Twisting braids and surfaces

Inaugural dissertation
of the Faculty of Science,
University of Bern

presented by

Levi Salomon Ryffel

from Stäfa

Supervisor of the doctoral thesis:
Prof. Dr. Sebastian Baader

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The Dean
Prof. Dr. Marco Herwegh

Introduction

The topics of this thesis are encompassed by the larger domain of research of the field of low-dimensional topology, more specifically, topology in dimensions two and three. The title of this thesis should be considered a pun, as the two twisting notions are otherwise unrelated: By "twisting surfaces" we mean studying Dehn twists as mapping classes of a surface, and by "twisting braids" we mean applying certain moves, called twist moves, to links. These notions divide this thesis into two parts, the first of which is concerned with studying subgroups of mapping class groups generated by Dehn twists, and the second with knots and links.

The study of mapping class groups is a broad and classical topic in the literature, and it admits questions that are easy to ask but not yet answered. For instance, it is unknown what groups arise as subgroups of mapping class groups generated by three Dehn twists. Dehn twists about three or more curves that intersect pairwise at most once, in a so-called circuit pattern (i.e. cyclically consecutive curves intersect, and other pairs do not), may or may not satisfy a cycle relation. Our main result in the first part of this thesis characterizes the situations in which said relation is satisfied. It turns out that the criterion is remarkably simple: a cycle relation is satisfied if and only if the curves bound an embedded closed disc, see Theorem 3.1.

The second part of the thesis is concerned with knots and links. More precisely, the theme is to transform links into other links by applying twist moves, a twisting notion introduced by Fox in the 1950's. For each $k \geq 2$, there is a version of a twist move, called a t_k-move, that replaces two parallel strands in a link diagram by k many crossings. Our main results come in two flavours, corresponding to the two chapters in this part. Chapter 4 is concerned with obstructions by Przytycki for two links to be related by twist moves. We use these obstructions to give a bound on the number of k such that a knot K with non-trivial HOMFLY polynomial may be unknotted using t_k-moves, in terms of the crossing number of K, see Theorem 4.3. Chapter 5 is less concerned with the qualitative question of whether two links are related by a sequence of twist moves, and rather with the more quantitative question of how quickly this may be achieved in a particular situation. Using ideas and intuitions from the previous chapter, we determine the cobordism distance between links of the form $T(3,m)$ and $T(2,k)^n$ explicitly up to a bounded error for $k \leq 6$, see Theorem 5.2.

The organisation of this thesis corresponds closely to the outline above. Chapter 1 is an introduction to the known theory of relations between Dehn twists in mapping class groups. We discuss known embeddings of Artin groups into mapping class groups, and known obstructions to the existence of such embeddings, most notably results by Labruère and Wajnryb that show that Artin groups of type $\widetilde{D}$ and E, respectively, cannot be geometrically embedded into mapping class groups of surfaces. Here, a geometric embedding is an injective homomorphism mapping generators to Dehn twists.

In Chapter 2, we study subgroups of mapping class groups generated by bouquets, which are systems of curves intersecting pairwise once in a common point. We characterize bouquets in terms of the cycle relation between Dehn twists about triples of curves, see Theorem 2.1. We then determine the possible isomorphism types of subgroups of the mapping class group of a surface without punctures generated by Dehn twists about a bouquet.

The final chapter of the first part, Chapter 3, is about circuits of curves. As mentioned above, the main result characterizes the cycle relation in terms of the surrounding surface. Our main tool for carrying out this characterization is the Birman-Hilden theorem, which allows us to show that certain groups generated by Dehn twists about circuits are Artin groups of type $\widetilde{A}$, contrary to a miscitation of Labruère's work by Wajnryb [Waj99, Theorem 2]. Mortada had previously refuted this misunderstanding by constructing geometric embeddings of Artin groups of type $\widetilde{A}$, and we improve upon Mordada's work by constructing many more. Such Artin groups are useful to us because their generators do not satisfy a cycle relation. A case distinction at the end of the chapter finally proves the main theorem, Theorem 3.1.

The second part starts off with Chapter 4, where we recall and reprove Przytycki's specializations, and then apply them to certain classes of knots. The final two sections of this chapter contain results exclusive to this thesis, perhaps most significantly a stronger version of a special case of Coxeter's result that the braid group B_n modulo a k-th power of a generator is finite if and only if $1/k+1/n > 1/2$. Our improvement concerns the case $1/k+1/n = 1/2$, for which we prove that the set of closures of n-braids contains infinitely many links that are pairwise not related by t_k-moves, see Theorem 4.6. The final section of this chapter provides a method to find twist-invariant specializations like Przytycki's for a more general class of twist moves.

In the final chapter, Chapter 5, we remove powers of generators from braids whose closures are torus links in order to obtain estimates of the cobordism distance from three strand torus links to iterated connected sums of trefoils, hexafoils, and other two strand torus links. We succeed to compute the cobordism distance of the closure of an arbitrary positive braid on three strands to iterated connected sums of trefoil knots, see Theorem 5.1. We also estimate the cobordism distance of three strand torus links to iterated connected sums of the torus link $T(2,k)$ for $k = 4, 5, 6$, see Theorem 5.2. Finally, we obtain an estimate for the cobordism distance between $T(3,m)$ and $T(2,k)^n$ for $k \geq 7$ and m large (or small) compared to n, see Theorem 5.3.

Acknowledgements. I am grateful to my advisor, Sebastian Baader, for sticking with me through tough times and supporting me wholeheartedly during the process of writing this thesis. He draws from a deep well of inspiration and is not shy to share its content widely, which is where most if not all good ideas in this thesis originate. Sebastian's neverending passion for mathematics made it easy to rekindle my own, time and time again.

The time I spent at the mathematical institute of the university of Bern was very enjoyable thanks to the time I got to spend with the people there. I especially wish to thank first and foremost Sebastian, and also Alejandro, Frieder, Jan, Livio, Naomi, Simon, and Tobias for all the fascinating discussions about mathematics we were able to have. The Bern-Fribourg graduate seminar organised by Damaris, Elia, Livio, and Ulrik was an invaluable resource of fun and wide mathematical exchange, and I thank the organisers and the participants of this seminar for expanding my view beyond, but also into, the horizon of low-dimensional topology.

Last but not least, I wish to thank my friends and family for all the support they had to offer. I am particularly grateful for the love I received from Julia and from Claire and Mina.

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Part I

Surfaces

Chapter 1

Relations in mapping class groups

This chapter covers the basics of the classical theory of mapping class groups of surfaces, with a focus on relations between Dehn twists, or the absence thereof. This focus is intended to establish some context around the results in the next few chapters of this thesis, which are all about one particular relation in mapping class groups: the so-called cycle relation.

Because the material in the first two sections is well-known, we go through it quite quickly and refer the reader to [FM11] for the details. The author attempted to keep the notation and terminology consistent with this reference.

1.1 Classical relations

For us, a *surface* is an oriented connected 2-dimensional manifold with or without boundary, and the *mapping class group* of a surface S, abbreviated by the symbol $\mathrm{MCG}(S)$, is the set of isotopy classes of orientation-preserving homeomorphisms of S that fix the boundary ∂S pointwise, with composition as the group operation. A *simple closed curve* in a surface S, or just *curve* for short, is a continuous injection $\alpha\colon S^1 \to S$. We usually consider curves up to isotopy, and also use the word *curve* to mean an isotopy class of simple closed curves. We do, however, attempt to distinguish the two meanings by using Greek letters for actual curves, and the corresponding Latin letters (a for α, b for β, and c for γ) for isotopy classes of curves. One further convention we adopt is that of minimal intersection numbers: whenever we say that two isotopy classes *intersect* a given number of times, say i, what we mean is that there exist representatives that intersect i times, and all other pairs of representatives intersect at least i times.

To any curve a in a surface S we may associate a mapping class T_a in $\mathrm{MCG}(S)$, called the *Dehn twist* about a, as follows. Let α be a representative of a and let A be a regular neighbourhood of α. The mapping class T_a is represented by the map that is the identity outside the interior of A, and has the effect of "twisting" the annulus A as described pictorially in Figure 1.1. The orientation of S makes a difference here, without it there is no way to distinguish a Dehn twist from its inverse. It turns out that if two Dehn twists T_a and T_b are isotopic, then $a = b$.

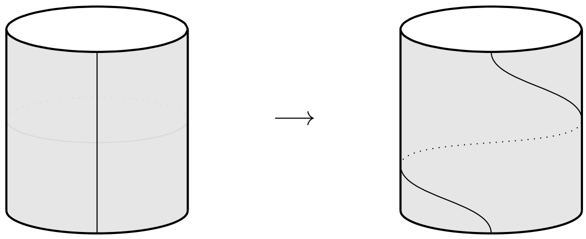


Figure 1.1: The effect of a Dehn twist about the core curve α of an annulus on a segment of a curve intersecting α.

The rest of this section is devoted to the most well-known relations between Dehn twists.

1.1.1 Commutation and braid relations

Let a and b be disjoint simple closed curves. Then T_a and T_b have representatives with disjoint supports, leading to the *commutation relation*

$$T_a T_b = T_b T_a.$$

Note that if a curve is isotopic a boundary component, then it is disjoint, up to isotopy, from any other curve. This observation shows that Dehn twists about boundary curves of a surface S lie in the centre of $\mathrm{MCG}(S)$, because mapping class groups are generated by Dehn twists (and half-twists in the case the surface has punctures).

For any mapping class f and any curve c we have $f T_c f^{-1} = T_{f(c)}$ [FM11, Fact 3.7]. We will henceforth refer to this result as the *conjugation formula*. Now suppose a and b intersect precisely once. Let α and β be representatives of a and b, respectively, such that α and β intersect exactly once. Let N be a regular neighbourhood of $\alpha \cup \beta$. Note that we have $T_a(b) = T_b^{-1}(a)$, see Figure 1.2. Using the conjugation formula, we obtain

$$T_b^{-1} T_a T_b = T_{T_b^{-1}(a)} = T_{T_a(b)} = T_a T_b T_a^{-1},$$

which is equivalent to the *braid relation*

$$T_a T_b T_a = T_b T_a T_b.$$

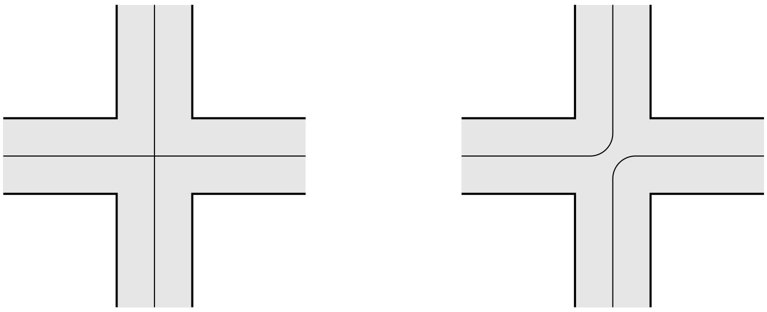


Figure 1.2: The curves a (horizontal) and b (vertical) on the left, and the curve $T_a(b) = T_b^{-1}(a)$ on the right. In these sorts of images, we implicitly identify the ends of the bands with their opposite side, so that the surface depicted in this particular example is a torus with one boundary component.

1.1.2 The chain relation

A *chain* in a surface S is a family of curves $c_1, \ldots, c_n$ in S such that each c_i intersects c_{i+1} exactly once, and c_i is disjoint from c_j whenever $|i-j| \geq 2$. Let N be a regular neighbourhood of $\gamma_1 \cup \cdots \cup \gamma_n$, where the γ_i are representatives of the c_i that realise the minimal intersection numbers. If n is even, then N has one boundary component which we call δ, and if n is odd, then N has two boundary components, which we call δ_1 and δ_2. The *chain relation* asserts that if n is even, then

$$(T_{c_1} \cdots T_{c_n})^{2n+2} = T_d,$$

and if n is odd, then

$$(T_{c_1} \cdots T_{c_n})^{n+1} = T_{d_1} T_{d_2}.$$

A pictorial way to understand this relation uses the fact that there exists an isomorphism from the group $G = \langle T_{c_1}, \cdots, T_{c_n} \rangle$ to the braid group on $n+1$ strands. The remainder of this subsection is devoted to obtaining a visual understanding of said isomorphism, while leaving the technical details to the literature.

As an intermediate step, we first try to understand why G is isomorphic to $\mathrm{MCG}(\Delta_{n+1})$, where Δ_{n+1} is an $(n+1)$-times punctured disc. The left side of Figure 1.3 depicts a decomposition of N into cross-shaped pieces. Rotating each of the pieces by 180 degrees around their centres yields a well-defined involution ι of N, and the quotient N/ι is a disc. The effect ι has on the boundary is

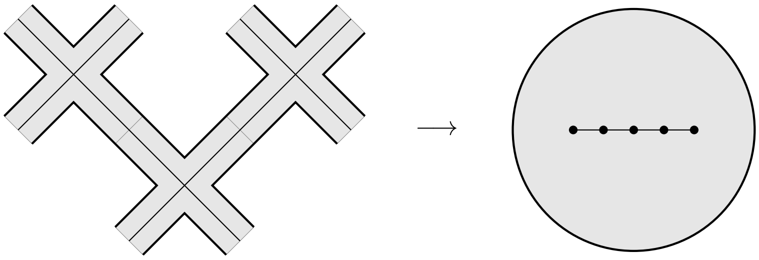


Figure 1.3: A neighbourhood N of a chain of four curves, decomposed into cross-shaped pieces, and the same surface modulo a half-rotation of each of the crosses. The small circles on the disc are the images of the fixed points of the involution, of which there are five: three in the centres of the crosses, and two on the edge of the outermost crosses.

different depending on the parity of n: If n is even, then ι restricts to a 180 degree rotation of the single boundary curve, and if n is odd, ι exchanges the two boundary components. Note that on the complement of the fixed points of ι, the quotient map $N \to N/\iota$ is a covering map. We call a homeomorphism φ of N *symmetric* if it commutes with ι. Note that a symmetric homeomorphism φ of N passes down to a homeomorphism $\overline{\varphi}$ of Δ_{n+1}. A famous result by Birman-Hilden which we will not reprove in this thesis, asserts that this yields a homomorphism from the subgroup of $\mathrm{MCG}(N)$ consisting of mapping classes with symmetric representatives to $\mathrm{MCG}(\Delta_{n+1})$. Note that this result is non-trivial. Indeed, whenever two symmetric homeomorphisms of N are isotopic, the isotopy might pass through non-symmetric homeomorphisms, which do not pass down to homeomorphisms of Δ_{n+1}. However, it turns out that under mild assumptions, isotopies of symmetric homeomorphisms may be symmetrised: this is the content of the so-called Birman-Hilden theorem, which asserts the following in the special cases appearing in this thesis.

Lemma 1.1 ([BH73, Theorem 1])**.** *Let ι be an orientation-preserving involution of a surface S with finitely many fixed points. Assume that S is neither a closed sphere nor a closed torus. Let f be an orientation-preserving symmetric homeomorphism of S that is isotopic to the identity. Then f is isotopic to the identity via symmetric homeomorphisms.*

A close look at Figure 1.3 might reveal to the reader that the image of a Dehn twist T_{c_i} is a *half-twist*, depicted in Figure 1.4. The next thing to understand is that $\mathrm{MCG}(\Delta_{n+1})$ is isomorphic to the braid group on $n+1$ strands. Formally, the isomorphism comes from the Birman exact sequence

$$1 \longrightarrow \pi_1(\mathrm{CS}(\Delta, n+1)) \longrightarrow \mathrm{MCG}(\Delta_{n+1}) \longrightarrow \mathrm{MCG}(\Delta) \longrightarrow 1,$$

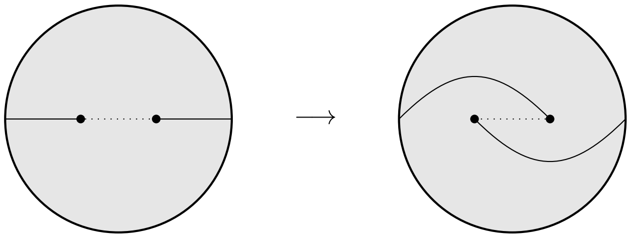


Figure 1.4: The effect of a half-twist about the grey arc γ.

where the group $\pi_1(\mathrm{CS}(\Delta, n+1))$ is the braid group, and $\mathrm{MCG}(\Delta)$ is trivial. We explain this formal reasoning in Subsection 1.3.1. Here, we will give visual description of this isomorphism, usually called the *push map*, and it goes as follows. To obtain a mapping class from a braid, one may imagine the disc being made of dough, and the punctures being some nails in the dough. The z-axis of the braid can be imagined as time, and the position of the strands at a given time can be interpreted as the position of the nails in the dough, so that going up the braid corresponds to moving the nails. The final state of the dough is the result of applying the mapping class associated to the braid. Conversely, starting with a mapping class f of Δ_{n+1}, there exists an isotopy transforming it to the identity when interpreting f as a map on the disc Δ without punctures. The paths the punctures follow during this isotopy give rise to a braid. An example for the case $n = 1$ is drawn in Figure 1.4, which depicts the effect of the mapping class associated to the generator $a = \sigma_1$ of the braid group on two strands. It might be worth pointing out that there is a slight clash of orientation conventions: The generators of the braid group are usually taken to have the left strand pass over the right, whereas the figure depicts the opposite. We will sweep this conflict firmly under the rug, as it will not affect any arguments in this thesis.

Putting these two steps together yields an isomorphism from G to the braid group on $n+1$ strands. It should be reasonably clear that if n is even, the double full twist lifts to the boundary twist T_d, and if n is odd, the full twist lifts to the product $T_{d_1}T_{d_2}$. Indeed, the full twist in $\mathrm{MCG}(\Delta_{n+1})$ is the Dehn twist $T_{\overline{d}}$, where $\overline{d}$ is the boundary curve of Δ_{n+1}. This shows the chain relation.

1.1.3 The inclusion homomorphism

The inclusion of a subsurface into a bigger surface induces a homomorphism on the mapping class groups. This works because we assumed the mapping classes to fix the boundaries pointwise, so that homeomorphisms of a subsurface may be extended with the identity to a homeomorphism of the full surface. It will prove useful for us to iteratively glue discs and other surfaces to regular neighbourhoods of curve systems, so we formulate the *inclusion homomorphism theorem* with this in mind.

Lemma 1.2 ([FM11, Theorem 3.18]). *Suppose S and S' are closed and connected subsurfaces of a surface $S \cup S'$ with disjoint interiors. Let K be the kernel of the inclusion-induced homomorphism* $\mathrm{MCG}(S) \to \mathrm{MCG}(S \cup S')$. *Then:*

(i) *If $S' = \Delta_1$ is a once-punctured disc with $\partial\Delta_1 \subset \partial S$, then K is cyclically generated by the Dehn twist about the boundary curve of Δ_1.*

(ii) *If $S' = Z$ is an annulus with $\partial Z \subset \partial S$, then K is cyclically generated by $T_a T_b^{-1}$, where T_a and T_b are Dehn twists about the boundary curves a and b of Z, respectively.*

(iii) *If S' is neither a disc, a once-marked disc, nor an annulus, then K is trivial.*

In particular, if S' has no component homeomorphic to a disc, then K is a subgroup of the centre of $\mathrm{MCG}(S)$.

It is worth pointing out that the inclusion homomorphism theorem makes no assertion about the case that S' is a disc. Indeed, the following examples of inclusion-induced homomorphisms show that the effect of gluing in discs is quite unpredictable.

It is a classical result that if two curves a and b intersect (at least) twice, then T_a and T_b generate a free group [FM11, Theorem 3.14]. If we now glue in a disc that decreases the intersection number to zero, the kernel of the inclusion-induced homomorphism does not lie in the centre, because the commutator $T_a T_b T_a^{-1} T_b^{-1}$ does not commute with T_a (nor with T_b) in the free group. An example of a disc that decreases the intersection number to zero can be found in Figure 1.5.

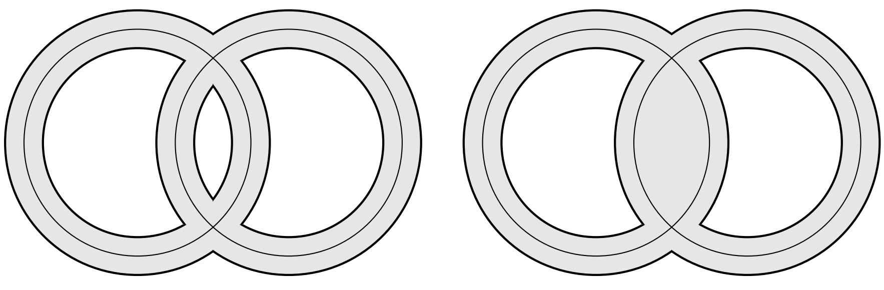


Figure 1.5: Two curves intersecting twice, but intersecting zero times after gluing in a disc.

The chain relation can also lead to further relations upon gluing in discs. As we have seen in Subsection 1.1.2, the Dehn twists about the curves a, b on the left of Figure 1.2 generate a subgroup of the mapping class group isomorphic to the braid group on three strands. By the chain relation, the mapping class $(T_aT_b)^6$ is the boundary twist. If we cap off the boundary with a disc to turn the regular neighbourhood into a torus without boundary, we obtain the relation $(T_aT_b)^6 = 1$, because the boundary curve becomes null-homotopic. It is well-known that this relation is not satisfied in the braid group. One possible justification could be that the relation is "inhomogeneous", see Chapter 3. The same thing occurs when capping off all boundaries of a regular neighbourhood of a chain of arbitrary length.

A less obvious example where discs cause a surprising relation is the *lantern relation* [FM11, Proposition 5.1], where four discs cause a relation between three Dehn twists: The curves on the left hand side of Figure 1.6 generate a free group [Hum89, Theorem 1.1], whereas the curves on the right hand side satisfy the (simplified) lantern relation $T_aT_bT_c = T_bT_cT_a = T_cT_aT_b$. The curves are labelled a, b, c in the clockwise manner.

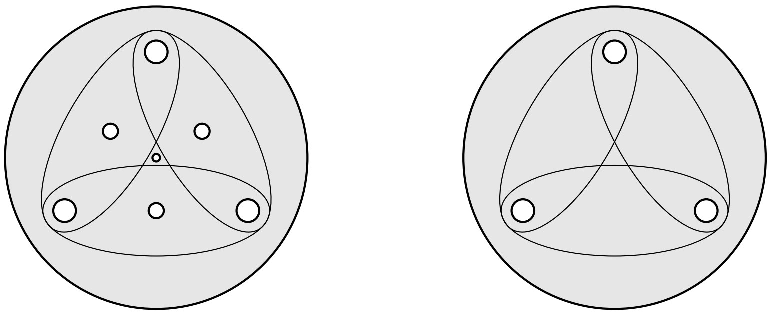


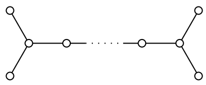
Figure 1.6: Three curves whose associated Dehn twists generate a free group on the left, and satisfy the lantern relation on the right.

Over the course of the next two chapters, we will see many more instances of the observation that gluing in discs may introduce new relations. More precisely, in Chapter 2 we shall see how gluing in a disc into a triangle of curves introduces a cycle relation of three curves, and in Chapter 3 we shall generalise this observation to many curves. The latter chapter goes on to explore how circuits of curves can sit in surfaces, and investigates how the surrounding surface has an effect on whether the curves satisfy further relations in addition to the pairwise braid and commutation relations that are expected just from their intersection pattern. As we shall see, the complication commonly arises when the complement of a regular neighbourhood of curves contains discs. The following two subsections are exceptions to this principle.

1.1.4 Labruère's relation

To a system of curves intersecting pairwise at most once, we associate a graph Γ called its *intersection pattern*, where each vertex corresponds to a curve and each edge corresponds to an intersection. In [Lab97], Labruère found a surprising relation between Dehn twists whose intersection pattern was that of a certain tree

$$\widetilde{D}_{n+3} =$$



with $n+4$ vertices, and subsequently showed that this relation does not hold in the Artin group associated to that graph. Suppose the left leaves of $\widetilde{D}_{n+3}$ correspond to the curves a_1, a_2, the middle nodes to the curves $b_1, \ldots, b_n$, and the right leaves to the curves c_1, c_2. In the case $n = 1$ the graph is a single central node with four leaves attached, and we simply write $b = b_1$. *Labruère's relation* asserts that

$$(T_{c_1} f) T_{c_2} (T_{c_1} f)^{-1} = f T_{c_2} f^{-1},$$

where $f = T_{b_n} \cdots T_{b_1} T_{a_1} T_{a_2}^{-1} T_{b_1}^{-1} \cdots T_{b_n}^{-1}$. Using the conjugation formula and the fact that $T_a = T_b$ if and only if $a = b$, Labruère's relation is equivalent to the requirement that $T_{c_1} f(c_2) = f(c_2)$. But this just means that c_1 and $f(c_2)$ are disjoint. The case $n = 5$ is sketched in Figure 1.7, and the general case can be shown similarly.

