Pseudo numerical ranges, Schur complement dominant operator matrices, Schrödinger operators with accretive potentials in weighted spaces and applications to damped wave equations

Inauguraldissertation
der Philosophisch-naturwissenschaftlichen Fakultät
der Universität Bern

vorgelegt von

Borbála Mercédesz Gerhát

von Österreich

Leiterin der Arbeit:
Prof. Dr. Christiane Tretter
Mathematisches Institut, Universität Bern

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Bern, 30.11.2021

Der Dekan
Prof. Dr. Zoltán Balogh

To my parents Márta and Zoltán.

Preface

This doctoral thesis was concluded under the supervision of Prof. Dr. Christiane Tretter at the Mathematical Institute, University of Bern, Switzerland. During my studies and research work, I have been generously supported by the Swiss National Science Foundation (grant no. 169104) and by the University of Bern. Moreover, I have received the financial support of the SEMP programme during a student exchange at the Queen's University Belfast, UK.

The present thesis is cumulative and contains results from the following three research papers, which are subsequently included in the appendix.

[Ger21] B. Gerhat, *Schur complement dominant operator matrices*

[GT21] B. Gerhat and C. Tretter, *Pseudo numerical ranges and spectral enclosures*

[GS21] B. Gerhat and P. Siegl, *Schrödinger operators with accretive potentials in weighted spaces*

While I am the single author of the first paper, the second one is co-authored with my supervisor. The third paper is a collaboration with Dr. Petr Siegl (Queen's University Belfast, UK), whose co-author statement is included in the appendix. The common theme of this cumulative work can be described as

"Analysis of non-selfadjoint operator matrices in the absence of standard dominance patterns".

The subsequent four chapters form an introduction to this dissertation. They are intended to put the articles listed above in a common context and give a comprehensive overview of the underlying research work. To enhance readability, not all results are included and the presented ones are largely simplified; however, the exact references to the corresponding parts of the articles (in the appendix) are always given, where full proofs and further results and references to related work can be found.

During the time I was financially supported by the University of Bern, I have moreover co-authored the article [GIS20], which is not included in this thesis.

Acknowledgments

I would like to first express my gratitude to my supervisor Christiane Tretter, to whom I owe the opportunity for this dissertation and who has continuously supported me over the last years. In particular, I thank her for the freedom she has given me in exploring mathematics, for encouraging my ideas and my research, even when it diverted from the original direction of this thesis. I am grateful for her honest advice, both professional and personal, especially for advising me to take my time when I felt under a lot of pressure.

My sincere gratitude goes to Petr Siegl, who in course of the last years has been a good colleague, friend and mentor for me. I truly appreciate our many fruitful mathematical discussions, sometimes just for the sake of mathematics, and I am grateful for his valuable advice and reliable support.

Moreover, I would like to thank Orif Ibrogimov, my academic brother, with whom I share a special connection beyond mathematics. Not only am I grateful for our good collaboration aside from this thesis, but also for his friendship and support.

Most importantly, I would like to express my deepest gratitude to my loving family, in particular my parents and my sister, for their strength, their wisdom, infinite support, patience and understanding. Without them, I would not be where I am today and none of my achievements would have been possible.

Finally, I would like to thank my friends, who have not abandoned me even though I have been absent and permanently busy during the last years. My special thanks go to my flatmates in Bern, who have over the last years not only been a pleasure to live with, but helped me to take my mind off work when needed.

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CHAPTER 1

Introduction

The first three sections of this chapter constitute a brief (non-exhaustive) overview of well-known results and previously developed methods in the spectral theory of non-selfadjoint operators and operator matrices. To put the mathematical contributions in this thesis into a context within the area of research, they are shortly introduced in Section 1.4 and further illustrated in Section 1.5 based on their outlined application to a particular spectral problem. While the subsequent chapters contain a more substantial summary of the results achieved in the course of this thesis and are thus more technical, the present chapter is intended for a reader who might not be fully familiar with spectral theory.

1.1. Mathematical physics and spectral theory

In the past centuries, fundamental research has led to an incredible progress in science by interpreting phenomena and patterns in the nature surrounding us. The ground to this understanding is for instance laid in theoretical physics, where certain processes are modeled with systems of differential equations. To gain further insight, it is then crucial to determine whether or not these equations have solutions and describe the latter, if existent. This is one instance when mathematics and physics interact, when mathematicians are given an equation to analyse it in a suitable context or setting.

For ordinary differential equations involving functions of only one variable, questions like solvability, stability of the solutions or their continuous dependence with respect to the data are historically well understood in a certain classical sense. However, for partial differential equations involving functions of multiple variables, analogous problems are much harder to address, especially as there is already quite some freedom in the mathematical definition of a solution.

Certain types of linear (partial) differential equations can be written as a so-called abstract Cauchy problem

$$\partial_t u(x,t) = Au(x,t), \qquad x \in \Omega \subseteq \mathbb{R}^n, \qquad t > 0, \qquad u(\cdot,0) = u_0. \tag{1.1}$$

The above can be viewed as an initial value problem in the time variable t with a differential expression A on the right hand side, which contains all derivatives with respect to the spatial variable x and is understood together with a boundary condition, like e.g. a so-called Dirichlet boundary condition

$$u(x,t) = 0, \qquad x \in \partial\Omega, \qquad t \geq 0. \tag{1.2}$$

In case that $A \in \mathbb{C}$ is only a scalar, the solution of (1.1) can easily be found by means of the exponential function as

$$u(t,x) = \mathrm{e}^{tA}\, u_0(x), \qquad x \in \Omega, \qquad t \geq 0. \tag{1.3}$$

Following this idea, the standard approach to the analysis of (1.1) is to associate a linear operator with the differential expression A which acts in a suitable function space, whilst respecting the boundary condition (1.2); an appropriate notion of solvability for (1.1) can then be related to a generalisation of the formula (1.3). The existence of the latter depends on qualitative and quantitative properties of the differential operator A, in particular on the structure and location of its spectrum and the behaviour of its resolvent norm.

While the described approach leaves quite some room for flexibility, the choice of the functional space and the specific realisation of A therein are particularly challenging as they can dramatically influence the (spectral) properties of the resulting operator. In fact, since the considered differential expressions typically lead to unbounded operators, choosing their domain of definition is already a crucial part of the analysis. In the sequel, we describe a selection of important aspects of spectral theory in relation to the Cauchy problem (1.1) and highlight some of the challenges therein, in particular the ones addressed in this thesis.

1.2. Non-selfadjoint spectral theory

For further details about the following well-known results, we refer the reader to the standard literature on operator and semigroup theory, see e.g. [Kat95, EE87, EN00, Dav07, Hel13].

1.2.1. The selfadjoint case. Consider again the Cauchy problem (1.1) and suppose that the differential expression A therein, together with the underlying boundary condition, is formally symmetric in a Hilbert space $\mathcal{H}$. Then one can give rise to a sensible solution by finding a self-adjoint realisation of A (if it exists), i.e. by identifying a suitable domain on which A becomes selfadjoint in $\mathcal{H}$.

Indeed, already on abstract level, the spectral theorem for selfadjoint operators has numerous powerful implications, which can go very far when applied to a specific problem. In particular, the selfadjoint functional calculus for A justifies (1.3) as a solution of the Cauchy problem, provided that the initial condition u_0 lies in the domain of A. The norm and in-/stability in time evolution (with respect to perturbations of the initial value u_0) of this solution are governed by

$$\|e^{tA}\| = e^{ta}, \qquad a = \sup_{\lambda \in \sigma(A)} \lambda, \qquad t \geq 0, \tag{1.4}$$

including the case $a = \infty$ where the exponential above is an unbounded operator. Note that similarly, if $A = -\mathrm{i}\, B$ with a selfadjoint operator B (which is the case e.g. for the Schrödinger equation in quantum mechanics), formula (1.3) defines a unitary solution operator, which can be understood even for negative times.

Once a suitable selfadjoint realisation of A is found, by (1.4) the further analysis essentially reduces to locating the various parts of $\sigma(A)$. It is well-known that the spectrum of a selfadjoint operator is a closed, non-empty subset of $\mathbb{R}$, consists only of point and continuous spectrum, and is purely discrete if and only if the resolvent is a compact operator. Moreover, due to the identity

$$\|(A-\lambda)^{-1}\| = \frac{1}{\operatorname{dist}(\lambda, \sigma(A))}, \qquad \lambda \in \rho(A),$$

finding the location of the spectrum is equivalent to determining the behaviour of the resolvent norm. Finally, in the semi-bounded case, the powerful min-max principle, see [Hel13, Thm. 11.7], even provides a variational characterisation of the eigenvalues below or above the essential spectrum (which can have intuitive interpretations in particular in quantum physics).

1.2.2. The non-selfadjoint case. While self-adjoint problems are motivated by quantum physics and have historically been most relevant, the spectral theory of non-selfadjoint operators is recently gaining importance driven by several applications; these for instance include magnetohydrodynamics, fluid mechanics, damped systems, superconductivity or even MRI, see e.g. [BT20, MT07, GGHT12, AHP13, GH18], as well as [Tre08] and the references therein. In absence of the spectral theorem, however, for non-selfadjoint (or rather non-normal) operators the situation becomes much more complex. Nevertheless, an analogous connection between A and the Cauchy problem (1.1) still prevails via the following result due to Feller, Miyadera and Phillips (which generalises the classical Hille-Yosida theorem for contraction semigroups, see [EN00, Thm. II.3.5]).

Theorem 1.1 ([EN00, Thm. II.3.8])**.** *Let $M \geq 1$ and $a \in \mathbb{R}$. A linear operator A in a Hilbert space $\mathcal{H}$ generates a strongly continuous one-parameter semigroup*

$$T(t) = e^{tA}, \qquad \|T(t)\| \leq Me^{ta}, \qquad t \geq 0, \tag{1.5}$$

if and only if A is closed, its domain is dense in $\mathcal{H}$ and the powers of its resolvent are bounded on a semi-axis by

$$\|(A-\lambda)^{-m}\| \leq \frac{M}{(a-\lambda)^m}, \qquad \lambda \in (-\infty, a) \subseteq \rho(A), \qquad m \in \mathbb{N}.$$

While a suitable Hilbert space $\mathcal{H}$ might be indicated by the physical model itself, the first truly non-trivial task in view of the above is to determine an appropriate dense domain for the differential expression A such that (together with the boundary condition) the resulting operator is closed and has non-empty resolvent set. Further information about the possible semigroup generation and the behaviour of the solutions of the Cauchy problem can then be obtained from the location of the spectrum and the behaviour of the resolvent norm. The strong relation between the latter and the long time behaviour of the semigroup is enforced by the classical Gearhart-Prüss Theorem (which was further generalised, e.g. in [BBT16]).

Theorem 1.2 ([Hel13, Thm. 13.26])**.** *If the resolvent of the generator A of a strongly continuous semigroup on $\mathcal{H}$ is uniformly bounded on a half plane, i.e. if*

$$H_a := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq a\} \subseteq \rho(A), \qquad \sup_{\lambda \in H_a} \|(A-\lambda)^{-1}\| < \infty,$$

with some $a \in \mathbb{R}$, then there exists $M > 0$ such that (1.5) *holds.*

Note that unlike in the selfadjoint case, the spectral properties of a non-selfadjoint operator can be arbitrarily non-trivial within the framework described above. Some pathologies can for instance be illustrated based on Schrödinger operators of the form

$$A = -\partial_x^2 + \mathrm{i}\, x^k$$

in the space $L^2(\mathbb{R})$, starting with the imaginary Airy operator ($k = 1$) which has empty spectrum and generates a strongly continuous semigroup whose norm decays super-exponentially in time, see e.g. [Hel13, Sec. 14.3]. Moreover, the Davies oscillator ($k = 2$) shows that the resolvent norm cannot be bounded above in terms of the distance to the spectrum as in the selfadjoint case, see [Dav99]. For the imaginary cubic oscillator ($k = 3$), even though A has real and purely discrete spectrum, the operator $-\mathrm{i}\, A$ does not generate a semigroup (and thus certainly not a unitary group as in the selfadjoint case), see [KSTV15].

1.2.3. Form representation theorems. A standard and widely successful approach in constructing operators with the above described desirable properties is the representation by their sesquilinear form. Roughly speaking, the idea is to consider the sesquilinear form associated with the differential expression A (with respect to the underlying inner product) on a suitable set of functions and to formally integrate by parts. If one can associate the resulting formula with a sesquilinear form in $\mathcal{H}$ having certain properties, the latter gives rise to a closed, densely defined linear operator in $\mathcal{H}$ with non-empty resolvent set. A classical method to achieve this connection is the following Lax-Milgram theorem.

Theorem 1.3 ([Hel13, Thm. 3.6])**.** *Let $\mathcal{V}$ and $\mathcal{H}$ be Hilbert spaces such that $\mathcal{V} \subseteq \mathcal{H}$ is continuously embedded and dense in $\mathcal{H}$, and let $\mathbf{a}$ be a bounded sesquilinear form on $\mathcal{V}$. If $\mathbf{a}$ is coercive, i.e. if there exists $m > 0$ such that*

$$|\mathbf{a}(f,f)| \geq m\|f\|_{\mathcal{V}}^2, \qquad f \in \mathcal{V},$$

then the operator A in $\mathcal{H}$ defined by

$$\begin{aligned} \operatorname{dom} A &:= \{f \in \mathcal{V} : \exists\, \eta_f \in \mathcal{H},\ \forall\, g \in \mathcal{V},\ \mathbf{a}(f,g) = \langle \eta, g\rangle_{\mathcal{H}}\}, \\ Af &:= \eta_f, \end{aligned} \tag{1.6}$$

is boundedly invertible and its domain is dense in $\mathcal{V}$ and $\mathcal{H}$.

The above theorem can for instance be employed to introduce (non-selfadjoint) Dirichlet realisations in the space $L^2(\Omega)$ of the Schrödinger operator

$$A = -\Delta + V$$

with a sectorial complex-valued potential V, see e.g. [Kat95]. For accretive potentials, i.e. if the range of V is contained only in a complex half plane, the domain $\mathcal{V}$ of the form can in general not be chosen such that the latter becomes coercive. A generalisation of the Lax-Milgram theorem due to Almog and Helffer, which in particular covers the described case of accretive potentials, reads as follows.

Theorem 1.4 ([AH15, Thm. 2.1, Thm. 2.2])**.** *Let $\mathcal{V}$ and $\mathcal{H}$ be Hilbert spaces such that $\mathcal{V} \subseteq \mathcal{H}$ is continuously embedded and dense in $\mathcal{H}$, and let $\mathbf{a}$ be a bounded sesquilinear form on $\mathcal{V}$. If there exist $m > 0$ and $\Phi_1, \Phi_2 \in \mathcal{B}(\mathcal{V})$ which extend to bounded operators on $\mathcal{H}$ such that*

$$\begin{aligned} |\mathbf{a}(f,f)| + |\mathbf{a}(\Phi_1 f, f)| &\geq m\|f\|_{\mathcal{V}}^2, \\ |\mathbf{a}(f,f)| + |\mathbf{a}(f, \Phi_2 f)| &\geq m\|f\|_{\mathcal{V}}^2, \end{aligned} \qquad f \in \mathcal{V}, \tag{1.7}$$

then the operator A in $\mathcal{H}$ defined by (1.6) *is boundedly invertible and its domain is dense in $\mathcal{V}$ and $\mathcal{H}$.*

In case of the accretive Schrödinger operators in [AH15], the estimates in (1.7) are achieved by correcting the indefinite imaginary part of the potential by a multiplier $\Phi_1 = \Phi_2$ which is essentially the sign of $\operatorname{Im} V$. Further generalisations of the Lax-Milgram representation theorem can be found in [McI68, BBDCZ10, GKMV13, Sch15, tESV15].

1.2.4. The numerical range. Besides its obvious importance in numerical computations, the numerical range

$$W(A) = \{\langle Af, f\rangle_{\mathcal{H}} : f \in \operatorname{dom} A,\ \|f\|_{\mathcal{H}} = 1\} \subseteq \mathbb{C},$$

of a linear operator in a Hilbert space is also a powerful tool in its a priori spectral analysis. Knowing fairly little about the operator, it offers information on the location of the (approximate) point spectrum via the enclosures

$$\sigma_{\mathrm{p}}(\mathcal{A}) \subseteq W(A), \qquad \sigma_{\mathrm{ap}} \subseteq \overline{W(A)}. \tag{1.8}$$

Moreover, provided that every connected component of $\mathbb{C} \setminus \overline{W(A)}$ contains a point in the resolvent set, it gives an enclosure of the full spectrum and even an upper bound for the resolvent norm,

$$\sigma(A) \subseteq \overline{W(A)}, \qquad \|(A-\lambda)^{-1}\| \leq \frac{1}{\operatorname{dist}(\lambda, W(A))}, \qquad \lambda \in \mathbb{C} \setminus \overline{W(A)};$$

note that the numerical range is convex and the complement of its closure thus consists of at most two connected components. The above properties are in particular useful in non-selfadjoint spectral problems due to the non-trivial relation between spectrum and resolvent norm therein. For instance, considering the Hille-Yosida theorem, it follows that if $-A$ is m-accretive, i.e. if $\operatorname{Re} W(A) \leq 0$ and the open right half plane contains a point in the resolvent set, then A generates a strongly continuous contraction semigroup.

1.3. Operator matrices

Details on the following, as well as an extensive collection of results about operator matrices together with several examples can be found in the monograph [Tre08].

1.3.1. Spectral equivalence. Many applications in physics lead to systems of equations where the differential expression on the right hand side of the Cauchy problem is in fact given by a matrix acting on a two component vector function, see e.g. [Tre08] and the references therein. In these cases, the basic spectral analysis described in Section 1.2 becomes substantially more complicated. However, we point out that the challenges are not necessarily due to a lack of symmetry with respect to the underlying inner product, but rather intrinsic to the matrix structure of the problem. For instance, even if the differential expression is symmetric and the diagonal entries are semi-bounded, the resulting matrix expression need not be semi-bounded; no standard variational principles for the discrete spectrum are thus available, while analogous min-max principles for operator matrices are already quite involved, see e.g. [Tre08, Sec. 2.10].

More generally speaking, even having suitably well-behaved operator realisations of the entries A, B, C and D at hand (acting between the respective Hilbert spaces $\mathcal{H}_1$ and/or $\mathcal{H}_2$), the resulting operator matrix

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathcal{H} \supset \operatorname{dom} \mathcal{A} = (\operatorname{dom} A \cap \operatorname{dom} C) \oplus (\operatorname{dom} B \cap \operatorname{dom} D) \to \mathcal{H}, \tag{1.9}$$

defined naively on the domain above in the product space $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, can have arbitrary properties in general. Actually, even if the entries are all densely defined and closed, it is unclear if the matrix domain is dense in $\mathcal{H}$, or if it is large enough such that $\mathcal{A}$ is closed; in fact, $\mathcal{A}$ might not be closable at all. Several ill-behaved examples can be found in the pioneering work [Nag89] on operator matrices, or in [Tre08] and the references therein.

A successful approach to the above issues was proposed in [Nag89] and has been employed ever since in different variations. Inspired by the properties of scalar matrices, the author suggests to relate the qualitative and quantitative properties of the operator matrix to its Schur complements. The first one of the latter can be employed whenever D is invertible and is the following operator family in $\mathcal{H}_1$

$$\begin{aligned} S_1(\lambda) &= A - \lambda - B(D-\lambda)^{-1}C, \qquad \lambda \in \rho(D), \\ \operatorname{dom} S_1(\lambda) &= \{f \in \operatorname{dom} A \cap \operatorname{dom} C : (D-\lambda)^{-1}Cf \in \operatorname{dom} B\}; \end{aligned}$$

the second Schur complement $S_2(\cdot)$ is defined analogously. Indeed, for $\lambda \in \rho(D)$, a relation between the inverses of $\mathcal{A}-\lambda$ and $S_1(\lambda)$ is provided by the Frobenius-Schur factorisation of the matrix resolvent, which is the following formal identity

$$\begin{aligned} &(\mathcal{A}-\lambda)^{-1} = \qquad\qquad (1.10) \\ &= \begin{pmatrix} S_1(\lambda)^{-1} & -S_1(\lambda)^{-1}B(D-\lambda)^{-1} \\ -(D-\lambda)^{-1}CS_1(\lambda)^{-1} & (D-\lambda)^{-1} + (D-\lambda)^{-1}CS_1(\lambda)^{-1}B(D-\lambda)^{-1} \end{pmatrix}. \end{aligned}$$

In order to give sense to the above representation on operator level, typically suitable patterns of relative boundedness are assumed within the matrix entries. For the notion of relative boundedness, which essentially measures the unboundedness of an operator with respect to another, we refer the reader to e.g. [Kat95, Sec. IV.1.1]. The first result of this type employs both Schur complements under the assumption that the off-diagonal of $\mathcal{A}$ is relatively bounded with respect to its diagonal.

Theorem 1.5 ([Nag89, Thm. 2.4])**.** *Let $\mathcal{A}$ be closed and let C and B, respectively, be A- and D-bounded. Then on the common resolvent set of the diagonal elements, the spectra of $\mathcal{A}$ and both its Schur complements coincide, i.e.*

$$\sigma(\mathcal{A}) \cap \rho(A) \cap \rho(D) = \sigma(S_1(\cdot)) \cap \rho(A) = \sigma(S_2(\cdot)) \cap \rho(D). \qquad (1.11)$$

In the above, the spectrum of an operator family is defined as usual, e.g.

$$\sigma(S_1(\cdot)) = \{\lambda \in \rho(D) : 0 \in \sigma(S_1(\lambda))\}.$$

Without going into further details, we mention that imposing various other dominance patterns and employing one or both Schur complements, also the closedness and closability of $\mathcal{A}$, as well as different parts of its spectrum (point, continuous, residual, essential), can be related equivalently to the analogous notions for the considered Schur complement. Moreover, similar results hold employing the so-called quadratic complements of the operator matrix. For a concise collection of these spectral equivalence results, we refer the reader to [Tre08, Sec. 2.2 – 2.4].

Due to this approach to the analysis of the matrix via its Schur complements, one can find a closed operator matrix realisation of $\mathcal{A}$ with non-empty resolvent set for instance by applying the representation theorems in Section 1.2.3 to $S_1(\lambda)$ and $S_2(\lambda)$ with a suitable $\lambda \in \rho(A) \cap \rho(D)$. The various parts of the spectrum of the operator matrix can then further be analysed in terms of the Schur complements. Moreover, by formula (1.10) and its analogue for $S_2(\cdot)$, even the resolvent norm of $\mathcal{A}$ can be related to the latter.

1.4. Contributions

In this section, we briefly sketch the main ideas behind the works [Ger21, GT21, GS21] which constitute this thesis, and put them in relation to the issues and methods in the previous introductory section. A more extensive and detailed summary of the presented research work can be found in Chapters 2 – 4.

1.4.1. Schur complement dominance. It is crucial to observe that in order to rigorously justify the Frobenius-Schur factorisation of the resolvent, the matrix entries themselves do not need to be in direct relation to each other. Indeed, it suffices that e.g. the first Schur complement dominates the neighbouring factors in (1.10) in a suitable sense. This approach, which was applied successfully in [EL07, FST18, IST16, Ibr17, IT17] to particular problems, inspires us to introduce a new abstract framework for the study of operator matrices in Chapter 2 below. Besides the new non-standard dominance structure therein, we go even further and do not require the matrix entries to act as operators in the respective Hilbert spaces (which in particular allows for problems with distributional coefficients).

The idea of extending the entries to take values in spaces of distributions has been employed before in several matrix problems, e.g. for damped systems in [JTTV18, AN15]. In fact, this method is already hidden behind the representation theorems in Section 1.2.3 and has proven to be essential whenever a formal sum of unbounded operators needs to be implemented. However, so far only the work [EL07] seems to share our approach to determine the distributional spaces according to the properties of the Schur complement. Indeed, in other works the involved spaces have consistently been determined by the entries themselves according to the underlying patterns of relative boundedness (e.g. as the form domain of one of the entries and its dual space in [JTTV18, AN15]); see Section 2.1 for more details on the above.

We point out that the operator matrices constructed by our method (together with e.g. the representation theorems in Section 1.2.3) in general do not have a domain which decomposes as in (1.9) with respect to the underlying product space, compare also [Nag90] for operator matrices whose domain is non-diagonal due to the imposed boundary conditions.

1.4.2. The pseudo (quadratic) numerical range. Having established a spectral equivalence between an operator matrix $\mathcal{A}$ and e.g. its first Schur complement $S_1(\cdot)$ on $\rho(D)$ similarly to (1.11), there are several possibilities to employ numerical ranges for locating the spectrum of $\mathcal{A}$. The most straightforward approach is to directly study the numerical range of the matrix itself; nonetheless, since the latter is convex and contains the numerical ranges of the diagonal elements, it can be quite large and might not lead to satisfactory results.

Another (often more successful) possibility to locate at least the part of $\sigma(\mathcal{A})$ in $\rho(D)$ is to employ the numerical range of its Schur complement,

$$W(S_1(\cdot)) = \{\lambda \in \rho(D) : 0 \in W(S_1(\lambda))\}.$$

For non-trivial operator families, however, the relation between the numerical range and the spectrum can be more complicated than in the operator case discussed in Section 1.2.4. While the numerical range still contains the eigenvalues, its closure might fail to include the approximate point spectrum. In order to overcome this, we introduce the new notion of pseudo numerical range for families of linear operators.

With this new notion, indeed both the enclosure property for the approximate point spectrum, as well as the relation to the resolvent norm prevail in general, see Section 3.2.1 below.

For operator matrices with diagonal domain, i.e. if $\mathcal{A}$ admits a matrix representation as in (1.9) in the underlying product space $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, the quadratic numerical range can provide a tighter spectral enclosure than the numerical range. By its definition

$$W^2(\mathcal{A}) = \bigcup_{(f,g)\in \operatorname{dom}\mathcal{A},\, \|f\|_{\mathcal{H}_1}=\|g\|_{\mathcal{H}_2}=1} \sigma_{\mathrm{p}} \begin{pmatrix} \langle Af, f\rangle_{\mathcal{H}_1} & \langle Bg, f\rangle_{\mathcal{H}_1} \\ \langle Cf, g\rangle_{\mathcal{H}_2} & \langle Dg, g\rangle_{\mathcal{H}_2} \end{pmatrix}, \tag{1.12}$$

the quadratic numerical range exploits the matrix structure of $\mathcal{A}$ and is indeed contained in the usual numerical range. The above notion was originally introduced in [LT98] (for bounded off-diagonal entries, see also [Tre09, TW03, RT18] for further generalisations).

Nevertheless, even though the point spectrum remains inside the quadratic numerical range, the approximate point spectrum is in general no longer contained in the closure of the latter. Unlike for the numerical range, this might happen not only for families of operator matrices, but already in the operator matrix case. We resolve this issue by introducing the pseudo quadratic numerical range for operator matrix families (which trivially includes the operator matrix case) along the lines of the pseudo numerical range. Indeed, this new notion shares both the spectral enclosure property and the connection to the resolvent norm with its scalar analogue, see Section 3.2.4 below.

1.4.3. Schrödinger operators with accretive potentials in weighted spaces. In Chapter 4 below, we employ the form representation Theorem 1.4 to construct Dirichlet realisations of Schrödinger-type operators

$$A_w = -\nabla \cdot (P\nabla) + V \tag{1.13}$$

with sectorial matrix functions P and accretive potentials V in suitably weighted function spaces. In our analysis, the admissible weights w have to satisfy certain regularity and growth conditions related to the coefficients, see Assumption 4.1. Depending on the imposed admissible weight w, we investigate the spectral properties of the operators A_w; in particular, we study the Schatten class of their resolvent, prove the invariance of their discrete spectra and eigenfunctions with respect to the weight and describe their operator domain.

Concerning the spectral analysis of operator matrices, the constructed weighted operators are of significant importance. In applications where the Schur complement is given by operators of the type (1.13), the Frobenius Schur factorisation of the resolvent might contain terms of the form

$$w_1(-\nabla \cdot (P\nabla) + V)^{-1} w_2$$

with multiplication operators w_1 and w_2, see (1.10). In order to give rise to a resolvent operator by the latter formula, a successful approach can be to implement the Schur complement in a suitably weighted space, see also Theorem 4.4 where bounded extensions of operator compositions as above are discussed. We employ this particular method to a second order matrix differential operator in Section 4.3.1 below; notice that for the wave equation in Section 1.5.3, we realise the Schur complement in a weighted space for another reason, namely due to the fact that the underlying Hilbert space itself carries a weight.

1.5. Linearly damped wave equation

We illustrate the results achieved in this thesis by revisiting a particular spectral problem several times throughout the subsequent chapters, see Sections 2.2, 3.3, and 4.2 below. More precisely, we consider a linearly damped wave equation

$$\partial_t^2 u(t,x) + 2a(x)\partial_t u(t,x) = (\Delta_x - q(x))u(t,x), \quad t > 0, \quad x \in \Omega, \tag{1.14}$$

with non-negative damping a and potential q on an open set $\Omega \subseteq \mathbb{R}^n$. After standard transformations, the corresponding Cauchy problem reads

$$\partial_t \begin{pmatrix} u_1(t,x) \\ u_2(t,x) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Delta_x - q(x) & -2a(x) \end{pmatrix} \begin{pmatrix} u_1(t,x) \\ u_2(t,x) \end{pmatrix}, \tag{1.15}$$

where the matrix differential operator on its right hand side is traditionally implemented in a certain product Hilbert space $\mathcal{W}(\Omega) \oplus L^2(\Omega)$, see (2.2).

1.5.1. Semigroup generation. The first goal is to find a Dirichlet realisation of the operator matrix in (1.15) without imposing any relation between damping and potential, under the sole assumption that the latter are locally integrable (which is the natural minimal assumption to apply the standard form representation methods in Section 1.2.3 to the Schur complement).

While the case where the damping is relatively bounded with respect to $\Delta - q$ has been studied extensively, see e.g. [AN15, JTTV18], to our knowledge non-standard dominance patterns allowing stronger dampings have been considered only in the works [FST18, IT20] so far. In [FST18], the spectral equivalence to the dominant second Schur complement and the Lax-Milgram representation theorem were employed to define the operator matrix and establish its semigroup generation. However, since no distributional approach was taken, the operator domain of the Schur complement had to be described, resulting in seemingly unnatural restrictions on the regularity and growth of the damping, see [FST18, Asm. I]. In the second work [IT20] on the other hand, a different (non-spectral theoretic) approach was taken to construct a unique weak solution and show a polynomial decay of its norm in time.

Assuming merely the minimal assumptions mentioned above, we use our new results on Schur complement dominance to introduce a Dirichlet realisation of the operator matrix

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ \Delta - q & -2a \end{pmatrix}$$

in the space $\mathcal{W}(\Omega) \oplus L^2(\Omega)$. We therefore employ the spectral equivalence to its second Schur complement, which is the Dirichlet realisation (due to the Lax-Milgram theorem) of

$$S_2(\lambda) = -\frac{1}{\lambda}(-\Delta + q + 2\lambda a + \lambda^2), \qquad \lambda \in \mathbb{C} \setminus \{0\}, \tag{1.16}$$

acting in the space $L^2(\Omega)$. We show that not only is the resulting operator matrix closed, densely defined and has non-empty resolvent set, but that actually $-\mathcal{A}$ is m-accretive and $\mathcal{A}$ thus generates a strongly continuous contraction semigroup according to the Hille-Yosida theorem. Under the more restrictive assumptions imposed in [FST18], we show that the generator defined therein coincides with our realisation of the operator matrix; for more details, see Section 2.2 below.

1.5.2. Spectral enclosure. By the above described spectral equivalence, the matrix problem can be reduced to the analysis of the Schur complement in (1.16). In Section 3.3 below, we derive an enclosure for the spectrum of the latter by employing its pseudo numerical range. To this end, we assume a certain form subordination property (3.5) of the damping with respect to the term $-\Delta+q$. Notice that this assumption implies relative form-boundedness of order zero (see [Kat95, Sec. VI.1.5] for relative boundedness of forms) and is used to obtain an estimate for the pseudo numerical range of $S_2(\cdot)$, while no relation between damping and potential is assumed for the spectral correspondence and semigroup generation described in Section 1.5.1.

As mentioned before, instead of studying the Schur complement, one can of course directly investigate the operator matrix. In general, however, the numerical range of $\mathcal{A}$ is the whole left complex half plane (even if the damping is form-subordinate as described above). The quadratic numerical range of $\mathcal{A}$, on the other hand, can give much tighter enclosures, see [JTTV18] for the case of accretive $(-\Delta+q)$-bounded damping.

1.5.3. Accretive damping in a weighted space. Combining our results on Schur complement dominance with the weighted Schrödinger operators constructed in Chapter 4, we can study the case of accretive dampings in weighted spaces. More precisely, in Section 4.2 below, we consider another wave equation

$$u_{tt}(t,x) + 2(a_1(x) + \mathrm{i}a_2(x) - \nabla_x \cdot (a_0(x)\nabla_x))u_t(t,x) = \Delta_x u(t,x) \tag{1.17}$$

for $x \in \Omega$ and $t > 0$. While in comparison to (1.14), we choose the potential to be zero, on the other hand we allow accretive (differential) damping, i.e. we assume a_1 to be non-negative and the matrix function a_0 to be positive semi-definite. On the right hand side of the corresponding Cauchy problem, we find the following matrix differential expression

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ \Delta & -2(a_1 + \mathrm{i}a_2 - \nabla \cdot (a_0\nabla)) \end{pmatrix}.$$

For a suitable weight w, we introduce a Dirichlet realisation of $\mathcal{A}$ in the weighted product space $\mathcal{W}_w(\Omega) \oplus L^2_{w^2}(\Omega)$, see (4.13) for the weighted analogue of $\mathcal{W}(\Omega)$, and show that it generates a strongly continuous semigroup. We therefore employ the dominance of its second Schur complement

$$S_2(\lambda) = -\frac{1}{\lambda}(-\nabla \cdot ((I_{\mathbb{C}^n} + \lambda a_0)\nabla) + 2\lambda(a_1 + \mathrm{i}\,a_2) + \lambda^2), \qquad \lambda \in \mathbb{C} \setminus \{0\},$$

which we implement in the weighted space $L^2_{w^2}(\Omega)$ by means of Theorem 4.2 below. The arising conditions on the damping and the admissible weight can be found in (4.12), (4.14) and (4.16).

CHAPTER 2

Schur complement dominant operator matrices

The results in this chapter are based on the research paper [Ger21]. While in Section 2.1 the abstract results obtained in the latter are summarised, in Section 2.2 we sketch their application to the damped wave equation. Section 2.3 contains a layout of further applications to second order matrix differential operators with unbounded and/or singular coefficients and to Klein-Gordon and Dirac operators. We stress that not all results from [Ger21] are included below and the ones mentioned are widely simplified.

2.1. Main results

We implement (unbounded) operator matrices $\mathcal{A}_0$ acting in the orthogonal sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ of two complex Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. More precisely, we identify a domain $\operatorname{dom} \mathcal{A}_0$ such that the resulting operator is densely defined, closed and has a non-empty resolvent set. To this end, we establish an equivalence of (point and essential) spectra of $\mathcal{A}_0$ and its first Schur complement $S_0(\cdot)$, which is a family of (unbounded) operators acting in $\mathcal{H}_1$, see Theorem 2.3 below. Note that we employ only one Schur complement (which we denote by $S_0(\cdot)$). The presented results hold accordingly for the second Schur complement and can be translated in a straightforward way, see [Ger21, Rem. 2.4].

The core concepts of our approach are on one hand the dominance of the Schur complement over neighbouring terms in the Frobenius-Schur factorisation of the resolvent. On the other hand, we extend the action of the matrix entries and of the Schur complement to certain distributional triplets of Hilbert spaces

$$\mathcal{D}_S \oplus \mathcal{D}_2 \subseteq \mathcal{H} \subseteq \mathcal{D}_{-S} \oplus \mathcal{D}_{-2}, \qquad \mathcal{D}_S \subseteq \mathcal{H}_1 \subseteq \mathcal{D}_{-S},$$

before restricting the resulting operators to their maximal domains in the original spaces $\mathcal{H}$ and $\mathcal{H}_1$.

2.1.1. Assumptions.

Our simplified set of Assumptions engaging the first Schur complement is the following.

Assumption 2.1 ([Ger21, Asm. 3.1]). (i) Let $\mathcal{D}_S$, $\mathcal{D}_2$, $\mathcal{D}_{-S}$ and $\mathcal{D}_{-2}$ be complex Hilbert spaces satisfying the inclusions

$$\mathcal{D}_S \subseteq \mathcal{H}_1 \subseteq \mathcal{D}_{-S}, \qquad \mathcal{D}_2 \subseteq \mathcal{H}_2 \subseteq \mathcal{D}_{-2},$$

with continuous canonical embeddings which have dense ranges.

(ii) Let A, B and C be bounded between the following spaces

$$A \in \mathcal{B}(\mathcal{D}_S, \mathcal{D}_{-S}), \qquad B \in \mathcal{B}(\mathcal{D}_2, \mathcal{D}_{-S}), \qquad C \in \mathcal{B}(\mathcal{D}_S, \mathcal{D}_{-2}).$$

(iii) Let $D_0 \in \mathcal{C}(\mathcal{H}_2)$ such that $\operatorname{dom} D_0 \subseteq \mathcal{D}_2$ is dense in $\mathcal{D}_2$ and assume there exists an extension $D \supseteq D_0$ with
$$D \in \mathcal{B}(\mathcal{D}_2, \mathcal{D}_{-2}), \qquad (D-\lambda)^{-1} \in \mathcal{B}(\mathcal{D}_{-2}, \mathcal{D}_2), \qquad \lambda \in \rho(D_0) \neq \emptyset. \qquad //$$

Note that the assumptions in [Ger21, Asm. 3.1] are more general. In fact, we allow A and B to map to a larger space $\mathcal{D}_{-1} \supseteq \mathcal{D}_{-S}$ and thus widen the range of applicability of our results; see Sections 2.3.1 and 2.3.3 where $\mathcal{D}_{-S}$ is a proper subspace of $\mathcal{D}_{-1}$. Moreover, instead of (iii) above we merely assume that $D_0 - \lambda$ has a generalised inverse with a suitable extension property, see [Ger21, Equ. (3.2)].

While several applications in [Ger21, GS21] suggest that $\mathcal{D}_S$ and $\mathcal{D}_{-S}$, respectively, are given as the form domain of the Schur complement and its anti-dual space, this is not always the case. Indeed, this can be seen in [Ger21, Sec. 7.2] or Section 4.3.1 below, where in the latter, $\mathcal{D}_S$ is the operator domain of the Schur complement, which is implemented in the weighted space $\mathcal{D}_{-S}$.

2.1.2. Definition of matrix and Schur complement. The operator matrix and its Schur complement are initially defined as bounded operators (with values in distributional spaces). However, we point out that the resulting operators $\mathcal{A}_0$ and $S_0(\cdot)$ which act in $\mathcal{H}$ and $\mathcal{H}_1$ are in general unbounded.

Definition 2.2 ([Ger21, Def. 3.2])**.** We define the operator matrix
$$\mathcal{A} := \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{B}(\mathcal{D}_S \oplus \mathcal{D}_2, \mathcal{D}_{-S} \oplus \mathcal{D}_{-2}),$$
and its (first) Schur complement
$$S(\lambda) := A - B(D-\lambda)^{-1}C \in \mathcal{B}(\mathcal{D}_S, \mathcal{D}_{-S}), \qquad \lambda \in \rho(D_0),$$
acting in the respective triplets
$$\mathcal{D}_S \oplus \mathcal{D}_2 \subseteq \mathcal{H} \subseteq \mathcal{D}_{-S} \oplus \mathcal{D}_{-2}, \qquad \mathcal{D}_S \subseteq \mathcal{H}_1 \subseteq \mathcal{D}_{-S}.$$
The operators $\mathcal{A}_0$ and $S_0(\cdot)$ are then defined as the maximal restrictions of $\mathcal{A}$ and $S(\cdot)$ to the original spaces $\mathcal{H}$ and $\mathcal{H}_1$, respectively, i.e. to their respective domains
$$\begin{aligned} \operatorname{dom} \mathcal{A}_0 &:= \{(f,g) \in \mathcal{D}_S \times \mathcal{D}_2 : \mathcal{A}(f,g) \in \mathcal{H}\}, \\ \operatorname{dom} S_0(\lambda) &:= \{f \in \mathcal{D}_S : S(\lambda) f \in \mathcal{H}_1\}, \quad \lambda \in \rho(D_0). \end{aligned} \qquad //$$

2.1.3. Main result. Our main result is the equivalence of (point and essential) spectra between the operator matrix and its Schur complement, which allows to construct the desired densely defined, closed realisation of $\mathcal{A}_0$ with non-empty resolvent set; here we consider the second out of five definitions of essential spectra in the sense of [EE87, Chap. IX].

Theorem 2.3 ([Ger21, Cor. 3.4. (ii), Cor. 3.5–3.6])**.** *Let Assumption* 2.1 *be satisfied and $\mathcal{A}_0$ and $S_0(\cdot)$ be as in Definition* 2.2*. If, for every $\lambda \in \Sigma \subseteq \rho(D_0)$, there exists $z_\lambda \in \mathbb{C}$ such that*
$$(S(\lambda) - z_\lambda)^{-1} \in \mathcal{B}(\mathcal{D}_{-S}, \mathcal{D}_S), \tag{2.1}$$
then the (point and essential) spectra of $\mathcal{A}_0$ and $S_0(\cdot)$ coincide on Σ, i.e.
$$\sigma_{(\mathrm{p/e2})}(\mathcal{A}_0) \cap \Sigma = \sigma_{(\mathrm{p/e2})}(S_0(\cdot)) \cap \Sigma.$$
Moreover, if $\rho(S_0(\cdot)) \cap \Sigma \neq \emptyset$, then $\operatorname{dom} \mathcal{A}_0$ is dense both in $\mathcal{D}$ and $\mathcal{H}$. In particular, this implies that $\mathcal{A}_0$ is densely defined, closed and has non-empty resolvent set.

Notice that the equivalence of point spectra and one implication

$$\lambda \in \sigma(S_0(\cdot)) \implies \lambda \in \sigma(\mathcal{A}_0)$$

also holds for $\lambda \in \rho(D_0) \setminus \Sigma$ without (2.1), see [Ger21, Prop. 2.6, Thm. 2.7 (i)] and also Theorems 2.4 and 2.6 below. If $\mathcal{D}_S$ is the form domain of the Schur complement and $\mathcal{D}_{-S}$ is its anti-dual, then condition (2.1) can be established e.g. by the representation theorems in Section 1.2.3. The last claim of the theorem in particular holds if z_λ can be chosen zero for some $\lambda \in \Sigma$.

Actually, instead of (2.1) it suffices that $S_0(\lambda) - z_\lambda$ has a generalised inverse with a suitable extension property, see [Ger21, Cor. 2.9, Rem. 2.10]. Moreover, we derive another sufficient condition for the density of $\operatorname{dom} \mathcal{A}_0$ in [Ger21, Cor. 3.4 (i)].

2.2. Semigroup generation for damped wave equations with singular and/or unbounded damping and potential

For details concerning this section, see [Ger21, Sec. 4]. We employ Schur complement dominance to introduce a Dirichlet realisation of the matrix differential expression in (1.15) on $\Omega \subseteq \mathbb{R}^n$ in a suitable Hilbert space. Under the weak regularity assumption that the damping a and potential q are non-negative and locally integrable (which allows for strong dampings unrelated to the potential), the latter generates a strongly continuous contraction semigroup, see Theorem 2.4 below.

We implement $\mathcal{A}_0$ in the standard choice Hilbert space $\mathcal{H} = \mathcal{W}(\Omega) \oplus L^2(\Omega)$, where $\mathcal{W}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to

$$\langle f, g\rangle_{\mathcal{W}} := \langle \nabla f, \nabla g\rangle_{L^2} + \langle q^{\frac{1}{2}} f, q^{\frac{1}{2}} g\rangle_{L^2}, \qquad f, g \in C_0^\infty(\Omega); \tag{2.2}$$

the space $\mathcal{W}(\Omega) \oplus L^2(\Omega)$ was chosen also in [FST18] and it coincides with the considered Hilbert space in [AN15, Chap. 1.2.1] or [JTTV18] if the potential is uniformly bounded below. We define the matrix and its Schur complement as

$$\mathcal{A}_0 := \begin{pmatrix} 0 & I \\ \Delta - q & -2a \end{pmatrix}, \quad S_0(\lambda) := -\frac{1}{\lambda}(-\Delta + q + 2\lambda a + \lambda^2), \quad \lambda \in \mathbb{C} \setminus \{0\}, \tag{2.3}$$

in $\mathcal{W}(\Omega) \oplus L^2(\Omega)$ and $L^2(\Omega)$, respectively, on their domains

$$\begin{aligned} \operatorname{dom} \mathcal{A}_0 &:= \{(f, g) \in \mathcal{W}(\Omega) \times \mathcal{D}_S : (\Delta - q)f - 2ag \in L^2(\Omega)\}, \\ \operatorname{dom} S_0(\lambda) &:= \{f \in \mathcal{D}_S : (\Delta - q - 2\lambda a)f \in L^2(\Omega)\}. \end{aligned} \tag{2.4}$$

Here $\mathcal{D}_S$ is the form domain of the Schur complement

$$\mathcal{D}_S := H_0^1(\Omega) \cap \operatorname{dom} q^{\frac{1}{2}} \cap \operatorname{dom} a^{\frac{1}{2}}.$$

Theorem 2.4 ([Ger21, Thm. 4.2])**.** *Let $a, q \in L^1_{\mathrm{loc}}(\Omega)$ with $a, q \geq 0$ a.e. in Ω and let $\mathcal{A}_0$ and $S_0(\cdot)$ be as in* (2.3) *and* (2.4)*. Then $\mathcal{A}_0$ generates a C_0-contraction semigroup on $\mathcal{W}(\Omega) \oplus L^2(\Omega)$ and $\operatorname{dom} \mathcal{A}_0$ is dense in $\mathcal{W}(\Omega) \oplus \mathcal{D}_S$. Moreover, the (point and essential) spectra of $\mathcal{A}_0$ and $S_0(\cdot)$ coincide on $\mathbb{C} \setminus (-\infty, 0]$,*

$$\sigma_{(\mathrm{p/e2})}(\mathcal{A}_0) \setminus (-\infty, 0] = \sigma_{(\mathrm{p/e2})}(S_0(\cdot)) \setminus (-\infty, 0],$$

and on $(-\infty, 0)$ we have

$$\begin{aligned} \sigma(\mathcal{A}_0) \cap (-\infty, 0) &\supseteq \sigma(S_0(\cdot)) \cap (-\infty, 0), \\ \sigma_{\mathrm{p}}(\mathcal{A}_0) \cap (-\infty, 0) &= \sigma_{\mathrm{p}}(S_0(\cdot)) \cap (-\infty, 0). \end{aligned}$$

If [FST18, Asm. I] is satisfied, our realisation $\mathcal{A}_0$ coincides with the semigroup generator G introduced in [FST18], see [Ger21, Rem. 4.3]. We also mention that our approach allows for distributional dampings like e.g. $a(x) = \delta(x - x_0)$ considered in [AN15, Chap. 4].

2.3. Further applications

We demonstrate our results based on three other spectral problems addressed in [Ger21]. While Section 2.3.1 is more substantial, the applications in Sections 2.3.2 and 2.3.3 are more of illustrative nature and are only briefly mentioned.

2.3.1. Second order matrix differential operators with singular and/or unbounded coefficients. In [Ger21, Sec. 6], we study second order differential operator matrices of the form

$$\mathcal{A}_0 := \begin{pmatrix} -\Delta + q & \nabla \cdot \mathbf{b} \\ \mathbf{c} \cdot \nabla & d \end{pmatrix} \tag{2.5}$$

in the Hilbert space $L^2(\Omega) \oplus L^2(\Omega)$ with an open set $\Omega \subseteq \mathbb{R}^n$ and Dirichlet boundary conditions, see also [BM20, Ibr17, IST16, IT17, Kon98, KLN08] for problems of similar type. Without imposing further restrictions on the coefficients, the matrix entries lack any structure of relative boundedness between them, which in previous results enforced assumptions on either the particular structure or the regularity of the coefficients, see e.g. [IST16] where latter were essentially assumed to be continuously differentiable.

We employ Theorem 2.3 to establish suitable realisations of $\mathcal{A}_0$ and its first Schur complement

$$S_0(\lambda) := -\nabla \cdot (\boldsymbol{\pi}(\lambda)\nabla) + q - \lambda, \qquad \lambda \in \Theta \subseteq \mathbb{C} \setminus \operatorname{ess\,ran} d, \tag{2.6}$$

see (2.7), and in Theorem 2.6 we show the equivalence of their spectra under fairly general assumptions. In case that q and d are sectorial and $\mathbf{c} = \overline{\mathbf{b}}$, we show in Theorem 2.7 that $\mathcal{A}_0$ generates a strongly continuous contraction semigroup.

2.3.1.1. *Spectral equivalence.* Due to our distributional approach, we are able to reduce the imposed regularity of the coefficients to a minimum which is required to ensure a dominant Schur complement and the applicability of standard sesquilinear form methods to the latter.

Assumption 2.5 ([Ger21, Asm. 6.1]**).** Assume that the following hold.

(i) *Basic assumptions on coefficients*: Let $\mathbf{b}, \mathbf{c} : \Omega \to \mathbb{C}^n$ be measurable and
$$q \in L^1_{\mathrm{loc}}(\Omega), \qquad d \in L^\infty_{\mathrm{loc}}(\Omega).$$

(ii) *Regularity of $\boldsymbol{\pi}$ on Θ*: Let $\Theta \subseteq \mathbb{C} \setminus \operatorname{ess\,ran} d$ be connected and assume
$$\boldsymbol{\pi}(\lambda) := I_{\mathbb{C}^n} + (d - \lambda)^{-1}(\mathbf{b} \otimes \mathbf{c}) \in L^1_{\mathrm{loc}}(\Omega)^{n \times n}, \qquad \lambda \in \Theta. \tag{2.7}$$

(iii) *Sectoriality of Schur complement on Φ*: For the set $\emptyset \neq \Phi \subseteq \Theta$ and all $\lambda \in \Phi$, assume there exist $\omega_\lambda \in (-\pi, \pi]$ and $\gamma_\lambda \geq 0$ such that both
$$\widetilde{q}(\lambda) := \mathrm{e}^{\mathrm{i}\,\omega_\lambda}\, q + \gamma_\lambda, \qquad \widetilde{\boldsymbol{\pi}}(\lambda) := \mathrm{e}^{\mathrm{i}\,\omega_\lambda}\, \boldsymbol{\pi}(\lambda),$$

are sectorial (with vertex zero) in the sense of [Kat95, Sec. V.3.10] and that $\operatorname{Re}\widetilde{\boldsymbol{\pi}}(\lambda)$ is positive definite. For all $\lambda, \mu \in \Phi$, let $m_{\lambda,\mu}, M_{\lambda,\mu} > 0$ such that

$$\begin{aligned} m_{\lambda,\mu} \operatorname{Re}\widetilde{q}(\mu) \;&\leq\; \operatorname{Re}\widetilde{q}(\lambda) \;\leq\; M_{\lambda,\mu} \operatorname{Re}\widetilde{q}(\mu), \\ m_{\lambda,\mu} \operatorname{Re}\widetilde{\boldsymbol{\pi}}(\mu) \;&\leq\; \operatorname{Re}\widetilde{\boldsymbol{\pi}}(\lambda) \;\leq\; M_{\lambda,\mu} \operatorname{Re}\widetilde{\boldsymbol{\pi}}(\mu). \end{aligned} \tag{2.8}$$

(iv) *Dominance of Schur complement*: For all $\lambda \in \Phi$, assume that

$$(d-\lambda)^{-1} \max\Big(\big|(\operatorname{Re}\widetilde{\boldsymbol{\pi}}(\lambda))^{-\frac{1}{2}}\mathbf{b}\big|, \big|(\operatorname{Re}\widetilde{\boldsymbol{\pi}}(\lambda))^{-\frac{1}{2}}\overline{\mathbf{c}}\big| \Big) \in L^\infty(\Omega). \qquad //$$

The regularity assumptions in (i) and (ii) above essentially ensure that the sesquilinear form of the Schur complement is densely defined, while its sectoriality (after shift and rotation) for parameters $\lambda \in \Phi$ is provided by (iii). Property (2.1) can then be established for such λ by the standard representation theorem for m-sectorial operators, see [Kat95, Thm. VI.2.1]. The required dominance of the Schur complement, however, is guaranteed by the boundedness of the combination in (iv). Note that the local boundedness of d is assumed for the sake of simplicity and can be relaxed; see [Ger21, Rem. 6.2] for more details on the above.

Under the assumptions above, our realisation of the operator matrix $\mathcal{A}_0$ in (2.5) and its Schur complement $S_0(\cdot)$ in (2.6) are defined to act on the domains

$$\begin{aligned} \operatorname{dom}\mathcal{A}_0 &:= \big\{(f,g) \in \mathcal{D}_S \times \mathcal{D}_2 \,:\, (\Delta - q)f - \nabla\cdot\mathbf{b}g \in L^2(\Omega), \\ &\qquad\qquad\qquad\qquad\qquad \mathbf{c}\cdot\nabla f + dg \in L^2(\Omega)\big\}, \\ \operatorname{dom} S_0(\lambda) &:= \{f \in \mathcal{D}_S : (\nabla\cdot(\boldsymbol{\pi}(\lambda)\nabla) - q)f \in L^2(\Omega)\}; \end{aligned} \tag{2.9}$$

here the space $\mathcal{D}_S$ is the closure of $C_0^\infty(\Omega)$ with respect to

$$\|f\|_S^2 := \|(\operatorname{Re}\widetilde{\boldsymbol{\pi}}(\lambda_0))^{\frac{1}{2}}\nabla f\|_{L^2}^2 + \|(\operatorname{Re}\widetilde{q}(\lambda_0))^{\frac{1}{2}} f\|_{L^2}^2 + \|f\|_{L^2}^2, \quad f \in C_0^\infty(\Omega),$$

where $\lambda_0 \in \Phi$ is fixed arbitrarily and $\mathcal{D}_2$ is the weighted space

$$\mathcal{D}_2 := L^2(\Omega, |d-\lambda_0|^2\omega^{-2}), \qquad \omega := \max\Big(1, |(\operatorname{Re}\boldsymbol{\pi}_0)^{-\frac{1}{2}}\overline{\mathbf{c}}|\Big).$$

We point out that (as topological spaces) neither $\mathcal{D}_S$ nor $\mathcal{D}_2$, and thus neither $\mathcal{A}_0$ nor $S_0(\cdot)$, depend on the choice of λ_0, see (2.8) and [Ger21, Lem. 6.11].

Theorem 2.6 ([Ger21, Thm. 6.3]**).** *Under Assumption* 2.5*, the (point and essential) spectra of* $\mathcal{A}_0$ *and* $S_0(\cdot)$*, defined as in* (2.5)*,* (2.6) *and* (2.9)*, are equivalent on the set* Φ*, i.e.*

$$\sigma_{(\mathrm{p}/\mathrm{e}2)}(\mathcal{A}_0) \cap \Phi = \sigma_{(\mathrm{p}/\mathrm{e}2)}(S_0(\cdot)) \cap \Phi. \tag{2.10}$$

On the remaining part of Θ*, they satisfy*

$$\begin{aligned} \sigma_{\mathrm{p}}(\mathcal{A}_0) \cap (\Theta \setminus \Phi) &= (\Theta \setminus \Phi) \cap \sigma_{\mathrm{p}}(S_0(\cdot)), \\ \sigma(\mathcal{A}_0) \cap (\Theta \setminus \Phi) &\supset (\Theta \setminus \Phi) \cap \sigma(S_0(\cdot)). \end{aligned} \tag{2.11}$$

A sufficient condition on the density of $\operatorname{dom}\mathcal{A}_0$ in $L^2(\Omega) \oplus L^2(\Omega)$ can be found in [Ger21, Thm. 6.3].

2.3.1.2. *Semigroup generation.* We consider the case that $\mathbf{c} = \overline{\mathbf{b}}$ and that q and d, respectively, are sectorial with semi-angle θ_q and θ_d (and vertex zero) in the sense of [Kat95, Sec. V.3.10]. In this setting, the Schur complement is sectorial (without rotation or shift) and $\mathcal{A}_0$ and $S_0(\cdot)$ can be defined as above with the choice

$$\lambda_0 := -1, \qquad \widetilde{\boldsymbol{\pi}} := \boldsymbol{\pi}, \qquad \widetilde{q} := q. \tag{2.12}$$

The analogous spectral equivalence to Theorem 2.6 (on certain sectors Θ and Φ below) then leads to the m-accretivity of $-\mathcal{A}_0$, and thus to the semigroup generation.

Theorem 2.7 ([Ger21, Thm. 6.5]). *Let Assumption* 2.5 (i) *hold with* $\mathbf{c} = \overline{\mathbf{b}}$ *and sectorial q and d as above. Let $\mathcal{A}_0$ and $S_0(\cdot)$ be as in* (2.5), (2.6) *and* (2.9) *with* (2.12) *therein, where $\boldsymbol{\pi}$ is as in* (2.7). *If the following holds*

$$(d+1)^{-1}(\mathbf{b} \otimes \overline{\mathbf{b}}) \in L^1_{\mathrm{loc}}(\Omega)^{n \times n}, \quad (d+1)^{-1}(\operatorname{Re} \boldsymbol{\pi}(-1))^{-\frac{1}{2}} \mathbf{b} \in L^\infty(\Omega)^n, \tag{2.13}$$

then $\mathcal{A}_0$ generates a strongly continuous contraction semigroup on $L^2(\Omega) \oplus L^2(\Omega)$, its domain is dense in $L^2(\Omega) \oplus L^2(\Omega)$ and the relations (2.10) *and* (2.11) *hold with*

$$\Theta := \{\lambda \in \mathbb{C} : |\arg \lambda| > \theta_d\}, \qquad \Phi := \{\lambda \in \mathbb{C} : |\arg \lambda| > \max(\theta_q, \theta_d)\}.$$

We mention that, similarly to (2.13) above, for Theorem 2.6 it suffices that Assumption 2.5 (iv) holds only in an arbitrary point $\lambda_0 \in \Phi$, see [Ger21, Rem. 6.2 (iv)].

2.3.2. Klein-Gordon equations with purely imaginary potential. We consider the Klein-Gordon equation with purely imaginary potential, where the latter is only assumed to be locally square integrable, see [Ger21, Sec. 5]. Our analysis relies on relating the problem to a suitable wave equation and employing the previously established spectral equivalence in Theorem 2.4. We thereby show that the resulting operator matrix is closed, densely defined and boundedly invertible, see [Ger21, Thm. 5.1].

In [Ger21, Ex. 5.2], we conclude that for the special potential $V(x) = \mathrm{i}\,x$ in one dimension, the operator matrix has empty spectrum, which corresponds to the analogous result for the Airy operator in case of the Schrödinger equation.

2.3.3. Dirac operators with potentials satisfying a Hardy-Dirac inequality. Selfadjoint realisations of certain Dirac operators with Coulomb-type potentials were established in [EL07], which we show to be a particular case of our abstract construction. The coercivity of the Schur complement therein is provided by a Hardy-Dirac inequality, which was established in the earlier works [DES00, DELV04]. In [Ger21, Sec. 7.1], we illustrate the underlying spaces and operators in our framework, and thereby recover the result in [EL07]. In addition, we obtain the spectral equivalence to the first Schur complement, see [Ger21, Prop. 7.2, Rem. 7.3].

CHAPTER 3

Pseudo numerical ranges and spectral enclosures

This chapter is based on the work in [GT21] and contains a brief (and non-exhaustive) summary of the latter, where the stated results are mostly simplified. Section 3.2 consists of the main results, which are applied in Section 3.3 to the damped wave equation while in Section 3.4 further results from [GT21] are sketched.

3.1. Preliminaries

We recall crucial notions which were already briefly mentioned in Chapter 1. For a domain $\Theta \subseteq \mathbb{C}$ and a family of linear operators $T(\lambda)$, $\lambda \in \Theta$, in a Hilbert space $\mathcal{H}$, the spectrum and resolvent set are defined as

$$\sigma(T(\cdot)) = \{\lambda \in \Theta : 0 \in \sigma(T(\lambda))\}, \qquad \rho(T(\cdot)) = \mathbb{C} \setminus \sigma(T(\cdot)),$$

and analogously for various parts of the spectrum, as well as the numerical range, see e.g. [Mar88, Sec. 11.2, Sec. 26.3]. Analogously, for a family $\mathcal{L}(\lambda)$, $\lambda \in \Theta$, of linear operators admitting a (λ-dependent) matrix representation with respect to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ of the underlying Hilbert space, see (1.9), the quadratic numerical range is given as

$$W^2(\mathcal{L}(\cdot)) = \{\lambda \in \Theta : 0 \in W^2(\mathcal{L}(\lambda))\} \subseteq W(\mathcal{L}(\cdot));$$

the above notion was introduced in [Tre10] (for holomorphic families of bounded operator matrices), cf. [RTW14] for the block numerical range. Note that, considering the trivial families $T(\lambda) = A - \lambda$ and $\mathcal{L}(\lambda) = \mathcal{A} - \lambda$, $\lambda \in \mathbb{C}$, these definitions include the corresponding ones for linear operators and operator matrices.

3.2. Main results

In order to minimise the notations and best convey the idea behind the introduced notions, we restrict ourselves to the case $n = 1$ and $n = 2$, i.e. to the notions of pseudo numerical and pseudo quadratic numerical range. The definition and properties of the pseudo block numerical range for arbitrary size $n \in \mathbb{N}$ are a straightforward generalisation of the case $n = 2$ and can be found in [GT21, Sec. 4].

3.2.1. The pseudo numerical range. The convenient properties of the numerical range of an operator A in a Hilbert space $\mathcal{H}$ mentioned in Section 1.2.4 are based on the fact that its ε-approximate point spectra are included in the ε-neighbourhoods of the numerical range, i.e.

$$\begin{aligned} \sigma_{\mathrm{ap},\varepsilon}(A) = \{\lambda \in \mathbb{C} : \exists\, f \in \operatorname{dom} A,\ \|f\|_{\mathcal{H}} = 1, \\ \|(A-\lambda)f\|_{\mathcal{H}} < \varepsilon\} \subseteq B_\varepsilon\left(W(A)\right), \end{aligned} \tag{3.1}$$

see e.g [Nev93, Def. 2.2.5]. Aiming to recover the above property in a suitable sense for an operator family $T(\lambda)$, $\lambda \in \Theta$, we introduce the following notion; the corresponding one for families of sesquilinear forms is omitted here and can be found in [GT21, Sec. 2].

Definition 3.1 **([GT21, Def. 2.1]).** The pseudo numerical range of $T(\cdot)$ is defined as the intersection

$$W_\Psi(T(\cdot)) := \bigcap_{\varepsilon>0} W_\varepsilon(T(\cdot)), \quad W_\varepsilon(T(\cdot)) := \bigcup_{\|B\|<\varepsilon} W(T(\cdot)+B), \quad \varepsilon > 0. \qquad //$$

Like its classical counterpart, the pseudo numerical range is in general neither connected nor bounded (even for bounded operator values). By definition, it always contains the classical numerical range and the inclusion might be proper, see [GT21, Ex. 3.2].

The set $W_\varepsilon(T(\cdot))$ above in fact coincides with the ε-pseudo numerical range previously introduced in [ET17, Def. 4.1], leading to the equivalent characterisation (3.3) below. The advantages of our definition of $W_\varepsilon(T(\cdot))$ become evident in higher dimensions, see Section 3.2.4 below.

Proposition 3.2 **([GT21, Prop. 2.3]).** *For every $\varepsilon > 0$, we have*

$$W_\varepsilon(T(\cdot)) = \{\lambda \in \Theta : \exists\ f \in \operatorname{dom} T(\lambda),\ \|f\|_{\mathcal{H}} = 1,\ |\langle T(\lambda)f, f\rangle_{\mathcal{H}}| < \varepsilon\}, \tag{3.2}$$

which then leads to the equivalent characterisation

$$W_\Psi(T(\cdot)) = \{\lambda \in \Theta : 0 \in \overline{W(T(\lambda))}\}. \tag{3.3}$$

The identity (3.3) above provides the invariance of the pseudo numerical range with respect to closures and Friedrichs extensions, see [GT21, Prop. 2.4. (i), (iii)]. We point out that the set in (3.3) was recently used in [BM20] to study linear (non-monic) pencils of operators.

In the simplest case $T(\lambda) = A - \lambda$, $\lambda \in \mathbb{C}$, the pseudo numerical range by (3.3) coincides with the closure of the numerical range. For general families however, $W_\Psi(T(\cdot))$ is neither open nor closed in Θ, see [GT21, Ex. 3.2 (i), Ex. 2.9], and neither the closures nor the interiors of pseudo numerical and classical numerical range in the relative topology of Θ need to coincide, see [GT21, Ex. 3.2].

For the pseudo numerical range, the ε-neighbourhoods in (3.1) are replaced by the sets $W_\varepsilon(T(\cdot))$, implying that the enclosure (1.8) for the approximate point spectrum continues to hold not only for the trivial family $T(\lambda) = A - \lambda$, but also for arbitrary families.

Proposition 3.3 **([GT21, Prop. 3.1]).** *For any $\varepsilon > 0$, we have*

$$\sigma_{\mathrm{ap},\varepsilon}(T(\cdot)) \subseteq W_\varepsilon(T(\cdot)), \quad \|T(\lambda)^{-1}\| \le \frac{1}{\varepsilon}, \quad \lambda \in \rho(T(\cdot)) \setminus W_\varepsilon(T(\cdot)),$$

and hence for the approximate point spectrum that

$$\sigma_{\mathrm{app}}(T(\cdot)) \subseteq W_\Psi(T(\cdot)).$$

Like for the classical numerical range, more assumptions are needed such that $W_\Psi(T(\cdot))$ encloses the full spectrum $\sigma(T(\cdot))$; sufficient conditions for the latter are given in [GT21, Prop. 3.1, Rem. 3.7]

3.2.2. Relation between $W_\Psi(T(\cdot))$ and $\overline{W(T(\cdot))}\cap\Theta$. In general, there is no relation between the pseudo numerical range and the closure of the numerical range in Θ, see [GT21, Ex. 2.9, Ex. 3.2]. Nevertheless, one inclusion holds for families of m-sectorial operators with constant form domain under a certain continuity assumption.

Theorem 3.4 ([GT21, Thm. 2.8])**.** *Let $T(\cdot)$ be a family of m-sectorial operators such that the associated forms $\mathbf{t}(\cdot)$, see* [Kat95, Thm. VI.2.1], *have constant form domain $\mathcal{D}_\mathbf{t}$. Suppose that, for each $\lambda_0\in\Theta$, there exist r, $C>0$ and*

$$w: B_r(\lambda_0)\to[0,\infty), \qquad \lim_{\lambda\to\lambda_0} w(\lambda)=0,$$

such that for all $\lambda\in B_r(\lambda_0)$ and $f\in\mathcal{D}_\mathbf{t}$ we have

$$|\mathbf{t}(\lambda_0)(f,f)-\mathbf{t}(\lambda)(f,f)|\le w(\lambda)\left(|\operatorname{Re}\mathbf{t}(\lambda_0)(f,f)|+C\|f\|_{\mathcal{H}}^2\right).$$

Then the following inclusion holds

$$\overline{W(T(\cdot))}\cap\Theta\subseteq\overline{W(\mathbf{t}(\cdot))}\cap\Theta\subseteq W_\Psi(T(\cdot)).$$

Employing the analogous notion of pseudo numerical range for form families, the statement above can in fact be formulated more generally, see [GT21, Thm. 2.8] for details. However, the assumption on the constant form domain cannot be omitted even if the family is analytic, see [GT21, Ex. 2.9].

The reverse inclusion, which by Proposition 3.3 automatically gives a spectral enclosure, is true for operator polynomials whose leading coefficient is uniformly bounded below in a certain sense.

Proposition 3.5 ([GT21, Prop. 2.7])**.** *Let $T(\cdot)$ be an operator polynomial of order $n\in\mathbb{N}$ with (possibly unbounded) coefficients, i.e.*

$$T(\lambda)=\sum_{k=0}^{n}\lambda^k A_k, \quad \operatorname{dom}T(\lambda)=\bigcap_{k=0}^{n}\operatorname{dom}A_k, \quad \lambda\in\mathbb{C}.$$

Then the following holds

$$0\notin\overline{W(A_n)} \quad\Longrightarrow\quad \sigma_{\mathrm{app}}(T(\cdot))\subseteq W_\Psi(T(\cdot))\subseteq\overline{W(T(\cdot))}\cap\Theta.$$

This generalises the analogous result for bounded operator polynomials in [Mar88, Thm. 26.7]; see also [Wag07, Prop. 3.3] for the block numerical range.

3.2.3. Spectral enclosures for holomorphic families. We employ the pseudo numerical range to prove a spectral enclosure for type (B) holomorphic operator families in the sense of [Kat95, Sec. VII.4].

Theorem 3.6 ([GT21, Thm. 3.3])**.** *Let $T(\cdot)$ be type* (B) *holomorphic, associated with a type* (a) *holomorphic family $\mathbf{t}(\cdot)$. If there exist $k\in\mathbb{N}_0$ and $\mu\in\Theta$ such that*

$$0\notin\overline{W\big(\mathbf{t}^{(k)}(\mu)\big)}, \tag{3.4}$$

then the pseudo numerical range of $T(\cdot)$ and the closure of the standard numerical range of $\mathbf{t}(\cdot)$ in Θ are equal and the following spectral enclosure holds

$$\sigma(T(\cdot))\subseteq W_\Psi(T(\cdot))=\overline{W(\mathbf{t}(\cdot))}\cap\Theta.$$

If, in addition, $T(\cdot)$ has constant domain, then

$$\sigma(T(\cdot))\subseteq W_\Psi(T(\cdot))=\overline{W(T(\cdot))}\cap\Theta.$$

In fact, it suffices to assume (3.4) only on a core of $\mathbf{t}(\mu)$, and $W_\Psi(T(\cdot)) \neq \Theta$ implies (3.4) with $k = 0$, see [GT21, Equ. (3.2), Rem. 3.4. (i)]. For type (A) holomorphic families in the sense of [Kat95, Sec. VII.2], the analogous result holds without making use of the pseudo numerical range.

Theorem 3.7 ([GT21, Thm. 3.5]). *Let $T(\cdot)$ be type (A) holomorphic . If there exist $k \in \mathbb{N}_0$ and $\mu \in \Theta$ such that*

$$0 \notin \overline{W\big(T^{(k)}(\mu)\big)},$$

then the following spectral enclosure holds

$$\sigma_{\mathrm{app}}(T(\cdot)) \subseteq \overline{W(T(\cdot))} \cap \Theta.$$

Both Theorems 3.6 and 3.7 generalise the classical result [Mar88, Thm. 26.6] for bounded holomorphic families, see also [Wag07, Thm. 2.14] for its generalisation to the block numerical range. Examples where the assumptions above are violated and the spectral inclusion fails can be found in [GT21, Ex. 2.6, Ex. 3.2].

3.2.4. The pseudo quadratic numerical range. We introduce the pseudo quadratic numerical range of an operator matrix family.

Definition 3.8 ([GT21, Def. 4.1]). The pseudo quadratic numerical range is defined as the intersection

$$W^2_\Psi(\mathcal{L}(\cdot)) := \bigcap_{\varepsilon>0} W^2_\varepsilon(\mathcal{L}(\cdot)), \qquad W^2_\varepsilon(\mathcal{L}(\cdot)) := \bigcup_{\|\mathcal{B}\|<\varepsilon} W^2(\mathcal{L}(\cdot)+\mathcal{B}), \qquad \varepsilon > 0. \quad //$$

Note that the quadratic numerical range and the pseudo quadratic numerical range are clearly related in the same way as their classical counterparts, i.e.

$$W^2_\Psi(\mathcal{L}(\cdot)) \subseteq W_\Psi(\mathcal{L}(\cdot)),$$

see [GT21, Prop. 4.6 (i)]. The concept to obtain possibly tighter spectral enclosures by exploiting the matrix structure of $\mathcal{L}(\cdot)$ is thus transmitted from classical to pseudo numerical ranges.

Regarding the alternative characterisation of $W_\Psi(T(\cdot))$ in Proposition 3.2, several other ways of defining the pseudo quadratic numerical range might seem natural, see [GT21, Def. 4.3]. For instance, inspired by (3.3), one can consider

$$W^2_{\Psi,0}(\mathcal{L}(\cdot)) := \{\lambda \in \Theta : 0 \in \overline{W^2(\mathcal{L}(\lambda))}\}.$$

A straightforward generalisation of (3.2) would be

$$\begin{aligned} W^2_{\Psi,1}(\mathcal{L}(\cdot)) := \bigcap_{\varepsilon>0} \{\lambda \in \Theta : \exists\, (f,g) \in \operatorname{dom} \mathcal{L}(\lambda),& \\ \|f\|_{\mathcal{H}_1} = \|g\|_{\mathcal{H}_2} = 1,\ |\det(\mathcal{L}(\lambda)_{f,g})| < \varepsilon\},& \end{aligned}$$

where $\mathcal{L}(\lambda)_{f,g}$ denotes the (λ-dependent) scalar matrix analogous to the one in (1.12), see [GT21, Def. 4.1 (i)]. Another possibility is to consider only diagonal perturbations

$$W^2_{\Psi,2}(\mathcal{L}(\cdot)) := \bigcap_{\varepsilon>0} \bigcup_{\|B_i\|<\varepsilon} W^2(\mathcal{L}(\cdot) + \operatorname{diag}(B_1, B_2)).$$

Recall that for the pseudo numerical range, the analogous notions above coincide. In the quadratic case, they are in general only nested between the classical numerical range and the pseudo numerical range.

Proposition 3.9 ([GT21, Prop. 4.4]). *The following inclusions hold.*

$$W^2(\mathcal{L}(\cdot)) \subseteq W^2_{\Psi,1}(\mathcal{L}(\cdot)) \subseteq W^2_{\Psi,0}(\mathcal{L}(\cdot)) \subseteq W^2_{\Psi,2}(\mathcal{L}(\cdot)) \subseteq W^2_{\Psi}(\mathcal{L}(\cdot)).$$

In [GS21, Ex. 4.5], we show that these alternative notions all fail to enclose the approximate point spectrum of $\mathcal{L}(\cdot)$ in general. For the set $W^2_{\Psi,0}(\mathcal{L}(\cdot))$, and thus for $W^2_{\Psi,1}(\mathcal{L}(\cdot))$, this is also reflected in the fact that

$$W^2_{\Psi,0}(\mathcal{L}(\cdot)) = \overline{W^2(\mathcal{L}(\cdot))} = \overline{W^2(\mathcal{A})}$$

for the trivial family $\mathcal{L}(\lambda) = \mathcal{A} - \lambda$, $\lambda \in \mathbb{C}$. Indeed, this means that their spectral enclosure fails whenever it fails for the closure of the quadratic numerical range. However, by our particular choice from the non-equivalent definitions above, the pseudo quadratic numerical range exhibits precisely the same convenient properties regarding the spectrum and resolvent norm as the pseudo numerical range.

Theorem 3.10 ([GT21, Thm. 4.10]). *For any $\varepsilon > 0$, we have*

$$\sigma_{\mathrm{ap},\varepsilon}(\mathcal{L}(\cdot)) \subseteq W^2_\varepsilon(\mathcal{L}(\cdot)), \quad \|\mathcal{L}(\lambda)^{-1}\| \le \frac{1}{\varepsilon}, \quad \lambda \in \rho(\mathcal{L}(\cdot)) \setminus W^2_\varepsilon(\mathcal{L}(\cdot)),$$

and hence for the approximate point spectrum that

$$\sigma_{\mathrm{app}}(\mathcal{L}(\cdot)) \subseteq W^2_\Psi(\mathcal{L}(\cdot)).$$

As usual, additional assumptions are needed for the enclosure of the full spectrum, see [GT21, Rem. 3.7, Prop. 4.10] for sufficient conditions.

3.3. Spectral enclosure for damped wave equations with p-subordinate damping

We revisit the spectral problem for the damped wave equation, see Sections 1.5 and 2.2. Assuming that the damping satisfies a certain form-subordination condition (3.5) with respect to $-\Delta + q$, we establish an enclosure for the spectrum of the Schur complement of the semigroup generator $\mathcal{A}_0$ in Theorem 2.4. More precisely, we consider the following abstract spectral problem; details can be found in [GT21, Sec. 7].

Suppose that $\mathbf{t}_0$ and $\mathbf{a}$ are densely defined sesquilinear forms in $\mathcal{H}$ such that $\mathbf{t}_0$ is closed, $\mathbf{t}_0 \ge \kappa_0 \ge 0$ and $\mathbf{a} \ge \alpha_0 \ge 0$. Assume there exist $\kappa \le \kappa_0$, $p \in [0,1)$ and $C_p > 0$ such that

$$\mathbf{a}(f,f) \le C_p\big((\mathbf{t}_0 - \kappa)(f,f)\big)^p \big(\|f\|^2_{\mathcal{H}}\big)^{1-p}, \qquad f \in \operatorname{dom}\mathbf{t}_0 \subseteq \operatorname{dom}\mathbf{a}, \tag{3.5}$$

i.e. such that $\mathbf{a}$ is p-subordinate to $\mathbf{t}_0 - \kappa$ in the sense of forms, cf. [Mar88, Sec. 5.1] for operator subordinacy. Consider the following quadratic pencil of forms

$$\mathbf{t}(\lambda) := \mathbf{t}_0 + 2\lambda\mathbf{a} + \lambda^2, \qquad \operatorname{dom}\mathbf{t}(\lambda) := \operatorname{dom}\mathbf{t}_0, \qquad \lambda \in \mathbb{C}. \tag{3.6}$$

The correlation to the Schur complement $S_0(\cdot)$ in Theorem 2.4 is then given by considering $\mathcal{H} = L^2(\mathbb{R}^n)$ and the particular forms

$$\begin{aligned} \mathbf{a}(f,f) &:= \int_{\mathbb{R}^n} a|f|^2\,\mathrm{d}x, & \operatorname{dom}\mathbf{a} &:= \operatorname{dom} a^{\frac12}, \\ \mathbf{t}_0(f,f) &:= \int_{\mathbb{R}^n} |\nabla f|^2\,\mathrm{d}x + \int_{\mathbb{R}^n} q|f|^2\,\mathrm{d}x, & \operatorname{dom}\mathbf{t}_0 &:= H^1(\mathbb{R}^n) \cap \operatorname{dom} q^{\frac12}, \end{aligned} \tag{3.7}$$

such that for $\lambda \ne 0$ one has $\mathbf{t}(\lambda) = -\lambda\mathbf{s}_0(\lambda)$ on $\operatorname{dom}\mathbf{t}(\lambda) = \operatorname{dom}\mathbf{s}_0(\lambda) = \mathcal{D}_S$, where $\mathbf{s}_0(\cdot)$ denotes the family of forms associated with the Schur complement, see [Ger21, Prop. 4.11]. Notice that when assuming (3.5), which implies relative form-boundedness of order zero, the assumptions of Theorem 2.3, and thus the equivalence of (point and essential) spectra of $\mathcal{A}_0$ and $S_0(\cdot)$, hold on whole $\mathbb{C} \setminus \{0\}$. The point $\lambda = 0$, however, has to be considered separately (and exhibits different behaviour depending on the underlying potential q, see also the last part of Example 3.12).

Employing Theorem 3.6 to the operator family associated with (3.6) and using condition (3.5) to localise its pseudo numerical range, we obtain the following spectral enclosure. The latter in particular implies that the spectrum lies in the left half of the complex plane, which agrees with the m-accretivity of $-\mathcal{A}_0$ obtained in Theorem 2.4.

Theorem 3.11 ([GT21, Thm. 7.1])**.** *Let* $\mathbf{t}(\cdot)$ *be as in* (3.6) *such that* (3.5) *is satisfied. Then* $\mathbf{t}(\cdot)$ *is type* (a) *holomorphic in the sense of* [Kat95, Sec. VII.4] *and the corresponding type* (B) *holomorphic family* $T(\cdot)$ *satisfies the following.*

(i) *The non-real spectrum of* $T(\cdot)$ *is contained in*

$$\sigma(T(\cdot)) \setminus \mathbb{R} \subseteq \Bigg\{ z \in \mathbb{C} \,:\, \operatorname{Re} z \le -\alpha_0,\ |z| \ge \sqrt{\kappa_0},\\ |\operatorname{Im} z| \ge \sqrt{\max\Big\{0, C_p^{-\frac{1}{p}} |\operatorname{Re} z|^{\frac{1}{p}} - |\operatorname{Re} z|^2 + \kappa\Big\}} \Bigg\}.$$

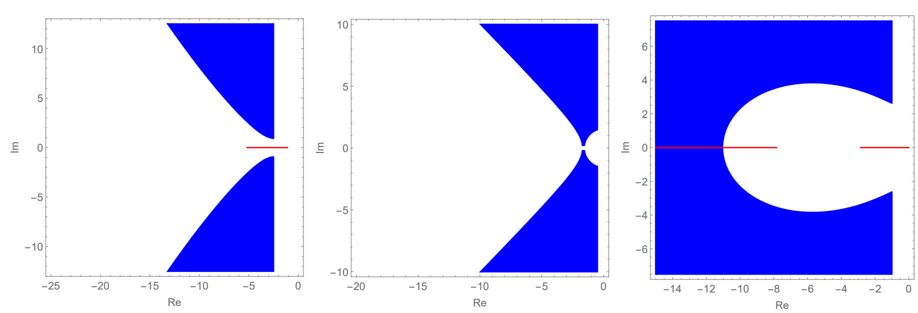
(ii) *If* $p < \frac{1}{2}$*, or if* $p = \frac{1}{2}$ *and* $C_{\frac{1}{2}} < 1$*, or if* $p = \frac{1}{2}$ *and* $C_{\frac{1}{2}} = 1$ *and* $\kappa > 0$*, then*

$$\sigma(T) \cap \mathbb{R} = \emptyset \qquad \vee \qquad \sigma(T) \cap \mathbb{R} \subseteq [s^-, s^+].$$

(iii) *If* $p > \frac{1}{2}$*, or if* $p = \frac{1}{2}$ *and* $C_{\frac{1}{2}} > 1$*, or if* $p = \frac{1}{2}$ *and* $C_{\frac{1}{2}} = 1$ *and* $\kappa \le 0$*, then*

$$\sigma(T) \cap \mathbb{R} \subseteq (-\infty, r^+] \cup [s^-, s^+] \qquad \vee \qquad \sigma(T) \cap \mathbb{R} \subseteq (-\infty, s^+].$$

In (ii) *and* (iii) *above, the interval bounds* $\infty < r^+ < s^- \le s^+ \le 0$ *depend on the problem parameters* p*,* C_p*,* κ_0 *and* κ*; see* [GT21, Thm. 7.1] *for more details, including a separate discussion of the case* $\kappa = 0$*.*



(A) $p = 0.4$, $C_p = 1.3$, $\kappa = -2$, $\alpha_0 = 2.5$, $\kappa_0 = 5$
(B) $p = 0.5$, $C_p = 0.7$, $\kappa = 3$, $\alpha_0 = 0.5, \kappa_0 = 6$
(C) $p = 0.65$, $C_p = 0.5$, $\kappa = -5$, $\alpha_0 = 1, \kappa_0 = 0$

FIGURE 1. Enclosures in Theorem 3.11 for $\sigma(T) \setminus \mathbb{R}$ in (i) (blue) and for $\sigma(T) \cap \mathbb{R}$ in (ii) and (iii) (red in (A) and (C), empty in (B)).

We remark that the enclosure for the non-real spectrum in [JT09, Thm. 3.2, Part 5] is a special case of Theorem 3.11 (i). The enclosure for the real spectrum in [JT09, Thm. 3.2, Part 5], however, can substantially be improved by Theorem 3.11 (ii), since the latter gives $\sigma(T(\cdot)) \cap \mathbb{R} = \emptyset$, see [GT21, Rem. 7.3].

