Images of Tensor Product Polynomials

Inauguraldissertation

der Philosophisch-naturwissenschaftlichen Fakultät
der Universität Bern

vorgelegt von

Andreas Blatter
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Leiter der Arbeit:
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Bern, 24. Mai 2024

**Der Dekan
Prof. Dr. Marco Herwegh**

Preface

This thesis builds up on Draisma's paper on Topological Noetherianity of Polynomial Functors ([Dra19]) and develops the theory of polynomial functors in three different directions.

Chapter 2 is based on [BDV23], and establishes a functorial analogue of Buchberger's algorithm. Chapter 3 proves a functorial analogue of Chevalley's Theorem on constructible sets, and it is intended to be published as a separate article. Chapter 4 is based on [BDR22b], and extends Draisma's Topological Noetherianity result, that had only be proven over infinite fields, to finite fields.

The thesis also contains an extensive introduction on the language of polynomial functors with material from all the above referenced papers, plus [Bik20], [FS97] and [Tou14].

I would like to thank Mateusz Michałek for refereeing my thesis and his insightful comments.

Acknowledgements

I would like to thank my supervisor Jan Draisma for risking part of his hard-fought grant money on me, always having his office door open for my questions, being open to my silly and not-so-silly ideas, infecting me with his passion on infinite-dimensional algebraic geometry, and being a perfect role model of a modern mathematician. I am very glad to have found a mentor who has all the mathematical skills that I would like to have, as I have realized once more when I received his final corrections on my thesis, and I found myself agreeing with every single suggestion he made.

Further thanks go to my other colleagues in Jan's research team, first of all the coauthors of the papers appearing in my thesis, Emanuele Ventura and Filip Rupniewski, for giving me more perspectives on the topics I was working on, but also Arthur Bik, Alejandro Vargas, Rob Eggermont, Azhar Farooq, Alessandro Danelon, Nafie Tairi, Tim Seynnaeve, Chris Chiu and Beni Biaggi, with whom I have enjoyed many productive Friday morning seminars.

I also thank all the professors and lecturers whose courses I had the pleasure to assist, giving me both some variety in my mathematical life, and a few greatly appreciated boosts in my self-confidence as a mathematician: George Metcalfe, Frank Kutzschebauch, Ian Banfield, Kinga Sipos, Christiane Tretter and Zoltán Balogh. A special thank you also to my master thesis advisor Sebastian Baader.

Finally, I thank all the people who do not fit in any of the above categories, but have made my time in Bern more valuable, be it by playing games, having lunch together, assisting courses together, or organizing student seminars: Elia, Feng, Kevin, Leandro, Levi, Livio, Naomi, Nicolas, Nicolà, Olim, Sarah, Simon, Simon, Stéphane, and anybody else I might have forgotten in this list.

Contents

Introduction

Tensor Product Polynomials

Let K be any field, $A \in (K^n)^{\otimes d_1}$ and $B \in (K^n)^{\otimes d_2}$. There is a natural way to multiply A and B by taking the tensor product: Writing $A = (a_{i_1 \dots i_{d_1}})_{i_1 \dots i_{d_1}}$, $B = (b_{j_1 \dots j_{d_2}})_{j_1 \dots j_{d_2}}$, then

$$A \otimes B = (a_{i_1 \dots i_{d_1}} b_{j_1 \dots j_{d_2}})_{i_1 \dots i_{d_1} j_1 \dots j_{d_2}} \in (K^n)^{\otimes d_1 + d_2}.$$

If $d_1 = d_2$, we can also add A and B simply by component-wise addition $A + B$. And we can also perform scalar multiplication $A \mapsto \lambda A$ for some $\lambda \in K$. Combining these operations we obtain a tensor product polynomial, say α, that maps from some direct sum of tensor spaces $(K^n)^{\otimes d_1} \oplus \dots \oplus (K^n)^{\otimes d_m}$ to some tensor space, say $(K^n)^{\otimes e}$.

Example. *Let d be fixed,*

$$\begin{aligned} \alpha_n^{(r)} : ((K^n)^{\otimes 1})^{\oplus d \cdot r} &\to (K^n)^{\otimes d} \\ (v_{11}, \dots, v_{1d}, \dots, v_{r1}, \dots, v_{rd}) &\mapsto \sum_{i=1}^{r} v_{i1} \otimes \dots \otimes v_{id}. \end{aligned}$$

For an element $A \in (K^n)^{\otimes d}$, the minimal r such that A is in the image of $\alpha_n^{(r)}$ is called the rank of A, also denoted $\mathrm{rk}(A)$. *The image of $\alpha_n^{(r)}$ is the set of tensors in $(K^n)^{\otimes d}$ of rank smaller or equal to r.*

Example. *Let*

$$\begin{aligned} \alpha_n : ((K^n)^{\otimes 2})^{\oplus 3} &\to (K^n)^{\otimes 4} \\ (A, B, C) &\mapsto A \otimes B - C \otimes C. \end{aligned}$$