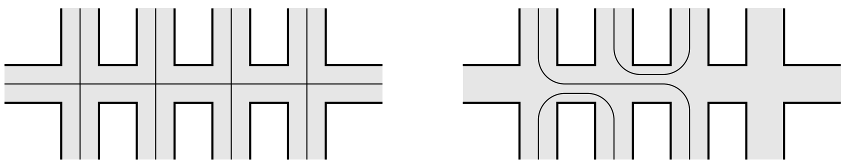


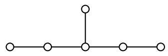
Figure 1.7: On the left hand side the vertical curves a_1, a_2 and c_2, c_1 from left to right, and the horizontal curve b. On the right hand side the curve $T_b T_{a_1} T_{a_2}^{-1} T_b^{-1}(c_2)$, which is disjoint from c_1.

Showing that Labruère's relation does not hold in the Artin group associated to $\widetilde{D}_{n+3}$ follows from the fact that it does not hold in the corresponding Coxeter group. Labruère uses the reflection representation to prove this, but it can also be seen as a consequence of Matsumoto's theorem [Mat64]. In our case, it asserts that two words that represent the same element in the Coxeter group under consideration may be transformed into each other by applying commutation and braid relations while preserving the length of the word, and cancellation of squares. Only in the word corresponding to f can we apply commutation relations, and nowhere can we apply braid relations. Thus, the two sides of

Labruère's relation represent different words in the associated Coxeter group.

1.1.5 Wajnryb's relation

Shortly after Labruère's discovery on graphs of type $\widetilde{D}$, Wajnryb [Waj99] published a very similar result on curves with intersection pattern the graph

$E_6 =$ 

Labelling the curve corresponding to the top leaf a and the curves corresponding to the bottom row $b_1, \ldots, b_5$, *Wajnryb's relation* asserts that

$$(fT_af^{-1})(T_afT_a^{-1}) = (T_afT_a^{-1})(fT_af^{-1}),$$

where $f = T_{b_3}T_{b_2}T_{b_4}T_{b_3}T_{b_1}T_{b_5}T_{b_2}T_{b_4}T_{b_3}$. Again using the conjugation formula, this amounts to showing that $T_{f(a)}$ commutes with $T_afT_a^{-1}$, which means that $T_afT_a^{-1}(f(a)) = f(a)$. Like in Labruère's case, showing that Wajnryb's formula does not hold in the Artin group associated to E_6 can be done by showing that it does not hold in the corresponding Coxeter group, and can be done by invoking Matsumoto's theorem, albeit in a more subtle way than in Labruère's case.

1.2 Artin and Coxeter groups

Artin groups are a class of groups introduced by Jacques Tits [Tit66] to generalise Artin's braid groups [Art25]. Artin groups are closely related to Coxeter groups [Cox35], and both classes of groups will be discussed frequently in this part of the thesis. We will start with some definitions.

A *Coxeter matrix* is a symmetric matrix M with coefficients in $\mathbb{N} \cup \{\infty\}$ with ones on the diagonal and all other coefficients being at least two. It is usually more convenient to encode Coxeter matrices as labelled graphs with some conventions: for a Coxeter matrix M of size $n \times n$ with coefficients m_{ij}, the *Coxeter-Dynkin diagram* of M is a graph with n vertices whose edges are labelled by the coefficients m_{ij}. We adopt the convention that an edge labelled 2 is not drawn, the label of an edge labelled 3 is suppressed, and an edge labelled 4 may instead be drawn as a double edge. For example, the Coxeter-Dynkin diagram of the $n \times n$ Coxeter matrix

$$M = \begin{pmatrix} 1 & 3 & 2 & \cdots & 2 \\ 3 & 1 & \ddots & \ddots & \vdots \\ 2 & \ddots & \ddots & \ddots & 2 \\ \vdots & \ddots & \ddots & 1 & 3 \\ 2 & \cdots & 2 & 3 & 1 \end{pmatrix}$$

is the graph

$$A_n = \circ\!\!-\!\!\!-\!\!\circ\!\!-\!\!\cdots\!\!-\!\!\circ\!\!-\!\!\!-\!\!\circ$$

with n vertices. To an $n \times n$ Coxeter matrix M or a Coxeter-Dynkin diagram Γ with n vertices we associate two groups, the *Artin group*

$$\mathrm{Art}(\Gamma) = \langle s_1, \ldots, s_n \mid R_{\mathrm{Art}}(\Gamma) \rangle$$

and the *Coxeter group*

$$\mathrm{Cox}(\Gamma) = \langle s_1, \ldots, s_n \mid R_{\mathrm{Cox}}(\Gamma) \rangle,$$

where the sets of relations R_{Art} and R_{Cox} are defined as follows. For each label $m_{ij} \neq \infty$ with $i \neq j$ we add the relation $(s_i s_j)^{m_{ij}/2} = (s_j s_i)^{m_{ij}/2}$ to $R_{\mathrm{Art}}(\Gamma)$. Here, we adopt the convention that $(st)^{(2k+1)/2} = (st)^k s$. For instance, a label of 2 corresponds to a commutation relation $st = ts$, and a label of 3 corresponds to a braid relation $sts = tst$. A label ∞ is to be interpreted as no relation. The set $R_{\mathrm{Cox}}(\Gamma)$ consist of the relations in $R_{\mathrm{Art}}(\Gamma)$, together with the additional relations $s_i^2 = 1$ for all i. Note that because $R_{\mathrm{Art}}(\Gamma)$ is a subset of $R_{\mathrm{Cox}}(\Gamma)$, we obtain a quotient map $\mathrm{Art}(\Gamma) \to \mathrm{Cox}(\Gamma)$.

1.2.1 The braid group

The most significant Artin group is undoubtedly Artin's *braid group*. It is defined to be the fundamental group of a certain configuration space $\mathrm{CS}(\Delta, n)$ (where Δ is a disc), which we briefly describe now. For a topological space X, its *configuration space* $\mathrm{CS}(\Delta, n)$ is the set of unordered subsets of X with n elements. It is equipped with a topology by viewing it as the set

$$\mathrm{CS}(\Delta, n) = Y / \mathrm{Sym}(n),$$

where the symmetric group $\mathrm{Sym}(n)$ acts on X^n by permuting the coordinates, and Y is the subset of X^n with pairwise distinct coordinates. A path in $\mathrm{CS}(X, n)$ may be viewed as a collection of n paths in X that never find themselves in the same point at the same time. In the case of $X = \Delta$, this leads to the

Artin's original interpretation of elements of $\mathrm{CS}(\Delta, n)$ as *braids* on n strands: We interpret the z-coordinate as the time, and plot the paths of the individual points through the disc, so that each of the n paths yields a strand of the braid.

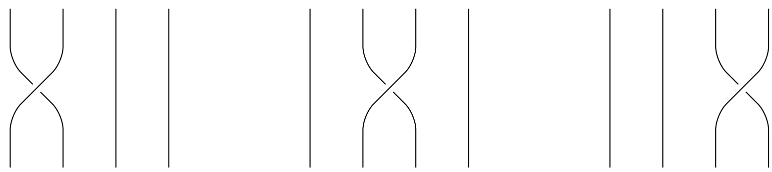


Figure 1.8: The generators σ_1, σ_2, σ_3 of the braid group $\pi_1(\mathrm{CS}(\Delta, 4))$ on four strands.

In [Art25] it was shown that the braid group is isomorphic to the Artin group $\mathrm{Art}(A_{n-1})$. The isomorphism $\mathrm{Art}(A_{n-1}) \to \pi_1(\mathrm{CS}(\Delta, n))$ sends the generators $s_1, \dots, s_{n-1}$ to the braids $\sigma_1, \dots, \sigma_{n-1}$, where σ_i has strand i cross over strand $i+1$, see Figure 1.8. Both the braid and commutation relations admit nice pictorial justifications, see Figure 1.9.

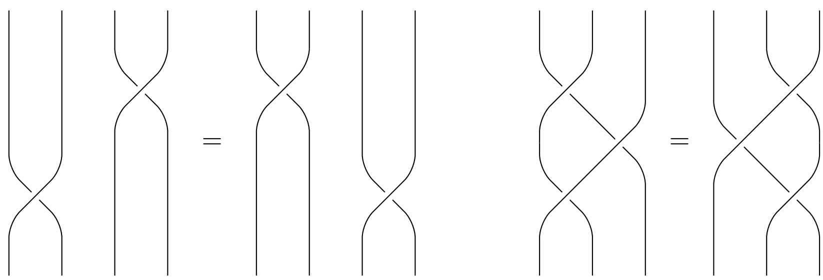


Figure 1.9: The commutation relation $\sigma_1\sigma_3 = \sigma_3\sigma_1$ and the braid relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ in $\pi_1(\mathrm{CS}(\Delta, 4))$.

The Coxeter group $\mathrm{Cox}(A_{n-1})$ is well-known to be isomorphic to the symmetric group $\mathrm{Sym}(n)$, and the isomorphism $\mathrm{Cox}(A_{n-1}) \to \mathrm{Sym}(n)$ maps the generator s_i to the transposition exchanging i with $i+1$. As a consequence, the quotient map $\mathrm{Art}(A_{n-1}) \to \mathrm{Cox}(A_{n-1})$ only remembers which strand goes where, but forgets all the over- and undercrossing information of a braid.

1.2.2 The annular braid group

In a very similar spirit to Artin's result, it was noticed by multiple mathematicians, for instance Crisp [Cri99, Section 5], that the Artin group $\mathrm{Art}(B_n)$ can also be understood as a kind of braid group. Here,

$B_n =$

is a graph with n vertices and a double edge indicates a label of four. It turns out that $\mathrm{Art}(B_n)$ is isomorphic to the *annular braid group* $\pi_1(\mathrm{CS}(Z, n))$, where Z is an annulus. Labelling the generators of $\mathrm{Art}(B_n)$ with the symbols $t, s_1, \ldots, s_{n-1}$, the isomorphism from $\mathrm{Art}(B_n)$ to $\pi_1(\mathrm{CS}(Z, n))$ maps t to the braid τ and s_i to the braid σ_i, see Figure 1.10.

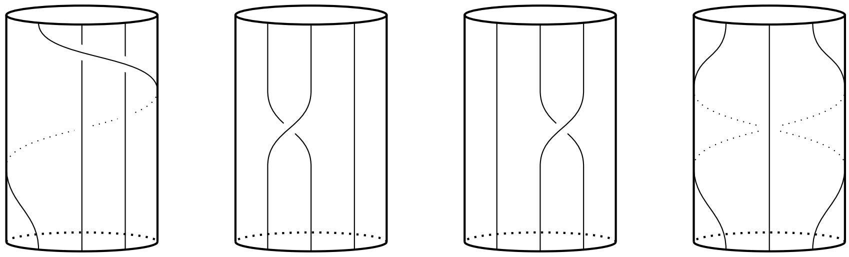
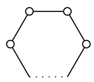


Figure 1.10: The generators τ, σ_1, σ_2 of the annular braid group $\pi_1(\mathrm{CS}(Z, 3))$ on three strands, as well as the element $\sigma_3 = \tau\sigma_1\sigma_2\sigma_1^{-1}\tau^{-1}$.

The Artin group $\mathrm{Art}(\widetilde{A}_{n-1})$, where

$$\widetilde{A}_{n-1} =$$



is a cyclic graph with n vertices, can be viewed as a normal subgroup of $\mathrm{Art}(B_n)$, namely as the kernel of the map $\mathrm{Art}(B_n) \to \mathbb{Z}$ sending t to 1 and all other generators to 0 [KP02]. The images of the generators $s_1, \ldots s_{n-1}$ of $\mathrm{Art}(\widetilde{A}_{n-1})$ in $\mathrm{Art}(B_n)$ are $s_1, \ldots s_{n-1}$, respectively, and the generator s_n maps to the element $ts_1 \cdots s_{n-2}s_{n-1}s_{n-2}^{-1} \cdots s_1^{-1}t^{-1}$, see Figure 1.10 again.

1.3 Artin groups in mapping class groups

Artin groups, especially those of finite or elliptical type, arise frequently as subgroups of the mapping class group. The most well-known connection from Artin groups to mapping class groups is the fact that the braid group can be viewed as the mapping class group of a multiply punctured disc [FM11, Theorem 9.1], as we have briefly outlined in Subsection 1.1.2. This leads to the marvellous result by Birman-Hilden that the braid group can be realised as a subgroup of a mapping class group generated by Dehn twist. Similarly, the Artin group $\mathrm{Art}(B_n)$ may be viewed as a subgroup of the mapping class group of a multiply punctured cylinder [Ryf23a, Lemma 2.3], which ultimately leads to the result that $\mathrm{Art}(\widetilde{A}_n)$ is also isomorphic to a subgroup of a mapping class group generated by Dehn twists, see Chapter 3.

This section covers a selection of the literature on such embeddings, with a focus on geometric embeddings. Here, a map $\mathrm{Art}(\Gamma) \to \mathrm{MCG}(S)$ is a *geometric* *embedding* if it is an injective homomorphism mapping generators of $\mathrm{Art}(\Gamma)$ to Dehn twists. Recall that pairs of Dehn twists either commute, satisfy a braid relation, or generate a free group. Thus, if an Artin group $\mathrm{Art}(\Gamma)$ geometrically embeds into $\mathrm{MCG}(S)$ for some surface S, the labels of Γ must lie in the set $\{2, 3, \infty\}$, and they determine the intersection pattern of the curves. Labruère's relation from Subsection 1.1.4 shows that Artin groups of type $\widetilde{D}$ do not geometrically embed into any mapping class group, because whenever a set of curves has intersection pattern $\widetilde{D}_{n+3}$ for some $n \geq 2$, the corresponding Dehn twists satisfy a relation that does not hold in $\mathrm{Art}(\widetilde{D}_{n+3})$. Similarly, Wajnryb's relation shows that $\mathrm{Art}(E_6)$ does not geometrically embed into any mapping class group. Van der Lek [vdL83, Theorem II.4.13] showed that a graph inclusion $\Gamma' \subset \Gamma$ yields an inclusion $\mathrm{Art}(\Gamma') \subset \mathrm{Art}(\Gamma)$, so no graph Γ that has E_6 or $\widetilde{D}_{n+3}$ as an induced subgraph for some $n \geq 2$ geometrically embeds into any Artin group. As far as the author of this thesis is aware, no further examples of Artin groups that do not geometrically embed into any mapping class group are known.

A vague justification for preferring geometric embeddings over general embeddings is that (many) Artin groups, as well as (many) mapping class groups, come with a distinguished conjugacy class, namely that of the generating system and that of Dehn twists, respectively. Indeed, if two generators s, t of an Artin group satisfy a braid relation $sts = tst$, then $s = (ts)t(ts)^{-1}$ is conjugate to t. Similarly, any two Dehn twists about curves intersecting exactly once are conjugate.

1.3.1 The Birman exact sequence

The main tool for establishing braid-like groups $\pi_1(\mathrm{CS}(S, n))$ as subgroups of mapping class groups is the Birman exact sequence, which first appeared in [Bir69]. A modern formulation [FM11, Theorem 9.1] asserts that if S is a surface such that the identity component of the space of orientation-preserving homeomorphisms of S fixing the boundary ∂S pointwise is simply connected (written $\pi_1(\mathrm{Homeo}_0^+(S, \partial S)) = 1$), then the sequence

$$1 \longrightarrow \pi_1(\mathrm{CS}(S, n)) \longrightarrow \mathrm{MCG}(S_n) \longrightarrow \mathrm{MCG}(S) \longrightarrow 1$$

is exact, where S_n is S with n points removed. The map $\pi_1(\mathrm{CS}(S, n))$ is the so-called *push map*, which can be visually thought of using the dough analogy from Subsection 1.1.2. The map $\mathrm{MCG}(S_n) \to \mathrm{MCG}(S)$ is obtained by filling in the punctures. It should be interpreted as the map that forgets that the punctures are missing. This makes sense because any homeomorphism of S_n can be uniquely extended to a homeomorphism of S. The main content of the result is that the push map is well-defined. We will not go into this here.

The fact that $\mathrm{Homeo}^+(\Delta, \partial\Delta)$ is contractible is essentially a consequence of the Alexander trick [FM11, Lemma 2.1]. The situation with the annulus Z is less transparent, but $\mathrm{Homeo}_0^+(Z)$ is also known to be contractible [Sco70, Lemma 0.10]. This establishes the annular braid group $\pi_1(\mathrm{CS}(Z, n))$, and hence $\mathrm{Art}(B_n)$ and $\mathrm{Art}(\widetilde{A}_{n-1})$, as a subgroup of $\mathrm{MCG}(Z_n)$, see Figure 1.11. Because half-twists are not Dehn twists, none of the embeddings of $\mathrm{Art}(A_{n-1})$, $\mathrm{Art}(B_n)$, or $\mathrm{Art}(\widetilde{A}_{n-1})$ constructed in this manner are geometric.

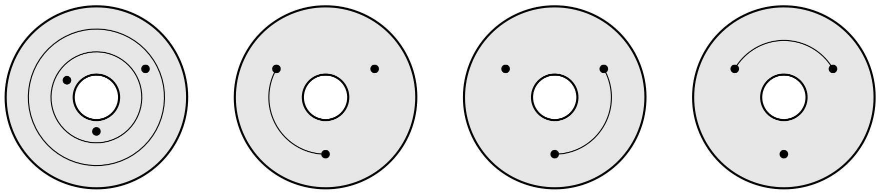


Figure 1.11: The images of the generators t, s_1, s_2 and of s_3 in $\mathrm{MCG}(Z_3)$. The generator t maps to the product $T_a T_b^{-1}$, where a is the inner curve and b is the outer curve.

1.3.2 Free groups and other right-angled Artin groups

A *right-angled Artin group* is an Artin group in which pairs of generators either commute or satisfy no relation. To make the statements in this section easier to formulate we associate a right-angled Artin group $\mathrm{Art}_{\mathrm{RA}}(\Gamma)$ to a graph Γ by interpreting the edges of Γ as commutation relations. In the extreme cases, for a graph Γ with no edges we have that $\mathrm{Art}_{\mathrm{RA}}(\Gamma)$ is a free group, and for a complete graph Γ we have that $\mathrm{Art}_{\mathrm{RA}}(\Gamma)$ is a free abelian group.

Just for this subsection, we say that the *intersection pattern* of a system of curves that are pairwise either disjoint or intersect at least twice is the graph Γ where each vertex corresponds to a curve, and two vertices are joined by an edge if and only if the corresponding curves are disjoint. Humphries [Hum89, Theorem 2.1] proved that if the intersection pattern Γ of a system of curves $c_1, \ldots, c_n$ is a disjoint union of complete graphs, and the complement of $c_1 \cup \cdots \cup c_n$ contains no discs, then the subgroup of the mapping class group generated by $T_{c_1}, \ldots, T_{c_n}$ is isomorphic to $\mathrm{Art}_{\mathrm{RA}}(\Gamma)$ via an isomorphism mapping the generators s_i of $\mathrm{Art}_{\mathrm{RA}}(\Gamma)$ to the corresponding Dehn twists T_{c_i}. In particular, every such right-angled Artin group, for instance every free group and every free abelian group, geometrically embeds into a mapping class group.

Many works in the literature manage to embed arbitrary right-angled Artin groups into mapping class groups [CP01, Lö10, Kob12, CMM21]. However, none of these embeddings are geometric. The author is unaware of any curve system whose complement does not contain discs and whose curves are either disjoint or intersect at least twice, such that the group generated by the Dehn twists about the curves in the system do not generate a right-angled Artin group.

Chapter 2

Bouquets of curves

This chapter seeks to characterise so-called bouquets of curves in terms of the relations the corresponding Dehn twists satisfy in the mapping class group. Most of the material in this chapter originates from an article written jointly with Sebastian Baader and Peter Feller [BFR23].

A *circuit* of n curves is a curve system with intersection pattern $\widetilde{A}_{n-1}$. This chapter begins with the special case $n = 3$ of our main result in Chapter 3, asserting that the Dehn twists about a circuit of n curves satisfy a so-called cycle relation if and only if the circuit bounds an *embedded closed disc*, i.e., a disc in $S \setminus (\gamma_1 \cup \cdots \cup \gamma_n)$ whose closure in S is also a disc. Here, $\gamma_1, \ldots, \gamma_n$ are minimally intersecting representatives of a circuit $c_1, \ldots, c_n$ in a surface S, with no triple intersections. See Figure 2.1 for an example of a circuit that bounds a disc which does not satisfy this condition, since its boundary is not embedded. Indeed, its closure is a disc with three pairs of antipodal points identified.

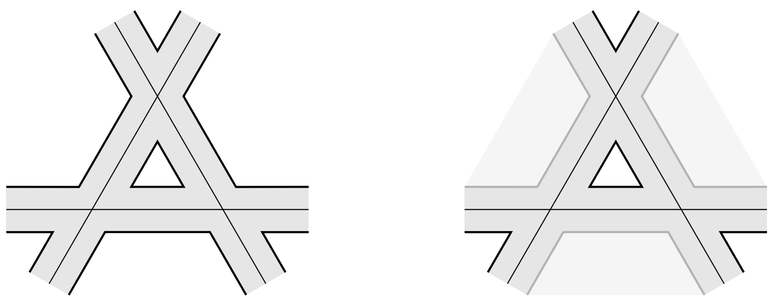


Figure 2.1: A regular neighbourhood of a circuit, as well as a (light grey) disc that does not cause a cycle relation. Capping off any one of the two other boundary components does cause a cycle relation.

If a circuit of three curves bounds an embedded closed disc, it is isotopic to a bouquet of three curves. Here, a *bouquet* is a system of n curves $c_1, \ldots, c_n$ that intersect pairwise exactly once, such that there are representatives $\gamma_1, \ldots, \gamma_n$ of $c_1, \ldots, c_n$, respectively, that intersect in one common point and have no other intersections, see Figure 2.2.

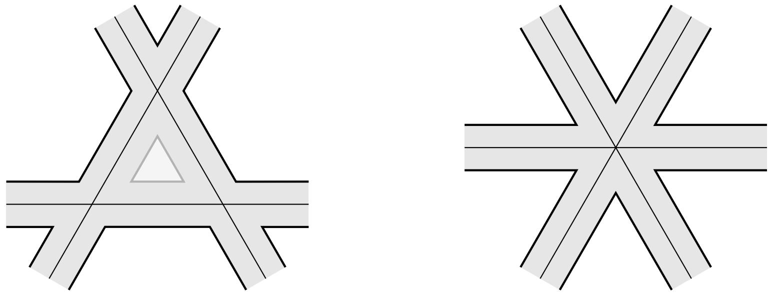


Figure 2.2: Gluing in a disc to a circuit of three curves to form a bouquet.

As we shall see in this chapter, the relations in the group generated by the Dehn twists about a curve system $c_1, \ldots, c_n$ detect whether $c_1, \ldots, c_n$ form a bouquet. More precisely, we say that a bouquet of n curves has *cyclic order* $c_1, \ldots, c_n$ if the curves appear in the order $c_1, \ldots, c_n$ when going clockwise around the common intersection point starting at c_1. In this situation, we also say that $c_1, \ldots, c_n$ is an *oriented bouquet*. Note that a cyclic order $c_1, \ldots, c_n$ induces a cyclic order on each triple c_i, c_j, c_k with i, j, k pairwise distinct, by removing all curves except c_i, c_j, c_k from the cyclic order. A precise formulation of our main theorem in this chapter asserts the following.

Theorem 2.1 ([BFR23, Theorem 1])**.** *Let S be a compact oriented surface and $n \geq 2$. Consider a system of simple closed curves $c_1, \ldots, c_n$ in S. For all i, Let T_i be the Dehn twist about c_i. Then, the system forms a bouquet with cyclic order $c_1, \ldots, c_n$ if and only if the Dehn twists $T_1, \ldots, T_n$ are not all equal and satisfy*

(i) *pairwise braid relations $T_iT_jT_i = T_jT_iT_j$,*

(ii) *triplewise cycle relations $T_iT_jT_kT_i = T_jT_kT_iT_j$, where c_i, c_j, c_k is the induced cyclic order from $c_1, \ldots, c_n$.*

We will prove Theorem 2.1 by induction. The case $n = 2$ follows from our discussions about pairs of curves in Subsection 1.1.1. The case $n = 3$ will be used in the induction step, so we treat it as the base case and devote the entire next section to it.

2.1 Bouquets of three curves

This section consists of two parts that together lead to the pleasant fact that cycle relations detect whether a circuit of three curves forms a bouquet or not.

We show this by elementary means. First we describe the regular neighbourhood of a circuit of three curves, which turns out to be unique up to homeomorphism, but not up to orientation-preserving homeomorphism. We proceed to show that the cycle relation holds for a circuit of three curves if and only if the circuit forms a bouquet.

2.1.1 Neighbourhoods of three curves

Let a, b, c be curves intersecting pairwise once, and let N be a regular neighbourhood of $\alpha \cup \beta \cup \gamma$, where α, β, γ are minimally intersecting representatives of a, b, c, respectively, that do not intersect in a common point.

