.i.d. copies of $X = f(A)$. Hence,

$$\frac{\varepsilon_t - d_{\lfloor \nu t\rfloor}}{c_{\lfloor \nu t\rfloor}} \stackrel{d}{=} \frac{\bigvee_{i=1}^{J_t} X_i - d_{\lfloor \nu t\rfloor}}{c_{\lfloor \nu t\rfloor}}.$$

Since $J_t = o_P(t)$, for every fixed $\delta > 0$ and t large enough, we have $\mathbf{P}\{J_t > \delta t\} < \delta$. For a measurable $B = \{J_t > \delta t\}$ we have

$$\begin{aligned}\mathbf{P}\left\{\frac{\bigvee_{i=1}^{J_t} X_i - d_{\lfloor \nu t\rfloor}}{c_{\lfloor \nu t\rfloor}} > x\right\} &\le \mathbf{P}\{B\} + \mathbf{P}\left\{\left\{\frac{\bigvee_{i=1}^{J_t} X_i - d_{\lfloor \nu t\rfloor}}{c_{\lfloor \nu t\rfloor}} > x\right\} \cap B^c\right\} \\ &< \delta + \mathbf{P}\left\{\frac{\bigvee_{i=1}^{\lfloor \delta t\rfloor} X_i - d_{\lfloor \nu t\rfloor}}{c_{\lfloor \nu t\rfloor}} > x\right\},\end{aligned}$$

which converges to 0, as $\delta \to 0$. □

As we have seen above, it is relatively easy to determine asymptotic behaviour of the maximal claim size $M(t)$ as long as one can determine tail properties of the random variables H_i and the number of points in the leftover effect at time t, J_t in (4.14). By applying Theorem 4.2.4, we can directly obtain the following corollary.

Corollary 4.3.2. *Let $J_t = o_P(t)$ and let $(X_{i,j})$ satisfy* (1.1) *and the assumptions from the proof of Theorem 4.2.4. Then* (4.10) *holds with (c_n) and (d_n) defined by*

$$(c_n) = (a_{\lfloor (\mathbb{E}[K]+1)\cdot n\rfloor}), \quad (d_n) = (b_{\lfloor (\mathbb{E}[K]+1)\cdot n\rfloor}). \tag{4.19}$$

4.4 Examples

Although the results presented in the previous sections may appear straightforward, proving that $J_t = o_P(t)$ holds can be a challenging task. Nevertheless, this can be accomplished for several commonly used cluster models, as illustrated in the following examples. For an application of these results to the case of marked Hawkes process, we refer the reader to [4].

Example 4.4.1. [Mixed binomial cluster model] Assume that the renewal counting process which generates the arrival times $(\Gamma_i)_{i\in\mathbb{N}}$ in the parent process is a homogeneous Poisson process with mean measure $\nu\,\mathrm{Leb}$ for $\nu > 0$, on the state space $\mathbb{R}_+$, and that the individual clusters have the following form

$$G_i = \sum_{j=1}^{K_i} \delta_{V_{i,j}, A_{i,j}},$$

where K is a non-negative integer-valued random variable with finite expectation representing the size of a cluster. To emphasize the simplicity of the cluster structure in this model compared to the second example, we use the notation $V_{i,j}$ instead of $T_{i,j}$. Assume that $(K_i, (V_{i,j})_{j\in\mathbb{N}}, (A_{i,j})_{j\in\mathbb{Z}_+})_{i\in\mathbb{N}}$ constitutes an i.i.d. sequence with the following properties for fixed $i \in \mathbb{N}$:

- $(A_{i,j})_{j\in\mathbb{Z}_+}$ are i.i.d.,
- $(V_{i,j})_{j\in\mathbb{N}}$ are conditionally i.i.d. given $A_{i,0}$,
- $(A_{i,j})_{j\in\mathbb{N}}$ are independent of $(V_{i,j})_{j\in\mathbb{N}}$,
- K_i is a stopping time with respect to the filtration generated by the $(A_{i,j})_{j\in\mathbb{Z}_+}$, i.e. for every $k \in \mathbb{Z}_+$, $\{K_i = k\} \in \sigma(A_{i,0}, \dots A_{i,k})$.