Example 3.12 **(**[GT21, Ex. 7.5]**).** We apply Theorem 3.11 to a particular example. In the space $\mathcal{H} = L^2(\mathbb{R}^n)$ with $n \geq 3$, we consider (3.7) with locally integrable a and q, such that the damping is non-zero and bounded by

$$a(x) \leq \sum_{j=1}^{m} |x - x_j|^{-t} + u(x) + v(x),$$

where $m \in \mathbb{N} \cup \{0\}$, $t \in [0, 2)$ and $u \in L^s(\mathbb{R}^n)$. The first two terms are then subordinate to the gradient in $\mathbf{t}_0$, and we assume the third term to be subordinate to the potential by

$$v(x) \leq c_1 q(x)^r + c_2,$$

see [GT21, Ex. 7.5] for details. We prove that the assumptions of Theorem 3.11 are satisfied and provide formulas for the parameters p, κ and C_p therein (depending on the constants involved), see [GT21, Equ. (7.17)]. In the particular case that

$$a(x) = |x|^k, \qquad k \in [0, 2), \qquad q(x) = |x|^2, \qquad x \in \mathbb{R}^n,$$

the above simplify and we determine the precise enclosures emerging from Theorem 3.11, see the last part of [GT21, Ex. 7.5]. Finally, we mention that in this case $\lambda = 0$ is in the resolvent set of the semigroup generator $\mathcal{A}_0$ in Theorem 2.4; this can be shown analogously as in the proof of [Ger21, Thm. 5.1], using that the harmonic oscillator $-\Delta + q$ in $L^2(\mathbb{R}^n)$ is boundedly invertible.

3.4. Further applications

The remaining results in [GT21, Sec. 5, Sec. 6] are spectral enclosures for operator matrix (families) under the assumption of suitable relative boundedness relations within their entries. Since this thesis is centered around non-standard dominance patterns, these results will be mentioned only briefly in this section.

3.4.1. Spectral enclosures in terms of pseudo numerical ranges of Schur complements. The detailed results in this section can be found in [GT21, Sec. 5]. Assuming that a family of operator matrices $\mathcal{L}(\cdot)$ is pointwise either diagonally dominant or off-diagonally dominant with boundedly invertible off-diagonal elements, see [Tre08, Sec. 2.2] for the notion of (off-)diagonal dominance, we show that

$$\sigma_{\mathrm{app}}(\mathcal{L}(\cdot)) \cap \rho(A(\cdot)) \cap \rho(D(\cdot)) \subseteq W_\Psi(S_1(\cdot)) \cup W_\Psi(S_2(\cdot)) \subseteq W^2_{\Psi,2}(\mathcal{L}(\cdot)),$$

see [GT21, Thm. 5.1]. Moreover, we show that if $\dim \mathcal{H}_1 \geq 2$ and $\mathcal{L}(\cdot)$ has pointwise symmetric corners, i.e. $C(\cdot) \subseteq B^*(\cdot)$, and if $A(\cdot)$ and $-D(\cdot)$, respectively, are accretive and sectorial in the sense of [Kat95, Sec. V.3.10], then

$$\sigma_{\mathrm{app}}(\mathcal{L}(\cdot)) \cap \rho(D(\cdot)) \subseteq W_\Psi(S_1(\cdot)) \cup W_\Psi(D(\cdot)) \subseteq W^2_{\Psi,2}(\mathcal{L}(\cdot)),$$

see [GT21, Thm. 5.3] for analogous results assuming similar structures. The above spectral enclosures are possibly tighter than the general one by the pseudo quadratic numerical range in Theorem 3.10. We point out that our results do not restrict the size of involved relative bounds.

3.4.2. Structured operator matrices. We apply the results from the previous section to operator matrices $\mathcal{A}$ with a particular structure in [GT21, Sec. 6]. More precisely, we therein assume that $C \subseteq B^*$ and that A and $-D$ are at least accretive (possibly sectorial) and their numerical ranges are separated by a strip

$$\operatorname{Re} W(D) \leq \delta < 0 < \alpha \leq \operatorname{Re} W(A).$$

see [GT21, Equ. (6.2)] for the detailed assumption. Supposing for instance that $\mathcal{A}$ is diagonally dominant and that A and $-D$ are m-accretive, i.e. essentially that the imaginary axis is in both their resolvent sets, we show in [GT21, Thm. 6.1] that the approximate point spectrum of $\mathcal{A}$ satisfies the enclosure

$$\sigma_{\mathrm{app}}(\mathcal{A}) \subseteq (-\Sigma \cup \Sigma) \cap \{z \in \mathbb{C} : \operatorname{Re} z \notin (\delta, \alpha)\};$$

here the set Σ is the smallest sector containing both $W(A)$ and $W(-D)$. This generalises [Tre09, Thm. 5.2], where the order of diagonal dominance was assumed to be zero. Finally, in [GT21, Prop. 6.5], we give sufficient conditions in terms of the relative bounds for the enclosure of the whole spectrum in the above double sector. Several other dominance and structural patterns arising from [GT21, Thm. 5.1, Thm. 5.3] are treated analogously.

CHAPTER 4

Schrödinger operators with accretive potentials in weighted spaces

This chapter is based on the work [GS21], where more details on the following can be found. In Section 4.1 we summarise our main results and in Section 4.2 we illustrate them on the example of a wave equation with accretive differential damping in a weighted space. Moreover, brief outlines of various further applications can be found in Section 4.3.

4.1. Main results

Let $\Omega \subseteq \mathbb{R}^n$ be open and non-empty. For a suitable weight $w : \Omega \to \mathbb{R}_+$, we introduce a realisation of the differential expression

$$T_w = -\nabla \cdot (P\nabla) + V$$

with Dirichlet boundary conditions in the weighted space $L^2_{w^2}(\Omega)$. Here the matrix $P : \Omega \to \mathbb{C}^{n\times n}$ is sectorial, the potential $V : \Omega \to \mathbb{C}$ is accretive and the admissibility of a weight w depends on P and V, see Assumption 4.1 below. In the sequel, we denote

$$P_1 := \operatorname{Re} P = \frac{1}{2}(P + P^*), \qquad P_2 := \operatorname{Im} P = \frac{1}{2\mathrm{i}}(P - P^*). \tag{4.1}$$

The weighted operator T_w is introduced by means of the generalised form representation methods in [AH15]. More precisely, we consider the sesquilinear form

$$\mathbf{t}_w(f, g) := \langle P\nabla f, \nabla(gw^2)\rangle_{L^2} + \langle wVf, wg\rangle_{L^2}, \qquad \operatorname{dom} \mathbf{t}_w := \mathcal{V}_w, \tag{4.2}$$

where the Hilbert space $\mathcal{V}_w$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|f\|^2_{\mathcal{V}_w} := \|P_1^{\frac{1}{2}}\nabla f\|^2_{L^2_{w^2}} + \||V|^{\frac{1}{2}} f\|^2_{L^2_{w^2}} + \|f\|^2_{L^2_{w^2}}, \tag{4.3}$$

equipped with the corresponding inner product; cf. [GS21, Lem. 4.1] where we show that $\mathbf{t}_w$ is continuous with respect to $\|\cdot\|_{\mathcal{V}_w}$ on $C_0^\infty(\Omega)$ and can thus be extended uniquely to $\mathcal{V}_w$. The result in Theorem 4.2 below is then obtained by splitting the real and imaginary parts of the potential into their regular and singular parts, see Assumption 4.1 below,

$$\operatorname{Re} V = U_r + U_s, \qquad \operatorname{Im} V = V_r + V_s, \tag{4.4}$$

and applying Theorem 1.4, in particular establishing the lower estimates (1.7) therein, with the following multiplier

$$\Phi_1 = \Phi_2 = \Phi := \frac{V_r}{\sqrt{1 + V_r^2 + U_r^2}} \in L^\infty(\Omega; \mathbb{R}); \tag{4.5}$$

see [GS21, Thm. 3.2] for more details.

4.1.1. Assumptions.

Assumption 4.1 **([GS21, Asm. 3.1]).** Assume the following regularity of P, (the regular part of) the potential V and the weight w, see (4.4),

$$P \in L^1_{\mathrm{loc}}(\Omega)^{d\times d}, \quad V \in L^1_{\mathrm{loc}}(\Omega), \quad U_r, V_r \in W^{1,\infty}_{\mathrm{loc}}(\overline{\Omega};\mathbb{R}), \quad w \in W^{1,\infty}_{\mathrm{loc}}(\overline{\Omega};\mathbb{R}_+).$$

Moreover, suppose that the following hold, see (4.1) and (4.5).

(i) *Sectoriality of* P: There exists $C_P \geq 0$ such that

$$\langle P_1\xi,\xi\rangle_{\mathbb{C}^n} \geq 0, \qquad |\langle P_2\xi,\xi\rangle_{\mathbb{C}^n}| \leq C_P\langle P_1\xi,\xi\rangle_{\mathbb{C}^n}, \qquad \xi \in \mathbb{C}^n.$$

(ii) *Accretivity of* V*, sectoriality of* $\operatorname{Re} V + \mathrm{i}V_s$: There exists $C_s \geq 0$ such that

$$\operatorname{Re} V \geq 0, \qquad |V_s| \leq C_s \operatorname{Re} V.$$

(iii) *Control of* ∇U_r *and* ∇V_r: For every $\varepsilon > 0$, there exists $C_\varepsilon \geq 0$ such that

$$\begin{aligned} &U_r|V_r| \max\{|P_1^{-\frac12}P\nabla U_r|, |P_1^{-\frac12}P^*\nabla U_r|\} \\ &\qquad\qquad \leq (1+V_r^2+U_r^2)^{\frac32}(\varepsilon(\operatorname{Re} V)^{\frac12} + \varepsilon(\Phi V_r)^{\frac12} + C_\varepsilon), \\ &(1+U_r^2)\max\{|P_1^{-\frac12}P\nabla V_r|, |P_1^{-\frac12}P^*\nabla V_r|\} \\ &\qquad\qquad \leq (1+V_r^2+U_r^2)^{\frac32}(\varepsilon(\operatorname{Re} V)^{\frac12} + \varepsilon(\Phi V_r)^{\frac12} + C_\varepsilon). \end{aligned} \tag{4.6}$$

(iv) *Admissibility of* w: There exist $\kappa_w, \sigma_w > 0$ and $C_w \geq 0$ such that

$$|P_1^{-\frac12}P^*\nabla(w^2)| \leq w^2(\kappa_w(\operatorname{Re} V)^{\frac12} + \sigma_w(\Phi V_r)^{\frac12} + C_w). \qquad //$$

We point out that our assumptions include sectorial potentials $V = U_s + \mathrm{i}V_s$ and mention that, in fact, in (iii) above it suffices to assume (4.6) with ε smaller than a certain critical value $\varepsilon_{\mathrm{crit}} > 0$, see [GS21, Rem. 3.3 (i)]. Moreover, in the generic case $V = \mathrm{i}V_r$, the condition (4.6) simplifies substantially; indeed, it reduces to assuming that for every $\varepsilon > 0$, there exists $C_\varepsilon \geq 0$ such that

$$\max\{|P_1^{-\frac12}P\nabla V_r|, |P_1^{-\frac12}P^*\nabla V_r|\} \leq \varepsilon|V_r|^{\frac72} + C_\varepsilon,$$

see [GS21, Rem. 3.3 (ii)].

4.1.2. Definition of T_w.

Our first and most fundamental result is the definition of the object we study in this chapter.

Theorem 4.2 **([GS21, Thm. 3.2]).** *Let Assumption* 4.1 *be satisfied with* κ_w*,* σ_w*,* C_P *and* C_s *small enough such that there exists* $0 < \beta < \min\{1/C_P, 1/C_s\}$ *satisfying the inequality*

$$\beta\kappa_w^2 + (1-\beta C_s)\sigma_w^2 < \frac{4\beta(1-\beta C_P)(1-\beta C_s)}{(1+\beta)^2}. \tag{4.7}$$

Let $\mathbf{t}_w$ *and* $\mathcal{V}_w$ *be as in* (4.2) *and* (4.3)*, respectively. Then the operator*

$$\begin{aligned} \operatorname{dom} T_w &:= \Big\{ f \in \mathcal{V}_w \,:\, \exists\, \eta_f \in L^2_{w^2}(\Omega),\, \forall\, g \in \mathcal{V}_w,\, \mathbf{t}_w(f,g) = \langle \eta_f, g\rangle_{L^2_{w^2}} \Big\}, \\ T_w f &:= \eta_f, \end{aligned}$$

acting in the weighted space $L^2_{w^2}(\Omega)$*, has non-empty resolvent set and dense domain both in* $\mathcal{V}_w$ *and* $L^2_{w^2}(\Omega)$*.*

In our setting, condition (4.7) is equivalent to the applicability of Theorem 1.4; see [GS21, Rem. 4.4] for more details where the origin of (4.7) is explained. For $V = \mathrm{i}V_r$, we show in [GS21, Rem. 3.3 (ii)] that (4.7) simplifies to

$$\sigma_w < \frac{1}{1+C_P}.$$

Notation 4.3. In what follows, we say that a weight w is admissible if it satisfies the assumptions of the above theorem (with respect to P and V). By T_w we denote the therein defined Dirichlet realisation of $-\nabla\cdot(P\nabla)+V$ acting in $L^2_{w^2}(\Omega)$ and write $T := T_1$; notice that $w = 1$ is admissible for any P and V. //

4.1.3. Boundedness of compositions. Employing a suitable weighted operator T_w, we construct bounded extensions for certain types of operator compositions. The latter appear in the Frobenius-Schur resolvent factorisation of various (Schur complement dominant) operator matrices.

Theorem 4.4 ([GS21, Thm. 3.4]). *Let $m_1, m_2, V : \Omega \to \mathbb{C}$ and $P : \Omega \to \mathbb{C}^{n\times n}$ be measurable and let*

$$w := \frac{(|V|+1)^{\frac{1}{2}}}{|m_2|} \in W^{1,\infty}_{\mathrm{loc}}(\overline{\Omega}, \mathbb{R}_+) \tag{4.8}$$

be an admissible weight with respect to P and V, i.e. let the assumptions of Theorem 4.2 be satisfied with P, V and the weight in (4.8). Assume that there exists $C > 0$ such that

$$|m_1 m_2| \le C(|V|+1).$$

Then there exists $\lambda_0 \in \rho(T)$ and a bounded extension

$$m_1(T-\lambda_0)^{-1}m_2 \subseteq S_{\lambda_0} \in \mathcal{B}(L^2(\Omega)).$$

The operator S_{λ_0} above is constructed via the resolvent of T_w; a precise formula can be found in [GS21, Lem. 4.7]. Its extension property is far from trivial and relies on the fact that the T_w are compatible for comparable weights, i.e. that we have $T_{w_1} \supseteq T_{w_2}$ if the weights satisfy $w_1 \lesssim w_2$, see [GS21, Lem. 4.6].

4.1.4. Schatten class. In Theorem 4.5 below we show that if the form domain of T is embedded in $L^2(\Omega)$ with an embedding in a certain Schatten class $\mathcal{S}_{2p}$, then the resolvent of the weighted operator T_w, for any admissible weight, is of Schatten class $\mathcal{S}_p$. The proof is conducted by showing that the family of transformed operators

$$S_\alpha := w^\alpha T_{w^{\mathrm{Re}\,\alpha}} w^{-\alpha}, \qquad \alpha \in \mathbb{C}, \tag{4.9}$$

in $L^2(\Omega)$, which are unitarily equivalent to $T_{w^{\mathrm{Re}\,\alpha}}$, emerges from a family of forms via Theorem 1.4 on the constant form domain $\mathcal{V}_1$, see [GS21, Lem. 4.9, Lem.4.10].

Theorem 4.5 ([GS21, Thm. 3.5]). *Let the assumptions of Theorem 4.2 hold and let $0 < p \le \infty$. Then*

$$\mathrm{id}_{\mathcal{V}_1} \in \mathcal{S}_{2p}(\mathcal{V}_1, L^2(\Omega)) \implies \left((T_w-\lambda)^{-1} \in \mathcal{S}_p(L^2_{w^2}(\Omega)),\ \ \lambda \in \rho(T_w)\right).$$

In particular, the above theorem gives a sufficient condition for T_w to have compact resolvent.

4.1.5. Invariance of discrete spectra and eigenfunctions. Our next result contains a sufficient condition for the discrete spectra (including algebraic multiplicities) of T and T_w to coincide. Moreover, we show that generalised eigenfunctions of T are also generalised eigenfunctions of T_w, which in particular implies a certain decay of the eigenfunctions of T.

The proof of Theorem 4.6 below relies on the analyticity of the family in (4.9) on a sufficiently large domain, see [GS21, Lem. 4.10], Theorem 4.5 together with a compactness argument based on Rellich's criterion, see [GS21, Rem. 3.7 (i)], the identity theorem for holomorphic functions and a slight modification of the Agmon type estimates in [KRRS17].

Theorem 4.6 ([GS21, Thm. 3.6, Thm. 3.9])**.** *Let the assumptions of Theorem 4.2 be satisfied, $P_1 \geq \delta_P > 0$ a.e. in Ω and suppose that the potential satisfies*

$$\lim_{R\to\infty} \operatorname*{ess\,inf}_{|x|>R,\, x\in\Omega} |V(x)| = \infty. \tag{4.10}$$

Then both T and T_w have compact resolvent, their (discrete) spectra coincide and for all $\lambda \in \sigma(T) = \sigma(T_w)$ and $k \in \mathbb{N}$ we have

$$\psi \in \ker(T-\lambda)^k \qquad \Longrightarrow \qquad \psi \in \ker(T_w - \lambda)^k.$$

In particular, the (finite) algebraic multiplicities of λ as an eigenvalue of T and T_w coincide, and all generalised eigenfunctions of T lie in $L^2_{w^2}(\Omega)$.

The invariance of the discrete spectrum and algebraic multiplicities holds also without the additional assumptions on P and V, merely assuming that T and T_w have compact resolvent, see [GS21, Thm. 3.6]. For the result about the eigenfunctions, it is then sufficient to assume (4.10) in addition, see [GS21, Thm. 3.9].

We point out that without the assumption on compact resolvent, the invariance result of the discrete spectrum fails in general. Indeed, in [GS21, Ex. 3.8] we show that an isolated eigenvalue of the family S_α in (4.9) might disappear when touched by the essential spectrum, see also [GS21, Rem. 3.7 (ii)].

4.1.6. Domain and graph norm separation. Our final result is the domain and graph norm separation of the weighted operators T_w. In [GS21, Sec. 5.1], we employ Theorem 4.8 below to derive the completeness of the eigenfunctions of certain Schrödinger operators with regular purely imaginary potentials in weighted spaces. We also mention that separation results of this type are crucial for the convergence analysis of domain truncation methods, see [BST17, SS21].

The following additional set of assumptions is required.

Assumption 4.7 ([GS21, Asm. 3.10])**.** Let Assumption 4.1 hold, let

$$V \in W^{1,\infty}_{\mathrm{loc}}(\overline{\Omega})$$

and assume the following.

(i) *Combined accretivity of V and P*:

$$\operatorname{Re}\langle \mathrm{e}^{-\mathrm{i}\arg V} P\xi, \xi\rangle_{\mathbb{C}^n} \geq 0, \qquad \xi \in \mathbb{C}^n.$$

(ii) *Control of ∇V*: There exist $\varepsilon_V > 0$ and $C_V \geq 0$ such that

$$\max\{|P_1^{-\frac{1}{2}} P^* \nabla V|, |P_1^{-\frac{1}{2}} P^* \nabla |V||\} \leq \varepsilon_V |V|^{\frac{3}{2}} + C_V.$$

(iii) *Admissibility of* w: There exist $\varepsilon_w > 0$ and $C_w \geq 0$ such that

$$|P_1^{-\frac{1}{2}} P^* \nabla(w^2)| \leq w^2(\varepsilon_w |V|^{\frac{1}{2}} + C_w). \qquad //$$

In a certain sense, Assumption 4.7 is stronger than Assumption 4.1, see [GS21, Rem. 3.11] for a comparison. In particular, in the latter we justify that for the generic case $V = \mathrm{i}V_r$, Assumption 4.1 follows from Assumption 4.7.

Theorem 4.8 ([GS21, Thm. 3.12])**.** *Let the assumptions of Theorem* 4.2 *hold, let Assumption* 4.7 *hold with* $\varepsilon_V + \varepsilon_w < 2 - \sqrt{2}$*, suppose that* $P \in W^{1,\infty}(\Omega)^{n\times n}$ *and that* $P_1 \geq \delta_P > 0$*. Then there exists* $a_{V,w} > 0$ *such that, for all* $f \in \operatorname{dom} T_w$*,*

$$\|T_w f\|_{L^2_{w^2}} + \|f\|_{L^2_{w^2}} \geq a_{V,w} \left(\|\nabla \cdot (P\nabla f)\|_{L^2_{w^2}} + \|Vf\|_{L^2_{w^2}} + \|f\|_{L^2_{w^2}} \right)$$

and as a consequence

$$\operatorname{dom} T_w = \{ f \in \mathcal{V}_w \, : \, \nabla \cdot (P\nabla f) \in L^2_{w^2}(\Omega), \, Vf \in L^2_{w^2}(\Omega) \}.$$

4.2. Damped wave equations in weighted spaces with accretive unbounded damping

For more details on the following, we refer the reader to [GS21, Sec. 5.4], see also Section 1.5.3. We consider the matrix differential operator

$$\mathcal{A}_0 := \begin{pmatrix} 0 & I \\ \Delta & -2(a_1 + \mathrm{i}a_2 - \nabla \cdot a_0 \nabla) \end{pmatrix} \tag{4.11}$$

arising from the damped wave equation in (1.17) on an open set $\Omega \subseteq \mathbb{R}^n$ in the space $\mathcal{H}_w := \mathcal{W}_w(\Omega) \oplus L^2_{w^2}(\Omega)$, where we choose a suitable weight

$$w \in W^{1,\infty}_{\mathrm{loc}}(\overline{\Omega}; \mathbb{R}_+) \tag{4.12}$$

and $\mathcal{W}_w(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|f\|^2_{\mathcal{W}_w} := \|\nabla f\|^2_{L^2_{w^2}} + \|f\|^2_{L^2_{w^2}}, \qquad f \in C_0^\infty(\Omega). \tag{4.13}$$

We assume that the coefficients in the damping satisfy

$$a_1 \in L^1_{\mathrm{loc}}(\Omega), \quad a_1 \geq 0, \quad a_2 \in W^{1,\infty}_{\mathrm{loc}}(\overline{\Omega}; \mathbb{R}), \quad a_0 \in L^1_{\mathrm{loc}}(\Omega)^{d\times d}, \quad a_0 \geq 0, \tag{4.14}$$

and introduce a Dirichlet realisation of (4.11) which generates a semigroup on $\mathcal{H}_w$. To this end, we follow the procedure in Section 2.2 and employ the dominance of the second Schur complement together with the weighted differential operators defined in Section 4.1. The domain of the resulting operator matrix $\mathcal{A}_0$ is then given by

$$\begin{aligned} \operatorname{dom} \mathcal{A}_0 := \{ (f,g) \in \mathcal{W}_w(\Omega) \times \mathcal{D}_S \, : \, & \Delta f - 2(a_1 + \mathrm{i}\, a_2) g \\ & + 2\nabla \cdot (a_0 \nabla g) \in L^2_{w^2}(\Omega) \}, \end{aligned} \tag{4.15}$$

where $\mathcal{D}_S$ is the closure of $C_0^\infty(\Omega)$ with respect to

$$\|f\|^2_S := \|(I_{\mathbb{C}^n} + a_0)^{\frac{1}{2}} \nabla f\|^2_{L^2_{w^2}} + \||a_1 + \mathrm{i}a_2|^{\frac{1}{2}} f\|^2_{L^2_{w^2}} + \|f\|^2_{L^2_{w^2}}, \quad f \in C_0^\infty(\Omega).$$

Proposition 4.9 ([GS21, Prop. 5.3])**.** *Assume that* a_0, a_1, a_2 *and* w *satisfy* (4.14) *and* (4.12)*, and in addition that*

$$\begin{aligned} |(I_{\mathbb{C}^n} + a_0)^{\frac{1}{2}} \nabla a_2| &\leq c(1 + |a_2|^3)(a_1^{\frac{1}{2}} + |a_2|^{\frac{1}{2}} + 1), \\ |\nabla(w^2)| &\leq c w^2 (a_1^{\frac{1}{2}} + |a_2|^{\frac{1}{2}} + 1), \\ |a_0^{\frac{1}{2}} \nabla(w^2)| &\leq \sqrt{2}\varepsilon_0 w^2 (a_1 + c_0)^{\frac{1}{2}}, \end{aligned} \tag{4.16}$$

with $c > 0$, $\varepsilon_0 \in (0, 2)$ and $c_0 \geq 0$. Then the operator matrix $\mathcal{A}_0$ in (4.11), (4.15) *generates a C_0-semigroup in $\mathcal{H}_w$ and its domain is dense in $\mathcal{W}_w(\Omega) \oplus \mathcal{D}_S$ and $\mathcal{H}_w$.*

4.3. Further applications

In this section, we briefly sketch three applications from [GS21] which illustrate the previously stated results therein. In particular, the matrix differential operator in Section 4.3.1 shows the connection between the works [Ger21] and [GS21].

4.3.1. A Schur complement dominant matrix differential operator with highly non-symmetric corners. For more details on the results in this section, we refer the reader to [GS21, Sec. 5.2]. We therein employ Schur complement dominance (with respect to the first Schur complement) to obtain a densely defined and closed realisation of the differential operator matrix

$$\mathcal{A}_0 := \begin{pmatrix} -\partial_x^2 + \mathrm{i}\sinh(5x^2) & \mathrm{e}^{x^2} \\ \mathrm{e}^x\,\partial_x + \mathrm{e}^{3x^2} & 0 \end{pmatrix} \tag{4.17}$$

in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ with non-empty resolvent set. Indeed, this can be achieved for the matrix $\mathcal{A}_0$ defined on

$$\operatorname{dom}\mathcal{A}_0 := \{(f, g) \in \mathcal{D}_S \times L^2(\mathbb{R}) \,:\, f'' + \mathrm{i}\sinh(5x^2)f + \mathrm{e}^{x^2}\,g \in L^2(\mathbb{R})\}, \tag{4.18}$$

where $\mathcal{D}_S$ is the closure of $C_0^\infty(\mathbb{R})$ with respect to the norm

$$\|f\|_S^2 := \|\,\mathrm{e}^{-x^2}\,f''\|_{L^2}^2 + \|\,\mathrm{e}^{4x^2}\,f\|_{L^2}^2, \qquad f \in C_0^\infty(\mathbb{R}),$$

by applying Theorem 2.3 to obtain spectral equivalence on $\mathbb{C} \setminus \{0\}$ to its Schur complement. The latter is the following operator family acting in $L^2(\mathbb{R})$

$$\begin{aligned} S_0(\lambda) &:= -\partial_x^2 + \mathrm{i}\sinh(5x^2) - \lambda + \frac{1}{\lambda}\,\mathrm{e}^{x^2}(\mathrm{e}^x\,\partial_x + \mathrm{e}^{3x^2}), \\ \operatorname{dom} S_0(\lambda) &:= W^{2,2}(\mathbb{R}) \cap \operatorname{dom}\mathrm{e}^{5x^2}, \end{aligned} \qquad \lambda \in \mathbb{C} \setminus \{0\}. \tag{4.19}$$

Proposition 4.10 ([GS21, Prop. 5.2])**.** *The operator matrix $\mathcal{A}_0$ in* (4.17), (4.18) *is closed, has non-empty resolvent set and its domain is dense in $\mathcal{D}_S \oplus L^2(\mathbb{R})$ and $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. Moreover, with $S_0(\cdot)$ as in* (4.19)*, we have*

$$\sigma_{\mathrm{p}}(\mathcal{A}_0) \supseteq \sigma(\mathcal{A}_0) \setminus \{0\} = \sigma(S_0(\cdot)) = \sigma_{\mathrm{p}}(S_0(\cdot)).$$

We point out that the off-diagonal of $\mathcal{A}_0$ is highly non-symmetric and the Schur complement has to be realised in a suitable weighted space to satisfy the conditions of Theorem 2.3. The remaining spaces therein are given by

$$\mathcal{D}_{-S} := L^2_{\mathrm{e}^{-2x^2}}(\mathbb{R}), \qquad \mathcal{D}_2 := \mathcal{D}_{-2} := L^2(\mathbb{R}).$$

In fact, for the result above we employ Theorems 4.2 and 4.8 to originally define the Schur complement in the weighted space $\mathcal{D}_{-S}$ with operator domain $\mathcal{D}_S$.

4.3.2. Diagonally dominant matrix Schrödinger operator in a weighted space. The precise results in this section can be found in [GS21, Sec. 5.3]. Similarly as in the previous section, one can employ Schur complement dominance to find a closed, densely defined realisation of the operator matrix

$$\mathcal{A} = \begin{pmatrix} -\partial_x^2 + \mathrm{i}x^3 & x \\ x^4 & -\partial_x^2 + x^6 \end{pmatrix}$$

in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ with non-empty resolvent set. Following a different approach, however, one can consider the product weighted space

$$L^2_{w_1^2}(\mathbb{R}) \oplus L^2_{w_2^2}(\mathbb{R}), \qquad w_1 := \langle x \rangle, \qquad w_2 := \langle x \rangle^{-1}, \qquad x \in \mathbb{R},$$

in which $\mathcal{A}$ in fact naturally becomes diagonally dominant of order zero; see [GS21, Sec. 5.3] where Theorem 4.2 is employed to introduce the Dirichlet realisations of the diagonal elements in the respective weighted spaces and the resulting diagonal dominance is shown using Theorem 4.8.

Placing $\mathcal{A}$ in a product weighted space can equivalently be understood as a conjugation with the (unbounded) diagonal matrix $\operatorname{diag}(w_1, w_2)$, see also [RT18] where suitable constants $w_1, w_2 > 0$ were chosen to balance off-diagonal terms.

4.3.3. Completeness of eigensystems for Schrödinger operators in weighted spaces. For more details about the following, see [GS21, Sec. 5.1]. Let the assumptions of Theorem 4.2 be satisfied with $\Omega = \mathbb{R}^n$, $P = I_{\mathbb{C}^n}$, a regular purely imaginary potential $V = \mathrm{i}V_r$ such that

$$|V_r(x)| + 1 \gtrsim \langle x \rangle^{\gamma}, \qquad \gamma > 0,$$

and an admissible weight w. Using Theorem 4.5, we show in [GS21, Sec. 5.1] that for the Dirichlet realisation

$$T_w = -\Delta + \mathrm{i}V_r$$

in the weighted space $L^2_{w^2}(\mathbb{R}^n)$ it holds that

$$(T_w - \lambda)^{-1} \in \mathcal{S}_p(L^2_{w^2}(\mathbb{R}^n)), \qquad \lambda \in \rho(T_w), \qquad p > \frac{2+\gamma}{2\gamma} n.$$

Moreover, we use [GS21, Lem. 4.3] and [DS88, Cor. XI.9.31] to conclude that if $n = 1$ and $\gamma > 2$, then the eigensystem of T_w is complete in $L^2_{w^2}(\mathbb{R})$ for any of the admissible weights

$$w(x) = \exp(\pm \langle x \rangle^{\alpha}), \qquad 0 < \alpha < 1 + \frac{\gamma}{2}. \tag{4.20}$$

For the particular choice $V_r(x) = x^3$, the above generalises the completeness result for the eigensystem of the imaginary cubic oscillator in $L^2(\mathbb{R})$ which was established in [SK12] to weighted spaces with weights as in (4.20) such that $0 < \alpha < 1 + 3/2$.

Bibliography

[AH15] Y. Almog and B. Helffer, *On the spectrum of non-selfadjoint Schrödinger operators with compact resolvent*, Comm. Partial Differential Equations **40** (2015), 1441–1466.

[AHP13] Y. Almog, B. Helffer, and X.-B. Pan, *Superconductivity near the normal state in a half-plane under the action of a perpendicular electric current and an induced magnetic field*, Trans. Amer. Math. Soc. **365** (2013), 1183–1217.

[AN15] K. Ammari and S. Nicaise, *Stabilization of elastic systems by collocated feedback*, Lecture Notes in Mathematics, vol. 2124, Springer, Cham, 2015.

[BBDCZ10] A. S. Bonnet-Ben Dhia, P. Ciarlet, Jr., and C. M. Zwölf, *Time harmonic wave diffraction problems in materials with sign-shifting coefficients*, J. Comput. Appl. Math. **234** (2010), 1912–1919.

[BBT16] C. J. K. Batty, A. Borichev, and Y. Tomilov, *L^p-tauberian theorems and L^p-rates for energy decay*, J. Funct. Anal. **270** (2016), 1153–1201.

[BM20] S. Bögli and M. Marletta, *Essential numerical ranges for linear operator pencils*, IMA J. Numer. Anal. **40** (2020), 2256–2308.

[BST17] S. Bögli, P. Siegl, and C. Tretter, *Approximations of spectra of Schrödinger operators with complex potential on $\mathbb{R}^d$*, Comm. Partial Differential Equations **42** (2017), 1001–1041.

[BT20] S. Bögli and C. Tretter, *Eigenvalues of magnetohydrodynamic mean-field dynamo models: bounds and reliable computation*, SIAM J. Appl. Math. **80** (2020), 2194–2225.

[Dav99] E. B. Davies, *Pseudo-spectra, the harmonic oscillator and complex resonances*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **455** (1999), 585–599.

[Dav07] E. B. Davies, *Linear operators and their spectra*, Cambridge University Press, 2007.

[DELV04] J. Dolbeault, M. J. Esteban, M. Loss, and L. Vega, *An analytical proof of Hardy-like inequalities related to the Dirac operator*, J. Funct. Anal. **216** (2004), 1–21.

[DES00] J. Dolbeault, M. J. Esteban, and E. Séré, *On the eigenvalues of operators with gaps. Application to Dirac operators*, J. Funct. Anal. **174** (2000), 208–226.

[DS88] N. Dunford and J. T. Schwartz, *Linear Operators, Part 2*, John Wiley & Sons, Inc., New York, 1988.

[EE87] D. E. Edmunds and W. D. Evans, *Spectral theory and differential operators*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1987, Oxford Science Publications.

[EL07] M. J. Esteban and M. Loss, *Self-adjointness for Dirac operators via Hardy-Dirac inequalities*, J. Math. Phys. **48** (2007), 112107, 8.

[EN00] K.-J. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*, Springer-Verlag, New York, 2000.

[ET17] C. Engström and A. Torshage, *Enclosure of the numerical range of a class of non-selfadjoint rational operator functions*, Integral Equations Operator Theory **88** (2017), 151–184.

[FST18] P. Freitas, P. Siegl, and C. Tretter, *The damped wave equation with unbounded damping*, J. Differential Equations **264** (2018), 7023–7054.

[Ger21] B. Gerhat, *Schur complement dominant operator matrices.*

[GGHT12] F. Gesztesy, J. A. Goldstein, H. Holden, and G. Teschl, *Abstract wave equations and associated Dirac-type operators*, Ann. Mat. Pura Appl. (4) **191** (2012), 631–676.

[GH18] D. S. Grebenkov and B. Helffer, *On spectral properties of the Bloch-Torrey operator in two dimensions*, SIAM J. Math. Anal. **50** (2018), 622–676.

[GIS20] B. Gerhat, O. Ibrogimov, and P. Siegl, *On the point spectrum in the Ekman boundary layer problem*, 2020.

[GKMV13] L. Grubišić, V. Kostrykin, K. A. Makarov, and K. Veselić, *Representation theorems for Indefinite quadratic forms revisited*, Mathematika **59** (2013), 169–189.

[GS21] B. Gerhat and P. Siegl, *Schrödinger operators with accretive potentials in weighted spaces.*

[GT21] B. Gerhat and C. Tretter, *Pseudo numerical ranges and spectral enclosures.*

[Hel13] B. Helffer, *Spectral theory and its applications*, Cambridge University Press, 2013.

[Ibr17] O. O. Ibrogimov, *Essential spectrum of non-self-adjoint singular matrix differential operators*, J. Math. Anal. Appl. **451** (2017), 473–496.

[IST16] O. O. Ibrogimov, P. Siegl, and C. Tretter, *Analysis of the essential spectrum of singular matrix differential operators*, J. Differential Equations **260** (2016), 3881–3926.

[IT17] O. O. Ibrogimov and C. Tretter, *Essential spectrum of elliptic systems of pseudo-differential operators on $L^2(\mathbb{R}^N) \oplus L^2(\mathbb{R}^N)$*, J. Pseudo-Differ. Oper. Appl. **8** (2017), 147–166.

[IT20] R. Ikehata and H. Takeda, *Uniform energy decay for wave equations with unbounded damping coefficients*, Funkcialaj Ekvacioj **63** (2020), 133–152.

[JT09] B. Jacob and C. Trunk, *Spectrum and analyticity of semigroups arising in elasticity theory and hydromechanics*, Semigroup Forum **79** (2009), 79–100.

[JTTV18] B. Jacob, C. Tretter, C. Trunk, and H. Vogt, *Systems with strong damping and their spectra*, Math. Methods Appl. Sci. **41** (2018), 6546–6573.

[Kat95] T. Kato, *Perturbation theory for linear operators*, Classics in Mathematics, Springer-Verlag, Berlin, 1995, Reprint of the 1980 edition.

[KLN08] P. Kurasov, I. Lelyavin, and S. Naboko, *On the essential spectrum of a class of singular matrix differential operators. II. Weyl's limit circles for the Hain-Lüst operator whenever quasi-regularity conditions are not satisfied*, Proc. Roy. Soc. Edinburgh Sect. A **138** (2008), 109–138.

[Kon98] A. Yu. Konstantinov, *Spectral theory of some matrix differential operators of mixed order*, Ukraïn. Mat. Zh. **50** (1998), 1064–1072.

[KRRS17] D. Krejčiřík, N. Raymond, J. Royer, and P. Siegl, *Non-accretive Schrödinger operators and exponential decay of their eigenfunctions*, Israel J. Math. **221** (2017), 779–802.

[KSTV15] D. Krejčiřík, P. Siegl, M. Tater, and J. Viola, *Pseudospectra in non-Hermitian quantum mechanics*, J. Math. Phys. **56** (2015), 103513.

[LT98] H. Langer and C. Tretter, *Spectral decomposition of some nonselfadjoint block operator matrices*, J. Operator Theory **39** (1998), no. 2, 339–359.

[Mar88] A. S. Markus, *Introduction to the spectral theory of polynomial operator pencils*, American Mathematical Society, Providence, 1988.

[McI68] A. McIntosh, *Representation of bilinear forms in Hilbert space by linear operators*, Trans. Amer. Math. Soc. **131** (1968), 365–377.

[MT07] M. Marletta and C. Tretter, *Essential spectra of coupled systems of differential equations and applications in hydrodynamics*, J. Differential Equations **243** (2007), 36–69.

[Nag89] R. Nagel, *Towards a "matrix theory" for unbounded operator matrices*, Math. Z. **201** (1989), 57–68.

[Nag90] ———, *The spectrum of unbounded operator matrices with nondiagonal domain*, J. Funct. Anal. **89** (1990), 291–302.

[Nev93] O. Nevanlinna, *Convergence of iterations for linear equations*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 1993.

[RT18] T. H. Rasulov and C. Tretter, *Spectral inclusion for unbounded diagonally dominant $n \times n$ operator matrices*, Rocky Mountain J. Math. **48** (2018), 279–324.

[RTW14] A. Radl, C. Tretter, and M. Wagenhofer, *The block numerical range of analytic operator functions*, Oper. Matrices **8** (2014), 901–934.

[Sch15] S. Schmitz, *Representation theorems for indefinite quadratic forms without spectral gap*, Integral Equations Operator Theory **83** (2015), 73–94.

[SK12] P. Siegl and D. Krejčiřík, *On the metric operator for the imaginary cubic oscillator*, Phys. Rev. D **86** (2012), 121702(R).

[SS21] I. Semorádová and P. Siegl, *Diverging eigenvalues in domain truncations of Schrödinger operators with complex potentials*, arXiv:2107.10557 [math.SP], 2021.

[tESV15] A. F. M. ter Elst, M. Sauter, and H. Vogt, *A generalisation of the form method for accretive forms and operators*, J. Funct. Anal. **269** (2015), 705–744.

[Tre08] C. Tretter, *Spectral Theory of Block Operator Matrices and Applications*, Imperial College Press, 2008.

[Tre09] C. Tretter, *Spectral inclusion for unbounded block operator matrices*, J. Funct. Anal. **256** (2009), no. 11, 3806–3829.

[Tre10] ______, *The quadratic numerical range of an analytic operator function*, Complex Anal. Oper. Theory **4** (2010), 449–469.

[TW03] C. Tretter and M. Wagenhofer, *The block numerical range of an $n \times n$ block operator matrix*, SIAM J. Matrix Anal. Appl. **24** (2003), 1003–1017.

[Wag07] M. Wagenhofer, *Block numerical ranges*, Ph.D. thesis, University of Bremen, 2007.

APPENDIX A

Research papers

SCHUR COMPLEMENT DOMINANT OPERATOR MATRICES

BORBALA GERHAT

Abstract. In mathematical physics, matrix differential operators arise naturally in applications as coupled systems of partial differential equations. Up to now, the spectral analysis of such problems has commonly been tackled assuming certain patterns of relative boundedness within the matrix entries. We propose to view operator matrices in a more general setting, which allows our results to abstain from perturbative arguments of this type. Rather than requiring the matrix to act in a Hilbert space $\mathcal{H}$, we extend its action to a suitable distributional triple $\mathcal{D} \subset \mathcal{H} \subset \mathcal{D}_-$ and restrict it to its maximal domain in $\mathcal{H}$. The crucial point in our approach is the choice of the spaces $\mathcal{D}$ and $\mathcal{D}_-$ which are essentially determined by the Schur complement of the matrix. We show spectral equivalence between the resulting operator matrix in $\mathcal{H}$ and its Schur complement, eventually implying closedness and non-empty resolvent set of the operator matrix. Finally, we apply our abstract results to the damped wave equation with possibly unbounded and/or singular damping, as well as to second order matrix differential operators with certain minimal restrictions on their coefficients. By means of our methods, the previously used regularity assumptions can be weakened substantially in both cases.

1. Introduction

Motivated by a wide range of applications, operator matrices emerge from coupled systems of linear partial differential equations and have been of considerable interest, see e.g. the pioneering work [26], the monograph [29] and the references therein. Typically, the spectral analysis of such problems is rather challenging, starting with the non-trivial task of determining a suitable domain of definition on which the resulting operator matrix is closed and has non-empty resolvent set. An often fruitful approach is to establish a certain spectral correspondence between the operator matrix

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathcal{H} \supset \operatorname{dom} \mathcal{A} \to \mathcal{H}, \qquad \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2,$$

and one of its (two) so-called Schur complements $S(\cdot)$, the (scalar) operator family

$$S(\lambda) = A - \lambda - B(D-\lambda)^{-1}C : \mathcal{H}_1 \supset \operatorname{dom} S(\lambda) \to \mathcal{H}_1, \qquad \lambda \in \rho(D).$$

2010 *Mathematics Subject Classification.* 35L05, 35P05, 47A56, 47D06.

Key words and phrases. Schur complement, operator matrices, distributional triplets, essential spectrum, semi-Fredholmness, damped wave equation.

The author gratefully acknowledges the support of the *Swiss National Science Foundation*, grant no. 169104. She would like to thank the *SEMP* student exchange program for their support, as well as the *Queen's University Belfast* for their hospitality. Moreover, she thanks her supervisor *C. Tretter*, University of Bern, and expresses her gratitude to *O. Ibrogimov* and *P. Siegl*, Queen's University of Belfast, for fruitful mathematical discussions and to *J.-C. Cuenin*, Loughborough University, for drawing her attention to the relevant reference [12].

Although the latter gives rise to a non-linear spectral problem, in general, more methods are available for its analysis.

Up to now, the spectral correspondence described above has mainly been achieved by taking advantage of certain patterns of relative boundedness between the entries of the operator matrix, see e.g. [25]. There are several results in [12, 13, 15, 16, 17], however, which seem to abstain from this type of perturbative argument. Inspired by their methods, we propose a more general framework for operator matrices and allow a systematic approach to the spectral analysis of a wider class of problems. We point out that even though our approach was inspired by the conceptual observations and ideas in [13, 15, 16, 17], its scope goes beyond the latter; not only are our results due to their abstract nature much more versatile, but even applied to particular problems in the mentioned references, they allow much weaker natural and in some sense minimal (even distributional) regularity of the coefficients, see the applications in Sections 4 and 6.

Our method combines a distributional setting with the assumption that, in a certain sense, the Schur complement dominates all other terms in the Frobenius-Schur factorisation of the resolvent. Said distributional approach consists of extending the operator matrix to a suitable triplet of Hilbert spaces

$$\mathcal{D} \subset \mathcal{H} \subset \mathcal{D}_-,$$

where each of the above inclusions represents a continuous embedding with dense range. More precisely, we define the action of $\mathcal{A}$ on a space of test functions $\mathcal{D}$ with values in a space of distributions $\mathcal{D}_-$ and consider its restriction $\mathcal{A}_0$ to the maximal domain

$$\operatorname{dom} \mathcal{A}_0 = \{\mathbf{x} \in \mathcal{D} : \mathcal{A}\mathbf{x} \in \mathcal{H}\}.$$

This method has been employed successfully in the past; however, except in [12], the spaces of test functions and distributions have consistently been determined by the underlying patterns of relative boundedness within the operator matrix, e.g. as form domain of some entry and its dual space in [4, Chap. 1.2.1] or [19].

The key novelty in our approach is to choose the spaces $\mathcal{D}$ and $\mathcal{D}_-$ in a way that the Schur complement $S(\cdot)$ consists of bounded and boundedly invertible operators between their first components $\mathcal{D}_S$ and $\mathcal{D}_{-S}$. This choice guarantees the required dominance of the Schur complement and allows us to relate invertibility and semi-Fredholmness of $\mathcal{A}_0$ to invertibility and semi-Fredholmness of $S_0(\cdot)$ defined as family of maximal operators in $\mathcal{H}_1$. We thus obtain equivalence of their (point and essential) spectra, which in applications might eventually lead to desired properties like closedness and non-empty resolvent set of the operator matrix and its semigroup generation, see Sections 4 and 6.

Although not closely related to our framework, we mention another non-standard approach in [27] towards the spectral analysis of operator matrices. The setting therein, however, covers problems of different type and essentially aims at incorporating mixing boundary conditions. Moreover, from a structural point of view, the latter is more restrictive than our approach and requires diagonal dominance of the underlying operator matrix; note that the dominance assumption therein was relaxed in [5] to less restrictive patterns of relative boundedness within the matrix entries.

We apply our abstract results to the linearly damped wave equation on $\Omega \subset \mathbb{R}^n$ with Dirichlet boundary conditions and non-negative damping a and potential q, which gives rise to the operator matrix

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ \Delta - q & -2a \end{pmatrix} \tag{1.1}$$

in a suitable Hilbert space. Unlike in most of the previously existing results, see e.g. [4, 14, 19], we allow the damping and potential to be singular and/or unbounded at infinity and do not require the damping to be relatively bounded with respect to $\Delta - q$. To the best of our knowledge, this case has only been covered in the works [13, 18]. In [13], assuming that $a \in W^{1,\infty}_{\mathrm{loc}}(\overline{\Omega})$ and essentially that for every $\varepsilon > 0$ there exists $C_\varepsilon \geq 0$ with $|\nabla a| \leq \varepsilon a^{\frac{3}{2}} + C_\varepsilon$, see [13, Asm. I] for the precise more general assumptions, the spectral equivalence of $\mathcal{A}$ to its Schur complement

$$S(\lambda) = -\frac{1}{\lambda}(-\Delta + q + 2\lambda a + \lambda^2), \qquad \lambda \in \mathbb{C} \setminus \{0\},$$

as an operator family in $L^2(\Omega)$ was established, leading to the generation of a contraction semigroup and thus existence and uniqueness of the solutions to the underlying equation. In [18] on the other hand, an approximation procedure is performed to construct a unique weak solution, whose norm and total energy are shown to decay polynomially in time.

Merely assuming $a, q \in L^1_{\mathrm{loc}}(\Omega)$, our methods provide spectral correspondence between matrix and Schur complement, as well as the generation of a contraction semigroup. We thereby significantly generalise [13], where technical assumptions on growth and regularity of the damping are needed in order to describe the operator domain of the Schur complement. Notice that under the latter more restrictive assumptions, our realisation of the operator matrix (1.1) coincides with the one defined in [13], see Remark 4.3. Moreover, we point out that our method can equally be employed to realise distributional dampings as considered e.g. in [4], see Remark 4.4.

As another application of our results, we present second order matrix differential operators of the form

$$\mathcal{A} = \begin{pmatrix} -\Delta + q & \nabla \cdot \mathbf{b} \\ \mathbf{c} \cdot \nabla & d \end{pmatrix} \tag{1.2}$$

on $\Omega \subset \mathbb{R}^n$ with Dirichlet boundary conditions and low regularity coefficients. Problems of this type arise in areas like magnetohydrodynamics or astrophysics and have been previously studied in e.g. [7, 15, 16, 17, 21, 22] and the references therein. Our methods allow us to avoid typical technical assumptions like $q \in C(\Omega)$, $\mathbf{b}, \mathbf{c} \in C^1(\Omega)^n$ and $d \in C^1(\Omega)$, see e.g [16]. Under certain *natural weak regularity conditions* on the coefficients, see Assumption 6.1, we are able to define a closed realisation of the operator matrix (1.2) in the space $L^2(\Omega) \oplus L^2(\Omega)$ with non-empty resolvent set. We show spectral equivalence to its Schur complement and, imposing additional assumptions on the structure of the matrix, that it generates a contraction semigroup.

We further apply our results to study Klein-Gordon equations with purely imaginary, (merely) locally square integrable potentials. The physically relevant case of real potentials, in which the problem exhibits an indefinite structure, was studied in many works using several different methods, see e.g. [23, 24] where Krein spaces were employed or [30] where for linear potentials a different approach via oscillatory integrals was taken. The case of (purely) imaginary potentials, however, seems to have not been considered so far. In fact, the problem is then equivalent to a wave equation with suitable damping and potential and can be treated by applying our previously obtained methods. We thereby show the spectral equivalence between matrix an Schur complement and conclude that the resulting operator matrix is densely defined and boundedly invertible, see Theorem 5.1. In the particular case of the potential $V(x) = \mathrm{i}\,x$ in one dimension, we show in Example 5.2 that the spectrum of the operator matrix is empty; this is not surprising considering similar results related to the complex Airy operator in e.g. [2, 3].

In order to further illustrate the impact of our construction, we consider Dirac operators with certain Coulomb type potentials. A self-adjoint realisation of the latter which was introduced in [12] can indeed be recovered as a special case of our abstract framework. The key step in the analysis in [12] is a Hardy-Dirac inequality for a class of Coulomb-like potentials, see (7.2), which was derived in [9, 10]. We translate the existing results to our setting, where the mentioned Dirac-Hardy inequality ensures that the Schur complement is bounded and boundedly invertible between the required spaces $\mathcal{D}_S$ and $\mathcal{D}_{-S}$.

Finally, we show that our method is not limited to applications where the spaces $\mathcal{D}_S$ and $\mathcal{D}_{-S}$ are given by the form domain of the Schur complement and its dual space. We demonstrate this fact based on a constant coefficient differential operator matrix where $\mathcal{D}_S = H^1(\mathbb{R}^n)$ and $\mathcal{D}_{-S} = H^{-2}(\mathbb{R}^n)$, while the natural form domain of the Schur complement

$$S(\lambda) = \Delta - \lambda + \Delta^2(\Delta - \lambda)^{-1}\sqrt{-\Delta}, \qquad \lambda \in \mathbb{C} \setminus (-\infty, 0],$$

is $H^{\frac{3}{2}}(\mathbb{R}^n)$, see Section 7.2. We stress that this example serves an illustrative purpose only and is kept as simple as possible; relevant problems with similar structure can be found in [15, 16, 17].

This paper is organised as follows. In Section 2, we present our main abstract results and lay the ground for the spectral equivalence between $\mathcal{A}_0$ and S_0; the main theorems are Theorem 2.7 and 2.8, where we show that invertibility or semi-Fredholmness of $\mathcal{A}_0$ and S_0, respectively, imply invertibility or semi-Fredholmness of S_0 and $\mathcal{A}_0$. In Section 3, we translate the results of Section 2 into several corollaries providing spectral equivalence between the operator matrix and its Schur complement as an operator family. In Section 4, we apply the established spectral equivalence to the damped wave equation with unbounded and/or singular damping and potential; in particular, the main Theorem 4.2 states the generation of a contraction semigroup for the underlying problem. While Section 5 concerns the spectrum of the Klein-Gordon equation with purely imaginary potential, Section 6 contains the application of our results to second order matrix differential operators with low regularity coefficients; the main results in the latter are Theorem 6.3, which states spectral equivalence between matrix and Schur complement, and Theorem 6.5, which shows the generation of a contraction semigroup under some additional structural assumptions. Finally, Section 7 further illustrates the nature of our results on Dirac operators with Coulomb type potentials and a constant coefficient matrix differential operator.

1.1. **Notation and preliminaries.** Let $n \in \mathbb{N}$ be the spatial dimension, let $\langle \cdot, \cdot \rangle_{\mathbb{C}^n}$ be the inner product and $|\cdot|$ the euclidean norm on $\mathbb{C}^n$. We denote the real scalar product and tensor product on $\mathbb{C}^n$, respectively, by

$$\xi \cdot \eta = \sum_{j=1}^{n} \xi_j \eta_j, \qquad \xi \otimes \eta = (\xi_j \eta_k)_{j,k} \in \mathbb{C}^{n\times n}, \qquad \xi, \eta \in \mathbb{C}^n.$$

For $\Omega \subset \mathbb{R}^n$ and a measurable function $m : \Omega \to \mathbb{C}$, we denote by $\operatorname{dom} m$ the maximal domain of the corresponding multiplication operator in $L^2(\Omega)$. Moreover, recall the definition of the essential range

$$\operatorname{ess\,ran} m = \{z \in \mathbb{C} : \lambda^n(m^{-1}(B_\varepsilon(z))) > 0 \text{ for any } \varepsilon > 0\},$$

where λ^n is the n-dimensional Lebesgue measure.

The space of matrix-valued locally integrable functions will be denoted

$$L^1_{\mathrm{loc}}(\Omega)^{n\times n} = \{M = (M_{jk})_{j,k} : \Omega \to \mathbb{C}^{n\times n} : M_{jk} \in L^1_{\mathrm{loc}}(\Omega),\ 1 \le j,k \le n\};$$

note that this is equivalent to $\|M\| \in L^1_{\mathrm{loc}}(\Omega)$ for any norm $\|\cdot\|$ on $\mathbb{C}^{n\times n}$. The spaces $L^\infty(\Omega)^n$ and $L^2_{\mathrm{loc}}(\Omega)^n$ shall be defined analogously. Moreover, $L^2(\Omega;\omega)$ shall denote the weighted L^2-space on Ω with non-negative measurable weight ω.

Let $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}$ be Hilbert spaces. The set of closed linear operators from $\mathcal{G}_1$ to $\mathcal{G}_2$ will be denoted by $\mathcal{C}(\mathcal{G}_1,\mathcal{G}_2)$, the space of everywhere defined and bounded operators by $\mathcal{B}(\mathcal{G}_1,\mathcal{G}_2)$ and the space of compact operators by $\mathcal{K}(\mathcal{G}_1,\mathcal{G}_2)$. Moreover, as usual, we write $\mathcal{C}(\mathcal{G}) = \mathcal{C}(\mathcal{G},\mathcal{G})$, $\mathcal{B}(\mathcal{G}) = \mathcal{B}(\mathcal{G},\mathcal{G})$ and $\mathcal{K}(\mathcal{G}) = \mathcal{K}(\mathcal{G},\mathcal{G})$.

We denote the duality pairing and anti-dual space of $\mathcal{G}$ by

$$(\phi, f)_{\mathcal{G}^*\times\mathcal{G}} = \phi(f), \quad f \in \mathcal{G}, \quad \phi \in \mathcal{G}^* = \{\phi : \mathcal{G} \to \mathbb{C} : \phi \text{ antilinear, bounded}\};$$

note that we hereby adopt the conventions in [11] and work with *antilinear* functionals, which allows us to identify bounded sesquilinear forms on $\mathcal{G}$ with bounded linear operators $\mathcal{G} \to \mathcal{G}^*$. This convention includes defining the space of distributions on $\Omega \subset \mathbb{R}^n$, denoted $\mathcal{D}'(\Omega)$, as *antilinear* continuous functionals on the space of test functions $C_0^\infty(\Omega)$.

The resolvent set, spectrum and point spectrum of a linear operator T in $\mathcal{G}$ will be denoted by $\rho(T)$, $\sigma(T)$ and $\sigma_{\mathrm{p}}(T)$, respectively. We use the following (one of five in general non-equivalent) definitions of the essential spectrum

$$\sigma_{\mathrm{e}2}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \mathcal{F}_+(\mathcal{G})\};$$

here the semi-Fredholm operators in $\mathcal{G}$ with finite nullity and deficiency, respectively, are defined in the following way

$$\begin{aligned}\mathcal{F}_+(\mathcal{G}) &= \{T \in \mathcal{C}(\mathcal{G}) : \operatorname{ran} T \text{ closed},\ \dim\ker T < \infty\},\\ \mathcal{F}_-(\mathcal{G}) &= \{T \in \mathcal{C}(\mathcal{G}) : \operatorname{ran} T \text{ closed},\ \dim\operatorname{coker} T < \infty\}.\end{aligned}$$

Here the co-kernel of T is the closed subspace $\operatorname{coker} T = (\operatorname{ran} T)^\perp$. If $T \in \mathcal{C}(\mathcal{G})$ and $\operatorname{ran} T$ is closed, one can define the generalised inverse of T as

$$T^\# = (T|_{\operatorname{dom} T\cap\ker T^\perp})^{-1} \in \mathcal{B}(\mathcal{G}); \tag{1.3}$$

recall that, if P and Q, respectively, denote the orthogonal projections on $\ker T$ and $\operatorname{coker} T$, then

$$\operatorname{ran} T^\# \subset \operatorname{dom} T, \qquad TT^\# = I - Q, \qquad T^\# T \subset I - P. \tag{1.4}$$

Note that, since T is closed, both $\ker T$ and $\operatorname{coker} T$ are closed and the projections above are bounded.

Throughout the entire paper, $\mathcal{H}_1$ and $\mathcal{H}_2$ denote complex Hilbert spaces and $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ their orthogonal sum. The canonical projections of $\mathcal{H}$ on $\mathcal{H}_1$ and

$\mathcal{H}_2$, respectively, are denoted by π_1 and π_2; their adjoints π_1^* and π_2^* are the corresponding canonical embeddings of $\mathcal{H}_1$ and $\mathcal{H}_2$ in $\mathcal{H}$.

Finally, we write $\lesssim$ or $\gtrsim$ if the respective inequalities hold with a multiplicative constant depending only on quantities which are fixed and the dependence on which is thus irrelevant.

2. Main abstract results

We present a new abstract setting for the spectral analysis of operator matrices. Under fairly general assumptions, our approach allows to establish a correspondence between semi-Fredholmness and invertibility of an operator matrix and one of its Schur complements; the latter provides a relation between (point and essential) spectra of matrix and Schur complement as an operator family, see Section 3.

Our strategy is the following. Rather than directly defining an operator matrix in the product space $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, we consider its entries acting in suitable triplets (2.1) and restrict the matrix action to the maximal domain in $\mathcal{H}$, see Definition 2.3. A crucial point of our construction is the choice of spaces $\mathcal{D}_S$ and $\mathcal{D}_{-S}$, such that the Schur complement $S : \mathcal{D}_S \to \mathcal{D}_{-S}$ is bounded and boundedly invertible. Notice also that we construct a generalised Schur complement, which unlike the classical definition does not require the entry D to be boundedly invertible but merely to have a generalised inverse, see Assumption 2.1 (iii) and (1.3), (1.4).

Proofs of the results in this section can be found in Section 2.5.

2.1. Assumptions and definitions.

We work exclusively with the first Schur complement; clearly, all results in this section hold under analogous assumptions for the second Schur complement, see Remark 2.4.

Assumption 2.1. (i) Let $\mathcal{D}_S$, $\mathcal{D}_2$, $\mathcal{D}_{-S}$, $\mathcal{D}_{-1}$ and $\mathcal{D}_{-2}$ be complex Hilbert spaces. Assume that

$$\mathcal{D}_S \subset \mathcal{H}_1 \subset \mathcal{D}_{-S}, \qquad \mathcal{D}_2 \subset \mathcal{H}_2 \subset \mathcal{D}_{-2}, \tag{2.1}$$

where the corresponding canonical embeddings are continuous and have dense ranges. Moreover, let $\mathcal{D}_{-S} \subset \mathcal{D}_{-1}$ be continuously embedded.

(ii) Assume that the operators A, B and C are bounded between the spaces

$$A \in \mathcal{B}(\mathcal{D}_S, \mathcal{D}_{-1}), \qquad B \in \mathcal{B}(\mathcal{D}_2, \mathcal{D}_{-1}), \qquad C \in \mathcal{B}(\mathcal{D}_S, \mathcal{D}_{-2}).$$

(iii) Let $D_0 \in \mathcal{C}(\mathcal{H}_2)$ have closed range in $\mathcal{H}_2$, let $\operatorname{dom} D_0 \subset \mathcal{D}_2$ be dense in $\mathcal{D}_2$ and assume that there exist extensions

$$D_0 \subset D \in \mathcal{B}(\mathcal{D}_2, \mathcal{D}_{-2}), \qquad D_0^{\#} \subset D^{\ddagger} \in \mathcal{B}(\mathcal{D}_{-2}, \mathcal{D}_2). \qquad //$$

Remark 2.2. We point out that $\mathcal{D}_{-S} = \mathcal{D}_{-1}$ in the applications in Sections 4, 5 and 7.2. However, allowing A and B to map to a larger space widens the range of applicability of our assumptions, see e.g. Sections 6 and 7.1 where $\mathcal{D}_{-S} \subsetneq \mathcal{D}_{-1}$. While $\mathcal{D}_{-S}$ plays an essential role and is determined by the Schur complement, the auxiliary space $\mathcal{D}_{-1}$ merely provides an environment for the operations needed in order to define the latter; notice that in Sections 6 and 7.1 it is clear from the construction that $\mathcal{D}_{-1}$ can even be chosen arbitrarily large in a certain sense.

The operator matrix $\mathcal{A}_0$ and its first Schur complement S_0 are defined as the following maximal operators in the underlying Hilbert spaces. We emphasise that, although their extensions $\mathcal{A}$ and S are assumed to be bounded between suitable spaces, $\mathcal{A}_0$ and S_0 are in general unbounded in $\mathcal{H}$ and $\mathcal{H}_1$, respectively.

Definition 2.3. Let Assumption 2.1 be satisfied. We define

$$\mathcal{A} := \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{B}(\mathcal{D}, \mathcal{D}_-), \qquad S := A - BD^\ddagger C \in \mathcal{B}(\mathcal{D}_S, \mathcal{D}_{-1}), \tag{2.2}$$

where $\mathcal{D} := \mathcal{D}_S \oplus \mathcal{D}_2$ and $\mathcal{D}_- := \mathcal{D}_{-1} \oplus \mathcal{D}_{-2}$. Moreover, let the corresponding maximal operators $\mathcal{A}_0 := \mathcal{A}|_{\operatorname{dom}\mathcal{A}_0}$ in $\mathcal{H}$ and $S_0 := S|_{\operatorname{dom} S_0}$ in $\mathcal{H}_1$ be defined on their respective domains

$$\operatorname{dom}\mathcal{A}_0 := \{(f,g) \in \mathcal{D} : \mathcal{A}(f,g) \in \mathcal{H}\}, \qquad \operatorname{dom} S_0 := \{f \in \mathcal{D}_S : Sf \in \mathcal{H}_1\}. \quad //$$

Notice that if $0 \in \rho(D_0)$, then $D^\ddagger = D^{-1}$ and the definition of S reduces to the standard formula for the Schur complement, see Lemma 2.12 below.

Remark 2.4. All results in the present section hold analogously for the second Schur complement. The assumptions have to be translated in a straightforward way as follows. One assumes the following inclusions between the Hilbert spaces

$$\mathcal{D}_1 \subset \mathcal{H}_1 \subset \mathcal{D}_{-1}, \qquad \mathcal{D}_S \subset \mathcal{H}_2 \subset \mathcal{D}_{-S} \subset \mathcal{D}_{-2},$$

where the corresponding canonical embeddings are continuous and all except the embedding $\mathcal{D}_{-S} \hookrightarrow \mathcal{D}_{-2}$ are assumed to have dense range. Moreover,

$$B \in \mathcal{B}(\mathcal{D}_S, \mathcal{D}_{-1}), \qquad C \in \mathcal{B}(\mathcal{D}_1, \mathcal{D}_{-2}), \qquad D \in \mathcal{B}(\mathcal{D}_S, \mathcal{D}_{-2}).$$

The operator $A_0 \in \mathcal{C}(\mathcal{H}_1)$ has closed range in $\mathcal{H}_1$, $\operatorname{dom} A_0$ is dense in $\mathcal{D}_1$ and

$$A_0 \subset A \in \mathcal{B}(\mathcal{D}_1, \mathcal{D}_{-1}), \qquad A_0^\# \subset A^\ddagger \in \mathcal{B}(\mathcal{D}_{-1}, \mathcal{D}_1).$$

Then the matrix $\mathcal{A}$ is defined as in (2.2) with $\mathcal{D} = \mathcal{D}_1 \oplus \mathcal{D}_S$ and the second Schur complement is given by

$$S = D - CA^\ddagger B \in \mathcal{B}(\mathcal{D}_S, \mathcal{D}_{-2}).$$

With domains analogous to the ones in Definition 2.3, $\mathcal{A}_0$ and S_0, respectively, are the corresponding maximal operators in $\mathcal{H}$ and $\mathcal{H}_2$. //

2.2. **Dense domain and point spectrum.** We start our analysis of the relation between $\mathcal{A}_0$ and S_0 by providing sufficient conditions for the matrix $\mathcal{A}_0$ to be densely defined in $\mathcal{H}$.

Proposition 2.5. *Let Assumption* 2.1 *be satisfied.*

(i) *Let* $\operatorname{coker} D_0 = \{0\}$, *let* $\operatorname{dom} S_0$ *be dense in* $\mathcal{H}_1$ *and assume with*

$$\operatorname{dom} B_0 := \{f \in \mathcal{D}_2 : Bf \in \mathcal{H}_1\} \tag{2.3}$$

that $\operatorname{dom} B_0 \cap \operatorname{dom} D_0$ *is dense in* $\mathcal{H}_2$. *Then* $\operatorname{dom}\mathcal{A}_0$ *is dense in* $\mathcal{H}$.

(ii) *Let* $\mathcal{D}_{-S} = \mathcal{D}_{-1}$, *let* $0 \in \rho(D_0) \cap \rho(S_0)$ *and let* $\operatorname{dom} S_0$ *be dense in* $\mathcal{D}_S$. *Moreover, assume there exists* $z \in \mathbb{C}$ *such that* $\operatorname{ran}(S_0 - z)$ *is closed in* $\mathcal{H}_1$ *with* $\operatorname{coker}(S_0 - z) = \{0\}$ *and such that there exists an extension*

$$(S_0 - z)^\# \subset S_z^\ddagger \in \mathcal{B}(\mathcal{D}_{-S}, \mathcal{D}_S). \tag{2.4}$$

Then $\operatorname{dom}\mathcal{A}_0$ *is dense in both* $\mathcal{D}$ *and* $\mathcal{H}$.

If D_0 is boundedly invertible, the kernels and thus point spectra of $\mathcal{A}_0$ and S_0 are related as follows.

Proposition 2.6. *Let Assumption* 2.1 *be satisfied and* $0 \in \rho(D_0)$. *Then*

$$0 \in \sigma_{\mathrm{p}}(\mathcal{A}_0) \iff 0 \in \sigma_{\mathrm{p}}(S_0)$$

and the following identities hold

$$\ker \mathcal{A}_0 = \{(f, -D^{-1}Cf) : f \in \ker S_0\}, \qquad \dim \ker \mathcal{A}_0 = \dim \ker S_0.$$

2.3. **Semi-Fredholmness and bounded invertibility.** We proceed by establishing a relation between bounded invertibility/semi-Fredholmness of the matrix and bounded invertibility/semi-Fredholmness of its Schur complement. This in turn provides a connection between the (essential) spectra of $\mathcal{A}_0$ and $S_0(\cdot)$ as an operator family, see Section 3. Proofs of the following results can be found in Section 2.5.

Theorem 2.7. *Let Assumption* 2.1 *be satisfied.*

(i) *Let* $0 \in \rho(D_0)$. *Then*

$$0 \in \rho(\mathcal{A}_0) \implies 0 \in \rho(S_0).$$

(ii) *Let* $S_0 \in \mathcal{C}(\mathcal{H}_1)$ *and assume either that* $\operatorname{coker} D_0 = \{0\}$ *or that* $D_0 \in \mathcal{F}_-(\mathcal{H}_2)$ *with* $\operatorname{coker} D_0 \subset \operatorname{dom} C_0^*$ *and* $\operatorname{dom} C_0$ *is dense in* $\mathcal{D}_S$, *where*

$$C_0 := C|_{\operatorname{dom} C_0}, \qquad \operatorname{dom} C_0 := \{f \in \mathcal{D}_S : Cf \in \mathcal{H}_2\}.$$

Then

$$\mathcal{A}_0 \in \mathcal{F}_+(\mathcal{H}) \implies S_0 \in \mathcal{F}_+(\mathcal{H}_1).$$

We establish the reverse implication, i.e. bounded invertibility/semi-Fredholmness of S_0 implying bounded invertibility/semi-Fredholmness of $\mathcal{A}_0$.

Theorem 2.8. *Let Assumption* 2.1 *be satisfied and assume that* $S \in \mathcal{B}(\mathcal{D}_S, \mathcal{D}_{-S})$ *and* $B(\operatorname{dom} D_0) \subset \mathcal{D}_{-S}$.

(i) *Let* $0 \in \rho(D_0)$. *If* $S^{-1} \in \mathcal{B}(\mathcal{D}_{-S}, \mathcal{D}_S)$ (*which by the continuity of the embeddings* $\mathcal{D}_S \hookrightarrow \mathcal{H}_1 \hookrightarrow \mathcal{D}_{-S}$ *implies that* $0 \in \rho(S_0)$), *then* $0 \in \rho(\mathcal{A}_0)$.

(ii) *Let* $\operatorname{dom} S_0$ *be dense in* $\mathcal{D}_S$, *let* $D_0 \in \mathcal{F}_+(\mathcal{H}_2)$ *and* $S_0 \in \mathcal{F}_+(\mathcal{H}_1)$ *such that there exists an extension* $S^\ddagger \supset S_0^\#$ *with* $S^\ddagger \in \mathcal{B}(\mathcal{D}_{-S}, \mathcal{D}_S)$. *Then* $\mathcal{A}_0 \in \mathcal{F}_+(\mathcal{H})$.

Finally, we provide a sufficient condition for the existence of the extension $S^\ddagger$ above; a corollary of Theorem 2.8 then reads as follows.

Corollary 2.9. *Let Assumption* 2.1 *be satisfied, let* $S \in \mathcal{B}(\mathcal{D}_S, \mathcal{D}_{-S})$, *let* $\operatorname{dom} S_0$ *be dense in* $\mathcal{D}_S$ *and* $B(\operatorname{dom} D_0) \subset \mathcal{D}_{-S}$. *Moreover, assume there exists* $z \in \mathbb{C}$ *such that* $\operatorname{ran}(S_0 - z)$ *is closed in* $\mathcal{H}_1$ *with* $\operatorname{coker}(S_0 - z) = \{0\}$ *and such that there exists an extension* $S_z^\ddagger$ *as in* (2.4).

(i) *Let* $0 \in \rho(D_0)$. *Then*

$$0 \in \rho(S_0) \implies 0 \in \rho(\mathcal{A}_0).$$

(ii) *Let* $D_0 \in \mathcal{F}_+(\mathcal{H}_2)$. *Then*

$$S_0 \in \mathcal{F}_+(\mathcal{H}_1) \implies \mathcal{A}_0 \in \mathcal{F}_+(\mathcal{H}).$$

Remark 2.10. The claims of both Proposition 2.5 (ii) and Corollary 2.9 remain true for finite dimensional $\operatorname{coker}(S_0 - z) \neq \{0\}$ if the orthogonal projection on $\operatorname{coker}(S_0 - z)$ in $\mathcal{H}_1$ has an extension in $\mathcal{B}(\mathcal{D}_{-S}, \mathcal{H}_1)$. Analogously, Proposition 2.5 (i) still holds if $\operatorname{coker} D_0 \neq \{0\}$ is finite dimensional and the orhtogonal projection on $\operatorname{coker} D_0$ in $\mathcal{H}_2$ has a bounded extension in $\mathcal{B}(\mathcal{D}_{-2}, \mathcal{H}_2)$. //

2.4. **Technical lemmas.** We start by stating and proving some auxiliary results.

Lemma 2.11. *Let $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ be Hilbert spaces such that $\mathcal{G}_2 \subset \mathcal{G}_3$ is continuously embedded. If $T \in \mathcal{B}(\mathcal{G}_1, \mathcal{G}_3)$ and $\operatorname{ran} T \subset \mathcal{G}_2$, then $T \in \mathcal{B}(\mathcal{G}_1, \mathcal{G}_2)$.*

Proof. Since $T \in \mathcal{C}(\mathcal{G}_1, \mathcal{G}_3)$ and $\mathcal{G}_2$ is continuously embedded in $\mathcal{G}_3$, we also have $T \in \mathcal{C}(\mathcal{G}_1, \mathcal{G}_2)$. The claim now follows from the closed graph theorem. □

Lemma 2.12. *Let $\mathcal{G}_1$, $\mathcal{G}$ and $\mathcal{G}_{-1}$ be Hilbert spaces such that*

$$\mathcal{G}_1 \subset \mathcal{G} \subset \mathcal{G}_{-1}$$

and the corresponding canonical embeddings are continuous. Let $T_0 \in \mathcal{C}(\mathcal{G})$ have closed range in $\mathcal{G}$, let $\operatorname{dom} T_0 \subset \mathcal{G}_1$ and assume there exist extensions

$$T_0 \subset T \in \mathcal{B}(\mathcal{G}_1, \mathcal{G}_{-1}), \qquad T_0^{\#} \subset T^{\ddagger} \in \mathcal{B}(\mathcal{G}_{-1}, \mathcal{G}_1).$$

Let P and Q, respectively, denote the orthogonal projections on $\ker T_0$ and $\operatorname{coker} T_0$ in $\mathcal{G}$. Then

$$TT^{\ddagger}|_{\mathcal{G}} = I - Q$$

and, if $\mathcal{G}$ is dense in $\mathcal{G}_{-1}$, then $\operatorname{coker} T_0 = \{0\}$ implies $TT^{\ddagger} = I$. If $\operatorname{dom} T_0$ is dense $\mathcal{G}_1$, then

$$T^{\ddagger}T = I - P|_{\mathcal{G}_1}$$

and $\ker T_0 = \{0\}$ implies $T^{\ddagger}T = I$. In particular, if $0 \in \rho(T_0)$, $\mathcal{G}$ is dense in $\mathcal{G}_{-1}$ and $\operatorname{dom} T_0$ is dense in $\mathcal{G}_1$, then $T^{\ddagger} = T^{-1}$.

Proof. Using relation (1.4) and since $\operatorname{ran} T_0^{\#} \subset \operatorname{dom} T_0$, we obtain

$$TT^{\ddagger}|_{\mathcal{G}} = T_0 T_0^{\#} = I - Q.$$

If $\mathcal{G}$ is dense in $\mathcal{G}_{-1}$ and $\operatorname{coker} T_0 = \{0\}$, then $TT^{\ddagger} = I$ follows from the above identity and from $TT^{\ddagger} \in \mathcal{B}(\mathcal{G}_{-1})$.

Let $\operatorname{dom} T_0$ be dense in $\mathcal{G}_1$. From Lemma 2.11, we obtain $P \in \mathcal{B}(\mathcal{G}, \mathcal{G}_1)$. Consequently, again using (1.4), we see that $T^{\ddagger}T \in \mathcal{B}(\mathcal{G}_1)$ and $I - P|_{\mathcal{G}_1} \in \mathcal{B}(\mathcal{G}_1)$ satisfy

$$T^{\ddagger}T|_{\operatorname{dom} T_0} = T_0^{\#} T_0 = (I - P)|_{\operatorname{dom} T_0} = (I - P|_{\mathcal{G}_1})|_{\operatorname{dom} T_0}.$$

Hence, $T^{\ddagger}T = I - P|_{\mathcal{G}_1}$ is a consequence of the density of $\operatorname{dom} T_0$ in $\mathcal{G}_1$ and clearly, $T^{\ddagger}T = I$ holds if $\ker T_0 = \{0\}$. The last claim is now immediate. □

Lemma 2.13. *If Assumption 2.1 holds, then $\ker D = \ker D_0$ and $\operatorname{ran} D^{\ddagger} \perp_{\mathcal{H}_2} \ker D_0$.*

Proof. We first show $\ker D = \ker D_0$. Let $f \in \ker D$. Since $\operatorname{dom} D_0$ is dense in $\mathcal{D}_2$, there exists a sequence $\{f_m\}_m \subset \operatorname{dom} D_0$ such that $f_m \to f$ in $\mathcal{D}_2$ as $m \to \infty$. Consequently, from $D_0^{\#} D_0 \subset D^{\ddagger} D \in \mathcal{B}(\mathcal{D}_2)$ and $Df = 0$, with $g_m := (I - D_0^{\#} D_0) f_m$ we conclude

$$\|g_m - f\|_{\mathcal{H}_2} \lesssim \|g_m - f\|_{\mathcal{D}_2} \le \|f_m - f\|_{\mathcal{D}_2} + \|D^{\ddagger} D f_m\|_{\mathcal{D}_2} \to 0, \quad m \to \infty.$$

Note that $I - D_0^{\#} D_0$ is the orthogonal projection on $\ker D_0$ in $\mathcal{H}_2$, see (1.4). Hence, the closedness of D_0 and $g_m \in \ker D_0$, $m \in \mathbb{N}$, imply $f \in \operatorname{dom} D_0$ and $D_0 f = 0$.

It remains to show $\operatorname{ran} D^{\ddagger} \perp_{\mathcal{H}_2} \ker D_0$. However, since $\mathcal{H}_2$ is dense in $\mathcal{D}_{-2}$ and $D^{\ddagger} : \mathcal{D}_{-2} \to \mathcal{D}_2$ is continuous, we have

$$\operatorname{ran} D^{\ddagger} = D^{\ddagger}\left(\overline{\mathcal{H}_2}^{\mathcal{D}_{-2}}\right) \subset \overline{D^{\ddagger}(\mathcal{H}_2)}^{\mathcal{D}_2}. \tag{2.5}$$

Moreover, from $\mathcal{D}_2$ being continuously embedded in $\mathcal{H}_2$ and (1.3), we conclude

$$\overline{\operatorname{ran} D_0^{\#}}^{\mathcal{D}_2} \subset \overline{\operatorname{ran} D_0^{\#}}^{\mathcal{H}_2} \perp_{\mathcal{H}_2} \ker D_0. \tag{2.6}$$

Considering $D^{\ddagger}|_{\mathcal{H}_2} = D_0^{\#}$, the claim now follows from (2.5) and (2.6). □

2.5. **Proofs.** We start by proving Proposition 2.5 (i); for technical reasons, the proof of (ii) is given at the end of this section.

Proof of Proposition 2.5 (i). By Lemma 2.12, we have that $DD^{\ddagger} = I$. Moreover, it is straightforward to check the following inclusion

$$\{(f, h - D^{\ddagger}Cf) : f \in \operatorname{dom} S_0,\ h \in \operatorname{dom} B_0 \cap \operatorname{dom} D_0\} \subset \operatorname{dom} \mathcal{A}_0. \tag{2.7}$$

Let $(u,v) \in \mathcal{H}$ with $(u,v) \perp_{\mathcal{H}} \operatorname{dom} \mathcal{A}_0$; we need to show $u = v = 0$. By (2.7),

$$\langle f, u\rangle_{\mathcal{H}_1} + \langle h - D^{\ddagger}Cf, v\rangle_{\mathcal{H}_2} = 0, \quad f \in \operatorname{dom} S_0, \quad h \in \operatorname{dom} B_0 \cap \operatorname{dom} D_0. \tag{2.8}$$

Setting $f = 0$ in (2.8), from the density of $\operatorname{dom} B_0 \cap \operatorname{dom} D_0$ in $\mathcal{H}_2$ it follows that $v = 0$. Hence, (2.8) implies $u \perp \operatorname{dom} S_0$ and we obtain $u = 0$ from the density of $\operatorname{dom} S_0$ in $\mathcal{H}_1$. □

Proof of Proposition 2.6. By Lemma 2.12, $0 \in \rho(D_0)$ implies $D^{\ddagger} = D^{-1}$. Let $(f,g) \in \ker \mathcal{A}_0$, i.e.

$$Af + Bg = 0, \qquad Cf + Dg = 0,$$

where $f \in \mathcal{D}_S$ and $g \in \mathcal{D}_2$. Applying D^{-1} to the second equation and inserting it in the first one, we obtain

$$g = -D^{-1}Cf, \qquad 0 = Af - BD^{-1}Cf = Sf = S_0 f.$$

Conversely, $f \in \ker S_0$ and $g = -D^{-1}Cf$ clearly imply $0 = \mathcal{A}(f,g) = \mathcal{A}_0(f,g)$. The equality of dimensions now follows immediately. □

Proof of Theorem 2.7. (i) We show that $S_0^{-1} = L := \pi_1 \mathcal{A}_0^{-1}\pi_1^* \in \mathcal{B}(\mathcal{H}_1)$, thus implying $0 \in \rho(S_0)$; recall that π_1 and π_2, respectively, denote the canonical projections from $\mathcal{H}$ onto $\mathcal{H}_1$ and $\mathcal{H}_2$. Let therefore $f \in \operatorname{dom} S_0$. Since $D^{\ddagger} = D^{-1}$ by $0 \in \rho(D_0)$ and Lemma 2.12, we have $(f, -D^{-1}Cf) \in \operatorname{dom} \mathcal{A}_0$ and

$$\mathcal{A}_0(f, -D^{-1}Cf) = (S_0 f, 0) = \pi_1^* S_0 f.$$

By applying $\pi_1\mathcal{A}_0^{-1}$ to the above identity, one obtains $f = LS_0 f$.

Conversely, let $f \in \mathcal{H}_1$ be arbitrary. Then

$$(f,0) = \mathcal{A}_0\mathcal{A}_0^{-1}\pi_1^* f = (ALf + B\pi_2\mathcal{A}_0^{-1}\pi_1^* f, CLf + D\pi_2\mathcal{A}_0^{-1}\pi_1^* f). \tag{2.9}$$

Applying D^{-1} to the second component of (2.9), we obtain

$$\pi_2\mathcal{A}_0^{-1}\pi_1^* f = -D^{-1}CLf.$$

We insert this identity in the first component of (2.9) to conclude

$$\mathcal{H}_1 \ni f = ALf - BD^{-1}CLf = SLf,$$

which in turn implies that $Lf \in \operatorname{dom} S_0$ and $S_0 Lf = f$.

(ii) Let us proceed by contraposition, i.e. let us show that if $S_0 \notin \mathcal{F}_+(\mathcal{H}_1)$, then $\mathcal{A}_0 \notin \mathcal{F}_+(\mathcal{H})$. Assume that $\mathcal{A}_0$ is closed, since otherwise $\mathcal{A}_0 \notin \mathcal{F}_+(\mathcal{H})$ by definition. Since S_0 is closed, there exists a sequence $\{f_m\}_m \subset \operatorname{dom} S_0$ with $\|f_m\|_{\mathcal{H}_1} = 1$, $m \in \mathbb{N}$, such that

$$S_0 f_m \xrightarrow{\mathcal{H}_1} 0, \qquad f_m \xrightarrow{w} 0 \ \text{ in } \ \mathcal{H}_1, \qquad m \to \infty, \tag{2.10}$$

see [11, Thm. IX.1.3 (i)] and the preceding paragraph (note that the density of the domain is only needed for part (ii) of the Theorem).