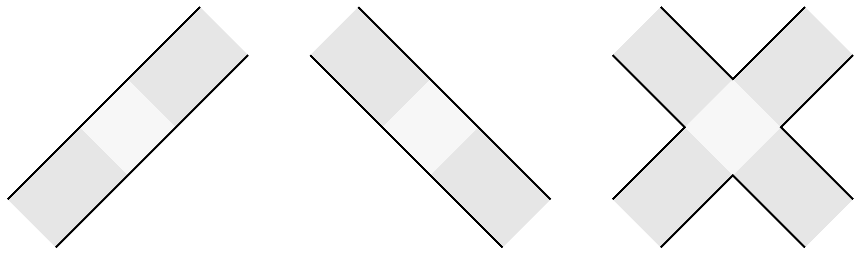


Figure 2.3: The bands B_α and B_β, both with dedicated squares, as well as the result from plumbing B_α to B_β along this square.

The neighbourhood N can be obtained in the following way. For each curve α, β, γ, consider annuli $B_\alpha, B_\beta, B_\gamma$ with core curves corresponding to α, β, γ, respectively. First, *plumb* B_α to B_β along a square, see Figure 2.3. Second, plumb B_γ to the result along a square disjoint from the previous one. Third, plumb the result to itself along another square disjoint from the previous two. Note that there is a unique way to do the first two plumbings, but there are two for the third, see Figure 2.4. Incidentally, the two neighbourhoods are related by an orientation-reversing homeomorphism preserving the curves α, β, γ. This homeomorphism can be vaguely described as sliding one of the bottom intersections on the right of Figure 2.4 over the end of the corresponding band, and then rotating the result to obtain the left hand side of the same figure.

As far as we are concerned, the difference between the two neighbourhoods is as follows. Both of the regular neighbourhoods have three boundary components, one of which is distinguished in that capping it off does not make α, β, γ bound an embedded closed disc. In our drawings, this boundary components has the appearance of a hexagon, as opposed to a triangle. We will refer to such a boundary component as a boundary component of *hexagonal type*. The other two boundary components are of *triangular type*. Travelling along any of the two boundary components of triangular type in the counter-clockwise direction, the curves appear in the order α, β, γ in one neighbourhood, and in the order

γ, β, α in the other. In the former case, we denote the neighbourhood N by the symbol $N^3_{\circlearrowright}$, and by $N^3_{\circlearrowleft}$ in the latter case. We sometimes abbreviate $N^3_{\circlearrowright}$ by the symbol N^3.

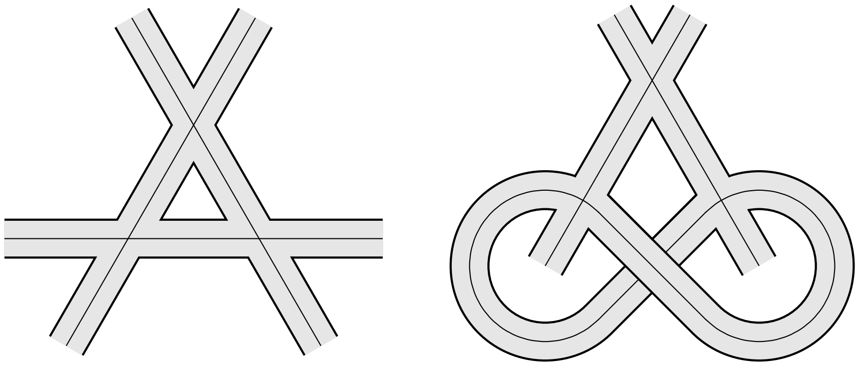


Figure 2.4: Two regular neighbourhoods of a circuit of three curves.

2.1.2 Cycle relations and discs

The goal of this subsection is to establish a connection between cycle relations between Dehn twists about a circuit of three curves, and embedded closed discs. As we have seen in the introduction to this chapter, the presence of such a closed embedded disc turns a circuit of three curves into a bouquet. This allows us to formulate our result as follows.

Proposition 2.2. *Let a, b, c be a circuit, and let α, β, γ be representatives of a, b, c, respectively, that intersect pairwise exactly once, but not all in the same point. Let N be a regular neighbourhood of $\alpha \cup \beta \cup \gamma$. Then the Dehn twists T_a, T_b, T_c satisfy the cycle relation*

$$T_c T_b T_a T_c = T_b T_a T_c T_b$$

if and only if a, b, c form an oriented bouquet.

Proof. Because $T_c T_b T_c = T_b T_c T_b$, it follows that $T_b^{-1} T_c T_b = T_c T_b T_c^{-1}$. Thus, the cycle relation is equivalent to

$$(T_c T_b T_c^{-1}) T_a = T_a (T_c T_b T_c^{-1}),$$

which holds if and only if a and $T_c(b)$ are disjoint. Figure 2.5 depicts the curve a as well as the curve $T_c(b)$. By the bigon criterion [FM11, Proposition 1.7], the curves in Figure 2.5 must bound a bigon.

Let us first consider $N^3_{\circlearrowright}$. Capping off any of the two boundary components of triangular type makes such a bigon appear. Both discs lead to a surface in

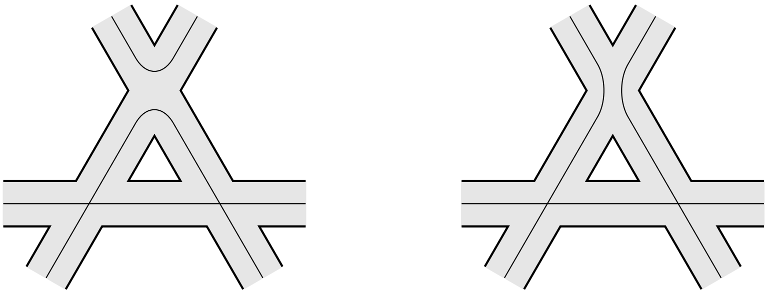


Figure 2.5: The curve a on the bottom and $T_c(b)$ in the two regular neighbourhoods $N^3_{\circlearrowright}$ and in $N^3_{\circlearrowleft}$.

which a, b, c form a bouquet. This should be evident in the case of the boundary component in the centre of Figure 2.5, but slightly less so for the other boundary component of triangular type. To understand this case, we construct an orientation-preserving homeomorphism of N that exchanges the two boundary components, called the *cross-involution*. Partition N into three cross-shaped pieces as in Figure 2.6. Rotating each of these three pieces exchanges the two boundary components under consideration. Capping off the boundary component of hexagonal type with a disc, however, neither adds a bigon nor turns a, b, c into a bouquet.

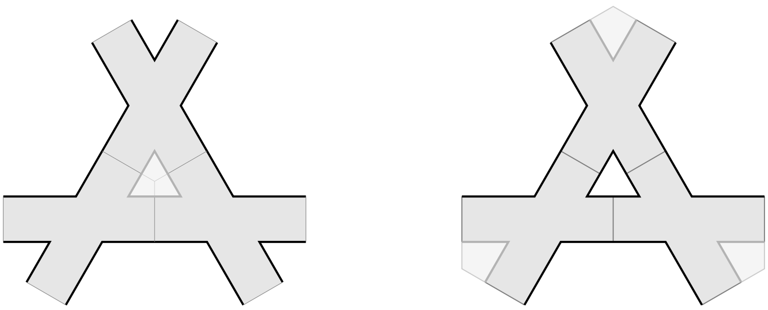


Figure 2.6: The cross-involution of a circuit of three curves.

On the other hand, no bigon can be added by capping off any boundary component in $N^3_{\circlearrowleft}$. This shows that the cycle relation $T_cT_bT_aT_c = T_bT_aT_cT_b$ holds if and only if a, b, c form an oriented bouquet. □

As a side note, Proposition 2.2 shows that the cycle relation

$$T_cT_bT_aT_c = T_bT_aT_cT_b$$

between Dehn twists is equivalent to the cyclic permutations

$$T_bT_aT_cT_b = T_aT_cT_bT_a \text{ and } T_aT_cT_bT_a = T_cT_bT_aT_c,$$

since a, b, c is an oriented bouquet if and only if c, a, b and b, c, a are.

2.2 Larger bouquets

In this section, we prove Theorem 2.1 by induction on n. As the following result shows, only a single cycle relation is needed to ensure that a bouquet remains a bouquet after the introduction of one additional curve.

Proposition 2.3 ([BFR23, Proposition 2])**.** *Let $n \geq 2$. Let $c_1, \ldots, c_n, c_{n+1}$ be simple closed curves in a surface S such that $c_1, \ldots, c_n$ form an oriented bouquet. If the Dehn twists T_i about c_i satisfy*

(i) *the braid relations $T_i T_{n+1} T_i = T_{n+1} T_i T_{n+1}$ for all $i \leq n$,*

(ii) *the cycle relation $T_n T_1 T_{n+1} T_n = T_1 T_{n+1} T_n T_1$,*

then $c_1, \ldots, c_n, c_{n+1}$ forms an oriented bouquet.

Proof. Let $\gamma_1, \ldots, \gamma_n$ be representatives of $c_1, \ldots, c_n$, respectively, that intersect pairwise once in a common point, and let γ_{n+1} be a representative of c_{n+1} that intersects γ_1 and γ_n in different points. Figure 2.7 depicts a regular neighbourhood N of $\gamma_1 \cup \gamma_n \cup \gamma_{n+1}$. The arrow in said Figure points from γ_1 to γ_n over $\gamma_2, \ldots, \gamma_{n-1}$, and γ_n is drawn as the horizontal curve. As we have seen in the proof of Proposition 2.2, N must indeed be of type $N^3_{\circlearrowleft}$ with respect to $\alpha = \gamma_1, \beta = \gamma_n, \gamma = \gamma_{n+1}$, because otherwise there would be no way for c_1, c_n, c_{n+1} to satisfy a cycle relation. Now Proposition 2.2 shows that S must contain a disc bounded by one of the two boundary components of N of triangular type, for instance the one indicated on the right of Figure 2.7. In both of these cases, $c_1, \ldots, c_n, c_{n+1}$ form an oriented bouquet. $\square$

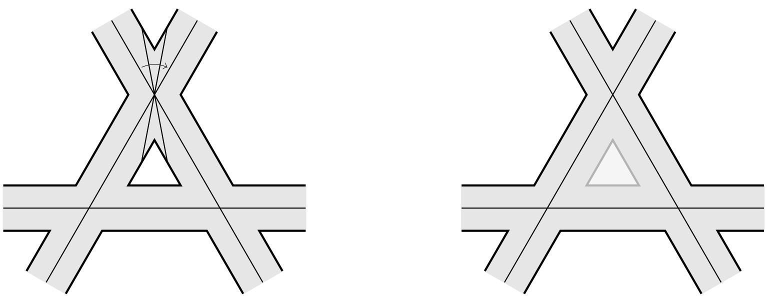


Figure 2.7: A regular neighbourhood N of $\gamma_1, \gamma_n, \gamma_{n+1}$ with segments of the curves $\gamma_2, \ldots, \gamma_{n-1}$, as well as one of the two possible discs in S bounded by a boundary component of N of triangular type.

Proof of Theorem 2.1. Proposition 2.2 covers the direction from left to right for all n, as well as the direction from right to left for $n = 3$. Since there are more relations in the assumptions of Theorem 2.1 than in those of Proposition 2.3, the latter implies the former. □

2.3 Bouquet groups

The main activity of this part of the thesis is to study groups generated by Dehn twists, which we have not yet done in this chapter. This section seeks to remedy this by studying *bouquet groups* generated by Dehn twists about bouquets of curves. We first discuss the relationship of bouquets to chains and then extract the description of bouquet groups from the description of *chain groups* generated by Dehn twists about chains of curves, which were previously considered in the literature.

2.3.1 Bouquets and chains

The goal of this subsection is to explain that bouquet groups and chain groups are the same. This is because a bouquet can be transformed into a chain by applying Dehn twists about curves in a bouquet to other curves in the bouquet.

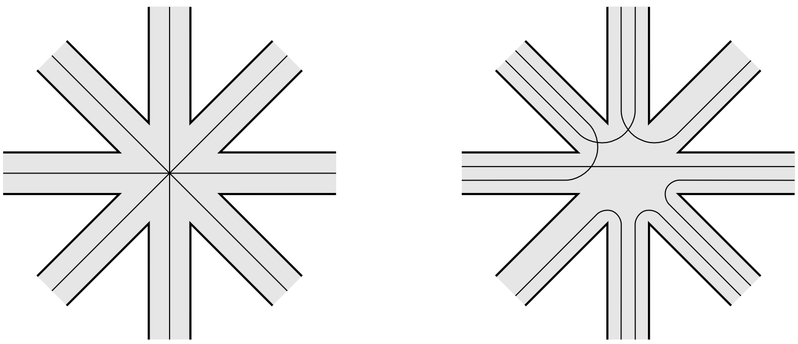


Figure 2.8: A bouquet of four curves $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ enumerated in the clockwise order, and the resulting chain $\gamma_1', \gamma_2', \gamma_3', \gamma_4'$. The curves γ_1 and γ_1' are drawn horizontally.

Proposition 2.4. *Each bouquet group is a chain group and vice versa.*

Proof. Let $c_1, \ldots, c_n$ be an oriented bouquet, and let $\gamma_1, \ldots, \gamma_n$ be minimally intersecting representatives of $c_1, \ldots, c_n$, respectively, that intersect in a common point. Let N be a regular neighbourhood of $\gamma_1 \cup \cdots \cup \gamma_n$. Consider the curves $c_1' = c_1$ and $c_i' = T_{i-1}^{-1}(c_i)$ for $i \geq 2$, where T_{i-1} is the Dehn twist about c_{i-1}. Then $c_1', \ldots, c_n'$ forms a chain admitting representatives $\gamma_1', \ldots, \gamma_n'$ such that N

is a regular neighbourhood of $\gamma_1' \cup \cdots \cup \gamma_n'$, see Figure 2.8 for the case $n = 4$. Because $T_1' = T_1$ and $T_i' = T_{i-1}^{-1} T_i T_{i-1}$ for $i \geq 2$, we obtain that the subgroup of $\mathrm{MCG}(S)$ generated by the T_i' coincides with the subgroup generated by the original Dehn twists T_i. □

2.3.2 Chains of three curves

A regular neighbourhood N of a chain $c_1, \ldots, c_n$ of $n \geq 3$ curves in a surface S has one or two boundary components, depending on whether n is odd or even, respectively. The case $n = 3$ is further distinguished because gluing in a single disc makes the two end curves isotopic, see Figure 2.9. This allows us to give an explicit description of all the groups that can arise as subgroups of the mapping class group generated by Dehn twists about a chain or bouquet of three curves.

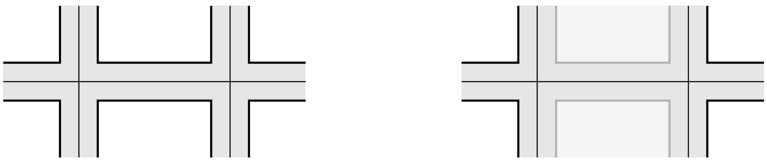


Figure 2.9: Capping off one boundary component of a regular neighbourhood of a chain of three curves makes the two end curves, drawn vertically, isotopic.

We say that two groups G, H generated by the conjugates of g, h, respectively, are *geometrically isomorphic* if there exists an isomorphism $G \to H$ mapping the conjugacy class of g to the conjugacy class of h. Many groups we consider come with such distinguished conjugacy classes: Mapping class groups are generated by the Dehn twists about non-separating curves, and all of these Dehn twists are conjugate. Similarly, Artin groups $\mathrm{Art}(\Gamma)$ are generated by the conjugacy class of one of the standard generators, provided Γ is connected.

Proposition 2.5. *Let a, b, c be a chain of curves in a surface S without punctures, and let α, β, γ be minimally intersecting representatives of a, b, c, respectively. Let N be a regular neighbourhood of S, and let G be the group generated by the Dehn twists about a, b, c. Then*

(i) *If $S \setminus N$ has no disc components and is not an annulus, then G is geometrically isomorphic to* $\mathrm{Art}(A_3)$.

(ii) *If $S \setminus N$ has exactly one disc component, then G is geometrically isomorphic to* $\mathrm{Art}(A_2)$.

(iii) *If $S \setminus N$ is an annulus, then G is geometrically isomorphic to* $\mathrm{Art}(A_3)/C$, *where C is the centre of* $\mathrm{Art}(A_3)$.

(iv) *If $S \setminus N$ has exactly two disc components, then S is a torus and G is isomorphic to* $\mathrm{SL}(2, \mathbb{Z})$.

Proof. By Lemma 1.1 and the surrounding discussion in Subsection 1.1.2, we obtain points (i) and (ii) for the two surfaces depicted in Figure 2.9. By Lemma 1.2, gluing in surfaces that are neither discs nor annuli nor punctured discs to N to obtain S does not change the resulting subgroup of $\mathrm{MCG}(S)$, proving points (i) and (ii) for the required surfaces.

For point (iii), note that Lemma 1.2 asserts that the kernel of the inclusion-induced homomorphism $\mathrm{MCG}(N) \to \mathrm{MCG}(S)$ is generated by $T_{d_1} T_{d_2}^{-1}$ where d_1, d_2 are the boundary curves of N. But by the chain relation from Subsection 1.1.2, we have that $T_{d_1} T_{d_2}^{-1}$ is equal to $(T_a T_b T_c)^4$. This is a generator of the centre C.

For point (iv), it is easy to see from Figure 2.9 that S is a torus. Recall that $\mathrm{MCG}(S)$ is isomorphic to $\mathrm{SL}(2, \mathbb{Z})$, and the Dehn twists T_a, T_b map to the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

respectively, which generate $\mathrm{SL}(2, \mathbb{Z})$. □

Allowing punctures in Proposition 2.5 is possible, but makes the formulation of the result more clumsy. The only interesting cases are the ones in which one or two components of $S \setminus N$ are punctured discs. By Lemma 1.2, we may assume without loss that all components of $S \setminus N$ are either discs, once-punctured discs, or annuli. Table 2.1 lists all possibilities for S after this restriction, as well as the subgroup of $\mathrm{MCG}(S)$ generated by T_a, T_b, T_c.

S	$\langle T_a, T_b, T_c \rangle$
N	$\mathrm{Art}(A_3)$
$N \cup \Delta_1$	$\mathrm{Art}(A_3)$
$N \cup 2\Delta_1$	$\mathrm{Art}(A_3)/C$
$N \cup Z$	$\mathrm{Art}(A_3)/C$
$N \cup \Delta$	$\mathrm{Art}(A_2)$
$N \cup \Delta \cup \Delta_1$	$\mathrm{SL}(2, \mathbb{Z})$
$N \cup 2\Delta$	$\mathrm{SL}(2, \mathbb{Z})$

Table 2.1: All groups generated by Dehn twists about a chain a, b, c of three curves in a surface S. The symbols Δ and Δ_1 denote a disc and a once-punctured disc, respectively, that have a common boundary component with N. The symbols 2Δ and $2\Delta_1$ denote a disjoint union of two such discs. The symbol Z denotes a cylinder, and C is the centre of $\mathrm{Art}(A_3)$.

The cases $S = N \cup 2\Delta_1, N \cup \Delta \cup \Delta_1$ can be dealt with as in the proof of Proposition 2.5. The case $S = N \cup \Delta_1$ requires some additional explanation.

Let $G = \langle T_a, T_b, T_c \rangle \subset \mathrm{MCG}(S)$. By the inclusion homomorphism theorem, the kernel K of the inclusion-induced homomorphism $\mathrm{MCG}(N) \to \mathrm{MCG}(S)$ is generated by T_{d_1}, where d_1 is the boundary curve of Δ_1. Because K is central in $\mathrm{MCG}(N)$, it is also central in $G_N = \langle T_a, T_b, T_c \rangle \subset \mathrm{MCG}(N)$. But the centre of that group is generated by the element $(T_a T_b T_c)^4 = T_{d_1} T_{d_2}^{-1}$, where d_2 is the other boundary component of N. So the kernel K' of the restriction to G_N is contained in the intersection of $\langle T_{d_1} \rangle$ and $\langle T_{d_1} T_{d_2}^{-1} \rangle$, which is trivial. Thus, the obviously surjective restriction $G_N \to G$ of the inclusion-induced homomorphism is injective.

2.3.3 Longer chains

As usual, let $c_1, \ldots, c_n$ be a chain of $n \geq 4$ curves in a surface S, and let $\gamma_1, \ldots, \gamma_n$ be minimally intersecting representatives of $c_1, \ldots, c_n$. Let N be a regular neighbourhood of $\gamma_1 \cup \cdots \cup \gamma_n$. We will study the subgroup G of $\mathrm{MCG}(S)$ generated by the Dehn twists $T_1, \ldots, T_n$ about $c_1, \ldots, c_n$, respectively.

The case of longer chains is more subtle than the case of chains of three curves, and there are many subcases for which we do not arrive at an explicit description of the subgroup of the mapping class group generated by Dehn twists about the chain, namely the case of the complement $S \setminus N$ containing at least one disc component. The remaining cases are listed in Table 2.2. The only case not analogous to the case of three curves is $S = N \cup \Delta_1$ for even n, so we display it as its own result.

Proposition 2.6. *Let $c_1, \ldots, c_n$ be a chain of $n \geq 4$ curves in a surface S with even n. Let N be a regular neighbourhood of minimally intersecting representatives $\gamma_1, \ldots, \gamma_n$ of $c_1, \ldots, c_n$. If $S \setminus N = \Delta_1 \cup S'$ for some possibly empty surface S' which is not a disc nor a once-punctured disc, then the group G generated by the Dehn twists $T_1, \ldots, T_n$ about $c_1, \ldots, c_n$ is isomorphic to $\mathrm{Art}(A_n)/C^2$, where C^2 consists of squares of central elements in $\mathrm{Art}(A_n)$.*

Proof. Write $G_N = \langle T_1, \ldots, T_n \rangle \subset \mathrm{MCG}(N)$. By Lemma 1.2, the kernel of the inclusion-induced homomorphism $\mathrm{MCG}(N) \to \mathrm{MCG}(S)$ is generated by $T_d = (T_1 \ldots T_n)^{2n+2}$, which lies in G_N. Because G_N is isomorphic to $\mathrm{Art}(A_n)$ via an isomorphism mapping $T_1, \ldots, T_n$ to the standard generators, the Dehn twist T_d maps to the double full twist in $\mathrm{Art}(A_n)$, which is the square of a generator of the centre C of $\mathrm{Art}(A_n)$. Restricting the inclusion-induced homomorphism above to G_N and precomposing with the isomorphism $\mathrm{Art}(A_n) \to G_N$ yields a surjective homomorphism $\mathrm{Art}(A_n) \to G$ whose kernel is C^2. $\square$

If $S \setminus N$ contains a disc, then all we are able to say is that $G = \langle T_1, \ldots, T_n \rangle$ is not geometrically isomorphic to an Artin group. We do this explicitly in

S	n	$\langle T_1, \ldots, T_n \rangle$
N	any	$\mathrm{Art}(A_n)$
$N \cup \Delta_1$	odd	$\mathrm{Art}(A_n)$
$N \cup 2\Delta_1$	odd	$\mathrm{Art}(A_n)/C$
$N \cup Z$	odd	$\mathrm{Art}(A_n)/C$
$N \cup \Delta_1$	even	$\mathrm{Art}(A_n)/C^2$

Table 2.2: Groups generated by Dehn twists $T_1, \ldots, T_n$ about a chain $c_1, \ldots, c_n$ of $n \geq 4$ curves. The symbols N, Δ, Δ_1, $2\Delta_1$, Z and C have the same meaning as in Table 2.1, and C^2 is the group consisting of squares of central elements in $\mathrm{Art}(A_n)$.

Chapter 3, where what is called N here is called $N^n \cup 2\Delta$. The reason we are able to make such a statement is that capping off any boundary component of N with a disc introduces an inhomogeneous relation (that is, a relation in $\mathrm{MCG}(S)$ with a different number of Dehn twists on either side), and Artin groups satisfy no such relations.

2.4 Sequences of discs

This last section is a small application of Proposition 2.2 where we construct long sequences of surfaces $S^0 \subset S^1 \subset \cdots \subset S^k$ all of the same genus (but different numbers of boundary components) containing a certain system of curves $\gamma_1, \ldots, \gamma_{2g+1}$ that intersect pairwise once, such that none of the inclusion-induced homomorphisms $\mathrm{MCG}(S^i) \to \mathrm{MCG}(S^{i+1})$ are injective. We construct the surface S^0 as a subsurface of the standard closed surface of genus g. Then we describe how to pass from S^i to S^{i+1} and subsequently explain why the corresponding inclusion-induced homomorphism is not injective.

In order to describe S^0, we construct a specific realisation of a bouquet $c_1, \ldots, c_n$ in the standard surface of genus g that is not the standard realisation which intersects in one point. Consider an arbitrary but fixed genus $g \geq 2$. Let S be the standard surface of genus g, thought of as a $(4g+2)$-gon P with opposite sides identified. Start with the curves $\gamma'_1, \ldots, \gamma'_{2g+1}$ connecting the centres of opposing pairs of edges of P in a straight line. Note that upon identifying opposite sides, the vertices of P are partitioned into two equivalence classes, and that both equivalence classes have a representative in each edge. Pick one such equivalence class and move $\gamma'_1, \ldots, \gamma'_{2g+1}$ half-way towards it while keeping the curve straight to obtain the *polygonal bouquet* $\gamma_1, \ldots, \gamma_{2g+1}$ for genus g, illustrated in Figure 2.10. We are now ready to state the result of this section.