Hence, we allow dependence between the random variables K_i and $(A_{i,j})_{j\in\mathbb{Z}_+}$, as well as between $(V_{i,j})_{j\in\mathbb{N}}$ and the ancestral mark $A_{i,0}$ (and consequently K_i). The marked point process N, whose clusters are G_i for $i \in \mathbb{N}$, is a marked version of the so–called Neyman–Scott process, see for example Example 6.3 (a) in [15]. An illustration of a simulation of this process can be found in Figure 4.2.

Corollary 4.4.2. *Assume that* $f(A) = X$ *belongs to* MDA(G) *with normalising sequences* $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *so that* (1.1) *holds. Then* (4.16) *holds for* (c_n) *and* (d_n) *defined in* (4.19).

Proof. By applying Theorem 4.2.4, we can deduce that the maximum H of all claims in a cluster belongs to MDA of the same extreme value distribution G as X. Then, using Proposition 4.3.1 and the fact that $J_t = o_P(t)$, we can conclude that (4.16) holds. To verify that $J_t = o(t)$, we need to show that $\mathbb{E}[J_t] = o(t)$. This can be done using Markov's inequality,

$$\begin{aligned}\mathbb{E}[J_t] &= \mathbb{E}\left[\#\{(i,j) : 0 \le \Gamma_i \le t,\, t < \Gamma_i + V_{i,j}\}\right] \\ &= \mathbb{E}\left[\sum_{0\le\Gamma_i\le t}\sum_{j=1}^{K_i} \mathbf{1}_{t<\Gamma_i+V_{i,j}}\right].\end{aligned}$$

The term in square brackets equals $\sum_{i=1}^{\infty} f(\Gamma_i)$, for $f(s) = \mathbf{1}_{0\le s\le t}\sum_{j=1}^{K_i}\mathbf{1}_{V_{i,j}>t-s}$. Moreover, this expectation can be rewritten as

$$\mathbb{E}[N(f)] = \mathbb{E}\left[\int_0^t f(s)N(ds)\right], \quad \text{where } N = \sum_{i=1}^{\infty}\delta_{\Gamma_i} = \mathrm{PRM}(\nu\,\mathrm{Leb}).$$

Furthermore, Lemma 7.2.12. in [42] and calculation similar as in the proofs of Corollaries 5.1. and 5.3. in [9], we have

$$\mathbb{E}[J_t] = \int_0^t \mathbb{E}\left[\sum_{j=1}^{K_i}\mathbf{1}_{V_{i,j}>t-s}\right]\nu ds = \int_0^t \mathbb{E}\left[\sum_{j=1}^{K_i}\mathbf{1}_{V_{i,j}>x}\right]\nu dx.$$

Use the following facts

$$\sum_{j=1}^{K_i} \mathbf{1}_{V_{i,j}>x} \leq K_i \mathbf{1}_{\max\{V_{i,j}:1\leq j\leq K_i\}>x},$$

$$\begin{aligned} K_i \mathbf{1}_{\max\{V_{i,j}:1\leq j\leq K_i\}>x} &\xrightarrow{a.s.} 0 \quad \text{as } x\to\infty, \\ |K_i \mathbf{1}_{\max\{V_{i,j}:1\leq j\leq K_i\}>x}| &\leq K_i, \\ \mathbb{E}[K_i] &< \infty, \end{aligned}$$

to conclude that by the dominated convergence theorem, as $x\to\infty$, we have

$$\mathbb{E}\left[K_i \mathbf{1}_{\max\{V_{i,j}:1\leq j\leq K_i\}>x}\right] \to 0,$$