We construct a singular sequence for $\mathcal{A}_0$ and conclude $\mathcal{A}_0 \notin \mathcal{F}_+(\mathcal{H})$ by [11, Thm. IX.1.3 (i)]. Define the sequence $\{\mathbf{x}_m := (f_m, -D^\ddagger C f_m)\}_m \subset \mathcal{D}$, then

$$\|\mathbf{x}_m\|_{\mathcal{H}} \geq \|f_m\|_{\mathcal{H}_1} = 1, \qquad \mathcal{A}\mathbf{x}_m = (S_0 f_m, (I - DD^\ddagger)Cf_m), \qquad m \in \mathbb{N}. \tag{2.11}$$

We show that $\mathbf{x}_m \in \operatorname{dom} \mathcal{A}_0$ and $\mathcal{A}_0 \mathbf{x}_m \to 0$ in $\mathcal{H}$ as $m \to \infty$. If $\operatorname{coker} D_0 = \{0\}$, then Lemma 2.12 implies $DD^\ddagger = I$ and consequently,

$$\mathcal{A}\mathbf{x}_m = (S_0 f_m, 0) = \mathcal{A}_0 \mathbf{x}_m \xrightarrow{\mathcal{H}} 0, \qquad m \to \infty.$$

Consider the case that

$$\{0\} \neq \operatorname{coker} D_0 = \operatorname{span}\{\phi_j\}_{1 \leq j \leq k} \subset \operatorname{dom} C_0^*;$$

recall that $\dim \operatorname{coker} D_0 < \infty$ as $D_0 \in \mathcal{F}_-(\mathcal{H}_2)$. Then, using that $Cf \in \mathcal{H}_2$ for $f \in \operatorname{dom} C_0 \subset \mathcal{D}_S$, that $DD^\ddagger|_{\mathcal{H}_2} = D_0 D_0^{\#}$ and relation (1.4), we obtain the identity

$$(I - DD^\ddagger)Cf = \sum_{j=1}^{k} \langle Cf, \phi_j \rangle_{\mathcal{H}_2} \phi_j = \sum_{j=1}^{k} \langle f, C_0^* \phi_j \rangle_{\mathcal{H}_1} \phi_j, \quad f \in \operatorname{dom} C_0. \tag{2.12}$$

By density of $\operatorname{dom} C_0$ in $\mathcal{D}_S$ and since both left and right hand side of (2.12) are continuous in f with respect to $\|\cdot\|_S$ (with values in $\mathcal{D}_{-2}$), formula (2.12) is valid for all $f \in \mathcal{D}_S$. We can thus extend

$$(I - DD^\ddagger)C \subset K := \sum_{j=1}^{k} \langle \cdot, C_0^* \phi_j \rangle_{\mathcal{H}_1} \phi_j \in \mathcal{K}(\mathcal{H}_1, \mathcal{H}_2),$$

hence (2.11) gives $\mathbf{x}_m \in \operatorname{dom} \mathcal{A}_0$ and (2.10) implies that $\mathcal{A}_0 \mathbf{x}_m = (S_0 f_m, K f_m) \to 0$ strongly in $\mathcal{H}$ as $m \to \infty$. Since $\|\mathbf{x}_m\|_{\mathcal{H}} \geq 1$, in both cases $\operatorname{coker} D_0 = \{0\}$ and $\operatorname{coker} D_0 \neq \{0\}$, we obtain an $\mathcal{H}$-normalised sequence

$$\widetilde{\mathbf{x}}_m := \frac{\mathbf{x}_m}{\|\mathbf{x}_m\|_{\mathcal{H}}} \in \operatorname{dom} \mathcal{A}_0, \qquad \mathcal{A}_0 \widetilde{\mathbf{x}}_m \xrightarrow{\mathcal{H}} 0, \qquad m \to \infty. \tag{2.13}$$

It remains to show that $\{\widetilde{\mathbf{x}}_m\}_m$ has no convergent subsequence in $\mathcal{H}$. We assume the opposite, i.e. that there exists $(f, g) \in \mathcal{H}$ and a subsequence (again denoted by $\{\widetilde{\mathbf{x}}_m\}_m$) such that $\widetilde{\mathbf{x}}_m \to (f, g)$ in $\mathcal{H}$ as $m \to \infty$. Consequently, from $\|\mathbf{x}_m\|_{\mathcal{H}} \geq 1$ and (2.10) it follows for arbitrary $u \in \mathcal{H}_1$ that

$$|\langle f, u \rangle_{\mathcal{H}_1}| = \lim_{m \to \infty} \frac{|\langle f_m, u \rangle_{\mathcal{H}_1}|}{\|\mathbf{x}_m\|_{\mathcal{H}}} \leq \lim_{m \to \infty} |\langle f_m, u \rangle_{\mathcal{H}_1}| = 0,$$

thus $f = 0$. Since $\mathcal{A}_0$ is closed, (2.13) and $\widetilde{\mathbf{x}}_m \to (0, g)$ imply

$$(0, g) \in \operatorname{dom} \mathcal{A}_0, \qquad \mathcal{A}_0(0, g) = (Bg, Dg) = 0,$$

in particular $g \in \ker D = \ker D_0$ by Lemma 2.13. Since by construction $\pi_2 \widetilde{\mathbf{x}}_m \in \operatorname{ran} D^\ddagger$, Lemma 2.13 ultimately leads to the contradiction

$$1 = \lim_{m \to \infty} \|\widetilde{\mathbf{x}}_m\|_{\mathcal{H}}^2 = \|g\|_{\mathcal{H}_2}^2 = \lim_{m \to \infty} \langle \pi_2 \widetilde{\mathbf{x}}_m, g \rangle_{\mathcal{H}_2} = 0. \qquad \square$$

The proof of Theorem 2.8 is based on constructing a (left approximate) inverse for $\mathcal{A}_0$. Its main ingredient is the following proposition.

Proposition 2.14. *Let the assumptions of Theorem* 2.8 (ii) *be satisfied. Then*

$$\mathcal{L} := \begin{pmatrix} S_0^{\#} & -S^{\ddagger}BD_0^{\#} \\ -D^{\ddagger}CS_0^{\#} & D_0^{\#} + D^{\ddagger}CS^{\ddagger}BD_0^{\#} \end{pmatrix} \in \mathcal{B}(\mathcal{H},\mathcal{D}) \tag{2.14}$$

satisfies $\mathcal{L}\mathcal{A}_0 = \mathcal{I} + \mathcal{K}|_{\operatorname{dom}\mathcal{A}_0}$, *where*

$$\mathcal{K} := \begin{pmatrix} -P_S & S^{\ddagger}BP_D \\ D^{\ddagger}CP_S & -P_D - D^{\ddagger}CS^{\ddagger}BP_D \end{pmatrix} \in \mathcal{K}(\mathcal{H},\mathcal{D}) \tag{2.15}$$

has finite rank. Here P_S *and* P_D *denote the orthogonal projections on* $\ker S_0$ *in* $\mathcal{H}_1$ *and on* $\ker D_0$ *in* $\mathcal{H}_2$, *respectively.*

Proof. From Lemma 2.11, it follows that $P_S \in \mathcal{B}(\mathcal{H}_1,\mathcal{D}_S)$ and $P_D \in \mathcal{B}(\mathcal{H}_2,\mathcal{D}_2)$. Moreover, $B(\operatorname{dom} D_0) \subset \mathcal{D}_{-S}$ implies $\operatorname{ran} BP_D \subset \mathcal{D}_{-S}$ and again by Lemma 2.11 we obtain $BP_D \in \mathcal{B}(\mathcal{H}_2,\mathcal{D}_{-S})$. Altogether, we have $\mathcal{K} \in \mathcal{B}(\mathcal{H},\mathcal{D})$. Since $\ker S_0$ and $\ker D_0$ are finite dimensional, $\mathcal{K}$ has finite rank and is thus compact.

We next show that $\mathcal{L}$ is well defined and in $\mathcal{B}(\mathcal{H},\mathcal{D})$. Notice therefore that, by Lemma 2.11 and (1.3),

$$D_0^{\#} = D^{\ddagger}|_{\mathcal{H}_2} \in \mathcal{B}(\mathcal{H}_2,\mathcal{D}_2), \qquad S_0^{\#} = S^{\ddagger}|_{\mathcal{H}_1} \in \mathcal{B}(\mathcal{H}_1,\mathcal{D}_S).$$

Hence, from $B(\operatorname{dom} D_0) \subset \mathcal{D}_{-S}$, we conclude that $\operatorname{ran} BD_0^{\#} \subset \mathcal{D}_{-S}$ and, applying Lemma 2.11, that $BD_0^{\#} \in \mathcal{B}(\mathcal{H}_2,\mathcal{D}_{-S})$. This shows the claimed boundedness.

It remains to prove that

$$\mathcal{L}\mathcal{A}_0(f,g) = (f,g) + \mathcal{K}(f,g), \qquad (f,g) \in \operatorname{dom}\mathcal{A}_0.$$

Since $S_0^{\#} \subset S^{\ddagger}$ and $D_0^{\#} \subset D^{\ddagger} : \mathcal{D}_{-2} \to \mathcal{D}_2$, and since $Cf, Dg \in \mathcal{D}_{-2}$, we can write

$$\pi_1\mathcal{L}\mathcal{A}_0(f,g) = S^{\ddagger}(Af + Bg - BD^{\ddagger}(Cf + Dg)) = S^{\ddagger}(Sf + Bg - BD^{\ddagger}Dg). \tag{2.16}$$

Applying Lemma 2.12 to S_0 and D_0, we obtain the identities

$$S^{\ddagger}S = I - P_S|_{\mathcal{D}_S}, \qquad D^{\ddagger}D = I - P_D|_{\mathcal{D}_2}.$$

Using this, identity (2.16), $S^{\ddagger} : \mathcal{D}_{-S} \to \mathcal{D}_S$ and $Sf, BP_Dg \in \mathcal{D}_{-S}$ gives

$$\pi_1\mathcal{L}\mathcal{A}_0(f,g) = S^{\ddagger}(Sf + BP_Dg) = f - P_Sf + S^{\ddagger}BP_Dg = f + \pi_1\mathcal{K}(f,g).$$

The proof of $\pi_2\mathcal{L}\mathcal{A}_0(f,g) = g + \pi_2\mathcal{K}(f,g)$ is analogous. □

Employing the above proposition, we are now able to prove Theorem 2.8.

Proof of Theorem 2.8. (i) We apply Proposition 2.14. Note that all the assumptions are satisfied with $S^{\ddagger} = S^{-1}$; the density of $\operatorname{dom} S_0$ in $\mathcal{D}_S$ follows from the density of $\mathcal{H}_1$ in $\mathcal{D}_{-S}$ and $S^{-1} \in \mathcal{B}(\mathcal{D}_{-S},\mathcal{D}_S)$.

From $0 \in \rho(D_0)$, we conclude $D_0^{\#} = D_0^{-1}$, $P_D = 0$ and $D^{\ddagger} = D^{-1}$ by Lemma 2.12. Moreover, Lemma 2.12 gives $S_0^{\#} = S_0^{-1}$ and $P_S = 0$ since $0 \in \rho(S_0)$. Hence $\mathcal{K} = 0$, i.e. the operator matrix

$$\mathcal{L} = \begin{pmatrix} S_0^{-1} & -S^{-1}BD_0^{-1} \\ -D^{-1}CS_0^{-1} & D_0^{-1} + D^{-1}CS^{-1}BD_0^{-1} \end{pmatrix}$$

from Proposition 2.14 is a left inverse for $\mathcal{A}_0$. We show that $\mathcal{L} \in \mathcal{B}(\mathcal{H},\mathcal{D})$ is also a right inverse for $\mathcal{A}_0$; as $\mathcal{D} \subset \mathcal{H}$ is continuously embedded, $0 \in \rho(\mathcal{A}_0)$ then follows from

$$\mathcal{A}_0^{-1} = \mathcal{L} \in \mathcal{B}(\mathcal{H}).$$

Let $(f,g) \in \mathcal{H}$, then we have $\mathcal{L}(f,g) \in \mathcal{D}$ and, using $D_0^{-1} \subset D^{-1}$ and $S_0^{-1} \subset S^{-1}$, we can write

$$\begin{aligned}\pi_1 \mathcal{A}\mathcal{L}(f,g) = A(&S^{-1}f - S^{-1}BD^{-1}g) \\ &+ B(-D^{-1}CS^{-1}f + D^{-1}g + D^{-1}CS^{-1}BD^{-1}g).\end{aligned}$$

Since $A : \mathcal{D}_S \to \mathcal{D}_{-1}$ and $B : \mathcal{D}_2 \to \mathcal{D}_{-1}$, we can break the parentheses in the above identity and obtain

$$\pi_1 \mathcal{A}\mathcal{L}(f,g) = (A - BD^{-1}C)S^{-1}f + (I - AS^{-1} + BD^{-1}CS^{-1})BD^{-1}g$$

and further conclude

$$\pi_1 \mathcal{A}\mathcal{L}(f,g) = SS^{-1}f + (I - SS^{-1})BD^{-1}g = f.$$

In the same way, one shows that $\pi_2 \mathcal{A}\mathcal{L}(f,g) = g$. Hence, $\mathcal{A}\mathcal{L}(f,g) = (f,g)$, which implies $\operatorname{ran}\mathcal{L} \subset \operatorname{dom}\mathcal{A}_0$ and $\mathcal{A}_0\mathcal{L} = \mathcal{I}$.

(ii) By Proposition 2.14, the identity

$$\mathcal{L}\mathcal{A}_0 = \mathcal{I} + \mathcal{K}|_{\operatorname{dom}\mathcal{A}_0} \tag{2.17}$$

holds true with $\mathcal{L} \in \mathcal{B}(\mathcal{H},\mathcal{D})$ and $\mathcal{K} \in \mathcal{B}(\mathcal{H},\mathcal{D})$ as in (2.14) and (2.15). We first show that $\mathcal{A}_0$ is closed in $\mathcal{H}$. Let therefore $\{(f_m,g_m)\}_m \subset \operatorname{dom}\mathcal{A}_0$ and $(f,g),(u,v) \in \mathcal{H}$ such that $(f_m,g_m) \to (f,g)$ and $\mathcal{A}_0(f_m,g_m) \to (u,v)$ in $\mathcal{H}$ as $m \to \infty$. The continuity of $\mathcal{L}$ and $\mathcal{K}$ imply

$$(f_m,g_m) = \mathcal{L}\mathcal{A}_0(f_m,g_m) - \mathcal{K}(f_m,g_m) \xrightarrow{\mathcal{D}} \mathcal{L}(u,v) - \mathcal{K}(f,g), \quad m \to \infty,$$

and so the sequence $\{(f_m,g_m)\}_m$ is convergent in both $\mathcal{H}$ and $\mathcal{D}$. Since $\mathcal{D} \subset \mathcal{H}$ is continuously embedded, the limits must coincide, thus $(f_m,g_m) \to (f,g)$ in $\mathcal{D}$ as $m \to \infty$. Consequently, $\mathcal{A} \in \mathcal{B}(\mathcal{D},\mathcal{D}_-)$ implies

$$\mathcal{A}_0(f_m,g_m) = \mathcal{A}(f_m,g_m) \xrightarrow{\mathcal{D}_-} \mathcal{A}(f,g), \quad m \to \infty.$$

Hence, $\{\mathcal{A}_0(f_m,g_m)\}_m$ is convergent in both $\mathcal{H}$ and $\mathcal{D}_-$, which, since $\mathcal{H} \subset \mathcal{D}_-$ is continuously embedded, gives the equality $\mathcal{A}(f,g) = (u,v) \in \mathcal{H}$. This in turn implies $(f,g) \in \operatorname{dom}\mathcal{A}_0$ and $\mathcal{A}_0(f,g) = (u,v)$.

By the continuity of the embedding $\mathcal{D} \subset \mathcal{H}$, we have $\mathcal{L},\mathcal{K} \in \mathcal{B}(\mathcal{H})$. Moreover, it follows from Proposition 2.14 that $\mathcal{K}$ has finite rank, thus $\mathcal{K} \in \mathcal{K}(\mathcal{H})$ and from (2.17) we see that $\mathcal{L}$ is a bounded left approximate inverse for $\mathcal{A}_0$. Finally, from the closedness of $\mathcal{A}_0$, (2.17) and [11, Thm. I.3.13], we conclude $\mathcal{A}_0 \in \mathcal{F}_+(\mathcal{H})$. □

Proof of Corollary 2.9. Let $S_0 \in \mathcal{F}_+(\mathcal{H}_1)$; note that this in particular holds if $0 \in \rho(S_0)$. We start by deriving a resolvent type identity for $S_0^\#$ in order to construct an extension

$$S_0^\# \subset S^\ddagger \in \mathcal{B}(\mathcal{D}_{-S},\mathcal{D}_S).$$

Let P denote the orthogonal projection on $\ker S_0$ in $\mathcal{H}_1$. From (1.4), we conclude (notice that $\operatorname{coker} S_0 = \{0\}$ and thus $Q = 0$)

$$(S_0-z)^\# - S_0^\# = (S_0^\# S_0 + P)(S_0-z)^\# - S_0^\#(S_0-z)(S_0-z)^\# = (zS_0^\# + P)(S_0-z)^\#.$$

We can thus define an extension

$$S_0^\# \subset S_z^\ddagger - (zS_0^\# + P)S_z^\ddagger =: S^\ddagger.$$

By Lemma 2.11, one has $zS_0^\# + P \in \mathcal{B}(\mathcal{H}_1,\mathcal{D}_S)$. Moreover, $S_z^\ddagger \in \mathcal{B}(\mathcal{D}_{-S},\mathcal{H}_1)$ since $\mathcal{D}_S$ is continuously embedded in $\mathcal{H}_1$. Altogether, $S^\ddagger \in \mathcal{B}(\mathcal{D}_{-S},\mathcal{D}_S)$ is shown.

The statements (i) and (ii), respectively, now immediately follow from Theorem 2.8 (i) and (ii); notice that in (i) the assumption $0 \in \rho(S_0)$ together with Lemma 2.12 implies $S^{-1} = S^{\ddagger}$. □

It remains to prove Proposition 2.5 (ii).

Proof of Proposition 2.5 (ii). Notice first that the assumptions of Theorem 2.8 (i) are satisfied and that $S^{-1} \in \mathcal{B}(\mathcal{D}_{-S}, \mathcal{D}_S)$, see the proof of Corollary 2.9 (i). Moreover, since $\mathcal{D}_{-S} = \mathcal{D}_{-1}$, the inverse $\mathcal{L}$ in the proof of Theorem 2.8 (i) has the following bounded extension

$$\widetilde{\mathcal{L}} := \begin{pmatrix} S^{-1} & -S^{-1}BD^{-1} \\ -D^{-1}CS^{-1} & D^{-1} + D^{-1}CS^{-1}BD^{-1} \end{pmatrix} \in \mathcal{B}(\mathcal{D}_-, \mathcal{D}).$$

It is straightforward to check that $\widetilde{\mathcal{L}} = \mathcal{A}^{-1}$, which by the density of $\mathcal{H}$ in $\mathcal{D}_-$ immediately implies the density of $\operatorname{dom} \mathcal{A}_0$ in $\mathcal{D}$. The density of $\operatorname{dom} \mathcal{A}_0$ in $\mathcal{H}$ then follows from the density of $\mathcal{D}$ in $\mathcal{H}$ and the continuity of its canonical embedding. □

3. Spectral equivalence between operator matrix and Schur complement

We apply the results from Section 2 in order to obtain spectral correspondence between the matrix $\mathcal{A}_0$ and its Schur complement $S_0(\cdot)$ as an operator family. More precisely, we assume that $\Theta \subset \mathbb{C}$ is a set such that, for a fixed $\lambda \in \Theta$, Assumption 2.1 is satisfied with $A - \lambda$, $D_0 - \lambda$ and $D - \lambda$ instead of A, D_0 and D, respectively. (In this case, we will say that the assumptions of the theorem are satisfied in a certain point $\lambda \in \Theta$.) Our main theorems (which correspond to the case $\lambda = 0$) then apply to the matrix $\mathcal{A} - \lambda$ and its (generalised) Schur complement $S(\lambda)$, see (3.1). This relates semi-Fredholmness/bounded invertibility of $\mathcal{A}_0 - \lambda$ to semi-Fredholmness/bounded invertibility of $S_0(\lambda)$ and thereby provides an equivalence of the type

$$\lambda \in \sigma_{(\mathrm{p}\,/\,\mathrm{e}2)}(\mathcal{A}_0) \iff 0 \in \sigma_{(\mathrm{p}\,/\,\mathrm{e}2)}(S_0(\lambda)) \iff \lambda \in \sigma_{(\mathrm{p}\,/\,\mathrm{e}2)}(S_0(\cdot)).$$

Recall that the spectrum of the operator family $\{S_0(\lambda) : \lambda \in \Theta\}$ is defined as

$$\sigma(S_0(\cdot)) = \{\lambda \in \Theta : 0 \in \sigma(S_0(\lambda))\};$$

its resolvent set, essential spectrum and point spectrum are defined analogously.

Assumption 3.1. Throughout this section, we assume the following.

(i) Let Assumption 2.1 (i) and (ii) be satisfied.
(ii) Let $D_0 \in \mathcal{C}(\mathcal{H}_2)$ such that $\operatorname{dom} D_0 \subset \mathcal{D}_2$ is dense in $\mathcal{D}_2$ and assume that there exists an extension
$$D_0 \subset D \in \mathcal{B}(\mathcal{D}_2, \mathcal{D}_{-2}). \qquad //$$

The assumptions above allow us to introduce the operator matrix $\mathcal{A}$ and its maximal restriction $\mathcal{A}_0$ in $\mathcal{H}$ analogously to Definition 2.3. Moreover, we can define the Schur complement $S(\lambda)$ for a subset Θ of spectral parameters $\lambda \in \mathbb{C}$ such that $D_0 - \lambda$ and its extension $D - \lambda$ satisfy Assumption 2.1 (iii). Notice that, in line with Assumption 2.1 (iii), we do not require $0 \in \rho(D_0)$ but allow $\lambda \in \mathbb{C}$ such that $D_0 - \lambda$ merely has a generalised inverse, see (1.3) and (1.4).

Definition 3.2. Let Assumption 3.1 be satisfied and define $\mathcal{A} \in \mathcal{B}(\mathcal{D}, \mathcal{D}_-)$, $\operatorname{dom} \mathcal{A}_0$ and $\mathcal{A}_0$ as in Definition 2.3. Moreover, define the following family of operators

$$S(\lambda) := A - \lambda - BD^{\ddagger}(\lambda)C \in \mathcal{B}(\mathcal{D}_S, \mathcal{D}_{-1}), \qquad \lambda \in \Theta, \tag{3.1}$$

where the set $\Theta \subset \mathbb{C}$ satisfies

$$\begin{aligned} \Theta \ \subset \ \{\lambda \in \mathbb{C} : \operatorname{ran}(D_0 - \lambda) \text{ closed in } \mathcal{H}_2, \\ \exists \text{ extension } (D_0 - \lambda)^{\#} \subset D^{\ddagger}(\lambda) \in \mathcal{B}(\mathcal{D}_{-2}, \mathcal{D}_2)\}. \end{aligned} \tag{3.2}$$

Let the corresponding family of maximal operators in $\mathcal{H}_1$ be defined as

$$S_0(\lambda) := S(\lambda)|_{\operatorname{dom} S_0(\lambda)}, \quad \operatorname{dom} S_0(\lambda) := \{f \in \mathcal{D}_S : S(\lambda)f \in \mathcal{H}_1\}, \quad \lambda \in \Theta. \quad //$$

Remark 3.3. Notice that the spaces in Assumption 3.1 (i) are chosen such that, independently of $\lambda \in \Theta$, the results of Section 2 apply to $\mathcal{A} - \lambda$ and $S(\lambda)$. However, if the spaces and operators in Assumption 3.1 may depend on $\lambda \in \Theta$, our method defines an operator matrix valued family

$$\mathcal{L}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \in \mathcal{B}(\mathcal{D}(\lambda), \mathcal{D}_-(\lambda)), \quad \lambda \in \Theta,$$

whose domain $\operatorname{dom} \mathcal{L}_0(\lambda) \subset \mathcal{H}$ in general depends on the spectral parameter.

Let us state the first of four corollaries, which are a translation of the results in Section 2 to the present setting. Namely, if the assumptions of Proposition 2.5 are satisfied in one point $\lambda_0 \in \Theta$, then $\mathcal{A}_0$ is densely defined in $\mathcal{H}$ and/or $\mathcal{D}$.

Corollary 3.4. *Let Assumption* 3.1 *be satisfied and let $\mathcal{A}_0$, S_0 and Θ be as in Definition* 3.2.

(i) *Let $\operatorname{dom} B_0 \cap \operatorname{dom} D_0$ (with $\operatorname{dom} B_0$ defined as in* (2.3)) *be dense in $\mathcal{H}_2$. Assume there exists $\lambda_0 \in \Theta$ such that $\operatorname{coker}(D_0 - \lambda_0) = \{0\}$ and such that $\operatorname{dom} S_0(\lambda_0)$ is dense in $\mathcal{H}_1$. Then $\operatorname{dom} \mathcal{A}_0$ is dense in $\mathcal{H}$.*

(ii) *Let $\mathcal{D}_{-S} = \mathcal{D}_{-1}$ and assume that there exists $\lambda_0 \in \Theta$ such that $D_0 - \lambda_0$ and $S_0(\lambda_0)$ satisfy the assumptions of Proposition* 2.5 (ii). *Then $\operatorname{dom} \mathcal{A}_0$ is dense in both $\mathcal{D}$ and $\mathcal{H}$.*

Proof. The assumptions imply that Assumption 2.1 is satisfied with $A-\lambda_0$, $D_0-\lambda_0$ and $D - \lambda_0$ instead of A, D_0 and D, respectively. Hence, the claims in (i) and (ii) follow from Proposition 2.5 (i) and (ii) applied to $\mathcal{A}_0 - \lambda_0$ and $S_0(\lambda_0)$ and from

$$\operatorname{dom} \mathcal{A}_0 = \operatorname{dom}(\mathcal{A}_0 - \lambda_0), \qquad \operatorname{dom} D_0 = \operatorname{dom}(D_0 - \lambda_0). \qquad \square$$

The point spectra of $\mathcal{A}_0$ and $S_0(\cdot)$ correspond on $\Theta \cap \rho(D_0)$.

Corollary 3.5. *Let Assumption* 3.1 *be satisfied and let $\mathcal{A}_0$, S_0 and Θ be as in Definition* 3.2. *Then*

$$\sigma_{\mathrm{p}}(\mathcal{A}_0) \cap \rho(D_0) \cap \Theta = \sigma_{\mathrm{p}}(S_0(\cdot)) \cap \rho(D_0).$$

Proof. Let $\lambda \in \Theta \cap \rho(D_0)$. By Proposition 2.6 applied to $\mathcal{A}_0 - \lambda$ and $S_0(\lambda)$, we conclude that $0 \in \ker(\mathcal{A}_0 - \lambda)$ if and only if $0 \in \ker S_0(\lambda)$. Hence,

$$\sigma_{\mathrm{p}}(\mathcal{A}_0) \cap \rho(D_0) \cap \Theta = \sigma_{\mathrm{p}}(S_0(\cdot)) \cap \rho(D_0) \cap \Theta = \sigma_{\mathrm{p}}(S_0(\cdot)) \cap \rho(D_0). \qquad \square$$

On subsets of Θ where the assumptions of Theorem 2.7 are satisfied, we obtain an inclusion of the (essential) spectrum of $S_0(\cdot)$ in the (essential) spectrum of $\mathcal{A}_0$.

Corollary 3.6. *Let Assumption* 3.1 *be satisfied and let $\mathcal{A}_0$, S_0 and Θ be defined as in Definition* 3.2.

(i) *The following inclusion holds*

$$\sigma(S_0(\cdot)) \cap \rho(D_0) \subset \sigma(\mathcal{A}_0) \cap \rho(D_0) \cap \Theta.$$

(ii) *If the subset* $\Sigma \subset \Theta$ *is such that* $\mathcal{A}_0 - \lambda$ *and* $S_0(\lambda)$ *satisfy the assumptions of Theorem* 2.7 (ii) *for all* $\lambda \in \Sigma$, *then*

$$\sigma_{e2}(S_0(\cdot)) \cap \Sigma \subset \sigma_{e2}(\mathcal{A}_0) \cap \Sigma.$$

Proof. The statements in (i) and (ii), respectively, follow similarly to the proof of Corollary 3.5 by applying Theorem 2.7 (i) and (ii) to $\mathcal{A}_0 - \lambda$ and $S_0(\lambda)$ (with fixed $\lambda \in \rho(D_0) \cap \Theta$ in (i) and $\lambda \in \Sigma$ in (ii)). □

The reverse inclusions are obtained from Corollary 2.9.

Corollary 3.7. *Let Assumption* 3.1 *be satisfied and let* $\mathcal{A}_0$, S, S_0 *and* Θ *be defined as in Definition* 3.2.

(i) *If the subset* $\Sigma \subset \Theta$ *is such that* $\mathcal{A}_0 - \lambda$, $S(\lambda)$ *and* $S_0(\lambda)$ *satisfy the assumptions of Corollary* 2.9 (i) *for all* $\lambda \in \Sigma$, *then*

$$\sigma(\mathcal{A}_0) \cap \Sigma \subset \sigma(S_0(\cdot)) \cap \Sigma.$$

(ii) *If the subset* $\Sigma \subset \Theta$ *is such that* $\mathcal{A}_0 - \lambda$, $S(\lambda)$ *and* $S_0(\lambda)$ *satisfy the assumptions of Corollary* 2.9 (ii) *for all* $\lambda \in \Sigma$, *then*

$$\sigma_{e2}(\mathcal{A}_0) \cap \Sigma \subset \sigma_{e2}(S_0(\cdot)) \cap \Sigma.$$

Proof. The statements follow similarly to the proof of Corollary 3.5 by applying Corollary 2.9 to $\mathcal{A}_0 - \lambda$ and $S_0(\lambda)$ (with fixed $\lambda \in \Sigma$). □

4. Damped wave equation with irregular damping and potential

As an application of our results, we consider the linearly damped wave equation

$$\partial_t^2 u(t,x) + 2a(x)\partial_t u(t,x) = (\Delta_x - q(x))u(t,x), \qquad t > 0, \quad x \in \Omega, \tag{4.1}$$

on $\Omega \subset \mathbb{R}^n$, subject to Dirichlet boundary conditions, and with non-negative and possibly singular and/or unbounded damping a and potential q. Equation (4.1) can be written as the following first order Cauchy problem

$$\partial_t \begin{pmatrix} u_1(t,x) \\ u_2(t,x) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Delta_x - q(x) & -2a(x) \end{pmatrix} \begin{pmatrix} u_1(t,x) \\ u_2(t,x) \end{pmatrix}. \tag{4.2}$$

Spectral properties of the operator matrix above determine existence, uniqueness and behaviour of the solutions to (4.1) and have been studied extensively. While the majority of results relies on relative boundedness of a with respect to $\Delta - q$, see e.g. [4, 14, 19], the recent results [13, 18] do not follow standard patterns and allow stronger damping. Combining a distributional approach similar to [4, 14, 19] with structural observations from [13], our method enables us to not only omit the assumption on relative boundedness of the damping but also to substantially lower the regularity of the coefficients. We thereby close a gap which was left open between the technical assumptions in [13] and the minimal ones suggested by the applied sesquilinear form techniques therein, see Remark 4.3.

The main result of this section is Theorem 4.2; assuming only that $a, q \in L^1_{\mathrm{loc}}(\Omega)$, we therein define an m-accretive realisation of the operator matrix on the right hand side of (4.2) in a suitable (standard choice) Hilbert space and show spectral equivalence to its first Schur complement. By semigroup theory, this guarantees existence and uniqueness of the solutions to the underlying Cauchy problem. We thereby cover the essential part of [13], where the generation of a semigroup was shown under more restrictive assumptions, see Remark 4.3. Moreover, we point out that our method can also be employed to realise distributional dampings as considered e.g. in [4], see Remark 4.4.

4.1. **Assumptions and main result.** We make the following natural *low regularity assumptions.*

Assumption 4.1. Let $\Omega \subset \mathbb{R}^n$ be open and let $a, q \in L^1_{\mathrm{loc}}(\Omega)$ such that $a, q \geq 0$ almost everywhere in Ω. //

In the following, we denote $\|\cdot\| := \|\cdot\|_{L^2(\Omega)}$ and $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{L^2(\Omega)}$ and use the same notation for norm and inner product in $L^2(\Omega)^n$ if no confusion can arise.

4.1.1. *Spectral correspondence and generation of contraction semigroup.* We establish the operator theoretic framework behind the Cauchy problem (4.2) under Assumption 4.1. Let $\mathcal{W}(\Omega)$ be the Hilbert space completion of $C_0^\infty(\Omega)$ with respect to the inner product

$$\langle f, g\rangle_{\mathcal{W}} := \int_\Omega \nabla f \cdot \overline{\nabla g}\,\mathrm{d}x + \int_\Omega q f \overline{g}\,\mathrm{d}x, \qquad f,\, g \in C_0^\infty(\Omega); \tag{4.3}$$

recall that $q \geq 0$ and $a \geq 0$ a.e. in Ω. Moreover, define

$$\mathcal{D}_S := H_0^1(\Omega) \cap \operatorname{dom} q^{\frac{1}{2}} \cap \operatorname{dom} a^{\frac{1}{2}}. \tag{4.4}$$

Let $\mathcal{A}_0$ and $S_0(\cdot)$ be the (family of) maximal operators in $\mathcal{W}(\Omega) \oplus L^2(\Omega)$ and $L^2(\Omega)$, respectively, corresponding to the differential expressions

$$\mathcal{A} := \begin{pmatrix} 0 & I \\ \Delta - q & -2a \end{pmatrix}, \quad S(\lambda) := -\frac{1}{\lambda}(-\Delta + q + 2\lambda a + \lambda^2), \quad \lambda \in \mathbb{C} \setminus \{0\}, \tag{4.5}$$

on their respective maximal domains

$$\begin{aligned} \operatorname{dom} \mathcal{A}_0 &:= \{(f, g) \in \mathcal{W}(\Omega) \times \mathcal{D}_S : (\Delta - q)f - 2ag \in L^2(\Omega)\}, \\ \operatorname{dom} S_0(\lambda) &:= \{f \in \mathcal{D}_S : (\Delta - q - 2\lambda a)f \in L^2(\Omega)\}. \end{aligned} \tag{4.6}$$

Here the above operations are understood in a standard (antilinear) distributional sense; see Definitions 4.5, 4.8, 4.10 and Remark 4.12 below for details. Notice that the Schur complement $S_0(\cdot)$ coincides with the usual definition by means of its quadratic form.

The main result of this section reads as follows and is proven in Section 4.3.

Theorem 4.2. *Let Assumption* 4.1 *be satisfied and let $\mathcal{A}_0$ and $S_0(\cdot)$ be as in* (4.5) *and* (4.6)*. Then $-\mathcal{A}_0$ is m-accretive, thus $\mathcal{A}_0$ generates a strongly continuous contraction semigroup on $\mathcal{W}(\Omega) \oplus L^2(\Omega)$ and its domain is dense both in $\mathcal{W}(\Omega) \oplus \mathcal{D}_S$ and in $\mathcal{W}(\Omega) \oplus L^2(\Omega)$. Moreover, the* (*point and essential*) *spectra of $\mathcal{A}_0$ and $S_0(\cdot)$ are equivalent on $\mathbb{C} \setminus (-\infty, 0]$,*

$$\sigma_{(\mathrm{p/e2})}(\mathcal{A}_0) \setminus (-\infty, 0] = \sigma_{(\mathrm{p/e2})}(S_0(\cdot)) \setminus (-\infty, 0], \tag{4.7}$$

and the following relations hold on $(-\infty, 0)$

$$\sigma(\mathcal{A}_0) \cap (-\infty, 0) \supset \sigma(S_0(\cdot)) \cap (-\infty, 0), \tag{4.8}$$

$$\sigma_{\mathrm{p}}(\mathcal{A}_0) \cap (-\infty, 0) = \sigma_{\mathrm{p}}(S_0(\cdot)) \cap (-\infty, 0). \tag{4.9}$$

Remark 4.3. If the assumptions on damping and potential in [13] are satisfied, the operator G introduced therein coincides with $\mathcal{A}_0$. Assuming essentially that for every $\varepsilon > 0$ there exists $C_\varepsilon \geq 0$ such that

$$a \in W^{1,\infty}_{\mathrm{loc}}(\overline{\Omega}), \qquad |\nabla a| \leq \varepsilon a^{\frac{3}{2}} + C_\varepsilon,$$

see [13, Asm. I] for the precise more general assumptions, the authors define G as the closure of the operator matrix G_0 in $\mathcal{W}(\Omega) \oplus L^2(\Omega)$ given by

$$G_0 = \begin{pmatrix} 0 & I \\ \Delta - q & -2a \end{pmatrix}, \qquad \operatorname{dom} G_0 = (\operatorname{dom}(-\Delta + q)_{L^2(\Omega)} \cap \operatorname{dom} a)^2;$$

here $(-\Delta + q)_{L^2(\Omega)}$ shall denote the Friedrichs extension of $-\Delta + q$ on $C_0^\infty(\Omega)$. They show that, independently of $\lambda \in \mathbb{C} \setminus (-\infty, 0]$,

$$\operatorname{dom} S_0(\lambda) = \operatorname{dom}(-\Delta + q)_{L^2(\Omega)} \cap \operatorname{dom} a \subset \mathcal{W}(\Omega) \cap \mathcal{D}_S.$$

From this it follows easily that $\operatorname{dom} G_0 \subset \operatorname{dom} \mathcal{A}_0$ and their actions clearly coincide. Since both $-G$ and $-\mathcal{A}_0$ are m-accretive, see [13, Thm. 2.2], their equality already follows from the inclusion $G \subset \mathcal{A}_0$, see e.g. [20, §V.3.10]. //

Remark 4.4. Our setting is more general than the perturbative framework in e.g. [4, 19] and can equally be employed to cover distributional dampings studied therein, see e.g. [4, Chap. 4] where $a(x) = \delta(x - x_0)$ is discussed in one dimension. //

4.2. **Realisation of matrix and Schur complement.** In line with Sections 2 and 3, we present our approach to the underlying spectral problem. Note that since we use the second Schur complement, the roles of the spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, as well as the matrix entries acting in them, are reversed correspondingly, see Remark 2.4.

4.2.1. *Definition of spaces.* We introduce the spaces for Assumption 3.1 (i).

Definition 4.5. Let Assumption 4.1 hold, let $\mathcal{H}_1 := \mathcal{W}(\Omega)$ with $\mathcal{W}(\Omega)$ as in Theorem 4.2 and $\mathcal{H}_2 := L^2(\Omega)$. Let the space $\mathcal{D}_S$ defined in (4.4) be equipped with the inner product

$$\langle f, g\rangle_S := \int_\Omega \nabla f \cdot \overline{\nabla g} \, \mathrm{d}x + \int_\Omega q f \overline{g} \, \mathrm{d}x + \int_\Omega a f \overline{g} \, \mathrm{d}x + \int_\Omega f \overline{g} \, \mathrm{d}x, \quad f, g \in \mathcal{D}_S; \tag{4.10}$$

notice that $q, a \geq 0$ a.e. in Ω. Moreover, define

$$\mathcal{D}_1 = \mathcal{D}_{-1} := \mathcal{W}(\Omega), \qquad \mathcal{D}_{-2} = \mathcal{D}_{-S} := \mathcal{D}_S^*. \qquad //$$

Proposition 4.6. *Under Assumption* 4.1*, the spaces in Definition* 4.5 *satisfy Assumption* 3.1 (i) *according to Remark* 2.4*. Moreover, $C_0^\infty(\Omega)$ is dense in $\mathcal{D}_S$ and one can embed $\mathcal{D}_S \subset \mathcal{W}(\Omega)$ continuously.*

Proof. Since $q \in L^1_{\mathrm{loc}}(\Omega)$, the Hilbert space completion of $C_0^\infty(\Omega)$ with respect to the inner product in (4.3) is well-defined. Clearly, all other spaces in Definition 4.5 are also well-defined and Hilbert. The inclusion $C_0^\infty(\Omega) \subset \mathcal{D}_S$ is clear from the assumption $a, q \in L^1_{\mathrm{loc}}(\Omega)$; its density can be shown using standard techniques. It thus follows from the construction of $\mathcal{W}(\Omega)$ and the inequality $\|\cdot\|_{\mathcal{W}} \leq \|\cdot\|_S$ that one can clearly embed $\mathcal{D}_S \subset \mathcal{W}(\Omega)$ continuously.

It remains to show that $\mathcal{D}_S \subset L^2(\Omega) \subset \mathcal{D}_S^*$, where the corresponding embeddings are continuous with dense range. The density of $\mathcal{D}_S$ in $L^2(\Omega)$ is a consequence of $C_0^\infty(\Omega) \subset \mathcal{D}_S$, and since we have $\|\cdot\| \leq \|\cdot\|_S$ by construction, it is continuously embedded. Upon identification of $L^2(\Omega)$ with its anti-dual space, we further obtain

$$\mathcal{D}_S \subset L^2(\Omega) \equiv L^2(\Omega)^* \subset \mathcal{D}_S^*. \tag{4.11}$$

The second inclusion in (4.11) is realised by the continuous embedding

$$L^2(\Omega) \ni f \equiv \langle f, \cdot \rangle \mapsto \langle f, \cdot \rangle|_{\mathcal{D}_S} \in \mathcal{D}_S^*. \tag{4.12}$$

Indeed, since $\mathcal{D}_S$ is continuously embedded in $L^2(\Omega)$, the map in (4.12) is well-defined and bounded; its injectivity follows from the density of $\mathcal{D}_S$ in $L^2(\Omega)$.

For Assumption 3.1 (i), it remains to show that $L^2(\Omega)$ is dense in $\mathcal{D}_S^*$. It suffices, however, to observe that the embedding in (4.12) is nothing but the adjoint operator

$$I_{\mathcal{D}_S}^* : L^2(\Omega)^* \to \mathcal{D}_S^*$$

and that $\operatorname{ran} I_{\mathcal{D}_S}^*$ is dense in $\mathcal{D}_S^*$ since $\ker I_{\mathcal{D}_S} = \{0\}$, cf. [11, p. 170]. □

Remark 4.7. The restriction of every functional in $\mathcal{D}_S^*$ to $C_0^\infty(\Omega) \subset \mathcal{D}_S$ is a distribution, i.e. one can embed $\mathcal{D}_S^* \subset \mathcal{D}'(\Omega)$; indeed, since $a, q \in L^1_{\mathrm{loc}}(\Omega)$, the convergence of test functions in $\mathcal{D}(\Omega)$ clearly implies their convergence in $\mathcal{D}_S$. Moreover, the density of $C_0^\infty(\Omega)$ guarantees injectivity of the embedding $\mathcal{D}_S^* \to \mathcal{D}'(\Omega)$.

If $n \geq 3$ then we also have the inclusion $\mathcal{W}(\Omega) \subset \mathcal{D}'(\Omega)$; this follows from

$$\mathcal{W}(\Omega) \subset \dot{H}^1(\Omega) \subset L^{\frac{2n}{n-2}}(\Omega) \subset L^1_{\mathrm{loc}}(\Omega),$$

where $\dot{H}^1(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ with respect to $\|\nabla \cdot\|$, i.e. the first order homogeneous Sobolev space, and the second inclusion is a consequence of the Gagliardo-Nirenberg-Sobolev inequality. //

4.2.2. *Definition of matrix entries.* We introduce the remaining objects needed for Assumption 3.1.

Definition 4.8. Let Assumption 4.1 be satisfied and define the following operators

$$\begin{aligned} A_0 = A &:= \quad 0 && \in \mathcal{B}(\mathcal{W}(\Omega)), & B &:= \quad I && \in \mathcal{B}(\mathcal{D}_S, \mathcal{W}(\Omega)), \\ C &:= \Delta - q && \in \mathcal{B}(\mathcal{W}(\Omega), \mathcal{D}_S^*), & D &:= -2a && \in \mathcal{B}(\mathcal{D}_S, \mathcal{D}_S^*); \end{aligned}$$

see Definition 4.5. Here the operators $\Delta - q$ and a are the unique extensions of

$$\left.\begin{aligned} ((\Delta - q)f, g)_{\mathcal{D}_S^* \times \mathcal{D}_S} &:= -\int_\Omega \nabla f \cdot \overline{\nabla g} \,\mathrm{d}x - \int_\Omega q f \overline{g} \,\mathrm{d}x, \\ (af, g)_{\mathcal{D}_S^* \times \mathcal{D}_S} &:= \int_\Omega a f \overline{g} \,\mathrm{d}x, \end{aligned}\right. \qquad f, g \in C_0^\infty(\Omega), \tag{4.13}$$

see Proposition 4.9 below for details. //

Proposition 4.9. *Under Assumption* 4.1*, the operators defined in Definition* 4.8 *are well-defined and satisfy Assumption* 3.1 *according to Remark* 2.4*.*

Proof. Since $\mathcal{D}_S \subset \mathcal{W}(\Omega)$ is continuously embedded by Proposition 4.6, clearly

$$0 \in \mathcal{B}(\mathcal{W}(\Omega)), \qquad I \in \mathcal{B}(\mathcal{D}_S, \mathcal{W}(\Omega)).$$

It remains to show that C and D are well-defined by (4.13) and bounded between the claimed spaces. For $f, g \in C_0^\infty(\Omega)$, we have the inequality

$$|((\Delta - q)f, g)_{\mathcal{D}_S^* \times \mathcal{D}_S}| \leq \|f\|_{\mathcal{W}} \|g\|_{\mathcal{W}} \leq \|f\|_{\mathcal{W}} \|g\|_S. \tag{4.14}$$

Taking into account (4.14) and the density of $C_0^\infty(\Omega)$ in $\mathcal{D}_S$, see Proposition 4.6, the formula (4.13) determines a unique bounded antilinear functional on $\mathcal{D}_S$

$$(\Delta - q)f \in \mathcal{D}_S^*, \qquad f \in C_0^\infty(\Omega).$$

Moreover, from (4.14) it follows that $f \mapsto (\Delta - q)f$ is a bounded map from $\mathcal{W}(\Omega)$ to $\mathcal{D}_S^*$ and therefore, by density of $C_0^\infty(\Omega)$ in $\mathcal{W}(\Omega)$, has a unique extension

$$\Delta - q \in \mathcal{B}(\mathcal{W}(\Omega), \mathcal{D}_S^*).$$

In the same way, by deriving the estimate

$$|(af, g)_{\mathcal{D}_S^* \times \mathcal{D}_S}| \le \|a^{\frac{1}{2}} f\| \|a^{\frac{1}{2}} g\| \le \|f\|_S \|g\|_S, \quad f, g \in C_0^\infty(\Omega),$$

one shows that $a \in \mathcal{B}(\mathcal{D}_S, \mathcal{D}_S^*)$ is well-defined. □

4.2.3. *Definition of matrix and Schur complement.* We proceed analogously to Definition 3.2, cf. Remark 2.4.

Definition 4.10. Let Assumption 4.1 be satisfied. Define the operator matrix

$$\mathcal{A} := \begin{pmatrix} 0 & I \\ \Delta - q & -2a \end{pmatrix} \in \mathcal{B}(\mathcal{W}(\Omega) \oplus \mathcal{D}_S, \mathcal{W}(\Omega) \oplus \mathcal{D}_S^*)$$

and its second Schur complement

$$S(\lambda) := -2a - \lambda + \frac{1}{\lambda}(\Delta - q)|_{\mathcal{D}_S} \in \mathcal{B}(\mathcal{D}_S, \mathcal{D}_S^*), \qquad \lambda \in \Theta := \mathbb{C} \setminus \{0\}; \qquad (4.15)$$

see Definitions 4.5 and 4.8, as well as Definition 3.2 and notice that Θ satisfies (3.2) therein. Let $\mathcal{A}_0$ and $S_0(\cdot)$, respectively, be the corresponding (family of) maximal operators in $\mathcal{W}(\Omega) \oplus L^2(\Omega)$ and $L^2(\Omega)$; more precisely, $\mathcal{A}_0 := \mathcal{A}|_{\operatorname{dom} \mathcal{A}_0}$ and $S_0(\lambda) := S(\lambda)|_{\operatorname{dom} S_0(\lambda)}$ with their respective domains

$$\operatorname{dom} \mathcal{A}_0 := \{(f, g) \in \mathcal{W}(\Omega) \times \mathcal{D}_S : \mathcal{A}(f, g) \in \mathcal{W}(\Omega) \times L^2(\Omega)\},$$
$$\operatorname{dom} S_0(\lambda) := \{f \in \mathcal{D}_S : S(\lambda) f \in L^2(\Omega)\}. \qquad //$$

The following proposition shows that (4.15) agrees with the standard definition of the Schur complement via its quadratic form.

Proposition 4.11. *Let Assumption* 4.1 *hold and let* $S(\cdot)$ *be as in Definition* 4.10. *Then, for every* $\lambda \in \mathbb{C} \setminus \{0\}$ *and* $f, g \in \mathcal{D}_S$,

$$(S(\lambda)f, g)_{\mathcal{D}_S^* \times \mathcal{D}_S} = -\frac{1}{\lambda}\left(\int_\Omega \nabla f \overline{\nabla g} \, \mathrm{d}x + \int_\Omega q f \overline{g} \, \mathrm{d}x + 2\lambda \int_\Omega a f \overline{g} \, \mathrm{d}x + \lambda^2 \int_\Omega f \overline{g} \, \mathrm{d}x\right).$$

Proof. Considering Definitions 4.8 and 4.10, it is clear that the claimed identity holds for $f, g \in C_0^\infty(\Omega)$. Since $C_0^\infty(\Omega)$ is dense in $\mathcal{D}_S$ and both sides of the formula are continuous with respect to convergence in $\mathcal{D}_S$, it remains valid for $f, g \in \mathcal{D}_S$. □

Remark 4.12. Under Assumption 4.1, the actions of $\mathcal{A}$ and $S(\cdot)$ introduced in Definition 4.10 coincide with their standard distributional definitions, cf. Remark 4.7. Indeed, taking into account that $q \in L^1_{\mathrm{loc}}(\Omega)$, clearly

$$\Delta f - q f \in L^1_{\mathrm{loc}}(\Omega), \qquad f \in C_0^\infty(\Omega), \qquad (4.16)$$

is a regular distribution and coincides with $Cf \in \mathcal{D}_S^*$ as defined in (4.13). The action of C on the completion $\mathcal{W}(\Omega)$, however, is constructed by continuous extension, see (4.13), and is thus given as a limit of regular distributions of the form (4.16); notice that convergence of functionals in $\mathcal{D}_S^*$ implies their convergence in $\mathcal{D}'(\Omega)$. Finally, the following distributions

$$ag \in L^1_{\mathrm{loc}}(\Omega), \quad -\frac{1}{\lambda}\left(-\Delta f + q f + 2\lambda a f + \lambda^2\right) \in \mathcal{D}'(\Omega), \quad \lambda \in \mathbb{C} \setminus \{0\}, \quad f, g \in \mathcal{D}_S,$$

are well-defined and coincide with $Dg \in \mathcal{D}_S^*$ and $S(\lambda)f \in \mathcal{D}_S^*$ as in (4.13) and (4.15), respectively; see Proposition 4.11 and notice that $\mathcal{D}_S \subset \operatorname{dom} a^{\frac{1}{2}} \cap \operatorname{dom} q^{\frac{1}{2}}$. //

4.3. **Proof of Theorem 4.2.** We first formulate a crucial ingredient for the proof of Theorem 4.2 as a lemma. For every $\lambda \in \mathbb{C} \setminus (\infty, 0]$, it provides the existence of an extension as in (2.4); this is needed in order to apply Corollaries 3.4 (ii) and 2.9.

Lemma 4.13. *Let Assumption* 4.1 *be satisfied and let* $S_0(\cdot)$ *be as in Definition* 4.10*. For all* $\lambda \in \mathbb{C} \setminus (-\infty, 0]$*, the domain* $\operatorname{dom} S_0(\lambda)$ *is dense in* $\mathcal{D}_S$ *and there exists* $z_\lambda \in \rho(S_0(\lambda))$ *such that*

$$(S(\lambda) - z_\lambda)^{-1} \in \mathcal{B}(\mathcal{D}_S^*, \mathcal{D}_S). \tag{4.17}$$

Moreover, if $\operatorname{Re} \lambda > 0$*, then one can choose* $z_\lambda = 0$*.*

Proof. Fix $\lambda \notin (-\infty, 0]$. Analogously to the proof of [13, Lem. 2.3], one can show that there exists $z_\lambda \in \mathbb{C}$ such that $S(\lambda) - z_\lambda$ corresponds to a bounded and coercive sesquilinear form on $\mathcal{D}_S$; cf. the proof of Theorem 5.1 and also the proof of Lemma 6.15, where the analogous statement is proven in a different setting. By the Lax-Milgram Theorem, see e.g. [11, Cor. IV.1.2], we conclude (4.17), $z_\lambda \in \rho(S_0(\lambda))$ and density of the maximal domain $\operatorname{dom}(S_0(\lambda) - z_\lambda) = \operatorname{dom} S_0(\lambda)$ in $\mathcal{D}_S$. Finally, from the proof of [13, Lem. 2.3], it is clear that one can choose $z_\lambda = 0$ in case that $\operatorname{Re} \lambda > 0$. □

Proof of Theorem 4.2. We start by pointing out that, by Propositions 4.6 and 4.9, Assumption 3.1 is satisfied, the objects in Definition 4.10 are well-defined and the results of Section 3 applicable. Moreover, $\mathcal{A}_0$ and $S_0(\cdot)$ as in Definition 4.10 coincide with their definition in (4.5), see Remark 4.12. Considering this, the description of their domains in (4.6) follows immediately from Definition 4.10.

Let us show the identities (4.7) - (4.9). To this end, we refer to Remark 2.4 and apply the results from Section 3 correspondingly. First, since $\rho(A_0) = \Theta = \mathbb{C} \setminus \{0\}$, Corollary 3.5 implies

$$\sigma_{\mathrm{p}}(\mathcal{A}_0) \setminus \{0\} = \sigma_{\mathrm{p}}(S_0(\cdot)). \tag{4.18}$$

We proceed by showing that the assumptions of Corollary 3.7 are satisfied. Let $\lambda \in \Sigma := \mathbb{C} \setminus (-\infty, 0]$ be arbitrary. Since $\mathcal{D}_{-S} = \mathcal{D}_{-2}$, one clearly has

$$S(\lambda) \in \mathcal{B}(\mathcal{D}_S, \mathcal{D}_{-S}), \qquad C(\operatorname{dom} A_0) \subset \mathcal{D}_{-S}.$$

Moreover, according to Lemma 4.13, $\operatorname{dom} S_0(\lambda)$ is dense in $\mathcal{D}_S$ and there exists $z_\lambda \in \rho(S_0(\lambda))$ such that

$$(S_0(\lambda) - z_\lambda)^{-1} \subset S_z^\ddagger(\lambda) := (S(\lambda) - z_\lambda)^{-1} \in \mathcal{B}(\mathcal{D}_{-S}, \mathcal{D}_S). \tag{4.19}$$

Considering that $\Sigma \subset \rho(A_0)$, we can thus apply Corollary 3.7 to conclude

$$\sigma_{(\mathrm{e}2)}(\mathcal{A}_0) \setminus (-\infty, 0] \subset \sigma_{(\mathrm{e}2)}(S_0(\cdot)) \setminus (-\infty, 0]. \tag{4.20}$$

Furthermore, from Corollary 3.6 (i) with $\rho(A_0) = \Theta = \mathbb{C} \setminus \{0\}$, we obtain

$$\sigma(\mathcal{A}_0) \setminus \{0\} \supset \sigma(S_0(\cdot)). \tag{4.21}$$

Moreover, since $\Sigma \subset \rho(A_0)$ and since, for all $\lambda \in \Sigma$, Lemma 4.13 implies that $\rho(S_0(\lambda)) \neq \emptyset$ and thus $S_0(\lambda) \in \mathcal{C}(\mathcal{H}_2)$, the assumptions of Corollary 3.6 (ii) are satisfied and we conclude

$$\sigma_{\mathrm{e}2}(\mathcal{A}_0) \setminus (-\infty, 0] \supset \sigma_{\mathrm{e}2}(S_0(\cdot)) \setminus (-\infty, 0]. \tag{4.22}$$

In summary, (4.7) follows from (4.18), (4.20), (4.21) and (4.22); the inclusion (4.8) is a consequence of (4.21) and the identity (4.9) follows from (4.18).

We continue by showing that $-\mathcal{A}_0$ is accretive, i.e. that

$$\operatorname{Re}\langle \mathcal{A}_0(f,g),(f,g)\rangle_{\mathcal{H}} \le 0, \qquad (f,g)\in \operatorname{dom}\mathcal{A}_0.$$

To this end, let $(f,g)\in\operatorname{dom}\mathcal{A}_0$. By the density of $C_0^\infty(\Omega)$ in $\mathcal{D}_S$, see Proposition 4.6, and by construction of $\mathcal{W}(\Omega)$, there exist $\{f_m\}_m,\{g_m\}_m\subset C_0^\infty(\Omega)$ such that $f_m\to f$ in $\mathcal{W}(\Omega)$ and $g_m\to g$ in $\mathcal{D}_S$ as $m\to\infty$. Moreover, $\mathcal{D}_S\subset\mathcal{W}(\Omega)$ is continuously embedded, see Proposition 4.6, and we conclude

$$\langle g,f\rangle_{\mathcal{W}} = \lim_{m\to\infty}\left(\langle\nabla g_m,\nabla f_m\rangle + \langle q^{\frac12}g_m, q^{\frac12}f_m\rangle\right). \tag{4.23}$$

One can easily show from the definitions in (4.13) that

$$((\Delta-q)f-2ag,g)_{\mathcal{D}_S^*\times\mathcal{D}_S} = -\lim_{m\to\infty}\Big(\langle\nabla f_m,\nabla g_m\rangle + \langle q^{\frac12}f_m,q^{\frac12}g_m\rangle \\ +2\langle a^{\frac12}g_m,a^{\frac12}g_m\rangle\Big).$$

Using this, (4.23) and $(h,\cdot)_{\mathcal{D}_S^*\times\mathcal{D}_S}=\langle h,\cdot\rangle$ for $h\in L^2(\Omega)$, we further derive

$$\begin{aligned}\langle\mathcal{A}_0(f,g),(f,g)\rangle_{\mathcal{H}} &= \langle g,f\rangle_{\mathcal{W}} + \langle(\Delta-q)f-2ag,g\rangle\\ &= -2\,\mathrm{i}\lim_{m\to\infty}\operatorname{Im}\left(\langle\nabla f_m,\nabla g_m\rangle+\langle q^{\frac12}f_m,q^{\frac12}g_m\rangle\right)\\ &\qquad -2\lim_{m\to\infty}\langle a^{\frac12}g_m,a^{\frac12}g_m\rangle.\end{aligned}$$

Finally, the accretivity of $-\mathcal{A}_0$ then follows from

$$\operatorname{Re}\langle\mathcal{A}_0(f,g),(f,g)\rangle_{\mathcal{H}} = -2\lim_{m\to\infty}\langle a^{\frac12}g_m,a^{\frac12}g_m\rangle\le 0.$$

Next we note that, by Lemma 4.13, we have $S(\lambda)^{-1}\in\mathcal{B}(\mathcal{D}_S^*,\mathcal{D}_S)$ and thus $0\in\rho(S_0(\lambda))$ whenever $\operatorname{Re}\lambda>0$. By taking complements in (4.20), we conclude

$$\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda>0\}\subset\rho(S_0(\cdot))\subset\rho(\mathcal{A}_0).$$

This implies that $-\mathcal{A}_0$ is m-accretive; the generation of a strongly continuous contraction semigroup then follows from a standard result in semigroup theory, see e.g. [20, §IX.1].

It remains to show that $\operatorname{dom}\mathcal{A}_0$ is dense in $\mathcal{W}(\Omega)\oplus\mathcal{D}_S$ and $\mathcal{W}(\Omega)\oplus L^2(\Omega)$. This, however, follows from $\mathcal{D}_{-S}=\mathcal{D}_S^*=\mathcal{D}_{-2}$ and Corollary 3.4 (ii), since we have shown already that the assumptions of Proposition 2.5 (ii) are satisfied in any point $\lambda\in\mathbb{C}\setminus(\infty,0]$, see (4.19). □

Remark 4.14. The density of $\operatorname{dom}\mathcal{A}_0$ in $\mathcal{W}(\Omega)\oplus L^2(\Omega)$ (which already follows from its m-accretivity, see [20, §V.3.10]) can equally be shown by employing Proposition 2.5 (i). Let therefore $(\Delta-q)_{L^2(\Omega)}$ denote the Friedrichs extension of $\Delta-q$ on $C_0^\infty(\Omega)$. One can show that

$$\operatorname{dom}(\Delta-q)_{L^2(\Omega)}\subset\operatorname{dom}C_0=\operatorname{dom}A_0\cap\operatorname{dom}C_0,$$

where we used $\operatorname{dom}A_0=\mathcal{W}(\Omega)$ and where $\operatorname{dom}C_0$ is defined analogously to (2.3). Moreover, one can prove that $\operatorname{dom}(\Delta-q)_{L^2(\Omega)}$ is dense in $\mathcal{W}(\Omega)$, which by means of Proposition 2.5 (i) gives the claimed density. //

5. Klein-Gordon equation with purely imaginary potential

We consider a Klein-Gordon equation on $\mathbb{R}^n$ with potential V and mass $m > 0$

$$(\partial_t - \mathrm{i}\,V(x))^2 u(x,t) - \Delta_x u(x,t) + m^2 u(x,t) = 0, \qquad x \in \mathbb{R}^n, \quad t \geq 0;$$

here the involved physical constants are normalised for the sake of simplicity, see e.g. [30] for the full generality. The equation above has been studied in a large number of works, employing various (operator theoretic) approaches, see e.g. [23, 24, 30]. After suitable transformations, one arrives at the following first order Cauchy problem

$$\partial_t \begin{pmatrix} u_1(t,x) \\ u_2(t,x) \end{pmatrix} = \mathrm{i} \begin{pmatrix} 0 & 1 \\ -\Delta_x + m^2 - V(x)^2 & 2V(x) \end{pmatrix} \begin{pmatrix} u_1(t,x) \\ u_2(t,x) \end{pmatrix}. \tag{5.1}$$

We mention that also another system of equations arising by means of different transformations has been of interest, for instance in [24, 30]. Motivated by the underlying physical problem, the potential is assumed to be real-valued in all works above. This results in a certain indefiniteness of the problem, which makes its spectral analysis less straightforward, see e.g. [23] where Krein spaces together with a smallness condition for the potential were employed.

For purely imaginary potentials $V = \mathrm{i}\,W$ with real-valued W, however, the problem can be reduced to a suitable wave equation, see (5.8) below; note that the latter has a special structure since the damping W is relatively bounded with respect to the term $-\Delta + m^2 + W^2$. Assuming only that the potential is locally square integrable, we define the matrix expression

$$\mathcal{A} := \begin{pmatrix} 0 & I \\ -\Delta + m^2 + W^2 & 2\,\mathrm{i}\,W \end{pmatrix} \tag{5.2}$$

on the right hand side of (5.1) as a densely defined, boundedly invertible operator in a suitable Hilbert space and show spectral equivalence to its second Schur complement

$$S(\lambda) := \frac{1}{\lambda}\left(-\Delta + m^2 + (W + \lambda\,\mathrm{i})^2\right), \qquad \lambda \in \mathbb{C} \setminus \{0\}, \tag{5.3}$$

see Theorem 5.1. Moreover, in Example 5.2 below, we consider the special case $W(x) = x$ in one dimension and show that the spectrum of the resulting operator matrix is empty, which is in line with the analogous results for the Airy operator in case of the Schrödinger equation.

We denote $\|\cdot\| := \|\cdot\|_{L^2(\mathbb{R}^n)}$ and $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{L^2(\mathbb{R}^n)}$ and, if its meaning is clear from the context, adapt the same notation for norm and inner product in $L^2(\mathbb{R}^n)^n$. The operator matrix and its Schur complement are defined as follows

$$\mathcal{A}_0 := \mathcal{A}|_{\operatorname{dom}\mathcal{A}_0}, \qquad S_0(\lambda) := S(\lambda)|_{\operatorname{dom} S_0}, \qquad \lambda \in \mathbb{C} \setminus \{0\}, \tag{5.4}$$

acting in the underlying Hilbert spaces $\mathcal{W}(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$, respectively, with $\mathcal{A}$ and $S(\lambda)$ as in (5.2) and (5.3) understood in the standard distributional sense, on their respective domains

$$\begin{aligned} \operatorname{dom}\mathcal{A}_0 &:= \operatorname{dom} S_0 \times \mathcal{W}(\mathbb{R}^n), \\ \operatorname{dom} S_0 &:= \{f \in \mathcal{W}(\mathbb{R}^n) : (\Delta f - W^2) f \in L^2(\mathbb{R}^n)\}; \end{aligned} \tag{5.5}$$

notice that the domain of $S_0(\lambda)$ is independent of $\lambda \in \mathbb{C} \setminus \{0\}$ and that the domain of $\mathcal{A}_0$ is diagonal. In the above, the first component of the product space is

$$\mathcal{W}(\mathbb{R}^n) := H^1(\mathbb{R}^n) \cap \operatorname{dom} W$$

considered as Hilbert space equipped with the inner product

$$\langle f, g\rangle_{\mathcal{W}}^2 := \int_{\mathbb{R}^n} \nabla f \overline{\nabla g}\, \mathrm{d}x + \int_{\mathbb{R}^n} W^2 f\overline{g}\, \mathrm{d}x + \int_{\mathbb{R}^n} f\overline{g}\, \mathrm{d}x, \qquad f, g \in \mathcal{W}(\mathbb{R}^n). \tag{5.6}$$

Theorem 5.1. *Assume that $W \in L^2_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R})$ and let $\mathcal{A}_0$ and $S_0(\cdot)$ be as in* (5.4) *and* (5.5)*. Then $\mathcal{A}_0$ is closed with $0 \in \rho(\mathcal{A}_0)$ and its domain is dense both in $\mathcal{W}(\mathbb{R}^n) \oplus \mathcal{W}(\mathbb{R}^n)$ and in $\mathcal{W}(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$. Moreover, the* (*point and essential*) *spectra of $\mathcal{A}_0$ and $S_0(\cdot)$ are equivalent, more precisely,*

$$\sigma_{(\mathrm{p/e2})}(\mathcal{A}_0) = \sigma_{(\mathrm{p/e2})}(S_0(\cdot)). \tag{5.7}$$

Proof. We relate the problem to a certain damped wave equation and apply the results in Section 4. In detail, for $\lambda \in \mathbb{C}$ we have

$$\mathcal{A}_0 - \lambda = \mathrm{i}\, \mathrm{diag}(1, \mathrm{i})^{-1} (\widetilde{\mathcal{A}}_0 - \mathrm{i}\,\lambda)\, \mathrm{diag}(1, \mathrm{i})$$

with $\widetilde{\mathcal{A}}_0$ being the following linear operator in $\mathcal{W}(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$

$$\widetilde{\mathcal{A}}_0 := \begin{pmatrix} 0 & I \\ \Delta - (m^2 + W^2) & -2W \end{pmatrix}, \qquad \operatorname{dom} \widetilde{\mathcal{A}}_0 := \operatorname{dom} \mathcal{A}_0. \tag{5.8}$$

This clearly gives the equivalence

$$\lambda \in \sigma_{(\mathrm{p/e2})}(\mathcal{A}_0) \iff \mathrm{i}\,\lambda \in \sigma_{(\mathrm{p/e2})}(\widetilde{\mathcal{A}}_0). \tag{5.9}$$

We apply Theorem 4.2 to $\widetilde{\mathcal{A}}_0$ and its second Schur complement $\widetilde{S}_0(\cdot)$ in a suitable way, i.e. with $\Omega := \mathbb{R}^n$, the potential $q := m^2 + W^2$ and the damping $a := W$ therein. Even though the latter might be indefinite, it is relatively bounded with bound zero with respect to the potential in the sense of quadratic forms, and the spectral equivalence can thus be implemented analogously. Indeed, merely the following adjustments have to be made.

Instead of (4.3), (4.4) and (4.10), define $\mathcal{W}(\Omega) := \mathcal{W}(\mathbb{R}^n)$ and $\|\cdot\|_{\mathcal{W}}$ as in (5.6) and set $\mathcal{D}_S := \mathcal{W}(\mathbb{R}^n)$; notice that (4.3) and (5.6) give equivalent norms and $\mathcal{W}(\mathbb{R}^n)$ is in fact independent of choosing either one of them. Taking into account these modifications, Propositions 4.6, 4.9 and 4.11 remain valid. Moreover, since $\mathcal{W}(\mathbb{R}^n) \subset \operatorname{dom} W$, the domain of $\widetilde{\mathcal{A}}_0$ defined as in (4.6) indeed coincides with $\operatorname{dom} \mathcal{A}_0$ in (5.5). In view of Lemma 4.13, we consider the Schur complement

$$\widetilde{S}(\lambda) = -\frac{1}{\lambda} \left(-\Delta + m^2 + W^2 + 2\lambda W + \lambda^2\right) \in \mathcal{B}(\mathcal{W}(\mathbb{R}^n), \mathcal{W}(\mathbb{R}^n)^*),$$

and the restriction $\widetilde{S}_0(\cdot)$ to its maximal domain in $L^2(\mathbb{R}^n)$, see (4.5) and (4.6), which since $\mathcal{W}(\mathbb{R}^n) \subset \operatorname{dom} W$, is given by

$$\operatorname{dom} \widetilde{S}_0(\lambda) = \operatorname{dom} S_0, \quad \lambda \in \mathbb{C} \setminus \{0\}.$$

In order to apply Theorem 4.2, it thus suffices to justify that $\operatorname{dom} S_0$ is dense in $L^2(\mathbb{R}^n)$ and that, for every $\lambda \neq 0$, there exists $z_\lambda \in \mathbb{C}$ such that (4.17) holds and $z_\lambda = 0$ can be chosen if $\lambda > 0$ is sufficiently large. This, however, follows in a straightforward way from the order zero form-relative boundedness of the damping with respect to the potential. More precisely, using Cauchy-Schwarz' and Young's inequalities, one shows that for every $\delta > 0$ it holds that

$$\int_{\mathbb{R}^n} |W||f|^2 \,\mathrm{d}x \le \frac{\delta}{2}\int_{\mathbb{R}^n} W^2|f|^2 \,\mathrm{d}x + \frac{1}{2\delta}\int_{\mathbb{R}^n} |f|^2 \,\mathrm{d}x, \quad f \in \operatorname{dom}(W^2).$$

Hence, for $\lambda \in \mathbb{C}\setminus\{0\}$ and $f \in \mathcal{W}(\mathbb{R}^n)$, the above gives

$$\begin{aligned}\operatorname{Re}(-\lambda\widetilde{S}(\lambda)f,f)_{\mathcal{W}(\mathbb{R}^n)\times\mathcal{W}(\mathbb{R}^n)^*} \ge \|\nabla f\|^2 &+ (1-|\lambda|\delta)\|Wf\|^2 \\ &+ (m^2+\operatorname{Re}(\lambda^2) - |\lambda|\delta^{-1})\|f\|^2,\end{aligned}$$

implying the existence of $z_\lambda \in \mathbb{C}$ such that $\widetilde{S}(\lambda) - z_\lambda$ is coercive on $\mathcal{W}(\mathbb{R}^n)$ and that $z_\lambda = 0$ is a possible choice for any $\lambda > 0$. The claimed invertibility in (4.17) now follows from the Lax-Milgram-Theorem, see [11, Cor. IV.1.2]; cf. the proofs of Lemmas 4.13 and 6.15. Combining the above, Theorem 4.2 implies the claimed density of $\operatorname{dom}\mathcal{A}_0 = \operatorname{dom}\widetilde{\mathcal{A}}_0$ and

$$0 \ne \lambda \in \sigma_{(\mathrm{p/e2})}(\widetilde{\mathcal{A}}_0) \iff \lambda \in \sigma_{(\mathrm{p/e2})}(\widetilde{S}_0(\cdot)). \tag{5.10}$$

It remains to show that $\mathcal{A}_0$ is boundedly invertible. The equivalence (5.7) then follows from (5.9), (5.10) and since for $\lambda \ne 0$ clearly

$$0 \in \sigma_{(\mathrm{p/e2})}(\widetilde{S}_0(\mathrm{i}\,\lambda)) \iff 0 \in \sigma_{(\mathrm{p/e2})}(S_0(\lambda)).$$

To show $0 \in \rho(\mathcal{A}_0)$, in view of (5.9), it suffices to prove that

$$\widetilde{\mathcal{A}}_0^{-1} = \mathcal{R} := \begin{pmatrix} -2T_0^{-1}W & -T_0^{-1} \\ I & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{W}(\mathbb{R}^n), L^2(\mathbb{R}^n))$$

where T_0 is the linear operator in $L^2(\mathbb{R}^n)$ given by

$$T_0 := -\Delta + m^2 + W^2, \qquad \operatorname{dom} T_0 := \operatorname{dom} S_0.$$

However, the claimed boundedness of $\mathcal{R}$ readily follows from

$$W \in \mathcal{B}(\mathcal{W}(\mathbb{R}^n), L^2(\mathbb{R}^n)), \qquad T_0^{-1} \in \mathcal{B}(L^2(\mathbb{R}^n), \mathcal{W}(\mathbb{R}^n)),$$

and the proof of $\widetilde{\mathcal{A}}_0\mathcal{R} = \mathcal{I}_{\mathcal{W}(\mathbb{R}^n)\oplus L^2(\mathbb{R}^n)}$ and $\mathcal{R}\widetilde{\mathcal{A}}_0 = \mathcal{I}_{\operatorname{dom}\mathcal{A}_0}$ is straightforward. □

Example 5.2. For the one dimensional purely imaginary potential $V = \mathrm{i}\,x$, our realisation of the Klein-Gordon Cauchy problem (5.1) has empty spectrum. More precisely, we show that if $n = 1$ and $W(x) := x$, $x \in \mathbb{R}$, then $\sigma(\mathcal{A}_0) = \emptyset$.

By Theorem 5.1, we have that $\sigma(\mathcal{A}_0) = \sigma(S_0(\cdot))$, where

$$S_0(\lambda) = \frac{1}{\lambda}\left(-\partial_x^2 + m^2 + (x+\mathrm{i}\,\lambda)^2\right), \qquad \lambda \in \mathbb{C}\setminus\{0\},$$

is the operator family in $L^2(\mathbb{R})$ on the (λ-independent) domain

$$\operatorname{dom} S_0 = H^2(\mathbb{R}) \cap \operatorname{dom}(x^2);$$

see [8, Prop. 2.6. (i)] for the domain separation property and note that

$$\operatorname{dom}(m^2 + (x+\mathrm{i}\,\lambda)^2) = \operatorname{dom}(x^2).$$

The claim thus follows if we show that $\sigma(S_0(\cdot)) = \emptyset$. It is easy to see that

$$T(\lambda) := \lambda S_0(\lambda), \quad T(0) := -\partial_x^2 + m^2 + x^2, \quad \operatorname{dom} T(\lambda) := \operatorname{dom} S_0, \quad \lambda \in \mathbb{C},$$

is a holomorphic family of type (A) in the sense of [20, Sec. VII.2]; notice that in fact $T(\cdot)$ is also type (B) holomorphic, see [20, Sec. VII.4], with form domain $\mathcal{W}(\mathbb{R})$ as in (5.6). Since for every $\lambda \ne 0$, the operator $T(\lambda)$ is a bound zero perturbation of $T(0)$ and the latter has compact resolvent, see [28, Thm. XIII.67], also $T(\lambda)$ has compact resolvent; see [20, Thm. IV.1.16] and note that

$$\|(T(0)-\mu)^{-1}\| \to 0, \qquad \mu \to -\infty.$$

Due to the analyticity of $T(\cdot)$, the isolated eigenvalues (of finite multiplicity) of $T(\lambda)$ depend analytically on $\lambda \in \mathbb{C}$, see [20, Sec. VII.1.3, Thm. VII.1.8]. Considering that, for $\lambda \in \mathrm{i}\,\mathbb{R}_+$, by unitary equivalence we have

$$\sigma(T(0)) = \sigma(-\partial_x^2 + m^2 + (x-|\lambda|)^2) = \sigma(T(\lambda)),$$

the spectrum of $T(\lambda)$ remains unchanged in $\lambda \in \mathbb{C}$ and thus, for $\lambda \neq 0$,

$$\lambda\sigma(S_0(\lambda)) = \sigma(T(\lambda)) = \sigma(T(0)).$$

Since it is well-known that $\sigma(T(0))$ does not contain zero, this gives $\sigma(S_0(\cdot)) = \emptyset$.

6. Singular coefficient matrix differential operators

As another application, we study the spectra of generic second order differential operator matrices of the form

$$\begin{pmatrix} -\Delta + q & \nabla \cdot \mathbf{b} \\ \mathbf{c} \cdot \nabla & d \end{pmatrix} \tag{6.1}$$

with irregular coefficients acting in the Hilbert space $L^2(\Omega) \oplus L^2(\Omega)$ with $\Omega \subset \mathbb{R}^n$ while imposing Dirichlet boundary conditions. Operator matrices with this particular structure appear in several problems in mathematical physics, see e.g. [7, 15, 16, 17, 21, 22]. Due to the apparent independence of their entries, however, their spectral analysis is not straightforward and previous results typically rely on the regularity or special form of coefficients. Our approach merges the conceptual idea of a dominant Schur complement in [15, 16, 17] with a distributional framework, which enables us to substantially reduce the required regularity of the coefficients. In particular, we allow the latter to be singular and/or unbounded, as long as they satisfy certain (in some sense minimal) conditions arising from our setting in Section 3, see Assumption 6.1.

The main results in this section are Theorems 6.3 and 6.5; the former provides spectral equivalence of our realisation of the operator matrix (6.1) and its Schur complement, whereas in the latter we show that if q and d are sectorial and $\mathbf{c} = \overline{\mathbf{b}}$, then the resulting operator matrix is m-accretive and generates a strongly continuous contraction semigroup.

6.1. Assumptions and main results.

We impose the following *low regularity assumptions* on the coefficients.

Assumption 6.1. Let $\Omega \subset \mathbb{R}^n$ be open.

(i) *Basic assumptions on coefficients*: Let $\mathbf{b}, \mathbf{c} : \Omega \to \mathbb{C}^n$ be measurable and

$$q \in L^1_{\mathrm{loc}}(\Omega), \qquad d \in L^\infty_{\mathrm{loc}}(\Omega).$$

(ii) *Definition and regularity of* $\boldsymbol{\pi}$ *on* Θ: Denote by $\mathbf{I} \in \mathbb{C}^{n\times n}$ the identity matrix, let $\Theta \subset \mathbb{C} \setminus \operatorname{ess\,ran} d$ be connected and let

$$\boldsymbol{\pi}(\lambda) := \mathbf{I} + (d-\lambda)^{-1}(\mathbf{b} \otimes \mathbf{c}) \in L^1_{\mathrm{loc}}(\Omega)^{n\times n}, \qquad \lambda \in \Theta. \tag{6.2}$$

(iii) *Sectoriality of q and $\boldsymbol{\pi}$ on Φ (after rotation and shift)*: Let $\emptyset \neq \Phi \subset \Theta$. For all $\lambda \in \Phi$, assume there exist $\omega_\lambda \in (-\pi, \pi]$ and $\gamma_\lambda \geq 0$ such that both

$$\widetilde{q}(\lambda) := \mathrm{e}^{\mathrm{i}\,\omega_\lambda}\, q + \gamma_\lambda, \qquad \widetilde{\boldsymbol{\pi}}(\lambda) := \mathrm{e}^{\mathrm{i}\,\omega_\lambda}\, \boldsymbol{\pi}(\lambda), \tag{6.3}$$

are sectorial; more precisely, a.e. in Ω, let $\operatorname{Re} \widetilde{q}(\lambda) \geq 0$, let the matrix $\operatorname{Re} \widetilde{\boldsymbol{\pi}}(\lambda)$ be positive definite and let $C_\lambda > 0$ be such that

$$\begin{aligned} &|\operatorname{Im} \widetilde{q}(\lambda)| \leq C_\lambda \operatorname{Re} \widetilde{q}(\lambda), \\ &|\operatorname{Im} \widetilde{\boldsymbol{\pi}}(\lambda)\xi \cdot \overline{\xi}| \leq C_\lambda \operatorname{Re} \widetilde{\boldsymbol{\pi}}(\lambda)\xi \cdot \overline{\xi}, \qquad \xi \in \mathbb{C}^n. \end{aligned}$$

Moreover, for all $\lambda, \mu \in \Phi$, assume there exist constants $m_{\lambda,\mu}, M_{\lambda,\mu} > 0$ such that a.e. in Ω the following holds

$$\begin{aligned} m_{\lambda,\mu} \operatorname{Re} \widetilde{q}(\mu) \ &\leq\ \operatorname{Re} \widetilde{q}(\lambda) \ \leq\ M_{\lambda,\mu} \operatorname{Re} \widetilde{q}(\mu), \\ m_{\lambda,\mu} \operatorname{Re} \widetilde{\boldsymbol{\pi}}(\mu) \ &\leq\ \operatorname{Re} \widetilde{\boldsymbol{\pi}}(\lambda) \ \leq\ M_{\lambda,\mu} \operatorname{Re} \widetilde{\boldsymbol{\pi}}(\mu). \end{aligned} \tag{6.4}$$

(iv) *Dominance of Schur complement*: For all $\lambda \in \Phi$, assume that

$$(d-\lambda)^{-1} \max\left(\left|(\operatorname{Re} \widetilde{\boldsymbol{\pi}}(\lambda))^{-\frac{1}{2}} \mathbf{b}\right|, \left|(\operatorname{Re} \widetilde{\boldsymbol{\pi}}(\lambda))^{-\frac{1}{2}} \overline{\mathbf{c}}\right| \right) \in L^\infty(\Omega). \qquad //$$

Remark 6.2. (i) The assumptions $q \in L^1_{\mathrm{loc}}(\Omega)$ and $\boldsymbol{\pi}(\lambda) \in L^1_{\mathrm{loc}}(\lambda)^{n\times n}$ are naturally minimal and guarantee that the generalised quadratic form of the Schur complement is densely defined for $\lambda \in \Theta$. Assumption 6.1 (iii), however, translates into sectoriality (after shift and rotation) of the Schur complement on the set of parameters Φ.

(ii) The assumption $d \in L^\infty_{\mathrm{loc}}(\Omega)$ is made for the sake of simplicity and can be relaxed. It is a sufficient condition for the following

$$\omega(d-\lambda)^{-1}\mathbf{b} \in L^2_{\mathrm{loc}}(\Omega)^n, \qquad \omega \in L^2_{\mathrm{loc}}(\Omega), \qquad (d-\lambda)\omega^{-1} \in L^2_{\mathrm{loc}}(\Omega),$$

where $\lambda \in \Phi$ and ω is defined in (6.6) below. The first two of the conditions above guarantee that our realisation of the operator matrix in (6.1) and its Schur complement coincide with their standard definition in the distributional sense, see Remark 6.14. The third condition ensures that $C_0^\infty(\Omega) \subset \mathcal{D}_2$, where $\mathcal{D}_2$ is defined in (6.6) below.

(iii) Assumption 6.1 (iv) is essential; it ensures that, on the set of parameters Φ, the Schur complement dominates the neighbouring factors of the Schur-Frobenius factorisation of the resolvent in a suitable way, cf. Corollary 3.7.