Theorem 2.7. *Let $\gamma_1, \ldots, \gamma_{2g+1}$ be the polygonal bouquet for genus $g \geq 2$. Let S^0 be a regular neighbourhood of $\gamma_1 \cup \cdots \cup \gamma_{2g+1}$. Then there exists a sequence*

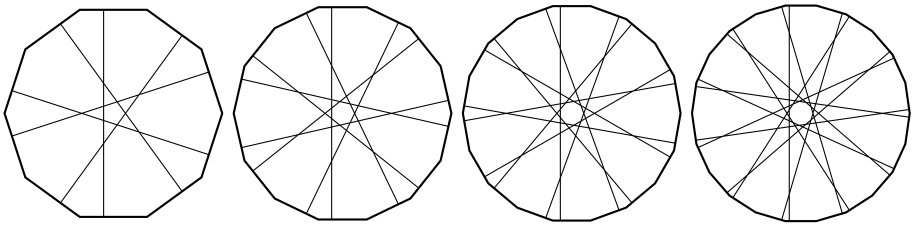


Figure 2.10: The polygonal bouquet for genus $g = 2, 3, 4, 5$.

of discs $\Delta^1, \ldots, \Delta^{(g-1)(2g+1)}$ *such that for the surfaces* $S^i = S^{i-1} \cup \Delta^i$*, none of the inclusion-induced homomorphisms* $\mathrm{MCG}(S^i) \to \mathrm{MCG}(S^{i+1})$ *are injective.*

Proof. The only restriction to the order in which the discs are filled in is that the innermost triangles go first, then the adjacent quadrilaterals, then the quadrilaterals adjacent to the previous ones, and so on, see Figure 2.11. As suggested by this restriction, we organise the discs into layers containing $2g+1$ discs each: the triangles belong to layer 1, the adjacent quadrilaterals to layer 2, and so on up to layer $g-1$.

Number the curves $\gamma_1, \ldots, \gamma_{2g+1}$ in the counter-clockwise order around the innermost $(2g+1)$-gon, and let $T_1, \ldots, T_{2g+1}$ be the corresponding Dehn twists. The triangles at layer 1 are bounded by the curves $\gamma_i, \gamma_{i+1}, \gamma_{i+2}$, where indices are taken modulo $2g+1$ until the end of the proof. This introduces a cycle relation between T_i, T_{i+1}, T_{i+2} into $\mathrm{MCG}(S^{j+1})$ from filling in the corresponding triangle compared to $\mathrm{MCG}(S^j)$, see Proposition 2.2.

In layer 2, gluing in a quadrilateral has the effect of introducing a cycle relation between T_i, T_{i+1}, T_{i+3} and between T_i, T_{i+2}, T_{i+3}. More generally, gluing in a quadrilateral introduces new cycle relations between T_i, T_{i+j}, T_{i+k+1} for all $j \in \{1, \ldots, k\}$. This shows that none of the inclusion-induced homomorphisms are injective. $\square$

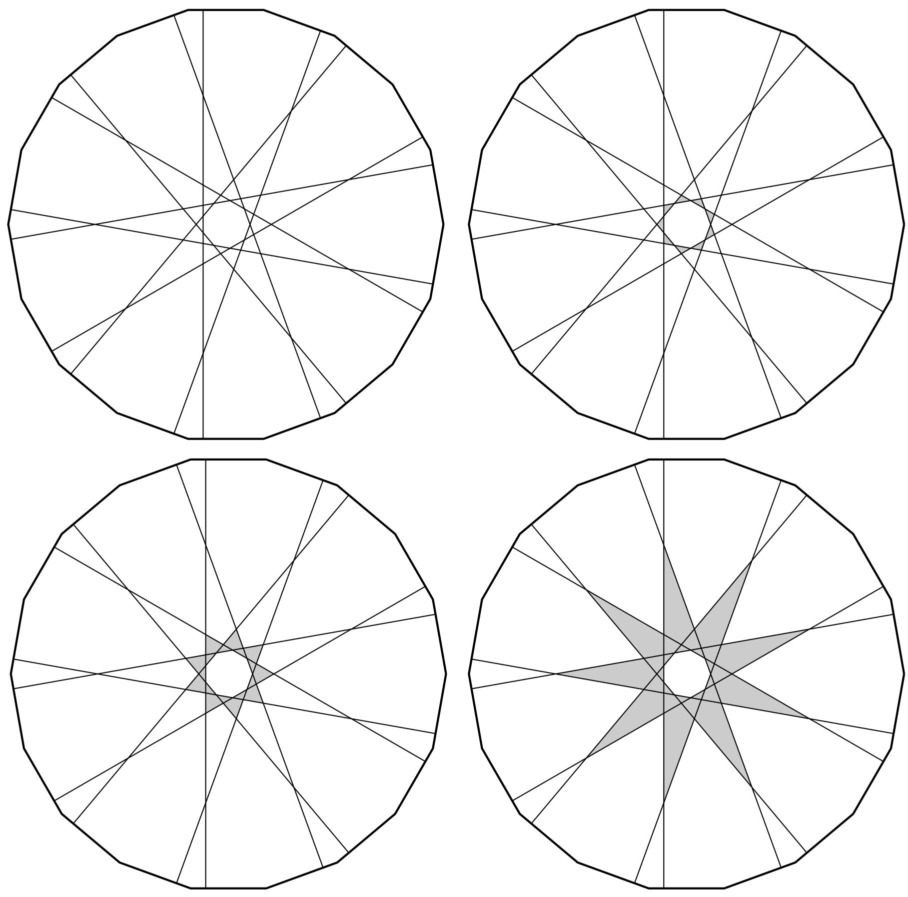


Figure 2.11: The polygonal bouquet for genus $g = 4$, the innermost triangles, the adjacent quadrilaterals, and the outermost quadrilaterals glued in successively.

Chapter 3

Circuits of curves

In this chapter, we study certain relations between Dehn twists about circuits, and what consequences the presence or absence of these relations have for the surrounding surface. The relation that is of particular interest to us is the cycle relation, a generalisation of the same named relation that featured in the previous chapter. We will show that a cycle relation holds if and only if the circuit bounds an embedded closed disc, generalising Proposition 2.2 to arbitrary circuits. A substantial part of the left-to-right direction was proved by Labruère [Lab97, Proposition 2], so most of the work in this Chapter goes into proving the converse. The way we approach it is to show that for any regular neighbourhood N of a circuit, the subgroup of $\mathrm{MCG}(N)$ generated by the Dehn twists about the circuit is geometrically isomorphic to a group in which the cycle relation does not hold, namely an Artin group of type $\widetilde{A}$. We achieve this result by exploiting a symmetry of N, and using it to invoke Birman-Hilden theory. By the end of Section 3.2, we will have proven this result. The effect of gluing in discs to regular neighbourhoods of circuits on the relations satisfied by the corresponding Dehn twists is the topic of Section 3.3, where we have the additional focus of answering the question when a circuit group is geometrically isomorphic to an Artin group. We proceed to prove the main result in Section 3.4. The material in this chapter was published in [Ryf23a].

3.1 The cycle relation

A system of n curves $c_1, \ldots, c_n$ is said to form a *circuit* if c_i intersects c_{i+1} exactly once for all i, where indices are taken modulo n. Let $c_1, \ldots, c_n$ be a circuit in a surface S. The *(standard) cycle relation* between Dehn twists $T_1, \ldots, T_n$

about $c_1, \dots, c_n$ is the relation

$$T_n \cdots T_1 T_n \cdots T_3 = T_{n-1} \cdots T_1 T_n \cdots T_2.$$

Our results in Chapter 2, specifically Proposition 2.2, enable us to understand fully when a cycle relation holds for a circuit of three curves, namely if and only if the curves c_1, c_2, c_3 form an oriented bouquet. This is equivalent to the circuit c_1, c_2, c_3 bounding an oriented embedded closed disc. In general, we say that a circuit $c_1, \dots, c_n$ bounds an *oriented embedded closed disc* if there are representatives $\gamma_1, \dots, \gamma_n$ of $c_1, \dots, c_n$, respectively, segments of which bound a disc Δ, such that when going around Δ in the counter-clockwise direction, the curves on the boundary appear in the order $\gamma_1, \dots, \gamma_n$. It turns out that the characterisation of the cycle relation in terms of oriented embedded closed discs generalises to larger circuits, as our main theorem asserts.

Theorem 3.1 ([Ryf23a, Theorem 1.1])**.** *Let $c_1, \dots, c_n$ be a circuit of $n \geq 3$ curves in a surface S. Let $T_1, \dots, T_n$ be the corresponding Dehn twists. Then $c_1, \dots, c_n$ bound an embedded closed disc Δ in S if and only if one of the two cycle relations*

$$T_n \cdots T_1 T_n \cdots T_3 = T_{n-1} \cdots T_1 T_n \cdots T_2$$

or

$$T_1 \cdots T_n T_1 \cdots T_{n-2} = T_2 \cdots T_n T_1 \cdots T_{n-1}$$

holds. The first relation corresponds to the curves $c_1, \dots, c_n$ admitting representatives appearing in the cyclic order $\gamma_1, \dots, \gamma_n$ when travelling around Δ in the counter-clockwise direction, whereas the second relation corresponds to the opposite cyclic order.

Proving Theorem 3.1 involves multiple key steps, each of which is interesting in its own right. Perhaps the simplest yet the most significant step is to determine the subgroup of $\mathrm{MCG}(N)$ generated by the Dehn twists about a circuit in a regular neighbourhood N. This step proves a conjecture by Mortada [Mor11, Conjecture 5.5.5], who has previously proved Theorem 3.2 in the case $N = M^n$ [Mor11, Theorem 5.5.4], where M^n is constructed below in Section 3.2.

Theorem 3.2 ([Ryf23a, Theorem 1.2])**.** *Let N be a regular neighbourhood of minimally intersecting representatives of a circuit $c_1, \dots, c_n$ of $n \geq 3$ curves. Then the subgroup of $\mathrm{MCG}(N)$ generated by the Dehn twists T_i about c_i is geometrically isomorphic to $\mathrm{Art}(\widetilde{A}_{n-1})$.*

The reason the latter theorem is helpful towards proving the former is that relatively elementary theory of Artin groups can be used to prove that the cycle relation does not hold in Artin groups of type $\widetilde{A}$. Of course, Theorem 3.1 makes a much stronger observation, the proof of which requires an exhaustive investigation of surfaces in which the circuit under consideration can lie. Considering all these surfaces leads us to discover various relations between Dehn twists about curves in circuits, depending on the surrounding surface.

3.2 Neighbourhoods of circuits

This section is concerned with the proof of Theorem 3.2. Let $c_1, \ldots c_n$ be a circuit in a surface S with minimally intersecting representatives $\gamma_1, \ldots, \gamma_n$. Up to orientation-preserving homeomorphism, there are two possible regular neighbourhoods of the union $\gamma_1 \cup \cdots \cup \gamma_n$, see Figure 3.1. One way to see this is as follows. There is only one possible regular neighbourhood of the chain $\gamma_1 \cup \cdots \cup \gamma_{n-1}$. Now the curve γ_n might sit in the regular neighbourhood in two different ways.

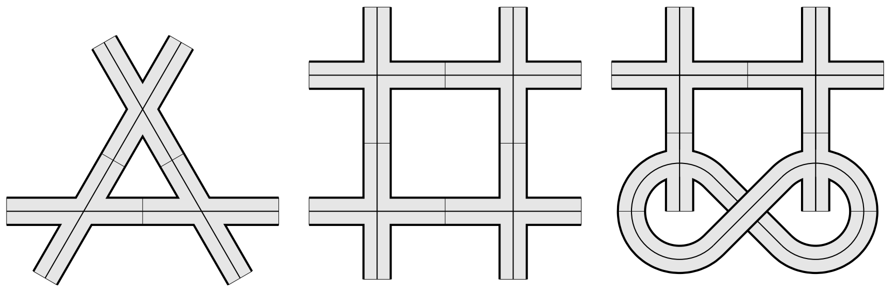


Figure 3.1: Each of the regular neighbourhoods N^3, N^4, M^4 can be subdivided into cross-shaped pieces. As usual, opposite ends of the strips are identified.

If n is odd, those two ways lead to regular neighbourhoods $N^n_{\circlearrowleft}$ and $N^n_{\circlearrowright}$ that are related by an orientation-reversing homeomorphism. For brevity, we will abbreviate $N^n_{\circlearrowleft}$ by the symbol N^n and usually not talk about $N^n_{\circlearrowright}$ explicitly, as all the results about N^n carry over to $N^n_{\circlearrowright}$ by enumerating the c_i in the opposite order. The left-hand side of Figure 3.7 below displays a drawing of the surface N^n embedded into $\mathbb{R}^3$.

If n is even, these two ways lead to regular neighbourhoods N^n and M^n that are related to themselves (but not to each other) via an orientation-reversing homeomorphism, so orientation is less of a concern in this case. The two neighbourhoods N^n and M^n differ, for example, in their number of boundary components: N^n has four and M^n just two. The genus also differs: M^n has genus $n/2$, and N^n has genus $n/2 - 1$. A drawing of the surfaces N^n and M^n embedded into $\mathbb{R}^3$ can be found in Figure 3.2.

Now let S be any of the above regular neighbourhoods of $\gamma_1 \cup \cdots \cup \gamma_n$.

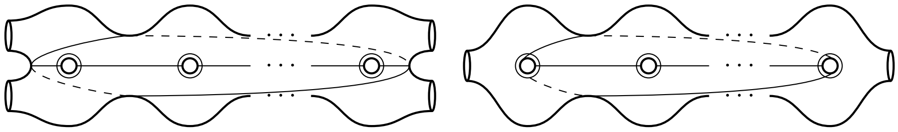


Figure 3.2: Another view of the surfaces N^n and M^n for even n.

Notice that in each case, S can be thought of as a union of n cross-shaped pieces, see Figure 3.1. Turning all those pieces by an angle of π yields a well-defined involution ι of S. We will call ι the *cross-involution*. In the drawings from Figures 3.2 and 3.7, the cross-involution is a rotation about the x-axis by an angle of π.

Proof of Theorem 3.2. Let S be any of the regular neighbourhoods N^n or M^n of $\gamma_1 \cup \cdots \cup \gamma_n$, and let ι be the cross-involution of S. We write S/ι for the orbit space of S without the fixed points of ι modulo ι. Note that S/ι is homeomorphic to an n times punctured annulus Z_n. Let $\mathrm{CG}(S)$ be the group generated by the Dehn twists $T_1, \ldots, T_n$ about $c_1, \ldots, c_n$, respectively. Because $c_1, \ldots, c_n$ are preserved by ι, the Dehn twists T_i all commute with ι. Thus, Lemma 1.1 shows that There is a well-defined homomorphism $\mathrm{CG}(S) \to \mathrm{MCG}(S/\iota)$ such that the images of the Dehn twists T_i are half-twists about arcs arranged such as in the second, third, and fourth pictures in Figure 1.11. In Subsection 1.2.2 we explained that the group generated by these half-twists is isomorphic to the subgroup $\mathrm{Art}(\widetilde{A}_{n-1})$ of the annular braid group via an isomorphism mapping the half-twists to generators. Composing these two maps yields a geometric isomorphism between $\mathrm{CG}(S)$ and $\mathrm{Art}(\widetilde{A}_{n-1})$. □

3.3 Gluing in discs

In this section, we consider all possible discs we could glue in to fill a boundary component of a regular neighbourhood of a circuit $c_1, \ldots, c_n$. To this end, we first name all these discs.

3.3.1 Disc notation

From now on, we adopt the perspective that the curves $c_1, \ldots, c_n$ along with their minimally intersecting representatives $\gamma_1, \ldots, \gamma_n$ stay fixed, while the surrounding surface S varies. We will write $\mathrm{CG}(S)$ for the subgroup of $\mathrm{MCG}(S)$ generated by the Dehn twists T_i about c_i.

The boundary components of neighbourhoods of $\gamma_1 \cup \cdots \cup \gamma_n$ can be qualitatively distinguished as follows. Considering intersection points between the γ_i as vertices of polygons allows us to make the following observation. If n is odd, then two boundary components of N^n are exchanged by the cross-involution. They both have the property that if they are capped by a disc, then $\gamma_1, \dots, \gamma_n$ bound an n-gon. We will denote such a disc by Δ^1, and the union of two such discs by $2\Delta^1$. Both of these will make the circuit bound embedded closed discs. The third boundary component can be capped off by a disc Δ^2. The curves $\gamma_1, \dots, \gamma_n$ bound a $2n$-gon in $S \cup \Delta^2$. Note that the circuit does not bound an embedded closed disc in $N^n \cup \Delta^2$. For brevity, we will sometimes say that the boundary of Δ^1 *is* an n-gon and that the boundary of Δ^2 *is* a $2n$-gon. See Figure 3.3 for a visual description of the various discs.

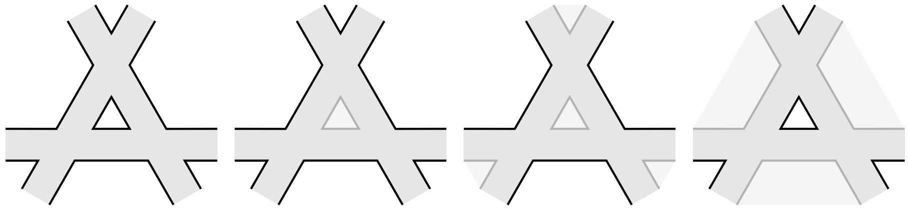


Figure 3.3: The surfaces N^3, $N^3 \cup \Delta^1$, $N^3 \cup 2\Delta^1$, $N^3 \cup \Delta^2$.

If n is even, the situation is qualitatively different. All four boundary components of N^n are n-gons, and both boundary components of M^n are $2n$-gons. But they differ in the following way. Travelling around the boundary component in the counter-clockwise direction, as seen from the centre of the disc, the curves might appear in the order $\gamma_1, \dots, \gamma_n$ or the other way around. In the first case, we will write $\Delta_{\circlearrowleft}$, and $\Delta_{\circlearrowright}$ otherwise, with the appropriate superscript numbers. See Figure 3.4 for the different discs in N^n. We will usually abbreviate $\Delta_{\circlearrowleft}$ by Δ with the appropriate superscript number, and we will abbreviate $\Delta^2_{\circlearrowleft} \cup \Delta^2_{\circlearrowright}$ by $2\Delta^2$. A list of all surfaces that are obtained by gluing in discs to a regular neighbourhood of $\gamma_1 \cup \dots \cup \gamma_n$ can be found in Table 3.1. Note that gluing in any disc into N^n makes the circuit bound an embedded closed disc, and no disc in M^n has this effect.

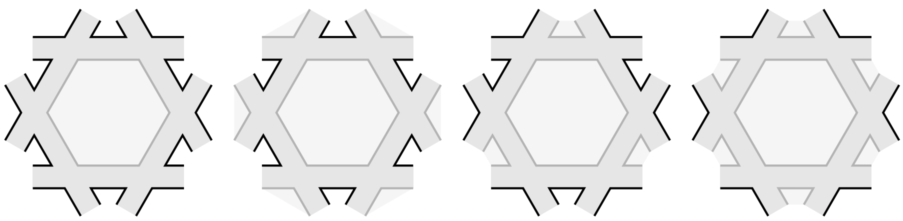


Figure 3.4: The surfaces $N^6 \cup \Delta^1_{\circlearrowleft}$, $N^6 \cup 2\Delta^1_{\circlearrowleft}$, $N^6 \cup \Delta^1_{\circlearrowleft} \cup \Delta^1_{\circlearrowright}$, $N^6 \cup \Delta^1_{\circlearrowleft} \cup 2\Delta^1_{\circlearrowright}$

S	$\mathrm{CG}(S)$	Reference
N^n	$\mathrm{Art}(\widetilde{A}_{n-1})$	Theorem 3.2
$N^n \cup \Delta^1$	$\mathrm{Art}(D_n)$	Proposition 3.7
$N^n \cup 2\Delta^1$	$\mathrm{Art}(A_{n-1})$	Proposition 3.4
$N^{2k+1} \cup \Delta^2$	$\mathrm{Art}(\widetilde{A}_{2k})$	Proposition 3.3
$N^{2k+3} \cup \Delta^1 \cup \Delta^2$	not Artin	Proposition 3.17
$N^{2k+1} \cup 2\Delta^1 \cup \Delta^2$	not Artin	Proposition 3.10
$N^{2k+4} \cup \Delta^1_{\circlearrowleft} \cup \Delta^1_{\circlearrowright}$	not Artin	Proposition 3.11
$N^{2k+4} \cup 2\Delta^1_{\circlearrowleft} \cup \Delta^1_{\circlearrowright}$	not Artin	Proposition 3.11
$N^{2k+4} \cup 2\Delta^1_{\circlearrowleft} \cup 2\Delta^1_{\circlearrowright}$	not Artin	Proposition 3.11
$N^3 \cup \Delta^1 \cup \Delta^2$	$\mathrm{Art}(A_2)$	Proposition 2.5
$N^4 \cup \Delta^1_{\circlearrowleft} \cup \Delta^1_{\circlearrowright}$	$\mathrm{Art}(A_3)$	Proposition 3.12
$N^4 \cup 2\Delta^1_{\circlearrowleft} \cup \Delta^1_{\circlearrowright}$	$\mathrm{Art}(A_2)$	Proposition 3.12
$N^4 \cup 2\Delta^1_{\circlearrowleft} \cup 2\Delta^1_{\circlearrowright}$	$\mathrm{SL}(2,\mathbb{Z})$	Proposition 3.12
M^{2k+2}	$\mathrm{Art}(\widetilde{A}_{2k+1})$	Theorem 3.2
$M^{2k+2} \cup \Delta^2$	$\mathrm{Art}(\widetilde{A}_{2k+1})$	Proposition 3.3
$M^{2k+2} \cup 2\Delta^2$	$\mathrm{Art}(\widetilde{A}_{2k+1})$	Proposition 3.3

Table 3.1: Various surfaces S obtained from regular neighbourhoods of a circuit $\gamma_1, \ldots, \gamma_n$ by gluing in discs. In this table, $n \geq 3$ and $k \geq 1$. Moreover, "not Artin" is short for "not geometrically isomorphic to an Artin group".

3.3.2 Extending the cross involution

It turns out that for certain surfaces S containing a neighbourhood of $\gamma_1, \ldots, \gamma_n$, the cross-involution of the neighbourhood extends to S. In these cases, by very similar reasoning as in the proof of Theorem 3.2, we are able to determine the group $\mathrm{CG}(S)$.

Proposition 3.3. *Let S be a regular neighbourhood of a circuit of $n \geq 3$ curves $\gamma_1, \ldots \gamma_n$, and let S' be a $2n$-gon Δ^2 (such a disc does not exist if n is even and $S = N^n$), or the union $2\Delta^2$ of two $2n$-gons (such a union only exists for even n and $S = M^n$). Then $\mathrm{CG}(S \cup S')$ is geometrically isomorphic to $\mathrm{Art}(\widetilde{A}_{n-1})$.*

Proof. In each case, the cross-involution ι extends to $S \cup S'$ in a straightforward fashion. In the case that $S' = \Delta^2$ is a $2n$-gon, ι gets one additional fixed point, so $(S \cup S')/\iota$ is a disc Δ_{n+1} with $n+1$ marked points. Again, the homomorphism $\varphi\colon\ \mathrm{CG}(S) \to \mathrm{MCG}(\Delta_{n+1})$ is well-defined by Lemma 1.1. The images of the T_i under φ fix the last puncture. The subgroup of $\pi_1(C(\Delta, n+1))$ fixing one strand is isomorphic to $\pi_1(C(Z, n))$ because Δ minus one point is homotopy equivalent to the annulus Z. By our discussions in Subsection 1.2.2, the image of φ is isomorphic to $\mathrm{Art}(\widetilde{A}_{n-1})$. Since the inverse homomorphism is well-defined, the result follows for this case.

Similarly, if $S' = 2\Delta^2$, we get a well-defined homomorphism φ between $\mathrm{CG}(S)$ and $\mathrm{MCG}(\Sigma_{n+2})$, where Σ_{n+2} is a sphere with $n+2$ marked points. The images of the Dehn twists T_i fix two of the punctures. The group of orientation-preserving homeomorphisms of Σ is not simply connected, so we cannot apply the Birman exact sequence directly to get a description of $\mathrm{MCG}(\Sigma_{n+2})$. However, it is relatively straightforward to show that the kernel of the push map $\pi_1(C(\Sigma, n+2)) \to \mathrm{MCG}(\Sigma_{n+2})$ is generated by the map rotating the $n+2$ marked points by a full twist [FM11, Section 9.1]. Because such a full twist cannot be expressed by just n generators, we have that the image of φ is isomorphic to the subgroup of $\pi_1(C(\Sigma, n+2))$ fixing two strands. Note that Σ minus two points is homotopy equivalent to Z, so we may conclude that the image of φ is isomorphic to $\mathrm{Art}(\widetilde{A}_{n-1})$, as desired. □

Recall that for odd n we abbreviate $N^n_{\circlearrowright}$ by N^n, and for all n, we abbreviate $\Delta^1_{\circlearrowright}$ by Δ^1. Using these conventions allows us to concisely state the following result which is essentially equivalent to a version of the Birman-Hilden theorem stated in the book by Farb and Margalit [FM11, Theorem 9.2].