and thus,

$$\mathbb{E}\left[\sum_{j=1}^{K_i} \mathbf{1}_{V_{i,j}>x}\right] \to 0.$$

An application of Cesàro argument yields now that $\mathbb{E}[J_t]/t \to 0$. □

Example 4.4.3 (Renewal cluster model). Assume next that the clusters G_i have the following representation

$$G_i = \sum_{j=1}^{K_i} \delta_{T_{i,j},A_{i,j}},$$

where $(T_{i,j})$ represents the sequence such that

$$T_{i,j} = V_{i,1} + \cdots + V_{i,j}, \quad 1\leq j\leq K_i.$$

We keep all the other assumptions from the model in Example 4.4.1.

A general unmarked model of the similar type is called Bartlett–Lewis model and is analysed in [15], see Example 6.3 (b). See also [23] for an application of a similar point process to modelling of teletraffic data. By adapting the arguments from the Corollary 4.4.2 we can easily obtain the next corollary.

Corollary 4.4.4. *Assume that $f(A) = X$ belongs to* MDA(G) *with normalising sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ so that* (1.1) *holds. Then* (4.16) *holds for (c_n) and (d_n) defined in* (4.19).

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Summary

This thesis focuses on regular variation in Polish spaces equipped with the general notion of scaling, bornology, and modulus. The bornology represents the collection of bounded sets, while the modulus generalises the metric to an arbitrary continuous Borel homogeneous function. We define and characterise regular variation and give numerous examples that show how the choice of scaling and bornology affects this definition.

The main part of the thesis concerns continuous maps of regularly varying elements. We examine continuous bornologically consistent morphisms and show that, in many cases, they preserve the regular variation property. We start with a regularly varying random element ξ with a tail measure μ, defined on the Polish space $\mathbb{X}$ endowed with continuous scaling and topologically and scaling consistent bornology with countable base. We then map it continuously by a bornologically consistent morphism ψ to a space $\mathbb{Y}$ with the same properties as $\mathbb{X}$. We show that if $\psi\mu$ is nontrivial on the bornology on $\mathbb{Y}$ that satisfies given properties, then $\psi\xi$ is regularly varying in $\mathbb{Y}$. This result is further applied to the polar decomposition map and quotient mapping, among others. As in some simpler spaces, we decompose ξ into an "angular" part that belongs to the set of all points with modulus one, and into a "modular" part and show that this modular part is also regularly varying. Conditionally on modulus being large, we obtain a nontrivial limit of their joint distribution. This limit is expressed as a product measure of the Pareto-type measure and the spectral measure of ξ.

For the continuous quotient map, we show that even though ξ is regularly varying, its equivalent class as a closed subset of $\mathbb{X}$ can contain both regularly varying and non-regularly varying selections. We show that if the selection map is chosen to be homogeneous with respect to scaling and if it minimises the modulus of the whole equivalence class, then the selected element is also regularly varying.

In the last chapter, we consider an independent and identically distributed sequence (X_n) of random variables that belong to some maximum domain of attraction. We consider $H = \max\{X_i : i = 1, 2, \dots, K\}$, where K is a positive random integer with finite expectation. We show that as long as K is a stopping time with respect to the filtration generated by the sequence (X_n) and possibly some random element independent of the sequence (X_n), then H is in the same maximum domain of attraction as X_1. We apply this result to some marked renewal cluster processes where we observe the maximum of all the observations that arrived until moment $t > 0$. Since this problem often arises in insurance models, it can be interpreted as a maximal claim problem until time t. We show that after proper

normalisation, the maximal claim until time t converges in distribution to the distribution G if X_1 is in the maximum domain of attraction of distribution G.

Declaration of consent

on the basis of Article 18 of the PromR Phil.-nat. 19

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Bachelor ☐ Master ☐ Dissertation ✔

Title of the thesis: Regular variation on Polish spaces, continuous maps and compound maxima

Supervisor: Prof. Dr. Ilya Molchanov, Prof. Dr. Bojan Basrak

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