(iv) By (6.4) and Lemma 6.11 below, it is equivalent to assume (6.2) or Assumption 6.1 (iv), respectively, only in an arbitrary point $\lambda_0 \in \Theta$ or $\lambda_0 \in \Phi$. //

From now on, we write $\|\cdot\| := \|\cdot\|_{L^2(\Omega)}$ and $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{L^2(\Omega)}$; the same notation will be used for norm and inner product on $L^2(\Omega)^n$ if no confusion can arise.

6.1.1. *Spectral correspondence.* Under Assumption 6.1, we place the spectral problem for the operator matrix (6.1) in a suitable setting and apply the results from Section 3. Fix an arbitrary point $\lambda_0 \in \Phi$ and set $q_0 := \widetilde{q}(\lambda_0)$ and $\boldsymbol{\pi}_0 := \widetilde{\boldsymbol{\pi}}(\lambda_0)$, see (6.3). Let $\mathcal{D}_S$ be the closure of $C_0^\infty(\Omega)$ with respect to the inner product

$$\langle f, g \rangle_S := \int_\Omega \operatorname{Re} \boldsymbol{\pi}_0 \nabla f \cdot \overline{\nabla g}\, \mathrm{d}x + \int_\Omega \operatorname{Re} q_0 f \overline{g}\, \mathrm{d}x + \int_\Omega f \overline{g}\, \mathrm{d}x, \quad f, g \in C_0^\infty(\Omega); \tag{6.5}$$

recall that $q_0 \geq 0$ and $\operatorname{Re} \boldsymbol{\pi}_0 > 0$ is positive definite a.e. in Ω. Moreover, define

$$\mathcal{D}_2 := L^2(\Omega, |d-\lambda_0|^2 \omega^{-2}), \qquad \omega := \max\left(1, |(\operatorname{Re} \boldsymbol{\pi}_0)^{-\frac{1}{2}} \overline{\mathbf{c}}|\right). \tag{6.6}$$

We emphasise that, as topological spaces, $\mathcal{D}_S$ and $\mathcal{D}_2$ do not depend on the choice of λ_0. Indeed, by (6.4) and Lemma 6.11 below, their respective inner products generate equivalent norms for distinct choices of $\lambda_0 \in \Phi$.

Let $\mathcal{A}_0$ and $S_0(\cdot)$ be the (family of) maximal operators in $L^2(\Omega) \oplus L^2(\Omega)$ and $L^2(\Omega)$, respectively, corresponding to the differential expressions

$$\mathcal{A} := \begin{pmatrix} -\Delta + q & \nabla \cdot \mathbf{b} \\ \mathbf{c} \cdot \nabla & d \end{pmatrix}, \qquad S(\lambda) := -\nabla \cdot \boldsymbol{\pi}(\lambda)\nabla + q - \lambda, \quad \lambda \in \Theta, \tag{6.7}$$

see (6.2), on their respective maximal domains

$$\begin{aligned} \operatorname{dom} \mathcal{A}_0 &:= \big\{(f,g) \in \mathcal{D}_S \times \mathcal{D}_2 \, : \, (\Delta - q)f - \nabla \cdot \mathbf{b}g \in L^2(\Omega), \\ &\qquad\qquad \mathbf{c} \cdot \nabla f + dg \in L^2(\Omega)\big\}, \\ \operatorname{dom} S_0(\lambda) &:= \{f \in \mathcal{D}_S : (\nabla \cdot \boldsymbol{\pi}(\lambda)\nabla - q)f \in L^2(\Omega)\}. \end{aligned} \tag{6.8}$$

Here the above operations are understood in a standard (antilinear) distributional sense, see Definitions 6.6, 6.9, 6.12 and Remark 6.14 below for details. Note that $S_0(\cdot)$ coincides with the standard definition of the Schur complement by means of its quadratic form on the set of parameters Φ were the latter is sectorial.

The first main result in this section, the spectral correspondence between operator matrix and its Schur complement, reads as follows; its proof can be found in Section 6.3.

Theorem 6.3. *Let Assumption* 6.1 *be satisfied and let* $\mathcal{A}_0$ *and* $S_0(\cdot)$ *be as in* (6.7) *and* (6.8)*. Then the following identities and inclusions hold*

$$\sigma_{\mathrm{p}}(\mathcal{A}_0) \cap \Theta = \sigma_{\mathrm{p}}(S_0(\cdot)) \cap \Theta, \tag{6.9}$$

$$\sigma(\mathcal{A}_0) \cap \Theta \supset \sigma(S_0(\cdot)) \cap \Theta, \tag{6.10}$$

$$\sigma(\mathcal{A}_0) \cap \Phi \subset \sigma(S_0(\cdot)) \cap \Phi, \tag{6.11}$$

$$\sigma_{\mathrm{e2}}(\mathcal{A}_0) \cap \Phi = \sigma_{\mathrm{e2}}(S_0(\cdot)) \cap \Phi. \tag{6.12}$$

If $\operatorname{dom} B_0 \cap \operatorname{dom} d$ *is dense in* $L^2(\Omega)$*, where*

$$\operatorname{dom} B_0 = \{f \in \mathcal{D}_2 : \nabla \cdot \mathbf{b}f \in L^2(\Omega)\},$$

then $\operatorname{dom} \mathcal{A}_0$ *is dense in* $L^2(\Omega) \oplus L^2(\Omega)$*.*

Remark 6.4. If e.g. $\mathbf{b} \in W^{1,2}_{\mathrm{loc}}(\Omega)$, then both $\operatorname{dom} d$ and $\operatorname{dom} B_0$ contain $C_0^\infty(\Omega)$, thus $\operatorname{dom} B_0 \cap \operatorname{dom} d$ is dense in $L^2(\Omega)$ and $\operatorname{dom} \mathcal{A}_0$ is dense in $L^2(\Omega) \oplus L^2(\Omega)$. //

6.1.2. *Generation of contraction semigroup.* When imposing additional assumptions on the structure of the problem, the resulting operator matrix proves to be m-accretive and thus generates a strongly continuous contraction semigroup, see Theorem 6.5 below.

More precisely, besides Assumption 6.1 (i), we require that $\mathbf{c} = \overline{\mathbf{b}}$ and that q and d are sectorial, i.e. that there exist $0 \le \theta_q, \theta_d < \pi/2$ such that a.e. in Ω

$$\begin{aligned} \operatorname{Re} q &\ge 0, \qquad & |\operatorname{Im} q| &\le \tan \theta_q \operatorname{Re} q, \\ \operatorname{Re} d &\ge 0, \qquad & |\operatorname{Im} d| &\le \tan \theta_d \operatorname{Re} d. \end{aligned} \tag{6.13}$$

The above structural assumptions imply Assumption 6.1 (iii) with certain sets

$$\Theta := \{\lambda \in \mathbb{C} : |\arg \lambda| > \theta_d\}, \quad \Phi := \{\lambda \in \mathbb{C} : |\arg \lambda| > \max(\theta_q, \theta_d)\}, \tag{6.14}$$

where $\arg : \mathbb{C} \setminus \{0\} \to (-\pi, \pi]$ and $\arg 0 = 0$. Moreover, the regularity and dominance Assumptions 6.1 (iii) and (iv), respectively, reduce to

$$\boldsymbol{\pi}_0 := \mathbf{I} + (d+1)^{-1}(\mathbf{b} \otimes \overline{\mathbf{b}}) \in L^1_{\mathrm{loc}}(\Omega)^{n\times n}, \quad (d+1)^{-1}(\operatorname{Re} \boldsymbol{\pi}_0)^{-\frac{1}{2}}\mathbf{b} \in L^\infty(\Omega)^n; \tag{6.15}$$

see Remark 6.2 (iv) and notice that $\operatorname{Re} \boldsymbol{\pi}_0 \geq \mathbf{I}$ is positive definite, see (6.33) below. Applying Theorem 6.3, we thereby obtain the following theorem; its proof can be found in Section 6.4.

Theorem 6.5. *Let Assumption* 6.1 (i) *hold, let* $\mathbf{c} = \overline{\mathbf{b}}$*, let* q *and* d *be sectorial with semi-angle* $\theta_q, \theta_d \in [0, \pi/2)$*, see* (6.13)*, and let* (6.15) *be satisfied. Let* Θ *and* Φ *be defined as in* (6.14) *and let* $\mathcal{A}_0$ *and* $S_0(\cdot)$ *be as in* (6.7) *and* (6.8)*, where* $\boldsymbol{\pi}$ *is as in* (6.2)*, and we set* $\lambda_0 := -1$*,* $q_0 := q$ *in the definitions* (6.5), (6.6)*. Then* (6.9) - (6.12) *hold and* $-\mathcal{A}_0$ *is m-accretive, thus* $\mathcal{A}_0$ *generates a strongly continuous contraction semigroup on* $L^2(\Omega) \oplus L^2(\Omega)$*. Its domain is dense in* $L^2(\Omega) \oplus L^2(\Omega)$ *and satisfies*

$$\pi_2 \operatorname{dom} \mathcal{A}_0 \subset \operatorname{dom} |d|^{\frac{1}{2}}. \tag{6.16}$$

6.2. **Realisation of matrix and Schur complement.** In the more general setting of Assumption 6.1, we provide the appropriate framework to the spectral problem for the operator matrix (6.1) such that the results of Section 3 apply. We therefore need to define the objects in Assumption 3.1 in a suitable way.

6.2.1. *Definition of spaces.* We introduce the spaces needed for Assumption 3.1 (i).

Definition 6.6. Let Assumption 6.1 be satisfied and let $\mathcal{H}_1 := \mathcal{H}_2 := L^2(\Omega)$. Moreover, let $\mathcal{D}_2$ and ω be as in (6.6) and let $\mathcal{D}_S$ and $\mathcal{D}_1$, respectively, be the closure of $C_0^\infty(\Omega)$ with respect to the inner products in (6.5) and

$$\langle f, g\rangle_1 := \int_\Omega |d-\lambda_0|^{-2}\omega^2(\mathbf{b}\otimes\overline{\mathbf{b}})\nabla f \cdot \overline{\nabla g}\,\mathrm{d}x + \int_\Omega \Delta f \overline{\Delta g}\,\mathrm{d}x + \langle f, g\rangle_S, \qquad f, g \in C_0^\infty(\Omega);$$

notice that $(\mathbf{b}\otimes\overline{\mathbf{b}}) \geq 0$ is positive semi-definite a.e. in Ω. Finally, define

$$\mathcal{D}_{-S} := \mathcal{D}_S^*, \qquad \mathcal{D}_{-1} := \mathcal{D}_1^*, \qquad \mathcal{D}_{-2} := L^2(\Omega, \omega^{-2}). \qquad //$$

Proposition 6.7. *Under Assumption* 6.1*, the spaces in Definition* 6.6 *are well-defined and satisfy Assumption* 3.1 (i)*.*

Proof. By assumption, $\operatorname{Re} \boldsymbol{\pi}_0 \in L^1_{\mathrm{loc}}(\Omega)^{n\times n}$ is positive definite and $\operatorname{Re} q_0 \in L^1_{\mathrm{loc}}(\Omega)$ is non-negative a.e. in Ω. Considering this, it is easy to see that $\mathcal{D}_S$ is a well-defined Hilbert space. Moreover, from $d \in L^\infty_{\mathrm{loc}}(\Omega)$ and Assumption 6.1 (iv) it follows that

$$(\operatorname{Re} \boldsymbol{\pi}_0)^{-\frac{1}{2}}\overline{c} = (d-\lambda_0)(d-\lambda_0)^{-1}(\operatorname{Re} \boldsymbol{\pi}_0)^{-\frac{1}{2}}\overline{c} \in L^\infty_{\mathrm{loc}}(\Omega)^n, \tag{6.17}$$

which in turn implies $\omega \in L^\infty_{\mathrm{loc}}(\Omega)$. From $\boldsymbol{\pi}_0 \in L^1_{\mathrm{loc}}(\Omega)^{n\times n}$ and Assumption 6.1 (iv), we further derive that

$$\omega(d-\lambda_0)^{-1}\mathbf{b} = \omega(\operatorname{Re} \boldsymbol{\pi}_0)^{\frac{1}{2}}(d-\lambda_0)^{-1}(\operatorname{Re} \boldsymbol{\pi}_0)^{-\frac{1}{2}}\mathbf{b} \in L^2_{\mathrm{loc}}(\Omega)^n, \tag{6.18}$$

and thus also $\mathcal{D}_1$ is a well-defined Hilbert space.

Analogously to the proof of Proposition 4.6, from the density of $C_0^\infty(\Omega)$ in $L^2(\Omega)$ and since $\|\cdot\| \leq \|\cdot\|_S$ by construction, we obtain the inclusions

$$\mathcal{D}_S \subset L^2(\Omega) \subset \mathcal{D}_S^*$$

where the corresponding embeddings are continuous and have dense range. Similarly, it follows that $\mathcal{D}_1 \subset \mathcal{D}_S$, and thus $\mathcal{D}_S^* \subset \mathcal{D}_1^*$, are dense and continuously embedded.

Clearly, the weighted spaces $\mathcal{D}_2$ and $\mathcal{D}_{-2}$ are Hilbert spaces. From $\lambda_0 \notin \operatorname{ess\,ran} d$ and Assumption 6.1 (iv), it follows that

$$|d-\lambda_0|^2\omega^{-2} = \min\left(|d-\lambda_0|^2, |(d-\lambda_0)^{-1}(\operatorname{Re}\boldsymbol{\pi}_0)^{-\frac{1}{2}}\overline{\mathbf{c}}|^{-2}\right) \gtrsim 1,$$

thus $\mathcal{D}_2 \subset L^2(\Omega)$ is continuously embedded. Moreover, this implies that, as a topological vector space, $\mathcal{D}_2$ coincides with the weighted space $L^2(\Omega, 1+|d-\lambda_0|^2\omega^{-2})$, i.e. with $\operatorname{dom}((d-\lambda_0)\omega^{-1})$. Since the maximal domain of a multiplication operator by a measurable function is dense, see e.g. [6, Ex. 4.3.3 (a)], the density of $\mathcal{D}_2$ in $L^2(\Omega)$ follows. Analogously, from $\omega \geq 1$ a.e. in Ω we conclude that $L^2(\Omega) \subset \mathcal{D}_{-2}$ is dense and continuously embedded. □

Remark 6.8. Due to Assumption 6.1, the spaces $\mathcal{D}_1^*$ and $\mathcal{D}_{-2}$ in Definition 6.6 can both be understood as subspaces of $\mathcal{D}'(\Omega)$, cf. Remark 4.7. Indeed, since ω is locally bounded by (6.17), we have

$$\mathcal{D}_{-2} = L^2(\Omega, \omega^{-2}) \subset L^1_{\mathrm{loc}}(\Omega) \subset \mathcal{D}'(\Omega).$$

The embedding of $\mathcal{D}_1^*$ in $\mathcal{D}'(\Omega)$ can be shown analogously to Remark 4.7. //

6.2.2. *Definition of matrix entries.* We introduce the operators needed for Assumption 3.1 (ii) and (iii).

Definition 6.9. Let Assumption 6.1 be satisfied. Define the following operators

$$\begin{aligned} A &:= -\Delta + q \in \mathcal{B}(\mathcal{D}_S, \mathcal{D}_1^*), \qquad & B &:= \nabla\cdot\mathbf{b} \in \mathcal{B}(\mathcal{D}_2, \mathcal{D}_1^*), \\ C &:= \mathbf{c}\cdot\nabla f \in \mathcal{B}(\mathcal{D}_S, \mathcal{D}_{-2}), \qquad & D &:= d \in \mathcal{B}(\mathcal{D}_2, \mathcal{D}_{-2}), \end{aligned}$$

see Definition 6.6 for the spaces involved. Here A and B are determined uniquely by the defining identities

$$\begin{aligned} (Af, g)_{\mathcal{D}_1^*\times\mathcal{D}_1} &:= \int_\Omega \nabla f\cdot\overline{\nabla g}\,\mathrm{d}x + \int_\Omega qf\overline{g}\,\mathrm{d}x, \qquad & f &\in C_0^\infty(\Omega), \quad g \in C_0^\infty(\Omega), \\ (Bf, g)_{\mathcal{D}_1^*\times\mathcal{D}_1} &:= -\int_\Omega \mathbf{b}f\cdot\overline{\nabla g}\,\mathrm{d}x, & f &\in \mathcal{D}_2, \quad g \in C_0^\infty(\Omega), \end{aligned} \tag{6.19}$$

and the operator C is the unique extension in $\mathcal{B}(\mathcal{D}_S, \mathcal{D}_{-2})$ of

$$Cf := \mathbf{c}\cdot\nabla f, \qquad f \in C_0^\infty(\Omega), \tag{6.20}$$

see Proposition 6.10 for details. Finally, define $D_0 := D|_{\operatorname{dom} d}$ as the maximal multiplication operator by d in $L^2(\Omega)$. //

Proposition 6.10. *Let Assumption* 6.1 *hold. Then the operators introduced in Definition* 6.9 *are well-defined and satisfy Assumption* 3.1 (ii) *and* (iii). *Moreover,*

$$(D-\lambda)^{-1} \in \mathcal{B}(\mathcal{D}_{-2}, \mathcal{D}_2), \qquad \lambda \in \Theta, \tag{6.21}$$

where $\Theta \subset \mathbb{C}\setminus\operatorname{ess\,ran} d$ *is the connected set in Assumption* 6.1 (ii)*.*

The following elementary lemma will be needed several times, including for the proof of (6.21) in Proposition 6.10.

Lemma 6.11. *Let* $\Theta \subset \mathbb{C}\setminus\operatorname{ess\,ran} d$ *be connected and* $\lambda, \mu \in \Theta$*. Then there exist constants* $C_{\lambda,\mu}, C'_{\lambda,\mu} > 0$ *such that a.e. in* Ω

$$C_{\lambda,\mu}|d-\mu| \leq |d-\lambda| \leq C'_{\lambda,\mu}|d-\mu|.$$

Proof. It is easy to see that, for every fixed $\lambda_0 \in \Theta$, there exists a set $N \subset \Omega$ with measure zero and a radius $\delta > 0$ such that

$$\frac{2}{3}|d(x) - \nu| \leq |d(x) - \lambda_0| \leq 2|d(x) - \nu|, \qquad \nu \in B_\delta(\lambda_0), \quad x \in \Omega \setminus N. \tag{6.22}$$

The claim of the lemma then follows from a standard compactness argument by connecting λ and μ with a path in Θ and using (6.22) locally. $\square$

Proof of Proposition 6.10. Integration by parts and sectoriality of q_0 give

$$|(Af, g)_{\mathcal{D}_1^* \times \mathcal{D}_1}| \lesssim \|f\|\|\Delta g\| + \|(\operatorname{Re} q_0)^{\frac{1}{2}} f\|\|(\operatorname{Re} q_0)^{\frac{1}{2}} g\| + \|f\|\|g\| \leq \|f\|_S \|g\|_1$$

for $f, g \in C_0^\infty(\Omega)$, see Assumption 6.1 (iii). Analogously to the proof of Proposition 4.9, this implies that $A \in \mathcal{B}(\mathcal{D}_S, \mathcal{D}_1^*)$ is uniquely well-defined by (6.19). Similarly, for $f \in \mathcal{D}_2$ and $g \in C_0^\infty(\Omega)$,

$$|(Bf, g)_{\mathcal{D}_1^* \times \mathcal{D}_1}| \leq \|f(d - \lambda_0)\omega^{-1}\|\|(d - \lambda_0)^{-1}\omega \overline{\mathbf{b}} \cdot \nabla g\| \leq \|f\|_{\mathcal{D}_2}\|g\|_1$$

implies that $B \in \mathcal{B}(\mathcal{D}_2, \mathcal{D}_1^*)$ is uniquely well-defined by (6.19). For any $f \in C_0^\infty(\Omega)$, we moreover obtain the chain of inequalities

$$\begin{aligned} \|Cf\|_{\mathcal{D}_{-2}}^2 &\leq \int_\Omega |\langle \overline{\mathbf{c}}, \nabla f \rangle_{\mathbb{C}^n}|^2 \omega^{-2} \,\mathrm{d}x \\ &\leq \int_\Omega |(\operatorname{Re} \boldsymbol{\pi}_0)^{-\frac{1}{2}} \overline{\mathbf{c}}|^2 |(\operatorname{Re} \boldsymbol{\pi}_0)^{\frac{1}{2}} \nabla f|^2 \omega^{-2} \,\mathrm{d}x \\ &\leq \|\omega^{-1}(\operatorname{Re} \boldsymbol{\pi}_0)^{-\frac{1}{2}} \overline{\mathbf{c}}\|_{L^\infty(\Omega)^n}^2 \|f\|_S^2; \end{aligned} \tag{6.23}$$

notice that the right hand side of the above inequality is finite by definition of ω. It now follows from the density of $C_0^\infty(\Omega)$ in $\mathcal{D}_S$ that the operator defined in (6.20) has a unique extension $C \in \mathcal{B}(\mathcal{D}_S, \mathcal{D}_{-2})$.

We next show that D and D_0 satisfy Assumption 3.1 (ii). From the definition of $\mathcal{D}_2$ and $\mathcal{D}_{-2}$, it is obvious that $D \in \mathcal{B}(\mathcal{D}_2, \mathcal{D}_{-2})$ and, as maximal multiplication operator in $L^2(\Omega)$, its restriction D_0 is closed in $L^2(\Omega)$. Moreover, since both $|d - \lambda_0|$ and ω are uniformly positive a.e. in Ω, the topological vector spaces $L^2(\Omega, |d|^2 + 1)$ and $L^2(\Omega, (\omega^2 + 1)|d - \lambda_0|^2\omega^{-2})$ coincide. It follows that $\operatorname{dom} d$ can be understood as the maximal domain $\operatorname{dom}_{\mathcal{D}_2} \omega$ of the multiplication operator by the measurable function ω in the weighted space $\mathcal{D}_2$ and is thus a dense subspace of the latter, see e.g. [6, Ex. 4.3.3 (a)] and note that the weighted Lebesgue measure with weight $(d - \lambda_0)^2 w^{-2} \in L^1_{\mathrm{loc}}(\Omega)$ is σ-finite. cf. Remark 6.2 (ii). Finally, for fixed $\lambda \in \Theta$, relation (6.21) holds since by Lemma 6.11 with $\mu = \lambda_0$ for all $f \in \mathcal{D}_{-2}$ we have

$$\|(d - \lambda)^{-1} f\|_{\mathcal{D}_2}^2 = \int_\Omega |d - \lambda|^{-2}|f|^2|d - \lambda_0|^2\omega^{-2} \,\mathrm{d}x \lesssim \int_\Omega |f|^2\omega^{-2} \,\mathrm{d}x = \|f\|_{\mathcal{D}_{-2}}^2. \quad \square$$

6.2.3. *Definition of matrix and Schur complement.* Analogously to Definition 3.2, we now introduce our realisation of the operator matrix (6.1) and its Schur complement. Note that, by Proposition 6.10, the set Θ satisfies inclusion (3.2).

Definition 6.12. We define the operator matrix

$$\mathcal{A} := \begin{pmatrix} -\Delta + q & \nabla \cdot \mathbf{b} \\ \mathbf{c} \cdot \nabla & d \end{pmatrix} \in \mathcal{B}(\mathcal{D}_S \oplus \mathcal{D}_2, \mathcal{D}_1^* \oplus \mathcal{D}_{-2})$$

and its first Schur complement

$$S(\lambda) := -\Delta + q - \lambda - (\nabla \cdot \mathbf{b})(d - \lambda)^{-1}(\mathbf{c} \cdot \nabla) \in \mathcal{B}(\mathcal{D}_S, \mathcal{D}_1^*), \qquad \lambda \in \Theta,$$

see Definitions 6.6 and 6.9 for the spaces and operators involved. Let the (family of) maximal operators $\mathcal{A}_0$ in $L^2(\Omega) \oplus L^2(\Omega)$ and $S_0(\cdot)$ in $L^2(\Omega)$, respectively, be defined as the restrictions of $\mathcal{A}$ and $S(\cdot)$ to their respective maximal domains

$$\operatorname{dom} \mathcal{A}_0 := \{(f,g) \in \mathcal{D}_S \times \mathcal{D}_2 : \mathcal{A}(f,g) \in L^2(\Omega) \times L^2(\Omega)\}, \tag{6.24}$$
$$\operatorname{dom} S_0(\lambda) := \{f \in \mathcal{D}_S : S(\lambda) f \in L^2(\Omega)\}. \qquad //$$

The proposition below shows that the Schur complement acts according to the formula obtained from naively integrating by parts. On the set of parameters Φ where it is sectorial, our definition of the Schur complement coincides with its standard definition by means of its quadratic form with form domain $\mathcal{D}_S$ and core $C_0^\infty(\Omega)$ and is bounded between $\mathcal{D}_S$ and its anti-dual $\mathcal{D}_S^*$.

Proposition 6.13. *Let Assumption* 6.1 *be satisfied and let* $S(\cdot)$ *be as in Definition* 6.12*. Then, for all* $\lambda \in \Theta$ *and* $f, g \in C_0^\infty(\Omega)$*,*

$$(S(\lambda)f, g)_{\mathcal{D}_1^* \times \mathcal{D}_1} = \int_\Omega \boldsymbol{\pi}(\lambda) \nabla f \cdot \overline{\nabla g} \, \mathrm{d}x + \int_\Omega q f \overline{g} \, \mathrm{d}x - \lambda \int_\Omega f \overline{g} \, \mathrm{d}x. \tag{6.25}$$

Moreover, if $\lambda \in \Phi$*, then* $S(\lambda) \in \mathcal{B}(\mathcal{D}_S, \mathcal{D}_S^*)$*.*

Proof. Fix $\lambda \in \Theta$ and let $f, g \in C_0^\infty(\Omega)$. Since $(d-\lambda)^{-1} \mathbf{c} \cdot \nabla f \in \mathcal{D}_2$, it follows by definition that

$$\begin{aligned}(S(\lambda)f, g)_{\mathcal{D}_1^* \times \mathcal{D}_1} = \int_\Omega \nabla f \cdot \overline{\nabla g} \, \mathrm{d}x + \int_\Omega q f \overline{g} \, \mathrm{d}x - \lambda \int_\Omega f \overline{g} \, \mathrm{d}x \\ + \int_\Omega (d-\lambda)^{-1} (\mathbf{c} \cdot \nabla f)(\mathbf{b} \cdot \overline{\nabla g}) \, \mathrm{d}x.\end{aligned}$$

Formula (6.25) is then easily derived from the definition of $\boldsymbol{\pi}(\lambda)$ and

$$(\mathbf{c} \cdot \xi)(\mathbf{b} \cdot \overline{\xi}) = (\mathbf{b} \otimes \mathbf{c}) \xi \cdot \overline{\xi}, \qquad \xi \in \mathbb{C}^n.$$

Now let $\lambda \in \Phi$ be fixed. Using formula (6.25), multiplying it by the unimodular factor $\mathrm{e}^{\mathrm{i}\,\omega_\lambda}$ and employing Assumption 6.1 (iii), we conclude

$$|(S(\lambda)f, f)_{\mathcal{D}_1^* \times \mathcal{D}_1}| \lesssim \int_\Omega \operatorname{Re} \widetilde{\boldsymbol{\pi}}(\lambda) \nabla f \cdot \overline{\nabla f} \, \mathrm{d}x + \int_\Omega \operatorname{Re} \widetilde{q}(\lambda) |f|^2 \, \mathrm{d}x + \int_\Omega |f|^2 \, \mathrm{d}x \lesssim \|f\|_S^2$$

for $f \in C_0^\infty(\Omega)$. By a straightforward polarisation argument, it follows that

$$|(S(\lambda)f, g)_{\mathcal{D}_1^* \times \mathcal{D}_1}| \lesssim \|f\|_S \|g\|_S, \quad f, g \in C_0^\infty(\Omega). \tag{6.26}$$

Since $C_0^\infty(\Omega)$ is dense both in $\mathcal{D}_1$ and in $\mathcal{D}_S$, $S(\lambda) \in \mathcal{B}(\mathcal{D}_S, \mathcal{D}_1^*)$ and $\mathcal{D}_1 \subset \mathcal{D}_S$ is continuously embedded, the inequality (6.26) remains valid for $f \in \mathcal{D}_S$ and $g \in \mathcal{D}_1$. However, this implies $S(\lambda) \in \mathcal{B}(\mathcal{D}_S, \mathcal{D}_S^*)$ for all $\lambda \in \Phi$. $\square$

Remark 6.14. The actions of $\mathcal{A}$ and $S(\cdot)$ can be understood in a standard distributional sense, cf. Remark 6.8 and also Remark 4.12 for the damped wave equation. More precisely, by Assumption 6.1 (i), the distributions

$$-\Delta f + q f \in L^1_{\mathrm{loc}}(\Omega), \quad \nabla \cdot \boldsymbol{\pi}(\lambda) \nabla f + q f - \lambda f \in \mathcal{D}'(\Omega), \qquad f \in C_0^\infty(\Omega), \tag{6.27}$$

are well-defined and coincide with the functionals $Af \in \mathcal{D}_1^*$ and $S(\lambda) f \in \mathcal{D}_1^*$, respectively, see definitions of the latter in (6.19) and formula (6.25). Moreover,

$$Cf = \mathbf{c} \cdot \nabla f \in \mathcal{D}_{-2} \subset L^1_{\mathrm{loc}}(\Omega), \qquad f \in C_0^\infty(\Omega). \tag{6.28}$$

Since the actions of A, C and $S(\lambda)$ on $\mathcal{D}_S$ are obtained by continuous extension, see (6.19) and (6.20), they are given as limits of distributions of the form (6.27) and (6.28); notice that the convergence of functionals in $\mathcal{D}_1^*$ or $\mathcal{D}_{-2} = L^2(\Omega, w^2)^*$ implies their convergence in $\mathcal{D}'(\Omega)$. From (6.18), we see that

$$\mathbf{b}g = \omega(d-\lambda_0)^{-1}\mathbf{b}g(d-\lambda_0)\omega^{-1} \in L^1_{\mathrm{loc}}(\Omega)^n, \quad g \in \mathcal{D}_2 = L^2(\Omega, |d-\lambda_0|^2\omega^{-2}).$$

Hence, the distributional divergence $\nabla \cdot \mathbf{b}g \in \mathcal{D}'(\Omega)$ is well-defined and clearly coincides with $Bg \in \mathcal{D}_1^*$ as in (6.19). Finally, it is clear that $D = d$ is a standard multiplication operator between the weighted spaces $\mathcal{D}_2$ and $\mathcal{D}_{-2}$. //

6.3. **Proof of Theorem 6.3.** The statement of Theorem 6.3 can be obtained from the results in Section 3; the following lemma is needed in order to apply Corollaries 3.6 and 3.7 therein.

Lemma 6.15. *Let Assumption 6.1 be satisfied and $S_0(\cdot)$ as in Definition 6.12. Then, for every $\lambda \in \Phi$, there exists $z_\lambda \in \rho(S_0(\lambda))$ such that*

$$(S(\lambda) - z_\lambda)^{-1} \in \mathcal{B}(\mathcal{D}_S^*, \mathcal{D}_S)$$

and $\operatorname{dom} S_0(\lambda) = \operatorname{dom}(S_0(\lambda) - z_\lambda)$ *is dense in $\mathcal{D}_S$.*

Proof. Let $\lambda \in \Phi$ be arbitrary but fixed, then $S(\lambda) \in \mathcal{B}(\mathcal{D}_S, \mathcal{D}_S^*)$ by Proposition 6.13. Moreover, multiplying (6.25) with the unimodular factor $\mathrm{e}^{\mathrm{i}\,\omega_\lambda}$, we obtain

$$\begin{aligned}\operatorname{Re}\big(\mathrm{e}^{\mathrm{i}\,\omega_\lambda} S(\lambda)f, f\big)_{\mathcal{D}_S^* \times \mathcal{D}_S} \geq \int_\Omega \operatorname{Re}\widetilde{\boldsymbol{\pi}}(\lambda)\nabla f \cdot \overline{\nabla f}\,\mathrm{d}x \\ + \int_\Omega \operatorname{Re}\widetilde{q}(\lambda)|f|^2\,\mathrm{d}x - |\gamma_\lambda + \mathrm{e}^{\mathrm{i}\,\omega_\lambda}\lambda|\|f\|^2\end{aligned}$$

for $f \in C_0^\infty(\Omega)$. Using (6.4), we further derive

$$\operatorname{Re}\big(\mathrm{e}^{\mathrm{i}\,\omega_\lambda} S(\lambda)f, f\big)_{\mathcal{D}_S^* \times \mathcal{D}_S} \geq C_1\|f\|_S^2 - C_2\|f\|^2, \qquad f \in C_0^\infty(\Omega),$$

where $C_1, C_2 > 0$ depend on λ but not on f. From this it is easy to see that there exists $z_\lambda \in \mathbb{C}$ such that

$$|\mathrm{e}^{\mathrm{i}\,\omega_\lambda}\,((S(\lambda) - z_\lambda)f, f)_{\mathcal{D}_S^* \times \mathcal{D}_S}| \gtrsim \|f\|_S^2, \qquad f \in C_0^\infty(\Omega). \tag{6.29}$$

By the density of $C_0^\infty(\Omega)$ in $\mathcal{D}_S$, the continuity of the embedding $\mathcal{D}_S \subset L^2(\Omega)$ and $S(\lambda) \in \mathcal{B}(\mathcal{D}_S, \mathcal{D}_S^*)$, the above inequality remains valid for $f \in \mathcal{D}_S$, i.e. $S(\lambda) - z_\lambda$ corresponds to a bounded and coercive sesquilinear form on $\mathcal{D}_S$. By the Lax-Milgram Theorem, see e.g. [11, Thm. IV.1.1], we conclude $z_\lambda \in \rho(S_0(\lambda))$,

$$(S(\lambda) - z_\lambda)^{-1} \in \mathcal{B}(\mathcal{D}_S^*, \mathcal{D}_S)$$

and density of the maximal domain $\operatorname{dom}(S_0(\lambda) - z_\lambda) = \operatorname{dom} S_0(\lambda)$ in $\mathcal{D}_S$. □

Proof of Theorem 6.3. By Propositions 6.7 and 6.10, Assumption 3.1 is satisfied, the objects in Definition 6.12 are well-defined and the results of Section 3 are applicable. We point out that our realisations of $\mathcal{A}_0$ and $S_0(\cdot)$ in Definition 6.12 coincide with their standard distributional definitions in (6.7), see Remark 6.14. The description of their domains in (6.8) then follows from their definition in (6.24).

We show the claims in (6.9) - (6.12). Since $\Theta \subset \mathbb{C} \setminus \operatorname{ess\,ran} d = \rho(D_0)$, the identity (6.9) is a direct consequence of Corollary 3.5. Let us proceed with using Corollary 3.7 in order to show (6.11) and

$$\sigma_{\mathrm{e}2}(\mathcal{A}_0) \cap \Phi \subset \sigma_{\mathrm{e}2}(S_0(\cdot)) \cap \Phi. \tag{6.30}$$

To this end, let $f \in \operatorname{dom} d$ and $g \in C_0^\infty(\Omega)$. We can estimate

$$\begin{aligned}|(Bf,g)_{\mathcal{D}_1^*\times\mathcal{D}_1}| &\le \int_\Omega |f||\langle \mathbf{b}, \nabla g\rangle_{\mathbb{C}^n}|\,\mathrm{d}x \\ &\le \int_\Omega |(d-\lambda_0)f||d-\lambda_0|^{-1}|(\operatorname{Re}\boldsymbol{\pi}_0)^{-\frac12}\mathbf{b}||(\operatorname{Re}\boldsymbol{\pi}_0)^{\frac12}\nabla g|\,\mathrm{d}x \qquad (6.31)\\ &\le \|(d-\lambda_0)f\|\|(d-\lambda_0)^{-1}(\operatorname{Re}\boldsymbol{\pi}_0)^{-\frac12}\mathbf{b}\|_{L^\infty(\Omega)^n}\|g\|_S;\end{aligned}$$

notice that the right hand side of the last inequality is finite due to Assumption 6.1 (iv). By the density of $C_0^\infty(\Omega)$ in $\mathcal{D}_S$ and since $\mathcal{D}_1 \subset \mathcal{D}_S$ is continuously embedded, both left and right hand side of (6.31) are continuous in g with respect to $\|\cdot\|_S$. Hence, $B(\operatorname{dom} d) \subset \mathcal{D}_S^*$ and (6.31) holds for all $f \in \operatorname{dom} d$ and $g \in \mathcal{D}_S$. The remaining assumptions of Corollary 3.7 with $\Sigma = \Phi \subset \Theta$ are a consequence of $\Phi \subset \rho(D_0)$, Proposition 6.13, Lemma 6.15 and

$$(S_0(\lambda) - z_\lambda)^{-1} \subset S_z^\ddagger(\lambda) := (S(\lambda) - z_\lambda)^{-1} \in \mathcal{B}(\mathcal{D}_S^*, \mathcal{D}_S).$$

The inclusions (6.11) and (6.30), respectively, follow from Corollary 3.7 (i) and (ii).

The statement in (6.10) is obtained immediately from $\Theta \subset \rho(D_0)$ and Corollary 3.6 (i). Moreover, the inclusion

$$\sigma_{\mathrm{e}2}(\mathcal{A}_0) \cap \Phi \supset \sigma_{\mathrm{e}2}(S_0(\cdot)) \cap \Phi$$

follows from Corollary 3.6 (ii) with $\Sigma = \Phi$; here it suffices to note that $\Phi \subset \rho(D_0)$ and that, for every $\lambda \in \Phi$, $S_0(\lambda)$ has non-empty resolvent set by Lemma 6.15 and is therefore closed in $L^2(\Omega)$. The claims (6.9) - (6.12) are thus proven.

Finally, assume that $\operatorname{dom} B_0 \cap \operatorname{dom} d$ is dense in $L^2(\Omega)$. Then Lemma 6.15 with $\lambda = \lambda_0$ implies that $\operatorname{dom} S_0(\lambda_0)$ is dense in $\mathcal{D}_S$. Consequently, since $\mathcal{D}_S$ is dense and continuously embedded in $L^2(\Omega)$, it follows from Corollary 3.4 that $\operatorname{dom}\mathcal{A}_0$ is dense in $L^2(\Omega) \oplus L^2(\Omega)$. □

6.4. **Proof of Theorem 6.5.** In preparation for the proof of Theorem 6.5, we show that the hypotheses of the latter imply Assumption 6.1.

Lemma 6.16. *Let the assumptions of Theorem* 6.5 *be satisfied. Then Assumption* 6.1 *holds with* Θ, Φ *as in* (6.14).

Proof. We only need to show Assumption 6.1 (ii)-(iv). However, (ii) is clearly satisfied since the set $\Theta \subset \mathbb{C} \setminus \operatorname{ess\,ran} d$ in (6.14) is connected and (6.2) follows from the first assumption in (6.15), see Remark 6.2 (iv). Recall the formula (6.2), which in this case reads

$$\boldsymbol{\pi}(\lambda) = \mathbf{I} + (d-\lambda)^{-1}\mathbf{B}, \qquad \lambda \in \Theta,$$

where $\mathbf{B} := (\mathbf{b} \otimes \overline{\mathbf{b}}) \ge 0$ is easily verified to be positive semi-definite a.e. in Ω.

For (iii), let $\lambda \in \Phi \subset \Theta$ be arbitrary but fixed. Set $\omega_\lambda := 0$ if $|\arg\lambda| > \pi/2$ and

$$\omega_\lambda := -\frac{\operatorname{sgn}(\arg\lambda)}{2}(\pi - \max(\theta_q, \theta_d) - |\arg\lambda|), \qquad |\arg\lambda| \le \pi/2,$$

where the above is well-defined since $\lambda \ne 0 \notin \Phi$; recall that $\arg : \mathbb{C} \setminus \{0\} \to (-\pi, \pi]$ in our convention and notice that $|\omega_\lambda| < \pi/2$ since $\max(\theta_q, \theta_d) < \pi/2$. It then follows from elementary geometrical considerations that

$$\widetilde{d}(\lambda) := \mathrm{e}^{\mathrm{i}\,\omega_\lambda}(d-\lambda)^{-1}, \qquad \widetilde{q}(\lambda) := \mathrm{e}^{\mathrm{i}\,\omega_\lambda}\, q,$$

are both sectorial; in particular, we can set $\gamma_\lambda := 0$ in view of Assumption 6.1 (iii). Moreover, by the sectoriality of $\widetilde{d}(\lambda)$ and $|\omega_\lambda| < \pi/2$, also

$$\widetilde{\boldsymbol{\pi}}(\lambda) := \mathrm{e}^{\mathrm{i}\,\omega_\lambda}\,\boldsymbol{\pi}(\lambda) = \mathrm{e}^{\mathrm{i}\,\omega_\lambda}\,\mathbf{I} + \widetilde{d}(\lambda)\mathbf{B} \tag{6.32}$$

is sectorial and its real part is positive definite a.e. in Ω since we have

$$\operatorname{Re}\widetilde{\boldsymbol{\pi}}(\lambda) \geq \cos\omega_\lambda \mathbf{I} > 0. \tag{6.33}$$

It remains to show (6.4). The first chain of inequalities therein, i.e. the equivalence of $\operatorname{Re}\widetilde{q}(\lambda)$ for different $\lambda \in \Phi$, is easily derived from the sectoriality of $\widetilde{q}(\lambda)$ and

$$\operatorname{Re}\widetilde{q}(\lambda) = |q|\cos(\arg\widetilde{q}(\lambda)), \qquad \lambda \in \Phi.$$

Using Lemma 6.11, one obtains the analogous inequalities for $\widetilde{d}(\lambda)$ in a similar way. The second chain of inequalities in (6.4) then follows easily from the decomposition (6.32) and $|\omega_\lambda| < \pi/2$.

Finally, Assumption 6.1 (iv) follows from the second assumption in (6.15), see Remark 6.2 (iv). □

By the lemma above, Theorem 6.3 is applicable in the present setting and can be used in order to prove Theorem 6.5. Before doing so, we first describe $\mathcal{D}_S$ and certain actions defined on it.

Lemma 6.17. *Let the assumptions of Theorem* 6.5 *be satisfied. Then*

$$\mathcal{D}_S = \{f \in H_0^1(\Omega) : (\operatorname{Re}\boldsymbol{\pi}_0)^{\frac{1}{2}}\nabla f \in L^2(\Omega)^n,\ (\operatorname{Re} q)^{\frac{1}{2}} f \in L^2(\Omega)\}. \tag{6.34}$$

The formulas (6.20) *and* (6.25) *remain valid with* $f = g \in \mathcal{D}_S$ *and* $\lambda = -1$. *Moreover, for* $f \in \operatorname{dom} d$ *and* $g \in \mathcal{D}_S$*, the action of* $Bg \in \mathcal{D}_S^*$ *is given by*

$$(Bf, g)_{\mathcal{D}_S^* \times \mathcal{D}_S} = -\int_\Omega f\,\mathbf{b}\cdot\overline{\nabla g}\,\mathrm{d}x. \tag{6.35}$$

Proof. In order to show (6.34), we point out that, by (6.33), we have

$$\|\nabla f\|^2 + \|(\operatorname{Re} q)^{\frac{1}{2}} f\|^2 + \|f\|^2 \lesssim \|f\|_S^2, \qquad f \in C_0^\infty(\Omega). \tag{6.36}$$

Since $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, this implies $\mathcal{D}_S \subset H_0^1(\Omega)$. The claims

$$(\operatorname{Re}\boldsymbol{\pi}_0)^{\frac{1}{2}}\nabla f \in L^2(\Omega)^n, \qquad (\operatorname{Re} q)^{\frac{1}{2}} f \in L^2(\Omega), \qquad f \in \mathcal{D}_S,$$

then follow by standard arguments from the definition of $\mathcal{D}_S$; indeed, if

$$f \in \mathcal{D}_S, \qquad \{f_m\}_m \subset C_0^\infty(\Omega), \qquad \|f_m - f\|_S \to 0, \qquad m \to \infty, \tag{6.37}$$

then by (6.36) also $f_m \to f$ in $H^1(\Omega)$ as $m \to \infty$. Since $\{(\operatorname{Re}\boldsymbol{\pi}_0)^{\frac{1}{2}}\nabla f_m\}_m$ is Cauchy in $L^2(\Omega)^n$ by (6.37), it has a limit h in $L^2(\Omega)^n$. For any test function $\varphi \in C_0^\infty(\Omega)^n$, we derive the following

$$\begin{aligned}\langle h, \varphi\rangle &= \lim_{m\to\infty}\langle(\operatorname{Re}\boldsymbol{\pi}_0)^{\frac{1}{2}}\nabla f_m, \varphi\rangle \\ &= \lim_{m\to\infty}\langle\nabla f_m, (\operatorname{Re}\boldsymbol{\pi}_0)^{\frac{1}{2}}\varphi\rangle = \langle\nabla f, (\operatorname{Re}\boldsymbol{\pi}_0)^{\frac{1}{2}}\varphi\rangle = \langle(\operatorname{Re}\boldsymbol{\pi}_0)^{\frac{1}{2}}\nabla f, \varphi\rangle.\end{aligned}$$

This in turn implies $(\operatorname{Re}\boldsymbol{\pi}_0)^{\frac{1}{2}}\nabla f = h \in L^2(\Omega)$ by density. Analogously, one shows that $(\operatorname{Re} q)^{\frac{1}{2}} f_m \to (\operatorname{Re} q)^{\frac{1}{2}} f \in L^2(\Omega)$ converges in $L^2(\Omega)$ as $m \to \infty$. Formula (6.25) with $\lambda = -1$ then immediately extends to $f = g \in \mathcal{D}_S$, given that the sectoriality of $\boldsymbol{\pi}_0$ and q implies continuity of both sides in f with respect to convergence in $\mathcal{D}_S$.

In order to show that (6.20) holds with $f \in \mathcal{D}_S$, consider a sequence as in (6.37). By construction of C and since $(\operatorname{Re}\boldsymbol{\pi}_0)^{\frac{1}{2}}\nabla f_m \to (\operatorname{Re}\boldsymbol{\pi}_0)^{\frac{1}{2}}\nabla f$ in $L^2(\Omega)^n$ as $n \to \infty$, for all test functions $\varphi \in C_0^\infty(\Omega)$ we have

$$\begin{aligned}\langle \omega^{-1} C f, \varphi\rangle = \lim_{m\to\infty} \langle \omega^{-1}\overline{\mathbf{b}} \cdot \nabla f_m, \varphi\rangle &= \lim_{m\to\infty} \langle (\operatorname{Re}\boldsymbol{\pi})^{\frac{1}{2}}\nabla f_m, (\operatorname{Re}\boldsymbol{\pi})^{-\frac{1}{2}}\mathbf{b}\omega^{-1}\varphi\rangle \\ &= \langle (\operatorname{Re}\boldsymbol{\pi})^{\frac{1}{2}}\nabla f, (\operatorname{Re}\boldsymbol{\pi})^{-\frac{1}{2}}\mathbf{b}\omega^{-1}\varphi\rangle \\ &= \langle \omega^{-1}\overline{\mathbf{b}} \cdot \nabla f, \varphi\rangle.\end{aligned}$$

By density, one concludes $\omega^{-1} C f = \overline{\omega}^{-1}\mathbf{b} \cdot \nabla f$, i.e. that (6.20) holds. The claim (6.35) can be shown similarly, cf. (6.31). ☐

Proof of Theorem 6.5. By Lemma 6.16, Assumption 6.1 is fully satisfied and the claims (6.9) - (6.12) follow from Theorem 6.3. It only remains to show that $\mathcal{A}_0$ is m-accretive in $L^2(\Omega) \oplus L^2(\Omega)$ with (6.16); the density of $\operatorname{dom}\mathcal{A}_0$ and the generation of the semigroup then follow from classical results, see e.g. [20, §V.3.10, §IX.1].

In order to show that $\mathcal{A}_0$ is accretive, let $(f, g) \in \operatorname{dom}\mathcal{A}_0$. Then from the sectoriality of $\boldsymbol{\pi}_0 = \mathbf{I} + (d+1)^{-1}\mathbf{B}$, from $f \in \mathcal{D}_S$ and (6.34), it follows that

$$|(d+1)^{-1}\mathbf{B}\nabla f \cdot \overline{\nabla f}| \lesssim \operatorname{Re}\boldsymbol{\pi}_0 \nabla f \cdot \overline{\nabla f} + |\nabla f|^2 \in L^1(\Omega). \tag{6.38}$$

Since $(D+1)^{-1} C f + g \in \operatorname{dom} d$ as $(f, g) \in \operatorname{dom}\mathcal{A}_0$, applying formula (6.35) and (6.20) according to Lemma 6.16, we further conclude

$$\begin{aligned}\big(B((D+1)^{-1}Cf + g), f\big)_{\mathcal{D}_S^* \times \mathcal{D}_S} &= -\int_\Omega ((d+1)^{-1}\overline{\mathbf{b}} \cdot \nabla f + g)\mathbf{b} \cdot \overline{\nabla f}\,\mathrm{d}x \\ &= -\int_\Omega (d+1)^{-1}\mathbf{B}\nabla f \cdot \overline{\nabla f}\,\mathrm{d}x + \int_\Omega g\mathbf{b} \cdot \overline{\nabla f}\,\mathrm{d}x;\end{aligned}$$

here in the second equality we used (6.38), which then also gives $g\mathbf{b} \cdot \overline{\nabla f} \in L^1(\Omega)$. Consequently, formula (6.25) according to Lemma 6.16 gives

$$\begin{aligned}\langle Af + Bg, f\rangle &= (S(-1)f, f)_{\mathcal{D}_S^* \times \mathcal{D}_S} + \big(B((D+1)^{-1}Cf + g), f\big)_{\mathcal{D}_S^* \times \mathcal{D}_S} \\ &= \|\nabla f\|^2 + \int_\Omega q|f|^2\,\mathrm{d}x + \int_\Omega g\mathbf{b} \cdot \overline{\nabla f}\,\mathrm{d}x.\end{aligned} \tag{6.39}$$

Using $g\mathbf{b} \cdot \overline{\nabla f} \in L^1(\Omega)$, it follows from (6.20) with Lemma 6.16 that

$$\langle Cf + Dg, g\rangle = \int_\Omega \overline{\mathbf{b}} \cdot \nabla f \overline{g}\,\mathrm{d}x + \int_\Omega d|g|^2\,\mathrm{d}x; \tag{6.40}$$

this in turn gives $d|g|^2 \in L^1(\Omega)$, thus $g \in \operatorname{dom}(\operatorname{Re} d)^{\frac{1}{2}}$ by the sectoriality of d and (6.16) is shown. Combining (6.39) and (6.40), we obtain accretivity of $\mathcal{A}_0$ as follows

$$\operatorname{Re}\langle \mathcal{A}_0(f, g), (f, g)\rangle_{\mathcal{H}} = \|\nabla f\|^2 + \int_\Omega \operatorname{Re} q|f|^2\,\mathrm{d}x + \int_\Omega \operatorname{Re} d|g|^2\,\mathrm{d}x \geq 0.$$

In order to conclude m-accretivity of $\mathcal{A}_0$, we show that $\rho(\mathcal{A}_0) \cap (-\infty, 0) \neq 0$, see e.g. [20, §V.3.10]. This however, follows from taking complements in (6.11) and the proof of Lemma 6.15, where one can easily show that, if $\lambda < 0$ is small enough, then (6.29) is satisfied with $z_\lambda = 0$, i.e. that $S(\lambda)$ is coercive on $\mathcal{D}_S$ and thus $0 \in \rho(S_0(\lambda))$ for sufficiently small $\lambda \in (-\infty, 0)$. ☐

7. Further examples

We schematically illustrate our results based on two more examples.

7.1. **Dirac operators with Coulomb type potentials.** In [12], an approach essentially corresponding to our method has been taken to construct self-adjoint realisations of Dirac operators with certain Coulomb type potentials. The key tool in their analysis is a Hardy-Dirac inequality (7.2) which was derived in [9, 10] and, translated to our framework, ensures coercivity of the first Schur complement on its form domain. Since our method widely agrees with what is proven in [12], we only sketch the underlying structure of the problem according to our setting in Section 3.

In this section, $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ shall denote the inner product and norm on $L^2(\mathbb{R}^3)^2$.

Example 7.1. We define a self-adjoint realisation of the Dirac operator

$$\mathcal{A} = \begin{pmatrix} V+1 & -\mathrm{i}\,\sigma\cdot\nabla \\ -\mathrm{i}\,\sigma\cdot\nabla & V-1 \end{pmatrix} \tag{7.1}$$

in the Hilbert space $L^2(\mathbb{R}^3)^2 \oplus L^2(\mathbb{R}^3)^2$ with a real-valued potential satisfying a certain Hardy type inequality (7.2). As usual, we denote

$$\sigma\cdot\nabla = \sum_{j=1}^{3} \sigma_j \partial_j$$

with the Pauli matrices σ_j, $j=1,2,3$, given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We assume that the potential $V \in L^2_{\mathrm{loc}}(\mathbb{R}^3,\mathbb{R})$ is bounded above with

$$\Gamma := \operatorname*{ess\,sup}_{x\in\mathbb{R}^3} V(x) < \infty$$

and that there exists a constant $\Lambda > \Gamma - 1$ such that the following holds true

$$\int_{\mathbb{R}^3} \frac{|(\sigma\cdot\nabla)f|^2}{1+\Lambda-V}\,\mathrm{d}x + \int_{\mathbb{R}^3} (V+1-\Lambda)|f|^2\,\mathrm{d}x \geq 0, \qquad f \in C_0^\infty(\mathbb{R}^3)^2. \tag{7.2}$$

It was shown in [10, 9] that the above Hardy-Dirac inequality is satisfied for certain Coulomb type potentials, in particular for $V(x) = -\nu/|x|$ with $\nu \in (0,1]$; in the latter case, $\Gamma = 0$ and $\Lambda = \sqrt{1-\nu}$, see [9, Thm. 1, Cor. 3].

By means of the construction in Section 3, the self-adjoint realisation in [12] of the operator matrix (7.1) can be reproduced. Moreover, the spectral equivalence to its first Schur complement on $(\Gamma-1,\infty)$ can be established, see Remark 7.3.

Proposition 7.2. *Let $\mathcal{A}_0 := \mathcal{A}|_{\operatorname{dom}\mathcal{A}_0}$ with $\mathcal{A}$ given in* (7.1) *where its action shall be understood in the standard distributional sense and*

$$\begin{aligned} \operatorname{dom}\mathcal{A}_0 := \{\, (f,g) \in \mathcal{D}_S \times L^2(\mathbb{R}^3)^2 \;:\; \\ Vf - \mathrm{i}(\sigma\cdot\nabla)g,\ -\mathrm{i}(\sigma\cdot\nabla)f + Vg \in L^2(\mathbb{R}^3)^2 \,\}. \end{aligned} \tag{7.3}$$

Here the space $\mathcal{D}_S$ is defined as the closure

$$\mathcal{D}_S := \overline{C_0^\infty(\mathbb{R}^3)^2}^{\|\cdot\|_S}, \qquad \|f\|_S^2 := \mathbf{s}_{\lambda_0}[f] + \|f\|^2, \qquad f \in C_0^\infty(\mathbb{R}^3)^2,$$

where $\lambda_0 \in (\Gamma-1,\Lambda)$ is arbitrary and

$$\mathbf{s}_{\lambda_0}[f] := \int_{\mathbb{R}^3} \frac{|(\sigma\cdot\nabla)f|^2}{1+\lambda_0-V}\,\mathrm{d}x + \int_{\mathbb{R}^3} (V+1-\lambda_0)|f|^2\,\mathrm{d}x \geq 0. \tag{7.4}$$

Then $\mathcal{A}_0$ is independent of the choice of λ_0 in (7.4). Moreover, it is densely defined and self-adjoint in $L^2(\mathbb{R}^3)^2 \oplus L^2(\mathbb{R}^3)^2$ and has a spectral gap

$$\sigma(\mathcal{A}_0) \cap (\Gamma - 1, \Lambda) = \emptyset. \tag{7.5}$$

Sketch of proof. For $\lambda \in (\Gamma - 1, \infty)$, let the quadratic forms $\mathbf{s}_\lambda$ be defined on $C_0^\infty(\mathbb{R}^3)^2$ analogously to (7.4). We first make the essential observation that (7.2) translates into $\mathbf{s}_\lambda$ being non-negative for $\lambda \in (\Gamma-1, \Lambda]$ and that the norms $\mathbf{s}_\lambda + \|\cdot\|^2$ are locally equivalent for $\lambda \in (\Gamma - 1, \Lambda)$. The latter holds true since, similarly to Lemma 6.11, one can show that for all $\lambda, \mu \in (\Gamma-1, \infty)$ there exist $m_{\lambda,\mu}, M_{\lambda,\mu} > 0$ such that a.e. in $\mathbb{R}^3$ and for all $f \in C_0^\infty(\mathbb{R}^3)^2$ the following holds

$$m_{\lambda,\mu}\mathbf{s}_\mu[f] + (\mu - \lambda)\|f\|^2 \leq \mathbf{s}_\lambda[f] \leq M_{\lambda,\mu}\mathbf{s}_\mu[f] + (\mu - \lambda)\|f\|^2. \tag{7.6}$$

Most importantly, the above chain of inequalities guarantees that $\mathcal{D}_S$ and thus $\mathcal{A}_0$ are independent of the choice of λ_0.

In order to apply the results in Section 3, we proceed by indicating the remaining spaces and operators needed for Assumption 3.1. Clearly, $\mathcal{H}_1 = \mathcal{H}_2 = L^2(\mathbb{R}^3)^2$ and we set $\mathcal{D}_{-S} := \mathcal{D}_S^*$ and $\mathcal{D}_{-1} := \mathcal{D}_1^*$ with $\mathcal{D}_1$ defined as the closure

$$\mathcal{D}_1 := \overline{C_0^\infty(\mathbb{R}^3)^2}^{\|\cdot\|_1}, \quad \|f\|_1^2 := \|f\|_S^2 + \|(V+1)f\|^2 + \|(\sigma\cdot\nabla)f\|^2, \quad f \in C_0^\infty(\mathbb{R}^3)^2,$$

and further introduce the spaces

$$\mathcal{D}_2 := L^2(\mathbb{R}^3)^2, \qquad \mathcal{D}_{-2} := L^2(\mathbb{R}^3; (1 + \lambda_0 - V)^{-2})^2.$$

Analogously to Propositions 4.6 and 6.7, one can show that the above defined spaces are well-defined and satisfy Assumption 2.1 (i). Notice that $\mathbf{s}_{\lambda_0} \geq s_0 > 0$ such that indeed $\mathcal{D}_S \subset L^2(\mathbb{R}^3)^2$. Similarly to Lemma 6.11, one shows that also $\mathcal{D}_{-2}$, and hence all appearing spaces, do not depend on λ_0.

We proceed to defining the action of the entries of $\mathcal{A}$. Analogously to the constructions (4.13) and (6.19) in Sections 4 and 6, the operators

$$A := V + 1 \in \mathcal{B}(\mathcal{D}_S, \mathcal{D}_1^*), \qquad B := -\mathrm{i}\,\sigma\cdot\nabla \in \mathcal{B}(L^2(\mathbb{R}^3)^2, \mathcal{D}_1^*),$$

shall be defined as unique bounded extensions of

$$\left.\begin{aligned} (Af, g)_{\mathcal{D}_1^* \times \mathcal{D}_1} &:= \int_{\mathbb{R}^3} (V+1)f \cdot \overline{g}\,\mathrm{d}x, \\ (Bf, g)_{\mathcal{D}_1^* \times \mathcal{D}_1} &:= \mathrm{i}\int_{\mathbb{R}^3} f \cdot \overline{(\sigma\cdot\nabla)g}\,\mathrm{d}x, \end{aligned}\right. \qquad f, g \in C_0^\infty(\mathbb{R}^3)^2.$$

The entry $C := -\mathrm{i}\,\sigma\cdot\nabla \in \mathcal{B}(\mathcal{D}_S, \mathcal{D}_{-2})$ is given, analogously to (6.20), by the unique bounded extension of

$$Cf := -\mathrm{i}(\sigma\cdot\nabla)f, \qquad f \in C_0^\infty(\mathbb{R}^3)^2,$$

see [12, Prop. 6] for the proof of the inequality corresponding to (6.23). Finally, let

$$D := (V-1) \in \mathcal{B}(L^2(\mathbb{R}^3)^2, \mathcal{D}_{-2}), \quad D_0 := D|_{\operatorname{dom} D_0}, \quad \operatorname{dom} D_0 := (\operatorname{dom} V)^2.$$

One can show that the operators defined above satisfy Assumptions 2.1 (ii) and 3.1 (ii), cf. Propositions 4.9 and 6.10, thus Assumption 3.1 holds. Moreover, we set

$$D^\ddagger(\lambda) := (D-\lambda)^{-1} \in \mathcal{B}(\mathcal{D}_{-2}, L^2(\mathbb{R}^3)^2), \qquad \lambda \in \Theta := (\Gamma - 1, \infty) \subset \rho(D_0),$$

and point out that Θ satisfies the inclusion (3.2), cf. Lemma 6.11 regarding the claimed boundedness of D and $D^\ddagger(\lambda)$.

The matrix $\mathcal{A}_0$ and its first Schur complement $S_0(\cdot)$ are well-defined by Definition 3.2 with $\operatorname{dom}\mathcal{A}_0$ as in (7.3). It is obvious that the action of $\mathcal{A}_0$ coincides with its standard definition by distributional operations and it thus remains to explain that $\mathcal{A}_0$ is densely defined, self-adjoint and that (7.5) holds. Since the numerical range of $\mathcal{A}_0$ is a subset of $\mathbb{R}$, see the proof of [12, Thm. 4], it suffices to employ Corollaries 3.4 (i) and 3.7 (i) in order to conclude the density of $\operatorname{dom}\mathcal{A}_0$ and (7.5); the self-adjointness of $\mathcal{A}_0$ then follows readily from (7.7) below. To this account, we sketch the proofs of the required assumptions.

Let $\lambda \in (\Gamma-1,\infty)$ be arbitrary but fixed. Considering the inequalities in (7.6), one shows that

$$S(\lambda) \in \mathcal{B}(\mathcal{D}_S, \mathcal{D}_S^*), \qquad (S(\lambda)f,f)_{\mathcal{D}_S^*\times\mathcal{D}_S} = \mathbf{s}_\lambda[f], \qquad f \in C_0^\infty(\mathbb{R}^3)^2,$$

cf. Proposition 6.13. Moreover, (7.6) can be employed to show that there exists a shift $z_\lambda \in \mathbb{R}$ such that

$$|((S(\lambda)-z_\lambda)f,f)_{\mathcal{D}_S^*\times\mathcal{D}_S}| \gtrsim \|f\|_S^2, \qquad f \in C_0^\infty(\mathbb{R}^3)^2,$$

which implies that $S(\lambda)-z_\lambda$ is coercive on $\mathcal{D}_S$; here it is important to note that if $\lambda \in (\Gamma-1,\lambda_0)$, then obviously $z_\lambda = 0$ can be chosen. Hence, by the Lax-Milgram Theorem, see e.g. [11, Thm. IV.1.1], we conclude

$$S_z^\ddagger(\lambda) := (S(\lambda)-z_\lambda)^{-1} \in \mathcal{B}(\mathcal{D}_S^*, \mathcal{D}_S), \qquad z_\lambda \in \rho(S_0(\lambda)),$$

and that $\operatorname{dom} S(\lambda)$ is dense in $\mathcal{D}_S$ and thus in $L^2(\mathbb{R}^3)^2$; in particular, this implies $(\Gamma-1,\lambda_0) \subset \rho(S_0)$. Moreover, $\mathcal{A}_0$ is thus densely defined by Corollary 3.4 (i) and

$$C_0^\infty(\mathbb{R}^3)^2 \subset \operatorname{dom} B_0 \cap (\operatorname{dom} V)^2.$$

For the remaining assumptions of Corollary 3.7 (i), it suffices to point out that

$$(\sigma\cdot\nabla)(V-1-\lambda_0)^{-1} \in \mathcal{B}(L^2(\mathbb{R}^3)^2, \mathcal{D}_S^*),$$

cf. [12, Lem. 7], which implies $B((\operatorname{dom} V)^2) \subseteq \mathcal{D}_S^*$. Finally, combining the above and applying Corollary 3.7 (i) with $\Sigma := (\Gamma-1,\lambda_0)$ gives

$$\sigma(\mathcal{A}_0) \cap (\Gamma-1,\lambda_0) \subset \sigma(S_0) \cap (\Gamma-1,\lambda_0) = \emptyset; \tag{7.7}$$

since λ_0 was arbitrary and $\mathcal{A}_0$ is independent of λ_0, (7.5) follows. □

Remark 7.3. Besides the spectral gap (7.5) found in [12] in order to establish the self-adjointness of $\mathcal{A}_0$, our method provides the equivalence of (point and essential) spectra of $\mathcal{A}_0$ and its first Schur complement on $[\Lambda,\infty)$, i.e.

$$\sigma_{(\mathrm{p}\,/\,\mathrm{e}2)}(\mathcal{A}_0) \cap [\Lambda,\infty) = \sigma_{(\mathrm{p}\,/\,\mathrm{e}2)}(S_0(\cdot)) \cap [\Lambda,\infty),$$

see Corollaries 3.5, 3.6, 3.7 and the arguments sketched for the proof of (7.5). Note that, due to the self-adjointness of $\mathcal{A}_0$, we have only considered real spectral parameters. For $\lambda \in (\Gamma-1,\infty)$, our realisation of the Schur complement is thereby defined as the restriction $S_0(\lambda) := S(\lambda)|_{\operatorname{dom} S_0(\lambda)}$ where

$$\begin{aligned} S(\lambda) &:= V + 1 - \lambda + (\sigma\cdot\nabla)(V-1-\lambda)^{-1}(\sigma\cdot\nabla) \\ \operatorname{dom} S_0(\lambda) &:= \{f \in \mathcal{D}_S : Vf + (\sigma\cdot\nabla)(V-1-\lambda)^{-1}(\sigma\cdot\nabla)f \in L^2(\mathbb{R}^3)^2\}. \end{aligned}$$

We point out that, by construction, the above operations can be understood in the standard distributional sense; more precisely, the action of $S(\lambda)$ on $\mathcal{D}_S$ is obtained by continuous extension, i.e. as a limit in $\mathcal{D}'(\mathbb{R}^3)^2$ of distributions of the form

$$Vf + f - \lambda f + (\sigma\cdot\nabla)(V-1-\lambda)^{-1}(\sigma\cdot\nabla)f \in \mathcal{D}'(\mathbb{R}^3)^2, \quad f \in C_0^\infty(\mathbb{R}^3)^2,$$

cf. Remarks 4.12 and 6.14. //

7.2. **A simple constant coefficient differential operator matrix.** In many applications, e.g. in Sections 4 – 6 or in case of the Dirac operator in the previous subsection, the spaces $\mathcal{D}_S$ and $\mathcal{D}_{-S}$ are given by the form domain of the Schur complement and its anti-dual. However, the latter is not always the case, as we demonstrate in a model problem where $\mathcal{D}_S = H^1(\mathbb{R}^n)$ and $\mathcal{D}_{-S} = H^{-2}(\mathbb{R}^n)$, while $H^{\frac{3}{2}}(\mathbb{R}^n)$ is the form domain of the Schur complement. We point out that this example is of illustrative purpose and chosen as simple as possible; examples of similar structure can be found e.g. in [17] where the entries are more general pseudodifferential operators.

Example 7.4. In the Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 = L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$, we consider the operator matrix

$$\mathcal{A} = \begin{pmatrix} \Delta & -\Delta^2 \\ \sqrt{-\Delta} & \Delta \end{pmatrix} \tag{7.8}$$

and the corresponding first Schur complement

$$S(\lambda) = \Delta - \lambda + \Delta^2(\Delta - \lambda)^{-1}\sqrt{-\Delta}, \qquad \lambda \in \mathbb{C} \setminus (-\infty, 0], \tag{7.9}$$

acting in $L^2(\mathbb{R}^n)$. Note that in this particular case, it is not difficult to explicitly determine the spectra of the operator matrix and its Schur complement above; however, we emphasise that the purpose of this example is not their spectral analysis, but the illustration of spaces and operators underlying the spectral correspondence developed in Section 3.

Proposition 7.5. *Let $\mathcal{A}_0 := \mathcal{A}|_{\operatorname{dom} \mathcal{A}_0}$ with $\mathcal{A}$ as in (7.8) and*

$$\operatorname{dom} \mathcal{A}_0 := \{(f, g) \in H^1(\mathbb{R}^n) \times H^2(\mathbb{R}^n) : \Delta f - \Delta^2 g \in L^2(\mathbb{R}^n)\}. \tag{7.10}$$

Moreover, let $S_0(\lambda) := S(\lambda)|_{\operatorname{dom} S_0(\lambda)}$, $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, be the family of maximal operators in $L^2(\mathbb{R}^n)$ where $S(\lambda)$ is as in (7.9) and

$$\operatorname{dom} S_0(\lambda) := \{f \in H^1(\mathbb{R}^n) : \Delta f + \Delta^2(\Delta - 1)^{-1}\sqrt{-\Delta} f \in L^2(\mathbb{R}^n)\}. \tag{7.11}$$

In the above, the square root of $-\Delta$ is defined via functional calculus. Then $\mathcal{A}_0$ is densely defined and closed in $L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$ with

$$\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > 0\} \subset \rho(\mathcal{A}_0) \tag{7.12}$$

and the following relations for the (point and essential) spectra hold

$$\sigma_{(\mathrm{p}\,/\,\mathrm{e}2)}(\mathcal{A}_0) \setminus (-\infty, 0] = \sigma_{(\mathrm{p}\,/\,\mathrm{e}2)}(S_0(\cdot)). \tag{7.13}$$

Sketch of proof. We indicate the objects needed for Assumption 3.1. Let

$$\begin{aligned} \mathcal{D}_S &:= H^1(\mathbb{R}^n), & \mathcal{D}_{-S} = \mathcal{D}_{-1} &:= H^{-2}(\mathbb{R}^n), \\ \mathcal{D}_2 &:= H^2(\mathbb{R}^n), & \mathcal{D}_{-2} &:= L^2(\mathbb{R}^n). \end{aligned}$$

Using the distributional Fourier transform, one can easily check that the following operators are bounded between the claimed spaces

$$\begin{aligned} A &:= \Delta \in \mathcal{B}(H^1(\mathbb{R}^n), H^{-1}(\mathbb{R}^n)), & B &:= -\Delta^2 \in \mathcal{B}(H^2(\mathbb{R}^n), H^{-2}(\mathbb{R}^n)), \\ C &:= \sqrt{-\Delta} \in \mathcal{B}(H^1(\mathbb{R}^n), L^2(\mathbb{R}^n)), & D &:= \Delta \in \mathcal{B}(H^2(\mathbb{R}^n), L^2(\mathbb{R}^n)), \end{aligned}$$

see e.g. [1, Thm. 3.41]. Since $H^{-1}(\mathbb{R}^n)$ embeds continuously in $H^{-2}(\mathbb{R}^n)$, Assumption 3.1 is satisfied with $D_0 := D$ and we can define $\mathcal{A}_0$ and $S_0(\cdot)$ as in Definition 3.2 with

$$D^{\ddagger}(\lambda) := (D-\lambda)^{-1}, \qquad \lambda \in \Theta := \rho(D_0) = \mathbb{C} \setminus (-\infty, 0].$$

One easily verifies that $\operatorname{dom} \mathcal{A}_0$ and $\operatorname{dom} S_0(\lambda)$ coincide with (7.10) and (7.11).

For the proof of (7.13), we outline the assumptions of Corollaries 3.6 and 3.7. Let therefore $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ be arbitrary but fixed. It is not difficult to see that there exists $z_\lambda > 0$ such that the inverse of $S(\lambda) - z_\lambda$ is bounded on $H^{-2}(\mathbb{R}^n)$; this follows from the fact that $S(\lambda)$ is unitarily equivalent to the multiplication operator by the symbol

$$s_\lambda(\xi) := -|\xi|^2 - \lambda - \frac{|\xi|^5}{|\xi|^2+\lambda}, \quad \xi \in \mathbb{R}^n,$$

in the Fourier space. Indeed, it is elementary to prove the lower bound

$$|s_\lambda(\xi) - z_\lambda| \gtrsim |\xi|^3 + 1, \qquad \xi \in \mathbb{R}^n,$$

if $z_\lambda > 0$ is large enough, which implies

$$S_z^{\ddagger}(\lambda) := (S(\lambda) - z_\lambda)^{-1} \in \mathcal{B}(H^{-2}(\mathbb{R}^n), H^1(\mathbb{R}^n)).$$

Moreover, $\operatorname{dom} S_0(\lambda)$ contains $C_0^\infty(\mathbb{R}^n)$ and is thus dense in $H^1(\mathbb{R}^n)$. The relations in (7.13) then follow from Corollary 3.5 and Corollaries 3.6 and 3.7 with

$$\Sigma := \Theta = \mathbb{C} \setminus (-\infty, 0] = \rho(D_0).$$

The density of $\operatorname{dom} \mathcal{A}_0$ in $L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$ is a consequence of Corollary 3.4 (i) and the fact that both $\operatorname{dom} B_0$ and $\operatorname{dom} D_0$ clearly contain $C_0^\infty(\mathbb{R}^n)$. Finally, if $\operatorname{Re} \lambda > 0$, then one can easily derive the following lower bound

$$|s_\lambda(\xi)| \geq \operatorname{Re} \lambda > 0, \qquad \xi \in \mathbb{R}^n,$$

which implies $0 \in \rho(S_0(\lambda))$ and in turn (7.12) by (7.13). □

References

[1] Abels, H. *Pseudodifferential and singular integral operators.* De Gruyter Graduate Lectures. De Gruyter, Berlin, 2012. An introduction with applications.

[2] Almog, Y. The stability of the normal state of superconductors in the presence of electric currents. *SIAM J. Math. Anal. 40*, 2 (2008), 824–850.

[3] Almog, Y., Grebenkov, D. S., and Helffer, B. On a Schrödinger operator with a purely imaginary potential in the semiclassical limit. *Comm. Partial Differential Equations 44*, 12 (2019), 1542–1604.

[4] Ammari, K., and Nicaise, S. *Stabilization of elastic systems by collocated feedback*, vol. 2124 of *Lecture Notes in Mathematics*. Springer, Cham, 2015.

[5] Bátkai, A., Binding, P., Dijksma, A., Hryniv, R., and Langer, H. Spectral problems for operator matrices. *Math. Nachr. 278*, 12-13 (2005), 1408–1429.

[6] Blank, J., Exner, P., and Havlíček, M. *Hilbert space operators in quantum physics*, second ed. Theoretical and Mathematical Physics. Springer, New York; AIP Press, New York, 2008.

[7] Bögli, S., and Marletta, M. Essential numerical ranges for linear operator pencils. *IMA J. Numer. Anal. 40*, 4 (2020), 2256–2308.

[8] Bögli, S., Siegl, P., and Tretter, C. Approximations of spectra of Schrödinger operators with complex potentials on $\mathbb{R}^d$. *Comm. Partial Differential Equations 42*, 7 (2017), 1001–1041.

[9] Dolbeault, J., Esteban, M. J., Loss, M., and Vega, L. An analytical proof of Hardy-like inequalities related to the Dirac operator. *J. Funct. Anal. 216*, 1 (2004), 1–21.

[10] Dolbeault, J., Esteban, M. J., and Séré, E. On the eigenvalues of operators with gaps. Application to Dirac operators. *J. Funct. Anal. 174*, 1 (2000), 208–226.

[11] Edmunds, D. E., and Evans, W. D. *Spectral theory and differential operators.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1987. Oxford Science Publications.

[12] Esteban, M. J., and Loss, M. Self-adjointness for Dirac operators via Hardy-Dirac inequalities. *J. Math. Phys. 48*, 11 (2007), 112107, 8.

[13] Freitas, P., Siegl, P., and Tretter, C. The damped wave equation with unbounded damping. *J. Differential Equations 264*, 12 (2018), 7023–7054.

[14] Gesztesy, F., Goldstein, J. A., Holden, H., and Teschl, G. Abstract wave equations and associated Dirac-type operators. *Ann. Mat. Pura Appl. (4) 191*, 4 (2012), 631–676.

[15] Ibrogimov, O. O. Essential spectrum of non-self-adjoint singular matrix differential operators. *J. Math. Anal. Appl. 451*, 1 (2017), 473–496.

[16] Ibrogimov, O. O., Siegl, P., and Tretter, C. Analysis of the essential spectrum of singular matrix differential operators. *J. Differential Equations 260*, 4 (2016), 3881–3926.

[17] Ibrogimov, O. O., and Tretter, C. Essential spectrum of elliptic systems of pseudo-differential operators on $L^2(\mathbb{R}^N) \oplus L^2(\mathbb{R}^N)$. *J. Pseudo-Differ. Oper. Appl. 8*, 2 (2017), 147–166.

[18] Ikehata, R., and Takeda, H. Uniform energy decay for wave equations with unbounded damping coefficients. *Funkcial. Ekvac. 63*, 1 (2020), 133–152.

[19] Jacob, B., Tretter, C., Trunk, C., and Vogt, H. Systems with strong damping and their spectra. *Math. Methods Appl. Sci. 41*, 16 (2018), 6546–6573.

[20] Kato, T. *Perturbation theory for linear operators.* Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[21] Konstantinov, A. Y. Spectral theory of some matrix differential operators of mixed order. *Ukraïn. Mat. Zh. 50*, 8 (1998), 1064–1072.

[22] Kurasov, P., Lelyavin, I., and Naboko, S. On the essential spectrum of a class of singular matrix differential operators. II. Weyl's limit circles for the Hain-Lüst operator whenever quasi-regularity conditions are not satisfied. *Proc. Roy. Soc. Edinburgh Sect. A 138*, 1 (2008), 109–138.

[23] Langer, H., Najman, B., and Tretter, C. Spectral theory of the Klein-Gordon equation in Pontryagin spaces. *Comm. Math. Phys. 267*, 1 (2006), 159–180.

[24] Langer, H., Najman, B., and Tretter, C. Spectral theory of the Klein-Gordon equation in Krein spaces. *Proc. Edinb. Math. Soc. (2) 51*, 3 (2008), 711–750.

[25] Langer, M., and Strauss, M. Spectral properties of unbounded J-self-adjoint block operator matrices. *J. Spectr. Theory 7*, 1 (2017), 137–190.

[26] Nagel, R. Towards a "matrix theory" for unbounded operator matrices. *Math. Z. 201*, 1 (1989), 57–68.

[27] Nagel, R. The spectrum of unbounded operator matrices with nondiagonal domain. *J. Funct. Anal. 89*, 2 (1990), 291–302.

[28] Reed, M., and Simon, B. *Methods of modern mathematical physics. IV. Analysis of operators.* Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978.

[29] Tretter, C. *Spectral theory of block operator matrices and applications.* Imperial College Press, London, 2008.

[30] Veselić, K. A spectral theory of the Klein-Gordon equation involving a homogeneous electric field. *J. Operator Theory 25*, 2 (1991), 319–330.

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PSEUDO NUMERICAL RANGES AND SPECTRAL ENCLOSURES

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Abstract. We introduce the new concepts of pseudo numerical range for operator functions and families of sesquilinear forms as well as the pseudo block numerical range for $n\times n$ operator matrix functions. While these notions are new even in the bounded case, we cover operator polynomials with unbounded coefficients, unbounded holomorphic form families of type (a) and associated operator families of type (B). Our main results include spectral inclusion properties of pseudo numerical ranges and pseudo block numerical ranges. For diagonally dominant and off-diagonally dominant operator matrices they allow us to prove spectral enclosures in terms of the pseudo numerical ranges of Schur complements that no longer require dominance order 0 and not even <1. As an application, we establish a new type of spectral bounds for linearly damped wave equations with possibly unbounded and/or singular damping.

1. Introduction

Spectral problems depending non-linearly on the eigenvalue parameter arise frequently in applications, see e.g. the comprehensive collection in [2] or the monograph [19]. The dependence ranges from quadratic in problems originating in second order Cauchy problems such as damped wave equations, see e.g. [13], [11], to rational as in electromagnetic problems with frequency dependent materials such as photonic crystals, see e.g. [8], [1]. In addition, if energy dissipation is present due to damping or lossy materials, then the values of the corresponding operator functions need not be selfadjoint.

While for operator functions $T(\lambda)$, $\lambda\in\Omega\subseteq\mathbb{C}$, with unbounded operator values in a Hilbert space $\mathcal{H}$ the notion of numerical range $W(T)$ exists,

$$W(T) := \{\lambda\in\Omega : 0\in W(T(\lambda))\}\,, \tag{1.1}$$

a spectral inclusion result $\sigma_{\mathrm{ap}}(T)\subseteq\overline{W(T)}\cap\Omega$ for the approximate point spectrum is lacking. Even in the case of bounded values $T(\lambda)$, spectral inclusion only holds under a certain condition that is not easy to verify. Moreover, spectral inclusion results are even lacking for the most important case of quadratic operator polynomials with unbounded coefficients, one of the most relevant cases for applications.

In the present paper we fill these gaps. To this end, we introduce the novel concept of *pseudo numerical range* of operator functions $T(\lambda)$, $\lambda\in\Omega\subseteq\mathbb{C}$, with unbounded values,

$$W_\Psi(T) := \bigcap\nolimits_{\varepsilon>0} W_\varepsilon(T), \quad W_\varepsilon(T) := \bigcup\nolimits_{\substack{B\in L(\mathcal{H})\\ \|B\|<\varepsilon}} W(T+B), \quad \varepsilon>0,$$

and analogously for families of unbounded quadratic forms $\mathbf{t}(\lambda)$, $\lambda\in\Omega\subseteq\mathbb{C}$. The sets $W_\varepsilon(T)$, $\varepsilon>0$, can be shown to have the equivalent form

$$W_\varepsilon(T)=\{\lambda\in\Omega:\exists\ f\in\operatorname{dom}T(\lambda),\ \|f\|=1,\ |(T(\lambda)f,f)|<\varepsilon\};$$

hence they coincide with the so-called ε-pseudo numerical range first considered in [9]. As a consequence, the pseudo numerical range $W_\Psi(T)$ can equivalently be described as

$$W_\Psi(T)=\left\{\lambda\in\Omega:0\in\overline{W(T(\lambda))}\right\}=:W_{\Psi,0}(T). \tag{1.2}$$

One could be tempted to think that the condition $0\in\overline{W(T(\lambda))}$ in $W_{\Psi,0}(T)$ is equivalent to $\lambda\notin\overline{W(T)}$, but this is neither true for operator functions with bounded values, as already noted in [29], nor for non-monic linear operator pencils for which the set $W_{\Psi,0}(T)$ was used recently in [3].

One of the crucial properties of the pseudo numerical range is that, *without any assumptions on the operator family*,

$$\sigma_{\mathrm{ap}}(T)\subseteq W_\Psi(T),$$

see Theorem 3.1, and that the norm of the resolvent of T can be estimated by

$$\left\|T(\lambda)^{-1}\right\|\le\varepsilon^{-1},\quad \lambda\in\rho(T)\setminus W_\varepsilon(T)\subseteq\rho(T)\setminus W_\Psi(T).$$

Not only from the analytical point of view, but also from a computational perspective, the pseudo numerical range seems to be more convenient since it is much easier to determine whether a number is small rather than zero.

Like the numerical range of an operator function, but in contrast to the classical numerical range of an operator, the pseudo numerical range need not be convex. An exception is the trivial case of a monic linear operator pencil $T(\lambda)=A-\lambda I$, $\lambda\in\mathbb{C}$, where the pseudo numerical range is simply the closure of the numerical range, $W_\Psi(T)=\overline{W(T)}=\overline{W(A)}$. In general, we only have the obvious enclosure $W(T)\subseteq W_\Psi(T)$. Neither the interiors nor the closures in Ω of $W_\Psi(T)$ and $W(T)$ need to coincide andthere is also no inclusion either way between $W_\Psi(T)$ or its closure $\overline{W_\Psi(T)}\cap\Omega$ in Ω and the closure $\overline{W(T)}\cap\Omega$ of $W(T)$ in Ω; we give various counter-examples to illustrate these effects.