Proposition 3.4. *For $n \geq 3$, the group $\mathrm{CG}(N^n \cup 2\Delta^1)$ is geometrically isomorphic to the braid group $\mathrm{Art}(A_{n-1})$ on n strands.*

Proof. Let us write $S = N^n \cup 2\Delta^1$. The cross-involution ι extends to the surface $N^n \cup 2\Delta^1$ with no additional fixed points. This yields a well-defined homomorphism $\varphi\colon \mathrm{CG}(S) \to \mathrm{MCG}(\Delta_n)$ mapping the T_i to half-twists by Lemma 1.1. Since these half-twists generate $\mathrm{MCG}(\Delta_n)$, we obtain the image of φ is isomorphic to $\mathrm{Art}(A_{n-1})$. □

Remark 3.5. It is not difficult to show that the kernel of the inclusion-induced homomorphism $\mathrm{CG}(N^n) \to \mathrm{CG}(N^n \cup 2\Delta^1)$ is normally generated by the relation $T_n \cdots T_2 = T_{n-1} \cdots T_1$. One possible strategy is to explicitly compute the kernel of the inclusion-induced homomorphism $\pi_1(C(Z, n)) \to \pi_1(C(\Delta, n))$.

3.3.3 Labruère's surface

The surface the current subsection is about is $N^n \cup \Delta^1$. Luckily for us, this case has almost entirely been solved by Labruère, and the rest can be extracted from work by Baader and Lönne.

If the circuit $\gamma_1, \ldots, \gamma_n$ bounds an n-gon $\Delta^1_{\circlearrowright}$ or $\Delta^1_{\circlearrowleft}$, we will say that the curves $c_1, \ldots, c_n$ form a *cycle*. The *standard cycle relation* between the Dehn twists $T_1, \ldots, T_n$ is $T_n \cdots T_1 T_n \cdots T_3 = T_{n-1} \cdots T_1 T_n \cdots T_2$. One can show that this relation is equivalent to the commutation relation $T_1 f = f T_1$ where

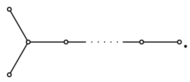
$$f = (T_n \cdots T_3) T_2 (T_n \cdots T_3)^{-1},$$

see [BL21, Section 1]. Using this representation of the standard cycle relation it becomes a routine task to verify that the standard cycle relation holds in the surface $N^n \cup \Delta^1$. Similarly, the *reverse cycle relation* $T_1 \cdots T_n T_1 \cdots T_{n-2} = T_2 \cdots T_n T_1 \cdots T_{n-1}$ holds in the surface $N^n_{\circlearrowleft} \cup \Delta^1_{\circlearrowleft}$. Labruère made an even stronger observation.

Lemma 3.6 ([Lab97, Proposition 2])**.** *The kernel of the standard homomorphism* $\mathrm{Art}(\widetilde{A}_{n-1}) \to \mathrm{CG}(N^n \cup \Delta^1)$ *mapping the standard generators s_i to T_i is normally generated by the cycle relation.*

Proposition 3.7. *For $n \geq 3$, the group* $\mathrm{CG}(N^n \cup \Delta^1)$ *is geometrically isomorphic to* $\mathrm{Art}(D_n)$*.*

Proof. Let $s_1, \dots, s_n$ be the standard generators of $\mathrm{Art}(D_n)$ read from left to right in the Dynkin diagram

$D_n =$ .

Using Lemma 3.6, one can verify that an explicit isomorphism $\mathrm{Art}(D_n) \to \mathrm{CG}(N^n \cup \Delta^1)$ is given by $s_1 \mapsto (T_n \cdots T_3)^{-1} T_1 (T_n \cdots T_3)$ and $s_i \mapsto T_i$ for $i \geq 2$. □

Remark 3.8. Baader and Lönne prove the considerably more general but also more involved result that the so-called secondary braid group is invariant via a geometric isomorphism under elementary conjugation [BL21, Section 4]. Indeed, by Lemma 3.6, the group $\mathrm{CG}(N^n \cup \Delta^1)$ is geometrically isomorphic to the secondary braid group [BL21, Definition 1] associated to the positive braid word $\sigma_1 \sigma_2 \sigma_1^{n-2} \sigma_2 \sigma_1$, whereas $\mathrm{Art}(D_n)$ is geometrically isomorphic to the group associated to $\sigma_1^2 \sigma_2 \sigma_1^{n-2} \sigma_2$.

3.3.4 Inhomogeneous relations

So far, we have been successful in determining the circuit group $\mathrm{CG}(S)$ obtained from gluing in discs to a regular neighbourhood of a circuit. Sadly, we do not manage to compute the remaining groups $\mathrm{CG}(S)$ up to isomorphism. We will, however, get to know the groups well enough to exclude the possibility of them being geometrically isomorphic to an Artin group.

A relation $t = t'$ is called *homogeneous* if the exponent sums of t and t' agree. Otherwise, the relation $t = t'$ is called *inhomogeneous*. The strategy in the current subsection will be to find inhomogeneous relations in $\mathrm{CG}(S)$. The following elementary result allows us to conclude that $\mathrm{CG}(S)$ is not geometrically isomorphic to an Artin group.

Lemma 3.9. *If* $\mathrm{CG}(S)$ *has an inhomogeneous relation, then* $\mathrm{CG}(S)$ *is not geometrically isomorphic to an Artin group.*

Proof. We argue contrapositively: The map sending each generator of an Artin group $\mathrm{Art}(\Gamma)$ to one extends to a homomorphism $\mathrm{Art}(\Gamma) \to \mathbb{Z}$ because all relations in $\mathrm{Art}(\Gamma)$ are homogeneous, and hence also hold in $\mathbb{Z}$. $\square$

An effective way to produce inhomogeneous relations is to apply the chain relation. Recall from Subsection 1.1.2 that a *chain* is a family $c_1, \ldots, c_n$ of n curves such that c_i intersects c_j exactly once if $j = i \pm 1$ and zero times otherwise. The chain relation then asserts that:

(i) If n is even, then $(T_n \cdots T_1)^{2n+2} = T_\beta$.

(ii) If n is odd, then $(T_n \cdots T_1)^{n+1} = T_{\beta_1} T_{\beta_2}$.

Here, $T_1, \ldots, T_n$ are the Dehn twists about $c_1, \ldots, c_n$, respectively.

Proposition 3.10. *For odd* $n \geq 3$*, the relation* $(T_{n-1} \cdots T_1)^{2n} = 1$ *holds in the group* $\mathrm{CG}(N^n \cup 2\Delta^1 \cup \Delta^2)$*. In particular, it is not geometrically isomorphic to an Artin group.*

Proof. Note that the boundary of a regular neighbourhood of the shorter chain $c_1, \ldots, c_{n-1}$ is null-homotopic. The proposition now follows from the chain relation and Lemma 3.9. $\square$

Proposition 3.11. *The group* $\mathrm{CG}(N^n \cup \Delta^1_{\circlearrowleft} \cup \Delta^1_{\circlearrowright})$ *is not geometrically isomorphic to an Artin group if* $n \geq 6$*, and neither is* $\mathrm{CG}(S)$ *for any supersurface* S *of* $N^n \cup \Delta^1_{\circlearrowleft} \cup \Delta^1_{\circlearrowright}$.

Proof. Suppose the curves are arranged as in Figure 3.5. Let

$$\beta = T_n \cdots T_3 T_3 \cdots T_n(\gamma_1).$$

Then $\beta \cup \gamma_1$ is the boundary of a regular neighbourhood of $\gamma_3 \cup \cdots \cup \gamma_{n-1}$, see Figure 3.5. By the chain relation, the relation $(T_{n-1} \cdots T_3)^{n-2} = T_1 T_\beta$ follows. Because T_β is conjugate to T_1 by the formula $T_{f(\gamma_1)} = f T_1 f^{-1}$ [FM11, Fact 3.7] for $f = T_n \cdots T_3 T_3 \cdots T_n$, the relation in question is inhomogeneous for $n \geq 6$. Lemma 3.9 leads us to the desired conclusion.

If the indices of the curves are shifted by one from the ones in Figure 3.5, we instead end up with the relation $(T_{n-2} \cdots T_2)^{n-2} = T_n T_{\beta'}$ where

$$\beta' = T_{n-1} \cdots T_2 T_2 \cdots T_{n-1}(\gamma_n),$$

which is also inhomogeneous for $n \geq 6$.

The statement about supersurfaces follows from the fact that the inclusion-induced homomorphisms preserve inhomogeneous relations. □

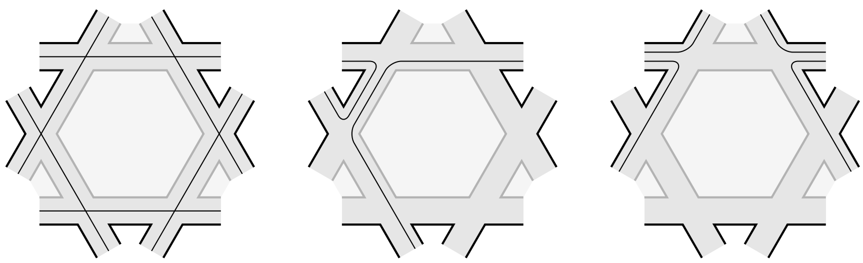


Figure 3.5: The curves $\gamma_1, \ldots, \gamma_6$, the curve $T_3T_4T_5T_6(\gamma_1)$, and the curve $\beta = T_6T_5T_4T_3^2T_4T_5T_6(\gamma_1)$, all in the surface $N^6 \cup \Delta^1_{\circlearrowleft} \cup \Delta^1_{\circlearrowleft}$

3.3.5 Four curve circuits

The case $n = 4$ becomes strange when too many discs are glued in, because some of the curves become isotopic. The relations from the proof of Proposition 3.11 do not reflect this, so we cover this case separately.

Proposition 3.12. *The following statements hold.*

(i) *The group* $\mathrm{CG}(N^4 \cup \Delta^1_{\circlearrowleft} \cup \Delta^1_{\circlearrowleft})$ *is geometrically isomorphic to* $\mathrm{Art}(A_3)$.

(ii) *The group* $\mathrm{CG}(N^4 \cup 2\Delta^1_{\circlearrowleft} \cup \Delta^1_{\circlearrowleft})$ *is geometrically isomorphic to* $\mathrm{Art}(A_2)$*,*

(iii) *The group* $\mathrm{CG}(N^4 \cup 2\Delta^1_{\circlearrowleft} \cup 2\Delta^1_{\circlearrowleft})$ *is isomorphic to* $\mathrm{SL}(2, \mathbb{Z})$*, and not geometrically isomorphic to an Artin group.*

Proof. Suppose the curves and discs are arranged as in Figure 3.6. We consider each surface S separately.

(i) Let $S = N^4 \cup \Delta^1_{\circlearrowleft} \cup \Delta^1_{\circlearrowleft}$ Because γ_2 and γ_4 are isotopic in S, we have that $\mathrm{CG}(S)$ is generated by T_1, T_2, T_3. Moreover, S is a regular neighbourhood of $\gamma_1, \gamma_2, \gamma_3$, so $\mathrm{CG}(S)$ is isomorphic to $\mathrm{Art}(A_3)$ [FM11, Section 9.4.1].

(ii) Let $S = N^4 \cup 2\Delta^1_{\circlearrowleft} \cup \Delta^1_{\circlearrowleft}$. In addition to γ_2 being isotopic to γ_4 from the previous case, γ_1 is also isotopic to γ_3. So $\mathrm{CG}(S)$ is generated by T_1, T_2. Moreover, S is a regular neighbourhood of $\gamma_1 \cup \gamma_2$. Hence, $\mathrm{CG}(S)$ is isomorphic to $\mathrm{Art}(A_2)$ [FM11, Theorem 9.2].

(iii) Let $S = N^4 \cup 2\Delta^1_{\circlearrowleft} \cup 2\Delta^1_{\circlearrowleft}$. Then S is just a torus with meridian γ_1 and γ_2. It is well-known that $\mathrm{MCG}(S)$ is generated by T_1, T_2, and that it is isomorphic to $\mathrm{SL}(2, \mathbb{Z})$ [FM11, Theorem 2.5]. Moreover, the inhomogeneous relation $(T_1T_2)^6 = 1$ (see [FM11, Section 3.5]) shows that $\mathrm{CG}(S)$ is not geometrically isomorphic to an Artin group.

If the indices of the curves are instead shifted by one, the same arguments hold. $\square$

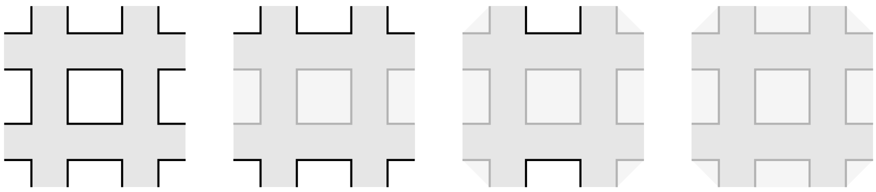


Figure 3.6: The surfaces N^4, $N^4 \cup \Delta^1_{\circlearrowleft} \cup \Delta^1_{\circlearrowright}$, $N^4 \cup 2\Delta^1_{\circlearrowleft} \cup \Delta^1_{\circlearrowright}$, $N^4 \cup 2\Delta^1_{\circlearrowleft} \cup 2\Delta^1_{\circlearrowright}$

3.3.6 A final surface

Up to orientation-reversing homeomorphism, we have now glued in every possible combination of discs, except one. For this final surface $S = N^n \cup \Delta^1 \cup \Delta^2$, the strategy of finding inhomogeneous relations failed, so the proof that $\mathrm{CG}(S)$ is not geometrically isomorphic to an Artin group turns out to be the most involved argument in this text. Toward a contradiction, we will assume that $\mathrm{CG}(S)$ is geometrically isomorphic to an Artin group $\mathrm{Art}(\Gamma)$. We then exclude all possibilities for the graph Γ. Lemma 3.13 below is a statement about Coxeter groups that helps achieve this for most graphs.

The *Coxeter group* $\mathrm{Cox}(\Gamma)$ is obtained from $\mathrm{Art}(\Gamma)$ by adding the relations $s^2 = 1$ for all generators s. If $\mathrm{Cox}(\Gamma)$ is finite, we will say that $\mathrm{Art}(\Gamma)$ is of *finite type*. Otherwise, $\mathrm{Art}(\Gamma)$ is of *infinite type*. The finite Coxeter groups were classified by Coxeter himself [Cox35, Theorem‡]. They are groups of the form $\mathrm{Cox}(\Gamma)$, where $\Gamma = A_n, B_n, D_n$ for arbitrary n, $\Gamma = E_n$ for $n = 6, 7, 8$, or a few more graphs that do not appear in this text.

Lemma 3.13 ([Max98, Theorem 0.4 and Table 3])**.** *Let $n \geq 3$ with $n \neq 4$. If there exists a surjective homomorphism $\mathrm{Cox}(D_n) \to \mathcal{C}(\Gamma)$, then Γ is either the one-vertex graph A_1, the graph A_{n-1}, or the graph D_n.*

Next, we need two results about the group-theoretic properties of the Artin groups $\mathrm{Art}(A_{n-1})$ and $\mathrm{Art}(D_n)$. The first result asserts that $\mathrm{Art}(A_{n-1})$ is "geometrically co-Hopfian".

Lemma 3.14. *Let $n \geq 2$. Every injective homomorphism from $\mathrm{Art}(A_{n-1})$ into itself such that the image of a standard generator is conjugate to a standard generator is an isomorphism.*

Proof. Think of $\mathrm{Art}(A_{n-1})$ as the group $\mathrm{CG}(S)$ generated by $\gamma_1, \ldots, \gamma_{n-1}$, where S is the surface $N^n \cup 2\Delta^1$, see Proposition 3.4. Then a homomorphism as in the assumption corresponds to an injective homomorphism $\varphi\colon \mathrm{CG}(S) \to \mathrm{CG}(S)$ mapping each T_i to a Dehn twist T_i' about a curve γ_i'. Because φ is injective, the γ_i' are pairwise non-isotopic. Moreover, the curves $\gamma_1', \ldots, \gamma_{n-1}'$ form a chain because consecutive curves satisfy the braid relation [FM11, Section 3.5.2]. Hence, by the change of coordinates principle [FM11, Section 1.3.3], there exists a homeomorphism f of S such that $\gamma_i' = f(\gamma_i)$. Thus, φ is given by conjugation by f, and hence is an isomorphism. □

Remark 3.15. Bell and Margalit in fact describe all the injective homomorphisms from the braid group $\mathrm{Art}(A_{n-1})$ to itself, even the non-geometric ones, for $n \geq 4$ [BM06, Main Theorem 1]. Their uniform description of these homomorphisms does not hold for $n = 3$ because $\mathrm{Art}(A_2)$ modulo its centre is not co-Hopfian (it is isomorphic to the free product $\mathbb{Z}/2 * \mathbb{Z}/3$).

Our final lemma in this Section asserts that finite type Artin groups are "Hopfian".

Lemma 3.16. *Every surjective homomorphism from a finite type Artin group onto itself is an isomorphism.*

Proof. Because finite type Artin groups are residually finite [BGJP18, Corollary 1.2] they are also Hopfian [LS01, Theorem IV.4.10]. □

Proposition 3.17. *For odd $n \geq 5$, the group $\mathrm{CG}(N^n \cup \Delta^1 \cup \Delta^2)$ is not geometrically isomorphic to an Artin group.*

Proof. Write $S = N^n \cup \Delta^1 \cup \Delta^2$. Suppose toward a contradiction that $\mathrm{CG}(S)$ is geometrically isomorphic to $\mathrm{Art}(\Gamma)$ for a graph Γ. Recall that by Proposition 3.7, the group $\mathrm{CG}(N^n \cup \Delta^1)$ is geometrically isomorphic to $\mathrm{Art}(D_n)$. Thus, the inclusion-induced homomorphism $\mathrm{CG}(N^n \cup \Delta^1) \to \mathrm{CG}(S)$ gives rise to a surjective homomorphism $\mathrm{Cox}(D_n) \to \mathcal{C}(\Gamma)$ (note that we use here that the isomorphism $\mathrm{Art}(D_n) \to \mathrm{CG}(N^n \cup \Delta^1)$ is geometric). From Lemma 3.13 it follows that Γ is either A_1, A_{n-1}, or D_n. We will now rule out each of those graphs.

We first argue that $\mathrm{CG}(S)$ contains a strict subgroup isomorphic to the braid group $\mathrm{Art}(A_{n-1})$. Consider the plastic view of N^n as on the left of Figure 3.7. Capping of the top and right boundary components with discs yields the surface S on the right. Now rotating about the x-axis by an angle of π yields an involution ι of S. Suppose the curves $\gamma_1, \ldots, \gamma_n$ are numbered such that γ_n is the right-most curve. Then ι preserves $\gamma_1, \ldots, \gamma_{n-1}$, but not γ_n. Thus, the strict subgroup of $\mathrm{CG}(S)$ generated by $T_1, \ldots, T_{n-1}$ is isomorphic to $\mathrm{Art}(A_{n-1})$. This excludes the case $\Gamma = A_1$ immediately, and an application of Lemma 3.14 excludes the case $\Gamma = A_{n-1}$.

Next, we show that the inclusion-induced homomorphism $\mathrm{CG}(N^n \cup \Delta) \to \mathrm{CG}(S)$ is not injective. To this end, consider the boundary curve β of the chain $\gamma_1, \dots, \gamma_{n-1}$ in $N^n \cup \Delta$. Then β intersects γ_n twice. But the image of β under the inclusion map $N^n \cup \Delta \to S$ does not intersect the image of γ_n. Hence, the commutator $T_\beta T_n T_\beta^{-1} T_n^{-1}$ is a non-trivial element of the kernel. By Lemma 3.16, Γ cannot be D_n, excluding all possibilities for Γ. $\square$

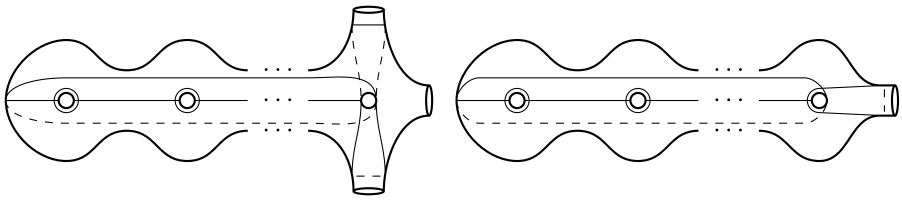


Figure 3.7: Another view of the surfaces N^n and $N^n \cup \Delta^1 \cup \Delta^2$ for odd n.

3.4 Revisiting the cycle theorem

This short final section is about gluing in punctured discs and annuli to the surfaces from Table 3.1 and collecting the relevant results in this text to prove Theorem 3.1.

Proposition 3.18. *Let S be a surface containing a circuit $\gamma_1, \dots, \gamma_n$. Suppose that $\mathrm{CG}(S)$ is geometrically isomorphic to $\mathrm{Art}(\widetilde{A}_{n-1})$. Let Δ_1 be a once-marked disc whose interior is disjoint from the interior of S, with $\partial\Delta_1 \subset \partial S$. Then the inclusion-induced homomorphism $\mathrm{CG}(S) \to \mathrm{CG}(S \cup \Delta_1)$ is an isomorphism. Similarly, if Z is an annulus whose interior is disjoint from the interior of S, with $\partial Z \subset \partial S$, then the inclusion-induced homomorphism $\mathrm{CG}(S) \to \mathrm{CG}(S \cup Z)$ is an isomorphism.*

Proof. Charney and Peifer show that for $n \geq 3$, the centre of $\mathrm{Art}(\widetilde{A}_{n-1})$ is trivial [CP03, Proposition 1.3]. It now follows from Lemma 1.2 that the inclusion-induced homomorphisms $\mathrm{CG}(S) \to \mathrm{CG}(S \cup \Delta_1)$ and $\mathrm{CG}(S) \to \mathrm{CG}(S \cup Z)$ are injective and hence isomorphisms. $\square$

Proof of Theorem 3.1. We prove the right-to-left implication, as the left-to-right implication follows from Labruère's work, specifically Lemma 3.6, after noticing that there is no way for a circuit in M^n to bound an embedded closed disc. Contrapositively, suppose that the circuit $\gamma_1, \dots, \gamma_n$ does not bound an embedded closed disc. In other words, the complement of a regular neighbourhood of $\gamma_1, \dots, \gamma_n$ in S is a union of surfaces that are not embedded discs. Let S' be the union of such a neighbourhood with all the non-embedded discs in its complement. Theorem 3.2 and Proposition 3.3 imply that $\mathrm{CG}(S')$ is geometrically isomorphic to $\mathrm{Art}(\widetilde{A}_{n-1})$. The complement of S' in S is a union of surfaces that are not discs, so by Proposition 3.18 and Lemma 1.2, it follows that also $\mathrm{CG}(S)$ is geometrically isomorphic to $\mathrm{Art}(\widetilde{A}_{n-1})$. But the cycle relation does not hold in this group. Indeed, as remarked above, the centre of $\mathrm{Art}(\widetilde{A}_{n-1})$ is trivial, whereas the quotient of $\mathrm{Art}(\widetilde{A}_{n-1})$ by the normal subgroup generated by the cycle relation is isomorphic to $\mathrm{Art}(D_n)$ (see Lemma 3.6 and Proposition 3.7), which has infinite cyclic centre [BS72, Satz 7.2]. $\square$

Part II

Braids

Chapter 4

Twist moves on links

In this chapter, we give a brief introduction to the twisting notion whose variations will be the topic of the remainder of the thesis. Said twisting notion is that of Fox's t_k-moves [Fox58], referred to as "twist moves" whenever we make the aesthetic choice of using plain letters such as in titles of sections.

For us, a *knot* is the oriented image of a smooth embedding of S^1 into S^3, and a *link* is a disjoint union of knots. Knots are links with one component. Two links L, L' are *isotopic* if there is an orientation-preserving diffeomorphism $S^3 \to S^3$ mapping L to L'. We consider links up to isotopy and do not distinguish between being equal and being isotopic. For a formal introduction into knot theory and its terminology, see established works like [Rol76] or [Lic97].

We say that a link L' is obtained from L by an *unoriented k-twist move* if L' can be obtained from L by replacing two parallel strands by the braid σ_1^k. We refer to the twist move as a t_k*-move* or a $\overline{t_k}$*-move* according to the orientations of the two strands, see Figure 4.1: if the two strands are oriented in the same direction, the twist move is a t_k-move, and otherwise it is a $\overline{t_k}$-move. In this thesis, we will focus almost exclusively on t_k-moves and refer the reader to [Prz06] and [ABFR22] for parallel results for $\overline{t_k}$-moves.

A well-studied, but not very well understood twist move is the t_2-move, which is the inverse of a $\overline{t_2}$-move, also known as *crossing changes*, see Figure 4.2. It is easy to understand that every knot can be transformed into the unknot using crossing changes, and the number of such moves required to turn a knot K into the unknot is called the *unknotting number* of K. More generally, the minimum number of crossing changes required to go from a knot K to a knot K' is called the *Gordian distance* of K to K'. Figuring out even just the unknotting number of a given knot is a notoriously hard problem, indicating that some creativity is required in order to make statements about t_2-moves, and t_k-moves in general.