Images of Tensor Product Polynomials

We now ask how we can describe the image of such a polynomial α_n. If we fix the integer n, then there already exists a satisfying answer, at least for certain fields: In case $K = \mathbb{C}$, or any other algebraically closed field, Chevalley's Theorem on constructible sets says that an image of a constructible set under a polynomial map is again constructible. A constructible set in some complex vector space, say V, is a set that can be described by a finite boolean

combination of polynomial equations and inequations, or in other words, it is a finite union of sets of the form

$$\{v \in V : f_1(v) = \ldots = f_k(v) = 0 \text{ and } g(v) \neq 0\}, \quad f_1, \ldots, f_k, g \in \mathbb{C}[V].$$

So, in conclusion, by Chevalley's Theorem, the image of α_n is constructible, since it is the image of a whole (constructible) vector space, under a polynomial map.

Additionally, using Buchberger's algorithm, it is possible to compute the equations of (the closure of) $\mathrm{im}(\alpha_n)$ (this actually works over any field), and there also exist algorithms to compute the inequations as well.

If $K = \mathbb{R}$, there is an analogous theorem to Chevalley's by Tarski-Seidenberg that says that an image of a semialgebraic set under a polynomial map is again semialgebraic. A semialgebraic set is also a set that can be described by finitely many equations and inequations, but when we use the word inequation in this setting, we do not just mean "$\neq$", but also "$>$" and "$\geq$". Also in this case, there are algorithms to calculate the equations and inequations describing $\mathrm{im}(\alpha_n)$.

There is a third, rather trivial, case, where we can fully describe $\mathrm{im}(\alpha_n)$, namely when K is a finite field. In this case, $\mathrm{im}(\alpha_n)$ is finite, so we can just list all the elements (and the algorithm to find these elements is also trivial).

Images of Infinite Collections of Tensor Product Polynomials

The point of this thesis is that we do not want n to be fixed, but we want to give a finite (implicit) description of the whole collection of the images, say $(\mathrm{im}(\alpha_n))_n$. A partial solution to this problem has been given by Draisma in [Dra19]. The main result in this paper basically implies that, if K is an infinite field, there exists $m \in \mathbb{N}$ such that $\overline{\mathrm{im}(\alpha_m)}$ (i.e. the closure of the image) already completely describes the whole collection $(\overline{\mathrm{im}(\alpha_n)})_n$, so the equations for any $\mathrm{im}(\alpha_n)$ can be pulled back from the equations for $\mathrm{im}(\alpha_m)$.

To make this more precise, note that if $\varphi : K^n \to K^m$ is a linear map, we get a linear map

$$\begin{aligned} \varphi^{\otimes e} : (K^n)^{\otimes e} &\to (K^m)^{\otimes e} \\ v_1 \otimes \ldots \otimes v_e &\mapsto \varphi(v_1) \otimes \ldots \otimes \varphi(v_e) \end{aligned}$$

It is easy to see (see Proposition 1.2.4) that $\overline{\mathrm{im}(\alpha_n)}$ is invariant under $\varphi^{\otimes e}$, meaning that for $\varphi : K^n \to K^m$ and $p \in \overline{\mathrm{im}(\alpha_n)}$ we have $\varphi^{\otimes e}(p) \in \overline{\mathrm{im}(\alpha_m)}$. By Draisma's result, there exists $m \in \mathbb{N}$ such that for every $n \in \mathbb{N}$

$$\overline{\mathrm{im}(\alpha_n)} = \{p \in (K^n)^{\otimes e} : \text{for every linear map } \varphi : K^n \to K^m, \ \varphi^{\otimes e}(p) \in \overline{\mathrm{im}(\alpha_m)}\}.$$

The Results

Draisma's result is not algorithmic, so while it was proven that this number m exists, it was not shown how to find it. Chapter 2 presents exactly this algorithm for the case where K is an algebraically closed field of charateristic 0, which is surprisingly non-trivial. This chapter is based on the paper [BDV23]. We should point out that this algorithm is completely theoretical, and has not served us yet to actually compute equations that had not been known before.

In Chapter 3, we show that the above result remains true if we discard the closure, i.e. that there exists $m \in \mathbb{N}$ such that $\mathrm{im}(\alpha_m)$ already completely describes the whole collection $(\mathrm{im}(\alpha_n))_n$, so both the equations and inequations for any $\mathrm{im}(\alpha_n)$ can be determined from the equations and inequations for $\mathrm{im}(\alpha_m)$ (although this does need some nontrivial quantifier elimination). We also prove this in the case $\mathrm{char}\, K = 0$. If $K = \mathbb{C}$, we can actually prove something stronger, and based on a natural analogue of constructible subsets, we give an analogue of Chevalley's Theorem.