In our first main result we use the pseudo numerical range of holomorphic form families $\mathbf{t}(\lambda)$, $\lambda\in\Omega$, of type (a) to prove the spectral inclusion for the associated holomorphic operator functions $T(\lambda)$, $\lambda\in\Omega$, of type (B) of m-sectorial operators $T(\lambda)$. More precisely, we show that if there exist $k\in\mathbb{N}_0$, $\mu\in\Omega$ and a core $\mathcal{D}$ of $\mathbf{t}(\mu)$ with

$$0\notin\overline{W\big(\mathbf{t}^{(k)}(\mu)\big|_{\mathcal{D}}\big)}, \tag{1.3}$$

then $\sigma(T)\subseteq W_\Psi(\mathbf{t})=\overline{W(\mathbf{t})}\cap\Omega$ and, if in addition, the operator family T has constant domain, then

$$\sigma(T)\subseteq W_\Psi(T)=\overline{W(T)}\cap\Omega, \tag{1.4}$$

see Theorem 3.3. Note that, due to (1.2), condition (1.3) for $k=0$, i.e. $0\notin\overline{W\big(\mathbf{t}(\mu)\big|_{\mathcal{D}}\big)}$ for some $\mu\in\mathbb{C}$, is equivalent to $W_\Psi(T)\neq\Omega$.

For operator polynomials $T(\lambda)=\sum_{k=0}^n\lambda^kA_k$ with domain $\operatorname{dom}T(\lambda)=\bigcap_{k=0}^n\operatorname{dom}A_k$, $\lambda\in\mathbb{C}$, we prove that, if $0\notin\overline{W(A_n)}$, then

$$\sigma_{\mathrm{ap}}(T)\subseteq W_\Psi(T)\subseteq\overline{W(T)}\cap\Omega,$$

see Proposition 2.7. The inclusion (1.4) follows if, in addition, $\sigma(T(\lambda)) \subseteq \overline{W(T(\lambda))}$, $\lambda \in \mathbb{C}$, which is a weaker condition than m-sectoriality of all $T(\lambda)$.

The second new concept we introduce in this paper is the *pseudo block numerical range* of operator functions $\mathcal{L}(\lambda)$, $\lambda \in \Omega$, that possess an operator matrix representation with respect to a decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$, $n \in \mathbb{N}$, of the given Hilbert space $\mathcal{H}$. This means that

$$\mathcal{L}(\lambda) = \big(L_{ij}(\lambda)\big)_{i,j=1}^n, \quad \operatorname{dom} \mathcal{L}(\lambda) = \bigoplus\nolimits_{j=1}^n \bigcap\nolimits_{i=1}^n \operatorname{dom} L_{ij}(\lambda),$$

with operator functions $L_{ij}(\lambda)$, $\lambda \in \Omega$, of densely defined and closable linear operators from $\mathcal{H}_j$ to $\mathcal{H}_i$, $i, j = 1, \ldots, n$.

Extending earlier concepts we first define the *block numerical range* of $\mathcal{L}$ as

$$W^n(\mathcal{L}) := \bigcup\nolimits_{\substack{(f_i) \in \operatorname{dom} \mathcal{L}(\lambda) \\ \|f_i\|=1}} \sigma_p\big(\mathcal{L}(\lambda)_{(f_i)}\big), \quad \mathcal{L}(\lambda)_{(f_i)} := (\mathcal{L}_{ij}(\lambda) f_j, f_i) \in \mathbb{C}^{n \times n};$$

for bounded values $\mathcal{L}(\lambda)$ see [21] and [26] for $n = 2$, for unbounded operator matrices $\mathcal{L}(\lambda) = \mathcal{A} - \lambda I_{\mathcal{H}}$ see [22]. Then we introduce the *pseudo block numerical range* of $\mathcal{L}$ as

$$W^n_\Psi(\mathcal{L}) := \bigcap\nolimits_{\varepsilon>0} W^n_\varepsilon(\mathcal{L}), \qquad W^n_\varepsilon(\mathcal{L}) := \bigcup\nolimits_{\substack{\mathcal{B} \in L(\mathcal{H}) \\ \|\mathcal{B}\| < \varepsilon}} W^n(\mathcal{L} + \mathcal{B}), \quad \varepsilon > 0.$$

For $n = 1$ both block numerical range and pseudo block numerical range coincide with the numerical range and pseudo numerical range of $\mathcal{L}$, respectively. For $n > 1$, the trivial inclusion $W^n(\mathcal{L}) \subseteq W^n_\Psi(\mathcal{L})$ and the characterisation (1.1), i.e.

$$W^n(\mathcal{L}) = \big\{\lambda \in \Omega : 0 \in W^n(\mathcal{L}(\lambda))\big\}, \quad n \in \mathbb{N},$$

and a resolvent norm estimate

$$\big\|\mathcal{L}(\lambda)^{-1}\big\| \le \varepsilon^{-1}, \quad \lambda \in \rho(\mathcal{L}) \setminus W^n_\varepsilon(\mathcal{L}) \subseteq \rho(\mathcal{L}) \setminus W^n_\Psi(\mathcal{L}), \quad n \in \mathbb{N},$$

see Theorem 4.10 for both, continue to hold, but otherwise not much carries over from the case $n = 1$. The first difference is that, for the simplest case $\mathcal{L}(\lambda) = \mathcal{A} - \lambda I_{\mathcal{H}}$, $\lambda \in \mathbb{C}$, we may have $W^n_\Psi(\mathcal{L}) \neq \overline{W^n(\mathcal{L})}$ for $n > 1$, see Example 4.5.

More importantly, for $n > 1$ the relation (1.2) need not hold for the pseudo block numerical range; here we only have the inclusion

$$W^n_\Psi(\mathcal{L}) \supseteq \Big\{\lambda \in \Omega : 0 \in \overline{W^n(\mathcal{L}(\lambda))}\Big\} =: W^n_{\Psi,0}(\mathcal{L}), \quad n \in \mathbb{N},$$

see Proposition 4.4. Therein we also assess two other candidates $W^n_{\Psi,i}(\mathcal{L}) = \bigcap_{\varepsilon>0} W^n_{\varepsilon,i}(\mathcal{L})$, $i = 1, 2$, for the pseudo block numerical range for which $W^n_{\varepsilon,1}(\mathcal{L})$ is defined by the scalar condition $\det \mathcal{L}(\lambda)_{(f_i)} < \varepsilon$ and $W^n_{\varepsilon,2}(\mathcal{L})$ by restricting to *diagonal* perturbations $\mathcal{B} \in L(\mathcal{H})$ with $\|\mathcal{B}\| < \varepsilon$. In fact, we show that

$$W^n(\mathcal{L}) \subseteq W^n_{\Psi,1}(\mathcal{L}) \subseteq W^n_{\Psi,0}(\mathcal{L}) \subseteq W^n_{\Psi,2}(\mathcal{L}) \subseteq W^n_\Psi(\mathcal{L}), \tag{1.5}$$

and that, like the pseudo numerical range, the pseudo block numerical range $W^n_\Psi(\mathcal{L})$ has the spectral inclusion property, i.e.

$$\sigma_{\mathrm{ap}}(T) \subseteq W^n_\Psi(\mathcal{L}) \subseteq W_\Psi(T), \quad n \in \mathbb{N},$$

but, in general, none of the subsets of $W^n_\Psi(\mathcal{L})$ in (1.5) is large enough to contain $\sigma_{\mathrm{ap}}(T)$, see Example 4.5.

Our second main result concerns the most important case $n=2$, the so-called *quadratic numerical range* and *pseudo quadratic numerical range*. Here we prove a novel type of spectral inclusion for diagonally dominant and off-diagonally dominant $\mathcal{L}(\lambda)=(L_{ij}(\lambda))_{i,j=1}^2$ in terms of the pseudo numerical ranges of the Schur complements S_1, S_2 and, further, the pseudo quadratic numerical range of $\mathcal{L}$,

$$\sigma_{\mathrm{ap}}(\mathcal{L}) \setminus (\sigma(L_{11}) \cup \sigma(L_{22})) \subseteq W_\Psi(S_1) \cup W_\Psi(S_2) \subseteq W_\Psi^2(\mathcal{L}),$$

see Theorem 5.1, where $S_1(\lambda)=L_{11}(\lambda)-L_{12}(\lambda)L_{22}(\lambda)^{-1}L_{21}(\lambda)$, $\lambda\in\rho(L_{22})$, and similarly for S_2 with the indices 1 and 2 reversed. For symmetric and anti-symmetric corners, i.e. $L_{21}(\lambda) \subseteq \pm L_{12}(\lambda)^*$, $\lambda\in\Omega$, we even show that

$$\sigma_{\mathrm{ap}}(\mathcal{L})\subseteq W_\Psi(S_1) \cup W_\Psi(L_{22}),$$

if $L_{11}(\lambda)$ is accretive, $\mp L_{22}(\lambda)$ is m-sectorial and $\operatorname{dom} L_{22}(\lambda)\subseteq \operatorname{dom} L_{12}(\lambda)$, see Theorem 5.3/Corollary 5.4, and similarly for the Schur complement S_2.

As an interesting consequence, we are able to establish spectral separation and inclusion theorems for unbounded 2×2 operator matrices $\mathcal{A} = (A_{ij})_{i,j=1}^2$ with 'separated' diagonal entries; here 'separated' means that the numerical ranges of A_{11} and A_{22} lie in half-planes and/or sectors in the right and left half-plane $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively, separated by a vertical strip $S:=\{z\in\mathbb{C} : \delta<\operatorname{Re} z<\alpha\}$ with $\delta<0<\alpha$ around $\mathrm{i}\mathbb{R}$. More precisely, *without* any bounds on the order of diagonal dominance or off-diagonal dominance we show that, if φ, $\psi\in[0,\frac{\pi}{2}]$ are the semi-angles of A_{11} and A_{22} and $\tau:=\max\{\varphi,\psi\}$, then

$$\sigma_{\mathrm{ap}}(\mathcal{A}) \subseteq (-\Sigma_\tau \cup \Sigma_\tau) \setminus S =: \Sigma, \quad \Sigma_\tau := \{z\in\mathbb{C} : |\arg z| \le \tau\},$$

and $\sigma(\mathcal{A}) \subseteq \Sigma$ if $\rho(\mathcal{A}) \cap (\mathbb{C}\setminus\Sigma)\neq\emptyset$, see Theorem 6.1. This result is a great step ahead compared to the earlier result [25, Thm. 5.2] where the dominance order had to be restricted to 0.

Moreover, even to ensure the condition $\rho(\mathcal{A})\cap(\mathbb{C}\setminus\Sigma)\neq\emptyset$ for the enclosure of the entire spectrum $\sigma(\mathcal{A})$ in Theorem 6.1, we do not have to restrict the dominance order as usual for perturbation arguments. Our new weak conditions involve only products of the columnwise relative bounds δ_1 in the first and δ_2 in the second column, see Proposition 6.5; in particular, either $\delta_1=0$ or $\delta_2=0$ guarantees $\rho(\mathcal{A})\cap(\mathbb{C}\setminus\Sigma)\neq\emptyset$ in Theorem 6.1 and hence $\sigma_{\mathrm{ap}}(\mathcal{A})\subseteq\Sigma$.

As an application of our results, we consider abstract quadratic operator polynomials $T(\lambda)$, $\lambda\in\mathbb{C}$, induced by forms $\mathbf{t}(\lambda)=\mathbf{t}_0+2\lambda\mathbf{a}+\lambda^2$ with $\operatorname{dom}\mathbf{t}(\lambda) = \operatorname{dom}\mathbf{t}_0$, $\lambda\in\mathbb{C}$, as they arise e.g. from linearly damped wave equations

$$u_{tt}(x,t) + 2a(x)u_t(x,t) = (\Delta_x - q(x))\, u(x,t), \quad x\in\mathbb{R}^d, \quad t>0, \tag{1.6}$$

where the non-negative potential q and damping a may be singular and/or unbounded, cf. [10, 11, 12, 13] where also accretive damping was considered, and for which it is well-known that the spectrum is symmetric with respect to $\mathbb{R}$ and confined to the closed left half-plane.

Here we use a finely tuned assumption on the 'unboundedness' of $\mathbf{a}$ with respect to $\mathbf{t}_0$, namely *p-subordinacy* for $p\in[0,1)$, comp. [18, § 5.1] or [27, Sect. 3] for the operator case. More precisely, if $\mathbf{t}_0 \geq \kappa_0 \geq 0$, $\mathbf{a} \geq \alpha_0 \geq 0$ with $\operatorname{dom} \mathbf{t}_0 \subseteq \operatorname{dom} \mathbf{a}$ and there exist $p \in [0,1)$ and $C_p > 0$ with

$$\mathbf{a}[f] \leq C_p \big(\mathbf{t}_0[f]\big)^p \big(\, \|f\|^2 \,\big)^{1-p}, \quad f \in \operatorname{dom} \mathbf{t}_0,$$

we use the enclosure $\sigma(T) \subseteq W_\Psi(T) = W_\Psi(\mathbf{t}) = \overline{W(\mathbf{t})}$ to prove that the non-real spectrum of T satisfies the bounds

$$\sigma(T) \setminus \mathbb{R} \subseteq \Big\{ z \in \mathbb{C} : \, |z| \geq \sqrt{\kappa_0}, \ \operatorname{Re} z \leq -\alpha_0, \\ |\operatorname{Im} z|^2 \geq \max\big\{0, C_p^{-\frac{1}{p}} |\operatorname{Re} z|^{\frac{1}{p}} - |\operatorname{Re} z|^2 \,\big\} \Big\}$$

and the real spectrum $\sigma(T) \cap \mathbb{R} \subset [-\infty, 0]$ is either empty or it is confined to one bounded interval, to one unbounded interval or to the union of a bounded and an unbounded interval, see Theorem 7.1 and Figure 7.2. Moreover, we describe both the thresholds for the transitions between these cases and the enclosures for $\sigma(T) \cap \mathbb{R}$ precisely in terms of p, C_p, κ and κ_0. As a concrete example, we consider the damped wave equation (1.6) with

$$a(x) \leq \sum_{j=1}^{n} |x - x_j|^{-t} + u(x) + v(x), \quad v(x) \leq c_1 q(x)^r + c_2 \quad \text{for almost all } x \in \mathbb{R}^d,$$

where $n \in \mathbb{N}_0$, $x_j \in \mathbb{R}^d$ for $j = 1, \dots, n$, $u \in L^s(\mathbb{R}^d)$ with $s > \frac{d}{2}$, $v \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, $t \in [0,2)$, c_1, $c_2 \geq 0$ and $r \in [0,1)$. For the special case $q(x) = |x|^2$, $a(x) = |x|^k$, $x \in \mathbb{R}^d$, with $k \in [0,2)$, the new spectral enclosure in Theorem 7.1 yields

$$\sigma(T) \setminus \mathbb{R} \subseteq \Big\{ z \in \mathbb{C} : \operatorname{Re} z \leq 0, \ |z| \geq \sqrt{d}, \ |\operatorname{Im} z| \geq \sqrt{\max\{0, |\operatorname{Re} z|^{\frac{2}{k}} - |\operatorname{Re} z|^2\}} \Big\}$$

and, with $t_0 = \max\big\{ \big(k(2-k)\big)^{-\frac{1}{k-1}}, d \big\}$,

$$\sigma(T) \cap \mathbb{R} \begin{cases} = \emptyset & \text{if } k \in [0,1), \\ \subseteq \big(-\infty, -\sqrt{d}\,\big] & \text{if } k = 1, \\ \subseteq \Big(-\infty, -\sqrt{t_0}^{\,k} + \sqrt{t_0^k - t_0}\,\Big] & \text{if } k \in (1,2). \end{cases}$$

The paper is organised as follows. In Section 2 we introduce the pseudo numerical range of operator functions and form functions and study the relation of $W_\Psi(T)$ and $\overline{W(T)} \cap \Omega$. In Section 3 we establish spectral inclusion results in terms of the pseudo numerical range. In Section 4 we define the block numerical range $W^n(\mathcal{L})$ and pseudo block numerical range $W^n_\Psi(\mathcal{L})$ of unbounded $n \times n$ operator matrix functions $\mathcal{L}$, investigate the differences to the special case $n = 1$ of the pseudo numerical range $W^1_\Psi(\mathcal{L}) = W_\Psi(\mathcal{L})$ and prove corresponding spectral inclusion theorems. In Section 5 we establish new enclosures of the approximate point spectrum of 2×2 operator matrix functions by means of the pseudo numerical ranges of their Schur complements. In Section 6 we apply them to prove spectral bounds for diagonally dominant and off-diagonally dominant operator matrices with symmetric or anti-symmetric corners without restriction on the dominance order. Finally, in Section 7, we apply our results to linearly damped wave equations with possibly unbounded and/or singular damping and potential.

Throughout this paper, $\mathcal{H}$ and $\mathcal{H}_i$, $i = 1, \dots, n$, denote Hilbert spaces, $L(\mathcal{H})$ denotes the space of bounded linear operators on $\mathcal{H}$ and $\Omega \subseteq \mathbb{C}$ is a domain.

2. The pseudo numerical range of operator functions and form functions

In this section, we introduce the new notion of pseudo numerical range for operator functions $\{T(\lambda) : \lambda \in \Omega\}$ and form functions $\{\mathbf{t}(\lambda) : \lambda \in \Omega\}$, respectively, briefly denoted by T and $\mathbf{t}$ if no confusion about Ω can arise. While the values $T(\lambda)$ and $\mathbf{t}(\lambda)$ may be bounded/unbounded linear operators and sesquilinear forms in a Hilbert space $\mathcal{H}$, the notion of pseudo numerical range is new also in the bounded case.

The *numerical range* of T and $\mathbf{t}$, respectively, are defined as

$$W(T) = \{\lambda \in \Omega : 0 \in W(T(\lambda))\}, \quad W(\mathbf{t}) = \{\lambda \in \Omega : 0 \in W(\mathbf{t}(\lambda))\},$$

comp. [18, § 26.6]. In the simplest case of a monic linear operator polynomial $T(\lambda) = T_0 - \lambda I_{\mathcal{H}}$, $\lambda \in \mathbb{C}$, this notion coincides with the numerical range $W(T_0)$ of the linear operator T_0, and analogously for forms.

The following new concept of pseudo numerical range employs the notion of ε-pseudo numerical range $W_\varepsilon(T)$, $\varepsilon > 0$, introduced in [9, Def. 4.1]; the equivalent original definition therein, see (2.1) below, was designed to obtain computable enclosures for spectra of rational operator functions.

Definition 2.1. We introduce the *pseudo numerical range* of an operator function T and a form function $\mathbf{t}$, respectively, as

$$W_\Psi(T) := \bigcap_{\varepsilon>0} W_\varepsilon(T), \qquad W_\Psi(\mathbf{t}) := \bigcap_{\varepsilon>0} W_\varepsilon(\mathbf{t}),$$

where

$$W_\varepsilon(T) := \bigcup_{B\in L(\mathcal{H}),\|B\|<\varepsilon} W(T+B), \quad W_\varepsilon(\mathbf{t}) := \bigcup_{\|\mathbf{b}\|<\varepsilon} W(\mathbf{t}+\mathbf{b}), \quad \varepsilon > 0;$$

here $\|\mathbf{b}\| = \sup_{\|f\|=\|g\|=1} |\mathbf{b}[f,g]|$ for a bounded sesquilinear form $\mathbf{b}$ in $\mathcal{H}$.

Clearly, for monic linear operator polynomials $T(\lambda) = A - \lambda I_{\mathcal{H}}$, $\lambda \in \mathbb{C}$, the pseudo numerical range is nothing but the closure of the classical numerical range $\overline{W(A)}$ of the linear operator A, and analogously for forms.

The pseudo numerical range of operator or form functions, is, like their numerical ranges, in general neither convex nor connected, and, even for families of bounded operators or forms, it may be unbounded.

Remark 2.2. (i) The following inclusions may be proper, see Example 3.2,

$$W(T) \subseteq W_\Psi(T), \qquad W(\mathbf{t}) \subseteq W_\Psi(\mathbf{t}).$$

(ii) In general, the pseudo numerical range need neither be open nor closed in Ω equipped with the relative topology, see Examples 3.2 (i) and 2.9, respectively.

(iii) Neither the closures nor the interiors with respect to the relative topology on Ω of the pseudo numerical range and the numerical range need to coincide, see Example 3.2 (i) and (ii).

The following alternative characterisation of the pseudo numerical range will be frequently used in the sequel.

Proposition 2.3. *For every $\varepsilon > 0$,*

$$W_\varepsilon(T) = \{\lambda \in \Omega : \exists\ f \in \operatorname{dom} T(\lambda),\ \|f\| = 1,\ |(T(\lambda)f, f)| < \varepsilon\}, \quad (2.1)$$
$$W_\varepsilon(\mathbf{t}) = \{\lambda \in \Omega : \exists\ f \in \operatorname{dom} \mathbf{t}(\lambda),\ \|f\| = 1,\ |\mathbf{t}(\lambda)[f]| < \varepsilon\},$$

and, consequently,

$$W_\Psi(T) = \left\{\lambda \in \Omega : 0 \in \overline{W(T(\lambda))}\right\}, \quad W_\Psi(\mathbf{t}) = \left\{\lambda \in \Omega : 0 \in \overline{W(\mathbf{t}(\lambda))}\right\}. \quad (2.2)$$

Proof. We show the claim for $W_\varepsilon(T)$; then the claim for $W_\Psi(T)$ is obvious by Definition 2.1. The proof for $W_\varepsilon(\mathbf{t})$ and $W_\Psi(\mathbf{t})$ is analogous.

Let $\varepsilon > 0$ be arbitrary and $\lambda \in W_\varepsilon(T)$. There exists a bounded operator B in $\mathcal{H}$ with $\|B\| < \varepsilon$ such that $\lambda \in W(T + B)$, i.e.

$$(T(\lambda)f, f) = -(Bf, f), \quad f \in \operatorname{dom} T(\lambda), \quad \|f\| = 1.$$

Hence, clearly, $|(T(\lambda)f, f)| \le \|B\| < \varepsilon$, thus λ is an element of the right hand side of (2.1).

Conversely, let $\lambda \in \Omega$ such that there exists $f \in \operatorname{dom} T(\lambda)$, $\|f\| = 1$, with $|(T(\lambda)f, f)| < \varepsilon$. Setting $B := -(T(\lambda)f, f)\, I$, this gives $\lambda \in W(T + B)$ and $\|B\| = |(T(\lambda)f, f)| < \varepsilon$, hence $\lambda \in W_\varepsilon(T)$. □

The following properties of the pseudo numerical range with respect to closures, form representations and Friedrichs extensions are immediate consequences of its alternative description (2.2).

Here an operator A or a form $\mathbf{a}$ is called *sectorial* if its numerical range lies in a sector $\{z \in \mathbb{C} : |\arg(z - \gamma)| \le \vartheta\}$ for some $\gamma \in \mathbb{R}$ and $\vartheta \in [0, \frac{\pi}{2})$, see [15, Sect. V.3.10, VI.1.2]; if, in addition, $\rho(A) \cap \{z \in \mathbb{C} : |\arg(z - \gamma)| > \vartheta\} \neq \emptyset$, then A is called m-sectorial.

Corollary 2.4. (i) *If the family T or $\mathbf{t}$, respectively, consists of closable operators or forms (and $\overline{T}$ or $\overline{\mathbf{t}}$ denotes the family of closures), then*

$$W_\Psi(T) = W_\Psi(\overline{T}), \qquad W_\Psi(\mathbf{t}) = W_\Psi(\overline{\mathbf{t}}).$$

(ii) *If the family $\mathbf{t}$ consists of densely defined closed sectorial forms and T denotes the family of associated m-sectorial operators, then*

$$W_\Psi(\mathbf{t}) = W_\Psi(T).$$

(iii) *If the family T consists of densely defined sectorial operators and T_F denotes the family of corresponding Friedrichs extensions then*

$$W_\Psi(T) = W_\Psi(T_F).$$

Proof. (i) The equalities follow from Proposition 2.3 and from the fact that $\overline{W(T(\lambda))} = \overline{W(\overline{T(\lambda)})}$ and $\overline{W(\mathbf{t}(\lambda))} = \overline{W(\overline{\mathbf{t}}(\lambda))}$ for $\lambda \in \Omega$, see [15, Prob. V.3.7, Thm. VI.1.18].

(ii) The equality follows from Proposition 2.3 and the identity $\overline{W(\mathbf{t}(\lambda))} = \overline{W(T(\lambda))}$ for $\lambda \in \Omega$, see [15, Cor. VI.2.3].

(iii) The claim is a consequence of (i) and (ii). □

The alternative characterisation (2.2) might suggest that there is a relation between the pseudo numerical range $W_\Psi(T)$ and the closure $\overline{W(T)} \cap \Omega$ of the numerical range $W(T)$ in Ω. However, in general, there is no inclusion either way between them, see e.g. Example 3.2 where $W_\Psi(T) \not\subseteq \overline{W(T)} \cap \Omega$ and Example 2.9 where $\overline{W(T)} \cap \Omega \not\subseteq W_\Psi(T)$.

In fact, it was already noted in [29, Prop. 2.9], for continuous functions of *bounded* operators and for the more general case of block numerical ranges, that, for $\lambda \in \Omega$,

$$\lambda \in \overline{W(T)} \implies 0 \in \overline{W(T(\lambda))};$$

the converse holds only under additional assumptions. More precisely, for families of bounded linear operators however, the following is known.

Theorem 2.5. [29, Prop. 2.9, Prop. 2.12, Thm. 2.14]

(i) *If T is a (norm-)continuous family of bounded linear operators, then*
$$\overline{W(T)} \cap \Omega \subseteq W_\Psi(T).$$

(ii) *If T is a holomorphic family of bounded linear operators and there exist $k \in \mathbb{N}_0$ and $\mu \in \Omega$ with*
$$0 \notin \overline{W(T^{(k)}(\mu))}, \tag{2.3}$$
then
$$\sigma(T) \subseteq \overline{W(T)} \cap \Omega = W_\Psi(T).$$

The following simple example from [29, Ex. 2.11], which is easily adapted to the unbounded case, shows that condition (2.3) is essential both for the equality $\overline{W(T)} \cap \Omega = W_\Psi(T)$ and for the spectral inclusion $\sigma(T) \subseteq \overline{W(T)} \cap \Omega$.

Example 2.6. Let $f : \Omega \to \mathbb{C}$ be holomorphic, $f \not\equiv 0$, A a bounded or unbounded linear operator in $\mathcal{H}$ with $0 \in \sigma(A)$, $0 \in \overline{W(A)} \setminus W(A)$ and consider

$$T(\lambda) := f(\lambda)A, \quad \operatorname{dom} T(\lambda) := \operatorname{dom} A, \quad \lambda \in \Omega.$$

Then (2.3) is violated because, for any $k \in \mathbb{N}_0$ and $\mu \in \Omega$, we have $T^{(k)}(\mu) = f^{(k)}(\mu)A$ with $\operatorname{dom} T^{(k)}(\lambda) = \operatorname{dom} A$, $\lambda \in \Omega$, and so $0 \in \overline{W(T^{(k)}(\mu))}$ since $0 \in \overline{W(A)}$. Further, it is easy to see that

$$\sigma(T) = \Omega, \quad W(T) = \overline{W(T)} \cap \Omega = \{z \in \Omega : f(z) = 0\} \neq \Omega, \quad W_\Psi(T) = \Omega.$$

Thus neither $\overline{W(T)} \cap \Omega = W_\Psi(T)$ nor the spectral inclusion $\sigma(T) \subseteq \overline{W(T)} \cap \Omega$ hold, while $\sigma(T) = W_\Psi(T)$.

In the sequel we generalise Theorem 2.5 (i) and (ii) to families of unbounded operators and/or forms, including operator polynomials and sectorial families with constant form domain. In the remaining part of this section, we study the relation between $W_\Psi(T)$ and $\overline{W(T)} \cap \Omega$; results containing spectral enclosures may be found in Section 3.

Proposition 2.7. *Let T be an operator polynomial in $\mathcal{H}$ of degree $n \in \mathbb{N}$ with (possibly unbounded) coefficients $A_k : \mathcal{H} \supseteq \operatorname{dom} A_k \to \mathcal{H}$, i.e.*

$$T(\lambda) := \sum_{k=0}^{n} \lambda^k A_k, \quad \operatorname{dom} T(\lambda) := \bigcap_{k=0}^{n} \operatorname{dom} A_k, \quad \lambda \in \mathbb{C}.$$

If $0 \notin \overline{W(A_n)}$*, then*

$$W_\Psi(T) \subseteq \overline{W(T)} \cap \Omega,$$

and analogously for form polynomials.

Proof. Let $\lambda_0 \in W_\Psi(T)$. By Proposition 2.3, there is a sequence $\{f_m\}_m \subseteq \operatorname{dom} T(\lambda_0)$ with $\|f_m\| = 1$, $m \in \mathbb{N}$, and $(T(\lambda_0)f_m, f_m) \to 0$ for $m \to \infty$. Since $0 \notin \overline{W(A_n)}$ by assumption, the complex polynomial

$$p_m(\lambda) := (T(\lambda)f_m, f_m) = \sum_{k=0}^{n} (A_k f_m, f_m)\lambda^k, \quad \lambda \in \mathbb{C},$$

has degree n for each $m \in \mathbb{N}$. Let $\lambda_1^m, \dots, \lambda_n^m \in \mathbb{C}$ denote its zeros. Then $\lambda_j^m \in W(T)$, $j = 1, \dots, n$, and p_m admits the factorisation

$$p_m(\lambda) = (A_n f_m, f_m) \prod_{j=1}^{n} (\lambda - \lambda_j^m), \quad \lambda \in \mathbb{C}, \quad m \in \mathbb{N}.$$

Since $p_m(\lambda_0) \to 0$ for $m \to \infty$ and $0 \notin \overline{W(A_n)}$, there exists $j_0 \in \{1, \dots, n\}$ with $\lambda_{j_0}^m \to \lambda_0$, $m \to \infty$, thus $\lambda_0 \in \overline{W(T)}$ and $\lambda_0 \in W_\Psi(T) \subseteq \Omega$. □

Next we generalise Theorem 2.5 (i) to families of sectorial forms with constant domain which satisfy a natural continuity assumption, see [15, Thm. VI.3.6]. This assumption is met, in particular, by holomorphic form families of type (a) and associated operator families of type (B).

Recall that a family $\mathbf{t}$ of densely defined closed sectorial sesquilinear forms in $\mathcal{H}$ is called holomorphic of type (a) if its domain is constant and the mapping $\lambda \mapsto \mathbf{t}(\lambda)[f]$ is holomorphic for every $f \in \mathcal{D}_{\mathbf{t}} := \operatorname{dom} \mathbf{t}(\lambda)$. The associated family T of m-sectorial operators is called holomorphic of type (B), see [15, Sect. VII.4.2] and also [28]. Sufficient conditions on form families to be holomorphic of type (a) can be found in [15, §VII.4].

Theorem 2.8. *Let* $\mathbf{t}$ *be a family of sectorial sesquilinear forms in* $\mathcal{H}$ *with constant domain* $\mathcal{D}_{\mathbf{t}} := \operatorname{dom} \mathbf{t}(\lambda)$, $\lambda \in \Omega$*. Assume that for each* $\lambda_0 \in \Omega$*, there exist* $r, C > 0$ *and* $w : B_r(\lambda_0) \to [0, \infty)$, $\lim_{\lambda \to \lambda_0} w(\lambda) = 0$*, such that*

$$|\mathbf{t}(\lambda_0)[f] - \mathbf{t}(\lambda)[f]| \le w(\lambda) \left(|\operatorname{Re} \mathbf{t}(\lambda_0)[f]| + C \|f\|^2 \right) \tag{2.4}$$

for all $\lambda \in B_r(\lambda_0)$ *and* $f \in \mathcal{D}_{\mathbf{t}}$*. Then*

$$\overline{W(\mathbf{t})} \cap \Omega \subseteq W_\Psi(\mathbf{t}).$$

In particular, if $\mathbf{t}$ *is a holomorphic form family of type* (a) *with associated holomorphic operator family* T *of type* (B) *in* $\mathcal{H}$*, then*

$$\overline{W(T)} \cap \Omega \subseteq W_\Psi(T), \qquad \overline{W(\mathbf{t})} \cap \Omega \subseteq W_\Psi(\mathbf{t}). \tag{2.5}$$

Proof. Let $\lambda_0 \in \overline{W(\mathbf{t})}$. Then there exist $\{\lambda_n\}_n \subseteq \Omega$ and $\{f_n\}_n \subseteq \mathcal{D}_{\mathbf{t}}$ with $\|f_n\| = 1$, $\mathbf{t}(\lambda_n)[f_n] = 0$, $n \in \mathbb{N}$, and $\lambda_n \to \lambda_0$, $n \to \infty$. We show that $\mathbf{t}(\lambda_0)[f_n] \to 0$ for $n \to \infty$ which, in view of (2.2), implies $\lambda_0 \in W_\Psi(\mathbf{t})$. By (2.4),

$$|\mathbf{t}(\lambda_0)[f_n]| = |\mathbf{t}(\lambda_0)[f_n] - \mathbf{t}(\lambda_n)[f_n]| \leq w(\lambda_n)\left(|\operatorname{Re}\mathbf{t}(\lambda_0)[f_n]| + C\right), \quad n \in \mathbb{N}.$$

Since $|\operatorname{Re}\mathbf{t}(\lambda_0)[f_n]| \leq |\mathbf{t}(\lambda_0)[f_n]|$ and $w(\lambda_n) \to 0$, $n \to \infty$, we obtain that, for $n \in \mathbb{N}$ sufficiently large,

$$|\mathbf{t}(\lambda_0)[f_n]| \leq C\frac{w(\lambda_n)}{1-w(\lambda_n)} \longrightarrow 0, \quad n \to \infty.$$

Now suppose that $\mathbf{t}$ and T are holomorphic families of type (a) and (B), respectively. We only need to show the second inclusion, the first one then follows from $W(T) \subseteq W(\mathbf{t})$ and Corollary 2.4 (ii). The second inclusion follows from what we already proved since for holomorphic form families of type (a), after a possible shift $\mathbf{t}+c$ where $c>0$ is sufficiently large to ensure $\operatorname{Re}\mathbf{t}(\lambda_0)\geq 1$, [15, Eqn. VII.(4.7)] shows that assumption (2.4) is satisfied. □

Theorem 2.5 (i) does not extend to analytic families of sectorial linear operators with non-constant form domains, as the following example inspired by [15, Ex. VII.1.4] illustrates.

Example 2.9. Let $\mathcal{H} = L^2(0,1)$. The family $T(\lambda)$, $\lambda \in \mathbb{C}$, given by

$$\begin{aligned} T(\lambda)f &:= -f'' - \lambda f, \\ \operatorname{dom} T(\lambda) &:= \left\{ f \in H^2(0,1) : f(0) = 0,\ \lambda f'(1) = f(1) \right\}, \end{aligned}$$

is a holomorphic family of m-sectorial operators, but not holomorphic of type (B). Below we will show that

$$0 \in \overline{W(T)} \subseteq \overline{W_\Psi(T)}, \qquad 0 \notin W_\Psi(T);$$

note that, since $\Omega=\mathbb{C}$, this implies that Theorem 2.5 (i) does not hold and that $W_\Psi(T)$ is not closed in $\mathbb{C}$.

Indeed, it is not difficult to check that the forms associated to $T(\lambda)$, $\lambda \in \mathbb{C}$,

$$\mathbf{t}(0)[f] = \|f'\|^2, \quad \mathbf{t}(\lambda)[f] = \|f'\|^2 - \lambda\,\|f\|^2 - \frac{1}{\lambda}|f(1)|^2, \quad \lambda \in \mathbb{C}\setminus\{0\},$$

are densely defined, closed and sectorial, but have λ-depending domain $\operatorname{dom}\mathbf{t}(0)=H_0^1(0,1)$ and $\operatorname{dom}\mathbf{t}(\lambda)=\left\{f \in H^1(0,1) : f(0) = 0\right\}$ for $\lambda \in\mathbb{C}\setminus\{0\}$. The holomorphy of the family follows from the holomorphy of the integral kernel, i.e. the Green's function, of $(T(\lambda) - \mu)^{-1}$, which, for $\lambda \in \mathbb{C}$ and $\mu \in \rho(T(\lambda)) \neq \emptyset$, is given by

$$G(x,y;\mu,\lambda) = \frac{\sin(\sqrt{\mu+\lambda}x)(\sin(\sqrt{\mu+\lambda}(1-y)) - \lambda\sqrt{\mu+\lambda}\cos(\sqrt{\mu+\lambda}(1-y)))}{\sqrt{\mu+\lambda}(\sin\sqrt{\mu+\lambda} - \lambda\sqrt{\mu+\lambda}\cos\sqrt{\mu+\lambda})}$$

for $0 \leq x \leq y \leq 1$ and $G(x,y;\mu,\lambda) = G(y,x;\mu,\lambda)$ for $0 \leq y \leq x \leq 1$, cf. [15, Ex. V.4.14, VII.1.5, VII.1.11] where the family $T(\lambda)+\lambda$, $\lambda \in \mathbb{C}$, was studied.

For fixed $\lambda \in \mathbb{C}$, the spectrum of $T(\lambda)$ is given by the singularities of the integral kernel $G(\cdot,\cdot;\mu,\lambda)$,

$$\sigma(T(\lambda)) = \sigma_{\mathrm{p}}(T(\lambda)) = \left\{\mu \in \mathbb{C} : \lambda\sqrt{\mu+\lambda} = \tan\sqrt{\mu+\lambda}\right\}, \quad \lambda \in \mathbb{C}.$$

For $\lambda \in (0,\infty)$ the operator $T(\lambda)$ is self-adjoint and unbounded from above, and for $\lambda\in(0,1)$ it has an eigenvalue $\mu_\lambda \in \sigma_{\mathrm{p}}(T(\lambda)) \subseteq W(T(\lambda))$ of the form

$\mu_\lambda = -\lambda - \kappa_\lambda^2 < 0$ where κ_λ is the unique positive solution of $\tanh\kappa = \lambda\kappa$. Thus $0 \in W(T(\lambda))$ for $\lambda \in (0,1)$ due to the convexity of $W(T(\lambda))$, which proves $\overline{(0,1)} \subseteq W(T) \subseteq W_\Psi(T)$ and thus $0 \in \overline{W(T)}$. On the other hand, $0 \notin \overline{W(T(0))} = [\pi^2, \infty)$ and so Proposition 2.3 implies $0 \notin W_\Psi(T)$.

3. Spectral enclosure via pseudo numerical range

In this section we derive spectral enclosures for families of unbounded linear operators $T(\lambda)$, $\lambda \in \Omega$, using the pseudo numerical range $W_\Psi(T)$. The latter is tailored to enclose the approximate point spectrum.

The spectrum and resolvent set of an operator family $T(\lambda)$, $\lambda \in \Omega$, respectively, are defined as

$$\sigma(T) := \{\lambda \in \Omega : 0 \in \sigma(T(\lambda))\} \subseteq \Omega, \quad \rho(T) := \Omega \setminus \sigma(T),$$

and analogously for the various subsets of the spectrum. In addition to the approximate point spectrum

$$\sigma_{\mathrm{ap}}(T) := \{\lambda \in \Omega : \exists \{f_n\}_n \subseteq \operatorname{dom} T(\lambda), \|f_n\| = 1, T(\lambda)f_n \to 0, n \to \infty\},$$

we introduce the ε-*approximate point spectrum*, see [20] for the operator case,

$$\sigma_{\mathrm{ap},\varepsilon}(T) := \{\lambda \in \Omega : \exists\, f \in \operatorname{dom} T(\lambda),\ \|f\| = 1,\ \|T(\lambda)f\| < \varepsilon\}. \tag{3.1}$$

The latter is a subset of the ε-pseudo spectrum

$$\sigma_\varepsilon(T) := \sigma_{\mathrm{ap},\varepsilon}(T) \cup \sigma(T),$$

which was defined for operator functions with unbounded closed values in [7, Sect. 9.2, (9.9)], comp. also [6].

Clearly, for monic linear polynomials $T(\lambda) = A - \lambda I_{\mathcal{H}}$, $\lambda \in \mathbb{C}$, these notions coincide with the spectrum, resolvent set, approximate point spectrum, ε-approximate point spectrum and ε-pseudo spectrum of the linear operator A.

Proposition 3.1. *For any operator family $T(\lambda)$, $\lambda \in \Omega$, and every $\varepsilon > 0$,*

$$\sigma_{\mathrm{ap},\varepsilon}(T) \subseteq W_\varepsilon(T), \quad \left\|T(\lambda)^{-1}\right\| \le \frac{1}{\varepsilon}, \quad \lambda \in \rho(T) \setminus W_\varepsilon(T),$$

and hence

$$\sigma_{\mathrm{ap}}(T) \subseteq W_\Psi(T).$$

If $\sigma(T(\lambda)) \subseteq \overline{W(T(\lambda))}$ for all $\lambda \in \Omega$, then

$$\sigma(T) \subseteq W_\Psi(T).$$

Proof. The claims follow easily from (3.1) and Definition 2.1 together with Cauchy-Schwarz' inequality and (2.1) in Proposition 2.3. □

The following simple examples illustrate some properties of $W_\Psi(T)$ versus $\overline{W(T)} \cap \Omega$, in particular, in view of spectral enclosures.

Example 3.2. (i) Let $A>0$ be self-adjoint in $\mathcal{H}$ with $0\in\sigma(A)$. Then, for the non-holomorphic family $T(\lambda)=A+|\sin\lambda|$, $\lambda\in\mathbb{C}$, it is easy to see that

$$W_\Psi(T)=\sigma(T)=\{k\pi : k\in\mathbb{Z}\}\not\subseteq\overline{W(T)}\cap\Omega=\emptyset;$$

notice that this implies $\overline{W_\Psi(T)}\cap\Omega\neq\overline{W(T)}\cap\Omega$, i.e. the closures of $W_\Psi(T)$ and $W(T)$ in Ω do not coincide.

(ii) Let A be bounded in $\mathcal{H}$ with $\operatorname{Re}W(A)>0$, $0\in\sigma(A)$ and $0\notin W(A)$. Consider the holomorphic family of bounded operators in $\mathcal{H}\oplus\mathcal{H}$

$$T(\lambda)=\begin{pmatrix}\lambda A & 0\\ 0 & \lambda\operatorname{Log}(\lambda+1)I_{\mathcal{H}}\end{pmatrix},\quad \lambda\in\Omega:=\mathbb{C}\setminus(-\infty,-1];$$

here $\operatorname{Log}:\mathbb{C}\setminus(-\infty,0]\to\{z\in\mathbb{C}:\operatorname{Im}z\in(-\pi,\pi]\}$ denotes the principal value of the complex logarithm.

This family does not satisfy condition (2.3) in Theorem 2.5 since $0\in\overline{W(A)}$ by assumption. It is not difficult to show that

$$W_\Psi(T)=\sigma(T)=\mathbb{C}\setminus(-\infty,-1]\not\subseteq\overline{W(T)}\cap\Omega\subseteq\overline{B_1(-1)}\setminus[-2,-1].$$

In fact, the claims for $W_\Psi(T)$ are obvious. If $\lambda\in W(T)$, then $\lambda\in\mathbb{C}\setminus(-\infty,-1]$ and there exists $x=(f,g)^{\mathrm{t}}\in\mathcal{H}\oplus\mathcal{H}$, $(f,g)^{\mathrm{t}}\neq(0,0)^{\mathrm{t}}$, with

$$\big(T(\lambda)x,x\big)=\lambda\big((Af,f)+(\ln|\lambda+1|+\mathrm{i}\arg(\lambda+1))(g,g)\big)=0$$

or, equivalently, noting that $\lambda\neq0$ implies $g\neq0$ as $0\notin W(A)$,

$$\lambda=0\ \vee\ \Big(|\lambda+1|=\exp\Big(-\frac{\operatorname{Re}(Af,f)}{(g,g)}\Big)\wedge\arg(\lambda+1)=-\frac{\operatorname{Im}(Af,f)}{(g,g)}\Big).$$

Hence, since $\operatorname{Re}W(A)>0$,

$$W(T)\setminus\{0\}\subseteq\big\{z\in\mathbb{C}\setminus(-\infty,-1] : |z+1|\in(0,1)\big\}\subseteq B_1(-1)\setminus(-2,-1].$$

Moreover, choosing $g=f$, we see that

$$\left(T\left(\exp\Big(-\frac{(Af,f)}{(f,f)}\Big)-1\right)\begin{pmatrix}f\\f\end{pmatrix},\begin{pmatrix}f\\f\end{pmatrix}\right)=0.$$

This shows that $\{\exp(-z)-1 : z\in W(A)\}\subseteq W(T)$ and since exp is entire and non-constant, $W(A)^\circ\neq\emptyset$ implies that $W(T)^\circ\neq\emptyset$ by the open mapping theorem for holomorphic functions. So in this case $W_\Psi(T)^\circ\neq W(T)^\circ$ and both are non-empty.

$$W_\Psi(T)^\circ=\mathbb{C}\setminus(-\infty,-1],\quad \emptyset\neq W(T)^\circ\subseteq B_1(-1)\setminus(-2,-1].$$

In the following, we generalise the spectral enclosure for bounded holomorphic families in Theorem 2.5 (ii) to holomorphic form families $\mathbf{t}$ of type (a) and associated operator families of type (B), i.e. $\mathbf{t}(\lambda)$ is sectorial with vertex $\gamma(\lambda)\in\mathbb{R}$, semi-angle $\vartheta(\lambda)\in[0,\frac{\pi}{2})$ and λ-independent domain $\operatorname{dom}\mathbf{t}(\lambda)=\mathcal{D}_{\mathbf{t}}$. Here, for $k\in\mathbb{N}_0$, we denote the k-th derivative of $\mathbf{t}$ by

$$\mathbf{t}^{(k)}(\lambda)[f]:=(\mathbf{t}(\cdot)[f])^{(k)}(\lambda),\quad f\in\operatorname{dom}\mathbf{t}^{(k)}(\lambda):=\mathcal{D}_{\mathbf{t}}=\operatorname{dom}\mathbf{t}(\lambda),\quad \lambda\in\Omega;$$

note that $\mathbf{t}^{(k)}(\lambda)$ need not be closable or sectorial if $k>0$.

Theorem 3.3. *Let* $\mathbf{t}$ *be a holomorphic form family of type* (a) *with associated holomorphic operator family* T *of type* (B) *in* $\mathcal{H}$. *If there exist* $k \in \mathbb{N}_0$, $\mu \in \Omega$ *and a core* $\mathcal{D}$ *of* $\mathbf{t}(\mu)$ *with*

$$0 \notin \overline{W\big(\mathbf{t}^{(k)}(\mu)\big|_{\mathcal{D}}\big)}, \tag{3.2}$$

then

$$\sigma(T) \subseteq W_\Psi(\mathbf{t}) = \overline{W(\mathbf{t})} \cap \Omega.$$

If, in addition, the operator family T *has constant domain, then*

$$\sigma(T) \subseteq W_\Psi(T) = \overline{W(T)} \cap \Omega.$$

Remark 3.4. (i) Since $\mathbf{t}(\lambda)$ is densely defined, closed and sectorial for all $\lambda \in \Omega$, condition (3.2) for $k = 0$ has the two equivalent forms

$$0 \notin \overline{W\big(\mathbf{t}(\mu)\big|_{\mathcal{D}}\big)} \iff 0 \notin \overline{W(T(\mu))};$$

hence, by Proposition 2.3 a sufficient condition for (3.2) is

$$W_\Psi(T) \neq \Omega.$$

(ii) For operator polynomials T, which are holomorphic and have constant domain by definition, see Proposition 2.7, no sectoriality assumption is needed for the enclosure

$$\sigma_{\mathrm{ap}}(T) \subseteq W_\Psi(T) \subseteq \overline{W(T)} \cap \Omega.$$

By Propositions 2.7 and 3.1, the above holds under the mere assumption that $0 \notin \overline{W(A_n)}$ where A_n is the leading coefficient of T; note that then (3.2) holds with $k = n$ and arbitrary $\mu \in \mathbb{C}$. This generalises the classical result [18, Thm. 26.7] for bounded operator polynomials; see also [29, Prop. 3.3] for the block numerical range.

(iii) In general, neither the assumption on holomorphy nor condition (3.2) in Theorem 3.3 can be omitted, see Examples 2.6 and 3.2.

Proof of Theorem 3.3. First we show that if condition (3.2) holds for some core $\mathcal{D}$ of $\mathbf{t}(\mu)$, it also holds for $\mathcal{D}$ replaced by $\mathcal{D}_{\mathbf{t}} = \operatorname{dom} \mathbf{t}(\lambda)$, $\lambda \in \Omega$. Without loss of generality, we may assume that $\operatorname{Re} \mathbf{t}(\mu) \geq 1$. From the proof of [15, Eqn. VII.(4.7)], it is easy to see that the second inequality therein holds for $\mathbf{t}^{(k)}$, i.e. there exists a constant $C_\mu > 0$ such that

$$\big|\mathbf{t}^{(k)}(\mu)[f,g]\big| \leq C_\mu \, |\mathbf{t}(\mu)[f]|^{\frac{1}{2}} \, |\mathbf{t}(\mu)[g]|^{\frac{1}{2}}, \quad f, g \in \mathcal{D}_{\mathbf{t}}. \tag{3.3}$$

To prove the claim stated at the beginning assume, to the contrary, that $0 \in \overline{W(\mathbf{t}^{(k)}(\mu))}$, i.e. that there exists a sequence $\{f_n\}_n \subseteq \mathcal{D}_{\mathbf{t}}$, $\|f_n\| = 1$, $n \in \mathbb{N}$, such that $\mathbf{t}(\mu)[f_n] \to 0$ as $n \to \infty$. By the core property of $\mathcal{D}$ for $\mathbf{t}[\mu]$ and by [15, Thm. VI.1.12], for fixed $n \in \mathbb{N}$, there exists $\{f_{n,m}\}_m \subseteq \mathcal{D}$ with

$$f_{n,m} \to f_n, \quad \mathbf{t}(\mu)[f_{n,m} - f_n] \to 0, \quad \mathbf{t}(\mu)[f_{n,m}] \to \mathbf{t}(\mu)[f_n], \quad m \to \infty. \tag{3.4}$$

Applying (3.3), we can estimate

$$\begin{aligned}\big|\mathbf{t}^{(k)}(\mu)[f_{n,m}] - \mathbf{t}^{(k)}(\mu)[f_n]\big| &\leq \big|\mathbf{t}^{(k)}(\mu)[f_{n,m}, f_{n,m} - f_n]\big| + \big|\mathbf{t}^{(k)}(\mu)[f_n - f_{n,m}, f_n]\big| \\ &\leq C_\mu \, |\mathbf{t}(\mu)[f_{n,m} - f_n]|^{\frac{1}{2}} \big(|\mathbf{t}(\mu)[f_{n,m}]|^{\frac{1}{2}} + |\mathbf{t}(\mu)[f_n]|^{\frac{1}{2}} \big).\end{aligned}$$

Since $\|f_n\|=1$, $n\in\mathbb{N}$, it follows from (3.4) and the above inequality that there exists $m_n \geq n$ such that

$$\|f_{n,m_n}\| \geq \frac{1}{2}, \quad \left|\mathbf{t}^{(k)}(\mu)[f_{n,m_n}]\right| < \left|\mathbf{t}^{(k)}(\mu)[f_n]\right| + \frac{1}{n}.$$

In view of $\mathbf{t}^{(k)}(\mu)[f_n] \to 0$, $n\to\infty$, this implies the required claim

$$0 \in \overline{W\big(\mathbf{t}^{(k)}(\mu)\big|_{\mathcal{D}}\big)}.$$

This completes the proof that (3.2) holds with $\mathcal{D}_{\mathbf{t}}$ instead of $\mathcal{D}$.

By Corollary 2.4 (ii), we have $W_\Psi(\mathbf{t}) = W_\Psi(T) \subseteq \Omega$. Thus, due to (2.5), for the claimed equalities between pseudo numerical and numerical ranges it is sufficient to show $W_\Psi(\mathbf{t}) \subseteq \overline{W(\mathbf{t})}$ and $W_\Psi(\mathbf{t}) \subseteq \overline{W(T)}$, respectively.

Let $\lambda_0 \in W_\Psi(\mathbf{t}) = W_\Psi(T)$. Then $0 \in \overline{W(T(\lambda_0))}$ by Proposition 2.3 and hence there exists $\{f_n\}_n \subseteq \operatorname{dom} T(\lambda_0) \subseteq \mathcal{D}_{\mathbf{t}}$ with $\|f_n\|=1$, $n\in\mathbb{N}$, such that

$$(T(\lambda_0)f_n, f_n) = \mathbf{t}(\lambda_0)[f_n] \to 0, \quad n\to\infty. \tag{3.5}$$

Define a sequence of holomorphic functions

$$\varphi_n(\lambda) := \mathbf{t}(\lambda)[f_n], \quad \lambda\in\Omega, \quad n\in\mathbb{N}.$$

Let $K\subseteq\Omega$ be an arbitrary compact subset and let $\gamma>0$ be such that $\operatorname{Re}(\mathbf{t}+\gamma)(\lambda_0)\geq 1$. By [15, Eqn. VII.(4.7)], there exists $b_K>0$ with

$$|(\mathbf{t}+\gamma)(\lambda)[f]| \leq b_K |(\mathbf{t}+\gamma)(\lambda_0)[f]|, \quad \lambda\in K, \ f\in\mathcal{D}_{\mathbf{t}}.$$

Using this, $\|f_n\|=1$ and (3.5), we find that, for all $\lambda\in K$,

$$|\varphi_n(\lambda)| \leq b_K\,|(\mathbf{t}+\gamma)(\lambda_0)[f_n]| + \gamma \leq b_K \sup_{n\in\mathbb{N}} |\mathbf{t}(\lambda_0)[f_n]| + (b_K+1)\gamma < \infty.$$

Consequently, $\{\varphi_n\}_n$ is uniformly bounded on compact subsets of Ω. By Montel's Theorem, see e.g. [4, §VII.2], there exists a subsequence $\{\varphi_{n_j}\}_j \subseteq \{\varphi_n\}_n$ that converges locally uniformly to a holomorphic function φ. Now assumption (3.2) with $\mathcal{D}_{\mathbf{t}}$, which we proved to hold in the first step, implies

$$\varphi^{(k)}(\mu) = \frac{\mathrm{d}^k}{\mathrm{d}\lambda^k} \lim_{j\to\infty} \varphi_{n_j}(\lambda)\bigg|_{\lambda=\mu} = \lim_{j\to\infty} \varphi_{n_j}^{(k)}(\mu) = \lim_{j\to\infty} \mathbf{t}^{(k)}(\mu)[f_{n_j}] \neq 0$$

and thus $\varphi\not\equiv 0$. By (3.5), we further conclude that $\varphi(\lambda_0)=0$. Then, by Hurwitz' Theorem, see e.g. [4, §VII.2], there exists a sequence $\{\lambda_j\}_j\subseteq\Omega$ with $\lambda_j\to\lambda_0$ for $j\to\infty$ and

$$0 = \varphi_{n_j}(\lambda_j) = \mathbf{t}(\lambda_j)[f_{n_j}], \quad j\in\mathbb{N}.$$

Hence, $\lambda_j\in W(\mathbf{t})$ for all $j\in\mathbb{N}$ and so $\lambda_0\in\overline{W(\mathbf{t})}\cap\Omega$, as required.

Now assume that the operator family T has constant domain. Then, in the above construction, we have $f_{n_j}\in\operatorname{dom}T(\lambda_0)=\operatorname{dom}T(\lambda_j)$ for every $j\in\mathbb{N}$. It follows that $\lambda_j\in W(T)$, $j\in\mathbb{N}$, and thus $\lambda_0\in\overline{W(T)}\cap\Omega$.

The enclosures of the spectrum follow from Proposition 3.1 and from the fact that $\sigma(T(\lambda))\subseteq\overline{W(T(\lambda))}$ since $T(\lambda)$ is m-sectorial for all $\lambda\in\Omega$. □

As forms are the natural objects regarding numerical ranges, it is not surprising that the inclusion $W_\Psi(T)\subseteq\overline{W(T)}\cap\Omega$ in Theorem 3.3 might cease to hold for more general analytic operator families where the connection to a family of forms is lost. Nevertheless, using an analogous idea as in the proof of Theorem 3.3, one can prove the corresponding inclusion for the approximate spectrum.

Recall that an operator family T in $\mathcal{H}$ is called holomorphic of type (A) if it consists of closed operators with constant domain and for each $f \in \mathcal{D}_T := \operatorname{dom} T(\lambda)$, the mapping $\lambda \mapsto T(\lambda)f$ is holomorphic on Ω. Here, for $k \in \mathbb{N}_0$, the k-th derivative of T is defined as

$$T^{(k)}(\lambda)f := (T(\cdot)f)^{(k)}(\lambda), \quad f \in \operatorname{dom} T^{(k)}(\lambda) := \mathcal{D}_T, \quad \lambda \in \Omega.$$

Theorem 3.5. *Let T be a holomorphic family of type* (A) *in $\mathcal{H}$. If there exist $k \in \mathbb{N}_0$, $\mu \in \Omega$ and a core $\mathcal{D}$ of $T(\mu)$ with*

$$0 \notin \overline{W\big(T^{(k)}(\mu)\big|_{\mathcal{D}}\big)}, \tag{3.6}$$

then

$$\sigma_{\mathrm{ap}}(T) \subseteq \overline{W(T)} \cap \Omega.$$

Proof. In the same way as in the proof of Theorem 3.3, using the analogue of [15, Eqn. VII.(2.3)] for the k-th derivative of T and Cauchy-Schwarz' inequality, one shows that (3.6) holds with $\mathcal{D}_T = \operatorname{dom} T(\lambda)$, $\lambda \in \Omega$, instead of $\mathcal{D}$.

We proceed similarly as in the proof of Theorem 3.3. Let $\lambda_0 \in \sigma_{\mathrm{ap}}(T)$. There exists a sequence $\{f_n\}_n \subseteq \mathcal{D}_T$ with $\|f_n\| = 1$, $n \in \mathbb{N}$, and $T(\lambda_0)f_n \to 0$ as $n \to \infty$. Define a sequence of holomorphic functions

$$\varphi_n(\lambda) := (T(\lambda)f_n, f_n), \quad \lambda \in \Omega, \quad n \in \mathbb{N}.$$

Analogously to the proof of Theorem 3.3, one uses Cauchy-Schwarz' inequality, equation [15, Eqn. VII.(2.2)], $\lim_{n\to\infty} T(\lambda_0)f_n = 0$ and (3.6) with $\mathcal{D}_T$ in order to show uniform boundedness of $\{\varphi_n\}_n$ on compacta, extract a locally uniformly converging subsequence with limit $\varphi \not\equiv 0$ and infer $\varphi(\lambda_0) = 0$. One then obtains $\lambda_0 \in \overline{W(T)} \cap \Omega$ in the same way as in Theorem 3.3. $\square$

Remark 3.6. Theorems 3.3 and 3.5 generalise the classical result [18, Thm. III. 26.6] for bounded holomorphic families (which follows from Theorem 2.5 (ii)).

Like for the numerical range of unbounded operators, cf. [15, Sct. V.3.2], additional conditions are needed for enclosing not only the approximate point spectrum, but the entire spectrum $\sigma(T)$ in $W_\Psi(T)$.

Remark 3.7. Let T be a family of closed operators in $\mathcal{H}$ and let T be continuous in the generalised sense. If $\sigma_{\mathrm{ap}}(T) \subseteq \Theta \subseteq \Omega$ and all connected components of $\Omega \setminus \Theta$ contain a point in the resolvent set of T, then $\sigma(T) \subseteq \Theta$. In particular, if all connected components of $\Omega \setminus W_\Psi(T)$ have non-empty intersection with $\rho(T)$, then

$$\sigma(T) \subseteq W_\Psi(T).$$

This follows from the fact that the index of $T(\lambda)$ is locally constant on the set of regular points, see [15, Thm. IV.5.17].

4. Pseudo block numerical ranges of operator matrix functions and spectral enclosures

In this section we introduce the pseudo block numerical range of $n \times n$ operator matrix functions for which the entries may have unbounded operator values. While we study its basic properties for $n \geq 2$, we study the most important case $n = 2$ in greater detail.

We suppose that with respect to a fixed decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$ with $n \in \mathbb{N}$, a family $\mathcal{L} = \{\mathcal{L}(\lambda) : \lambda \in \Omega\}$ of densely defined linear operators in $\mathcal{H}$ admits a matrix representation

$$\mathcal{L}(\lambda) = (L_{ij}(\lambda))_{i,j=1}^n : \mathcal{H} \supseteq \operatorname{dom} \mathcal{L}(\lambda) \to \mathcal{H};$$

here L_{ij} are families of densely defined and closable linear operators from $\mathcal{H}_j$ to $\mathcal{H}_i$, i, $j = 1, \ldots, n$, and $\operatorname{dom} \mathcal{L}(\lambda) = \mathcal{D}_1(\lambda) \oplus \cdots \oplus \mathcal{D}_n(\lambda)$,

$$\mathcal{D}_j(\lambda) := \bigcap_{i=1}^n \operatorname{dom} L_{ij}(\lambda), \quad j = 1, \ldots, n.$$

The following definition generalises, and unites, several earlier concepts: the block numerical range of $n \times n$ operator matrix families whose entries have bounded linear operator values, see [21], the block numerical range of unbounded $n \times n$ operator matrices, see [22], and in the special case $n = 2$, the quadratic numerical range for bounded analytic operator matrix families and unbounded operator matrices, see [26] and [17], [25], respectively. Further, we introduce the new concept of pseudo block numerical range.

Definition 4.1. (i) We define the *block numerical range* of $\mathcal{L}$ (with respect to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$) as

$$W^n(\mathcal{L}) := \bigcup_{f \in \operatorname{dom} \mathcal{L}(\lambda) \cap \mathcal{S}^n} \sigma\big(\mathcal{L}(\lambda)_f\big)$$

where $\mathcal{S}^n := \{f = (f_i)_{i=1}^n \in \mathcal{H} : \|f_i\| = 1, i = 1, \ldots, n\}$ and, for $f = (f_i)_{i=1}^n \in \operatorname{dom} \mathcal{L}(\lambda) \cap \mathcal{S}^n$ with $\lambda \in \Omega$,

$$\mathcal{L}(\lambda)_f := (\mathcal{L}_{ij}(\lambda) f_j, f_i) \in \mathbb{C}^{n \times n}.$$

(ii) We introduce the *pseudo block numerical range* of $\mathcal{L}$ as

$$W^n_\Psi(\mathcal{L}) := \bigcap_{\varepsilon > 0} W^n_\varepsilon(\mathcal{L}), \qquad W^n_\varepsilon(\mathcal{L}) := \bigcup_{\mathcal{B} \in L(\mathcal{H}), \|\mathcal{B}\| < \varepsilon} W^n(\mathcal{L} + \mathcal{B}), \quad \varepsilon > 0.$$

Note that, indeed, if $\mathcal{L}(\lambda) = \mathcal{A} - \lambda I_{\mathcal{H}}$, $\lambda \in \mathbb{C}$, with an (unbounded) operator matrix $\mathcal{A}$ in $\mathcal{H}$, then $W^n(\mathcal{L})$ coincides with the block numerical range $W^n(\mathcal{A})$ first introduced in [22] and, for $n = 2$, in [25]. While the pseudo numerical range also satisfies $W_\Psi(\mathcal{L}) = \overline{W(\mathcal{L})} = \overline{W(\mathcal{A})}$ this is no longer true for the pseudo block numerical range when $n > 1$; in fact, Example 4.5 below shows that $W^2_\Psi(\mathcal{L}) \neq \overline{W^2(\mathcal{L})} = \overline{W^2(\mathcal{A})}$ is possible.

Remark 4.2. It is not difficult to see that, for the block numerical range and the pseudo block numerical range of general operator matrix families,

$$\lambda \in W^n(\mathcal{L}) \iff 0 \in W^n(\mathcal{L}(\lambda)) \tag{4.1}$$

and $W^n(\mathcal{L}) \subseteq W^n_\Psi(\mathcal{L})$.

There are several other ways to define the pseudo block numerical range. In the following we show that, in general, they inevitably fail to contain the approximate point spectrum of an operator matrix family.

Definition 4.3. Define

$$W^n_{\Psi,0}(\mathcal{L}) := \Big\{\lambda \in \Omega : 0 \in \overline{W^n(\mathcal{L}(\lambda))}\Big\}, \quad W^n_{\Psi,i}(\mathcal{L}) := \bigcap_{\varepsilon>0} W^n_{\varepsilon,i}(\mathcal{L}),\ i=1,2,$$

where, for $\varepsilon > 0$,

$$W^n_{\varepsilon,1}(\mathcal{L}) := \{\lambda \in \Omega : \exists\, f \in \operatorname{dom} \mathcal{L}(\lambda) \cap \mathcal{S}^n, |\det(\mathcal{L}(\lambda)_f)| < \varepsilon\},$$
$$W^n_{\varepsilon,2}(\mathcal{L}) := \bigcup_{B_i \in L(\mathcal{H}_i), \|B_i\|<\varepsilon} W^n(\mathcal{L} + \operatorname{diag}(B_1, \dots, B_n)).$$

While for the pseudo numerical range, analogous concepts as in Definition 4.3 coincide by Proposition 2.3, this is not true for the pseudo block numerical range. Here, in general, we only have the following inclusions.

Proposition 4.4. *The pseudo block numerical range $W^n_\Psi(\mathcal{L})$ satisfies*

$$W^n(\mathcal{L}) \subseteq W^n_{\Psi,1}(\mathcal{L}) \subseteq W^n_{\Psi,0}(\mathcal{L}) \subseteq W^n_{\Psi,2}(\mathcal{L}) \subseteq W^n_\Psi(\mathcal{L}). \tag{4.2}$$

Proof. We consider the case $n = 2$; the proofs for $n > 2$ are analogous. The leftmost and rightmost inclusions are trivial by definition. For the remaining inclusions, it is sufficient to show that, for every $\varepsilon > 0$,

$$W^2_{\varepsilon,1}(\mathcal{L}) \subseteq \Big\{\lambda \in \Omega : 0 \in \mathrm{B}_{\sqrt{\varepsilon}}(W^2(\mathcal{L}(\lambda)))\Big\} \subseteq W^2_{\sqrt{\varepsilon},2}(\mathcal{L}). \tag{4.3}$$

Then the respective claims follow by taking the intersection over all $\varepsilon > 0$.

Let $\varepsilon > 0$ and $\lambda \in W^2_{\varepsilon,1}(\mathcal{L})$. Then there exists $f \in \operatorname{dom} \mathcal{L}(\lambda) \cap \mathcal{S}^2$ with

$$\sigma(\mathcal{L}(\lambda)_f) = \{\lambda_1, \lambda_2\} \subseteq W^2(\mathcal{L}(\lambda)), \qquad |\lambda_1|\,|\lambda_2| = |\det \mathcal{L}(\lambda)_f| < \varepsilon.$$

Now the first inclusion in (4.3) follows from

$$\operatorname{dist}(0, W^2(\mathcal{L}(\lambda))) \le \min\{|\lambda_1|, |\lambda_2|\} < \sqrt{\varepsilon}.$$

For the second inclusion, let $\lambda \in \Omega$ with $\operatorname{dist}(0, W^2(\mathcal{L}(\lambda))) < \sqrt{\varepsilon}$, i.e. there exists $\mu \in \mathbb{C}$, $|\mu| < \sqrt{\varepsilon}$ with $\mu \in W^2(\mathcal{L}(\lambda))$. By (4.1), the latter is equivalent to $0 \in W^2(\mathcal{L}(\lambda) - \mu\mathcal{I}_\mathcal{H})$ and hence

$$\lambda \in W^2(\mathcal{L} - \mu\mathcal{I}_\mathcal{H}) \subseteq W^2_{\sqrt{\varepsilon},2}(\mathcal{L}). \qquad \square$$

Clearly, in the simplest case $\mathcal{L}(\lambda) = \mathcal{A} - \lambda I_\mathcal{H}$, $\lambda \in \mathbb{C}$, with an $n \times n$ operator matrix $\mathcal{A}$ in $\mathcal{H}$ we have

$$W^n_{\Psi,0}(\mathcal{L}) = \overline{W^n(\mathcal{L})} = \overline{W^n(\mathcal{A})}; \tag{4.4}$$

this shows that $W^n_{\Psi,0}(\mathcal{L})$ fails to enclose the spectrum whenever $\overline{W^n(\mathcal{A})}$ does.

The following example shows that, already in this simple case, in fact *none* of the subsets $W^n_{\Psi,1}(\mathcal{L}) \subseteq W^n_{\Psi,0}(\mathcal{L}) \subseteq W^n_{\Psi,2}(\mathcal{L})$ of the pseudo block numerical range $W_\Psi^n(\mathcal{L})$, see (4.2), is large enough to contain the approximate point spectrum $\sigma_{\mathrm{ap}}(\mathcal{L})$.

Example 4.5. Let $\mathcal{H}=\ell^2(\mathbb{N})\oplus\ell^2(\mathbb{N})$ and $\mathcal{L}(\lambda)=\mathcal{A}-\lambda I_{\mathcal{H}}$, $\lambda\in\mathbb{C}$, with

$$\mathcal{A}:=\begin{pmatrix} 0 & \operatorname{diag}(m^2-1 : m\in\mathbb{N}) \\ 0 & 0 \end{pmatrix}, \quad \operatorname{dom}\mathcal{A}:=\ell^2(\mathbb{N})\oplus\operatorname{dom}\operatorname{diag}(m^2-1 : m\in\mathbb{N}),$$

where $\operatorname{diag}(m^2-1 : m\in\mathbb{N})$ is the unbounded maximal multiplication operator in $\ell^2(\mathbb{N})$ with domain

$$\operatorname{dom}\operatorname{diag}(m^2-1 : m\in\mathbb{N}) := \big\{\{x_m\}_m\in\ell^2(\mathbb{N}) : \{(m^2-1)x_m\}_m\in\ell^2(\mathbb{N})\big\}.$$

We will now show that

$$\{0\}=W^2_{\Psi,1}(\mathcal{L})=W^2_{\Psi,0}(\mathcal{L})=W^2_{\Psi,2}(\mathcal{L})\neq W^2_\Psi(\mathcal{L})=\sigma_{\mathrm{ap}}(\mathcal{L})=\mathbb{C}.$$

By (4.4) and since $\mathcal{A}$ is triangular, we have $W^2_{\Psi,0}(\mathcal{L})=\overline{W^2(\mathcal{L})}=\overline{W^2(\mathcal{A})}=W^2(\mathcal{A})=\{0\}$. Since $W^2_{\Psi,1}(\mathcal{L})\subseteq W^2_{\Psi,0}(\mathcal{L})$ by (4.2), the first and second equality from the left follow. The third equality from the left follows from the definition of $W^2_{\Psi,0}(\mathcal{L})$ and $W^2_{\Psi,2}(\mathcal{L})$ since $\mathcal{A}$ is triangular. To prove the two equalities on the right, and hence the claimed inequality, let $\lambda\in\mathbb{C}$ be arbitrary. If $\lambda=0$, then $\lambda\in W^2_\Psi(\mathcal{L})$ by (4.3). If $\lambda\neq0$, we define the bounded operator matrices

$$\mathcal{B}_k:=\begin{pmatrix} -\operatorname{diag}(\frac{\lambda}{m}\delta_{mk} : m\in\mathbb{N}) & 0 \\ -\operatorname{diag}(\frac{\lambda^2}{m^2}\delta_{mk} : m\in\mathbb{N}) & \operatorname{diag}(\frac{\lambda}{m}\delta_{mk} : m\in\mathbb{N}) \end{pmatrix}, \quad k\in\mathbb{N},$$

where δ_{mk} denotes the Kronecker delta. Then $\|\mathcal{B}_k\|\to0$ as $k\to\infty$ and a straightforward calculation shows that

$$(\mathcal{A}-\lambda I_{\mathcal{H}})f_k=\mathcal{B}_kf_k, \quad f_k:=\frac{\widetilde{f}_k}{\|\widetilde{f}_k\|}\in\operatorname{dom}\mathcal{A}, \quad \widetilde{f}_k=\begin{pmatrix}\frac{k(k+1)}{\lambda}e_k \\ e_k\end{pmatrix}, \quad k\in\mathbb{N}.$$

On the one hand, for arbitrary $\varepsilon>0$, this implies that there exists $N\in\mathbb{N}$ such that $\|\mathcal{B}_N\|<\varepsilon$ and $0\in\sigma(\mathcal{A}-\lambda-\mathcal{B}_N)=\sigma_{\mathrm{p}}(\mathcal{L}(\lambda)-\mathcal{B}_N)$, whence

$$\lambda\in\sigma_{\mathrm{p}}(\mathcal{L}-\mathcal{B}_N)\subseteq W^2(\mathcal{L}-\mathcal{B}_N)\subseteq W^2_\varepsilon(\mathcal{L})$$

and thus $\lambda\in W^2_\Psi(\mathcal{L})$ by intersection over all $\varepsilon>0$. On the other hand, $\lambda\in\sigma_{\mathrm{ap}}(\mathcal{L})$ since the normalised sequence $\{f_k\}_k\subseteq\operatorname{dom}\mathcal{L}(\lambda)$ satisfies

$$\|(\mathcal{A}-\lambda)f_k\|=\|\mathcal{B}_kf_k\|\le\|\mathcal{B}_k\|\to0, \quad k\to\infty.$$

With one exception, we now focus on the most important case $n=2$ for which the notation

$$\begin{aligned} &\mathcal{L}(\lambda):=\begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda)\end{pmatrix} \quad \text{in } \mathcal{H}=\mathcal{H}_1\oplus\mathcal{H}_2, \\ &\operatorname{dom}\mathcal{L}(\lambda):=\big(\operatorname{dom}A(\lambda)\cap\operatorname{dom}C(\lambda)\big)\oplus\big(\operatorname{dom}B(\lambda)\cap\operatorname{dom}D(\lambda)\big), \end{aligned} \tag{4.5}$$

is more customary. We establish various inclusions between the (pseudo) quadratic numerical range $W^2_{(\Psi)}(\mathcal{L})$ and the (pseudo) numerical ranges of the diagonal operator functions A, D, as well as between $W^2_{(\Psi)}(\mathcal{L})$ and the (pseudo) numerical ranges of the Schur complements of $\mathcal{L}$.

Proposition 4.6. (i) *The quadratic numerical range and the pseudo quadratic numerical range satisfy*

$$W^2(\mathcal{L}) \subseteq W(\mathcal{L}), \quad W^2_\Psi(\mathcal{L}) \subseteq W_\Psi(\mathcal{L}).$$

(ii) *Let* $\Omega_1 := \{\lambda \in \Omega : \mathcal{D}_1(\lambda) = \operatorname{dom} A(\lambda)\}$ *and suppose* $\dim \mathcal{H}_2 > 1$. *Then*

$$W(A) \cap \Omega_1 \subseteq W^2(\mathcal{L}), \quad W_\Psi(A) \cap \Omega_1 \subseteq W^2_{\Psi,2}(\mathcal{L}) \subseteq W^2_\Psi(\mathcal{L});$$

if $\mathcal{D}_1(\lambda) = \operatorname{dom} A(\lambda)$ *for all* $\lambda \in W(A)$ *or* $\lambda \in W_\Psi(A)$, *respectively, then*

$$W(A) \subseteq W^2(\mathcal{L}), \quad W_\Psi(A) \subseteq W^2_{\Psi,2}(\mathcal{L}) \subseteq W^2_\Psi(\mathcal{L}).$$

(iii) *Let* $\Omega_2 := \{\lambda \in \Omega : \mathcal{D}_2(\lambda) = \operatorname{dom} D(\lambda)\}$ *and suppose* $\dim \mathcal{H}_1 > 1$. *Then*

$$W(D) \cap \Omega_2 \subseteq W^2(\mathcal{L}), \quad W_\Psi(D) \cap \Omega_2 \subseteq W^2_{\Psi,2}(\mathcal{L}) \subseteq W^2_\Psi(\mathcal{L});$$

if $\mathcal{D}_2(\lambda) = \operatorname{dom} D(\lambda)$ *for all* $\lambda \in W(D)$ *or* $\lambda \in W_\Psi(D)$, *respectively, then*

$$W(D) \subseteq W^2(\mathcal{L}), \quad W_\Psi(D) \subseteq W^2_{\Psi,2}(\mathcal{L}) \subseteq W^2_\Psi(\mathcal{L}).$$

Proof. The claims for the quadratic numerical range are consequences of (4.1) and of the corresponding statements [25, Prop. 3.2, 3.3 (i),(ii)] for operator matrices. So it remains to prove the claims (i) and (ii) for the pseudo quadratic numerical range; the proof of claim (iii) is completely analogous.

(i) The inclusion for the quadratic numerical range in (i) applied to $\mathcal{L}+\mathcal{B}$ with $\|\mathcal{B}\| < \varepsilon$ yields $W^2_\varepsilon(\mathcal{L}) \subseteq W_\varepsilon(\mathcal{L})$ for any $\varepsilon > 0$. The claim for the pseudo quadratic numerical range follows if we take the intersection over all $\varepsilon > 0$.

(ii) Let $\lambda \in W_\varepsilon(A) \cap \Omega_1$ with $\varepsilon > 0$ arbitrary. Then there exists a bounded operator B_ε in $\mathcal{H}_1$ with $\|B_\varepsilon\| < \varepsilon$ and $\lambda \in W(A + B_\varepsilon)$. Since $\operatorname{dom}(A(\lambda) + B_\varepsilon)$ $= \operatorname{dom} A(\lambda) \subseteq \operatorname{dom} C(\lambda)$, the inclusion for the quadratic numerical range in (ii) applied to $\mathcal{L} + \operatorname{diag}(B_\varepsilon, 0_{\mathcal{H}_2})$ shows that

$$\lambda \in W^2(\mathcal{L} + \operatorname{diag}(B_\varepsilon, 0_{\mathcal{H}_2})) \subseteq W^2_{\varepsilon,2}(\mathcal{L}) \subseteq W^2_\varepsilon(\mathcal{L}).$$

By intersecting over all $\varepsilon > 0$, we obtain $\lambda \in W^2_{\Psi,2}(\mathcal{L}) \subseteq W^2_\Psi(\mathcal{L})$. The second claim is obvious from the first one since then $\Omega_1 \subseteq W_\Psi(A)$. □

Both qualitative and quantitative behaviour of operator matrices are closely linked to the properties of their so-called Schur complements, see e.g. [25]; the same is true for operator matrix functions, see e.g. [26] for the case of bounded operator values.

Definition 4.7. The Schur complements of the 2×2 operator matrix family $\mathcal{L} = \{\mathcal{L}(\lambda) : \lambda \in \Omega\}$ in $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ as in (4.5) are the families

$$S_1(\lambda) := A(\lambda) - B(\lambda)D(\lambda)^{-1}C(\lambda), \quad \lambda \in \rho(D),$$

$$S_2(\lambda) := D(\lambda) - C(\lambda)A(\lambda)^{-1}B(\lambda), \quad \lambda \in \rho(A),$$

of linear operators in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, with domains

$$\operatorname{dom} S_1(\lambda) := \left\{ f \in \mathcal{D}_1(\lambda) : D(\lambda)^{-1} C(\lambda) f \in \operatorname{dom} B(\lambda) \right\}, \quad \lambda \in \rho(D),$$
$$\operatorname{dom} S_2(\lambda) := \left\{ f \in \mathcal{D}_2(\lambda) : A(\lambda)^{-1} B(\lambda) f \in \operatorname{dom} C(\lambda) \right\}, \quad \lambda \in \rho(A).$$

The following inclusions between the numerical ranges and pseudo numerical ranges of the Schur complements S_1, S_2 and the quadratic numerical range and pseudo quadratic numerical range, respectively, of $\mathcal{L}$ hold.

Proposition 4.8. *The numerical ranges and pseudo numerical ranges of the Schur complements satisfy*

$$W(S_1) \cup W(S_2) \subseteq W^2(\mathcal{L}), \quad W_\Psi(S_1) \cup W_\Psi(S_2) \subseteq W^2_{\Psi,2}(\mathcal{L}) \subseteq W^2_\Psi(\mathcal{L}).$$

Proof. The first claim follows from (4.1) and the corresponding statement [24, Thm. 2.5.8] for unbounded operator matrices.

Using the first claim, the second claim can be proven in a similar way as the claim for the pseudo numerical range in Proposition 4.6 (ii). □

The following spectral enclosure properties of the block numerical range and pseudo block numerical range hold for operator matrix functions. They generalise results for the case of bounded operator values from [29], see also [26] for $n = 2$, as well as the results for the operator function case, i.e. $n = 1$, in Proposition 3.1.

Proposition 4.9. *Let $\mathcal{L}$ be a family of operator matrices. Then*

$$\sigma_{\mathrm{p}}(\mathcal{L}) \subseteq W^n(\mathcal{L}) \subseteq W^n_\Psi(\mathcal{L}).$$

Proof. The proof of the first inclusion is analogous to the bounded case, see [29, Thm. 2.14] or [26, Thm. 3.1] for $n = 2$; the second inclusion is obvious by definition. □

Theorem 4.10. *Let $\mathcal{L}$ be a family of operator matrices in $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$. For every $\varepsilon > 0$,*

$$\sigma_{\mathrm{ap},\varepsilon}(\mathcal{L}) \subseteq W^n_\varepsilon(\mathcal{L}), \qquad \left\| \mathcal{L}(\lambda)^{-1} \right\| \le \frac{1}{\varepsilon}, \quad \lambda \in \rho(\mathcal{L}) \setminus W^n_\varepsilon(\mathcal{L}), \tag{4.6}$$

and hence

$$\sigma_{\mathrm{ap}}(\mathcal{L}) \subseteq W^n_\Psi(\mathcal{L});$$

if, for all $\lambda \in \Omega$, $\sigma(\mathcal{L}(\lambda)) \subseteq \overline{W^n(\mathcal{L}(\lambda))}$, then

$$\sigma(\mathcal{L}) \subseteq W^n_{\Psi,0}(\mathcal{L}) \subseteq W^n_\Psi(\mathcal{L}).$$

Proof. First let $\lambda \in \sigma_{\mathrm{ap},\varepsilon}(\mathcal{L})$. Then there exists $f_\varepsilon \in \operatorname{dom} \mathcal{L}(\lambda)$, $\|f_\varepsilon\| = 1$, with $\|\mathcal{L}(\lambda) f_\varepsilon\| < \varepsilon$. The linear operator in $\mathcal{H}$ given by

$$\mathcal{B} f := \begin{cases} \mathcal{L}(\lambda) \mu f_\varepsilon & \text{if } f = \mu f_\varepsilon \in \operatorname{span} f_\varepsilon, \\ \quad 0 & \text{if } f \perp f_\varepsilon, \end{cases}$$

is bounded with $\|\mathcal{B}\| = \|\mathcal{L}(\lambda) f_\varepsilon\| < \varepsilon$ and $(\mathcal{L}(\lambda) – \mathcal{B}) f_\varepsilon = 0$, i.e. $\lambda \in \sigma_{\mathrm{p}}(\mathcal{L} – \mathcal{B})$. By Proposition 4.9 and since $\|\mathcal{B}\| < \varepsilon$, we conclude that $\lambda \in W^n(\mathcal{L} - \mathcal{B}) \subseteq W^n_\varepsilon(\mathcal{L})$, which proves the first claim.

The resolvent estimate in (4.6) follows from the first claim and from the definition of $\sigma_{\mathrm{ap},\varepsilon}(\mathcal{L})$, cf. the proof of Proposition 3.1.

Taking the intersection over all $\varepsilon > 0$ in the first claim, we obtain that $\sigma_{\mathrm{ap}}(\mathcal{L}) \subseteq W^n_\Psi(\mathcal{L})$.