This thesis makes no progress toward understanding t_2-moves, and instead

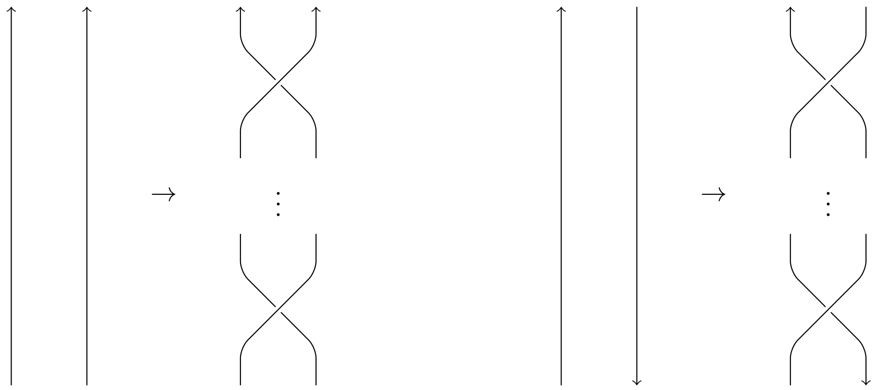


Figure 4.1: A t_k-move and a $\overline{t_k}$-move. The right hand sides contain k many half-twists.

Figure 4.2: A t_2-move and a $\overline{t_2}$-move, also known as crossing changes.

focuses on the question of which knots and links are related to each other via t_k-moves for fixed k. The tools we use partially apply to $\overline{t_k}$-moves too, which is why we would be able to make some assertions about both kinds of moves. However, we choose to focus on t_k-moves here because the strategies in the later sections of this chapter only apply to equally oriented strands.

In the first section of this chapter, we describe some obstructions, some of which are known to Przytycki [Prz06], to links being related via t_k-moves or via $\overline{t_k}$-moves. All of these obstructions come from the HOMFLY polynomial, a classical polynomial link invariant. These topics are all addressed in [ABFR22]. Subsequently, using a particular way of computing the HOMFLY polynomial, we are able to generalise a result about factor groups of the braid groups, which Coxeter attributes to G.A. Miller [Cox57].

4.1 Twist obstructions

For a link L, its *HOMFLY polynomial* is a Laurent polynomial $P_L \in \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ in two variables v, z, determined by the skein relation

$$v^{-1}P_{\nwarrow\!\!\!\!\nearrow} - vP_{\nearrow\!\!\!\!\nwarrow} = zP_{)(},$$

where the symbols $\nwarrow\!\!\!\!\nearrow, \nearrow\!\!\!\!\nwarrow,)($ stand for link diagrams that agree everywhere except for a single region that looks like the symbol itself, normalized so that the HOMFLY polynomial P_O of the unknot satisfies $P_O(v,z) = 1$. The fact that the HOMFLY polynomial is a well-defined link invariant is a non-trivial but well-established result [FYH+85]. The goal of this chapter is to relate certain specializations $z = z_0$ for some $z_0 \in \mathbb{C}$ of the HOMFLY polynomial to twist moves.

4.1.1 Two strand twist specializations

The result of this subsection is a version, and a slight improvement over a very similar statement by Przytycki [Prz06, Corollary 1.2] (although a different skein relation is used), as well as of the version in [ABFR22, Proposition 2]. For a natural number n, we write ζ_n for a primitive n-th root of unity. The obstructions we will use throughout the rest of this section are as follows.

Theorem 4.1. *Let L' be obtained from a link L via a positive t_k-move for some $k \geq 3$. Then,*

(i) *if $k = 2\ell + 1$ is odd, then $P_{L'}(v, z_0) = iv^k P_L(v, z_0)$ for $z_0 = \zeta_{4k} - \zeta_{4k}^{-1}$,*

(ii) *if $k = 4\ell$ is doubly even, then $P_{L'}(v, z_0) = -v^k P_L(v, z_0)$ for $z_0 = \zeta_{2k} - \zeta_{2k}^{-1}$,*

(iii) *if $k = 2\ell$ is even, then $P_{L'}(v, z_0) = v^k P_L(v, z_0)$ for $z_0 = \zeta_k - \zeta_k^{-1}$.*

Proof. Let L be a link, and consider a region in L as on the very left of Figure 4.1. Let $L_0, L_1, \ldots$ be the links such that L_k is obtained from $L_0 = L$ via a t_k-move associated to that region. Let P_k be the HOMFLY polynomial of L_k. The skein relation of the HOMFLY polynomial shows that

$$P_{k+1}(v, z) = vzP_k(v, z) + v^2 P_{k-1}(v, z),$$

which can be written in matrix form as

$$\begin{pmatrix} P_{k+1}(v, z) \\ P_k(v, z) \end{pmatrix} = \begin{pmatrix} vz & v^2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} P_k(v, z) \\ P_{k-1}(v, z) \end{pmatrix}.$$

Let M be the iteration matrix

$$M = \begin{pmatrix} vz & v^2 \\ 1 & 0 \end{pmatrix}$$

and compute $\det M = -v^2$ and $\operatorname{tr} M = vz$.

(i) Suppose k is odd. Substitute $z = \zeta_{4k} - \zeta_{4k}^{-1}$ and notice that under this substitution, M is diagonalisable (because $k \geq 3$) and has the same trace and determinant as the matrix

$$D = v \begin{pmatrix} \zeta_{4k} & 0 \\ 0 & -\zeta_{4k}^{-1} \end{pmatrix}$$

which satisfies $D^k = iv^k$ because k is odd. We conclude that $D^k = M^k$ because M and D are conjugate and D^k is central. Thus,

$$\begin{pmatrix} P_{k+1} \\ P_k \end{pmatrix} = M^k \begin{pmatrix} P_1 \\ P_0 \end{pmatrix} = D^k \begin{pmatrix} P_1 \\ P_0 \end{pmatrix},$$

and, in particular, $P_k(v, z) = iv^k P_0(v, z)$. The remaining cases are similar, so we omit the details that are analogous.

(ii) Suppose k is doubly even. Substitute $z = \zeta_{2k} - \zeta_{2k}^{-1}$ and diagonalise M to obtain

$$D = iv \begin{pmatrix} \zeta_{2k} & 0 \\ 0 & \zeta_{2k}^{-1} \end{pmatrix},$$

which satisfies $D^k = -v^k$ because k is doubly even.

(iii) Suppose k is even. Substitute $z = \zeta_k - \zeta_k^{-1}$ and diagonalise M to obtain

$$D = v \begin{pmatrix} \zeta_k & 0 \\ 0 & -\zeta_k^{-1} \end{pmatrix},$$

which satisfies $D^k = v^k$ because k is even. □

Upon determination of the relevant specializations of the HOMFLY polynomial, this result allows us to compute the set of k such that two strand torus knots or non-trivial twist knots can be unknotted using t_k-moves.

Example 4.2 ([ABFR22, Proposition 4])**.** For all $n \geq 1$, the set of k such that $T(2, 2n + 1)$ can be unknotted using positive or negative t_k-moves coincides with the set of divisors of n and $n + 1$. In contrast, there is no $k \geq 2$ such that the twist knot K_n can be unknotted using positive or negative t_k-moves. See Figure 4.3 for diagrams of the mentioned knots.

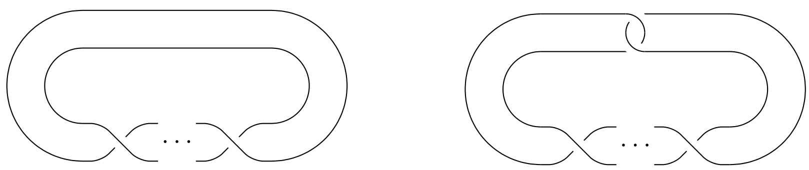


Figure 4.3: The two strand torus knot $T(2, 2k + 1)$ and the twist knot K_n. The twist regions contain $2k + 1$ and $2n$ half-twists, respectively.

We draw particular attention to the invariant specializations for t_3-moves, t_4-moves, and t_6-moves, as we will use these particular specializations to prove Theorem 4.6 below, see Table 4.1.

k	z_0	factor
3	$\sqrt{-1}$	iv^3
4	$\sqrt{-2}$	$-v^4$
6	$\sqrt{-3}$	v^6

Table 4.1: Invariant specializations for t_k-moves for some small k.

4.1.2 Performance on low crossing numbers

Theorem 4.1 immediately provides an obstruction for two links to belong to the same t_k-equivalence class, just by looking at the coefficients of the invariant specializations of their HOMFLY polynomials. It turns out that this obstruction perform quite well on low-crossing examples, see Table 4.2. In addition to ignoring powers of v^k when comparing the polynomials, it is important to consider HOMFLY polynomials equivalent if they are obtained from each other by replacing v by v^{-1}. This is because if $-K$ is the mirror image of K, then $P_{-K}(v, z) = P_K(v^{-1}, z)$.

c	t_4-classes	t_6-classes
8	7	23
9	$[15, 17]$	$[53, 55]$
10	$[42, 47]$	$[134, 145]$
11	$[54, 98]$	$[392, 470]$

Table 4.2: Effectiveness of obstructing t_4-equivalence and t_6-equivalence using the specializations $z_0 = \sqrt{-2}$ and $z_0 = \sqrt{-3}$, respectively. Each row gives bounds on the number of equivalence classes containg a knot of crossing number at most c. The lower bound comes from the specialization, and the upper bound comes from brute forcing equivalence among knots of at most 12 crossings. The upper bound given is quite ad-hoc and gets much worse as c increases.

The explicitly determined t_4-equivalence classes of knots up to 8 crossings are

(1) $0_1, 3_1, 5_1, 6_2, 7_1, 7_3, 7_5, 8_4, 8_5, 8_7, 8_9, 8_{10}, 8_{16}, 8_{20}$,

(2) $4_1, 5_2, 6_3, 8_2, 8_6, 8_{17}, 8_{18}, 8_{19}, 8_{21}$,

(3) $6_1, 7_2, 7_6, 8_8, 8_{11}, 8_{15}$,

(4) $7_4, 7_7, 8_{13}, 8_{14}$,

and the knots 8_1, 8_3, 8_{12} that do not appear in the lists are each in their own t_4-equivalence class. Similarly, the explicitly determined t_6-equivalence classes are

(1) $0_1, 5_1, 7_1, 8_2$,

(2) $4_1, 7_3$,

(3) $5_2, 6_2, 8_7$,

(4) $6_3, 7_5, 8_9$,

(5) $7_2, 8_4$,

(6) $7_6, 8_6$,

(7) $8_5, 8_{19}, 8_{20}$,

(8) $8_{10}, 8_{21}$,

and the knots $3_1, 6_1, 7_4, 7_7, 8_1, 8_3, 8_8, 8_{11}, 8_{12}, 8_{13}, 8_{14}, 8_{15}, 8_{16}, 8_{17}, 8_{18}$ that do not appear in the lists are each in their own t_6-equivalence class.

4.1.3 Crossing number and braid index

For a knot K we write $c(K)$ for its *crossing number*, defined to be the minimal number of crossings m such that K admits a diagram with m crossings. Similarly, the *braid index* of a knot K is the minimal number of strands m such that K is the closure of a braid in B_m. The HOMFLY polynomial can be used to bound the crossing number and braid index of a knot or link, as exhibited by the crossing number and braid index inequalities

$$\deg_z P_K \leq c(K) - 1,$$
$$\operatorname{span}_v P_K \leq 2b(K) - 2$$

by Franks-Williams [FW87, Corollary 1.10] and Morton [Mor86, Corollary 1], where $\deg_z$ is the highest exponent m of a monomial z^m appearing in P_K, and span_v is the difference between the largest and the smallest exponent of v. These inequalities remain true after specialization of one variable, which allows us to deduce certain things from Theorem 4.1.

Note that by the skein relation of the HOMFLY polynomial, we have that the specialization $P_K(v, v^{-1} - v)$ is the constant function 1. This implies that $P_K(v, z)$ cannot be of the form $v^m f(z)$. Indeed, if $P_K(v, z) = v^m f(z)$, then

$$v^m f(v^{-1} - v) = 1$$

for all v, implying $f(v^{-1} - v) = v^{-m}$ for all v. But $f(v^{-1} - v)$ is a polynomial whose v-span is $2 \deg_z f$, so f must be constant. But this shows that $m = 0$, so $P_K(v, z)$ is trivial. We can use this to prove the following pleasant result.

Theorem 4.3 ([ABFR22, Theorem 2])**.** *Let K be a knot with non-trivial HOMFLY polynomial. Then the set of $k \geq 3$ such that K can be transformed to the (one component) unknot using positive or negative t_k-moves has at most $c(K)-1$ elements.*

Proof. Let n, m be integers with $n < m$ and $f, g \in \mathbb{Z}[z^{\pm 1}]$ be polynomials such that the terms of lowest and highest v-degree in $P_K(v,z)$ are $v^n f(z)$ and $v^m g(z)$, respectively. Suppose that K can be unknotted using t_k-moves for some $k \geq 3$. Because the unknot O has trivial HOMFLY polynomial $P_O(v,z) = 1$, Theorem 4.1 implies that $P_K(v, z_0)$ is a monomial in v, for some z_0 depending on k. Thus, either $f(z_0)$ or $g(z_0)$ must be zero.

Moreover, note that the z_0 from Theorem 4.1 determines k. Because of the degree bound

$$\deg_z fg = \deg_z f + \deg_z g \leq 2 \deg_z P_K,$$

and the degree of the minimal polynomial of each z_0 is at least two, we obtain that there are at most $\deg_z P_K \leq c(K) - 1$ many k such that K can be transformed to the unknot using t_k-moves. □

Using the other inequality by Franks-Williams and Morton, we are able to make a statement about the braid index as well.

Theorem 4.4 ([ABFR22, Proposition 3])**.** *Let K be a knot and $k \geq 3$. Then every knot that is related to K by a finite sequence of positive or negative t_k-moves satisfies*

$$\operatorname{span}_v P_K(\cdot, z_0) \leq 2b(K') - 2,$$

where $z_0 = \zeta_{4k} - \zeta_{4k}^{-1}$ if k is odd, $z_0 = \zeta_{2k} - \zeta_{2k}^{-1}$ if k is doubly even, and $z_0 = \zeta_k - \zeta_k^{-1}$ if k is even.

Proof. Note that $\operatorname{span}_v P_K(\cdot, z_0)$ is invariant under multiplication by powers of v. Thus, we have

$$\operatorname{span}_v P_K(\cdot, z_0) = \operatorname{span}_v P_{K'}(\cdot, z_0) \leq \operatorname{span}_v P_{K'} \leq 2b(K') - 2. \qquad \square$$

Corollary 4.5 ([ABFR22, Proposition 1])**.** *Let K be a two-bridge or an alternating fibered knot. For all but finitely many $k \in \mathbb{N}$, all knots K' that are related to K by a finite sequence of positive or negative t_k-moves satisfy $b(K') \geq b(K)$.*

Proof. For a two-bridge or fibered alternating knot K, the equality

$$\operatorname{span}_v P_K = 2b(K) + 2$$

holds [Mur91, Theorems A and B]. Moreover, for all but finitely many k, the equality

$$\operatorname{span}_v P_K(v, z_0) = \operatorname{span}_v P_K$$

holds, where z_0 depends on k as in Theorem 4.1. □

4.2 Infinite twist orbits

Let B_n be the braid group on n strands. Coxeter [Cox57] writes that the quotient of B_n modulo the normal subgroup generated by the k-th power σ_1^k of a standard generator σ_1 is finite if and only if

$$\frac{1}{k} + \frac{1}{n} > \frac{1}{2},$$

citing results by G.A. Miller. In particular, if $1/k + 1/n > 1/2$, then the set $\mathcal{L}_n$ of closures of n-braids only contains representatives of finitely many distinct t_k-equivalence classes. In this section, we aim to provide a converse of this observation for the particular case $1/k + 1/n = 1/2$.

Theorem 4.6. *For $n \in \mathbb{N}$, let $\mathcal{L}_n$ be the set of closures of n-braids.*

(i) *$\mathcal{L}_3$ contains representatives of infinitely many t_6-equivalence classes.*

(ii) *$\mathcal{L}_4$ contains representatives of infinitely many t_4-equivalence classes.*

(iii) *$\mathcal{L}_6$ contains representatives of infinitely many t_3-equivalence classes.*

Our approach for proving Theorem 4.6 is to compute the t_k-invariant specializations in Theorem 4.1 for certain torus knots, and noticing that the coefficients of the resulting polynomials in $\mathbb{C}[v^{\pm 1}]$ are unbounded. This computation will immediately imply the main theorem of this section. The explicit determination of the specializations will be carried out in the Hecke algebra H_n over $\mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$. As H_n has dimension $n!$, it is only feasible to do the computation by hand for $n = 3$. We will not do this and instead have a computer verify our formula for all three cases $n = 3, 4, 6$.

4.2.1 The Hecke algebra

Recall that the Coxeter group $\operatorname{Cox}(A_{n-1})$ is isomorphic to the symmetric group S_n on n elements, where the i-th generator is mapped to the transposition s_i exchanging i and $i+1$. Since the braid group B_n is isomorphic to the Artin group $\operatorname{Art}(A_{n-1})$, we obtain that B_n modulo the normal subgroup generated by the squares of generators is isomorphic to S_n. This observation allows us to construct an $n!$-dimensional algebra over $\mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ called the *Hecke algebra* H_n, together with a representation $\omega\colon B_n \to H_n$. As a module, H_n is the quotient of the free module generated by S_n over $\mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$. We turn H_n into an algebra by extending the relations

$$s_i \cdot s_i = v^2 + vzs_i$$

linearly. We obtain a well-defined homomorphism of monoids $\omega\colon B_n \to H_n$ mapping σ_i to s_i.

For a link L, let $P_L \in \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ be its HOMFLY polynomial. For a permutation $g \in S_n$, let $p(g) \in \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ be the HOMFLY polynomial of $\widehat{\beta_g}$, where β_g is a positive braid of minimal length such that $\omega(\beta_g) = g$. Then the homomorphism of modules $p\colon H_n \to \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ mapping an element π to the polynomial $p(\pi)$ describes the HOMFLY polynomial: for $\beta \in B_n$, we have

$$P_{\widehat{\beta}} = (p \circ \omega)(\beta).$$

Example 4.7. The Hecke algebra H_3 is a 6-dimensional algebra over the polynomial ring $\mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$. As a module, it admits a basis consisting of the elements $e, (1\ 2), (2\ 3), (1\ 3), (1\ 2\ 3), (3\ 2\ 1)$. Writing $a = \sigma_1, b = \sigma_2$ for the generators of B_3, we have $\beta_e = \varepsilon$, $\beta_{(1\ 2)} = a$, $\beta_{(2\ 3)} = b$, $\beta_{(1\ 3)} = aba$, $\beta_{(1\ 2\ 3)} = ab$, $\beta_{(3\ 2\ 1)} = ba$. Here, e is the trivial permutation in S_3 and ε is the empty braid in B_3. An example multiplication of basis vectors is

$$\begin{aligned}(1\ 3) \cdot (2\ 3) &= s_2 s_1 s_2 \cdot s_2 \\ &= s_2 s_1 (v^2 + vzs_2) \\ &= v^2 (3\ 2\ 1) + vz(1\ 3),\end{aligned}$$

and the homomorphism $p\colon H_3 \to \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ is determined by the images of the basis vectors and satisfies

$$\begin{aligned}p(e) &= v^2 z^{-2} - 2z^{-2} + v^{-2} z^{-2}, \\ p((1\ 2)) = p((2\ 3)) &= v^{-1} z^{-1} - vz^{-1}, \\ p((1\ 2\ 3)) = p((3\ 2\ 1)) &= 1.\end{aligned}$$

4.2.2 Coxeter edge cases

This subsection is devoted to showing that the sets $\mathcal{L}_3, \mathcal{L}_4, \mathcal{L}_6$ contain families of specializations $P(K, z_0)$ for $z_0 = \sqrt{-3}, \sqrt{-2}, \sqrt{-1}$, respectively, with unbounded coefficients. To this end, we determine some of these specializations explicitly.

Proposition 4.8. *The following formulas hold:*

(i) $P_{T(3,3\ell+1)}(v, \sqrt{-3}) = (\ell v^4 + \ell v^2 + (\ell + 1))v^{6\ell}$,

(ii) $P_{T(4,8\ell+1)}(v, \sqrt{-2}) = (2\ell v^6 + 2\ell v^4 + 2\ell v^2 + (2\ell + 1))v^{24\ell}$,

(iii) $P_{T(6,36\ell+1)}(v,\sqrt{-1}) = (6\ell v^{10} + 6\ell v^8 + 6\ell v^4 + 6\ell v^2 + (6\ell+1))v^{180\ell}$.

Proof. For $n = 3,4,6$, let $\delta_n = \sigma_1 \cdots \sigma_{n-1} \in B_n$. Then, $T(n,m)$ is the closure of the braid δ_n^m. For $z_0 \in \mathbb{C}$, let $H_n(z_0)$ be the Hecke algebra H_n with the second variable specialized to $z = z_0$. With a naive implementation of Hecke algebras, one may verify that

(i) $\omega(\delta_3^6) - \omega(\delta_3^3)v^6 = \omega(\delta_3^3)v^6 - v^{12}$ in $H_3(\sqrt{-3})$,

(ii) $\omega(\delta_4^{16}) - \omega(\delta_4^8)v^{24} = \omega(\delta_4^8)v^{24} - v^{48}$ in $H_4(\sqrt{-2})$,

(iii) $\omega(\delta_6^{72}) - \omega(\delta_6^{36})v^{180} = \omega(\delta_6^{36})v^{180} - v^{360}$ in $H_6(\sqrt{-1})$.

In principle, the first formula can be verified by hand. As the last case involves matrix multiplication of matrices with $6! = 720$ rows and columns, this is no longer feasible. A standard laptop, however, takes around half an hour to do the computation.

These results inductively give rise to the formulas

(i) $\omega(\delta_3^{3\ell+1}) - \omega(\delta_3)v^{6\ell} = \ell(\omega(\delta_3^3)v^{6\ell} - v^{6(\ell+1)})$ in $H_3(\sqrt{-3})$,

(ii) $\omega(\delta_4^{8\ell+1}) - \omega(\delta_4)v^{24\ell} = \ell(\omega(\delta_4^8)v^{24\ell} - v^{24(\ell+1)})$ in $H_4(\sqrt{-2})$,

(iii) $\omega(\delta_6^{36\ell+1}) - \omega(\delta_6)v^{180\ell} = \ell(\omega(\delta_6^{36})v^{180\ell} - v^{180(\ell+1)})$ in $H_6(\sqrt{-1})$.

Using that $P_{\widehat{\beta}}(v,z) = (p \circ \omega)(\beta)$, the same standard laptop now verifies the proposition in just a few seconds. All the computations were done using the author's software [Ryf23b]. □

Proof of Theorem 4.6. Recall from Theorem 4.1 that if K' is related to K via a t_k-move for $k = 6,4,3$, then $P_{K'}(v,z_0) = v^m P_K(v,z_0)$ for $z_0 = \sqrt{-3}, \sqrt{-2}, \sqrt{-1}$, respectively, which does not change the size of the coefficients of the specialised HOMFLY polynomial. By Proposition 4.8, the sets $\mathcal{L}_3, \mathcal{L}_4, \mathcal{L}_6$ contain knots with specialised HOMFLY polynomials by $z_0 = \sqrt{-3}, \sqrt{-2}, \sqrt{-1}$, respectively, of unbounded coefficients. □

4.3 Generalized twist moves

In this experimental final section of this chapter, we investigate some computational aspects of the Hecke algebra in more detail. Our goal here is to describe a systematic approach for finding specializations of the z-variable of the HOMFLY polynomial that are invariant, similar to those specializations in Theorem 4.1, under the application of a generalized twist move. We associate one of these twist moves to each braid $\beta \in B_n$: We say that a link L' is obtained from L by a *β-twist* if L can be transformed into L' by finding n parallel strands that are oriented in the same direction, and replacing them by β. See Figure 4.4

for a schematic representation of a β-twist. The twist moves from the previous sections are a special case: A t_k-move is a σ_1^k-twist.

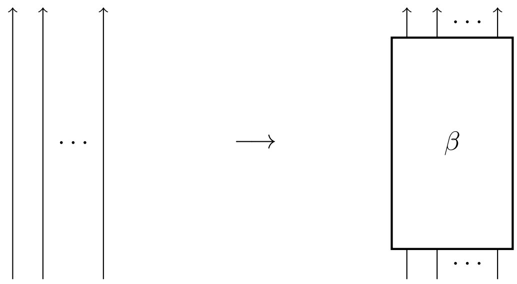


Figure 4.4: A β-twist.

4.3.1 Elimination of the first variable

Let $\ell\colon S_n \to \mathbb{N}$ be the length function of the Coxeter group S_n. Then, any element $g \in S_n$ and any generator s_i satisfy either $\ell(gs_i) = \ell(g) + 1$, or g can be written as a word of length $\ell(g)$ ending in s_i. This fact has the following consequence, showing that the variable v can be neglected when studying the images of braids in H_n.

Proposition 4.9. *Let $\eta = \omega(\beta) \in H_n$ be in the image of ω. Then the coefficients of η are all of the form $\pi_g(\eta) = \eta_g(z)v^{d(\beta)-\ell(g)}$ for some Laurent polynomial $\eta_g(z) \in \mathbb{Z}[z^{\pm 1}]$. Here, $d\colon B_n \to \mathbb{Z}$ is the homomorphism sending σ_i to 1 for all i, and $g \in S_n$ is a permutation.*

Proof. This is based on v-homogeneity of the skein relation. We prove the claim by induction on the length of β. The claim is certainly true for the empty braid word β: in this case, $\pi_e(\eta) = 1$ (where $e \in S_n$ is the identity), and $\pi_g(\eta) = 0$ for all $g \neq e$.

Now suppose the claim holds for β. Let σ_i be a standard generator. Then

$$\omega(\beta\sigma_i) = \sum_{g \in S_n} \pi_g(\beta\sigma_i) \cdot g = \sum_{g \in S_n} \pi_g(\beta) \cdot (g \cdot s_i) = \sum_{g \in S_n} \eta_g(z)v^{d(\beta)-\ell(g)} \cdot (g \cdot s_i).$$

We examine each summand individually. If $\ell(gs_i) = \ell(g) + 1$, then we have $d(\beta) - \ell(g) = d(\beta\sigma_i) - \ell(gs_i)$, so the summand becomes

$$\eta_g(z)v^{d(\beta\sigma_i)-\ell(g)} \cdot g \cdot s_i = \eta_g(z)v^{d(\beta\sigma_i)-\ell(gs_i)} \cdot gs_i,$$

which is of the desired form. On the other hand, if $\ell(gs_i) = \ell(g) - 1$, then we have $d(\beta) - \ell(g) + 2 = d(\beta\sigma_i) - \ell(gs_i)$, so

$$\begin{aligned}\eta_g(z)v^{d(\beta)-\ell(g)} \cdot (g \cdot s_i) &= \eta_g(z)v^{d(\beta)-\ell(g)} \cdot (v^2 gs_i + vzg)\\ &= (\eta_g(z)v^{d(\beta\sigma_i)-\ell(gs_i)} \cdot gs_i) + (\eta_g(z)zv^{d(\beta\sigma_i)-\ell(g)} \cdot g),\end{aligned}$$

which is also of the desired form.

Finally, suppose that the claim holds for β, and let σ_i^{-1} be the inverse of a standard generator. From $s_i \cdot s_i = v^2 + vzs_i$ we obtain $s_i^{-1} = v^{-2}s_i - v^{-1}z$. Again, compute

$$\begin{aligned}\omega(\beta\sigma_i^{-1}) &= \sum_{g \in S_n} \pi_g(\beta\sigma_i^{-1}) \cdot g\\ &= \sum_{g \in S_n} \pi_g(\beta) \cdot (g \cdot s_i^{-1})\\ &= \sum_{g \in S_n} \eta_g(z)v^{d(\beta)-\ell(g)} \cdot (v^{-2}(g \cdot s_i) - v^{-1}zg).\end{aligned}$$

Again, we examine each summand individually. If $\ell(gs_i) = \ell(g) + 1$, then we have $d(\beta) - \ell(g) - 2 = d(\beta\sigma_i^{-1}) - \ell(gs_i)$, and $d(\beta) - \ell(g) - 1 = d(\beta\sigma_i^{-1}) - \ell(g)$, so

$$\pi_g(\beta) \cdot (g \cdot s_i^{-1}) = (\eta_g(z)v^{d(\beta\sigma_i^{-1})-\ell(gs_i)} \cdot gs_i) + (-\eta_g(z)zv^{d(\beta\sigma_i^{-1})-d(g)} \cdot g),$$

which is of the desired form. On the other hand, if $\ell(gs_i) = \ell(g) - 1$, then $d(\beta) - \ell(g) = d(\beta\sigma_i^{-1}) - \ell(gs_i)$, so

$$\begin{aligned}\pi_g(\beta) \cdot (g \cdot s_i^{-1}) &= \eta_g(z)v^{d(\beta)-\ell(g)} \cdot (v^{-2}(v^2 gs_i + vzg) - v^{-1}zg)\\ &= \eta_g(z)v^{d(\beta\sigma_i^{-1})-\ell(gs_i)} \cdot gs_i,\end{aligned}$$

which is of the desired form. □

4.3.2 Braid-twist specializations

Due to Proposition 4.9, without losing much information, we may consider the *one-variable Hecke algebra* H'_n over the one-variable polynomial ring $\mathbb{Z}[z^{\pm 1}]$ obtained from H_n by substituting $v = 1$. Let $\eta \in H'_n$ be any element. We frequently consider the *standard decomposition*

$$\eta = \psi_\eta + \varphi_\eta \eta',$$

of η, where $e \in S_n$ is the identity, $\pi_e(\eta')$ is zero, $\psi_\eta(z) = \pi_e(\eta)$, and φ_η is a greatest common divisor of the coefficients of $\eta - \psi_\eta$ if it exists, and zero otherwise. In particular, $\varphi_\eta(0) = 0$ if and only if η is a multiple of 1 (in which case φ_η is the zero polynomial). Our next proposition is about substituting complex numbers for z, giving rise to a procedure for finding β-twist invariant specializations as in Theorem 4.1.

We write $H'_n(z_0)$ for the algebra over $\mathbb{C}$ obtained from H'_n by specializing the second variable to some non-zero $z = z_0$. Note that for each z_0, the one-variable Hecke algebra H'_n acts multiplicatively on $H'_n(z_0)$ in an obvious way. We are now ready to describe our procedure for finding β-twist invariant specializations of the HOMFLY polynomial.

Theorem 4.10. *Let $\beta_0 \in B_n$ be any braid, and let $\eta = \omega(\beta_0)$. Consider the decomposition $\eta = \psi_\eta + \varphi_\eta \eta' \in H_n$ and let $z_0 \in \mathbb{C}\backslash\{0\}$ be a zero of φ_η. Whenever a link L' is obtained from another link L by a β_0-twist, then*

$$P_{L'}(v, z_0) = \psi_\eta(z_0) v^{d(\beta_0)} P_L(v, z_0).$$

Proof. Let L be any link, and consider a diagram D of L with a region R of n parallel and equally oriented strands as on the left of Figure 4.4. The pair (D, R) leads to a non-standard way of obtaining a link $L_{D,R}(\beta)$ from a braid β by replacing the parallel strands in R with β. As an example, if D is a diagram of the unlink of n equally oriented nested components, and R is a region in D with n parallel strands, then $L_{D,R}(\beta)$ is the ordinary closure $\widehat{\beta}$, see Figure 4.5.

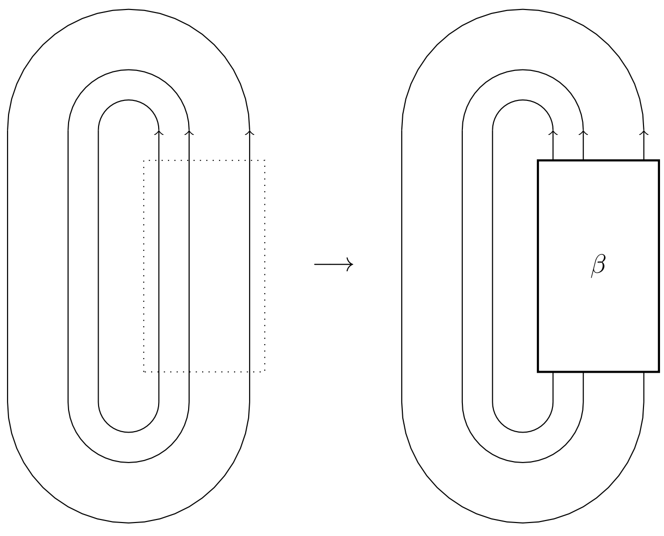


Figure 4.5: Ordinary braid closure can be expressed as (D, R)-closure for a certain diagram D and region R: The left hand side is a picture of D, and the region R is indicated as a dotted rectangle. The right hand side is the ordinary closure $L_{D,R}(\beta) = \widehat{\beta}$.

As another, more relevant example, $L_{D,R}(\beta_0)$ is the link L' obtained from

L by a β_0-twist in the region R. Let $p_{D,R}\colon H_n \to \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ be the homomorphism of modules mapping the basis vectors e_g to the HOMFLY polynomial of $L_{D,R}(\beta_g)$, where β_g is the positive braid of minimal length with $\omega(\beta_g) = e_g$. Note that $p_{D,R}$ describes the HOMFLY polynomial of (D, R)-closures and, in particular, we have $p_{D,R}(\beta_0) = P_{L'}$. Let $p'_{D,R}\colon H'_n \to \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ have the same definition as $p_{D,R}$. By Proposition 4.9, we have

$$p_{D,R}(\beta_0) = v^{d(\beta_0)} p'_{D,R}(\beta_0)$$

for all $\beta \in B_n$. Specializing to $z = z_0$, we obtain that

$$\begin{aligned}
P_{L'}(v, z_0) &= p_{D,R}(\eta)(v, z_0) \\
&= v^{d(\beta_0)} p'_{D,R}(\eta)(v, z_0) \\
&= \psi_\eta(z_0) v^{d(\beta_0)} p'_{D,R}(1)(v, z_0) \\
&= \psi_\eta(z_0) v^{d(\beta_0)} p_{D,R}(1)(v, z_0) \\
&= \psi_\eta(z_0) v^{d(\beta_0)} P_L(v, z_0),
\end{aligned}$$

where ε is the trivial braid in B_n. □

4.3.3 Examples of specializations

In this section, we apply Theorem 4.10 to some explicit braids. A specialization $z = z_0$ satisfying the conclusion of said result is called a *β-twist invariant specialization.* We first show how to recover Theorem 4.1 by studying the Hecke algebra H_2.

Proposition 4.11. *Let $F_k \in \mathbb{Z}[z]$ be the k-th Fibonacci polynomial satisfying the recurrence relations $F_0 = 0$, $F_1 = 1$, and $F_{k+2} = zF_{k+1} + F_k$. If $F_k(z_0) = 0$ for a non-zero complex number z_0, then $z = z_0$ is a t_k-invariant specialization.*

Proof. Write a for the generator of B_2, and let $\alpha = \omega(a)$. In the basis $\{e, (1\ 2)\}$ of H'_2, The element α can be represented as the vector $(0, 1)$, and we have $(f, g) \cdot \alpha = (g, zg + f)$ in H'_2. We thus get the recurrence relation $\varphi_1 = 0, \varphi_\alpha = 1$ and

$$\varphi_{\alpha^{k+2}}(z) = z\varphi_{\alpha^{k+1}}(z) + \varphi_{\alpha^k}(z)$$

in the notation of Theorem 4.10. This is precisely the recurrence relation of the Fibonacci polynomials. □

The zeroes of the Fibonacci polynomials have been explicitly determined to be of the form

$$z_0 = \zeta_{2k}^{j} - \zeta_{2k}^{-j}$$

for $j = 0, \ldots, k/2 - 1$ if k is even, and

$$z_0 = \zeta_{4k}^{2j+1} - \zeta_{4k}^{-(2j+1)}$$

for $j = 0, \ldots, (k-1)/2 - 1$ if k is odd [HB73]. The specializations in Theorem 4.1 are among these zeroes.

We end this section with a family of examples of braid twists on three strands.

Proposition 4.12. *Consider the sequence $b_{-1}, b_0, b_1, \ldots$ of polynomials in $\mathbb{Z}[z]$ satisfying $b_{-1} = 1$, $b_0 = 0, b_1 = 1, b_2 = 1$ and the recurrence relation*

$$b_{k+2} = (z^2 + 1)b_k - b_{k-2}.$$

If z_0 is a zero of b_k, then z_0 is a $(\sigma_1\sigma_2^{-1})^k$-twist invariant specialization. More precisely, if z_0 is a zero of b_k and L' is obtained from L by a $(\sigma_1\sigma_2^{-1})$-twist, then

$$P_{L'}(v, z_0) = P_L(v, z_0).$$

Proof. Let $\beta = \sigma_1\sigma_2^{-1} \in B_3$. In the basis $\{e, (2\ 3), (1\ 2), (1\ 2\ 3), (3\ 2\ 1), (1\ 3)\}$ of H_3', we may write $\omega(\beta) = (0, 0, -z, 0, 1, 0)$. We first prove by induction that for all k we have

$$b_k b_{k+3} + (-1)^k = b_{k+1}b_{k+2}.$$

The claim is certainly true for $k = -1$ and $k = 0$, and if it holds for k, then

$$\begin{aligned} b_{k+2}b_{k+5} + (-1)^{k+2} &= b_{k+2}((1 + z^2)b_{k+3} - b_{k+1}) + (-1)^k \\ &= (1 + z^2)b_{k+2}b_{k+3} - (b_{k+1}b_{k+2} + (-1)^{k+1}) \\ &= (1 + z^2)b_{k+2}b_{k+3} - b_k b_{k+3} \\ &= b_{k+3}b_{k+4}, \end{aligned}$$

By another induction, one may show that

$$\omega(\beta^k) = (-1)^k \begin{pmatrix} b_k(b_{k+1} - b_{k-1}) + (-1)^k \\ -zb_{k-1}b_k \\ zb_k b_{k+1} \\ b_{k-1}b_k \\ -b_k b_{k+1} \\ 0 \end{pmatrix}.$$

Indeed, the claim is true for $k = 0$, and if it is true for k then a computer verifies quickly that

$$\omega(\beta^{k+1}) = (-1)^{k+1} \begin{pmatrix} z^2 b_k b_{k+1} - b_{k-1} b_k \\ -z b_k b_{k+1} \\ z(z^2 b_k b_{k+1} - b_{k-1} b_k + b_k b_{k+1} + (-1)^k) \\ b_k b_{k+1} \\ -(z^2 b_k b_{k+1} - b_{k-1} b_k + b_k b_{k+1} + (-1)^k) \\ 0 \end{pmatrix}.$$

To cover the first component, compute

$$\begin{aligned} z^2 b_k b_{k+1} - b_{k-1} b_k &= b_k((1+z^2) b_{k+1} - b_{k-1}) - b_k b_{k+1} \\ &= b_k b_{k+3} - b_k b_{k+1} \\ &= b_{k+1} b_{k+2} + (-1)^{k+1} - b_k b_{k+1} \\ &= b_{k+1}(b_{k+2} - b_k) + (-1)^{k+1}. \end{aligned}$$

To cover the third and fifth components, compute

$$\begin{aligned} z^2 b_k b_{k+1} - b_{k-1} b_k + b_k b_{k+1} + (-1)^k &= b_k((1+z^2) b_k - b_{k-1}) + (-1)^k \\ &= b_k b_{k+3} + (-1)^k \\ &= b_{k+1} b_{k+2}. \end{aligned}$$

This concludes the induction step. Because b_k is a factor of every coefficient of $\omega(\beta) - \pi_e(\omega(\beta))$, Theorem 4.10 proves the claim. □

Because $b_3(z) = z^2$, this only leads to $(\sigma_1\sigma_2^{-1})^k$-twist invariant specializations for $k \geq 4$. A list of the polynomials b_k for small k is given in Table 4.3. As a by-product of expressing $\omega(\beta^k)$ explicitly in coordinates, we also obtain a formula for the HOMFLY polynomial of the closure L of $(\sigma_1\sigma_2^{-1})^k$, namely

$$P_L(v,z) = 1 + (-1)^k(b_{k-1}b_k - b_k b_{k+1} + (-1)^{k+1})(1 - v^2 z^{-2} + 2z^{-2} - v^{-2} z^{-2}).$$

k	b_k
4	z^2+1
5	z^4+z^2-1
6	$(z^2+2)z^2$
7	$z^6+2z^4-z^2-1$
8	$(z^4+2z^2-1)(z^2+1)$
9	$(z^6+3z^4-3)z^2$
10	$(z^4+3z^2+1)(z^4+z^2-1)$
11	$z^{10}+4z^8+2z^6-5z^4-2z^2+1$
12	$(z^4+2z^2-2)(z^2+2)(z^2+1)z^2$
13	$z^{12}+5z^{10}+5z^8-6z^6-7z^4+2z^2+1$
14	$(z^6+4z^4+3z^2-1)(z^6+2z^4-z^2-1)$
15	$(z^8+5z^6+5z^4-5z^2-5)(z^4+z^2-1)z^2$
16	$(z^8+4z^6+2z^4-4z^2-1)(z^4+2z^2-1)(z^2+1)$
17	$z^{16}+7z^{14}+14z^{12}-z^{10}-25z^8-9z^6+12z^4+3z^2-1$
18	$(z^6+3z^4-1)(z^6+3z^4-3)(z^2+2)z^2$

Table 4.3: Some factorized polynomials b_k whose zeroes are $(\sigma_1\sigma_2^{-1})^k$-twist invariant specializations.

Chapter 5

Cobordisms of positive 3-braids

The *cobordism distance* $d_\chi(K, K')$ between two links K, K' is defined to be the minimal Euler characteristic among all smooth cobordisms in $S^3 \times [0,1]$ without closed components between $K \subset S^3 \times \{0\}$ and $K' \subset S^3 \times \{1\}$, see [Baa12]. In this final chapter, we present joint work with Sebastian Baader on the cobordism distance of $T(3,m)$ torus links to n-th powers of $T(2,k)$ torus links [BR23].

In light of Coxeter's result that the quotient braid group B_3 on three strands modulo the normal subgroup generated by σ_1^k is finite if and only if $k \leq 5$, it might not come as a surprise to the reader that any positive braid in B_3^+ can be untwisted to a finite family of braids by using t_3-moves. Doing this explicitly allows us to compute the cobordism distance between the closure of an arbitrary positive braid on three strands to a connected sum power $T(2,3)^n$ of the trefoil knot.

For $k = 4, 5$ we can do the same to obtain a similar estimate of the cobordism distance between three strand torus links $T(3,m)$ and connected powers $T(2,k)^n$. Surprisingly, the argument extends to the case $k = 6$, even though as we have seen in Theorem 4.6, it is not possible to transform links of type $T(3,m)$ into a finite family of links using t_6-moves. For $k \geq 7$ our results are less strong, only giving an estimate of the cobordism for large or very small n compared to m.

Our first result may be stated as follows.

Theorem 5.1. *There exists a constant $C_3 \leq 17$ such that for all non-split positive braids $\beta \in B_3^+$ of length $\ell(\beta)$ and all $n \in \mathbb{N}$ we have*

$$d_\chi(\widehat{\beta}, T(2,3)^n) = \frac{\ell(\beta)}{3} + 2\left|n - \frac{\ell(\beta)}{3}\right| + E_3(\beta, n),$$

with an error term $E_3(\beta, n)$ satisfying $|E_3(\beta, n)| \leq C_3$.

In order to state our other two results, we need to specify the meaning of iterated connected sums of torus links $T(2,k)$ for even k, since the connected sum of links with more than one component depends on the chosen components. We fix one distinguished component of $T(2,k)$ and attach all copies of $T(2,k)$ to it.

Theorem 5.2. *There exist constants $C_4, C_5, C_6 < 44$ such that for all $m \geq 1$, all $n \geq 0$, and all $k \in \{4,5,6\}$ we have*

$$d_\chi(T(3,m), T(2,k)^n) = \frac{2m}{k} + (k-1)\left|n - \frac{2m}{k}\right| + E_k(m,n)$$

with an error term $E_k(m,n)$ satisfying $|E_k(m,n)| \leq C_k$.

Our final main result is related to a result of Feller [Fel16], who considered the cobordism distance between torus links of type $T(3,m)$ and $T(2,k)$, but no iterated connected sums of the latter. However, apart from the extreme case of large k, our result is independent from Feller's.

Theorem 5.3. *There exists a constant $C < 44$ such that for all $k \geq 7$ and all $n, m \geq 0$ there exists an error term $E_k(m,n)$ satisfying $|E_k(m,n)| \leq C$ we have:*

(i) *If $n \geq m/3$, then*

$$d_\chi(T(3,m), T(2,k)^n) = (k-1)n - 4m/3 + E_k(m,n).$$

(ii) *If $n \leq 5m/3k - (k+4)$, then*

$$d_\chi(T(3,m), T(2,k)^n) = 2m - (k-1)n + E_k(m,n).$$

The proof strategy for all three results is very similar. We start off this chapter by constructing cobordisms between our link L in question (either the closure of a positive three braid or a three strand torus link) and iterated connected sums $T(2,k)^n$ for small n. This is done by finding k-th powers of generators in a braid representative of L, and we carry out this work in Section 5.1. These explicit cobordisms give an upper bound on the cobordism distance. Section 5.2 is about lower bounds on the cobordism distance using the Levine-Tristram signature, which is then combined with the previously mentioned upper bounds in Section 5.3.

5.1 Constructing key cobordisms

The key technique in constructing minimal cobordisms is to find many subsurfaces of the right type in the Seifert surface of the braid in question. More precisely, we transform a braid β to the trivial braid by deleting powers a^k or b^k (corresponding to subsurfaces of type $T(2,k)$) of generators, for fixed $k \geq 3$, while at the cost of an error we are allowed to also delete single generators. We summarize this strategy in the following result.

Lemma 5.4. *Let $k \geq 1$, let $\beta \in B_3^+$ be a positive braid, and let β' be obtained from β by removing a k-th power a^k or b^k of a generator n times, and by removing an arbitrary number of single generators. Then*

$$d_\chi(\widehat{\beta}, T(2,k)^n) \leq \ell(\beta) - (k-1)n + 2.$$

Proof. We first construct a cobordism of Euler characteristic -1 between $\widehat{\beta}$ and $\widehat{\beta}'\#T(2,k)$, where β' is obtained from β by deleting a k-th power of a generator. Note that the notation $\widehat{\beta}'\#T(2,k)$ is ambiguous as soon as $\widehat{\beta}'$ has more than one component. This ambiguity will be resolved at the end of the proof. As indicated in Figure 5.1, a cobordism as described above can be obtained by cutting the canonical Seifert surface of β along an arc. Because $T(2,1)$ is the trivial knot, the special case $k=1$ shows in particular that the deletion of a generator yields a cobordism of Euler characteristic -1.

Because the summand $T(2,k)$ can be moved freely along the strand it is connected to, this allows iteration of this removal, so that if β' is obtained from β by deleting n instances of k-th powers of a or b and j instances of single generators, then there exists a cobordism of Euler characteristic $-(n+j)$ between $\widehat{\beta}$ and a connected sum of β' with n many copies of $T(2,k)$. Deleting the generators in β' one by one yields a split union of three powers of $T(2,k)$, which can be merged to the sum $T(2,k)^n$ using two saddle moves yielding a cobordism of Euler characteristic -2. We thus obtain a cobordism of Euler characteristic

$$\chi = -(n + j + \ell(\beta') + 2) = -(\ell(\beta) - (k-1)n + 2)$$

between $\widehat{\beta}$ and $T(2,k)^n$. □

5.1.1 Third powers

Our goal in this subsection is to start with a positive braid $\beta \in B_3^+$ and, by removing third powers of generators and a constant number of single generators, transform it into the trivial braid. We first iterate the procedure of writing β

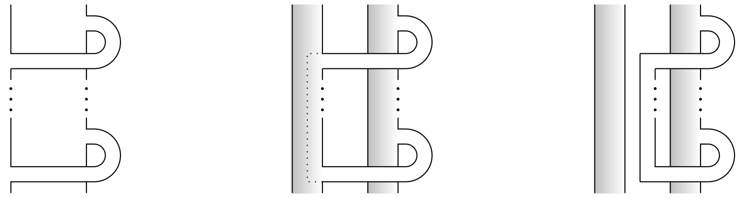


Figure 5.1: The braid a^k or b^k, its canonical Seifert surface, and the surface obtained from cutting along the dotted arc

in Garside normal form, and then removing all emerging third powers of generators. This process terminates once the Garside normal form has no third powers of generators. After conjugation and removing at most 3 single generators, the Garside normal form [Gar69] of the resulting braid can be assumed to be of the form $\Delta^i(a^2b^2)^j$, where $\Delta = aba$. Indeed, Proposition 3.2 in [Tru22] directly implies that, up to conjugation, any 3-braid can be written as a product $\Delta^i a^{p_1} b^{q_1} \cdots a^{p_r} b^{q_r} \beta'$, where $p_j, q_j \geq 2$ for all j, and where β' is either a power of a or equal to $a^p b$ for some $p \in \{1, 2, 3\}$.

Consider the braid $(a^2b^2)^\ell$. Upon deletion of a single generator, it can be transformed into

$$(a^2b^2)^{\ell_1} ab^2 (a^2b^2)^{\ell_2} = (a^2b^2)^{\ell_1 - 1} a \Delta^2 (a^2b^2)^{\ell_2} = \Delta^2 (a^2b^2)^{\ell_1 - 1} a (a^2b^2)^{\ell_2},$$

for any ℓ_1, ℓ_2 with $\ell_1 + \ell_2 = \ell - 1$. Removing a third power of a and then a third power of b similarly leads to

$$\Delta^2 (a^2b^2)^{\ell_1 - 2} a^2 b (a^2b^2)^{\ell_2 - 1} = \Delta^4 (a^2b^2)^{\ell_1 - 2} b (a^2b^2)^{\ell_2 - 2},$$

from which we may again remove two third powers of generators to obtain the braid $\Delta^4(a^2b^2)^{\ell_1-3}ab^2(a^2b^2)^{\ell_2-3}$. By suitable choice of ℓ_1 and ℓ_2, i.e. $\ell_1 = \ell_2$ or $\ell_1 = \ell_2 + 1$, this process can be iterated until we are left with a positive braid of the form $\Delta^{2i}\beta'$ for some i, where β' has length at most 5. In fact, if β' has length 3 or more, it contains one further half-twist Δ after conjugation. We conclude that after deleting at most 3 single generators, the braid $(a^2b^2)^\ell$ can be transformed into a braid of the form Δ^i for some i.