Finally, the assumption that $\sigma(\mathcal{L}(\lambda)) \subseteq \overline{W^n(\mathcal{L}(\lambda))}$ for all $\lambda \in \Omega$ implies that $\sigma(\mathcal{L}) \subseteq W^n_{\Psi,0}(\mathcal{L})$, see Definition 4.3. Now the second inequality in the last claim follows from the inclusion $W^n_{\Psi,0}(\mathcal{L}) \subseteq W^n_\Psi(\mathcal{L})$ by Proposition 4.4. □

5. Spectral enclosures by pseudo numerical ranges of Schur complements

In this section we establish a new enclosure of the approximate point spectrum of an operator matrix family $\mathcal{L}$ by means of the pseudo numerical ranges of the associated Schur complements and hence, by Proposition 4.8, in $W^2_{\Psi,2}(\mathcal{L})$ and in the pseudo quadratic numerical range $W^2_\Psi(\mathcal{L})$. Compared to earlier work, we no longer need restrictive dominance assumptions.

Theorem 5.1. *Let $\mathcal{L}$ be a family of operator matrices as in* (4.5). *If $\lambda \in \sigma_{\mathrm{ap}}(\mathcal{L}) \setminus (\sigma(A) \cup \sigma(D))$ is such that one of the conditions*

(i) *$C(\lambda)$ is $A(\lambda)$-bounded and $B(\lambda)$ is $D(\lambda)$-bounded;*
(ii) *$A(\lambda)$ is $C(\lambda)$-bounded, $D(\lambda)$ is $B(\lambda)$-bounded and both $C(\lambda)$ and $B(\lambda)$ are boundedly invertible;*

is satisfied, then $\lambda \in \sigma_{\mathrm{ap}}(S_1) \cup \sigma_{\mathrm{ap}}(S_2)$. If for all $\lambda \in \rho(A) \cap \rho(D)$ one of the conditions (i) *or* (ii) *is satisfied, then*

$$\begin{aligned} \sigma_{\mathrm{ap}}(\mathcal{L}) \setminus (\sigma(A) \cup \sigma(D)) &\subseteq \sigma_{\mathrm{ap}}(S_1) \cup \sigma_{\mathrm{ap}}(S_2) \\ &\subseteq W_\Psi(S_1) \cup W_\Psi(S_2) \subseteq W^2_{\Psi,2}(\mathcal{L}) \subseteq W^2_\Psi(\mathcal{L}). \end{aligned} \tag{5.1}$$

Proof. Let $\lambda \in \sigma_{\mathrm{ap}}(\mathcal{L})$. Then there exists a sequence $\{(u_n, v_n)\}_n \subseteq \operatorname{dom} \mathcal{L}(\lambda)$ with $\|u_n\|^2 + \|v_n\|^2 = 1$, $n \in \mathbb{N}$, and

$$A(\lambda)u_n + B(\lambda)v_n =: h_n \to 0, \quad n \to \infty, \tag{5.2}$$

$$C(\lambda)u_n + D(\lambda)v_n =: k_n \to 0, \quad n \to \infty. \tag{5.3}$$

The normalisation implies that $\liminf_{n\to\infty} \|u_n\| > 0$ or $\liminf_{n\to\infty} \|v_n\| > 0$. Let $\liminf_{n\to\infty} \|u_n\| > 0$, without loss of generality $\inf_{n\in\mathbb{N}} \|u_n\| > 0$. We show that, if $\lambda \in \rho(D)$, then $\lambda \in \sigma_{\mathrm{ap}}(S_1)$; if $\liminf_{n\to\infty} \|v_n\| > 0$, an analogous proof yields that, if $\lambda \in \rho(A)$, then $\lambda \in \sigma_{\mathrm{ap}}(S_2)$.

First we assume that λ satisfies (i). Since $\lambda \in \rho(D)$, (5.3) implies that

$$v_n = D(\lambda)^{-1}k_n - D(\lambda)^{-1}C(\lambda)u_n, \quad n \in \mathbb{N}.$$

Inserting this into (5.2) and using $\operatorname{dom} D(\lambda) \subseteq \operatorname{dom} B(\lambda)$, we conclude that

$$S_1(\lambda)u_n + B(\lambda)D(\lambda)^{-1}k_n = h_n \to 0, \quad n \to \infty. \tag{5.4}$$

Due to (i) $B(\lambda)D(\lambda)^{-1}$ is bounded and hence $B(\lambda)D(\lambda)^{-1}k_n \to 0$, $n \to \infty$. Then (5.4) yields that $S_1(\lambda)u_n \to 0$, $n \to \infty$. Because $\inf_{n\in\mathbb{N}} \|u_n\| > 0$, we can set

$$f_n := \frac{u_n}{\|u_n\|} \in \mathcal{D}_1(\lambda) = \operatorname{dom} S_1(\lambda), \quad n \in \mathbb{N},$$

and obtain that $S_1(\lambda)f_n \to 0$ for $n \to \infty$, which proves $\lambda \in \sigma_{\mathrm{ap}}(S_1)$.

Now assume that λ satisfies (ii). Since $C(\lambda)$ is invertible, (5.3) shows that

$$u_n = C(\lambda)^{-1}k_n - C(\lambda)^{-1}D(\lambda)v_n =: C(\lambda)^{-1}k_n - w_n, \quad n \in \mathbb{N}, \tag{5.5}$$

where $w_n := C(\lambda)^{-1}D(\lambda)v_n \in \operatorname{dom} S_1(\lambda)$ for $n \in \mathbb{N}$ since

$$w_n \in \mathcal{D}_1(\lambda) = \operatorname{dom} C(\lambda), \quad D(\lambda)^{-1}C(\lambda)w_n = v_n \in \mathcal{D}_2(\lambda) = \operatorname{dom} B(\lambda).$$

Inserting (5.5) into (5.2) and using $\operatorname{dom} C(\lambda) \subseteq \operatorname{dom} A(\lambda)$, we obtain that

$$A(\lambda)C(\lambda)^{-1}k_n - S_1(\lambda)w_n = h_n \;\to\; 0, \quad n \to \infty. \tag{5.6}$$

Since $C(\lambda)^{-1}$ is bounded, it follows that $C(\lambda)^{-1}k_n \to 0$, $n \to \infty$. Thus $\inf_{n\in\mathbb{N}} \|u_n\| > 0$ and (5.5) show that we can assume without loss of generality that $\inf_{n\in\mathbb{N}} \|w_n\| > 0$. Let us set

$$g_n := \frac{w_n}{\|w_n\|} \in \operatorname{dom} S_1(\lambda), \quad n \in \mathbb{N}.$$

By (ii) $A(\lambda)C(\lambda)^{-1}$ is bounded and hence $A(\lambda)C(\lambda)^{-1}k_n \to 0$, $n \to \infty$. Now (5.6) shows that $S_1(\lambda)w_n \to 0$ and thus $S_1(\lambda)g_n \to 0$, $n \to \infty$, which proves $\lambda \in \sigma_{\mathrm{ap}}(S_1)$.

Finally, the first inclusion in (5.1) is obvious from what was already shown; the second inclusion in (5.1) follows from Proposition 3.1 and the last two inclusions from Proposition 4.8. □

Remark 5.2. If under the assumptions of Theorem 5.1, the Schur complements S_1 and S_2 satisfy the assumptions of Theorem 3.3 or 3.5 on every connected component of $\rho(D)$ and $\rho(A)$, respectively, then

$$\sigma_{\mathrm{ap}}(\mathcal{L}) \setminus (\sigma(A) \cup \sigma(D)) \subseteq \overline{W(S_1)} \cup \overline{W(S_2)} \subseteq \overline{W^2(\mathcal{L})},$$

see Proposition 4.8 for the second inclusion.

For operator matrix families $\mathcal{L}$ with symmetric or anti-symmetric corners, we now establish conditions ensuring that the approximate point spectrum of $\mathcal{L}$ is contained in the union of the approximate point spectrum of one Schur complement and the pseudo numerical range of the corresponding diagonal entry, i.e. S_1 and D or S_2 and A.

Theorem 5.3. *Let $\mathcal{L}$ be an operator matrix family as in* (4.5)*.*

(i) *If $\lambda \in \sigma_{\mathrm{ap}}(\mathcal{L}) \setminus \sigma(D)$ is such that $C(\lambda) \subseteq \pm B(\lambda)^*$, $A(\lambda)$ is accretive, $\mp D(\lambda)$ sectorial and $B(\lambda)$ is $D(\lambda)$-bounded, then $\lambda \in \sigma_{\mathrm{ap}}(S_1) \cup W_\Psi(D)$. If these conditions hold for all $\lambda \in \rho(D)$, then*

$$\sigma_{\mathrm{ap}}(\mathcal{L}) \setminus \sigma(D) \subseteq \sigma_{\mathrm{ap}}(S_1) \cup W_\Psi(D) \subseteq W_\Psi(S_1) \cup W_\Psi(D); \tag{5.7}$$

if $\dim \mathcal{H}_1 > 1$*, then*

$$\sigma_{\mathrm{ap}}(\mathcal{L}) \setminus \sigma(D) \subseteq W^2_{\Psi,2}(\mathcal{L}) \subseteq W^2_\Psi(\mathcal{L}). \tag{5.8}$$

(ii) *If $\lambda \in \sigma_{\mathrm{ap}}(\mathcal{L}) \setminus \sigma(A)$ is such that $C(\lambda) \subseteq \pm B(\lambda)^*$, $A(\lambda)$ is sectorial, $\mp D(\lambda)$ accretive and $C(\lambda)$ is $A(\lambda)$-bounded, then $\lambda \in \sigma_{\mathrm{ap}}(S_2) \cup W_\Psi(A)$. If these conditions hold for all $\lambda \in \rho(A)$, then*

$$\sigma_{\mathrm{ap}}(\mathcal{L}) \setminus \sigma(A) \subseteq \sigma_{\mathrm{ap}}(S_2) \cup W_\Psi(A) \subseteq W_\Psi(S_2) \cup W_\Psi(A);$$

if $\dim \mathcal{H}_2 > 1$, *then*

$$\sigma_{\mathrm{ap}}(\mathcal{L}) \backslash \sigma(A) \subseteq W^2_{\Psi,2}(\mathcal{L}) \subseteq W^2_{\Psi}(\mathcal{L}).$$

The following corollary is immediate from Theorem 5.3 due to Proposition 4.6 and Proposition 4.8.

Corollary 5.4. *Under the assumptions of Theorem* 5.3, *if in* (i) *additionally* $\sigma(D) \subseteq W_\Psi(D)$, *then*

$$\sigma_{\mathrm{ap}}(\mathcal{L}) \subseteq \sigma_{\mathrm{ap}}(S_1) \cup W_\Psi(D) \subseteq W_\Psi(S_1) \cup W_\Psi(D) \subseteq W^2_{\Psi,2}(\mathcal{L}) \subseteq W^2_{\Psi}(\mathcal{L}),$$

and if in (ii) *additionally* $\sigma(A) \subseteq W_\Psi(A)$, *then*

$$\sigma_{\mathrm{ap}}(\mathcal{L}) \subseteq \sigma_{\mathrm{ap}}(S_2) \cup W_\Psi(A) \subseteq W_\Psi(S_2) \cup W_\Psi(A) \subseteq W^2_{\Psi,2}(\mathcal{L}) \subseteq W^2_{\Psi}(\mathcal{L}).$$

Proof of Theorem 5.3. We only prove (i); the proof of (ii) is analogous. Let $\lambda \in \sigma_{\mathrm{ap}}(\mathcal{L}) \setminus \sigma(D)$. In the same way as at the beginning of the proof of Theorem 5.1 we conclude that if $\liminf_{n\to\infty} \|u_n\| > 0$, then $\lambda \in \sigma_{\mathrm{ap}}(S_1)$. It remains to be shown that in the case $\liminf_{n\to\infty} \|v_n\| > 0$, without loss of generality $\inf_{n\in\mathbb{N}} \|v_n\| > 0$, it follows that $\lambda \in W_\Psi(D)$.

Taking the scalar product with u_n in (5.2) and with v_n in (5.3), respectively, we conclude that

$$(A(\lambda)u_n, u_n) + (B(\lambda)v_n, u_n) = (h_n, u_n), \quad n \in \mathbb{N}, \tag{5.9}$$

$$\pm(u_n, B(\lambda)v_n) + (D(\lambda)v_n, v_n) = (k_n, v_n), \quad n \in \mathbb{N}. \tag{5.10}$$

By subtracting from (5.9), or adding to (5.9), the complex conjugate of (5.10), we deduce that

$$(A(\lambda)u_n, u_n) \mp \overline{(D(\lambda)v_n, v_n)} = (h_n, u_n) \mp \overline{(k_n, v_n)} \to 0, \quad n \to \infty.$$

Taking real parts and using the accretivity of $A(\lambda)$ and $\mp D(\lambda)$, we obtain

$$0 \le \mathrm{Re}(\mp D(\lambda)v_n, v_n) \le \mathrm{Re}(A(\lambda)u_n, u_n) \mp \mathrm{Re}(D(\lambda)v_n, v_n) \to 0, \quad n \to \infty.$$

Since $\mp D(\lambda)$ is sectorial by assumption, this implies $(\mp D(\lambda)v_n, v_n) \to 0$ and hence $(D(\lambda)v_n, v_n) \to 0$, $n \to \infty$, which proves that $\lambda \in W_\Psi(D)$ by Proposition 2.3.

Finally, the first inclusion in (5.7) is obvious from what was already proved; the second inclusion in (5.7) follows from Proposition 3.1. The last claim in (5.8) is then a consequence of Propositions 4.6 (iii) and 4.8. □

Remark 5.5. (i) Sufficient conditions for the inclusions $\sigma(A) \subseteq W_\Psi(A)$ or $\sigma(D) \subseteq W_\Psi(D)$, respectively, may be found e.g. in Theorem 3.3 or Proposition 3.1.

(ii) An analogue of Remark 5.2 also holds for Theorem 5.3; the details of all possible combinations of assumptions and corresponding inclusions are left to the reader.

6. Application to structured operator matrices

In this section, we apply the results of the previous section to prove new spectral enclosures and resolvent estimates for non-selfadjoint operator matrix functions exhibiting a certain dichotomy.

More precisely, we consider a linear monic family $\mathcal{L}(\lambda) = \mathcal{A} - \lambda I_{\mathcal{H}}$, $\lambda \in \mathbb{C}$, with a densely defined operator matrix

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \operatorname{dom} \mathcal{A} = \big(\operatorname{dom} A \cap \operatorname{dom} C\big) \oplus \big(\operatorname{dom} B \cap \operatorname{dom} D\big) \quad (6.1)$$

with $C \subseteq B^*$ in $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. We assume that the entries of $\mathcal{A}$ are densely defined closable linear operators acting between the respective spaces $\mathcal{H}_1$ and/or $\mathcal{H}_2$.

In the following, we denote the closed sector with semi-axis $\mathbb{R}_+$ and semi-angle $\omega \in [0, \pi/2]$ by

$$\Sigma_\omega := \{z \in \mathbb{C} : |\arg z| \leq \omega\}, \quad \omega \in [0, \pi/2];$$

here $\arg : \mathbb{C} \to (-\pi, \pi]$ is the argument of a complex number with $\arg 0 = 0$.

The next theorem no longer requires bounds on the dominance orders among the entries in the columns of $\mathcal{A}$, in contrast to earlier results in [25, Thm. 5.2] where the relative bounds had to be 0.

Theorem 6.1. *Let $\mathcal{A}$ be an operator matrix as in* (6.1) *with $C \subseteq B^*$. Assume that A and $-D$ are uniformly accretive, i.e. there exist α, $\delta \in \mathbb{R}$ and semi-angles $\varphi, \psi \in [0, \pi/2]$ with*

$$\operatorname{Re} W(D) \leq \delta < 0 < \alpha \leq \operatorname{Re} W(A), \quad W(A) \subseteq \Sigma_\varphi, \quad W(D) \subseteq -\Sigma_\psi. \quad (6.2)$$

Suppose further that one of the following holds:

(i) *A, $-D$ are m-accretive, C is A-bounded, B is D-bounded,*
(ii) *A, $-D$ are m-accretive, A is C-bounded, D is B-bounded and B, C are boundedly invertible,*
(iii) *$-D$ is m-sectorial, i.e. $\psi < \pi/2$, and B is D-bounded,*
(iv) *A is m-sectorial, i.e. $\varphi < \pi/2$, and C is A-bounded.*

Then, with $\tau := \max\{\varphi, \psi\}$,

$$\sigma_{\mathrm{ap}}(\mathcal{A}) \subseteq (-\Sigma_\tau \cup \Sigma_\tau) \cap \{z \in \mathbb{C} : \operatorname{Re} z \notin (\delta, \alpha)\} =: \Sigma; \quad (6.3)$$

if, in addition, $\rho(\mathcal{A}) \cap \Sigma^{\mathrm{c}} \neq \emptyset$, then $\sigma(\mathcal{A}) \subseteq \Sigma$.

The proof of Theorem 6.1 relies on Theorems 5.1 and 5.3, and on the following enclosures for the pseudo numerical ranges of the Schur complements.

Lemma 6.2. *Let $\mathcal{A}$ be as in* (6.1) *with $C \subseteq B^*$ and let $\lambda \in \mathbb{C}$.*

(i) *Suppose A, $-D$ are uniformly accretive,*

$$\operatorname{Re} W(D) \leq \delta < 0 < \alpha \leq \operatorname{Re} W(A). \quad (6.4)$$

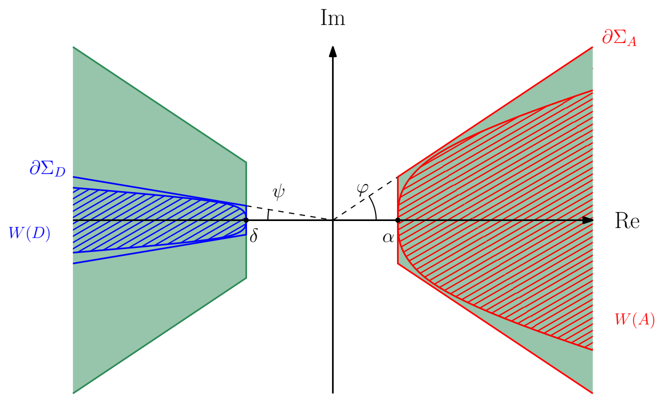


FIGURE 6.1. The set Σ (green) enclosing $\sigma_{\mathrm{ap}}(\mathcal{A})$, see (6.3); inside the sets $\Sigma_A := \Sigma_\varphi \setminus S$ (bounded by red line) enclosing $W(A)$ (red, dashed) and $\Sigma_D := -\Sigma_\psi \setminus S$ (bounded by blue line) enclosing $W(D)$ (blue, dashed), separated by $S := \{z \in \mathbb{C} : \operatorname{Re} z \in (\delta, \alpha)\}$, see (6.2).

If $\operatorname{Re}\lambda \in (\delta, \alpha)$, *then*

$$\lambda \in \rho(D) \implies \operatorname{Re}\overline{W(S_1(\lambda))} \geq \alpha - \operatorname{Re}\lambda > 0,$$
$$\lambda \in \rho(A) \implies \operatorname{Re}\overline{W(S_2(\lambda))} \leq \delta - \operatorname{Re}\lambda < 0.$$

(ii) *Suppose* $A, -D$ *are sectorial,*

$$W(A) \subseteq \Sigma_\varphi, \qquad W(D) \subseteq -\Sigma_\psi$$

with $\varphi, \psi \in [0, \pi/2)$ *and let* $\tau := \max\{\varphi, \psi\}$. *If* $\arg\lambda \in (\tau, \pi - \tau)$, *then*

$$\lambda \in \rho(D) \implies \arg(\overline{W(S_1(\lambda))} + \lambda) \in [-\arg\lambda, \tau],$$
$$\lambda \in \rho(A) \implies \arg(\overline{W(S_2(\lambda))} + \lambda) \in (-\pi, -\arg\lambda] \cup [\pi - \tau, \pi];$$

if $\arg\lambda \in (-\pi + \tau, -\tau)$, *then*

$$\lambda \in \rho(D) \implies \arg(\overline{W(S_1(\lambda))} + \lambda) \in [-\tau, -\arg\lambda],$$
$$\lambda \in \rho(A) \implies \arg(\overline{W(S_2(\lambda))} + \lambda) \in (-\pi, -\pi + \tau] \cup [-\arg\lambda, \pi].$$

Proof. We show the claims for S_1, the proofs for S_2 are analogous. It is easy to see that it suffices to prove the claimed non-strict inequalities for $W(S_1(\lambda))$. Let $\lambda \in \rho(D)$, $f \in \operatorname{dom} S_1(\lambda) \subseteq \operatorname{dom} A \cap \operatorname{dom} B^*$ with $\|f\| = 1$, and set $g := (D - \lambda)^{-1} B^* f$. Then

$$(S_1(\lambda)f, f) = (Af, f) - \lambda - \overline{(Dg, g)} + \overline{\lambda}\, \|g\|^2. \tag{6.5}$$

(i) If $\operatorname{Re}\lambda \in (\delta, \alpha)$, then (6.5) and (6.4) show that

$$\operatorname{Re}(S_1(\lambda)f, f) \geq \alpha - \operatorname{Re}\lambda + (-\delta + \operatorname{Re}\lambda)\, \|g\|^2 \geq \alpha - \operatorname{Re}\lambda > 0.$$

(ii) We consider $\arg\lambda \in (\tau, \pi - \tau)$, the case $\arg\lambda \in (-\pi + \tau, -\tau)$ can be shown analogously. By assumption, $|\arg(Af, f)| \leq \varphi \leq \tau$, $|\arg \overline{(-Dg, g)}| \leq \psi \leq \tau$. Together with $\arg(\overline{\lambda}\, \|g\|^2) = -\arg\lambda \in (-\pi + \tau, -\tau)$, it follows from (6.5) that

$$\arg\big((S_1(\lambda)f, f) + \lambda\big) = \arg\big((Af, f) + \overline{(-Dg, g)} + \overline{\lambda}\, \|g\|^2\big) \in [-\arg\lambda, \tau]. \qquad \square$$

Proof of Theorem 6.1. First we use Lemma 6.2 to show that if A or $-D$ are m-accretive, respectively, then

$$W_\Psi(S_2) \subseteq \Sigma \quad \text{or} \quad \mathrm{W}_\Psi(\mathrm{S}_1) \subseteq \Sigma. \tag{6.6}$$

We prove the claim for S_1 by taking complements; the proof for S_2 is analogous. To this end, let $\lambda \in \Sigma^{\mathrm{c}} \subseteq \rho(D)$. Then $\operatorname{Re}\lambda \in (\delta,\alpha)$ or $|\arg\lambda| \in (\tau, \pi-\tau)$; note that the latter case only occurs if both A and $-D$ are sectorial, i.e. if $\tau < \pi/2$. If $\operatorname{Re}\lambda \in (\delta,\alpha)$, Lemma 6.2 (i) implies $0 \notin \overline{W(S_1(\lambda))}$, i.e. $\lambda \notin W_\Psi(S_1)$ by (2.2). In the same way, if $|\arg\lambda| \in (\tau,\pi-\tau)$, then $\lambda \notin W_\Psi(S_1)$ follows from Lemma 6.2 (ii); indeed, otherwise we would have $0 \in \overline{W(S_1(\lambda))}$ and hence, e.g. if $\arg\lambda \in (\tau,\pi-\tau)$,

$$\arg(0+\lambda) = \arg\lambda \in [-\arg\lambda, \tau] \cap (\tau, \pi-\tau) = \emptyset,$$

and analogously for $\arg\lambda \in (-\pi+\tau, -\tau)$. This completes the proof of (6.6).

We show that assumptions (i) or (iii) imply (6.3); the proof when assumptions (ii) or (iv) hold is analogous.

Assume first that (i) holds and let $\lambda \in \sigma_{\mathrm{ap}}(\mathcal{A})$. If $\lambda \in \sigma(A) \cup \sigma(D) \subseteq \Sigma$, there is nothing to show. If $\lambda \notin \sigma(A) \cup \sigma(D)$, then Theorem 5.1 (i) shows that $\lambda \in W_\Psi(S_1) \cup W_\Psi(S_2)$ and we conclude $\lambda \in \Sigma$ from (6.6).

Now assume that (iii) is satisfied. Then $-D$ is m-sectorial and $\sigma(D) \subseteq \overline{W(D)} \subseteq \Sigma$. In order to prove (6.3), we show $\sigma_{\mathrm{ap}}(\mathcal{A}) \cap \Sigma^{\mathrm{c}} = \emptyset$. To this end, it suffices to prove that

$$\sigma_{\mathrm{ap}}(\mathcal{A}) \cap \Sigma^{\mathrm{c}} \subseteq W_\Psi(S_1) \cup W_\Psi(D - \cdot I_{\mathcal{H}_2}); \tag{6.7}$$

here, in the sequel, we write $D - \cdot I_{\mathcal{H}_2}$ for the operator family $D - \lambda I_{\mathcal{H}_2}$, $\lambda \in \mathbb{C}$. Indeed, if (6.7) holds, then $W_\Psi(D - \cdot I_{\mathcal{H}_2}) = \overline{W(D)} \subseteq \Sigma$ and (6.6) yield that $\sigma_{\mathrm{ap}}(\mathcal{A}) \cap \Sigma^{\mathrm{c}} \subseteq \Sigma$ and hence the claim.

For the proof of (6.7), we will use Theorem 5.3 (i). To this end, for $\lambda \in \Sigma^{\mathrm{c}}$, we define a rotation angle

$$\omega(\lambda) := \begin{cases} 0, & \operatorname{Re}\lambda \in (\delta,\alpha), \\ \operatorname{sgn}(\arg\lambda)\left|\frac{\pi}{2} - |\arg\lambda|\right|, & \operatorname{Re}\lambda \notin (\delta,\alpha) \wedge |\arg\lambda| \in (\tau, \pi-\tau); \end{cases}$$

note that the second case only occurs if A is sectorial, i.e. if $\tau < \pi/2$, and that then $\lambda \neq 0$ and $|\omega(\lambda)| \in (0, \pi/2 - \tau)$. It is easy to see that the rotated operator matrix family $\widetilde{\mathcal{L}}$ defined by

$$\widetilde{\mathcal{L}}(\lambda) := \operatorname{diag}\left(\mathrm{e}^{\mathrm{i}\omega(\lambda)}, \mathrm{e}^{-\mathrm{i}\omega(\lambda)}\right)(\mathcal{A} - \lambda\mathcal{I}_{\mathcal{H}}), \quad \operatorname{dom}\widetilde{\mathcal{L}}(\lambda) := \operatorname{dom}\mathcal{A}, \quad \lambda \in \Sigma^{\mathrm{c}},$$

satisfies $\sigma_{\mathrm{ap}}(\widetilde{\mathcal{L}}) = \sigma_{\mathrm{ap}}(\mathcal{A}) \cap \Sigma^{\mathrm{c}}$. It is not difficult to show that the angle $\omega(\lambda)$ is chosen such that $\mathrm{e}^{\mathrm{i}\omega(\lambda)}(A - \lambda I_{\mathcal{H}_1})$ is accretive, $-\mathrm{e}^{-\mathrm{i}\omega(\lambda)}(D - \lambda I_{\mathcal{H}_2})$ is sectorial and $\mathrm{e}^{-\mathrm{i}\omega(\lambda)}C \subseteq \mathrm{e}^{\mathrm{i}\omega(\lambda)}B^*$ for every $\lambda \in \Sigma^{\mathrm{c}}$. In fact, if $\operatorname{Re}\lambda \in (\delta,\alpha)$, the claim is obvious. If $\operatorname{Re}\lambda \notin (\delta,\alpha)$ and $|\arg\lambda| \in (\tau,\pi-\tau)$, elementary geometric considerations show that the sectoriality of A, $\operatorname{Re} W(A) \geq \alpha > 0$ and the choice of $\omega(\lambda)$ imply that $\mathrm{e}^{\mathrm{i}\omega(\lambda)}A$ is sectorial and, since $\operatorname{Re}(\mathrm{e}^{\mathrm{i}\omega(\lambda)}\lambda) \leq 0$, so is $\mathrm{e}^{\mathrm{i}\omega(\lambda)}(A - \lambda I_{\mathcal{H}_1})$. Because $-D$ is sectorial, $\operatorname{Re} W(-D) \geq \delta > 0$ and $\operatorname{Re}(\mathrm{e}^{-\mathrm{i}\omega(\lambda)}\lambda) \geq 0$, the proof for $-\mathrm{e}^{-\mathrm{i}\omega(\lambda)}(D - \lambda I_{\mathcal{H}_2})$ is analogous.

Therefore $\widetilde{\mathcal{L}}$ satisfies the assumptions of Theorem 5.3 (i) and, because $\sigma(\mathrm{e}^{-\mathrm{i}\omega}(D - \cdot I_{\mathcal{H}_2})) = \sigma(D) \cap \Sigma^{\mathrm{c}} = \emptyset$, (5.7) therein yields that

$$\sigma_{\mathrm{ap}}(\mathcal{A}) \cap \Sigma^{\mathrm{c}} = \sigma_{\mathrm{ap}}(\widetilde{\mathcal{L}}) \subseteq W_\Psi(\widetilde{S}_1) \cup W_\Psi(\mathrm{e}^{-\mathrm{i}\omega}(D - \cdot I_{\mathcal{H}_2})),$$

where $\widetilde{S}_1$ is the first Schur complement of $\widetilde{\mathcal{L}}$. Now the claim (6.7) follows from the above inclusion and from the fact that, since $\mathrm{e}^{\mathrm{i}\omega(\lambda)} \neq 0$,

$$0 \in \overline{W(\widetilde{S}_1(\lambda))} \iff 0 \in \overline{W(\mathrm{e}^{\mathrm{i}\omega(\lambda)}\, S_1(\lambda))} = \mathrm{e}^{\mathrm{i}\omega(\lambda)}\, \overline{W(S_1(\lambda))} \iff 0 \in \overline{W(S_1(\lambda))}$$

for $\lambda \in \Sigma^{\mathrm{c}}$, and analogously for the family $\mathrm{e}^{-\mathrm{i}\omega}(D - \cdot I_{\mathcal{H}_2})$. This completes the proof that (i) and (iii) imply (6.3).

Finally, if $\rho(\mathcal{A}) \cap \Sigma^{\mathrm{c}} \neq \emptyset$, then $\mathcal{A}$ is closed and $\sigma(\mathcal{A}) \subseteq \Sigma$ follows from $\sigma_{\mathrm{ap}}(\mathcal{A}) \subseteq \Sigma$, see (6.3), and from the stability of Fredholm index, see [15, Thm. IV.5.17]. □

In Proposition 6.5 below, we derive sufficient conditions for $\rho(\mathcal{A}) \cap \Sigma^{\mathrm{c}} \neq \emptyset$ in Theorem 6.1 for diagonally dominant and off-diagonally dominant operator matrices. For the latter, we use a result of [5], while for the former we employ the following lemma, inspired by an estimate in [15, Prob. V.3.31] for accretive operators.

Lemma 6.3. *Let T be an m-sectorial or m-accretive linear operator in $\mathcal{H}$, i.e. assume there exists $\omega \in [0, \pi/2]$ with $\sigma(T) \subseteq \overline{W(T)} \subseteq \Sigma_\omega$. Then*

$$\left\|T(T-\lambda)^{-1}\right\| \le \frac{1}{m_T(\arg\lambda)} := \begin{cases} \dfrac{1}{\sin(|\arg\lambda| - \omega)}, & |\arg\lambda| \in (\omega, \omega + \frac{\pi}{2}), \\ 1, & |\arg\lambda| \in [\omega + \frac{\pi}{2}, \pi], \end{cases} \quad \lambda \notin \Sigma_\omega.$$

Proof. Let $\lambda \notin \Sigma_\omega$ and $\varepsilon \in (0, |\lambda|)$ be arbitrary. Then $\lambda \in \rho(T)$, $-\varepsilon \in \rho(T)$, $\lambda \neq -\varepsilon$ and we can write

$$\begin{aligned} T(T-\lambda)^{-1} &= (T+\varepsilon)(T+\varepsilon - (\lambda+\varepsilon))^{-1} - \varepsilon(T-\lambda)^{-1}, \\ &= -(\lambda+\varepsilon)^{-1}\left((T+\varepsilon)^{-1} - (\lambda+\varepsilon)^{-1}\right)^{-1} - \varepsilon(T-\lambda)^{-1}. \end{aligned} \tag{6.8}$$

Since $\varepsilon > 0$, it is easy to see that $T+\varepsilon$ is m-accretive or m-sectorial with semi-angle ω and hence so is $(T+\varepsilon)^{-1}$, cf. [15, Prob. V.3.31] for the m-accretive case. Thus, by [15, Thm. V.3.2] and (6.8), we can estimate

$$\left\|T(T-\lambda)^{-1}\right\| \le \frac{|\lambda+\varepsilon|^{-1}}{\mathrm{dist}\left((\lambda+\varepsilon)^{-1}, \Sigma_\omega\right)} + \frac{\varepsilon}{\mathrm{dist}\left(\lambda, \Sigma_\omega\right)}.$$

The claim now follows by taking the limit $\varepsilon \to 0$ and using the estimate

$$\mathrm{dist}\left(\lambda^{-1}, \Sigma_\omega\right) \ge \begin{cases} \dfrac{\sin(|\arg\lambda| - \omega)}{|\lambda|}, & |\arg\lambda| \in \left(\omega, \omega + \frac{\pi}{2}\right), \\ \dfrac{1}{|\lambda|}, & |\arg\lambda| \in \left[\omega + \frac{\pi}{2}, \pi\right], \end{cases} \tag{6.9}$$

cf. [14, Thm. 2.2]. □

Remark 6.4. The inequality in Lemma 6.3 is optimal, equality is achieved e.g. for normal operators with spectrum on the boundary of Σ_ω.

Proposition 6.5. *Suppose that, under the assumptions of Theorem* 6.1, *we strengthen assumptions* (i) *and* (ii) *to*

(i′) *A, $-D$ are m-sectorial, C is A-bounded with relative bound δ_A and B is D-bounded with relative bound δ_D such that*

$$\delta_A\delta_D < \sin(\theta_0-\varphi)\sin(\theta_0+\psi) =: M_{\theta_0} \in (0,1]$$

where

$$\theta_0 := \begin{cases} \max\left\{\frac{\pi}{2}+\frac{\varphi-\psi}{2}, \tau\right\}, & \varphi\le\psi, \\ \min\left\{\frac{\pi}{2}+\frac{\varphi-\psi}{2}, \pi-\tau\right\}, & \psi<\varphi; \end{cases}$$

(ii′) *A, $-D$ are m-accretive, $C=B^*$, A is C-bounded with relative bound δ_C, D is B-bounded with relative bound δ_B with*

$$\delta_B\delta_C < 1,$$

B, C are boundedly invertible, and the relative boundedness constants a_C, $a_B\ge 0$, b_C, $b_B\ge 0$ in

$$\|Ax\|^2 \le a_C^2\|x\|^2 + b_C^2\|Cx\|^2, \quad x\in\operatorname{dom} C,$$
$$\|Dy\|^2 \le a_B^2\|y\|^2 + b_B^2\|By\|^2, \quad y\in\operatorname{dom} B,$$

satisfy

$$\sqrt{a_C^2\|B^{-1}\|^2+b_C^2}\sqrt{a_B^2\|B^{-1}\|^2+b_B^2} < 1.$$

Then $\rho(\mathcal{A})\cap\Sigma^{\mathrm{c}}\ne\emptyset$ and hence

$$\sigma(\mathcal{A}) \subseteq (-\Sigma_\tau\cup\Sigma_\tau)\cap\{z\in\mathbb{C} : \operatorname{Re} z\notin(\delta,\alpha)\} = \Sigma.$$

Proof. By Theorem 6.1, it suffices to show $\rho(\mathcal{A})\cap\Sigma^{\mathrm{c}}\ne\emptyset$.

Suppose that (i′) holds and let $\lambda = r\,\mathrm{e}^{\mathrm{i}\theta}$ with $r>0$ and $\theta\in(\tau,\pi-\tau)$ to be chosen later. Then $\lambda\in\rho(A)\cap\rho(D)$. Since $\frac{1}{M_{\theta_0}}\delta_A\delta_D<1$, there exists $\varepsilon>0$ so that

$$\frac{1}{M_{\theta_0}-\varepsilon}(\delta_A+\varepsilon)(\delta_D+\varepsilon) < 1. \tag{6.10}$$

Due to the relative boundedness assumption on C, there exist a_A, $b_A>0$, $b_A\in[\delta_A,\delta_A+\varepsilon)$ such that

$$\left\|C(A-\lambda)^{-1}\right\| \le a_A\left\|(A-\lambda)^{-1}\right\| + b_A\left\|A(A-\lambda)^{-1}\right\|. \tag{6.11}$$

Since A is m-sectorial with semi-angle φ, we have the estimate

$$\left\|(A-\lambda)^{-1}\right\| \le \frac{1}{\operatorname{dist}(\lambda, W(A))} \le \frac{1}{r m_A(\theta)}, \tag{6.12}$$

with $m_A(\theta)$ defined as in Lemma 6.3, see [15, Thm. V.3.2] or (6.9). Consequently, by (6.11), (6.12) and Lemma 6.3, we obtain

$$\left\|C(A-\lambda)^{-1}\right\| \le \frac{a_A}{r m_A(\theta)} + \frac{b_A}{m_A(\theta)}.$$

Similarly, since $-D$ is m-sectorial with semi-angle ψ and using Lemma 6.3 as well as (6.9) and $|\arg(-\lambda)| = \pi - \theta$, we conclude that there exist a_D, $b_D > 0$, $b_D \in [\delta_D, \delta_D + \varepsilon)$ with

$$\left\|B(D-\lambda)^{-1}\right\| \le \frac{a_D}{rm_{-D}(\pi-\theta)} + \frac{b_D}{m_{-D}(\pi-\theta)}$$

with $m_{-D}(\pi - \theta)$ defined as in Lemma 6.3 and hence

$$\|C(A-\lambda)^{-1}B(D-\lambda)^{-1}\| \le \frac{b_A b_D}{M_\theta}\Big(\frac{a_A}{rb_A}+1\Big)\Big(\frac{a_D}{rb_D}+1\Big). \tag{6.13}$$

Here the function

$$[\varphi, \pi-\psi] \to [0,1], \quad \theta \mapsto M_\theta := m_A(\theta) m_{-D}(\pi-\theta),$$

is continuous, monotonically increasing for $\theta \le \widetilde{\theta}_0 := \frac{\pi}{2} + \frac{\varphi-\psi}{2} \in [\varphi, \pi - \psi]$ and decreasing for $\theta \ge \widetilde{\theta}_0$. Hence, the restriction of $\theta \mapsto M_\theta$ to $[\tau, \pi - \tau]$ attains its maximum at θ_0 and we can choose $\delta > 0$ such that $M_{\theta_0} - \varepsilon < M_\theta$ for $\theta \in (\theta_0 - \delta, \theta_0 + \delta) \cap (\tau, \pi - \tau)$. Now we fix such a θ. Using (6.13) and (6.10), we conclude that there exists $r > 0$ so large that

$$\|C(A-\lambda)^{-1}B(D-\lambda)^{-1}\| \le \frac{(\delta_A+\varepsilon)(\delta_D+\varepsilon)}{M_{\theta_0}-\varepsilon}\Big(\frac{a_A}{rb_A}+1\Big)\Big(\frac{a_D}{rb_D}+1\Big) < 1.$$

This implies $1 \in \rho(C(A-\lambda)^{-1}B(D-\lambda)^{-1})$ and hence $\lambda \in \rho(\mathcal{A})$ by [24, Cor. 2.3.5].

Suppose that (ii′) is satisfied. By the assumptions on B, C, the operator $\mathcal{S} := \mathcal{S}_1$ is selfadjoint and has a spectral gap $(-\|B^{-1}\|^{-1}, \|B^{-1}\|^{-1})$ around 0. Then [5, Thm. 4.7] with $\beta_T = 1/\left\|B^{-1}\right\|$ therein implies that $\mathrm{i}\mathbb{R} \subseteq \rho(\mathcal{A})$. □

7. Application to damped wave equations in $\mathbb{R}^d$ with unbounded damping

In this section we use the results obtained in Section 3 to derive new spectral enclosures for linearly damped wave equations with non-negative possibly singular and/or unbounded damping a and potential q.

Our result covers a new class of unbounded dampings which are p-subordinate to $-\Delta + q$, a notion going back to [16, §I.7.1], [18, §5.1], cf. [27, Sect. 3].

Theorem 7.1. *Let* $\mathbf{t}$ *be a quadratic pencil of sesquilinear forms given by*

$$\mathbf{t}(\lambda) := \mathbf{t}_0 + 2\lambda\mathbf{a} + \lambda^2, \quad \operatorname{dom}\mathbf{t}(\lambda) := \operatorname{dom}\mathbf{t}_0, \quad \lambda \in \mathbb{C},$$

where $\mathbf{t}_0$ *and* $\mathbf{a}$ *are densely defined sesquilinear forms in* $\mathcal{H}$ *such that* $\mathbf{t}_0$ *is closed,* $\mathbf{t}_0 \ge \kappa_0 \ge 0$, $\mathbf{a} \ge \alpha_0 \ge 0$ *and* $\operatorname{dom}\mathbf{t}_0 \subseteq \operatorname{dom}\mathbf{a}$. *Suppose that there exist* $\kappa \le \kappa_0$ *and* $p \in [0, 1)$ *such that* $\mathbf{a}$ *is* p*-form-subordinate with respect to* $\mathbf{t}_0 - \kappa \ge 0$, *i.e. there is* $C_p > 0$ *with*

$$\mathbf{a}[f] \le C_p\big((\mathbf{t}_0-\kappa)[f]\big)^p\big(\,\|f\|^2\,\big)^{1-p}, \quad f \in \operatorname{dom}\mathbf{t}_0. \tag{7.1}$$

Then the family $\mathbf{t}$ *is holomorphic of type* (a)*. If* T *denotes the associated holomorphic family of type* (B)*, then*

$$\sigma(T) \subseteq W_\Psi(T) \subseteq \{z \in \mathbb{C} \colon \operatorname{Re} z \le 0\}$$

and the following more precise spectral enclosures hold:

(i) *The non-real spectrum of* T *is contained in*

$$\sigma(T) \setminus \mathbb{R} \subseteq W_\Psi(T) \setminus \mathbb{R} \subseteq \Big\{ z \in \mathbb{C} \colon \operatorname{Re} z \le -\alpha_0,\ |z| \ge \sqrt{\kappa_0}, \\ |\operatorname{Im} z| \ge \sqrt{\max\Big\{0, C_p^{-\frac{1}{p}} |\operatorname{Re} z|^{\frac{1}{p}} - |\operatorname{Re} z|^2 + \kappa\Big\}} \Big\};$$

(ii) *if* $p < \frac{1}{2}$ *or if* $p = \frac{1}{2}$ *and* $C_{\frac{1}{2}} < 1$ *or if* $p = \frac{1}{2}$ *and* $C_{\frac{1}{2}} = 1$ *and* $\kappa > 0$*, the real spectrum of* T *satisfies either*

$$\sigma(T) \cap \mathbb{R} = \emptyset \quad \textit{or} \quad \sigma(T) \cap \mathbb{R} \subseteq [s^-, s^+],$$

if $p > \frac{1}{2}$ *or if* $p = \frac{1}{2}$ *and* $C_{\frac{1}{2}} > 1$ *or if* $p = \frac{1}{2}$ *and* $C_{\frac{1}{2}} = 1$ *and* $\kappa \le 0$*, the real spectrum of* T *satisfies either*

$$\sigma(T) \cap \mathbb{R} \subseteq (-\infty, r^+] \cup [s^-, s^+] \quad \textit{or} \quad \sigma(T) \cap \mathbb{R} \subseteq (-\infty, s^+],$$

where $\infty < r^+ < s^- \le s^+ \le 0$ *depend on* p*,* C_p*,* κ_0 *and* κ*.*

(iii) *if* $\kappa = 0$ *and* $p < \frac{1}{2}$*, then*

$$\begin{aligned} &\sigma(T) \cap \mathbb{R} = \emptyset && \textit{if } (C_p^2)^{\frac{1}{1-2p}} < \kappa_0, \\ &\sigma(T) \cap \mathbb{R} \subseteq \Big[-C_p t_0^p - \sqrt{C_p^p t_0^{2p} - t_0}, -C_p \kappa_0^p + \sqrt{C_p^2 \kappa_0^{2p} - \kappa_0} \Big) \Big] \\ & && \textit{if } (C_p^2)^{\frac{1}{1-2p}} \ge \kappa_0, \end{aligned}$$

where $t_0 := \max\big\{ \big(4C_p^2 p(1-p)\big)^{-\frac{1}{2p-1}}, \kappa_0 \big\}$*;*

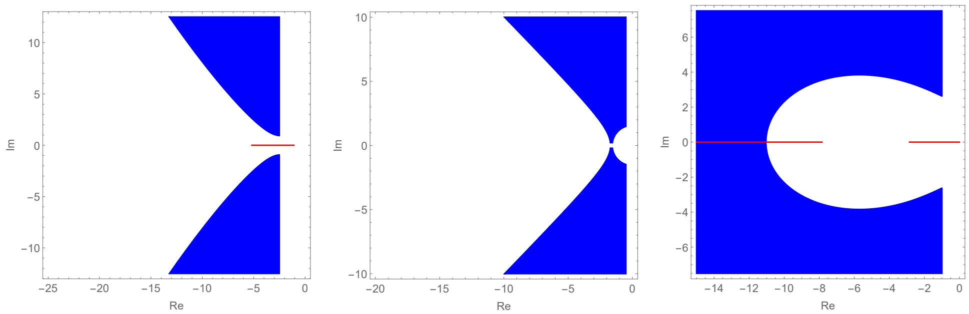
(iv) *if* $\kappa = 0$ *and* $p = \frac{1}{2}$*, then*

$$\begin{aligned} &\sigma(T) \cap \mathbb{R} = \emptyset && \textit{if } C_{\frac{1}{2}} < 1 \textit{ and } \kappa_0 > 0, \\ &\sigma(T) \cap \mathbb{R} \subseteq \{0\} && \textit{if } C_{\frac{1}{2}} < 1 \textit{ and } \kappa_0 = 0, \\ &\sigma(T) \cap \mathbb{R} \subseteq \Big(-\infty, -\Big(C_{\frac{1}{2}} - \sqrt{C_{\frac{1}{2}}^2 - 1} \Big) \kappa_0^{\frac{1}{2}} \Big] && \textit{if } C_{\frac{1}{2}} \ge 1; \end{aligned}$$

(v) *if* $\kappa = 0$ *and* $p > \frac{1}{2}$*, then*

$$\begin{aligned} &\sigma(T) \cap \mathbb{R} \subseteq \Big(-\infty, -C_p t_0^p + \sqrt{C_p^2 t_0^{2p} - t_0} \Big] && \textit{if } \kappa_0 > 0, \\ &\sigma(T) \cap \mathbb{R} \subseteq \Big(-\infty, -C_p t_0^p + \sqrt{C_p^2 t_0^{2p} - t_0} \Big] \cup \{0\} && \textit{if } \kappa_0 = 0, \end{aligned}$$

where $t_0 := \max\big\{ \big(4C_p^2 p(1-p)\big)^{-\frac{1}{2p-1}}, \kappa_0 \big\}$*.*



(a) $p=0.4, C_p=1.3, \kappa=-2,$ $\alpha_0=2.5, \kappa_0=5$

(b) $p=0.5, C_p=0.7, \kappa=3,$ $\alpha_0=0.5, \kappa_0=6$

(c) $p=0.65, C_p=0.5, \kappa=-5,$ $\alpha_0=1, \kappa_0=0$

FIGURE 7.2. Enclosures for $\sigma(T)\backslash\mathbb{R}$ in Theorem 7.1 (i) (blue) and for $\sigma(T)\cap\mathbb{R}$ in Theorem 7.1 (ii)-(v) (red in (a), (c), empty in (b)).

Proof of Theorem 7.1. Clearly, $\mathbf{t}$ is holomorphic. For arbitrary $\varepsilon>0$, applying Young's inequality to (7.1), we obtain

$$\begin{aligned}\mathbf{a}[f] &\leq \left(\frac{\varepsilon}{p}\right)^{p}\big((\mathbf{t}_0-\kappa)[f]\big)^{p}\left(\frac{p}{\varepsilon}\right)^{p} C_p\big(\,\|f\|^2\,\big)^{1-p}\\ &\leq \varepsilon\big((\mathbf{t}_0-\kappa)[f]\big)+(1-p)\left(\frac{p}{\varepsilon}\right)^{\frac{p}{1-p}} C_p^{\frac{1}{1-p}}\,\|f\|^2\end{aligned}$$

for all $f\in\operatorname{dom}\mathbf{t}_0$, i.e. $\mathbf{a}$ is $\mathbf{t}_0$-bounded with relative bound 0. Hence, for each $\lambda\in\mathbb{C}$, the form $\mathbf{t}(\lambda)$ is densely defined, sectorial and closed, see e.g. [15, Thm. VI.1.33]. This shows that $\mathbf{t}$ is a holomorphic family of type (a). Since all enclosing sets in Theorem 7.1 are closed and

$$\sigma(T)\subseteq W_\Psi(T)=W_\Psi(\mathbf{t})=\overline{W(\mathbf{t})}$$

by Theorem 3.3 with $k=2$ and $\mu\in\mathbb{C}$ arbitrary, it suffices to show that $W(\mathbf{t})\setminus\mathbb{R}$ and $W(\mathbf{t})\cap\mathbb{R}$ satisfy the claimed enclosures.

Let $\lambda_0\in W(\mathbf{t})$, i.e. there exists $f\in\operatorname{dom}\mathbf{t}_0$, $\|f\|=1$, with $\mathbf{t}(\lambda_0)[f]=0$. Taking real and imaginary part in this equation, we conclude that

$$\mathbf{t}_0[f]+2\operatorname{Re}\lambda_0\,\mathbf{a}[f]+(\operatorname{Re}\lambda_0)^2-(\operatorname{Im}\lambda_0)^2=0, \tag{7.2}$$

$$2\operatorname{Im}\lambda_0\,\mathbf{a}[f]+2\operatorname{Re}\lambda_0\operatorname{Im}\lambda_0=0. \tag{7.3}$$

First assume that $\lambda_0\in W(\mathbf{t})\setminus\mathbb{R}$. Then $\mathbf{a}[f]^2-\mathbf{t}_0[f]<0$. Dividing (7.3) by $2\operatorname{Im}\lambda_0$ ($\neq 0$) and inserting this into (7.2), we find

$$\begin{aligned}&\operatorname{Re}\lambda_0=-\mathbf{a}[f]\leq-\alpha_0\leq 0,\\ &|\lambda_0|^2=(\operatorname{Im}\lambda_0)^2+(\operatorname{Re}\lambda_0)^2=\mathbf{t}_0[f]\geq\kappa_0.\end{aligned}$$

Using these relations and assumption (7.1), we can further estimate

$$(\operatorname{Im}\lambda_0)^2=\mathbf{t}_0[f]-|\operatorname{Re}\lambda_0|^2\geq\max\{0,C_p^{-\frac{1}{p}}\,|\operatorname{Re}\lambda_0|^{\frac{1}{p}}-|\operatorname{Re}\lambda_0|^2+\kappa\},$$

and hence $\lambda_0\in W(\mathbf{t})\backslash\mathbb{R}$ satisfies all three claimed inequalities in (i).

Now assume that $\lambda_0 \in W(\mathbf{t}) \cap \mathbb{R}$. Then $\mathbf{a}[f]^2 - \mathbf{t}_0[f] \geq 0$ and thus, in particular, $\mathbf{a}[f] \geq \max\{\alpha_0, \sqrt{\kappa_0}\}$. Moreover, since $\operatorname{Im}\lambda_0 = 0$, equality (7.3) trivially holds and (7.2) implies $\lambda_0 = -\mathbf{a}[f] \pm \sqrt{\mathbf{a}[f]^2 - \mathbf{t}_0[f]} \leq 0$ because $\mathbf{t}_0 \geq 0$. This, together with $\mathbf{a} \geq \alpha_0$ and assumption (7.1), yields that

$$\max\left\{\alpha_0^2, \kappa_0\right\} \leq \max\{\alpha_0^2, \mathbf{t}_0[f]\} \leq \mathbf{a}[f]^2 \leq C_p^2\big((\mathbf{t}_0 - \kappa)[f]\big)^{2p}. \tag{7.4}$$

If we define

$$d(x) := C_p^{-\frac{1}{p}} x^{\frac{1}{2p}} - x + \kappa, \quad x \in [0,\infty), \quad D_{\leq 0} := \left\{x \in [\kappa_0, \infty) : d(x) \leq 0\right\},$$

then it is easy to see that

$$\mathbf{t}_0[f] \in D_{\leq 0}, \quad \lambda_0 = -\mathbf{a}[f] \pm \sqrt{\mathbf{a}[f]^2 - \mathbf{t}_0[f]}; \tag{7.5}$$

in particular, $D_{\leq 0} = \emptyset$ implies $W(\mathbf{t}) \cap \mathbb{R} = \emptyset$. An elementary analysis shows that d has either no zero, one simple zero or two (possibly coinciding) zeros on $[0,\infty)$, which we denote by x_+ and $x_- \leq x_+$, respectively, if they exist. Then

$$p < \frac{1}{2} \text{ or } p = \frac{1}{2}, C_{\frac{1}{2}} < 1 \text{ or } p = \frac{1}{2}, C_{\frac{1}{2}} = 1, \kappa > 0 \tag{7.6}$$

$$\Longrightarrow\ D_{\leq 0} = \emptyset \text{ or } D_{\leq 0} \text{ is bounded, } D_{\leq 0} = [\kappa_0, x_+] \text{ or } D_{\leq 0} = [x_-, x_+],$$

$$p > \frac{1}{2} \text{ or } p = \frac{1}{2}, C_{\frac{1}{2}} > 1 \text{ or } p = \frac{1}{2}, C_{\frac{1}{2}} = 1, \kappa \leq 0 \tag{7.7}$$

$$\Longrightarrow\ D_{\leq 0} \neq \emptyset \text{ is unbounded, } D_{\leq 0} = [\kappa_0, \infty) \text{ or } D_{\leq 0} = [x_+, \infty)$$
$$\text{or } D_{\leq 0} = [\kappa_0, x_-] \cup [x_+, \infty).$$

Which case prevails for fixed $p \in [0,1)$ can be characterised by means of inequalities involving the constants κ_0, κ and C_p. For estimating λ_0 in (7.5) while respecting the restrictions in (7.4), we consider the functions

$$f_\pm(s,t) := -s \pm \sqrt{s^2 - t}, \quad s \in [\alpha_0, \infty),\ t \in [\kappa_0, \infty),\ t \leq s^2 \leq C_p^2 (t-\kappa)^{2p}.$$

It is easy to check that f_+ is monotonically increasing in s and monotonically decreasing in t, while f_- is monotonically decreasing in s and monotonically increasing in t and hence, since $s \leq C_p(t-\kappa)^p$,

$$\begin{aligned} f_+(s,t) &\leq f_+(C_p(t-\kappa)^p, t) =: g_+(t), \\ f_-(s,t) &\geq f_-(C_p(t-\kappa)^p, t) =: g_-(t). \end{aligned} \tag{7.8}$$

We distinguish the two qualitatively different cases (7.6) and (7.7) and use (7.4), (7.5) and (7.8) to obtain enclosures for $W(\mathbf{t}) \cap \mathbb{R}$.

If (7.6) holds, then there are the following two possibilities:

(1) If d has no zeros on $[0,\infty)$ or if d has at least one zero and $x_+ < \kappa_0$, then $D_{\leq 0} = \emptyset$ and thus

$$W(\mathbf{t}) \cap \mathbb{R} = \emptyset.$$

(2) If d has at least one zero x_+ and $x_+ \geq \kappa_0$, then $D_{\leq 0}$ is one bounded interval and

$$W(\mathbf{t}) \cap \mathbb{R} \subseteq \left[s^-, s^+\right], \quad s^- := \min_{t \in D_{\leq 0}} g_-(t), \quad s^+ := \max_{t \in D_{\leq 0}} g_+(t);$$

here if d has only one zero x_+ or if d has two zeros $x_\pm$ and $x_- < \kappa_0$, then $D_{\leq 0} = [\kappa_0, x_+]$ and if d has two zeros and $x_- \geq \kappa_0$, then $D_{\leq 0} = [x_-, x_+]$.

If (7.7) holds, there are the following two possibilities:

(3) If d has two zeros $x_\pm$ on $[0, \infty)$ and $x_- \geq \kappa_0$, then $D_{\leq 0} = [\kappa_0, x_-] \cup [x_+, \infty)$ and we obtain

$$W(\mathbf{t}) \cap \mathbb{R} \subseteq \left(-\infty, r^+\right] \cup \left[s^-, s^+\right], \quad r^+ := \max_{t \in [x_+, \infty)} g_+(t), \quad s^+ := \max_{t \in [\kappa_0, x_-]} g_+(t),$$
$$s^- := \min_{t \in [\kappa_0, x_-]} g_-(t);$$

here g_+ attains a maximum on $[x_+, \infty)$ since $g_+(t)$ tends to $-\infty$ as $t \to \infty$, and analogously in the next case.

(4) If d has either at most one zero x_+ or two zeros $x_\pm$ on $[0, \infty)$ and $x_- < \kappa_0$, then $D_{\leq 0} = [\max\{\kappa_0, x_+\}, \infty)$ and we conclude that

$$W(\mathbf{t}) \cap \mathbb{R} \subseteq \left(-\infty, s^+\right], \quad s^+ := \max_{t \in [\max\{\kappa_0, x_+\}, \infty)} g_+(t).$$

This proves claim (ii).

Claim (iv) for $\kappa = 0$ and $p = \frac{1}{2}$ follows from cases (1), (2) and (4) above if we note that then $d(x) = (C_{\frac{1}{2}}^{-2} - 1)x$, $x \in [0, \infty)$, is either identically zero or has the only zero $x_+ = 0$ and, for case (4), $g_+(t) = -t^{\frac{1}{2}}\left(C_{\frac{1}{2}} + \sqrt{C_{\frac{1}{2}}^2 - 1}\right)$ is montonically decreasing so that $s^+ = g_+(\kappa_0)$.

Finally, if $\kappa = 0$ and $p \neq \frac{1}{2}$, the function d has the two zeros $x_- = 0$ and $x_+ = (C_p^2)^{\frac{1}{1-2p}}$ on $[0, \infty)$, and the respective bounds r^+, $s^\pm$ above can be determined explicitly to deduce claims (iii) and (v). More precisely, claim (iii) follows from cases (1) and (2) if we note that, in (2), $D_{\leq 0} = [\kappa_0, x_+]$, g_+ is monotonically decreasing on $[0, x_+]$ and g_- attains its minimum on $[0, x_+]$ at $t = \left(4C_p^2 p(1-p)\right)^{-\frac{1}{2p-1}}$. Claim (v) follows from cases (4) if $\kappa_0 > 0$ and (3) if $\kappa_0 = 0$; note that, for $\kappa = 0$, case (3) where $p > 1/2$ can only occur if $\kappa_0 = 0$. In both cases, we use that g_+ attains its maximum on $[x_+, \infty)$ at $t = \left(4C_p^2 p(1-p)\right)^{-\frac{1}{2p-1}}$. □

Remark 7.2. If (7.1) holds with $\kappa \leq \kappa_0$ and $p \in [0, 1)$, then it holds for every $q \in (p, 1)$ with $\kappa_1 \leq \kappa$ such that $\kappa_1 < \kappa_0$.

Indeed, then $\mathbf{t}_0 - \kappa \leq \mathbf{t}_0 - \kappa_1$ and $\mathbf{t}_0 - \kappa_1 \geq \kappa_0 - \kappa_1 > 0$ which implies that $(\|f\|^2)^{q-p} \leq (\kappa_0 - \kappa_1)^{p-q}\left((\mathbf{t}_0 - \kappa_1)[f]\right)^{q-p}$, $f \in \operatorname{dom} \mathbf{t}_0$. Hence (7.1) holds with q, κ_1 and $C_q = C_p(\kappa_0 - \kappa_1)^{p-q}$.

Remark 7.3. As a special case of Theorem 7.1 we obtain the enclosure for the non-real spectrum proved in [13, Thm. 3.2, Part 5] (where the damping was only assumed to be accretive) and we considerably improve the enclosure for the real spectrum therein since we obtain that the latter is, in fact, empty. The assumption in [13, Thm. 3.2, Part 5] is that

$$\nu := \sup_{f\in \operatorname{dom} \mathbf{t}_0\setminus\{0\}} \frac{2\mathbf{a}[f]}{\mathbf{t}_0[f]^{1/2}\|f\|} \in (0,2). \tag{7.9}$$

The parameters a_0, β and ν in [13, (5) and p. 83] correspond to the following special choices in Theorem 7.1 and assumption (7.1):

$$p = \frac{1}{2}, \quad C_{\frac{1}{2}} = \frac{\nu}{2}, \quad \kappa = 0, \quad \kappa_0 = a_0^2 > 0, \quad \alpha_0 = \frac{\beta}{2}.$$

Under the assumption (7.9) made in [13, Thm. 3.2, Part 5], Theorem 7.1 (i) yields the spectral enclosure

$$\sigma(T)\setminus\mathbb{R} \subseteq \left\{ z\in\mathbb{C} : \operatorname{Re} z \le -\frac{\beta}{2},\ |z|\ge a_0,\ |\operatorname{Im} z| \ge \sqrt{\frac{4}{\nu^2}-1}\ |\operatorname{Re} z| \right\}.$$

This enclosure is the same as in [13, Thm. 3.2, Part 5]. However, since $\nu<2$ is equivalent to $C_{\frac{1}{2}}<1$, the enclosure $\sigma(T)\cap\mathbb{R}\subseteq(-\infty, -\frac{a_0}{\nu}-\frac{4a_0}{\nu^3}]$ in [13, Thm. 3.2, Part 5] is considerably improved by Theorem 7.1 (iv) to

$$\sigma(T)\cap\mathbb{R} = \emptyset.$$

Remark 7.4. In the second case in Theorem 7.1 (ii), i.e. if $p>\frac{1}{2}$ or $p=\frac{1}{2}$, $C_{\frac{1}{2}}>1$ or $p=\frac{1}{2}$, $C_{\frac{1}{2}}=1$, $\kappa\le 0$, the set $W(\mathbf{t})\cap(-\infty,0]$ used to enclose the spectrum can, indeed, be unbounded if so is $\mathbf{t}_0$.

In fact, if $W(\mathbf{t}_0)=[\kappa_0,\infty)$, we can choose $\mathbf{a}=C_p(\mathbf{t}_0-\kappa)^p$. Then there exist $f_n\in\operatorname{dom}\mathbf{t}_0$, $\|f_n\|=1$, with $\mathbf{t}_0[f_n]\ge n$ for $n\in\mathbb{N}$. The conditions on p, C_p and κ ensure, comp. (7.7), that $C_p^2(\mathbf{t}_0[f_n]-\kappa)^{2p}-\mathbf{t}_0[f_n]\ge 0$ for sufficiently large $n\in\mathbb{N}$ and thus

$$W(\mathbf{t})\cap(-\infty,0] \ni \lambda_0 = -\mathbf{t}_0[f_n]^p - \sqrt{\mathbf{t}_0[f_n]^{2p}-\mathbf{t}_0[f_n]} \le -\mathbf{t}_0[f_n]^p \le -n^p \to -\infty,$$

and hence $\inf\left(W(\mathbf{t})\cap(-\infty,0]\right) = -\infty$.

In the next example we apply Theorem 7.1 to linearly damped wave equations with possibly unbounded and/or singular damping.

Example 7.5. Let $\mathcal{H}=L^2(\mathbb{R}^d)$ with $d\ge 3$ and a, $q\in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, $a\ne 0$ and $a,q\ge 0$ almost everywhere. If $\operatorname{dom} a^{\frac{1}{2}}$ and $\operatorname{dom} q^{\frac{1}{2}}$ denote the maximal domains of the multiplication operators $a^{\frac{1}{2}}$ and $q^{\frac{1}{2}}$ in $L^2(\mathbb{R}^d)$, respectively, we define the quadratic forms $\mathbf{a}$ and $\mathbf{t}_0$ in $L^2(\mathbb{R}^d)$ by

$$\mathbf{a}[f] := \int_{\mathbb{R}^d} a\,|f|^2\,\mathrm{d}x, \qquad \operatorname{dom}\mathbf{a} := \operatorname{dom} a^{\frac{1}{2}},$$

$$\mathbf{t}_0[f] := \int_{\mathbb{R}^d} |\nabla f|^2\,\mathrm{d}x + \int_{\mathbb{R}^d} q\,|f|^2\,\mathrm{d}x, \qquad \operatorname{dom}\mathbf{t}_0 := H^1(\mathbb{R}^d)\cap\operatorname{dom} q^{\frac{1}{2}}.$$

Suppose that, for almost all $x\in\mathbb{R}^d$,

$$a(x) \le \sum_{j=1}^n |x-x_j|^{-t} + u(x) + v(x), \qquad v(x)\le c_1 q(x)^r + c_2, \tag{7.10}$$

where $u \in L^s(\mathbb{R}^d)$ with $s > d/2$, $v \in L^1_{\rm loc}(\mathbb{R}^d)$, $t \in [0,2)$, $n \in \mathbb{N}_0$, $x_j \in \mathbb{R}^d$ for $j = 1, \dots, n$, c_1, $c_2 \geq 0$ and $r \in [0,1)$. Then $\mathbf{a}$, $\mathbf{t}_0$ are closed, $\mathbf{a}$, $\mathbf{t}_0 \geq 0$ and, without further assumptions, we only know that $\alpha_0 \geq 0$, $\kappa_0 \geq 0$ in Theorem 7.1. In order to verify (7.1), let $f \in \operatorname{dom} \mathbf{t}_0$ with $\|f\| = 1$. By Hölder's and Hardy's inequality, for $1 \leq j \leq n$,

$$\int_{\mathbb{R}^d} |x - x_j|^{-t} |f|^2 \, \mathrm{d}x \leq \left(\int_{\mathbb{R}^d} |x - x_j|^{-2} |f|^2 \, \mathrm{d}x \right)^{\frac{t}{2}} \leq \frac{2^t}{(d-2)^t} \|\nabla f\|^t. \tag{7.11}$$

Moreover, by Gagliardo-Nirenberg-Sobolev's inequality, there exists a constant $G_d > 0$ depending only on the dimension d such that

$$\|f\|_{L^{2^*}(\mathbb{R}^d)} \leq G_d \|\nabla f\|, \quad f \in H^1(\mathbb{R}^d), \quad 2^* := \frac{2d}{d-2},$$

where $2^* > 2$ is the critical Sobolev exponent for the embedding $H^1(\mathbb{R}^d) \hookrightarrow L^{2^*}(\mathbb{R}^d)$. Since $d/s \in (0,2)$, we can use Hölder's inequality with three terms to estimate

$$\int_{\mathbb{R}^d} u|f|^2 \mathrm{d}x \leq \|u\|_{L^s(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} |f|^{\frac{d}{s}\frac{2s}{d-2}} \mathrm{d}x \right)^{\frac{d-2}{2s}} \left(\int_{\mathbb{R}^d} |f|^{\left(2 - \frac{d}{s}\right)\frac{2s}{2s-d}} \mathrm{d}x \right)^{\frac{2s-d}{2s}}.$$

This inequality, together with the relations

$$\frac{d}{s}\frac{2s}{d-2} = 2^*, \quad \frac{d-2}{2s} = \frac{d}{2^* s}, \quad \left(2 - \frac{d}{s}\right)\frac{2s}{2s-d} = 2,$$

and $\|f\| = 1$, yields that

$$\int_{\mathbb{R}^d} u|f|^2 \mathrm{d}x \leq \|u\|_{L^s(\mathbb{R}^d)} \|f\|^{\frac{d}{s}}_{L^{2^*}(\mathbb{R}^d)} \leq \|u\|_{L^s(\mathbb{R}^d)} G_d^{\frac{d}{s}} \|\nabla f\|^{\frac{d}{s}}. \tag{7.12}$$

Next the bound on v in (7.10) with $r \in [0,1)$, Hölder's inequality with $1/r \in (1, \infty]$, $1/(1-r) \in [1, \infty)$ and $\|f\| = 1$ give

$$\int_{\mathbb{R}^d} v|f|^2 \mathrm{d}x \leq c_1 \int_{\mathbb{R}^d} q^r |f|^2 \, \mathrm{d}x + c_2 \leq c_1 \left(\int_{\mathbb{R}^d} q \, |f|^2 \, \mathrm{d}x \right)^r + c_2. \tag{7.13}$$

Combining the inequalities (7.11), (7.12) and (7.13), we arrive at

$$\begin{aligned} \mathbf{a}[f] &\leq \frac{n2^t}{(d-2)^t} \|\nabla f\|^t + \|u\|_{L^s(\mathbb{R}^d)} G_d^{\frac{d}{s}} \|\nabla f\|^{\frac{d}{s}} + c_1 \left(\int_{\mathbb{R}^d} q \, |f|^2 \, \mathrm{d}x \right)^r + c_2 \\ &=: \alpha_1 (\|\nabla f\|^2)^{\frac{t}{2}} + \alpha_2 (\|\nabla f\|^2)^{\frac{d}{2s}} + \alpha_3 \left(\int_{\mathbb{R}^d} q \, |f|^2 \, \mathrm{d}x \right)^r + \alpha_4. \end{aligned} \tag{7.14}$$

In order to further bound (7.14), we estimate $\alpha_1 x_1^{p_1} + \alpha_2 x_2^{p_2} + \alpha_3 x_3^{p_3} + \alpha_4$ with $x_i \geq 0$, $p_i \in [0,1)$, $i = 1,2,3$, and $\alpha_i \geq 0$, $i = 1,2,3,4$; note that $x_1 = x_2 = \|\nabla f\|^2$ in (7.14). If we set $p := \max\{p_1, p_2, p_3\}$ and maximise $\delta(x) := x^{p_i} - x^p$, $x \in [0,1]$, $i = 1,2,3$, we find that

$$x_i^{p_i} \leq x_i^p + \delta_i, \quad \delta_i := \begin{cases} 0 & \text{if } p_i = p, \\ \frac{p - p_i}{p} \left(\frac{p_i}{p} \right)^{\frac{p_i}{p - p_i}} & \text{if } p_i < p, \end{cases} \quad i = 1,2,3. \tag{7.15}$$

If $\max\{\alpha_1,\alpha_2,\alpha_3\}\neq 0$, then

$$\gamma_p := \alpha_1(1+\delta_1)+\alpha_2(1+\delta_2)+\alpha_3(1+\delta_3)+\alpha_4 \neq 0. \tag{7.16}$$

If we use (7.15), the concavity of $x\mapsto x^p$ on $[0,\infty)$ and $x_1=x_2$, we obtain

$$\begin{aligned}
&\alpha_1 x_1^{p_1}+\alpha_2 x_2^{p_2}+\alpha_3 x_3^{p_3}+\alpha_4 \;\le\; \alpha_1(x_1^p+\delta_1)+\alpha_2(x_2^p+\delta_2)+\alpha_3(x_3^p+\delta_3)+\alpha_4\\
&= \gamma_p\Big(\frac{\alpha_1}{\gamma_p}x_1^p+\frac{\alpha_2}{\gamma_p}x_2^p+\frac{\alpha_3}{\gamma_p}x_3^p+\frac{\alpha_1\delta_1+\alpha_2\delta_2+\alpha_3\delta_3+\alpha_4}{\gamma_p}\Big)\\
&\le \gamma_p\Big(\frac{\alpha_1}{\gamma_p}x_1+\frac{\alpha_2}{\gamma_p}x_2+\frac{\alpha_3}{\gamma_p}x_3+\frac{\alpha_1\delta_1+\alpha_2\delta_2+\alpha_3\delta_3+\alpha_4}{\gamma_p}\Big)^p\\
&= \gamma_p^{1-p}\big((\alpha_1+\alpha_2)x_1+\alpha_3x_3+\alpha_1\delta_1+\alpha_2\delta_2+\alpha_3\delta_3+\alpha_4\big)^p\\
&\le \gamma_p^{1-p}\max\{\alpha_1+\alpha_2,\alpha_3\}^p\Big(x_1+x_3+\frac{\alpha_1\delta_1+\alpha_2\delta_2+\alpha_3\delta_3+\alpha_4}{\max\{\alpha_1+\alpha_2,\alpha_3\}}\Big)^p.
\end{aligned}$$

If $\max\{n,\|u\|_{L^s(\mathbb{R}^d)},c_1\}\neq 0$, we can apply this estimate to (7.14) with $p_1=t/2$, $p_2=d/(2s)$, $p_3=r$, δ_i, $i=1,2,3$, as in (7.15) to obtain that $\operatorname{dom}\mathbf{t}_0\subseteq\operatorname{dom}\mathbf{a}$ and assumption (7.1) holds with the parameters

$$\begin{aligned}
&p=\max\Big\{\frac{t}{2},\frac{d}{2s},r\Big\},\quad C_p=\gamma_p^{1-p}\max\Big\{\frac{n2^t}{(d-2)^t}+\|u\|_{L^s(\mathbb{R}^d)}G_d^{\frac{d}{s}},c_1\Big\}^p,\\
&\kappa=-\frac{n2^t\delta_1+(d-2)^t(\|u\|_{L^s(\mathbb{R}^d)}C_*^{\frac{d}{s}}\delta_2+c_1\delta_3+c_2)}{\max\{n2^t+(d-2)^t\,\|u\|_{L^s(\mathbb{R}^d)}G_d^{\frac{d}{s}}(d-2)^t,\;c_1\}},
\end{aligned} \tag{7.17}$$

where, according to (7.16),

$$\gamma_p=\frac{n2^t}{(d-2)^t}(1+\delta_1)+\|u\|_{L^s(\mathbb{R}^d)}G_d^{\frac{d}{s}}(1+\delta_2)+c_1(1+\delta_3)+c_2.$$

If $\max\{n,\|u\|_{L^s(\mathbb{R}^d)},c_1\}=0$, i.e. $n=0$, $u\equiv 0$ and $c_1=0$, then the damping $a=v$ is bounded, our assumption $a\neq 0$ implies $c_2>0$ and (7.1) trivially holds with $p=0$, $C_0=c_2=\|a\|_\infty$ and $\kappa\le d=\kappa_0$ arbitrary.

The constants in (7.17) in the general case $\max\{n,\|u\|_{L^s(\mathbb{R}^d)},c_1\}\neq 0$ simplify substantially if either $n=0$, $u\equiv 0$ or $v\equiv 0$. If e.g. two of n, u or v vanish, the constants p, C_p and κ, which may be read off from (7.11), (7.12) or (7.13), are also obtained as special cases of (7.17). For instance,

$$\begin{array}{llll}
p=\dfrac{t}{2}, & C_{\frac{t}{2}}=\dfrac{n2^t}{(d-2)^t}, & \kappa=0 & \text{if } n\neq 0,\ u\equiv 0 \text{ and } v\equiv 0,\\
p=\dfrac{d}{2s}, & C_{\frac{d}{2s}}=\|u\|_{L^s(\mathbb{R}^d)}G_d^{\frac{d}{s}}, & \kappa=0 & \text{if } n=0,\ u\not\equiv 0 \text{ and } v\equiv 0,\\
p=r, & C_r=(c_1+c_2)^{1-r}c_1^r, & \kappa=-\dfrac{c_2}{c_1} & \text{if } n=0,\ u\equiv 0 \text{ and } v\not\equiv 0,\ c_1>0;
\end{array}$$

in (7.17) these are the 3 cases $\delta_1=0$ with $c_1=c_2=r=0$ and s sufficiently large such that $d/(2s)<r$, $\delta_2=0$ with $t=c_1=c_2=r=0$, and $\delta_3=0$ with $t=0$ and s sufficiently large, respectively. The cases where only one of n, u or v vanishes are similar and are left to the reader.

As a special case, we consider

$$a(x) = |x|^k \text{ with } k \in [0,2), \quad q(x) = |x|^2, \quad x \in \mathbb{R}^d.$$

Here $\alpha_0 = 0$ and we can choose $\kappa_0 > 0$ as the ground energy of the harmonic oscillator, cf. [23, Sec. XIII.12], i.e.

$$\kappa_0 = \inf_{f \in \operatorname{dom} \mathbf{t}_0} \frac{\mathbf{t}_0[f]}{\|f\|^2} = \frac{\mathbf{t}_0[f_0]}{\|f_0\|^2} = d,$$

where $f_0(x) = \exp(-|x|^2/2)$, $x \in \mathbb{R}^d$, is the (non-normalised) ground state of the harmonic oscillator. Moreover, in this special case a satisfies (7.10) with

$$n = 0, \quad t = 0, \quad u \equiv 0, \quad v \equiv a, \quad r = \frac{k}{2}, \quad c_1 = 1, \quad c_2 = 0,$$

and by what was shown above, condition (7.1) holds with

$$p = \frac{k}{2}, \quad C_p = 1, \quad \kappa = 0.$$

Hence the results in Theorem 7.1 (iii), (iv) and (v) yield that

$$\sigma(T) \backslash \mathbb{R} \subseteq \left\{ z \in \mathbb{C} : \operatorname{Re} z \le 0, \ |z| \ge \sqrt{d}, \ |\operatorname{Im} z| \ge \sqrt{\max\{0, |\operatorname{Re} z|^{\frac{2}{k}} - |\operatorname{Re} z|^2\}} \right\}$$

and

$$\sigma(T) \cap \mathbb{R} \begin{cases} = \emptyset & \text{if } k \in [0,1), \\ \subseteq (-\infty, -\sqrt{d}] & \text{if } k = 1, \\ \subseteq \left(-\infty, -\sqrt{t_0}^{\,k} + \sqrt{t_0^k - t_0}\,\right] & \text{if } k \in (1,2), \end{cases}$$

where in the latter case $t_0 = \max\left\{ \left(k(2-k)\right)^{-\frac{1}{k-1}}, d \right\}$.

Acknowledgements. The authors gratefully acknowledge the support of the Swiss National Science Foundation (SNF), grant no. 169104.

References

[1] Araújo C., J. C., and Engström, C. On spurious solutions encountered in Helmholtz scattering resonance computations in $\mathbb{R}^d$ with applications to nano-photonics and acoustics. *J. Comput. Phys. 429* (2021), 110024, 20.

[2] Betcke, T., Higham, N. J., Mehrmann, V., Schröder, C., and Tisseur, F. NLEVP: a collection of nonlinear eigenvalue problems. *ACM Trans. Math. Software 39*, 2 (2013), Art. 7, 28.

[3] Bögli, S., and Marletta, M. Essential numerical ranges for linear operator pencils. *IMA J. Numer. Anal. 40*, 4 (2020), 2256–2308.

[4] Conway, J. B. *Functions of one complex variable*, second ed., vol. 11 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1978.

[5] Cuenin, J.-C., and Tretter, C. Non-symmetric perturbations of self-adjoint operators. *J. Math. Anal. Appl. 441*, 1 (2016), 235–258.

[6] Davies, E. B. Semi-classical analysis and pseudo-spectra. *J. Differential Equations 216*, 1 (2005), 153–187.

[7] Davies, E. B. *Linear operators and their spectra*, vol. 106 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2007.

[8] Engström, C., Langer, H., and Tretter, C. Rational eigenvalue problems and applications to photonic crystals. *J. Math. Anal. Appl. 445*, 1 (2017), 240–279.

[9] Engström, C., and Torshage, A. Enclosure of the numerical range of a class of non-selfadjoint rational operator functions. *Integral Equations Operator Theory 88*, 2 (2017), 151–184.

[10] Freitas, P., Siegl, P., and Tretter, C. The damped wave equation with unbounded damping. *J. Differential Equations 264*, 12 (2018), 7023–7054.

[11] Jacob, B., Tretter, C., Trunk, C., and Vogt, H. Systems with strong damping and their spectra. *Math. Methods Appl. Sci. 41*, 16 (2018), 6546–6573.

[12] Jacob, B., and Trunk, C. Location of the spectrum of operator matrices which are associated to second order equations. *Oper. Matrices 1*, 1 (2007), 45–60.

[13] Jacob, B., and Trunk, C. Spectrum and analyticity of semigroups arising in elasticity theory and hydromechanics. *Semigroup Forum 79*, 1 (2009), 79–100.

[14] Kato, T. Fractional powers of dissipative operators. *J. Math. Soc. Japan 13* (1961), 246–274.

[15] Kato, T. *Perturbation theory for linear operators.* Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[16] Krein, S. G. *Linear differential equations in Banach space*, vol. 29 of *Translations of Mathematical Monographs.* American Mathematical Society, Providence, R.I., 1971.

[17] Langer, H., and Tretter, C. Spectral decomposition of some nonselfadjoint block operator matrices. *J. Operator Theory 39*, 2 (1998), 339–359.

[18] Markus, A. S. *Introduction to the spectral theory of polynomial operator pencils*, vol. 71 of *Translations of Mathematical Monographs.* American Mathematical Society, Providence, RI, 1988.

[19] Möller, M., and Pivovarchik, V. *Spectral theory of operator pencils, Hermite-Biehler functions, and their applications*, vol. 246 of *Operator Theory: Advances and Applications.* Birkhäuser/Springer, Cham, 2015.

[20] Nevanlinna, O. *Convergence of iterations for linear equations.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1993.

[21] Radl, A., Tretter, C., and Wagenhofer, M. The block numerical range of analytic operator functions. *Oper. Matrices 8*, 4 (2014), 901–934.

[22] Rasulov, T. H., and Tretter, C. Spectral inclusion for unbounded diagonally dominant $n \times n$ operator matrices. *Rocky Mountain J. Math. 48*, 1 (2018), 279–324.

[23] Reed, M., and Simon, B. *Methods of modern mathematical physics. IV. Analysis of operators.* Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978.

[24] Tretter, C. *Spectral theory of block operator matrices and applications.* Imperial College Press, London, 2008.

[25] Tretter, C. Spectral inclusion for unbounded block operator matrices. *J. Funct. Anal. 256*, 11 (2009), 3806–3829.

[26] Tretter, C. The quadratic numerical range of an analytic operator function. *Complex Anal. Oper. Theory 4*, 2 (2010), 449–469.

[27] Tretter, C., and Wyss, C. Dichotomous Hamiltonians with unbounded entries and solutions of Riccati equations. *J. Evol. Equ. 14*, 1 (2014), 121–153.

[28] Vogt, H., and Voigt, J. Holomorphic families of forms, operators and C_0-semigroups. *Monatsh. Math. 187*, 2 (2018), 375–380.

[29] Wagenhofer, M. *Block Numerical Ranges.* PhD thesis, University of Bremen, 2007.

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SCHRÖDINGER OPERATORS WITH ACCRETIVE POTENTIALS IN WEIGHTED SPACES

BORBALA GERHAT AND PETR SIEGL

Abstract. We analyse Schrödinger operators with accretive potentials in weighted spaces. We find conditions on potentials and weights for which the Dirichlet realisation, introduced by generalised form methods, has non-empty resolvent set. We establish a domain and graph norm separation property, as well as sufficient conditions for the compactness and Schatten class of the resolvent. Moreover, we investigate the relation between discrete spectra and eigenfunctions of operators in standard and weighted spaces. As applications we extend results on the completeness of eigensystems of operators with accretive potentials from standard to weighted spaces and analyse operator matrices exhibiting a Schur dominance property, in particular, related to a wave equation with strong accretive damping.

1. Introduction

Both in classical and more recent works, the spectral properties of Schrödinger operators

$$T = -\Delta + V$$

in the space $L^2(\Omega)$ with (possibly unbounded) complex potentials V on an open set $\Omega \subset \mathbb{R}^d$ have been studied extensively, see e.g. [29, 20, 16, 26, 44, 12, 21]. These operators arise in several applications, ranging from superconductivity [40, 2, 6], Bloch-Torrey equations [25, 3], hydrodynamics [23, 7, 5], optics with gains and losses [48, 18] to damped wave equations [22, 8, 28] and many more.