Next, consider the braid Δ^i. As indicated by the fact that

$$(ab)^6 = (aba)b(aba)b(aba)b = bab^3ab^3ab^2,$$

it is possible to delete a total of four powers of the form a^3 and b^3 from $\Delta^4 = (ab)^6$ to obtain the trivial braid. The fact that

$$(ab)^4 = (aba)b(aba)b = bab^3ab^2$$

shows that two third powers of a generator can be removed from $\Delta^3 = (ab)^4 a$, showing that after deletion of at most 3 single generators, the braid Δ^i can be transformed to the trivial braid by deleting third powers of generators.

Note that in order to reduce β to the trivial braid by removing third powers of generators, we had to delete at most $3+3+3 = 9$ generators. By Lemma 5.4, we have thus shown that for any positive braid $\beta \in B_3$, and any $n \geq 0$ with $3n \leq \ell(\beta) - 9$ we have

$$d_\chi(\widehat{\beta}, T(2,3)^n) \leq \ell(\beta) - 2n + 2.$$

5.1.2 Fourth powers

As shown in [BBL20], we have

$$\Delta^8 = (ab)^{12} = ab^3ab^5ab^3ab^4ab^4.$$

After removing three fourth powers of b, the braid is transformed into

$$ab^3(aba)b^3a^2 = ab^4ab^4a^2,$$

which is transformed into the trivial braid after further removal of three fourth powers of generators.

It is possible to remove 2 single generators from $(ab)^3$ to obtain b^4. Thus, one can remove 4 single generators from $(ab)^4$ to obtain b^4.

Similarly, note that

$$(ab)^5 = a(bab)(aba)(bab) = a^2(bab)(aba)ba = a^3(bab)ab^2a = a^4ba^2b^2a,$$

which, after removing a fourth power of a, is equal to ba^2b^2a. This braid is elementarily conjugate to

$$aba^2b^2 = babab^2 = b^2ab^3,$$

which again is elementarily conjugate to ab^5. It is thus possible to remove 2 single generators from $(ab)^5$, 4 single generators from $(ab)^6$, or 6 generators from $(ab)^7$ to obtain a braid that can be transformed into the trivial braid by removing two fourth powers of generators.

Next, similar techniques show that three fourth powers of generators can be removed from $(ab)^8$. It is thus enough to remove 4 single generators from $(ab)^8$ or 6 generators from $(ab)^9$. Finally, removing 4 single generators from $(ab)^{10}$ or 6 single generators from $(ab)^{11}$ is enough.

Thus, the braid $(ab)^m$ can be transformed into the trivial braid by deleting at most 6 single generators, the worst case being $(ab)^j$ for $j \in \{7, 9, 11\}$. Lemma 5.4 implies that for any $n \geq 0$ with $4n \leq 2m - 6$ that

$$d_\chi(T(3,m), T(2,4)^n) \leq 2m - 3n + 2.$$

5.1.3 Fifth powers

Again as shown in [BBL20],

$$\Delta^{4i} = (ab)^{6i} = ab^3(ab^5)^{i-1}ab^3ab^4(ab^5)^{i-2}ab^4$$

for all $i \geq 2$. Removing fifth powers from this braids results in $ab^3a^ib^3ab^4a^{i-1}b^4$, which after deleting at most 7 generators and further removal of fifth powers is equal to $abab^3 = bab^4$, which upon deletion of a further single generator is equal to the fifth power b^5.

Similarly,

$$\Delta^{4i+2} = (ab)^{6i+3} = ab^3(ab^5)^{i-1}ab^4ab^3(ab^5)^{i-1}ab^4$$

for all $i \geq 1$, which after deletion of at most 8 generators can be transformed, using deletion of fifth powers of generators, into ab^2ab^2, a braid of length 6.

Finally, an arbitrary braid $(ab)^m$ with $m \geq 6$ can be transformed either to Δ^{4i} or Δ^{4i+2} for some i using at most 5 deletions of single generators for a total error of at most $5 + 6 + 8 = 19$. The braid $(ab)^j$ for $j \leq 5$ is shorter than this maximal error, so we have shown that for all $n \geq 0$ with $5n \leq 2m - 19$ we have

$$d_\chi(T(3,m), T(2,5)^n) \leq 2m - 4n + 2.$$

5.1.4 Sixth powers

The formula $\Delta^{4i} = ab^3(ab^5)^{i-1}ab^3ab^4(ab^5)^{i-2}ab^4$ from the previous subsection shows that by deleting 12 single generators and removing one sixth power of a generator, the braid $(ab)^{6i}$ can be transformed into $(ab^5)^{2i-3}$. Removing a further sixth power yields $(ab^{-1})(ab^5)^{2i-3}$, and removing two further single generators transforms $(ab)^{6i}$ into $(ab^{-1})^{2i-4}$ using 14 deletions of single generators and $2i - 2$ deletions of sixth powers.

Similarly, the formula $\Delta^{4i+2} = ab^3(ab^5)^{i-1}ab^4ab^3(ab^5)^{i-1}ab^4$ shows that by deleting 12 single generators and removing $2i - 1$ sixth powers of a generator, the braid $(ab)^{6i+3}$ can be transformed into $(ab^{-1})^{2i-2}$.

The closures of the braids $(ab^{-1})^{2j}$ have cobordism distance at most two to the unknot. This is because two cobordisms of Euler characteristic -1 transform the closure of $(ab^{-1})^{2j}$ into $L\#L$, where L is the closure of $(ab^{-1})^j$. Because L

is isotopic to its mirror image, it follows that $L\#L$ is slice. Using this as well as Lemma 5.4 and the fact that four deletions of single generators are enough to turn any power of ab into Δ^{4i} or Δ^{4i+2} for some i, we obtain that for all $n \geq 0$ with $6n \leq 2m - 20$ we have

$$d_\chi(T(3,m), T(2,6)^n) \leq 2m - 5n + 2.$$

5.1.5 Larger powers

From here on we are not quite as efficient as in the previous cases, in the sense that we cannot, up to bounded error, turn an arbitrary power of ab into the trivial braid by removing k-th powers of generators for $k \geq 7$. We nonetheless find a substantial number of k-th powers of generators, although we make no claim that this number is optimal.

Note that the braid $(ab)^{6i} = ab^3(ab^5)^{i-1}ab^3ab^4(ab^5)^{i-2}ab^4$ can be transformed into b^{10i-1} by deleting all $2i+1$ occurrences of a. This finds $[(10i-1)/k]$ many k-th powers of generators in $(ab)^{6i}$.

Similarly, the braid $(ab)^{6i+3} = ab^3(ab^5)^{i-1}ab^4ab^3(ab^5)^{i-1}ab^4$ can be transformed into b^{10i+4} by deleting all $2i+2$ occurrences of a. This finds $[(10i+4)/k]$ many k-th powers of generators in $(ab)^{6i+3}$.

Finally, any power $(ab)^m$ can be turned into one of the braids under consideration by deleting at most four single generators. Using Lemma 5.4, we are now able to conclude that for all $n \geq 0$ with $n \leq 5m/3k - (k+4)$ we have

$$d_\chi(T(3,m), T(2,k)^n) \leq 2m - (k-1)n + 2.$$

5.2 Signatures

In this section, we provide lower bounds on the cobordism distance between $T(3,m)$ (or a positive braid β) and $T(2,k)^n$ for small m compared to n. To this end, we recall a result proven by Gambaudo-Ghys, namely Corollary 4.4 in [GG05].

Lemma 5.5. *Let $\beta \in B_3^+$ be a positive braid, and let $\omega = e^{2\pi i\theta}$ for a rational number θ with $0 < \theta < 1/3$. Then the Levine-Tristram signature at ω satisfies*

$$|\sigma_\omega(\widehat{\beta}) - 2\theta\ell(\beta)| \leq 2.$$

The formulas of Gambaudo-Ghys, specifically Proposition 5.1 in [GG05], can be used to explicitly determine the Levine-Tristram signatures of the links $T(2,k)$ for $k \in \{3,4,5\}$, see Figure 5.2 for the result. For a link L, we introduce the shorthand notations

(i) $\sigma_3(L) = \lim_{\theta \to 1/6_+} \sigma_{\exp(2\pi i\theta)}(L)$,

(ii) $\sigma_4(L) = \lim_{\theta \to 1/4_+} \sigma_{\exp(2\pi i\theta)}(L)$,

(iii) $\sigma_5(L) = \lim_{\theta \to 3/10_+} \sigma_{\exp(2\pi i\theta)}(L)$,

(iv) $\sigma_k(L) = \sigma_{-1}(L)$ for $k \geq 6$,

in order to facilitate comparison of the signature functions in specific neighbourhoods. Note that the limit in σ_k for $k \leq 5$ is taken down to the θ corresponding to the last jump in the signature function of $T(2,k)$, as this is where we expect to find the largest difference between $\sigma_\omega(T(3,m))$ and $\sigma_\omega(T(2,k)^n)$ for small m.

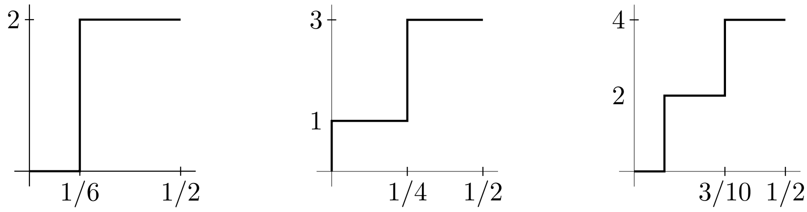


Figure 5.2: Levine-Tristram signatures of the $T(2,k)$ torus links for $k = 3, 4, 5$. The x-axis is θ, and the y-axis is the signature function evaluated at $\omega = e^{2\pi i\theta}$.

Lemma 5.6. *Let $\beta \in B_3^+$ be a positive braid of length $\ell(\beta)$ and let n be a non-negative integer. Then*

(i) $d_\chi(\widehat{\beta}, T(2,3)^n) \geq |2n - \ell(\beta)/3| - 2$,

(ii) $d_\chi(\widehat{\beta}, T(2,4)^n) \geq |3n - \ell(\beta)/2| - 2$,

(iii) $d_\chi(\widehat{\beta}, T(2,5)^n) \geq |4n - 3\ell(\beta)/5| - 2$,

(iv) $d_\chi(\widehat{\beta}, T(2,k)^n) \geq |(k-1)n - 2\ell(\beta)/3| - 4$ *for* $k \geq 6$ *if* $\widehat{\beta}$ *is a torus link.*

Proof. Let $k \geq 3$. For $k \leq 5$, Lemma 5.5 as well as the values $\sigma_k(T(2,k)) = k-1$ yield

$$\max_{\omega \in S^1} |\sigma_\omega(T(2,k)^n) - \sigma_\omega(\widehat{\beta})| \geq |\sigma_k(T(2,k)^n) - \sigma_k(\widehat{\beta})| \geq |(k-1)n - 2\theta_k \ell(\beta)| - 2,$$

where $\theta_3 = 1/6$, $\theta_4 = 1/4$, $\theta_5 = 3/10$. For $k \geq 6$ on the other hand, we have

$$\max_{\omega \in S^1} |\sigma_\omega(T(2,k)^n) - \sigma_\omega(\widehat{\beta})| \geq |(k-1)n - 2\ell(\beta)/3| - 4.$$

Here we use in addition that $|\sigma(T(3,m)) - 4m| \leq 4$, which follows from [GG05], specifically Proposition 5.2 and the fact that the homogeneous signature on B_n differs from the signature by at most $2n$, which is mentioned in their Section 5.1.

Alternatively, explicit computations for the classical signatures of torus knots can be gathered from [GLM81], specifically Theorem 5.2. The desired results now follow from the well-known inequality

$$d_\chi(L, L') \geq \max\left\{|\sigma_\omega(L) - \sigma_\omega(L')| \mid \omega \in S^1\right\}$$

for all links L, L'. $\square$

5.3 Proofs of main results

In this section, we combine the bounds from the previous two sections to construct essentially minimal cobordisms between the closure $\widehat{\beta}$ of a non-split positive braid $\beta \in B_3^+$ or $T(3,m)$ and $T(2,k)^n$. In the case $k \in \{3,4,5,6\}$, all these cobordisms factor through one specific link, approximately $T(2,k)^{[\ell(\beta)/k]}$ or $T(2,k)^{[2m/k]}$, respectively.

Proofs of Theorems 5.1, 5.2, and 5.3. We have $\chi(\widehat{\beta}) = -(\ell(\beta) - 2)$ for all non-split positive braid words $\beta \in B_3^+$, and $\chi(T(2,k)^n) = -n(k-1)$ for all $k \geq 3$. Therefore, by the slice-Bennequin inequality [Rud93], we obtain

$$d_\chi(\widehat{\beta}, T(2,k)^n) \geq |\chi(T(2,k)^n) - \chi(\widehat{\beta})| = |(k-1)n - \ell(\beta)| - 2. \tag{5.1}$$

Combining this inequality with Lemma 5.6 yields a piecewise linear lower bound for $d_\chi(\widehat{\beta}, T(2,k)^n)$ in n. A quick calculation shows

$$d_\chi(\widehat{\beta}, T(2,k)^n) \geq \ell(\beta)/k + (k-1)|n - \ell(\beta)/k| - c_k$$

for all $k \geq 3$, where $c_3 = c_4 = c_5 = 2$ and $c_6 = 4$. This takes care of the claimed lower bounds.

Next, we turn to the upper bounds. Let $k \in \{3,4,5,6\}$ and $\beta \in B_3^+$ be a non-split positive braid. If $k \geq 4$, we assume in addition that $\widehat{\beta}$ is a torus link $T(3,m)$, for which we have $\ell(\beta) = 2m$. Let $\widetilde{C}_3 = 9$, $\widetilde{C}_4 = 6$, $\widetilde{C}_5 = 19$, $\widetilde{C}_6 = 20$. Recall that we have shown in Section 5.1 for all $N \geq 0$ such that $kN \leq \ell(\beta) - \widetilde{C}_k$, that

$$d_\chi(\widehat{\beta}, T(2,k)^N) \leq \ell(\beta) - (k-1)N + 2. \tag{5.2}$$

Let N be the maximal integer satisfying this restriction, and let $n \geq N$ be arbitrary. We use the triangle inequality to derive the upper bound in question.

It is easy to see that $d_\chi(T(2,k)^N, T(2,k)^n) = (k-1)(n-N)$. We thus obtain

$$\begin{aligned} d_\chi(\widehat{\beta}, T(2,k)^n) &\leq d_\chi(\widehat{\beta}, T(2,k)^N) + d_\chi(T(2,k)^N, T(2,k)^n) \\ &\leq \ell(\beta) - (k-1)N + 2 + (k-1)(n-N) \\ &= \ell(\beta) + (k-1)n + 2 - 2(k-1)N. \end{aligned}$$

We have $kN \geq \ell(\beta) - (\widetilde{C}_k + k - 1)$ by maximality of N. This yields

$$d_\chi(\widehat{\beta}, T(2,k)^n) \leq (k-1)n - \frac{(k-2)\ell(\beta)}{k} + 2(k-1)\frac{\widetilde{C}_k + k - 1}{k} + 2.$$

Combining (5.2) with this inequality proves Theorems 5.1 and 5.2 for the constants $C_k = 2(k-1)(\widetilde{C}_k + k - 1)/k + 2$. We obtain the following explicit constants:

(i) $C_3 = 50/3$,

(ii) $C_4 = 31/2$,

(iii) $C_5 = 194/5$,

(iv) $C_6 = 131/3$. □

Proof of Theorem 5.3. The lower bounds come from Lemma 5.6 and Equation (5.1) (where we apply $m \geq n/3$). The upper bound for point (ii) was done in Section 5.1, so only the upper bound for point (i) is missing. To this end, we describe a two-step process to construct a cobordism from $T(3,m)$ to $T(2,k)^n$. Let $n \geq m/3$. First, a cobordism of Euler characteristic $-(k-6)n$ transforms $T(2,k)^n$ into $T(2,6)^n$. Next, by Theorem 5.2 we know that

$$d_\chi(T(3,m), T(2,6)^n) \leq 5n - 4m/3 + E_k(m,n).$$

Putting the two together we obtain the claimed bound

$$\begin{aligned} d_\chi(T(3,m), T(2,k)^n) &\leq d_\chi(T(3,m), T(2,6)^n) + d_\chi(T(2,6)^n, T(2,k)^n) \\ &\leq (k-1)n - 4m/3 + E_k(m,n), \end{aligned}$$

where $|E_k(m,n)| \leq |E_6(m,n)| \leq C_6$. □

Bibliography

[ABFR22] Lambert A'Campo, Sebastian Baader, Livio Ferretti, and Levi Ryffel, *Two strand twisting*, 2022, `arXiv:2209.06439`.

[Art25] Emil Artin, *Theorie der Zöpfe*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg **4** (1925), no. 1, 47–72.

[Baa12] Sebastian Baader, *Scissor equivalence for torus links*, Bulletin of the London Mathematical Society **44** (2012), no. 5, 1068–1078.

[BBL20] Sebastian Baader, Ian Banfield, and Lukas Lewark, *Untwisting 3-strand torus knots*, Bulletin of the London Mathematical Society **52** (2020), no. 3, 429–436.

[BFR23] S. Baader, P. Feller, and L. Ryffel, *Bouquets of curves in surfaces*, Glasgow Mathematical Journal **65** (2023), no. 1, 90–97.

[BGJP18] Rubén Blasco-García, Arye Juhász, and Luis Paris, *Note on the residual finiteness of Artin groups*, Journal of Group Theory **21** (2018), no. 3, 531–537.

[BH73] Joan S. Birman and Hugh M. Hilden, *On isotopies of homeomorphisms of Riemann surfaces*, The Annals of Mathematics **97** (1973), no. 3, 424.

[Bir69] Joan S. Birman, *On braid groups*, Communications on Pure and Applied Mathematics **22** (1969), 41–72.

[BL21] Sebastian Baader and Michael Lönne, *Secondary braid groups*, 2021, `arXiv:2001.09098`.

[BM06] Robert W. Bell and Dan Margalit, *Braid groups and the co-Hopfian property*, Journal of Algebra **303** (2006), no. 1, 275–294.

[BR23] Sebastian Baader and Levi Ryffel, *Trefoils and hexafoils in 3-braids*, 2023, `arXiv:2310.11836`.

[BS72] Egbert Brieskorn and Kyoji Saito, *Artin-Gruppen und Coxeter-Gruppen*, Inventiones Mathematicae **17** (1972), no. 4, 245–271.

[CMM21] Matt Clay, Johanna Mangahas, and Dan Margalit, *Right-angled Artin groups as normal subgroups of mapping class groups*, Compositio Mathematica **157** (2021), no. 8, 1807–1852.

[Cox35] H. S. M. Coxeter, *The complete enumeration of finite groups of the form* $R_i^2 = (R_iR_j)^{k_{ij}} = 1$, Journal of the London Mathematical Society **s1-10** (1935), no. 1, 21–25.

[Cox57] ———, *Factor groups of the braid groups*, Proceedings of the Fourth Can. Math. Cong. (1957), 95–122.

[CP01] John Crisp and Luis Paris, *The solution to a conjecture of Tits on the subgroup generated by the squares of the generators of an Artin group*, Inventiones Mathematicae **145** (2001), no. 1, 19–36.

[CP03] Ruth Charney and David Peifer, *The* $K(\pi, 1)$*-conjecture for the affine braid groups*, Commentarii Mathematici Helvetici **78** (2003), 584–600.

[Cri99] John Crisp, *Injective maps between Artin groups*, pp. 119–138, De Gruyter, Berlin, New York, 1999.

[Fel16] Peter Feller, *Optimal cobordisms between torus knots*, Communications in Analysis and Geometry **24** (2016), no. 5, 993–1025.

[FM11] Benson Farb and Dan Margalit, *A primer on mapping class groups*, Princeton University Press, December 2011.

[Fox58] Ralph H. Fox, *Congruence classes of knots*, Osaka Mathematical Journal **10** (1958), no. 1, 37 – 41.

[FW87] John Franks and R. F. Williams, *Braids and the jones polynomial*, Transactions of the American Mathematical Society **303** (1987), no. 1, 97–108.

[FYH+85] P. Freyd, D. Yetter, J. Hoste, W. B.R. Lickorish, K. Millett, and A. Ocneanu, *A new polynomial invariant of knots and links*, Bulletin of the American Mathematical Society **12** (1985), no. 2, 239–246 (English (US)).

[Gar69] F. A. Garside, *The braid group and other groups*, The Quarterly Journal of Mathematics **20** (1969), no. 1, 235–254.

[GG05] Jean-Marc Gambaudo and Étienne Ghys, *Braids and signatures*, Bull. Soc. math. France **133** (2005), no. 4, 541–579.

[GLM81] C. Mca. Gordon, R. A. Litherland, and K. Murasugi, *Signatures of covering links*, Canadian Journal of Mathematics **33** (1981), no. 2, 381–394.

[HB73] V. E. Hoggatt. and Marjorie Bicknell, *Roots of fibonacci polynomials*, Fibonacci Quart. **11** (1973), no. 3, 271–274.

[Hum89] Stephen P. Humphries, *Free products in mapping class groups generated by Dehn twists*, Glasgow Mathematical Journal **31** (1989), no. 2, 213–218.

[Kob12] Thomas Koberda, *Right-angled artin groups and a generalized isomorphism problem for finitely generated subgroups of mapping class groups*, Geometric and Functional Analysis **22** (2012), no. 6, 1541–1590.

[KP02] Richard P. Kent IV and David Peifer, *A geometric and algebraic description of annular braid groups*, Int. J. Algebra Comput. **12** (2002), 85–97.

[Lab97] Catherine Labruère, *Generalized braid groups and mapping class groups*, Journal of Knot Theory and Its Ramifications **06** (1997), no. 05, 715–726.

[Lic97] W. B. Raymond Lickorish, *An introduction to knot theory*, Springer New York, 1997.

[LS01] Roger C. Lyndon and Paul E. Schupp, *Combinatorial group theory*, Springer Berlin Heidelberg, 2001.

[Lö10] Michael Lönne, *Presentations of subgroups of the braid group generated by powers of band generators*, Topology and its Applications **157** (2010), no. 7, 1127–1135.

[Mat64] Hideya Matsumoto, *Générateurs et relations des groupes de Weyl généralisés*, Comptes rendus hebdomadaires des séances de l'Académie des sciences **258** (1964), no. 13, 3419.

[Max98] G. Maxwell, *The normal subgroups of finite and affine coxeter groups*, Proceedings of the London Mathematical Society **76** (1998), no. 2, 359–382.

[Mor86] H. R. Morton, *Seifert circles and knot polynomials*, Mathematical Proceedings of the Cambridge Philosophical Society **99** (1986), no. 1, 107–109.

[Mor11] Jamil W. Mortada, *Artin and Dehn twist subgroups of the mapping class group*, Ph.D. thesis, 2011, `https://purl.lib.fsu.edu/diginole/FSU_Mortada_2011_Spring`.

[Mur91] Kunio Murasugi, *On the braid index of alternating links*, Transactions of the American Mathematical Society **326** (1991), 237–260.

[Prz06] Jozef H. Przytycki, t_k*-moves on links*, arXiv (2006), `https://arxiv.org/abs/math/0606633`.

[Rol76] Dale Rolfsen, *Knots and links*, AMS Chelsea Publishing, American Mathematical Society, Providence, RI, 1976 (en).

[Rud93] Lee Rudolph, *Quasipositivity as an obstruction to sliceness*, Bulletin of the American Mathematical Society **29** (1993), 51–59.

[Ryf23a] Levi Ryffel, *Curves intersecting in a circuit pattern*, Topology and its Applications **332** (2023), 108522.

[Ryf23b] ———, *Hecke*, `https://github.com/raw-bacon/hecke`, 2023.

[Sco70] G.P. Scott, *The space of homeomorphisms of a 2-manifold*, Topology **9** (1970), no. 1, 97–109.

[Tit66] Jacques Tits, *Normalisateurs de tores I. Groupes de Coxeter étendus*, Journal of Algebra **4** (1966), no. 1, 96–116.

[Tru22] Paula Truöl, *The upsilon invariant at 1 of 3-braid knots*, 2022, `arXiv:2108.03674`.

[vdL83] Harm van der Lek, *The homotopy type of complex hyperplane complements*, Ph.D. thesis, Nijmegen, 1983.

[Waj99] Bronislaw Wajnryb, *Artin groups and geometric monodromy*, Inventiones mathematicae **138** (1999), 563–571.