In particular for accretive potentials, i.e. for $V : \Omega \to \mathbb{C}$ with $\operatorname{Re} V \geq 0$, which satisfy $V \in W^{1,\infty}_{\mathrm{loc}}(\overline{\Omega})$ and

$$\exists \varepsilon_\nabla \in [0, 2-\sqrt{2}), \quad \exists M_\nabla \geq 0, \quad |\nabla V| \leq \varepsilon_\nabla |V|^{\frac{3}{2}} + M_\nabla \quad \text{a.e. in } \Omega, \tag{1.1}$$

it is known that the Dirichlet realisation of $-\Delta + V$ in $L^2(\Omega)$ is m-accretive. Moreover, the operator domain has the separation property, i.e.

$$\operatorname{Dom}(T) = \operatorname{Dom}(-\Delta_{\mathrm{D}}) \cap \operatorname{Dom}(V),$$

where $\operatorname{Dom}(-\Delta_{\mathrm{D}})$ is the domain of the self-adjoint Dirichlet Laplacian in $L^2(\Omega)$, see (2.1), and the corresponding separation of the graph norm holds

$$\|Tf\|^2 + \|f\|^2 \geq a_\nabla (\|\Delta f\|^2 + \|Vf\|^2 + \|f\|^2), \quad f \in \operatorname{Dom}(T), \tag{1.2}$$

with $a_\nabla = a_\nabla(\varepsilon_\nabla, M_\nabla) > 0$; for details see [4, 11, 30, 27, 41] where also further extensions, e.g. for an additional magnetic field, can be found.

The separation (1.2) allows for reducing questions on the compactness and Schatten class of the resolvent of T to the properties of the self-adjoint operator $-\Delta + |V|$, which can be further employed to investigate the completeness of the eigensystem

2020 *Mathematics Subject Classification.* 35J10, 47A10, 35P10, 81Q12.

Key words and phrases. Schrödinger operators, complex potentials, weighted L^2-space, generalised coercivity, completeness of eigensystems, Schur complement dominant operator matrices.

B.G. gratefully acknowledges the support of the Swiss National Science Foundation (SNF), grant no. 169104, and expresses her gratitude to the SEMP student exchange program for their support, as well as to the Queen's University Belfast for their hospitality.

of T, see [42, 4, 45]. Moreover, (1.2) is also a key step in the analysis of the spectral convergence of domain truncations, see [11, 41].

In this paper we analyse $T_w = -\Delta + V$ with an accretive potential V in the weighted space $L^2_{w^2}(\Omega)$. The considered weights $w : \Omega \to (0, \infty)$ are assumed to satisfy an admissibility condition relating the weight and potential, see Assumption 3.1 for details, which allows for (super)-exponential weights if the potential is unbounded at infinity. It is crucial to observe that, due to the possible unboundedness of w and/or w^{-1}, the connection between the operators T_w in $L^2_{w^2}(\Omega)$ and T in $L^2(\Omega)$ can in general be quite loose. Equivalently, this applies to the relation between T and $S = wT_w w^{-1}$ in $L^2(\Omega)$, i.e. the unitary transformation of the operator T_w to $L^2(\Omega)$. On the level of differential expressions, however, the conjugation by w can be viewed as "unbounded" similarity transform (which can be also interpreted as adding a complex magnetic field, see e.g. [32]) and might suggest a closer relation between T and T_w.

In order to illustrate the possibly occurring non-trivial effects in the relation between T and T_w, consider first the one-dimensional operator

$$T = -\partial_x^2 + |x|^\beta, \quad \beta \geq 2,$$

which is self-adjoint in $L^2(\mathbb{R})$, and the corresponding weighted operator T_w in $L^2_{w^2}(\mathbb{R})$ with weight $w(x) = \exp(x)$, $x \in \mathbb{R}$; see [34] for a detailed study of the unitary transform $S = wT_w w^{-1}$ in $L^2(\mathbb{R})$ of the latter. The spectra of T and T_w coincide, consist of discrete simple real eigenvalues and the corresponding eigenfunctions are complete both in $L^2(\mathbb{R})$ and $L^2_{w^2}(\mathbb{R})$. Nevertheless, the eigenfunctions do not form a basis of $L^2_{w^2}(\mathbb{R})$ and the norms of the corresponding spectral projections P_k diverge at the rate

$$\lim_{k\to\infty} \frac{\log \|P_k\|}{k^{\frac{2}{2+\beta}}} = C_\beta > 0;$$

more details, e.g. the explicit constant C_β and analogous results for various potentials and weights (even of very slow growth), can be found in [34].

As a second ill-behaved example related to the advection-diffusion operator, see [38, 15, 31], consider the one-dimensional operator $T = -\partial_x^2 + V$ in $L^2(\mathbb{R})$ with compactly supported real-valued potential $V \in L^\infty(\mathbb{R};\mathbb{R})$ and the weights

$$w_\beta(x) = \exp(\beta x), \quad \beta \in \mathbb{R}, \quad x \in \mathbb{R}.$$

It can be readily seen that the essential spectra of T and the corresponding weighted operator T_{w_β} differ. Even more importantly, however, the point spectra of T and T_{w_β} might not coincide either, see Example 3.8 below for details.

The study of transformed operator problems as described above dates back to Whittaker [47] and Sommerfeld [43]. Among many others, it appears in the spectral analysis of the bi-stable potential in quantum mechanics [37], the hypoelliptic Laplacian studied by Bismut and Lebeau [10, Chap. 16], see also [35], the Hill operator with a two-term potential [17], the Ornstein-Uhlenbeck operator [33] or the Black-Scholes operator [9].

Our analysis is focused on fundamental properties of weighted Schrödinger operators T_w with accretive potentials V (and $-\Delta$ possibly generalised to $-\nabla \cdot (P\nabla)$ with a sectorial coefficient matrix P). We identify conditions on the coefficients V and P and on the weight w allowing us to find a Dirichlet realisation of

$$T_w = -\nabla \cdot P\nabla + V$$

in the weighted space $L^2_{w^2}(\Omega)$ with non-empty resolvent set, see Theorem 3.2, as well as to establish the graph norm separation, see Theorem 3.12. As the key technical ingredient we employ the notion of generalised coercivity of the associated sesquilinear form introduced in [4], see also Section 2.2. In Theorem 3.4, the boundedness of compositions of the type

$$w_1(-\nabla \cdot (P\nabla) + V + 1)^{-1} w_2$$

is addressed. Furthermore, in Theorem 3.5, we study the compactness and Schatten class of the resolvent of T_w. The invariance of discrete spectra and eigenfunctions is discussed in Theorems 3.6 and 3.9.

As examples of applications of our results, we first investigate the completeness of eigensystems of Schrödinger operators with imaginary potentials, see Section 5.1, in particular extending the results in [4]. Next, we show how our results enter the analysis of operator matrices with non-symmetric differential entries which exhibit Schur complement dominance, see Sections 5.2 – 5.4. In particular, the last example in Section 5.4 deals with a wave equation in a weighted space subject to strong accretive damping, for which we prove the generation of a C_0-semigroup (and thereby generalise results in [22, 28, 24]).

The paper is organised as follows. In Section 2, we collect the used notation and preliminaries. The main results are presented in Section 3 and their proofs are given in Section 4. Section 5 contains examples of applications.

2. Notation and Preliminaries

2.1. **Notation.**

- For an open set $\Omega \subset \mathbb{R}^d$ and a measurable function $m : \Omega \to \mathbb{C}$, we define the corresponding multiplication operator in $L^2(\Omega)$ on the maximal domain

$$\mathrm{Dom}(m) := \{f \in L^2(\Omega) \, : \, mf \in L^2(\Omega)\}.$$

- The Dirichlet Laplacian $-\Delta_{\mathrm{D}}$ is defined as usual via its quadratic form, i.e.

$$\mathrm{Dom}(\Delta_{\mathrm{D}}) := \{f \in W_0^{1,2}(\Omega) \, : \, \Delta f \in L^2(\Omega)\}. \tag{2.1}$$

- The norm and inner product in $L^2(\Omega)$ and $L^2(\Omega)^d$ are denoted by $\|\cdot\|$, $\langle \cdot, \cdot \rangle$.
- For a weight $w \in W^{1,\infty}_{\mathrm{loc}}(\overline{\Omega}; \mathbb{R}_+)$, we introduce

$$L^2_{w^2}(\Omega) := \left\{ f : \Omega \to \mathbb{C} \, : \, f \text{ is measurable and } \|f\|_{L^2_{w^2}} < \infty \right\},$$

$$\|f\|_{L^2_{w^2}} := \left(\int_\Omega |f|^2 w^2 \, \mathrm{d}x \right)^{\frac{1}{2}} = \|wf\|,$$

and the related inner product is denoted by

$$\langle f, g \rangle_{L^2_{w^2}} := \int_\Omega f \overline{g} w^2 \, \mathrm{d}x = \langle wf, wg \rangle.$$

- We write

$$\langle x \rangle := (1 + |x|^2)^{\frac{1}{2}}, \quad x \in \mathbb{R}^d.$$

- If not specified otherwise, all inequalities between measurable functions $\Omega \to \mathbb{R}$ are understood a.e. in Ω.
- The essential spectra of (non-selfadjoint) operators in a Hilbert space $\mathcal{H}$ are defined as in [20, Sec. IX.1]. We shall mainly use the second definition therein

$$\sigma_{\mathrm{e2}} := \{\lambda \in \mathbb{C} \, : \, T - \lambda \notin \mathcal{F}_+(\mathcal{H})\}.$$

Here $A \in \mathcal{F}_+(\mathcal{H})$ if A is closed, has closed range and $\dim \mathrm{Ker}(A) < \infty$.

- For $\Theta \subset \mathbb{C}$, the spectrum of an operator family $\{T(\lambda) \, : \, \lambda \in \Theta\}$ is defined as

$$\sigma(T(\cdot)) := \{\lambda \in \Theta \, : \, 0 \in \sigma(T(\lambda))\}$$

and analogously for the resolvent set and various other parts of the spectrum.

- We write $a \lesssim b$ if there exists a constant $C > 0$, independent of any relevant variable or parameter, such that $a \leq Cb$; the convention for $a \gtrsim b$ is analogous.

2.2. **Generalised coercivity.** We first recall the following generalised representation theorems from [4].

Theorem 2.1 ([4, Thm. 2.1])**.** *Let $\mathcal{V}$ be a Hilbert space and let* $\mathbf{a}$ *be a bounded sesquilinear form on $\mathcal{V}$. Assume that there exist $\Phi_1, \Phi_2 \in \mathcal{B}(\mathcal{V})$ and $m > 0$ such that for all $f \in \mathcal{V}$ we have*

$$\begin{aligned} |\mathbf{a}(f,f)| + |\mathbf{a}(\Phi_1 f, f)| &\geq m\|f\|_{\mathcal{V}}^2 \,, \\ |\mathbf{a}(f,f)| + |\mathbf{a}(f, \Phi_2 f)| &\geq m\|f\|_{\mathcal{V}}^2 \,. \end{aligned}$$

Then the corresponding bounded operator $\widehat{A}$ given by

$$\widehat{A} \in \mathcal{B}(\mathcal{V}, \mathcal{V}^*), \quad \langle \widehat{A} f, g \rangle_{\mathcal{V}^* \times \mathcal{V}} := \mathbf{a}(f,g), \quad f, g \in \mathcal{V}, \tag{2.2}$$

is boundedly invertible, i.e an isomorphism between $\mathcal{V}$ and $\mathcal{V}^*$.

If $\mathcal{V} \subset \mathcal{H}$ is continuously embedded and dense in another Hilbert space $\mathcal{H}$, then upon the standard identification of $\mathcal{H}$ and its (anti-)dual $\mathcal{H}^*$, one can consider

$$\mathcal{H} \ni f \equiv \langle f, \cdot \rangle_{\mathcal{H}} \in \mathcal{H}^*, \qquad \mathcal{V} \subset \mathcal{H} \equiv \mathcal{H}^* \subset \mathcal{V}^*.$$

In the above Hilbert space triplet, the operator $\widehat{A}$ in (2.2) then naturally defines a maximal restriction in $\mathcal{H}$ which is formally given by

$$A := (\mathrm{id}_{\mathcal{V}}^*)^{-1} \widehat{A} \, \mathrm{id}_{\mathcal{V}} \,. \tag{2.3}$$

Here $\mathrm{id}_{\mathcal{V}}$ denotes the continuous embedding $\mathcal{V} \hookrightarrow \mathcal{H}$ and its adjoint $\mathrm{id}_{\mathcal{V}}^*$ is the restriction operator $\mathcal{V}^* \supset \mathcal{H}^* \to \mathcal{H}^*$. Under additional assumptions, T is boundedly invertible in $\mathcal{H}$.

Theorem 2.2 ([4, Thm. 2.2])**.** *In addition to the Assumptions of Theorem* 2.1*, assume that $\mathcal{V} \subset \mathcal{H}$ is continuously embedded and dense in another Hilbert space $\mathcal{H}$ and that Φ_1, Φ_2 extend to bounded operators on $\mathcal{H}$. Then the operator A in $\mathcal{H}$ defined by*

$$\begin{aligned} \mathrm{Dom}(A) &:= \{ f \in \mathcal{V} \, : \, \exists \eta_f \in \mathcal{H}, \ \forall g \in \mathcal{V}, \ \mathbf{a}(f,g) = \langle \eta, g \rangle_{\mathcal{H}} \}, \\ Af &:= \eta_f , \end{aligned} \tag{2.4}$$

is boundedly invertible and its domain is dense in $\mathcal{V}$ and $\mathcal{H}$.

To obtain the results for Schrödinger operators with complex potentials V in $L^2(\Omega)$ mentioned in the introduction, Φ_1 and Φ_2 can be selected as the following multiplication operators

$$\Phi := \Phi_1 = \Phi_2 = \frac{\operatorname{Im} V}{\sqrt{1 + |V|^2}}.$$

2.3. **Schur complement dominant operator matrices.** We recall claims from [24] which are relevant for the applications in Sections 5.2 – 5.4. They allow us to introduce operator matrices with non-empty resolvent set in the product space $\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2$ of two complex Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$.

Assumption 2.3.
(i) Let $\mathcal{D}_S$, $\mathcal{D}_2$, $\mathcal{D}_{-S}$, and $\mathcal{D}_{-2}$ be complex Hilbert spaces such that the inclusions

$$\mathcal{D}_S \subset \mathcal{H}_1 \subset \mathcal{D}_{-S}, \qquad \mathcal{D}_2 \subset \mathcal{H}_2 \subset \mathcal{D}_{-2},$$

hold and the corresponding canonical embeddings are continuous and have dense ranges.

(ii) Suppose that the operators A, B and C are bounded between the spaces

$$A \in \mathcal{B}(\mathcal{D}_S, \mathcal{D}_{-S}), \qquad B \in \mathcal{B}(\mathcal{D}_2, \mathcal{D}_{-S}), \qquad C \in \mathcal{B}(\mathcal{D}_S, \mathcal{D}_{-2}).$$

(iii) Let $D_0 \in \mathcal{C}(\mathcal{H}_2)$ such that $\operatorname{Dom}(D_0) \subset \mathcal{D}_2$ is dense in $\mathcal{D}_2$ and assume that there exist an extension $D_0 \subset D$ and $\lambda_0 \in \rho(D_0)$ such that

$$D \in \mathcal{B}(\mathcal{D}_2, \mathcal{D}_{-2}), \qquad (D - \lambda_0)^{-1} \in \mathcal{B}(\mathcal{D}_{-2}, \mathcal{D}_2).$$

Under Assumption 2.3, we define the operator matrix $\mathcal{A}$ and its first Schur complement $\{S(\lambda) : \lambda \in \rho(D_0)\}$ as

$$\mathcal{A} := \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{B}(\mathcal{D}, \mathcal{D}_-), \quad S(\lambda) := A - B(D-\lambda)^{-1}C \in \mathcal{B}(\mathcal{D}_S, \mathcal{D}_{-S}), \tag{2.5}$$

where $\mathcal{D} := \mathcal{D}_S \oplus \mathcal{D}_2$ and $\mathcal{D}_- := \mathcal{D}_{-S} \oplus \mathcal{D}_{-2}$. We point out that Assumption 2.3 (iii) implies that $D - \lambda$ is boundedly invertible for any $\lambda \in \rho(D_0)$, such that the above formula for the Schur complement is indeed well-defined. Finally, we define the corresponding maximal operators $\mathcal{A}_0 := \mathcal{A}|_{\operatorname{Dom}(\mathcal{A}_0)}$ in $\mathcal{H}$ and $S_0(\lambda) := S(\lambda)|_{\operatorname{Dom}(S_0(\lambda))}$ in $\mathcal{H}_1$ on their respective domains

$$\operatorname{Dom}(\mathcal{A}_0) := \{(f,g) \in \mathcal{D} : \mathcal{A}(f,g) \in \mathcal{H}\}, \quad \operatorname{Dom}(S_0(\lambda)) := \{f \in \mathcal{D}_S : S(\lambda)f \in \mathcal{H}_1\}.$$

Then the spectra of $\mathcal{A}_0$ and $S_0(\cdot)$ are related by the following theorem.

Theorem 2.4 ([24, Cor. 3.4 (ii), Cor. 3.5, Cor. 3.6, Cor. 3.7])**.** *Let Assumption* 2.3 *be satisfied. Assume that, for every $\lambda \in \Sigma \subset \rho(D_0)$, there exists $z_\lambda \in \mathbb{C}$ such that*

$$(S(\lambda) - z_\lambda)^{-1} \in \mathcal{B}(\mathcal{D}_{-S}, \mathcal{D}_S).$$

Then the spectra of $\mathcal{A}_0$ and $S_0(\cdot)$ are equivalent on Σ, more precisely,

$$\sigma(\mathcal{A}_0) \cap \Sigma = \Sigma \cap \sigma(S_0(\cdot)),$$
$$\sigma_{\mathrm{p}}(\mathcal{A}_0) \cap \Sigma = \Sigma \cap \sigma_{\mathrm{p}}(S_0(\cdot)),$$
$$\sigma_{\mathrm{e2}}(\mathcal{A}_0) \cap \Sigma = \Sigma \cap \sigma_{\mathrm{e2}}(S_0(\cdot)).$$

Moreover, if $\rho(S_0(\cdot)) \cap \Sigma \neq \emptyset$, then $\operatorname{Dom}(\mathcal{A}_0)$ *is dense in both $\mathcal{D}$ and $\mathcal{H}$.*

3. Main results

We introduce a Dirichlet realisation of the second order partial differential operator

$$T_w = -\nabla \cdot (P\nabla) + V$$

with accretive potential V in the space $L^2_{w^2}(\Omega)$ with suitable weight $w : \Omega \to (0, \infty)$ and show that it has non-empty resolvent set. Employing the constructed weighted operators T_w, we discuss bounded extensions of certain compositions of the type

$$w_1(T - \lambda)^{-1} w_2, \qquad \lambda \in \rho(T).$$

We derive sufficient conditions for the Schatten class of the resolvent, as well as for the invariance of the discrete spectra and generalised eigenfunctions of T_w and the Dirichlet realisation $T = T_1$ in $L^2(\Omega)$. Finally, we give sufficient conditions for the domain and graph norm separation property of T_w and thereby generalise the result (1.2) for Schrödinger operators to more general second order operators and, most importantly, to weighted spaces.

Our first main set of assumptions is written below. It lays the ground for the following theorems on the Dirichlet realisation T_w, the Schatten class of its resolvent, the invariance of spectra and eigenfunctions, as well as the boundedness of mentioned compositions.

Assumption 3.1. Let $\emptyset \neq \Omega \subset \mathbb{R}^d$ be open. Let the real and imaginary part of V be decomposed into a regular and singular part, respectively, as follows

$$\begin{aligned} \operatorname{Re} V &= U_r + U_s, \\ \operatorname{Im} V &= V_r + V_s, \end{aligned} \qquad U_r, V_r \in W^{1,\infty}_{\mathrm{loc}}(\overline{\Omega}; \mathbb{R}), \qquad U_s, V_s \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}).$$

Moreover, let $P \in L^1_{\rm loc}(\Omega)^{d\times d}$ and write

$$P_1 := \operatorname{Re} P = \frac{1}{2}(P + P^*), \qquad P_2 := \operatorname{Im} P = \frac{1}{2\mathrm{i}}(P - P^*).$$

Let $w \in W^{1,\infty}_{\rm loc}(\overline{\Omega};\mathbb{R}_+)$ and suppose that the following conditions are satisfied.

(i) *Sectoriality of P*: Assume there exists $C_P \geq 0$ and a set $N_P \subset \Omega$ with zero measure such that for all $x \in \Omega \setminus N_P$ and all $\xi \in \mathbb{C}^d$

$$\langle P_1(x)\xi, \xi\rangle \geq 0, \qquad |\langle P_2(x)\xi,\xi\rangle| \leq C_P \langle P_1(x)\xi,\xi\rangle. \tag{3.1}$$

(ii) *Accretivity of V*: Let $U_r \geq 0$ and $U_s \geq 0$.

(iii) *Sectoriality of $V - \mathrm{i}V_r$*: Let $C_s \geq 0$ be such that $|V_s| \leq C_s \operatorname{Re} V$.

Define the multiplier

$$\Phi := \frac{V_r}{\sqrt{1 + V_r^2 + U_r^2}} \in L^\infty(\Omega;\mathbb{R}), \tag{3.2}$$

and further assume the following growth conditions on the admissible weight and the regular part of the potential.

(iv) *Control of ∇U_r and ∇V_r*: Suppose that for every $\varepsilon > 0$ there exists $C_\varepsilon \geq 0$ such that

$$\begin{aligned} &U_r |V_r| \max\{|P_1^{-\frac12} P \nabla U_r|, |P_1^{-\frac12} P^* \nabla U_r|\} \\ &\qquad\qquad \leq (1 + V_r^2 + U_r^2)^{\frac32} (\varepsilon (\operatorname{Re} V)^{\frac12} + \varepsilon(\Phi V_r)^{\frac12} + C_\varepsilon), \\ &(1 + U_r^2) \max\{|P_1^{-\frac12} P \nabla V_r|, |P_1^{-\frac12} P^* \nabla V_r|\} \\ &\qquad\qquad \leq (1 + V_r^2 + U_r^2)^{\frac32} (\varepsilon (\operatorname{Re} V)^{\frac12} + \varepsilon(\Phi V_r)^{\frac12} + C_\varepsilon). \end{aligned} \tag{3.3}$$

(v) *Admissibility of weight w*: Assume there exist $\kappa_w, \sigma_w > 0$ and $C_w \geq 0$ such that

$$|P_1^{-\frac12} P^* \nabla(w^2)| \leq w^2 (\kappa_w (\operatorname{Re} V)^{\frac12} + \sigma_w (\Phi V_r)^{\frac12} + C_w). \tag{3.4}$$

In Theorem 3.2 below, we introduce the weighted operator T_w using the representation theorems in Section 2.2. The latter is done via the form

$$\mathbf{t}_w(f,g) := \langle P\nabla f, \nabla(g w^2)\rangle + \langle wVf, wg\rangle, \quad \operatorname{Dom}(\mathbf{t}_w) := \mathcal{V}_w, \tag{3.5}$$

where the space $\mathcal{V}_w$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|f\|^2_{\mathcal{V}_w} := \|P_1^{\frac12}\nabla f\|^2_{L^2_{w^2}} + \||V|^{\frac12} f\|^2_{L^2_{w^2}} + \|f\|^2_{L^2_{w^2}}; \tag{3.6}$$

see Lemma 4.1 for the extension of (3.5) from $C_0^\infty(\Omega)$ to $\mathcal{V}_w$. In the sequel, we refer to T_w as the Dirichlet realisation of $-\nabla\cdot(P\nabla) + V$ in $L^2_{w^2}(\Omega)$ and write $T := T_1$.

Theorem 3.2 (Dirichlet realisation of T_w)**.** *Let Assumption* 3.1 *be satisfied with κ_w, σ_w, C_P and C_s small enough such that there exists $0 < \beta < \min\{1/C_P, 1/C_s\}$ satisfying the inequality*

$$\beta \kappa_w^2 + (1 - \beta C_s)\sigma_w^2 < \frac{4\beta(1-\beta C_P)(1-\beta C_s)}{(1+\beta)^2}. \tag{3.7}$$

Let the form $\mathbf{t}_w$ be as in (3.5)*, let $\mathcal{V}_w$ be as in* (3.6) *and let Φ be as in* (3.2)*. Then there exists $\lambda > 0$ such that Theorems* 2.1 *and* 2.2 *hold for*

$$\mathbf{a} := \mathbf{t}_w + \lambda\|\cdot\|^2_{L^2_{w^2}}, \quad \mathcal{V} := \mathcal{V}_w, \quad \Phi_1 = \Phi_2 := \beta\Phi, \quad \mathcal{H} := L^2_{w^2}(\Omega).$$

The operator $T_w := A - \lambda$ *in* $L^2_{w^2}(\Omega)$, *where* A *is obtained from Theorem* 2.2, *is closed, has non-empty resolvent set and its domain is dense both in* $\mathcal{V}_w$ *and* $L^2_{w^2}(\Omega)$. *Moreover,* T_w *is independent of* λ *with*

$$\begin{aligned} \operatorname{Dom}(T_w) &= \Big\{ f \in \mathcal{V}_w \, : \, \exists \; \eta_f \in L^2_{w^2}(\Omega), \, \forall \; g \in \mathcal{V}_w, \, \mathbf{t}_w(f,g) = \langle \eta_f, g \rangle_{L^2_{w^2}} \Big\}, \\ T_w f &= \eta_f. \end{aligned} \tag{3.8}$$

Remark 3.3. (i) For the statement of Theorem 3.2, it is sufficient to assume that (3.3) holds with $\varepsilon \in (0, \varepsilon_{\text{crit}})$ and $C_\varepsilon \geq 0$. Here the critical value $\varepsilon_{\text{crit}} > 0$ depends on the remaining parameters κ_w, σ_w, C_w, C_P and C_s, as well as β in (3.7), and can be obtained from a thorough analysis of the inequalities in the proof of Lemma 4.3; see also Lemma 4.12, where the condition $\varepsilon_V + \varepsilon_w < 2 - \sqrt{2}$ arises in a similar way.

(ii) For sectorial potentials $V \in L^1_{\text{loc}}(\Omega)$, i.e. in the case $U_r = V_r = 0$, Assumption 3.1 (iv) is trivially satisfied.

For regular purely imaginary potentials $V = \mathrm{i}V_r$, the condition (iv) in Assumption 3.1 above can be substantially simplified. Indeed, it reduces to assuming that for every $\varepsilon > 0$ there is $C_\varepsilon \geq 0$ such that

$$\max\{|P_1^{-\frac{1}{2}} P \nabla V_r|, |P_1^{-\frac{1}{2}} P^* \nabla V_r|\} \leq \varepsilon |V_r|^{\frac{7}{2}} + C_\varepsilon,$$

which improves the previously used assumption (1.1) (for $P = I_{\mathbb{C}^d}$ and $w = 1$). The condition (3.7) on the weight also simplifies to

$$\sigma_w < \max_{0<\beta<1/C_P} \frac{4\beta(1-\beta C_P)}{(1+\beta)^2} = \frac{1}{1+C_P}. \tag{3.9}$$

In other words, in this particular case, assuming that $\sigma_w < 1/(1+C_P)$ is sufficient for condition (3.7).

Employing a suitable weighted operator, the following theorem allows us to construct bounded extensions of certain conjugations of the resolvent of the non-weighted Dirichlet realisation T.

Theorem 3.4 (Boundedness of compositions)**.** *Let* $m_1, m_2, V : \Omega \to \mathbb{C}$ *and* $P : \Omega \to \mathbb{C}^{d\times d}$ *be measurable, let* m_2 *and* V *be such that*

$$w := \frac{(|V|+1)^{\frac{1}{2}}}{|m_2|} \in W^{1,\infty}_{\text{loc}}(\overline{\Omega}, \mathbb{R}_+)$$

is an admissible weight according to (3.4) *and let the assumptions of Theorem* 3.2 *be satisfied with* P, V *and* w. *Let* T *be the Dirichlet realisation of* $-\nabla \cdot (P\nabla) + V$ *in* $L^2(\Omega)$ *and assume that there exists* $C > 0$ *such that*

$$|m_1 m_2| \leq C(|V|+1). \tag{3.10}$$

Then there exists $\lambda_0 \in \rho(T)$ *and a bounded extension*

$$m_1(T-\lambda_0)^{-1} m_2 \subset S_{\lambda_0} \in \mathcal{B}(L^2(\Omega)),$$

see Lemma 4.7 *for the precise formula for* S_{λ_0}.

Our next result gives a sufficient condition for the Schatten class of the resolvent of T_w. Indeed, independently of the admissible weight, it is sufficient that the embedding of the form domain of the (non-weighted) Dirichlet realisation in $L^2(\Omega)$ is of the respective Schatten class.

Theorem 3.5 (Schatten class of resolvent)**.** *Let the assumptions of Theorem* 3.2 *be satisfied, let* T_w *be the Dirichlet realisation of* $-\nabla \cdot (P\nabla) + V$ *in* $L^2_{w^2}(\Omega)$, *let* $\mathcal{V}_1$ *be as in* (3.6) *with* $w = 1$ *and assume that*

$$\mathrm{id}_{\mathcal{V}_1} \in \mathcal{S}_{2p}(\mathcal{V}_1, L^2(\Omega))$$

for some $p \in (0,\infty]$. *Then*

$$(T_w - \lambda)^{-1} \in \mathcal{S}_p(L^2_{w^2}(\Omega)), \quad \lambda \in \rho(T_w).$$

If both the weighted operator T_w and the standard Dirichlet realisation T have compact resolvent, their (discrete) spectra coincide and the (finite) algebraic multiplicities of their eigenvalues agree.

Theorem 3.6 (Invariance of discrete spectra)**.** *Let the assumptions of Theorem* 3.2 *be satisfied, let* T *and* T_w*, respectively, be the Dirichlet realisations of* $-\nabla\cdot(P\nabla)+V$ *in* $L^2(\Omega)$ *and* $L^2_{w^2}(\Omega)$*. Suppose in addition that both* T *and* T_w *have compact resolvent. Then their (discrete) spectra coincide and*

$$m_a(T,\lambda) = m_a(T_w,\lambda), \qquad \lambda \in \sigma(T) = \sigma(T_w), \tag{3.11}$$

where $m_a(T,\lambda)$ *and* $m_a(T_w,\lambda)$*, respectively, denote the (finite) algebraic multiplicity of an eigenvalue* λ *of* T *and* T_w*.*

Remark 3.7. (i) The resolvent of T and T_w is compact e.g. if

$$\lim_{R\to\infty} \operatorname*{ess\,inf}_{|x|>R,\, x\in\Omega} |V(x)| = \infty \tag{3.12}$$

and $P_1 \geq \delta_P > 0$ a.e. in Ω. This follows from Theorem 3.5 and a standard compactness argument based on Rellich's criterion, see [39, Thm. XIII.65, XIII.67], which implies that $\mathcal{V}_1$ is compactly embedded in $L^2(\Omega)$.

(ii) The invariance of a discrete eigenvalue $\lambda \in \sigma(T)$ remains valid also when the resolvents of T and T_w are not compact if we assume in addition that λ stays separated from the rest of the spectrum of T_{w^α} for $\alpha \in (0,1]$ (see the proof of Theorem 3.6). However, a simple example below (see Example 3.8), related to an advection-diffusion operator, see [38, 15], shows that an eigenvalue can disappear when touched by the essential spectrum.

Example 3.8. We sketch and slightly adapt an example in [31, Sec. VII.C]. Consider the standard self-adjoint realisation of $T := -\partial_x^2 + V$ in $L^2(\mathbb{R})$ with $V \in L^\infty(\mathbb{R};\mathbb{R})$ supported in $[-1,1]$ and assume there exists a simple eigenvalue $0 > \lambda_0 \in \sigma_{\mathrm{disc}}(T)$; the existence of such potential V follows by well-known min-max arguments, see e.g. [39]. Note that it follows from the condition on the support of V that the eigenfunction ψ_0 corresponding to λ_0 satisfies

$$\psi_0(x) = \exp(-\sqrt{-\lambda_0}|x|), \quad |x| > 1.$$

Consider the family of admissible weights $w_\beta(x) = \exp(\beta x)$, $\beta, x \in \mathbb{R}$. We determine the essential spectrum of T_{w_β} by passing to the following family of unitarily equivalent operators in $L^2(\mathbb{R})$

$$S_\beta := w_\beta T_{w_\beta} w_\beta^{-1} = -\partial_x^2 + 2\beta\partial_x - \beta^2 + V, \quad \mathrm{Dom}(S_\beta) := W^{2,2}(\mathbb{R}).$$

Then the Fourier transform and a stability argument for essential spectra yield

$$\sigma_{\mathrm{e}i}(T_{w_\beta}) = \sigma_{\mathrm{e}i}(S_\beta) = \{k^2 - 2\beta\mathrm{i}k - \beta^2 \,:\, k \in \mathbb{R}\}, \quad i = 1,\dots,4,$$

see e.g. [20, Sec. IX] for details. Note that if $\beta^2 \geq |\lambda_0|$, then $\psi_0 \notin L^2_{w_\beta^2}(\mathbb{R})$. Since no other solution of $-\psi'' + V\psi = \lambda_0\psi$ lies in $L^2_{w_\beta^2}(\mathbb{R})$ for $\beta^2 \geq |\lambda_0|$, the eigenvalue λ_0 is lost when the essential spectrum of T_{w_β} touches λ_0 (which happens for $\beta^2 = |\lambda_0|$).

Our next result complements Theorem 3.9 on the invariance of discrete spectra. If, in addition to the assumptions of the latter, the potential satisfies the growth condition (3.12), then every generalised eigenfunction of T is also a generalised eigenfunction of T_w. In particular, this provides information on the decay of the eigenfunctions of the non-weighted operator T. The proof is based on a slight adaption of the Agmon type decay estimates in [30]. These can be extended also for cases without (3.12), see [30] and [41, Appendix].

Theorem 3.9 (Invariance of generalised eigenfunctions)**.** *Let the Assumptions of Theorem* 3.6 *be satisfied and suppose in addition that* V *satisfies* (3.12)*. If* $\lambda \in \sigma(T)$ *and* $k \in \mathbb{N}$*, then*

$$\psi \in \ker(T-\lambda)^k \qquad \Longrightarrow \qquad \psi \in \ker(T_w-\lambda)^k. \tag{3.13}$$

In particular, all generalised eigenfunctions of T *lie in* $L^2_{w^2}(\Omega)$*.*

Our last result on the domain and graph norm separation of the weighted operators T_w requires the following additional set of assumptions.

Assumption 3.10. Let Assumption 3.1 hold. In addition, assume the following.

(i) *Combined accretivity of* V *and* P*:* Suppose there exists a set $N_V \subset \Omega$ with zero measure such that for all $x \in \Omega \setminus N_V$ and all $\xi \in \mathbb{C}^d$

$$\operatorname{Re}\langle e^{-\mathrm{i}\arg V(x)} P(x)\xi, \xi\rangle \geq 0. \tag{3.14}$$

(ii) *Control of* ∇V*:* Let $V \in W^{1,\infty}_{\mathrm{loc}}(\overline{\Omega})$ and assume that there exist $\varepsilon_V > 0$ and $C_V \geq 0$ such that

$$\max\{|P_1^{-\frac12} P^* \nabla V|, |P_1^{-\frac12} P^* \nabla |V||\} \leq \varepsilon_V |V|^{\frac32} + C_V. \tag{3.15}$$

(iii) *Admissibility of the weight* w: Assume there exist $\varepsilon_w > 0$ and $C_w \geq 0$ with

$$|P_1^{-\frac12} P^* \nabla(w^2)| \leq w^2(\varepsilon_w |V|^{\frac12} + C_w). \tag{3.16}$$

Remark 3.11. If (3.16) is satisfied, then (3.4) holds with $\sigma_w = \kappa_w = \varepsilon_w$. Moreover, if $P = P^*$, then assumption (3.15) implies that (with $U_r = \operatorname{Re} V$ and $V_r = \operatorname{Im} V$)

$$\begin{aligned} U_r|V_r||P^{\frac12}\nabla U_r| &\leq (1+U_r^2+V_r^2)^{\frac32}\left(\frac{\delta_1\varepsilon_V}{2}U_r^{\frac12} + \frac{\varepsilon_V}{2\delta_1}(\Phi V_r)^{\frac12} + \frac{C_V}{2}\right), \\ (1+U_r^2)|P^{\frac12}\nabla V_r| &\leq (1+U_r^2+V_r^2)^{\frac32}(\varepsilon_V U_r^{\frac12} + \delta_2(\Phi V_r)^{\frac12} + \varepsilon_V + C_V), \end{aligned}$$

where $\delta_1, \delta_2 > 0$ can be arbitrary. The Assumptions of Theorem 3.2 are thus satisfied if ε_V and ε_w are small enough, see Remark 3.3 (i). If moreover $U_r = o(|V_r|+1)$ or $|V_r| = o(|U_r|+1)$ as $|x| \to \infty$ in Ω, then even condition (3.3) is satisfied (with arbitrarily small $\varepsilon > 0$).

In particular, in the special case $V = \mathrm{i}V_r$ considered in Remark 3.3 (ii), the above shows that assuming $\varepsilon_w < 1/(1+C_P)$ is sufficient for the assumptions of Theorem 3.2; notice that then κ_w can be selected arbitrarily small.

Theorem 3.12 (Domain and graph norm separation)**.** *Let the assumptions of Theorem* 3.2 *hold, let Assumption* 3.10 *be satisfied with* $\varepsilon_V + \varepsilon_w < 2 - \sqrt{2}$ *and suppose in addition that* $P \in W^{1,\infty}(\Omega)^{d\times d}$ *and that* $P_1 \geq \delta_P > 0$ *a.e. in* Ω*. Let* T_w *be the Dirichlet realisation of* $-\nabla\cdot(P\nabla)+V$ *in* $L^2_{w^2}(\Omega)$*. Then there exists* $a_{V,w} > 0$ *such that for all* $f \in \operatorname{Dom}(T_w)$

$$\|T_w f\|_{L^2_{w^2}} + \|f\|_{L^2_{w^2}} \geq a_{V,w}\left(\|\nabla\cdot(P\nabla f)\|_{L^2_{w^2}} + \|Vf\|_{L^2_{w^2}} + \|f\|_{L^2_{w^2}}\right)$$

and as a consequence

$$\operatorname{Dom}(T_w) = \{f \in \mathcal{V}_w \,:\, \nabla\cdot(P\nabla f) \in L^2_{w^2}(\Omega),\ Vf \in L^2_{w^2}(\Omega)\}.$$

4. Proofs

4.1. **Weighted coercivity.**

Lemma 4.1 (Boundedness of $\mathbf{t}_w$ on $\mathcal{V}_w$)**.** *Let Assumption* 3.1 *be satisfied and let* $\mathbf{t}_w$ *and* $\mathcal{V}_w$ *be as in* (3.5) *and* (3.6)*, respectively. Then* $\mathbf{t}_w$ *is a well-defined and bounded form on* $\mathcal{V}_w$*.*

Proof. We show the boundedness on $C_0^\infty(\Omega)$, the claim then follow by density and continuous extension. Let $f \in C_0^\infty(\Omega)$ and note that

$$\begin{aligned}\langle P\nabla f, \nabla(fw^2)\rangle &= \langle wP\nabla f, w\nabla f\rangle + \langle P\nabla f, f\nabla(w^2)\rangle \\ &= \|wP_1^{\frac{1}{2}}\nabla f\|^2 + \mathrm{i}\langle wP_2\nabla f, w\nabla f\rangle + \langle P\nabla f, f\nabla(w^2)\rangle.\end{aligned} \tag{4.1}$$

Using (3.1), we have

$$|\langle wP_2\nabla f, w\nabla f\rangle| \le C_P\|wP_1^{\frac{1}{2}}\nabla f\|^2 \tag{4.2}$$

and from (3.4), Cauchy-Schwarz' and Young's inequalities, we obtain

$$\begin{aligned}|\langle P\nabla f, f\nabla(w^2)\rangle| &\le \langle |P_1^{\frac{1}{2}}\nabla f|, |P_1^{-\frac{1}{2}}P^*\nabla(w^2)||f|\rangle \\ &\le \langle w|P_1^{\frac{1}{2}}\nabla f|, w(\kappa_w(\operatorname{Re}V)^{\frac{1}{2}} + \sigma_w(\Phi V_r)^{\frac{1}{2}} + C_w)|f|\rangle \\ &\lesssim \|wP_1^{\frac{1}{2}}\nabla f\|^2 + \|w(\operatorname{Re}V)^{\frac{1}{2}}f\|^2 + \|w|V_r|^{\frac{1}{2}}f\|^2 + \|wf\|^2.\end{aligned} \tag{4.3}$$

From (4.2) and (4.3), we see that the right hand side of (4.1) is bounded by $\|f\|_{\mathcal{V}_w}^2$. Hence, the boundedness of $\mathbf{t}_w$ on $\mathcal{V}_w$ finally follows from

$$|\langle wVf, wf\rangle| \le \|w|V|^{\frac{1}{2}}f\|^2 \le \|f\|_{\mathcal{V}_w}^2. \qquad \square$$

Lemma 4.2 (Boundedness of Φ on $\mathcal{V}_w$)**.** *Let Assumption* 3.1 *be satisfied and let the space* $\mathcal{V}_w$ *be defined in* (3.6)*. Then the multiplication by* Φ*, defined as in* (3.2)*, is a bounded operator on* $\mathcal{V}_w$*.*

Proof. We show the claimed boundedness on $C_0^\infty(\Omega)$, the full claim then follows by a density and continuity argument. Clearly $|\Phi| \le 1$, thus for all $f \in C_0^\infty(\Omega)$,

$$\|w|V|^{\frac{1}{2}}\Phi f\| + \|w\Phi f\| \le \|f\|_{\mathcal{V}_w},$$

and moreover

$$\|wP_1^{\frac{1}{2}}\nabla(\Phi f)\| \le \|wP_1^{\frac{1}{2}}\nabla f\| + \|wP_1^{\frac{1}{2}}(\nabla\Phi)f\|.$$

To estimate the second term above, one easily derives the identity

$$\nabla\Phi = \frac{(1+U_r^2)\nabla V_r - U_rV_r\nabla U_r}{(1+U_r^2+V_r^2)^{\frac{3}{2}}}$$

which using (3.3) implies that

$$\max\{|P_1^{-\frac{1}{2}}P\nabla\Phi|, |P_1^{-\frac{1}{2}}P^*\nabla\Phi|\} \le 2(\varepsilon(\operatorname{Re}V)^{\frac{1}{2}} + \varepsilon(\Phi V_r)^{\frac{1}{2}} + C_\varepsilon). \tag{4.4}$$

Hence, applying Cauchy-Schwarz' and Young's inequalities with $\delta_1, \delta_2, \delta_3 > 0$, we arrive at

$$\begin{aligned}\|wP_1^{\frac{1}{2}}(\nabla\Phi)f\|^2 &\le |\langle wP(\nabla\Phi)f, w(\nabla\Phi)f\rangle| \le \langle w|P_1^{\frac{1}{2}}(\nabla\Phi)f|, w|P_1^{-\frac{1}{2}}P^*\nabla\Phi||f|\rangle \\ &\le 2\langle w|P_1^{\frac{1}{2}}(\nabla\Phi)f|, w(\varepsilon(\operatorname{Re}V)^{\frac{1}{2}} + \varepsilon(\Phi V_r)^{\frac{1}{2}} + C_\varepsilon)|f|\rangle \\ &\le (\varepsilon\delta_1 + \varepsilon\delta_2 + C_\varepsilon\delta_3)\|wP_1^{\frac{1}{2}}(\nabla\Phi)f\|^2 + \frac{\varepsilon}{\delta_1}\|w(\operatorname{Re}V)^{\frac{1}{2}}f\|^2 \\ &\qquad + \frac{\varepsilon}{\delta_2}\|w(\Phi V_r)^{\frac{1}{2}}f\|^2 + \frac{C_\varepsilon}{\delta_3}\|wf\|^2.\end{aligned}$$

Note that the full generality of the above inequality will be useful only later in the proof of Lemma 4.3. Choosing δ_1, δ_2 and δ_3 small enough such that

$$0 < \eta_1 := \varepsilon\delta_1 + \varepsilon\delta_2 + C_\varepsilon\delta_3 < 1 \tag{4.5}$$

leads to the estimate

$$(1-\eta_1)\|wP_1^{\frac{1}{2}}(\nabla\Phi)f\|^2 \le \frac{\varepsilon}{\delta_1}\|w(\operatorname{Re}V)^{\frac{1}{2}}f\|^2 + \frac{\varepsilon}{\delta_2}\|w(\Phi V_r)^{\frac{1}{2}}f\|^2 + \frac{C_\varepsilon}{\delta_3}\|wf\|^2 \tag{4.6}$$

and thus to the boundedness of the multiplication by Φ on $\mathcal{V}_w$ by combining the estimates above. □

Lemma 4.3 (Generalised weighted coercivity of $\mathbf{t}_w$)**.** *Let the assumptions of Theorem 3.2 be satisfied. Then there exist $m_1, m_2, \gamma_1, \gamma_2 > 0$ (depending continuously on β and the parameters $\varepsilon, \kappa_w, \sigma_w, C_\varepsilon, C_w, C_P$ and C_s) such that, for all $f \in \mathcal{V}_w$,*

$$\begin{aligned}\operatorname{Re}\mathbf{t}_w(f,f) + \operatorname{Im}\mathbf{t}_w(\beta\Phi f, f) + \gamma_1\|wf\|^2 \ge m_1\|f\|^2_{\mathcal{V}_w},\\ \operatorname{Re}\mathbf{t}_w(f,f) + \operatorname{Im}\mathbf{t}_w(f, \beta\Phi f) + \gamma_2\|wf\|^2 \ge m_2\|f\|^2_{\mathcal{V}_w}.\end{aligned} \tag{4.7}$$

Remark 4.4. The proof of Lemma 4.3 provides more information on the constants in (4.7). More precisely, we therein prove the following sufficient claim. For every $\varepsilon > 0$ there exist $\gamma_1(\varepsilon), \gamma_2(\varepsilon) > 0$ such that

$$\begin{aligned}&\operatorname{Re}\mathbf{t}_w(f,f) + \operatorname{Im}\mathbf{t}_w(\beta\Phi f, f) + \gamma_1(\varepsilon)\|wf\|^2\\ &\quad\ge (1-\mu_1-\varepsilon)\|wP_1^{\frac{1}{2}}\nabla f\|^2 + (1-\mu_2-\varepsilon)\|w(\operatorname{Re}V)^{\frac{1}{2}}f\|^2\\ &\qquad + (\beta-\mu_3-\varepsilon)\|w(\Phi V_r)^{\frac{1}{2}}f\|^2,\\ &\operatorname{Re}\mathbf{t}_w(f,f) + \operatorname{Im}\mathbf{t}_w(f, \beta\Phi f) + \gamma_2(\varepsilon)\|wf\|^2\\ &\quad\ge (1-\mu_1-\varepsilon)\|wP_1^{\frac{1}{2}}\nabla f\|^2 + (1-\mu_2-\varepsilon)\|w(\operatorname{Re}V)^{\frac{1}{2}}f\|^2\\ &\qquad + (\beta-\mu_3-\varepsilon)\|w(\Phi V_r)^{\frac{1}{2}}f\|^2.\end{aligned}$$

Here the constants $\mu_1, \mu_2 \in (0,1)$ and $\mu_3 \in (0,\beta)$ are given by

$$\mu_1 := (1+\beta)(\delta\kappa_w + \delta'\sigma_w) + \beta C_P, \quad \mu_2 := \frac{(1+\beta)\kappa_w}{4\delta} + \beta C_s, \quad \mu_3 := \frac{(1+\beta)\sigma_w}{4\delta'},$$

where $\delta, \delta' > 0$ are such that the following inequalities are satisfied

$$\delta > \frac{\kappa_w(1+\beta)}{4(1-\beta C_s)}, \qquad \delta' > \frac{\sigma_w(1+\beta)}{4\beta}, \qquad \delta\kappa_w + \delta'\sigma_w < \frac{1-\beta C_P}{1+\beta}.$$

We point out that the existence of $\delta, \delta' > 0$ satisfying the above inequalities is equivalent to the assumption (3.7).

Proof. We show the estimates in (4.7) for $f \in C_0^\infty(\Omega)$, the full claim then follows by density and continuity. We start by estimating $\operatorname{Re}\mathbf{t}_w(f,f)$. Taking the real part of (3.5) and using (4.1), we obtain

$$\operatorname{Re}\mathbf{t}_w(f,f) \ge \|wP_1^{\frac{1}{2}}\nabla f\|^2 + \|w(\operatorname{Re}V)^{\frac{1}{2}}f\|^2 - |\langle P\nabla f, f\nabla(w^2)\rangle|. \tag{4.8}$$

By the second row of (4.3), we further have

$$|\langle P\nabla f, f\nabla(w^2)\rangle| \le \langle w|P_1^{\frac{1}{2}}\nabla f|, w(\kappa_w(\operatorname{Re}V)^{\frac{1}{2}} + \sigma_w(\Phi V_r)^{\frac{1}{2}} + C_w)|f|\rangle. \tag{4.9}$$

In order to estimate $\operatorname{Im}\mathbf{t}_w(\Phi f, f)$, we use Assumption 3.1 (iii) to derive

$$\begin{aligned}\operatorname{Im}\mathbf{t}_w(\Phi f, f) &= \operatorname{Im}\langle P\nabla(\Phi f), \nabla(fw^2)\rangle + \operatorname{Im}\langle wV\Phi f, wf\rangle\\ &\ge \|w(\Phi V_r)^{\frac{1}{2}}f\|^2 - C_s\|w(\operatorname{Re}V)^{\frac{1}{2}}f\|^2\\ &\qquad - |\operatorname{Im}\langle P\nabla(\Phi f), \nabla(fw^2)\rangle|\end{aligned} \tag{4.10}$$

and further using the product rule

$$
\begin{aligned}
|\operatorname{Im}\langle P\nabla(\Phi f), \nabla(fw^2)\rangle| \le\ & |\langle wP_2(\nabla f)\Phi, w\nabla f\rangle| + |\langle P(\nabla f)\Phi, f\nabla(w^2)\rangle| \\
& + |\langle wP(\nabla\Phi)f, w\nabla f\rangle| + |\langle P(\nabla\Phi)f, f\nabla(w^2)\rangle|.
\end{aligned}
\tag{4.11}
$$

We split $\Phi = \Phi_+ - \Phi_-$ where both $\Phi_\pm \ge 0$ and $\Phi_+ \cdot \Phi_- = 0$ and estimate the first term on the right of (4.11) using (3.1) and $\Phi_+ + \Phi_- = |\Phi| \le 1$

$$
\begin{aligned}
|\langle wP_2(\nabla f)\Phi, w\nabla f\rangle| &\le |\langle wP_2(\nabla f)\Phi_+, w\nabla f\rangle| + |\langle wP_2(\nabla f)\Phi_-, w\nabla f\rangle| \\
&\le C_P\langle wP_1(\nabla f)(\Phi_+ + \Phi_-), w\nabla f\rangle \\
&\le C_P\|wP_1^{\frac12}\nabla f\|^2.
\end{aligned}
\tag{4.12}
$$

Using that $|\Phi| \le 1$, the second term on the right of (4.11) can be estimated analogously to (4.3) and it gives

$$
|\langle P(\nabla f)\Phi, f\nabla(w^2)\rangle| \le \langle w|P_1^{\frac12}\nabla f|, w(\kappa_w(\operatorname{Re} V)^{\frac12} + \sigma_w(\Phi V_r)^{\frac12} + C_w)|f|\rangle. \tag{4.13}
$$

We estimate the third term on the right of (4.11) as follows using (4.4)

$$
\begin{aligned}
|\langle wP(\nabla\Phi)f, w\nabla f\rangle| &\le \langle w|P_1^{-\frac12}P\nabla\Phi||f|, w|P_1^{\frac12}\nabla f|\rangle \\
&\le 2\langle w(\varepsilon(\operatorname{Re} V)^{\frac12} + \varepsilon(\Phi V_r)^{\frac12} + C_\varepsilon)|f|, w|P_1^{\frac12}\nabla f|\rangle.
\end{aligned}
\tag{4.14}
$$

For the fourth term in (4.11), we use (3.4), Cauchy-Schwarz' and Young's inequalities with $\delta_4, \delta_5, \delta_6 > 0$ to obtain

$$
\begin{aligned}
|\langle P(\nabla\Phi)f, f\nabla(w^2)\rangle| &\le \langle |P_1^{\frac12}(\nabla\Phi)f|, |P_1^{-\frac12}P^*\nabla(w^2)||f|\rangle \\
&\le \langle w|P_1^{\frac12}(\nabla\Phi)f|, w(\kappa_w(\operatorname{Re} V)^{\frac12} + \sigma_w(\Phi V_r)^{\frac12} + C_w)|f|\rangle \\
&\le \left(\frac{\kappa_w}{4\delta_4} + \frac{\sigma_w}{4\delta_5} + \frac{C_w}{4\delta_6}\right)\|wP_1^{\frac12}(\nabla\Phi)f\|^2 + \kappa_w\delta_4\|w(\operatorname{Re} V)^{\frac12}f\|^2 \\
&\qquad + \sigma_w\delta_5\|w(\Phi V_r)^{\frac12}f\|^2 + C_w\delta_6\|wf\|^2.
\end{aligned}
$$

Using (4.6) with $\delta_1, \delta_2, \delta_3 > 0$, this leads to

$$
|\langle P(\nabla\Phi)f, f\nabla(w^2)\rangle| \le \eta_3\|w(\operatorname{Re} V)^{\frac12}f\|^2 + \eta_4\|w(\Phi V_r)^{\frac12}f\|^2 + \eta_5\|wf\|^2 \tag{4.15}
$$

where we require $\eta_1 < 1$, see (4.5), and we have set $\eta_2, \eta_3, \eta_4, \eta_5 > 0$ to be

$$
\begin{aligned}
\eta_2 &:= \frac{1}{1-\eta_1}\left(\frac{\kappa_w}{4\delta_4} + \frac{\sigma_w}{4\delta_5} + \frac{C_w}{4\delta_6}\right), \quad & \eta_3 &:= \kappa_w\delta_4 + \eta_2\frac{\varepsilon}{\delta_1}, \\
\eta_4 &:= \sigma_w\delta_5 + \eta_2\frac{\varepsilon}{\delta_2}, & \eta_5 &:= C_w\delta_6 + \eta_2\frac{C_\varepsilon}{\delta_3}.
\end{aligned}
$$

Combining (4.8) – (4.14) and (4.15), we arrive at

$$
\begin{aligned}
&\operatorname{Re}\mathbf{t}_w(f,f) + \operatorname{Im}\mathbf{t}_w(\beta\Phi f, f) \\
&\quad \ge (1-\beta C_P)\|wP_1^{\frac12}\nabla f\|^2 + (1-\beta(C_s+\eta_3))\|w(\operatorname{Re} V)^{\frac12}f\|^2 \\
&\qquad + \beta(1-\eta_4)\|w(\Phi V_r)^{\frac12}f\|^2 \\
&\qquad - (\kappa_w + \beta(\kappa_w + 2\varepsilon))\langle w|P_1^{\frac12}\nabla f|, w(\operatorname{Re} V)^{\frac12}|f|\rangle \\
&\qquad - (\sigma_w + \beta(\sigma_w + 2\varepsilon))\langle w|P_1^{\frac12}\nabla f|, w(\Phi V_r)^{\frac12}|f|\rangle \\
&\qquad - (C_w + \beta(C_w + 2C_\varepsilon))\langle w|P_1^{\frac12}\nabla f|, w|f|\rangle.
\end{aligned}
$$

Using Cauchy-Schwarz' and Young's inequalities with $\delta_7, \delta_8, \delta_9 > 0$, we therefrom finally obtain the estimate

$$\begin{aligned}&\operatorname{Re}\mathbf{t}_w(f,f) + \operatorname{Im}\mathbf{t}_w(\beta\Phi f, f) + \eta_9\|wf\|^2\\ &\qquad \geq (1-\eta_6)\,\|wP_1^{\frac12}\nabla f\|^2 + (1-\eta_7)\|w(\operatorname{Re}V)^{\frac12}f\|^2 + (\beta-\eta_8)\|w(\Phi V_r)^{\frac12}f\|^2\end{aligned}$$

with the constants $\eta_6, \eta_7, \eta_8, \eta_9 > 0$ given by

$$\begin{aligned}\eta_6 &:= \beta C_P + \delta_7(\kappa_w + \beta(\kappa_w + 2\varepsilon)) + \delta_8(\sigma_w + \beta(\sigma_w + 2\varepsilon)) + \delta_9(C_w + \beta(C_w + 2C_\varepsilon)),\\ \eta_7 &:= \frac{1}{4\delta_7}(\kappa_w + \beta(\kappa_w + 2\varepsilon)) + \beta(C_s + \eta_3),\\ \eta_8 &:= \frac{1}{4\delta_8}(\sigma_w + \beta(\sigma_w + 2\varepsilon)) + \beta\eta_4,\\ \eta_9 &:= \frac{1}{4\delta_9}(C_w + \beta(C_w + 2C_\varepsilon)).\end{aligned}$$

From (3.7) it follows elementarily that one can select δ_7 and δ_8 such that

$$\delta_7 > \frac{\kappa_w(1+\beta)}{4(1-\beta C_s)}, \quad \delta_8 > \frac{\sigma_w(1+\beta)}{4\beta}, \quad \delta_7\kappa_w + \delta_8\sigma_w < \frac{1-\beta C_P}{1+\beta} \tag{4.16}$$

are satisfied. We fix δ_7 and δ_8 as above. In view of Assumption 3.1 (iv), by an arbitrarily small choice of ε, it is not difficult to check that (for fixed $\delta_1, \dots, \delta_6$) the constants η_3 and η_4 can be selected arbitrarily small. Considering (4.16), one can thus achieve that $\eta_6, \eta_7 < 1$ and $\eta_8 < \beta$ by a sufficiently small choice of ε. The constants in (4.7) can then be chosen as

$$m_1 := \min\{1-\eta_6, 1-\eta_7, \beta-\eta_8\} > 0, \quad \gamma_1 := \eta_9 + m_1 > 0.$$

Their continuous dependence on the parameters therein is obvious.

To verify the second inequality in (4.7), we analogously to (4.10) estimate

$$\operatorname{Im}\mathbf{t}_w(f, \Phi f) \geq \|w(V_r\Phi)^{\frac12}f\|^2 - C_s\|w(\operatorname{Re}V)^{\frac12}f\|^2 - |\operatorname{Im}\langle P\nabla f, \nabla(\Phi f w^2)\rangle|. \tag{4.17}$$

Next we estimate the third term using the product rule

$$\begin{aligned}|\operatorname{Im}\langle P\nabla f, \nabla(\Phi f w^2)\rangle| \leq |\langle wP\nabla f, w(\nabla\Phi)f\rangle| + |\langle wP_2\nabla f, w\Phi\nabla f\rangle|\\ + |\langle P\nabla f, \Phi f\nabla(w^2)\rangle|.\end{aligned} \tag{4.18}$$

The second and third term on the right of (4.18) have appeared before in (4.12) and (4.13). For the first term, we use (4.4) to arrive at

$$\begin{aligned}|\langle wP\nabla f, w(\nabla\Phi)f\rangle| &\leq \langle w|P_1^{\frac12}\nabla f|, w|P_1^{-\frac12}P^*\nabla\Phi||f|\rangle\\ &\leq 2\langle w|P_1^{\frac12}\nabla f|, w(\varepsilon(\operatorname{Re}V)^{\frac12} + \varepsilon(\Phi V_r)^{\frac12} + C_\varepsilon)|f|\rangle.\end{aligned} \tag{4.19}$$

Combing the estimates (4.8), (4.9), (4.17) – (4.19), (4.12) and (4.13), we deduce

$$\begin{aligned}&\operatorname{Re}\mathbf{t}_w(f,f) + \operatorname{Im}\mathbf{t}_w(f, \beta\Phi f)\\ &\quad\geq (1-\beta C_P)\|wP_1^{\frac12}\nabla f\|^2 + (1-\beta C_s)\|w(\operatorname{Re}V)^{\frac12}f\|^2 + \beta\|w(\Phi V_r)^{\frac12}f\|^2\\ &\qquad - (\kappa_w + \beta(\kappa_w + 2\varepsilon))\langle w|P_1^{\frac12}\nabla f|, w(\operatorname{Re}V)^{\frac12}|f|\rangle\\ &\qquad - (\sigma_w + \beta(\sigma_w + 2\varepsilon))\langle w|P_1^{\frac12}\nabla f|, w(\Phi V_r)^{\frac12}|f|\rangle\\ &\qquad - (C_w + \beta(C_w + 2C_\varepsilon))\langle w|P_1^{\frac12}\nabla f|, w|f|\rangle.\end{aligned}$$

We can thus employ Cauchy-Schwarz' and Young's inequalities with $\delta_{10}, \delta_{11}, \delta_{12} < 0$ to finally arrive at

$$\begin{aligned}&\operatorname{Re}\mathbf{t}_w(f,f) + \operatorname{Im}\mathbf{t}_w(f, \beta\Phi f) + \eta_{13}\|wf\|^2\\ &\quad\geq (1-\eta_{10})\,\|wP_1^{\frac12}(\nabla f)\|^2 + (1-\eta_{11})\|w(\operatorname{Re}V)^{\frac12}f\|^2 + (\beta-\eta_{12})\|w(\Phi V_r)^{\frac12}f\|^2\end{aligned}$$

with the following constants

$$\begin{aligned}\eta_{10} &:= \beta C_P + \delta_{10}(\kappa_w + \beta(\kappa_w + 2\varepsilon)) + \delta_{11}(\sigma_w + \beta(\sigma_w + 2\varepsilon)) \\ &\qquad\qquad + \delta_{12}(C_w + \beta(C_w + 2C_\varepsilon)), \\ \eta_{11} &:= \frac{1}{4\delta_{10}}(\kappa_w + \beta(\kappa_w + 2\varepsilon)) + \beta C_s, \\ \eta_{12} &:= \frac{1}{4\delta_{11}}(\sigma_w + \beta(\sigma_w + 2\varepsilon)), \\ \eta_{13} &:= \frac{1}{4\delta_{12}}(C_w + \beta(C_w + 2C_\varepsilon)).\end{aligned}$$

The justification of the second inequality in (4.7) is now analogous to the first one. □

Proof of Theorem 3.2. Setting $\lambda := \max\{\gamma_1, \gamma_2\}$ and $m := \min\{m_1, m_2\}$, the assumptions of Theorem 2.1 are justified by the Lemmas 4.1, 4.2, 4.3, the observation

$$|\mathbf{a}(f,f)| + |\mathbf{a}(\beta\Phi f, f)| \geq \operatorname{Re}\mathbf{t}_w(f,f) + \operatorname{Im}\mathbf{t}_w(\beta\Phi f, f) + \lambda\|wf\|^2$$

and the analogous estimate for the second inequality in (4.7). Moreover, since the multiplication by the bounded function Φ extends boundedly to $L^2_{w^2}(\Omega)$, the assumptions of Theorem 2.2 are also satisfied. The description of T_w in (3.8) and the claimed independence on λ are then obvious from (2.4). □

4.2. **Boundedness of compositions.**

Lemma 4.5 (Inclusion of $W^{1,\infty}_{\mathrm{comp}}(\Omega)$ in $\mathcal{V}_w$)**.** *Let Assumption* 3.1 *be satisfied. Then*

$$W^{1,\infty}_{\mathrm{comp}}(\Omega) := \{f \in W^{1,\infty}(\Omega) \,:\, \operatorname{supp} f \text{ compact in } \Omega\} \tag{4.20}$$

is a (dense) subspace of $\mathcal{V}_w$ *and the formulas* (3.5)*,* (3.6) *remain valid on* $W^{1,\infty}_{\mathrm{comp}}(\Omega)$.

Proof. Fix $f \in W^{1,\infty}_{\mathrm{comp}}(\Omega)$. Let $\varepsilon > 0$, let ϕ_ε be a standard mollifier on $\mathbb{R}^d$ and set $f_\varepsilon := f * \phi_\varepsilon$, cf. [1, Def. 2.28, Thm. 2.29]. Then $f_\varepsilon \in C_0^\infty(\Omega)$ and $\operatorname{supp} f_\varepsilon \subset K$ for sufficiently small ε, where K is compact in Ω and independent of ε. Moreover, $f_\varepsilon \to f$ in $L^2(\Omega)$ as $\varepsilon \to 0$ and

$$\|f_\varepsilon\|_\infty \leq \|f\|_\infty, \qquad \|\nabla f_\varepsilon\|_\infty = \|\nabla f * \phi_\varepsilon\|_\infty \leq \|\nabla f\|_\infty \tag{4.21}$$

due to the boundedness of f and ∇f. While the local boundedness of ω clearly implies $wf_\varepsilon \to wf$ in $L^2(\Omega)$ as $\varepsilon \to 0$, the limits

$$\int_\Omega |V||f - f_\varepsilon|^2 w^2 \mathrm{d}x \to 0, \qquad \int \langle P_1\nabla(f - f_\varepsilon), \nabla(f - f_\varepsilon)\rangle_{\mathbb{C}^n} w^2 \mathrm{d}x \to 0, \quad \varepsilon \to 0$$

follow from the dominated convergence theorem, see e.g. [1, Thm. 1.50], the bounds in (4.21), $V \in L^1_{\mathrm{loc}}(\Omega)$, $P \in L^1_{\mathrm{loc}}(\Omega)^{d\times d}$, $\operatorname{supp} f_\varepsilon \subset K$ (and extracting an a.e. pointwise convergent subsequence of f_ε). Altogether, this proves that $f_\varepsilon \to f$ in $\mathcal{V}_w$ as $\varepsilon \to 0$, and in consequence $f \in \mathcal{V}_w$ with its norm given in (3.6). The validity of the formula (3.5) for $f, g \in W^{1,\infty}_{\mathrm{comp}}(\Omega)$ then follows from the continuity of both sides with respect to convergence in $\mathcal{V}_w$, cf. the proof of Lemma 4.1. □

Lemma 4.6 (Extension property for comparable weights)**.** *Let the assumptions of Theorem* 3.2 *be satisfied with two admissible weights* w_1 *and* w_2*. Assume that* $w_1 \leq Cw_2$ *with some* $C > 0$*. Then*

$$\mathbf{t}_{w_1}(f, w_2^2 g) = \mathbf{t}_{w_2}(f, w_1^2 g), \qquad f \in \mathcal{V}_{w_2}, \qquad g \in W^{1,\infty}_{\mathrm{comp}}(\Omega),$$

see (4.20)*, where* $\mathbf{t}_{w_1}$ *and* $\mathbf{t}_{w_2}$ *are as in* (3.5)*. Moreover, let* T_{w_1} *and* T_{w_2} *be the Dirichlet realisations of* $-\nabla\cdot(P\nabla) + V$ *in* $L^2_{w_1^2}(\Omega)$ *and* $L^2_{w_2^2}(\Omega)$*, respectively. Then*

$$T_{w_2} \subset T_{w_1}.$$

Proof. Clearly, the assumptions on w_1 and w_2 imply

$$g \in W^{1,\infty}_{\mathrm{comp}}(\Omega) \quad \Longrightarrow \quad w_1^2 g, w_2^2 g \in W^{1,\infty}_{\mathrm{comp}}(\Omega).$$

Due to Lemma 4.5, we thus have

$$\mathbf{t}_{w_1}(f, w_2^2 g) = \langle P\nabla f, \nabla(w_1^2 w_2^2 g)\rangle + \langle Vf, w_1^2 w_2^2 g\rangle = \mathbf{t}_{w_2}(f, w_1^2 g)$$

for $f, g \in W^{1,\infty}(\Omega)$ with compact support. Since $w_1 \le C w_2$ a.e. in Ω, by construction $\mathcal{V}_{w_2}$ is boundedly embedded in $\mathcal{V}_{w_1}$. Hence, the above identity is continuous in f with respect to $\|\cdot\|_{\mathcal{V}_{w_2}}$ and can thus be extended to all $f \in \mathcal{V}_{w_2}$. For $f \in \mathrm{Dom}(T_{w_2})$, this gives

$$\mathbf{t}_{w_1}(f, g) = \langle T_{w_2} f, w_1^2 w_2^{-2} g\rangle_{L^2_{w_2^2}} = \langle Tf, g\rangle_{L^2_{w_1^2}}, \quad g \in C_0^\infty(\Omega),$$

which, since $T_{w_2} f \in L^2_{w_2^2}(\Omega) \subset L^2_{w_1^2}(\Omega)$ and since $C_0^\infty(\Omega)$ is dense in $\mathcal{V}_{w_1}$, implies that $f \in \mathrm{Dom}(T_{w_1})$ with $T_{w_1} f = T_{w_2} f$. □

Lemma 4.7 (Construction of extension S_λ)**.** *Let the assumptions of Theorem* 3.4 *be satisfied and let T_w be the Dirichlet realisation of $-\nabla \cdot (P\nabla) + V$ in $L^2_{w^2}(\Omega)$. Then*

$$S_\lambda := m_1(\widehat{T}_w - \lambda \,\mathrm{id}^*_{\mathcal{V}_w} \mathrm{id}_{\mathcal{V}_w})^{-1} \widetilde{m}_2 \in \mathcal{B}(L^2(\Omega)), \quad \lambda \in \rho(T_w), \tag{4.22}$$

see also (2.2)*, where $\widetilde{m}_2$ is defined as*

$$\langle \widetilde{m}_2 f, \cdot\rangle_{\mathcal{V}_w^* \times \mathcal{V}_w} := \langle w m_2 f, w\cdot\rangle \in \mathcal{V}_w^*, \quad f \in L^2(\Omega). \tag{4.23}$$

Proof. We show that (4.23) indeed defines a bounded functional on $\mathcal{V}_w$. This follows from the fact that, for $f \in L^2(\Omega)$ and $g \in C_0^\infty(\Omega)$, the choice of w and Cauchy Schwarz' inequality give

$$|\langle \widetilde{m}_2 f, g\rangle_{\mathcal{V}_w^* \times \mathcal{V}_w}| \le \|f\| \|w(|V|+1)^{\frac{1}{2}} g\| \le \|f\| \|g\|_{\mathcal{V}_w}.$$

The above, however, implies $\widetilde{m}_2 \in \mathcal{B}(L^2(\Omega), \mathcal{V}_w^*)$ by the density of $C_0^\infty(\Omega)$ in $\mathcal{V}_w$.

We now fix $\lambda \in \rho(T_w)$. By Theorem 3.2, there exists $\lambda_0 \in \rho(T_w)$ with

$$(\widehat{T}_w - \lambda_0 \,\mathrm{id}^*_{\mathcal{V}_w} \mathrm{id}_{\mathcal{V}_w})^{-1} \in \mathcal{B}(\mathcal{V}_w^*, \mathcal{V}_w).$$

Using the resolvent identity,

$$(T_w - \lambda)^{-1} = (T_w - \lambda_0)^{-1} + (\lambda - \lambda_0)(T_w - \lambda)^{-1}(T_w - \lambda_0)^{-1} \subset R$$

where the extension $R \in \mathcal{B}(\mathcal{V}_w^*, \mathcal{V}_w)$ is given by

$$R := (\widehat{T}_w - \lambda_0 \,\mathrm{id}^*_{\mathcal{V}_w} \mathrm{id}_{\mathcal{V}_w})^{-1} + (\lambda - \lambda_0)(T_w - \lambda)^{-1}(\widehat{T}_w - \lambda_0 \,\mathrm{id}^*_{\mathcal{V}_w} \mathrm{id}_{\mathcal{V}_w})^{-1}.$$

From the boundedness of the compositions

$$R(\widehat{T}_w - \lambda \,\mathrm{id}^*_{\mathcal{V}_w} \mathrm{id}_{\mathcal{V}_w}) \in \mathcal{B}(\mathcal{V}_w), \quad (\widehat{T}_w - \lambda \,\mathrm{id}^*_{\mathcal{V}_w} \mathrm{id}_{\mathcal{V}_w}) R \in \mathcal{B}(\mathcal{V}_w^*),$$

and the fact that they equal $\mathrm{id}_{\mathcal{V}_w}$ on the dense subspace $\mathrm{Dom}(T_w)$ of $\mathcal{V}_w$, and $\mathrm{id}_{\mathcal{V}_w^*}$ on the dense subspace $L^2_{w^2}(\Omega)$ of $\mathcal{V}_w^*$, respectively, we infer

$$R = (\widehat{T}_w - \lambda \,\mathrm{id}^*_{\mathcal{V}_w} \mathrm{id}_{\mathcal{V}_w})^{-1} \in \mathcal{B}(\mathcal{V}_w^*, \mathcal{V}_w).$$

Finally, from (3.10) and the choice of w it follows that

$$\|m_1 f\| \lesssim \|w(|V|+1)^{\frac{1}{2}} f\| \le \|f\|_{\mathcal{V}_w}, \qquad f \in \mathcal{V}_w,$$

i.e. that $m_1 \in \mathcal{B}(\mathcal{V}_w, L^2(\Omega))$. Altogether, this shows that $S_\lambda \in \mathcal{B}(L^2(\Omega))$. □

Proof of Theorem 3.4. With S_λ defined as in (4.22), we show that

$$m_1(T-\lambda)^{-1}m_2 \subset S_\lambda, \quad \lambda \in \rho(T)\cap\rho(T_w).$$

To this end, it is clearly sufficient to show that

$$(T-\lambda)^{-1}m_2 \subset (\widehat{T}_w - \lambda\,\mathrm{id}_{\mathcal{V}_w}^*\,\mathrm{id}_{\mathcal{V}_w})^{-1}\widetilde{m}_2. \tag{4.24}$$

We define an auxiliary weighted operator with bounded weight

$$\widetilde{w} := \chi_1(w) \in W^{1,\infty}(\overline{\Omega};\mathbb{R}_+),$$

where χ_1 is the cut-off defined in (4.43) below. One can then easily show that $\widetilde{w}$ satisfies condition (3.4) (with the same constants κ_w, σ_w and C_w), is thus an admissible weight and the operator $T_{\widetilde{w}}$ in $L^2_{\widetilde{w}^2}(\Omega)$ is well-defined by Theorem 3.2. Moreover, $T \subset T_{\widetilde{w}}$ by Lemma 4.6.

In order to show (4.24), let $f \in \mathrm{Dom}((T-\lambda)^{-1}m_2)$, i.e. let $f \in L^2(\Omega)$ with $m_2 f \in L^2(\Omega)$. Since $\widetilde{w} \le w$, also $\mathcal{V}_w \subset \mathcal{V}_{\widetilde{w}}$ by construction and

$$u := (\widehat{T}_w - \lambda\,\mathrm{id}_{\mathcal{V}_w}^*\,\mathrm{id}_{\mathcal{V}_w})^{-1}\widetilde{m}_2 f \in \mathcal{V}_w \subset \mathcal{V}_{\widetilde{w}}.$$

For $g \in C_0^\infty(\Omega)$, we have $\widetilde{g} := \widetilde{w}^2 w^{-2} g \in W^{1,\infty}_{\mathrm{comp}}(\Omega) \subset \mathcal{V}_w$ and we can thus use Lemma 4.6 to derive the identity

$$\begin{aligned}
\langle(\widehat{T}_{\widetilde{w}} - \lambda\,\mathrm{id}_{\mathcal{V}_{\widetilde{w}}}^*\,\mathrm{id}_{\mathcal{V}_{\widetilde{w}}})u, g\rangle_{\mathcal{V}_{\widetilde{w}}^*\times\mathcal{V}_{\widetilde{w}}} &= \mathbf{t}_{\widetilde{w}}(u,g) - \lambda\langle u,g\rangle_{L^2_{\widetilde{w}^2}} \\
&= \mathbf{t}_w(u,\widetilde{g}) - \lambda\langle u,\widetilde{g}\rangle_{L^2_{w^2}} \\
&= \langle(\widehat{T}_w - \lambda\,\mathrm{id}_{\mathcal{V}_w}^*\,\mathrm{id}_{\mathcal{V}_w})u, \widetilde{g}\rangle_{\mathcal{V}_w^*\times\mathcal{V}_w} \\
&= \langle\widetilde{m}_2 f, \widetilde{g}\rangle_{\mathcal{V}_w^*\times\mathcal{V}_w} \\
&= \langle\widetilde{w} m_2 f, \widetilde{w} g\rangle.
\end{aligned} \tag{4.25}$$

Since $m_2 f \in L^2(\Omega) \subset L^2_{\widetilde{w}^2}(\Omega)$ due to the boundedness of $\widetilde{w}$, (4.25) implies

$$u \in \mathrm{Dom}(T_{\widetilde{w}}), \quad (T_{\widetilde{w}} - \lambda)u = m_2 f.$$

However, since $T \subset T_{\widetilde{w}}$ and $m_2 f \in L^2(\Omega) = \mathrm{Ran}(T-\lambda)$, we conclude

$$u \in \mathrm{Dom}(T), \quad (T-\lambda)u = m_2 f.$$

By the bijectivity of $T-\lambda$, this gives $u = (T-\lambda)^{-1}m_2 f$, and thus (4.24) is proven.

It remains to explain that the extension S_λ defined in Lemma 4.7 can always be found, i.e. that $\rho(T)\cap\rho(T_w) \neq \emptyset$. This however, can easily be seen from the proof of Theorem 3.2 and the application of Lemma 4.3 therein, since there it is clear that $\lambda \in \rho(T)\cap\rho(T_w)$ if $\lambda > 0$ is sufficiently large. □

4.3. **Schatten classes.**

Lemma 4.8 (Holomorphic families of generalised coercive forms)**.** *Let $\mathcal{V}$ be a Hilbert space and let $\{\mathbf{t}_\alpha : \alpha \in \Theta\}$ be a holomorphic family of bounded sesquilinear forms on $\mathcal{V}$. Suppose there exists $\lambda \in \mathbb{C}$ such that $\mathbf{a}_\alpha := \mathbf{t}_\alpha + \lambda\|\cdot\|^2_{\mathcal{H}}$ satisfies the assumptions of Theorem* 2.1 *for all $\alpha \in \Theta$ (where $\Phi_1^\alpha, \Phi_2^\alpha \in \mathcal{B}(\mathcal{V})$ and $m_\alpha > 0$ therein possibly depend on α). Let $\mathcal{V}$ be continuously embedded and dense in another Hilbert space $\mathcal{H}$ and assume that Φ_1^α, Φ_2^α extend boundedly to $\mathcal{H}$. Then the operators $T_\alpha := A_\alpha - \lambda$ in $\mathcal{H}$, where A_α is defined as in Theorem* 2.2*, form a holomorphic family on Θ (in the sense of* [29, Chap. 7]*).*

Proof. We first point out that the operators $\widehat{A}_\alpha \in \mathcal{B}(\mathcal{V},\mathcal{V}^*)$ corresponding to $\mathbf{a}_\alpha$, see also (2.2), clearly form a holomorphic family of bounded operators on Θ. The bounded inverse of the operator A_α given in Theorem 2.2 is then obtained by the composition

$$U_\alpha := (T_\alpha + \lambda)^{-1} = A_\alpha^{-1} = \mathrm{id}_{\mathcal{V}}\,\widehat{A}_\alpha^{-1}\,\mathrm{id}_{\mathcal{V}}^* \in \mathcal{B}(\mathcal{H}).$$

The above not only shows that T_α is closed in $\mathcal{H}$, but most importantly that the resolvent U_α is a holomorphic family of bounded operators on Θ, see [29, Sec. VII.1.1]. Setting $X = Y = Z := \mathcal{H}$ and observing that

$$T_\alpha U_\alpha = \mathrm{id}_{\mathcal{H}} - \lambda U_\alpha =: V_\alpha,$$

where V_α is clearly bounded holomorphic as well, we conclude that T_α is a holomorphic family according to the definition in [29, Sec. VII.1.2]. □

Lemma 4.9 (Boundedness of w^α and $w^{-\alpha}$)**.** *Let Assumption* 3.1 *be satisfied and let $\alpha \in \mathbb{C}$. Then $w^{\operatorname{Re}\alpha} \in W^{1,\infty}_{\mathrm{loc}}(\overline{\Omega};\mathbb{R}_+)$ satisfies Assumption* 3.1 (v)*, more precisely,*

$$|P_1^{-\frac{1}{2}} P^* \nabla(w^{2\operatorname{Re}\alpha})| \le w^{2\operatorname{Re}\alpha} |\operatorname{Re}\alpha| (\kappa_w (\operatorname{Re} V)^{\frac{1}{2}} + \sigma_w (\Phi V_r)^{\frac{1}{2}} + C_w). \tag{4.26}$$

Moreover, $\mathcal{V}_1$ and $\mathcal{V}_{w^{\operatorname{Re}\alpha}}$ defined as in (3.6) *are isomorphic via*

$$w^\alpha \in \mathcal{B}(\mathcal{V}_{w^{\operatorname{Re}\alpha}}, \mathcal{V}_1), \qquad w^{-\alpha} \in \mathcal{B}(\mathcal{V}_1, \mathcal{V}_{w^{\operatorname{Re}\alpha}}). \tag{4.27}$$

Proof. The claimed regularity of $w^{\operatorname{Re}\alpha}$ follows from $w \in W^{1,\infty}_{\mathrm{loc}}(\overline{\Omega};\mathbb{R}_+)$ and

$$\nabla(w^{2\operatorname{Re}\alpha}) = 2\operatorname{Re}\alpha w^{2\operatorname{Re}\alpha - 1} \nabla w = \operatorname{Re}\alpha w^{2(\operatorname{Re}\alpha - 1)} \nabla(w^2),$$

which, using the assumption (3.4), gives the bound in (4.26).

In order to prove the first boundedness claim in (4.27), we start by showing the required inequality for $f \in C_0^\infty(\Omega)$. We estimate

$$\begin{aligned} \|w^\alpha f\|^2_{\mathcal{V}_1} &= \|P_1^{\frac{1}{2}} \nabla(w^\alpha f)\|^2 + \||V|^{\frac{1}{2}} w^\alpha f\|^2 + \|w^\alpha f\|^2 \\ &\lesssim \|P_1^{\frac{1}{2}} \nabla(w^\alpha) f\|^2 + \|f\|^2_{\mathcal{V}_{w^{\operatorname{Re}\alpha}}} \end{aligned} \tag{4.28}$$

and notice that $w^\alpha f \in W^{1,\infty}_{\mathrm{comp}}(\Omega)$, such that the norm on the left hand side is well-defined by Lemma 4.5. Using

$$\nabla(w^\alpha) = \alpha w^{\alpha-1} \nabla w = \frac{\alpha}{2} w^{\alpha-2} \nabla(w^2), \tag{4.29}$$

relation (3.4), Cauchy Schwarz' and Young's inequalities with $\delta > 0$, it follows that

$$\begin{aligned} &\|P_1^{\frac{1}{2}} \nabla(w^\alpha) f\|^2 \\ &\quad \le |\langle P^* \nabla(w^\alpha) f, \nabla(w^\alpha) f \rangle| \\ &\quad \le \langle |P_1^{-\frac{1}{2}} P^* \nabla(w^\alpha)| |f|, |P_1^{\frac{1}{2}} \nabla(w^\alpha) f| \rangle \\ &\quad \le \frac{|\alpha|}{2} \langle w^{\operatorname{Re}\alpha} (\kappa_w (\operatorname{Re} V)^{\frac{1}{2}} + \sigma_w (\Phi V_r)^{\frac{1}{2}} + C_w) |f|, |P_1^{\frac{1}{2}} \nabla(w^\alpha) f| \rangle \\ &\quad \le \frac{3|\alpha|}{8\delta} \left(\kappa_w \|w^{\operatorname{Re}\alpha} (\operatorname{Re} V)^{\frac{1}{2}} f\|^2 + \sigma_w \|w^{\operatorname{Re}\alpha} (\Phi V_r)^{\frac{1}{2}} f\|^2 \right) \\ &\qquad\qquad + \frac{3|\alpha|}{8\delta} C_w \|w^{\operatorname{Re}\alpha} f\|^2 + \frac{3|\alpha|\delta}{2} \|P_1^{\frac{1}{2}} \nabla(w^\alpha) f\|^2. \end{aligned} \tag{4.30}$$

Hence, choosing δ small enough such that $3|\alpha|\delta < 2$ and combining (4.28) with (4.30), we find that there exists $C_\alpha > 0$ with

$$\|w^\alpha f\|^2_{\mathcal{V}_1} \le C_\alpha \|f\|^2_{\mathcal{V}_{w^{\operatorname{Re}\alpha}}}, \quad f \in C_0^\infty(\Omega). \tag{4.31}$$

The full claim in (4.27) now follows from a standard density argument. To see this, let $f \in \mathcal{V}_{w^{\operatorname{Re}\alpha}}$ such that $\{f_n\}_n \subset C_0^\infty(\Omega)$ converges to f in $\mathcal{V}_{w^{\operatorname{Re}\alpha}}$. Then $w^{\operatorname{Re}\alpha} f_n$ converges to $w^{\operatorname{Re}\alpha} f$ in $L^2(\Omega)$, which implies the $L^2(\Omega)$ convergence of

$$w^\alpha f_n = w^{\mathrm{i}\operatorname{Im}\alpha} w^{\operatorname{Re}\alpha} f_n \to w^\alpha f, \quad n \to \infty. \tag{4.32}$$

On the other hand, from (4.31) it follows that $w^\alpha f_n$ is a Cauchy sequence in $\mathcal{V}_1$ and thus has a limit $g \in \mathcal{V}_1$. In particular, $w^\alpha f_n$ converges to g in $L^2(\Omega)$, thus $w^\alpha f = g$ in view of (4.32). The inequality (4.31) hence extends to $\mathcal{V}_1$ due to continuity of both sides.

In order to show the second claim in (4.27), we proceed in a similar way. For $f \in C_0^\infty(\Omega)$, we estimate

$$\begin{aligned}\|w^{-\alpha} f\|^2_{\mathcal{V}_{w^{\operatorname{Re}\alpha}}} &= \|w^{\operatorname{Re}\alpha} P_1^{\frac{1}{2}} \nabla(w^{-\alpha} f)\|^2 + \|w^{-\mathrm{i}\operatorname{Im}\alpha} |V|^{\frac{1}{2}} f\|^2 + \|w^{-\mathrm{i}\operatorname{Im}\alpha} f\|^2 \\ &\lesssim \|w^{\operatorname{Re}\alpha} P_1^{\frac{1}{2}} \nabla(w^{-\alpha}) f\|^2 + \|f\|^2_{\mathcal{V}_1}.\end{aligned} \tag{4.33}$$

Analogously to (4.30), using (4.29) with $-\alpha$ instead of α, (3.4), Cauchy-Schwarz' and Young's inequalities with $\delta > 0$, we then obtain

$$\begin{aligned}&\|w^{\operatorname{Re}\alpha} P_1^{\frac{1}{2}} \nabla(w^{-\alpha}) f\|^2 \\ &\quad\le \frac{3|\alpha|\delta}{2} \|w^{\operatorname{Re}\alpha} P_1^{\frac{1}{2}} \nabla(w^{-\alpha}) f\|^2 + \frac{3|\alpha|}{8\delta} \Big(\kappa_w \|w^{-\mathrm{i}\operatorname{Im}\alpha} (\operatorname{Re} V)^{\frac{1}{2}} f\|^2 \\ &\qquad + \sigma_w \|w^{-\mathrm{i}\operatorname{Im}\alpha} (\Phi V_r)^{\frac{1}{2}} f\|^2 + C_w \|w^{-\mathrm{i}\operatorname{Im}\alpha} f\|^2 \Big).\end{aligned} \tag{4.34}$$

As before, the second claim in (4.27) then follows from (4.33) and (4.34) by density and continuity. □

Lemma 4.10 (Unitary equivalence of $T_{w^{\operatorname{Re}\alpha}}$ and S_α)**.** *Let the assumptions of Theorem 3.2 be satisfied, let $\mathcal{V}_1$ be as in (3.6) and define a family $\{\mathbf{s}_\alpha : \alpha \in \mathbb{C}\}$ of bounded sesquilinear forms on $\mathcal{V}_1$ by*

$$\mathbf{s}_\alpha(f, g) := \mathbf{t}_{w^{\operatorname{Re}\alpha}}(w^{-\alpha} f, w^{-\alpha} g), \qquad f, g \in \mathcal{V}_1,$$

where $\mathbf{t}_{w^{\operatorname{Re}\alpha}}$ is as in (3.5), cf. Lemma 4.9. Then there exist $\delta > 0$ and $\lambda \ge 0$ such that the forms

$$\mathbf{a}_\alpha := \mathbf{s}_\alpha + \lambda \|\cdot\|^2, \qquad \alpha \in M_\delta := \{z \in \mathbb{C} : |\operatorname{Re}\alpha| < 1 + \delta,\ |\operatorname{Im}\lambda| < \delta\},$$

satisfy the assumptions of Theorem 2.1. Moreover, the operators $\{S_\alpha : \alpha \in M_\delta\}$, where A_α in $L^2(\Omega)$ is defined according to Theorem 2.2 and $S_\alpha := A_\alpha - \lambda$, form an analytic family in the sense of [29, Chap. 7]. For fixed $\alpha \in M_\delta$, the operators S_α and $T_{w^{\operatorname{Re}\alpha}}$, where $T_{w^{\operatorname{Re}\alpha}}$ in $L^2_{w^{2\operatorname{Re}\alpha}}$ is defined as in Theorem 3.2, are unitarily equivalent via the identity

$$S_\alpha = w^\alpha T_{w^{\operatorname{Re}\alpha}} w^{-\alpha}. \tag{4.35}$$

Proof. Clearly, $w^{\operatorname{Re}\alpha}$ is an admissible weight by Lemma 4.9, in particular it satisfies condition (3.4) with κ_w, σ_w and C_w replaced by

$$\kappa_\alpha := |\operatorname{Re}\alpha| \kappa_w, \qquad \sigma_\alpha := |\operatorname{Re}\alpha| \sigma_w, \qquad C_\alpha := |\operatorname{Re}\alpha| C_w, \tag{4.36}$$

see (4.26). Hence, from Lemmas 4.1 and 4.9, it follows that the forms $\mathbf{s}_\alpha$ are well-defined and bounded on $\mathcal{V}_1$ for all $\alpha \in \mathbb{C}$.

We proceed by showing that the latter family of bounded (everywhere defined) forms depends analytically on the parameter $\alpha \in M_\delta$ (for arbitrary $\delta > 0$). To this end, it suffices to prove that the mapping

$$M_\delta \ni \alpha \mapsto \|\mathbf{s}_\alpha\| = \sup_{f, g \in \mathcal{V}_1} \frac{|\mathbf{s}_\alpha(f, g)|}{\|f\|_{\mathcal{V}_1} \|g\|_{\mathcal{V}_1}} \tag{4.37}$$

is locally bounded and that the scalar functions $\mathbf{s}_\alpha(f, f)$ are holomorphic in α for every $f \in C_0^\infty(\Omega)$, see [29, Sec. VII.1.1, Sec. III.3.1]. The local boundedness of (4.37) follows since by (4.36) the constants provided by the inequalities in the proofs of Lemmas 4.1, 4.9, and thus the norm $\|\mathbf{s}_\alpha\|$, depend continuously on α. To show the claimed holomorphy, let $f \in C_0^\infty(\Omega)$. We start by deriving

$$\begin{aligned}
&\langle P\nabla(w^{-\alpha}f), \nabla(fw^{\overline{\alpha}})\rangle \\
&\qquad = \langle P(-\alpha w^{-\alpha-1}(\nabla w)f + w^{-\alpha}\nabla f), (\nabla f)w^{\overline{\alpha}} + f\overline{\alpha}w^{\overline{\alpha}-1}\nabla w\rangle \\
&\qquad = \alpha\big(\langle P\nabla f, w^{-1}(\nabla w)f\rangle - \langle w^{-1}P(\nabla w)f, \nabla f\rangle\big) \\
&\qquad\qquad - \alpha^2\langle w^{-1}P(\nabla w)f, w^{-1}(\nabla w)f\rangle + \langle P\nabla f, \nabla f\rangle.
\end{aligned} \tag{4.38}$$

Since $w^{-\alpha}f \in W^{1,\infty}(\Omega)$ has compact support in Ω, we can use Lemma 4.5 to derive

$$\begin{aligned}
\mathbf{s}_\alpha(f,f) &= \int_\Omega \langle P\nabla(w^{-\alpha}f), \nabla(w^{-\alpha}fw^{2\operatorname{Re}\alpha})\rangle_{\mathbb{C}^d}\mathrm{d}x + \int_\Omega V|w^{-\alpha}f|^2 w^{2\operatorname{Re}\alpha}\mathrm{d}x \\
&= \langle P\nabla(w^{-\alpha}f), \nabla(fw^{\overline{\alpha}})\rangle_{\mathbb{C}^d}\mathrm{d}x + \int_\Omega V|f|^2\mathrm{d}x.
\end{aligned} \tag{4.39}$$

Combining (4.38) and (4.39) then yields the claim.

Next we choose $\delta > 0$ small enough such that there exists $\beta_0 > 0$ such that, for all $\alpha \in M_\delta$, condition (3.7) holds with κ_α and σ_α instead of κ_w and σ_w. In view of (4.36), this is possible by the continuity of (3.7) in κ_w, σ_w and β. Hence, by Theorem 3.2 applied to $\mathbf{t}_{w^{\operatorname{Re}\alpha}}$ for $\alpha \in M_\delta$, there exist $\lambda \geq 0$, $m > 0$ such that

$$\begin{aligned}
&|\mathbf{t}_{w^{\operatorname{Re}\alpha}}(f,f) + \lambda\|w^{\operatorname{Re}\alpha}f\|^2| \\
&\qquad + |\mathbf{t}_{w^{\operatorname{Re}\alpha}}(\beta_0\Phi f, f) + \lambda\langle w^{\operatorname{Re}\alpha}\beta_0\Phi f, w^{\operatorname{Re}\alpha}f\rangle| \geq m\|f\|^2_{\mathcal{V}_{w^{\operatorname{Re}\alpha}}}, \\
&|\mathbf{t}_{w^{\operatorname{Re}\alpha}}(f,f) + \lambda\|w^{\operatorname{Re}\alpha}f\|^2| \\
&\qquad + |\mathbf{t}_{w^{\operatorname{Re}\alpha}}(f, \beta_0\Phi f) + \lambda\langle w^{\operatorname{Re}\alpha}f, w^{\operatorname{Re}\alpha}\beta_0\Phi f\rangle| \geq m\|f\|^2_{\mathcal{V}_{w^{\operatorname{Re}\alpha}}},
\end{aligned} \tag{4.40}$$

for all $f \in \mathcal{V}_{w^{\operatorname{Re}\alpha}}$. Notice that since M_δ is compact and λ_α, m_α in Theorem 3.2 depend continuously on α, the constants λ and m in the above inequalities can be chosen independently of α, see also Lemma 4.3. By Lemma 4.9, $w^\alpha \in \mathcal{B}(\mathcal{V}_{w^{\operatorname{Re}\alpha}}, \mathcal{V}_1)$ is boundedly invertible, thus from (4.40) with $g = w^\alpha f \in \mathcal{V}_1$, we arrive at the lower estimates

$$\left.\begin{aligned}
&|\mathbf{s}_\alpha(g,g) + \lambda\|g\|^2| + |\mathbf{s}_\alpha(\beta_0\Phi g, g) + \lambda\langle\beta_0\Phi g, g\rangle| \geq m\|w^\alpha\|^{-2}\|g\|^2_{\mathcal{V}_1}, \\
&|\mathbf{s}_\alpha(g,g) + \lambda\|g\|^2| + |\mathbf{s}_\alpha(g, \beta_0\Phi g) + \lambda\langle g, \beta_0\Phi g\rangle| \geq m\|w^\alpha\|^{-2}\|g\|^2_{\mathcal{V}_1},
\end{aligned}\right. \quad g \in \mathcal{V}_1,$$

where $\|w^\alpha\|$ denotes the norm of the isomorphism $w^\alpha : \mathcal{V}_{w^{\operatorname{Re}\alpha}} \to \mathcal{V}_1$ and

$$\Phi = w^\alpha\Phi w^{-\alpha} \in \mathcal{B}(\mathcal{V}_1).$$

In other words, for every $\alpha \in M_\delta$, the form $\mathbf{a}_\alpha := \mathbf{s}_\alpha + \lambda\|\cdot\|^2$ satisfies the assumptions of Theorem 2.1. Since the multiplication by the bounded function Φ extends boundedly to $L^2(\Omega)$, the family $\{A_\alpha : \alpha \in M_\delta\}$ of associated operators in $L^2(\Omega)$ is well-defined and holomorphic by Lemma 4.8. The holomorphy of $S_\alpha = A_\alpha - \lambda$ is now immediate.

It remains to show the identity (4.35), which then, since

$$w^\alpha \in \mathcal{B}(L^2_{w^{2\operatorname{Re}\alpha}}(\Omega), L^2(\Omega))$$

is unitary, yields the unitary equivalence of S_α and $T_{w^{\operatorname{Re}\alpha}}$. To this end, we consider $f \in \operatorname{Dom}(S_\alpha)$ with $S_\alpha f =: \eta \in L^2(\Omega)$. By definition, this is equivalent to

$$\forall g \in \mathcal{V}_1, \quad \mathbf{s}_\alpha(f,g) = \langle\eta, g\rangle.$$

Using the definition of $\mathbf{s}_\alpha$, the bijectivity of $w^\alpha : \mathcal{V}_{w^{\operatorname{Re}\alpha}} \to \mathcal{V}_1$ by Lemma 4.9, setting $u := w^{-\alpha}f \in \mathcal{V}_{w^{\operatorname{Re}\alpha}}$ and $\xi := w^{-\alpha}\eta \in L^2_{w^{2\operatorname{Re}\alpha}}(\Omega)$, this in turn is equivalent to

$$\forall v \in \mathcal{V}_1, \quad \mathbf{t}_{w^{\operatorname{Re}\alpha}}(u,v) = \langle w^{\mathrm{i}\operatorname{Im}\alpha}\xi, w^{\mathrm{i}\operatorname{Im}\alpha}v\rangle_{L^2_{w^{2\operatorname{Re}\alpha}}} = \langle\xi, v\rangle_{L^2_{w^{2\operatorname{Re}\alpha}}}.$$

The above, however, by definition means $u \in \mathrm{Dom}(T_{w^{\mathrm{Re}\,\alpha}})$ with $T_{w^{\mathrm{Re}\,\alpha}} u = \xi$. We have thus shown that $f \in \mathrm{Dom}(S_\alpha)$ if and only if

$$f = w^\alpha u \in w^\alpha \mathrm{Dom}(T_{w^{\mathrm{Re}\,\alpha}}) = \mathrm{Dom}(w^\alpha T_{w^{\mathrm{Re}\,\alpha}} w^{-\alpha})$$

and that then

$$S_\alpha f = \eta = w^\alpha \xi = w^\alpha T_{w^{\mathrm{Re}\,\alpha}} u = w^\alpha T_{w^{\mathrm{Re}\,\alpha}} w^{-\alpha} f. \qquad \square$$

Proof of Theorem 3.5. Let T_w and $S_1 = w T_w w^{-1}$, respectively, be as in Theorem 3.2 and Lemma 4.10 and consider $\lambda \in \rho(S_1) = \rho(T_w)$. By construction, the resolvent of S_1 is given by

$$(S_1 - \lambda)^{-1} = \mathrm{id}_{\mathcal{V}_1} (\widehat{S}_1 - \lambda \,\mathrm{id}_{\mathcal{V}_1} \mathrm{id}_{\mathcal{V}_1}^*)^{-1} \mathrm{id}_{\mathcal{V}_1}^*, \tag{4.41}$$

see also (2.2). Since $\mathrm{id}_{\mathcal{V}} \in \mathcal{S}_{2p}(\mathcal{V}_1, L^2(\Omega))$ and thus $\mathrm{id}_{\mathcal{V}}^* \in \mathcal{S}_{2p}(L^2(\Omega), \mathcal{V}_1^*)$, it follows from the identity (4.41), the ideal property of Schatten classes and Hölder's inequality for Schatten classes, see e.g. [46, Thm. 3.23], that

$$\|(S_1 - \lambda)^{-1}\|_{\mathcal{S}_p} \le 2^{\frac{1}{p}} \|\,\mathrm{id}_{\mathcal{V}_1}\|_{\mathcal{S}_{2p}} \|(\widehat{S}_1 - \lambda \,\mathrm{id}_{\mathcal{V}_1} \mathrm{id}_{\mathcal{V}_1}^*)^{-1} \mathrm{id}_{\mathcal{V}_1}^*\|_{\mathcal{S}_{2p}} < \infty.$$

The claim is now immediate from the unitary equivalence of S_1 and T_w. $\square$

4.4. **Invariance of discrete spectra and eigenfunctions.**

Proof of Theorem 3.6. Let S_α, $\alpha \in M_\varepsilon$ be the analytic family in Lemma 4.10. From the unitary equivalence $T_w = w^{-1} S_1 w$ and the assumption on the compactness of the resolvents of T and T_w, we have

$$\sigma(T) = \sigma_{\mathrm{disc}}(T), \quad \sigma(T_w) = \sigma_{\mathrm{disc}}(T_w) = \sigma_{\mathrm{disc}}(S_1) = \sigma(S_1), \tag{4.42}$$

with coinciding algebraic multiplicities in the second identity. Moreover, for $\mu \in \mathbb{R}$, again by unitary equivalence it follows that

$$m_a(\lambda, S_{\mathrm{i}\mu}) = m_a(\lambda, T), \qquad \lambda \in \sigma_{\mathrm{disc}}(S_{\mathrm{i}\mu}) = \sigma_{\mathrm{disc}}(w^{\mathrm{i}\mu} T w^{-\mathrm{i}\mu}) = \sigma_{\mathrm{disc}}(T),$$

i.e. the discrete eigenvalues of S_α are constant for $\alpha \in \mathrm{i}R$. Since the latter are analytic functions in α due to the holomorphy of S_α, see [29, Sec. VII.1.3, Thm. VII.1.8], the discrete eigenvalues of S_α and their multiplicities remain constant on M_δ by the identity theorem for holomorphic functions. In view of (4.42), this in particular implies (3.11). $\square$

Proof of Theorem 3.9. We point out that, since P_1 is uniformly bounded below, one can show with a standard approximation argument that, for any admissible weight w,

$$\mathcal{V}_w = \{f \in L^2_{w^2}(\Omega) \,:\, \nabla f,\, P_1^{\frac{1}{2}} \nabla f \in L^2_{w^2}(\Omega)^d,\ |V|^{\frac{1}{2}} f \in L^2_{w^2}(\Omega)\}$$

holds and the formulas (3.6) and (3.5) remain valid for $f, g \in \mathcal{V}_w$.

In order to prove the claim (3.13), it suffices to show that if $\psi \in \mathrm{Dom}(T)$ and $\psi_0 \in L^2(\Omega) \cap L^2_{w^2}(\Omega)$ such that $(T_w - \lambda)\psi = \psi_0$, then $\psi \in \mathrm{Dom}(T_w)$ and $(T_w - \lambda)\psi = \psi_0$; the full claim then follows by an inductive argument. We approximate w with a sequence of bounded weights in a suitable sense. For $n \in \mathbb{N}$, define

$$w_n := \chi_n(w), \qquad \chi_n : \mathbb{R}_+ \to \mathbb{R}_+, \qquad \chi_n(x) := \begin{cases} x, & x \in (0, n], \\ n, & x \in (n, \infty), \end{cases} \tag{4.43}$$

then it is easy to see that $w_n \in W^{1,\infty}(\overline{\Omega}; \mathbb{R}_+) \cap L^\infty(\Omega; \mathbb{R}_+)$ satisfies condition (3.4) with the same constants κ_w, σ_w and C_w as the original weight w. By Lemma 4.6, the operator T_{w_n} is an extension of T and thus

$$\psi \in \mathrm{Dom}(T_{w_n}), \quad (T_{w_n} - \lambda)\psi = \psi_0 \in L^2_{w_n}(\Omega). \tag{4.44}$$

Hence, from Lemma 4.3 it follows that there exist $\mu \geq 0$ and $m > 0$ (depending on w and V but not on n) such that, with Φ as in (3.2), we have

$$\begin{aligned} m\|\psi\|^2_{\mathcal{V}_{w_n}} &\leq \operatorname{Re}\mathbf{t}_{w_n}(\psi,\psi) + \mu\|w_n\psi\|^2 + \operatorname{Im}\mathbf{t}_{w_n}(\psi,\Phi\psi) \\ &\leq (|\operatorname{Re}\lambda| + |\operatorname{Im}\lambda| + \mu + 1)\|w_n\psi\|^2 + \|w_n\psi_0\|^2 \end{aligned} \tag{4.45}$$

where for the second inequality we used (4.44). Let us set

$$C := |\operatorname{Re}\lambda| + |\operatorname{Im}\lambda| + \mu + 1 > 0,$$

then by assumption (3.12), there exist $R > 0$ and $m > \delta > 0$ such that

$$m(|V(x)| + 1) - C \geq \delta(|V(x)| + 1), \quad x \in \Omega \setminus B_R(0). \tag{4.46}$$

It thus follows from (4.45), (4.46) and $w_n \leq w$ that

$$\begin{aligned} &\delta\left(\|w_n P_1^{\frac{1}{2}}\nabla\psi\|^2 + \|w_n|V|^{\frac{1}{2}}\psi\|^2 + \|w_n\psi\|^2\right) \\ &\quad\leq m\int_\Omega |P_1^{\frac{1}{2}}\nabla\psi|^2 w_n^2 \mathrm{d}x + \int_{\Omega\setminus B_R(0)} (m(|V|+1) - C)|\psi|^2 w_n^2 \mathrm{d}x \\ &\qquad\qquad + m\int_{\Omega\cap B_R(0)} (|V|+1)|\psi|^2 w_n^2 \mathrm{d}x \\ &\quad\leq C\int_{\Omega\cap B_R(0)} |\psi|^2 w_n^2 \mathrm{d}x + \|w_n\psi_0\|^2 \\ &\quad\leq C\|w\|^2_{L^\infty(\Omega\cap B_R(0))}\|\psi\|^2 + \|w\psi_0\|^2. \end{aligned}$$

We can thus apply Fatou's Lemma to conclude $\psi \in \mathcal{V}_w$ from

$$\|wP_1^{\frac{1}{2}}\nabla\psi\|^2 + \|w|V|^{\frac{1}{2}}\psi\|^2 + \|w\psi\|^2 \leq \frac{C}{\delta}\|w\|^2_{L^\infty(\Omega\cap B_R(0))}\|\psi\|^2 + \frac{1}{\delta}\|w\psi_0\|^2.$$

Note that the right hand side of the above inequality is finite due to $w \in L^\infty_{\mathrm{loc}}(\overline{\Omega};\mathbb{R}_+)$ and $\psi_0 \in L^2_{w^2}(\Omega)$. Finally, for $\phi \in C_0^\infty(\Omega)$, the function $w^2\phi \in W^{1,\infty}(\Omega)$ has compact support, is thus an element of $\mathcal{V}_1$ by Lemma 4.5 and

$$\mathbf{t}_w(\psi,\phi) = \langle P\nabla\psi, \nabla(w^2\phi)\rangle + \langle V\psi, w^2\phi\rangle = \mathbf{t}(\psi, w^2\phi). \tag{4.47}$$

Since $\psi \in \operatorname{Dom}(T)$ and $T\psi = \lambda\psi + \psi_0$, we further conclude

$$\mathbf{t}(\psi, w^2\phi) = \langle T\psi, w^2\phi\rangle = \langle \lambda\psi + \psi_0, \phi\rangle_{L^2_{w^2}}. \tag{4.48}$$

Combining (4.47) and (4.48), as well as the density of $C_0^\infty(\Omega)$ in $\mathcal{V}_w$, we obtain $\psi \in \operatorname{Dom}(T_w)$ with $T_w\psi = \lambda\psi + \psi_0$. □

4.5. **Graph norm separation.** We point out that the assumptions of Theorem 3.12 guarantee that, for every $f \in \mathcal{V}_w$, the weak gradient of f is a regular distribution such that

$$P_1{}^{\frac{1}{2}}\nabla f \in L^2_{w^2}(\Omega)^d, \qquad |V|^{\frac{1}{2}} f \in L^2_{w^2}(\Omega).$$

The action of T_w then coincides with the corresponding differential expression in the standard distributional sense, i.e.

$$T_w f = -\nabla\cdot(P\nabla f) + Vf \in \mathcal{D}'(\Omega), \qquad f \in \operatorname{Dom}(T_w).$$

Lemma 4.11 (Core of T_w)**.** *Let the assumptions of Theorem* 3.12 *be satisfied. Then*

$$\mathcal{D}_w := \{f \in \operatorname{Dom}(T_w) : \operatorname{supp} f \text{ is bounded in } \mathbb{R}^d\} \tag{4.49}$$

is a core of T_w*.*

Proof. The claim can be justified by a standard cut-off strategy, see e.g. [30, proof of Lem. 3.6] or [14, proof of Thm. 8.2.1, Part 1]. □

Lemma 4.12 (Graph norm separation)**.** *Let the assumptions of Theorem* 3.12 *be satisfied and let* $\mathcal{D}_w$ *be as in* (4.49). *Then for every* $\varepsilon > 0$ *there exists* $C(\varepsilon) > 0$ *such that*

$$\|T_w f\|^2_{L^2_{w^2}} \geq (1-\mu_1-\varepsilon)\|\nabla\cdot(P\nabla f)\|^2_{L^2_{w^2}} + (1-\mu_2-\varepsilon)\|Vf\|^2_{L^2_{w^2}} - C(\varepsilon)\|f\|^2_{L^2_{w^2}}$$

for $f \in \mathcal{D}_w$, *where the constants* $\mu_1, \mu_2 \in (0,1)$ *are*

$$\mu_1 := \frac{2(\varepsilon_V+\varepsilon_w)\delta\delta'}{1-(\varepsilon_V+\varepsilon_w)\delta}, \quad \mu_2 := \frac{2(\varepsilon_V+\varepsilon_w)}{1-(\varepsilon_V+\varepsilon_w)\delta}\left(\frac{1}{4\delta}+\frac{\delta}{4\delta'}\right),$$

and $\delta, \delta' > 0$ *are the numbers*

$$\delta := \sqrt{\frac{1+\sqrt{2}}{4\sqrt{2}}}, \qquad \delta' := \frac{1+\sqrt{2}}{2}. \tag{4.50}$$

Remark 4.13. In the above inequality, we have $\mu_1, \mu_2 \in (0,1)$ if and only if

$$\varepsilon_V + \varepsilon_w < \min\left\{\frac{1}{2\delta\delta'+\delta}, \frac{2\delta\delta'}{\delta^2(1+2\delta')+\delta'}\right\} \leq 2-\sqrt{2}$$

where the maximum $2-\sqrt{2}$ is indeed attained with δ and δ' as in (4.50).

Proof of Lemma 4.12. We start by deriving an estimate for a relevant quantity in the graph norm of T_w, see (4.57). To this end, let $f \in \mathcal{D}_w$. Integration by parts gives the estimate

$$\begin{aligned}\|wP_1^{\frac{1}{2}}\nabla f\|^2 = \langle wP_1\nabla f, w\nabla f\rangle &\leq |\langle w^2 P\nabla f, \nabla f\rangle| \\ &\leq |\langle w\nabla\cdot(P\nabla f), wf\rangle| + |\langle P\nabla f, \nabla(w^2)f\rangle|.\end{aligned} \tag{4.51}$$

Moreover, using (3.16), Cauchy Schwarz' inequality and Young's inequality with $\delta_1, \delta_2 > 0$, we derive

$$\begin{aligned}|\langle P\nabla f, \nabla(w^2)f\rangle| &\leq \langle |P_1^{\frac{1}{2}}\nabla f|, |P_1^{-\frac{1}{2}}P^*\nabla(w^2)||f|\rangle \\ &\leq \langle w|P_1^{\frac{1}{2}}\nabla f|, w(\varepsilon_w|V|^{\frac{1}{2}}+C_w)|f|\rangle \\ &\leq \varepsilon_w\delta_1\|wP_1^{\frac{1}{2}}(\nabla f)|V|^{\frac{1}{2}}\|^2 + C_w\delta_2\|wP_1^{\frac{1}{2}}\nabla f\|^2 \\ &\quad + \left(\frac{\varepsilon_w}{4\delta_1}+\frac{C_w}{4\delta_2}\right)\|wf\|^2.\end{aligned} \tag{4.52}$$

Putting together (4.51), (4.52) and choosing δ_2 sufficiently small such that $C_w\delta_2 < 1$ eventually leads to

$$\begin{aligned}\|wP_1^{\frac{1}{2}}\nabla f\|^2 \leq \frac{1}{1-C_w\delta_2}\Bigg(&\varepsilon_w\delta_1\|wP_1^{\frac{1}{2}}(\nabla f)|V|^{\frac{1}{2}}\|^2 \\ &+ |\langle w\nabla\cdot(P\nabla f), wf\rangle| + \left(\frac{\varepsilon_w}{4\delta_1}+\frac{C_w}{4\delta_2}\right)\|wf\|^2\Bigg).\end{aligned} \tag{4.53}$$

Using Cauchy-Schwarz' inequality, Young's inequality with $\delta_3, \delta_4 > 0$ and (4.53) gives the following estimate

$$\begin{aligned}
&\langle w|P_1^{\frac12}\nabla f|, w((\varepsilon_V+\varepsilon_w)|V|^{\frac32}+C_w|V|+C_V)|f|\rangle\\
&\quad\le (\varepsilon_V+\varepsilon_w)\delta_3\|wP_1^{\frac12}(\nabla f)|V|^{\frac12}\|^2+\left(\frac{C_w}{4\delta_4}+\frac{C_V}{2}\right)\|wP_1^{\frac12}\nabla f\|^2\\
&\qquad\quad+\left(\frac{\varepsilon_V+\varepsilon_w}{4\delta_3}+C_w\delta_4\right)\|wVf\|^2+\frac{C_V}{2}\|wf\|^2\\
&\quad\le \eta_1\|wP_1^{\frac12}(\nabla f)|V|^{\frac12}\|^2+\left(\frac{C_w}{4\delta_4}+\frac{C_V}{2}\right)\frac{1}{1-C_w\delta_2}|\langle w\nabla\cdot(P\nabla f), wf\rangle|\\
&\qquad\quad+\left(\frac{\varepsilon_V+\varepsilon_w}{4\delta_3}+C_w\delta_4\right)\|wVf\|^2+\eta_2\|wf\|^2
\end{aligned}\tag{4.54}$$

where we have set $\eta_1, \eta_2 > 0$ to be

$$\begin{aligned}
\eta_1 &:= (\varepsilon_V+\varepsilon_w)\delta_3+\left(\frac{C_w}{4\delta_4}+\frac{C_V}{2}\right)\frac{\varepsilon_w\delta_1}{1-C_w\delta_2},\\
\eta_2 &:= \frac{C_V}{2}+\left(\frac{C_w}{4\delta_4}+\frac{C_V}{2}\right)\frac{1}{1-C_w\delta_2}\left(\frac{\varepsilon_w}{4\delta_1}+\frac{C_w}{4\delta_2}\right).
\end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}
\|wP_1^{\frac12}(\nabla f)|V|^{\frac12}\|^2 &= \langle wP_1(\nabla f)|V|^{\frac12}, w(\nabla f)|V|^{\frac12}\rangle\\
&\le |\langle w^2P(\nabla f)|V|, \nabla f\rangle|\\
&\le |\langle w\nabla\cdot(P\nabla f), w|V|f\rangle|\\
&\qquad + |\langle P\nabla f, \nabla(w^2)|V|f+w^2(\nabla|V|)f\rangle|.
\end{aligned}\tag{4.55}$$

Assumptions (3.15) and (3.16) further imply

$$\begin{aligned}
&|\langle P\nabla f, \nabla(w^2)|V|f+w^2(\nabla|V|)f\rangle|\\
&\qquad\le \langle |P_1^{\frac12}\nabla f|, |P_1^{-\frac12}P^*\nabla(w^2)||Vf|+|P_1^{-\frac12}P^*\nabla|V||w^2|f|\rangle\\
&\qquad\le \langle w|P_1^{\frac12}\nabla f|, w((\varepsilon_V+\varepsilon_w)|V|^{\frac32}+C_w|V|+C_V)|f|\rangle.
\end{aligned}\tag{4.56}$$

Combining (4.54), (4.55), (4.56), and subsequently using Cauchy-Schwarz' inequality and Young's inequality with $\delta_5, \delta_6 > 0$, we arrive at

$$\begin{aligned}
&\langle w|P_1^{\frac12}\nabla f|, w((\varepsilon_V+\varepsilon_w)|V|^{\frac32}+C_w|V|+C_V)|f|\rangle\\
&\quad\le \frac{1}{1-\eta_1}\Bigg\{\eta_1|\langle w\nabla\cdot(P\nabla f), w|V|f\rangle|\\
&\qquad\quad+\left(\frac{C_w}{4\delta_4}+\frac{C_V}{2}\right)\frac{1}{1-C_w\delta_2}|\langle w\nabla\cdot(P\nabla f), wf\rangle|\\
&\qquad\quad+\left(\frac{\varepsilon_V+\varepsilon_w}{4\delta_3}+C_w\delta_4\right)\|wVf\|^2+\eta_2\|wf\|^2\Bigg\}\\
&\quad\le \frac{1}{1-\eta_1}\Bigg\{\left(\eta_1\delta_5+\left(\frac{C_w}{4\delta_4}+\frac{C_V}{2}\right)\frac{\delta_6}{1-C_w\delta_2}\right)\|w\nabla\cdot(P\nabla f)\|^2\\
&\qquad\quad+\left(\frac{\varepsilon_V+\varepsilon_w}{4\delta_3}+C_w\delta_4+\frac{\eta_1}{4\delta_5}\right)\|wVf\|^2+\eta_3\|wf\|^2\Bigg\},
\end{aligned}\tag{4.57}$$

where we have set

$$\eta_3 := \eta_2 + \left(\frac{C_w}{4\delta_4} + \frac{C_V}{2}\right)\frac{1}{1 - C_w\delta_2}\frac{1}{4\delta_6} > 0.$$

We continue by estimating the graph norm

$$\begin{aligned} \|w(-\nabla\cdot(P\nabla) + V)f\|^2 = \|w\nabla\cdot(P\nabla f)\|^2 + \|wVf\|^2 \\ - 2\operatorname{Re}\langle w\nabla\cdot(P\nabla f), wVf\rangle. \end{aligned} \tag{4.58}$$

Integrating by parts leads to

$$-\langle\nabla\cdot(P\nabla f), Vfw^2\rangle = \langle P\nabla f, (\nabla V)fw^2 + V(\nabla f)w^2 + Vf\nabla(w^2)\rangle \tag{4.59}$$

and using the combined accretivity (3.14),

$$\operatorname{Re}\langle wP\nabla f, wV\nabla f\rangle = \operatorname{Re}\langle we^{-\mathrm{i}\arg V}P(\nabla f)|V|^{\frac{1}{2}}, w(\nabla f)|V|^{\frac{1}{2}}\rangle \geq 0. \tag{4.60}$$

Employing (3.15) and (3.16), we moreover derive

$$\begin{aligned} &|\langle P\nabla f, (\nabla V)fw^2 + Vf\nabla(w^2)\rangle| \\ &\quad\leq \langle |P_1^{\frac{1}{2}}\nabla f|, |P_1^{-\frac{1}{2}}P^*\nabla V||f|w^2 + |P_1^{-\frac{1}{2}}P^*\nabla(w^2)||Vf|\rangle \\ &\quad\leq \langle w|P_1^{\frac{1}{2}}\nabla f|, w((\varepsilon_v + \varepsilon_w)|V|^{\frac{3}{2}} + C_w|V| + C_V)|f|\rangle. \end{aligned} \tag{4.61}$$

Putting together (4.58), (4.59), (4.60), (4.61) and (4.57), we arrive at

$$\|wT_wf\|^2 \geq \eta_4\|w\nabla\cdot(P\nabla f)\|^2 + \eta_5\|wVf\|^2 - \frac{2\eta_3}{1-\eta_1}\|wf\|^2$$

where $\eta_4, \eta_5 > 0$ are the constants

$$\begin{aligned} \eta_4 &:= 1 - \frac{2}{1-\eta_1}\left(\eta_1\delta_5 + \left(\frac{C_w}{4\delta_4} + \frac{C_V}{2}\right)\frac{\delta_6}{1 - C_w\delta_2}\right), \\ \eta_5 &:= 1 - \frac{2}{1-\eta_1}\left(\frac{\varepsilon_V + \varepsilon_w}{4\delta_3} + C_w\delta_4 + \frac{\eta_1}{4\delta_5}\right). \end{aligned}$$

It is important to note that since $\varepsilon_V + \varepsilon_w < 2 - \sqrt{2}$, with the special choice $\delta_3 := \delta$ and $\delta_5 := \delta'$ as in (4.50), we can achieve $\mu_1, \mu_2 \in (0,1)$, see Remark 4.13, and one can thus select δ_1, δ_4 and δ_6 small enough such that μ_1 and μ_2 are positive. □

Proof of Theorem 3.12. The claim follows from the lower estimate in Lemma 4.12 and the density of $\mathcal{D}_w$ in $\operatorname{Dom}(T_w)$ with respect to the graph norm of T_w, see Lemma 4.11. □

5. Applications and examples

5.1. **Completeness of eigensystems of Schrödinger operators in $L^2_{w^2}(\Omega)$.** Suppose that Assumption 3.1 is satisfied with a weight w, a purely imaginary regular potential $V = \mathrm{i}V_r$ (and $P = I_{\mathbb{C}^d}$) and consider the corresponding Schrödinger operator

$$T_w = -\Delta + \mathrm{i}V_r$$

in the weighted space $L^2_{w^2}(\Omega)$. We employ [19, Cor. XI.9.31] to establish the completeness of the eigensystem of T_w in $L^2_{w^2}(\Omega)$. To this end, we need information on the Schatten class of $(T_w - \lambda)^{-1}$ and an estimate of the resolvent norm $\|(T_w - \lambda)^{-1}\|$ for λ diverging to infinity along a sufficient amount of rays in $\mathbb{C}$. In the scope of this example, we only aim to study imaginary potentials which in general lead to accretive operators in $L^2(\Omega)$; other special cases, like sectorial potentials, can be analysed in an analogous way.

We obtain the Schatten class of $(T_w - \lambda)^{-1}$ from Theorem 3.5 and

Theorem 5.1 ([4, Thm. 1.3], [13])**.** *Let $\Omega \subset \mathbb{R}^d$ be open and $\partial\Omega \in C^{2,\alpha}$ for some $\alpha > 0$. Let $Q \in L^2_{\mathrm{loc}}(\Omega)$, $Q \geq 0$, and for $p > 0$*

$$\int_{\Omega \times \mathbb{R}^d} (|\xi|^2 + Q(x) + 1)^{-p} \, \mathrm{d}x\mathrm{d}\xi < \infty. \tag{5.1}$$

Then for the self-adjoint Dirichlet realisation $S := -\Delta + Q$, we have

$$(S+1)^{-1} \in \mathcal{S}_p(L^2(\Omega)).$$

Note that $(S+1)^{-1} \in \mathcal{S}_p(L^2(\Omega))$ implies $(S+1)^{-1/2} \in \mathcal{S}_{2p}(L^2(\Omega))$. Moreover, by the second representation theorem [29, Thm. VI.2.23], we have

$$\|(S+1)^{\frac{1}{2}} \cdot \|^2 = \|\nabla \cdot \|^2 + \|Q^{\frac{1}{2}} \cdot \|^2 + \| \cdot \|^2.$$

In other words, if $Q := |V_r|$ is such that (5.1) is satisfied for some $p > 0$, then $\mathrm{id}_{\mathcal{V}_1} \in \mathcal{S}_{2p}(\mathcal{V}_1, L^2(\Omega))$ and hence Theorem 3.5 implies

$$(T_w - \lambda)^{-1} \in \mathcal{S}_p(L^2_{w^2}(\Omega)).$$

In particular, for $\Omega = \mathbb{R}^d$, $|V_r(x)| + 1 \gtrsim \langle x \rangle^\gamma$ with $\gamma > 0$ and any admissible weight w, we conclude for the corresponding weighted Schrödinger operator T_w that

$$(T_w - \lambda)^{-1} \in \mathcal{S}_p(L^2_{w^2}(\Omega)), \quad \lambda \in \rho(T_w), \quad p > p_{\gamma,d} := \frac{2+\gamma}{2\gamma} d;$$

the value of $p_{\gamma,d}$ is obtained from (5.1) and Young's inequality. In the sequel, we consider weights

$$w(x) = \exp(\pm \langle x \rangle^\alpha), \qquad 0 < \alpha < 1 + \frac{\gamma}{2},$$

for which (3.4) is satisfied with arbitrarily small $\sigma_w > 0$. Note that in such case, $\beta > 0$ in the conditions (3.9) or (3.7) can be selected arbitrarily small.

The resolvent estimates required for the completeness result [19, Cor. XI.9.31] then follow from Lemma 4.3. Replacing $\mathbf{t}_w$ by $\mathbf{t}_w - \lambda$ with $\operatorname{Re}\lambda < 0$ leads to an additional term $(|\operatorname{Re}\lambda| - \beta|\operatorname{Im}\lambda|)\|f\|^2_{\mathcal{V}_w}$ on the right hand sides of (4.7). Since one can assume β to be arbitrarily small, we conclude that (for the considered potentials and weights)

$$\|(T_w - re^{\mathrm{i}\omega})^{-1}\| \lesssim 1, \quad r \to +\infty,$$

for any $\omega \in (\pi/2, 3\pi/2)$. This in particular covers the well-known example of the imaginary cubic oscillator ($V_r(x) = x^3$, $d = 1$), for which the completeness of the eigensystem in $L^2(\mathbb{R})$ was established in [42]. We thereby extend the completeness result therein to weighted operators in $L^2_{w^2}(\mathbb{R})$ with weights $w(x) = \exp(\pm\langle x \rangle^\alpha)$ where $0 < \alpha < 1 + 3/2$.

5.2. **Schur complement dominant matrix differential operator.** We employ Theorems 3.2, 3.12 and the results about Schur complement dominance, see Section 2.3, to show that the operator matrix

$$\mathcal{A} = \begin{pmatrix} -\partial_x^2 + \mathrm{i}\sinh(5x^2) & e^{x^2} \\ e^x\partial_x + e^{3x^2} & 0 \end{pmatrix} \tag{5.2}$$

in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ can be realised as a closed, densely defined operator with non-empty resolvent set. Note that $\mathcal{A}$ is an example of an operator matrix with highly non-symmetric off-diagonal and without any usual (diagonal, off-diagonal) dominance pattern. Nonetheless, the off-diagonal entries can be controlled by the first Schur complement

$$S(\lambda) = -\partial_x^2 + \mathrm{i}\sinh(5x^2) - \lambda + \frac{1}{\lambda} e^{x^2}(e^x\partial_x + e^{3x^2}), \quad \lambda \in \mathbb{C} \setminus \{0\}, \tag{5.3}$$

in a suitable representation of the resolvent of $\mathcal{A}$. In order to satisfy the conditions for Schur complement dominance, $S(\lambda)$ will be realised in a weighted space.

Proposition 5.2. *Let $\mathcal{A}$ be the matrix differential expression in* (5.2). *Let $\mathcal{D}_S$ be the closure of $C_0^\infty(\mathbb{R})$ with respect to the norm*

$$\|f\|_{\mathcal{D}_S}^2 := \|e^{-x^2} f''\|^2 + \|e^{4x^2} f\|^2, \quad f \in C_0^\infty(\mathbb{R}). \tag{5.4}$$

Define the operator in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$

$$\begin{aligned} \mathcal{A}_0(f,g) &:= \mathcal{A}(f,g), \\ \operatorname{Dom}(\mathcal{A}_0) &:= \{(f,g) \in \mathcal{D}_S \oplus L^2(\mathbb{R}) \,:\, \mathcal{A}(f,g) \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R})\}. \end{aligned}$$

Then $\mathcal{A}_0$ is closed, has non-empty resolvent set and its domain is dense in both $\mathcal{D}_S \oplus L^2(\mathbb{R})$ and $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. Moreover, for $\lambda \in \mathbb{C} \setminus \{0\}$ and the operator

$$S_0(\lambda) f := S(\lambda) f, \quad \operatorname{Dom}(S_0(\lambda)) := W^{2,2}(\mathbb{R}) \cap \operatorname{Dom}(e^{5x^2}), \tag{5.5}$$

acting in $L^2(\mathbb{R})$, where $S(\lambda)$ is as in (5.3), *we have the spectral equivalence*

$$\lambda \in \sigma(\mathcal{A}_0) \quad \Longleftrightarrow \quad 0 \in \sigma(S_0(\lambda)) = \sigma_{\mathrm{p}}(S_0(\lambda)). \tag{5.6}$$

Moreover, every point $\lambda \neq 0$ in $\sigma(\mathcal{A}_0)$ is an eigenvalue of $\mathcal{A}_0$.

Proof. The claims follow by applying Theorem 2.4. In order to define $\mathcal{A}$ and $S(\cdot)$ as considered therein, we first specify the spaces and operators needed for Assumption 2.3. In our case, $\mathcal{D}_S$ is defined as in (5.4) and we have

$$\mathcal{H}_1 := \mathcal{H}_2 := L^2(\mathbb{R}), \qquad \mathcal{D}_2 := \mathcal{D}_{-2} := L^2(\mathbb{R}), \qquad \mathcal{D}_{-S} := L^2_{e^{-2x^2}}(\mathbb{R}).$$

The fact that $A := -\partial_x^2 + \mathrm{i} \sinh(5x^2) \in \mathcal{B}(\mathcal{D}_S, \mathcal{D}_{-S})$ follows from the inequality

$$\|Af\|_{\mathcal{D}_{-S}} \lesssim \|e^{-x^2} f''\| + \|e^{4x^2} f\| \lesssim \|f\|_{\mathcal{D}_S}, \quad f \in C_0^\infty(\mathbb{R}).$$

Moreover, $D := D_0 := 0$ is bounded and boundedly invertible on $L^2(\mathbb{R})$, and the multiplication operator $B := e^{x^2}$ is clearly bounded between $L^2(\mathbb{R})$ and $\mathcal{D}_{-S}$. For all $f \in C_0^\infty(\mathbb{R})$, integration by parts and Cauchy-Schwarz' and Young's inequalities (with $\delta_1, \delta_2 > 0$) yield that

$$\begin{aligned} \|e^x f'\|^2 &= \langle f', e^{2x} f' \rangle = -\langle f, 2e^{2x} f' + e^{2x} f'' \rangle \\ &\leq \delta_1 \|e^x f'\|^2 + \delta_2 \|e^{-x^2} f''\|^2 + C_{\delta_1, \delta_2} (\|e^x f\|^2 + \|e^{x^2 + 2x} f\|^2). \end{aligned}$$

The above implies that $C := e^x \partial_x + e^{3x^2} \in \mathcal{B}(\mathcal{D}_S, L^2(\mathbb{R}))$, more precisely we have (with $\delta_3, \delta_4 > 0$)

$$\|Cf\| \leq \delta_3 \|e^{-x^2} f''\| + C_{\delta_3} \|e^{3x^2} f\| \leq \delta_4 \|f\|_{\mathcal{D}_S} + C_{\delta_4} \|e^{-x^2} f\|, \quad f \in C_0^\infty(\mathbb{R}), \tag{5.7}$$

where the second inequality can be shown using Hölder's inequality and δ_4 therein can be taken arbitrarily small (which we use later).

Having introduced the matrix entries according to Assumption 2.3, we can define

$$\mathcal{A} \in \mathcal{B}(\mathcal{D}_S \oplus L^2(\mathbb{R}), \mathcal{D}_{-S} \oplus L^2(\mathbb{R})), \quad S(\lambda) \in \mathcal{B}(\mathcal{D}_S, \mathcal{D}_{-S}), \quad \lambda \in \mathbb{C} \setminus \{0\},$$

by the formulas in (2.5). We show that for each $\lambda \in \mathbb{C} \setminus \{0\}$, there is $z_\lambda \in \mathbb{C}$ with

$$(S(\lambda) - z_\lambda)^{-1} \in \mathcal{B}(\mathcal{D}_{-S}, \mathcal{D}_S); \tag{5.8}$$

moreover, one can choose $z_\lambda = 0$ for $\lambda < 0$ with $|\lambda|$ sufficiently large. The claims about the spectral equivalence between $\mathcal{A}_0$ and the maximal restriction $S_0(\cdot)$ of $S(\cdot)$ to $L^2(\mathbb{R})$, a well as the claim about the density of $\operatorname{Dom}(\mathcal{A}_0)$, then follow from Theorem 2.4 with $\Sigma = \mathbb{C} \setminus \{0\}$. Moreover, we show below that $S_0(\lambda)$ has compact resolvent, so the last equality in (5.6) follows.

To show (5.8), consider the Dirichlet realisation of the differential expression

$$T_w(\mu) := -\partial_x^2 + \mathrm{i} \sinh(5x^2) + \mu, \qquad \mu > 0,$$

in the space $\mathcal{D}_{-S} = L^2_{w^2}(\mathbb{R})$ with weight $w(x) := e^{-x^2}$, $x \in \mathbb{R}$. Here Assumptions 3.1 and also 3.10 are satisfied with

$$P := 1, \quad U_r := V_s := 0, \quad U_s := \mu, \quad V_r(x) := \sinh(5x^2), \quad x \in \mathbb{R},$$

where the constants in (3.3), (3.4), (3.15) and (3.16) can be selected independently of μ. In particular, it follows from the asymptotic relations, as $|x| \to \infty$,

$$\frac{|V_r'(x)|}{(1+V_r^2(x))^{\frac{3}{2}}} = o(|\sinh(5x^2)|^{\frac{1}{2}}), \qquad \frac{|(w^2)'(x)|}{w^2(x)} = o(|\sinh(5x^2)|^{\frac{1}{2}}),$$
$$|V_r'(x)| = o(|\sinh(5x^2)|^{\frac{3}{2}}),$$

that the constants ε, κ_w and σ_w in (3.3) and (3.4), as well as ε_V and ε_w in (3.15) and (3.16), can be chosen arbitrarily small and uniform in μ. Hence, $T_w(\mu)$ is indeed well-defined by Theorem 3.2. Moreover, Theorem 3.12 gives

$$\|T_w(\mu)f\|_{L^2_{w^2}} \geq c_1 \left(\|f''\|_{L^2_{w^2}} + \|(\mathrm{i}\sinh(5x^2)+\mu)f\|_{L^2_{w^2}} \right) - c_2\|f\|_{L^2_{w^2}}$$

for all $f \in C_0^\infty(\mathbb{R}) \subset \mathrm{Dom}(T_w(\mu))$, where the constants $c_1 > 0$ and $c_2 \geq 0$ are independent of μ. From the above it easily follows that

$$\|T_w(\mu)f\|_{\mathcal{D}_{-S}} \geq m_1\|f\|_{\mathcal{D}_S} + m_2(\mu - c)\|f\|_{\mathcal{D}_{-S}}, \quad f \in C_0^\infty(\mathbb{R}), \quad \mu > 0, \tag{5.9}$$

with $m_1, m_2 > 0$ and $c \geq 0$ all independent of μ. Notice that from Lemma 4.11 and an additional mollification argument, see [14, proof of Thm. 8.2.1, Part 3], it follows that $C_0^\infty(\mathbb{R})$ is a core of $T_w(\mu)$ and thus (5.9) remains valid on $\mathrm{Dom}(T_w(\mu)) = \mathcal{D}_S$.

Consider now $\lambda \in \mathbb{C} \setminus \{0\}$. Then $S(\lambda)$, viewed as a linear operator in $\mathcal{D}_{-S}$ with $\mathrm{Dom}(S(\lambda)) = \mathcal{D}_S$, is a relatively bounded perturbation of $T_w(\mu)$, for any $\mu > 0$, with relative bound zero. In detail, for all $f \in C_0^\infty(\mathbb{R})$, the estimate (5.7) gives

$$\|e^{x^2}(e^x\partial_x + e^{3x^2})f\|_{L^2_{e^{-2x^2}}} = \|Cf\| \leq \delta_4\|f\|_{\mathcal{D}_S} + C_{\delta_4}\|f\|_{L^2_{e^{-2x^2}}}, \tag{5.10}$$

with arbitrarily small $\delta_4 > 0$ and some $C_{\delta_4} \geq 0$. By density and continuity, the above extends to all $f \in \mathcal{D}_S = \mathrm{Dom}(T_w(\mu))$ and the claimed zero order relative boundedness follows from (5.9) and (5.10). Since the resolvent of $T_w(\mu)$ is compact by Theorems 3.5 and 5.1, and since by (5.9) the norm of $T_w(\mu)^{-1}$ decays as $\mu \to \infty$, also $S(\lambda)$ has compact resolvent, see [29, Thm. IV.1.16]. Consequently, the resolvent

$$(S_0(\lambda) - z)^{-1} = \mathrm{id}_{\mathcal{D}_S}(S(\lambda) - z\,\mathrm{id}_{L^2(\Omega)}\,\mathrm{id}_{\mathcal{D}_S})^{-1}\,\mathrm{id}_{L^2(\Omega)}, \quad z \in \rho(S_0(\lambda)),$$

of the maximal restriction $S_0(\lambda)$ in $L^2(\mathbb{R})$ is compact as well, see also (2.3).

In order to show (5.8), we compare $S(\lambda) - z_\lambda$ to $T_w(\mu)$ with a suitable $\mu > 0$. For $\lambda \in \mathbb{C} \setminus \{0\}$ and $f \in \mathcal{D}_S$, using (5.10), (5.9) and setting $z_\lambda := -(\mu + \lambda)$, we have

$$\begin{aligned} \|(S(\lambda) - z_\lambda)f\|_{\mathcal{D}_{-S}} &\geq \|T_w(\mu)f\|_{\mathcal{D}_{-S}} - \frac{\delta_4}{|\lambda|}\|f\|_{\mathcal{D}_S} - \frac{C_{\delta_4}}{|\lambda|}\|f\|_{\mathcal{D}_{-S}} \\ &\geq \left(m_1 - \frac{\delta_4}{|\lambda|}\right)\|f\|_{\mathcal{D}_S} - \left(m_2(\mu - c) - \frac{C_{\delta_4}}{|\lambda|}\right)\|f\|_{\mathcal{D}_{-S}}. \end{aligned} \tag{5.11}$$

Clearly, for fixed $\lambda \in \mathbb{C} \setminus \{0\}$, one can select δ_4 sufficiently small and μ sufficiently large such that both parentheses above are positive. It follows that z_λ is a regular point for $S(\lambda)$ and, since the latter has compact resolvent, that (5.8) holds. For $\lambda < 0$ however, choosing $\mu = -\lambda$ in the above estimate shows that, if $|\lambda|$ is sufficiently large, (5.8) is satisfied with $z_\lambda = 0$.

It remains to explain that $\mathrm{Dom}(S_0(\lambda))$ is as in (5.5), i.e. that $S_0(\lambda) = \widetilde{S}_0(\lambda)$ where the latter acts as (5.3) in $L^2(\mathbb{R})$ on the domain

$$\mathrm{Dom}(\widetilde{S}_0(\lambda)) := \mathcal{D}_0 := W^{2,2}(\mathbb{R}) \cap \mathrm{Dom}(e^{5x^2}), \quad \lambda \in \mathbb{C} \setminus \{0\}.$$

Analogously to before, one shows that $\widetilde{S}_0(\lambda)$ is a bound zero perturbation of $T_1(\mu)$ with $\operatorname{Dom}(T_1(\mu)) = \mathcal{D}_0$ for any $\mu > 0$. Moreover, analogously to (5.11), one can find $\mu > 0$ sufficiently large such that both

$$\widetilde{S}_0(-\mu)^{-1} \in \mathcal{B}(L^2(\mathbb{R})), \quad S_0(-\mu)^{-1} \in \mathcal{B}(L^2(\mathbb{R})).$$

For the claimed equality it thus suffices to show that $\mathcal{D}_0 \subset \operatorname{Dom}(S_0(\lambda))$. This, however, is obvious since

$$S(\lambda)f = \widetilde{S}_0(\lambda)f \in L^2(\mathbb{R}), \qquad f \in \mathcal{D}_0 \subset \mathcal{D}_S. \qquad \square$$

5.3. **Diagonally dominant matrix Schrödinger operator in a weighted space.** Similarly as in Section 5.2, a realisation of the matrix Schrödinger operator

$$\mathcal{A} = \begin{pmatrix} -\partial_x^2 + \mathrm{i}x^3 & x \\ x^4 & -\partial_x^2 + x^6 \end{pmatrix} \tag{5.12}$$

in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ with non-empty resolvent set can be found using Schur complement dominance (in fact, even with respect to both Schur complements). However, one can also select suitable weights w_1 and w_2 such that (5.12) becomes diagonally dominant in the product space $L^2_{w_1^2}(\mathbb{R}) \oplus L^2_{w_2^2}(\mathbb{R})$.

To be more precise, let $w_1 := \langle x \rangle$ and $w_2 := \langle x \rangle^{-1}$, $x \in \mathbb{R}$, and consider the Dirichlet realisations of

$$A_{w_1} := -\partial_x^2 + \mathrm{i}x^3, \qquad D_{w_2} := -\partial_x^2 + x^6$$

in the spaces $L^2_{w_1^2}(\mathbb{R})$ and $L^2_{w_2^2}(\mathbb{R})$, respectively. By Theorem 3.12, with $\mu > 0$ sufficiently large, we then have

$$\begin{aligned} \|(A_{w_1} + \mu)f\|_{L^2_{w_1^2}} &\gtrsim \|f''\|_{L^2_{w_1^2}} + \|\langle x \rangle^3 f\|_{L^2_{w_1^2}}, \qquad && f \in \operatorname{Dom}(A), \\ \|(D_{w_2} + \mu)g\|_{L^2_{w_2^2}} &\gtrsim \|g''\|_{L^2_{w_2^2}} + \|\langle x \rangle^6 g\|_{L^2_{w_2^2}}, \qquad && g \in \operatorname{Dom}(D). \end{aligned} \tag{5.13}$$

Moreover, for the multiplication operators

$$B := x : L^2_{w_2^2}(\mathbb{R}) \to L^2_{w_1^2}(\mathbb{R}), \qquad C := x^4 : L^2_{w_1^2}(\mathbb{R}) \to L^2_{w_2^2}(\mathbb{R}),$$

defined on their maximal domains (which by (5.13) contain the domains of D and A, respectively), using Hölder's and Young's inequalities, we arrive at

$$\begin{aligned} \|Cf\|_{L^2_{w_2^2}} &\lesssim \|\langle x \rangle^2 f\|_{L^2_{w_1^2}} \lesssim \varepsilon \|\langle x \rangle^3 f\|_{L^2_{w_1^2}} + C_\varepsilon \|f\|_{L^2_{w_1^2}}, \qquad && f \in \operatorname{Dom}(A), \\ \|Bg\|_{L^2_{w_1^2}} &\lesssim \|\langle x \rangle^3 g\|_{L^2_{w_2^2}} \lesssim \varepsilon \|\langle x \rangle^6 g\|_{L^2_{w_2^2}} + C_\varepsilon \|g\|_{L^2_{w_2^2}}, \qquad && g \in \operatorname{Dom}(D), \end{aligned} \tag{5.14}$$

with arbitrarily small $\varepsilon > 0$ and some $C_\varepsilon \geq 0$. Combining the inequalities (5.13) and (5.14), we indeed obtain diagonal dominance of order 0.

We remark that the effect of considering weighted spaces can be equivalently explained as a transformation of $\mathcal{A}$ from $L^2_{w_1^2}(\mathbb{R}) \oplus L^2_{w_2^2}(\mathbb{R})$ to $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$, i.e. by employing the conjugation

$$\operatorname{diag}(w_1, w_2) \mathcal{A} \operatorname{diag}(w_1, w_2)^{-1} = \begin{pmatrix} w_1(-\partial_x^2 + \mathrm{i}x^3)w_1^{-1} & w_1 x w_2^{-1} \\ w_2 x^4 w_1^{-1} & w_2(-\partial_x^2 + x^6)w_2^{-1} \end{pmatrix}.$$

Similarly to choosing suitable constants $w_1, w_2 > 0$ e.g. in [36], we select weights w_1 and w_2 in order to balance the off-diagonal terms.

5.4. **Damped wave equation in weighted space with accretive and unbounded damping.** We consider the following damped wave equation on an open set $\Omega \subset \mathbb{R}^d$

$$u_{tt}(t,x)+2(a_1(x)+\mathrm{i}a_2(x)-\nabla_x\cdot(a_0(x)\nabla_x))u_t=\Delta_x u(t,x),\quad x\in\Omega,\ t>0,\tag{5.15}$$

where we assume $a_1 \in L^1_{\mathrm{loc}}(\Omega)$, $a_1 \geq 0$, $a_2 \in W^{1,\infty}_{\mathrm{loc}}(\overline{\Omega};\mathbb{R})$ and that $a_0 \in L^1_{\mathrm{loc}}(\Omega)^{d\times d}$ is positive semi-definite almost everywhere in Ω. By a standard procedure, (5.15) can be rewritten as a system

$$\partial_t\begin{pmatrix}u_1\\u_2\end{pmatrix}=\mathcal{A}\begin{pmatrix}u_1\\u_2\end{pmatrix}$$

where $\mathcal{A}$ is the operator matrix

$$\mathcal{A}=\begin{pmatrix}0 & I\\ \Delta & -2(a_1+\mathrm{i}a_2-\nabla\cdot(a_0\nabla))\end{pmatrix}.\tag{5.16}$$

Our goal is to show that a suitable realisation of $-\mathcal{A}$ generates a semigroup in the product space $\mathcal{H}_w := \mathcal{W}_w \oplus L^2_{w^2}(\Omega)$, where $\mathcal{W}_w$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|f\|^2_{\mathcal{W}_w} := \|\nabla f\|^2_{L^2_{w^2}} + \|f\|^2_{L^2_{w^2}},\qquad f\in C_0^\infty(\Omega),\tag{5.17}$$

with a suitably chosen weight $w \in W^{1,\infty}_{\mathrm{loc}}(\overline{\Omega};\mathbb{R}_+)$. To this end, we employ Schur complement dominance, see Section 2.3.

At first we introduce the space $\mathcal{D}_S$ as the closure of $C_0^\infty(\Omega)$ with respect to

$$\|f\|^2_{\mathcal{D}_S} := \|(I_{\mathbb{C}^d}+a_0)^{\frac12}\nabla f\|^2_{L^2_{w^2}} + \||a_1+\mathrm{i}a_2|^{\frac12} f\|^2_{L^2_{w^2}} + \|f\|^2_{L^2_{w^2}}\tag{5.18}$$

and the following operator entries

$$\begin{aligned}&A := 0\in\mathcal{B}(\mathcal{W}_w), && B := I\in\mathcal{B}(\mathcal{D}_S,\mathcal{W}_w),\\ &C := \Delta\in\mathcal{B}(\mathcal{W}_w,\mathcal{D}_S^*), && D := -2(a_1+\mathrm{i}a_2-\nabla\cdot(a_0\nabla))\in\mathcal{B}(\mathcal{D}_S,\mathcal{D}_S^*).\end{aligned}\tag{5.19}$$

Here $\mathcal{D}_S^*$ is the (anti-)dual space of $\mathcal{D}_S$ and C, D are understood as the unique bounded extensions of

$$\begin{aligned}(\Delta f,g)_{\mathcal{D}_S^*\times\mathcal{D}_S} &:= \langle\Delta f,g\rangle_{L^2_{w^2}},\\ (-2(a_1+\mathrm{i}a_2-\nabla\cdot(a_0\nabla))f,g)_{\mathcal{D}_S^*\times\mathcal{D}_S} &:= -2\langle(a_1+\mathrm{i}a_2)f,g\rangle_{L^2_{w^2}}\\ &\quad - 2\langle a_0\nabla f,\nabla(w^2 g)\rangle,\end{aligned}\tag{5.20}$$

initially defined for $f,g\in C_0^\infty(\Omega)$; see the justification within the proof of Proposition 5.3 below. With these definitions, one can interpret the matrix in (5.16) as

$$\mathcal{A}\in\mathcal{B}(\mathcal{D},\mathcal{D}_-),\quad \mathcal{D}:=\mathcal{W}_w\oplus\mathcal{D}_S,\quad \mathcal{D}_- := \mathcal{W}_w\oplus\mathcal{D}_S^* = \mathcal{D}^*,\tag{5.21}$$

and introduce its second Schur complement (with $\lambda\in\mathbb{C}\setminus\{0\}$)

$$S(\lambda) := -2(a_1+\mathrm{i}a_2-\nabla\cdot(a_0\nabla))-\lambda+\frac{1}{\lambda}\Delta_{\restriction\mathcal{D}_S}\in\mathcal{B}(\mathcal{D}_S,\mathcal{D}_S^*).$$

Proposition 5.3. *Let $\mathcal{A}$ be as in* (5.16), (5.21)*, let $\mathcal{W}_w$, $\mathcal{D}_S$ be as in* (5.17), (5.18)*, respectively, and let $\mathcal{A}_0$ be the maximal restriction of $\mathcal{A}$ to $\mathcal{H}_w = \mathcal{W}_w \oplus L^2_{w^2}(\Omega)$, i.e. $\mathcal{A}_0 := \mathcal{A}_{\restriction\mathrm{Dom}(\mathcal{A}_0)}$ with*

$$\mathrm{Dom}(\mathcal{A}_0) := \{(f,g)\in\mathcal{W}_w\oplus\mathcal{D}_S\ :\ \mathcal{A}(f,g)\in\mathcal{H}_w\}.$$

Suppose that a_1, a_2, a_0 and w satisfy in addition that

$$\begin{aligned}|(I_{\mathbb{C}_d}+a_0)^{\frac12}\nabla a_2| &\leq c(1+|a_2|^3)(a_1^{\frac12}+|a_2|^{\frac12}+1),\\ |\nabla(w^2)| &\leq cw^2(a_1^{\frac12}+|a_2|^{\frac12}+1),\\ |a_0^{\frac12}\nabla(w^2)| &\leq \sqrt{2}\varepsilon_0 w^2(a_1+c_0)^{\frac12}\end{aligned}\tag{5.22}$$

with some $c > 0$, $\varepsilon_0 \in (0,2)$ and $c_0 \geq 0$. Then $\mathcal{A}_0$ generates a C_0-semigroup in $\mathcal{H}_w$ and its domain is dense in $\mathcal{W}_w \oplus \mathcal{D}_S$ and in $\mathcal{H}_w$.

Proof. We first justify that C and D in (5.19) are well-defined, i.e. that there are indeed unique bounded extensions of the operators in (5.20). To this end, employing the second assumption in (5.22), we obtain for all $f, g \in C_0^\infty(\Omega)$

$$|\langle \Delta f, g\rangle_{L^2_{w^2}}| \leq |\langle \nabla f, \nabla g\rangle_{L^2_{w^2}}| + |\langle \nabla f, w^{-2}\nabla(w^2) g\rangle_{L^2_{w^2}}| \lesssim \|f\|_{\mathcal{W}_w}\|g\|_{\mathcal{D}_S},$$

justifying the claimed boundedness of C. Moreover,

$$\begin{aligned}|\langle (a_1 + \mathrm{i}a_2 - \nabla\cdot(a_0\nabla))f, g\rangle_{L^2_{w^2}}| &\leq \||a_1+\mathrm{i}a_2|^{\frac12} f\|_{L^2_{w^2}} \||a_1+\mathrm{i}a_2|^{\frac12} g\|_{L^2_{w^2}} \\ &\quad + \|a_0^{\frac12}\nabla f\|_{L^2_{w^2}}\|a_0^{\frac12}\nabla g\|_{L^2_{w^2}} \\ &\quad + \|a_0^{\frac12}\nabla f\|_{L^2_{w^2}}\|w^{-2}a_0^{\frac12}\nabla(w^2) g\|_{L^2_{w^2}} \\ &\lesssim \|f\|_{\mathcal{D}_S}\|g\|_{\mathcal{D}_S},\end{aligned}$$

where we have used the last relation in (5.22) for the last inequality. Thus the claims on both bounded extensions follow.

Next we show that the matrix $-\mathcal{A}_0 + \mu$ is accretive in $\mathcal{H}_w$ if $\mu > 0$ is sufficiently large. For all $f_1, f_2 \in C_0^\infty(\Omega)$, we have

$$\begin{aligned}&(-\mathcal{A}(f_1,f_2),(f_1,f_2))_{\mathcal{D}^*\times\mathcal{D}} \\ &\qquad = -\langle f_2, f_1\rangle_{\mathcal{W}_w} - (\Delta f_1 - 2(a_1+\mathrm{i}a_2 - \nabla\cdot(a_0\nabla))f_2, f_2)_{\mathcal{D}_S^*\times\mathcal{D}_S} \\ &\qquad = -\langle \nabla f_2, \nabla f_1\rangle_{L^2_{w^2}} - \langle f_2, f_1\rangle_{L^2_{w^2}} + \langle \nabla f_1, \nabla f_2\rangle_{L^2_{w^2}} \\ &\qquad\qquad + \langle \nabla f_1, w^{-2}\nabla(w^2) f_2\rangle_{L^2_{w^2}} + 2\langle (a_1+\mathrm{i}a_2)f_2, f_2\rangle_{L^2_{w^2}} \\ &\qquad\qquad + \|a_0^{\frac12}\nabla f_2\|^2_{L^2_{w^2}} + \langle a_0^{\frac12}\nabla f_2, w^{-2}a_0^{\frac12}\nabla(w^2) f_2\rangle_{L^2_{w^2}}.\end{aligned}$$

Cauchy-Schwarz's and Young's inequalities (with $\delta_1 > 0$) then yield

$$\begin{aligned}&\mathrm{Re}((-\mathcal{A} + \mu\,\mathrm{id}_{\mathcal{D}}^*\,\mathrm{id}_{\mathcal{D}})(f_1,f_2),(f_1,f_2))_{\mathcal{D}^*\times\mathcal{D}} \\ &\qquad \geq -\frac12(\|f_2\|^2_{L^2_{w^2}} + \|f_1\|^2_{L^2_{w^2}}) - \frac{1}{4\delta_1}\|\nabla f_1\|^2_{L^2_{w^2}} - \delta_1\|w^{-2}\nabla(w^2) f_2\|^2_{L^2_{w^2}} \\ &\qquad\quad + 2\|a_1^{\frac12} f_2\|^2_{L^2_{w^2}} - \frac14\|w^{-2}a_0^{\frac12}\nabla(w^2) f_2\|^2_{L^2_{w^2}} \\ &\qquad\quad + \mu\left(\|\nabla f_1\|^2_{L^2_{w^2}} + \|f_1\|^2_{L^2_{w^2}} + \|f_2\|^2_{L^2_{w^2}}\right),\end{aligned}$$

hence using the third assumption in (5.22), choosing δ_1 sufficiently small and μ sufficiently large, we indeed obtain that the above quantity is non-negative. By (5.21) and the density of $C_0^\infty(\Omega)$ in both $\mathcal{D}_S$ and $L^2_{w^2}(\Omega)$, the latter can be extended to all $(f_1,f_2) \in \mathcal{D}$. For vectors $(f_1,f_2) \in \mathrm{Dom}(\mathcal{A}_0)$, this then gives

$$\begin{aligned}&\mathrm{Re}\langle (-\mathcal{A}+\mu)(f_1,f_2),(f_1,f_2)\rangle_{\mathcal{H}_w} \\ &\qquad = \mathrm{Re}((-\mathcal{A} + \mu\,\mathrm{id}_{\mathcal{D}}^*\,\mathrm{id}_{\mathcal{D}})(f_1,f_2),(f_1,f_2))_{\mathcal{D}^*\times\mathcal{D}} \geq 0.\end{aligned}$$

We proceed by showing that $S(\lambda)$ is boundedly invertible, if $\lambda > 0$ is sufficiently large, i.e. that $S(\lambda)^{-1} \in \mathcal{B}(\mathcal{D}_S^*, \mathcal{D}_S)$. To this end, we use that $-\lambda S(\lambda) = \widehat{T}_w(\lambda)$, where the latter denotes the operator associated with the form $\mathbf{t}_w(\lambda)$ of the Dirichlet realisation of

$$-\nabla\cdot((I_{\mathbb{C}^d} + \lambda a_0)\nabla) + 2\lambda(a_1+\mathrm{i}a_2) + \lambda^2, \quad \lambda > \lambda_0 > 0,$$

in the space $L^2_{w^2}(\Omega)$, see (2.2), with form domain $\mathcal{V}_w = \mathcal{D}_S$. Notice that while the coefficients, and thus the corresponding norms defined in (3.6), depend on λ, the norms are in fact all equivalent to $\|\cdot\|_S$ such that the form domain itself, as a topological space, remains constant. To verify the assumptions of Theorem 3.2, we take

$$P = I_{\mathbb{C}^d} + \lambda a_0, \quad U_r = \lambda^2, \quad U_s = 2\lambda a_1, \quad V_r = 2\lambda a_2, \quad V_s = 0.$$

Employing (5.22), it is elementary to see that the conditions (iv) and (v) in Assumption 3.1 are satisfied, where the constants therein depend on λ and exhibit the following asymptotic behaviour as $\lambda \to \infty$

$$\varepsilon = \mathcal{O}(\lambda^{-2}),\ C_\varepsilon = \mathcal{O}(\lambda^{-2}),\ \kappa_w = \varepsilon_0 + \mathcal{O}(\lambda^{-\frac{1}{2}}),\ \sigma_w = \mathcal{O}(\lambda^{-\frac{1}{2}}),\ C_w = \mathcal{O}(\lambda^{-\frac{1}{2}}).$$

It follows that there exists $\lambda_0 > 0$ such that the above constants can be selected sufficiently small and uniformly in λ, i.e. such that Assumption 3.1, as well as condition (3.7), are satisfied with the same constants therein for all $\lambda > \lambda_0$, see also Remark 3.3 (i). Theorem 3.2 thus indeed holds and in particular, we obtain the estimates in (4.7) with λ-independent constants m_1, m_2, γ_1 and γ_2. Since $|V| \geq \operatorname{Re} V \geq \lambda^2$, the latter estimates yield generalised coercivity of $\mathbf{t}_w(\lambda)$ if $\lambda > \lambda_0$ is chosen sufficiently large, implying that

$$S(\lambda)^{-1} = -\lambda \widehat{T}_w(\lambda)^{-1} \in \mathcal{B}(\mathcal{V}_w^*, \mathcal{V}_w) = \mathcal{B}(\mathcal{D}_S^*, \mathcal{D}_S).$$

Employing Theorem 2.4, we conclude that $\operatorname{Dom}(\mathcal{A}_0)$ is dense in both $\mathcal{W}_w \oplus \mathcal{D}_S$ and $\mathcal{H}_w$. Moreover, it gives $\rho(\mathcal{A}_0) \cap \mathbb{R}_+ \neq \emptyset$ and in consequence that $-\mathcal{A}_0 + \mu$ is m-accretive in $\mathcal{H}_w$. Hence, $-\mathcal{A}_0$ indeed generates a C_0-semigroup, see e.g. [16, Thm. 11.4.1]. □

References

[1] Adams, R. A., and Fournier, J. J. F. *Sobolev spaces*, 2nd ed. Elsevier, Amsterdam, 2003.

[2] Almog, Y. The Stability of the Normal State of Superconductors in the Presence of Electric Currents. *SIAM J. Math. Anal. 40* (2008), 824–850.

[3] Almog, Y., and Helffer, B. On the spectrum of some Bloch-Torrey vector operators. arXiv:2009.03036 [math-ph].

[4] Almog, Y., and Helffer, B. On the spectrum of non-selfadjoint Schrödinger operators with compact resolvent. *Comm. Partial Differential Equations 40* (2015), 1441–1466.

[5] Almog, Y., and Helffer, B. On the Stability of Laminar Flows Between Plates. *Arch. Rational Mech. Anal. 241* (2021), 1281–1401.

[6] Almog, Y., Helffer, B., and Pan, X.-B. Superconductivity near the normal state in a half-plane under the action of a perpendicular electric current and an induced magnetic field. *Trans. Amer. Math. Soc. 365* (2013), 1183–1217.

[7] Almog, Y., and Henry, R. Spectral analysis of a complex Schrödinger operator in the semiclassical limit. *SIAM J. Math. Anal. 48* (2016), 2962–2993.

[8] Arifoski, A., and Siegl, P. Pseudospectra of damped wave equation with unbounded damping. *SIAM J. Math. Anal. 52* (2020), 1343–1362.

[9] Baaquie, B. E. *Mathematical Methods and Quantum Mathematics for Economics and Finance*. Springer Singapore, 2020.

[10] Bismut, J.-M., and Lebeau, G. *The hypoelliptic Laplacian and Ray-Singer metrics*. Princeton University Press, Princeton, NJ, 2008.

[11] Bögli, S., Siegl, P., and Tretter, C. Approximations of spectra of Schrödinger operators with complex potential on $\mathbb{R}^d$. *Comm. Partial Differential Equations 42* (2017), 1001–1041.

[12] Caliceti, E., Graffi, S., and Maioli, M. Perturbation theory of odd anharmonic oscillators. *Comm. Math. Phys. 75* (1980), 51–66.

[13] Combes, J. M., Schrader, R., and Seiler, R. Classical bounds and limits for energy distributions of Hamilton operators in electromagnetic fields. *Ann. Physics 111* (1978), 1–18.

[14] Davies, E. B. *Spectral theory and differential operators*. Cambridge University Press, 1995.

[15] Davies, E. B. Non-Self-Adjoint Differential Operators. *Bull. Lond. Math. Soc. 34*, 5 (2002), 513–532.

[16] Davies, E. B. *Linear operators and their spectra*. Cambridge University Press, 2007.

[17] Djakov, P., and Mityagin, B. Asymptotics of instability zones of the Hill operator with a two term potential. *J. Funct. Anal. 242* (2007), 157–194.

[18] Dohnal, T., and Siegl, P. Bifurcation of eigenvalues in nonlinear problems with antilinear symmetry. *J. Math. Phys. 57* (2016), 093502.

[19] Dunford, N., and Schwartz, J. T. *Linear Operators, Part 2*. John Wiley & Sons, Inc., New York, 1988.

[20] Edmunds, D. E., and Evans, W. D. *Spectral Theory and Differential Operators*. Oxford University Press, New York, 1987.
[21] Exner, P. Complex-potential description of the damped harmonic oscillator. *J. Math. Phys. 24* (1983), 1129–1135.
[22] Freitas, P., Siegl, P., and Tretter, C. Damped wave equation with unbounded damping. *J. Differential Equations 264* (2018), 7023–7054.
[23] Gallagher, I., Gallay, T., and Nier, F. Spectral Asymptotics for Large Skew-Symmetric Perturbations of the Harmonic Oscillator. *Int. Math. Res. Not. 2009* (2009), 2147–2199.
[24] Gerhat, B. Schur complement dominant operator matrices, 2021.
[25] Grebenkov, D. S., and Helffer, B. On spectral properties of the Bloch-Torrey operator in two dimensions. *SIAM J. Math. Anal. 50* (2018), 622–676.
[26] Helffer, B. *Spectral theory and its applications*. Cambridge University Press, 2013.
[27] Helffer, B., and Nourrigat, J. On the Domain of a Magnetic Schrödinger Operator with Complex Electric Potential. In *Analysis and Operator Theory*. Springer International Publishing, 2019, pp. 149–165.
[28] Ikehata, R., and Takeda, H. Uniform energy decay for wave equations with unbounded damping coefficients. *Funkcial. Ekvac. 63*, 1 (2020), 133–152.
[29] Kato, T. *Perturbation theory for linear operators*. Springer-Verlag, Berlin, 1995.
[30] Krejčiřík, D., Raymond, N., Royer, J., and Siegl, P. Non-accretive Schrödinger operators and exponential decay of their eigenfunctions. *Israel J. Math. 221* (2017), 779–802.
[31] Krejčiřík, D., Siegl, P., Tater, M., and Viola, J. Pseudospectra in non-Hermitian quantum mechanics. *J. Math. Phys. 56* (2015), 103513.
[32] Krejčiřík, D. Complex magnetic fields: an improved Hardy-Laptev-Weidl inequality and quasi-self-adjointness. *SIAM J. Math. Anal. 51* (2019), 790–807.
[33] Metafune, G., Pallara, D., and Priola, E. Spectrum of Ornstein-Uhlenbeck operators in L^p spaces with respect to invariant measures. *J. Funct. Anal. 196* (2002), 40–60.
[34] Mityagin, B., Siegl, P., and Viola, J. Differential operators admitting various rates of spectral projection growth. *J. Funct. Anal. 272* (2017), 3129–3175.
[35] Mityagin, B., Siegl, P., and Viola, J. The Shifted Harmonic Oscillator and the Hypoelliptic Laplacian on the Circle. *Ann. Henri Poincaré* (2021).
[36] Rasulov, T. H., and Tretter, C. Spectral inclusion for unbounded diagonally dominant $n \times n$ operator matrices. *Rocky Mountain J. Math. 48* (2018), 279–324.
[37] Razavy, M. An exactly soluble Schrödinger equation with a bistable potential. *Am. J. Phys. 48* (1980), 285–288.
[38] Reddy, S. C., and Trefethen, L. N. Pseudospectra of the convection-diffusion operator. *SIAM J. Appl. Math. 54*, 6 (1994), 1634–1649.
[39] Reed, M., and Simon, B. *Methods of Modern Mathematical Physics, Vol. 4: Analysis of Operators*. Academic Press, New York-London, 1978.
[40] Rubinstein, J., Sternberg, P., and Zumbrun, K. The Resistive State in a Superconducting Wire: Bifurcation from the Normal State. *Arch. Rat. Mech. Anal. 195* (2010), 117–158.
[41] Semorádová, I., and Siegl, P. Diverging eigenvalues in domain truncations of Schrödinger operators with complex potentials. arXiv:2107.10557 [math.SP], 2021.
[42] Siegl, P., and Krejčiřík, D. On the metric operator for the imaginary cubic oscillator. *Phys. Rev. D 86* (2012), 121702(R).
[43] Sommerfeld, A. *Wave Mechanics*. New York, 1929.
[44] Trefethen, L. N., and Embree, M. *Spectra and Pseudospectra: The Behavior of Nonnormal Matrices and Operators*. Princeton University Press, 2005.
[45] Tumanov, S. Completeness theorem for the system of eigenfunctions of the complex Schrödinger operator $\mathcal{L}_c = -d^2/dx^2 + cx^{2/3}$. *J. Funct. Anal. 280* (2021), 108820.
[46] Weidmann, J. *Lineare Operatoren in Hilberträumen*. Vieweg+Teubner Verlag, 2003.
[47] Whittaker, E. T. On a class of differential equations whose solutions satisfy integral equations. *Proc. Edinb. Math. Soc. 33* (1914), 14–23.
[48] Zezyulin, D. A., and Konotop, V. V. Nonlinear modes in the harmonic $\mathcal{PT}$-symmetric potential. *Phys. Rev. A 85* (2012), 043840.

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SCHRÖDINGER OPERATORS WITH ACCRETIVE POTENTIALS IN WEIGHTED SPACES

by Petr Siegl

The first ideas for the paper originated as a question of B. Gerhat during her work on generalizations of numerical ranges in spring 2018. These ideas were consolidated and first results (on the operator domain separation) were jointly obtained in the follow up discussions, in particular during her short research visits at Queen's University Belfast (QUB) in September 2018 and February 2019. During her SEMP stay at QUB in 2019-2020, under my supervision, she worked on several remaining questions which constitute the main parts of the resulting paper. The work was essentially intensified in spring 2021 and finalized in summer 2021.

B. Gerhat's contribution in this work was essential, several parts (weighted coercivity, generalized holomorphic families, boundedness of compositions, applications to particular operator matrices) were worked out independently and originated in her ideas. Moreover, she successfully dealt with unforeseen technical difficulties along the way and significantly improved originally anticipated results.

October 2021 Petr Siegl

Erklärung

gemäss Art. 18 PromR Phil.-nat. 2019

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Studiengang: Mathematik

Bachelor ☐ Master ☐ Dissertation ☑

Titel der Arbeit: Analysis of non-selfadjoint operator matrices in the absence of standard dominance patterns

LeiterIn der Arbeit: Prof. Dr. Christiane Tretter

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