Recall that Draisma's result was only shown over infinite fields. In Chapter 4 we close this gap in our theory, and also prove it over finite fields (we actually show the complete analogue of the main result in [Dra19]). This chapter is based on the paper [BDR22b]. This has some nice applications, including a tensor restriction theorem and a result which says that the set of all asymptotic ranks is well-ordered. In Section 4.6, we give a simple argument that over $\mathbb{C}$ the set of all asymptotic ranks is countable. This is based on the note [BDR22a].

Our results are slightly more general than what is described above. Let us use coordinate-free notation from now on. Note that $V \mapsto (V)^{\otimes d_1} \oplus \ldots \oplus (V)^{\otimes d_m}$ is a functor from the category of finite-dimensional K-vector-spaces to itself. In general, a polynomial functor, P, is a functor of this form, or a subquotient thereof (e.g. S^2, which is a quotient of $V \mapsto V^{\otimes 2}$, is also a polynomial functor).

A polynomial transformation, α, from a polynomial functor Q to a polynomial functor P assigns to every vector space V a polynomial map $\alpha_V : Q(V) \to P(V)$ that behaves well with respect to the functoriality of Q and P, but essentially they are just tensor product polynomials as above. We want to describe the collection $(\mathrm{im}(\alpha_V))_V$ with finite information. We give a full introduction to the language of polynomial functors in Chapter 1.

Chapter 1

Polynomial Functors

For now, let K be a field of characteristic 0. We will also consider polynomial functors over finite fields in Chapter 4, but we will discuss the necessary modifications of the definitions there.

1.1 Definition and Description

1.1.1 Definition

Let **Vec** be the category of finite-dimensional K-vector spaces. We write $\mathrm{Hom}(U, V)$ for the space of K-linear maps $U \to V$.

Definition 1.1.1. A *polynomial functor* over K is a (covariant) functor $P : \mathbf{Vec} \to \mathbf{Vec}$ such that for any $U, V \in \mathbf{Vec}$ the map $P : \mathrm{Hom}(U, V) \to \mathrm{Hom}(P(U), P(V))$ is polynomial of degree at most some integer d that does not depend on U or V. The minimal such integer d is called the *degree* of P. ♢

The phrase "the map $P : \mathrm{Hom}(U, V) \to \mathrm{Hom}(P(U), P(V))$ is polynomial" means that when choosing bases for U, V, $P(U)$ and $P(V)$, the map P that maps the matrix representation of $\varphi \in \mathrm{Hom}(U, V)$ to the matrix representation of $P(\varphi) \in \mathrm{Hom}(P(U), P(V))$ must be polynomial. This notion is independent of the choice of bases.

Example 1.1.2.

(1) For a fixed $U \in \mathbf{Vec}$, the *constant functor* $P : V \mapsto U$, $\varphi \mapsto \mathrm{id}$, is a polynomial functor of degree 0.

(2) The *identity functor* $T : V \mapsto V$, $\varphi \mapsto \varphi$, is a polynomial functor of degree 1.

(3) The *d-th direct sum* $T^{\oplus d} : V \mapsto V^{\oplus d}$, $\varphi \mapsto \varphi^{\oplus d}$, is a polynomial functor of degree 1.

(4) The *d-th tensor power* $T^{\otimes d} : V \mapsto V^{\otimes d}$, $\varphi \mapsto \varphi^{\otimes d}$, is a polynomial functor of degree d. ♢

Remark 1.1.3. The requirement from Definition 1.1.1 that the degree of the maps $P(\varphi)$ must be universally bounded rules out examples like $V \mapsto \bigwedge^0(V) \oplus \bigwedge^1(V) \oplus \bigwedge^2(V) \oplus \dots$ ♢

Polynomial functors form an abelian category, where the morphisms are the following:

Definition 1.1.4. A natural (or linear) transformation $\alpha : Q \to P$ consists of a linear map $\alpha_V : Q(V) \to P(V)$ for each $V \in \mathbf{Vec}$, such that for all $\varphi \in \mathrm{Hom}(V, W)$ the following diagram commutes:

$$\begin{array}{ccc} Q(V) & \xrightarrow{\alpha_V} & P(V) \\ \downarrow{\scriptstyle Q(\varphi)} & & \downarrow{\scriptstyle P(\varphi)} \\ Q(W) & \xrightarrow{\alpha_W} & P(W) \end{array}$$

♢

1.1.2 Characterisation

The following definitions will allow us to give an intuitive characterisation of all polynomial functors.

Definition 1.1.5. Let $P : \mathbf{Vec} \to \mathbf{Vec}$ be any functor. A functor $Q : \mathbf{Vec} \to \mathbf{Vec}$ is called a *subfunctor* of P if there exists an injective natural transformation $\alpha : Q \to P$ (i.e. α_V is injective for every V), or, equivalently, Q is isomorphic to a polynomial functor Q', such that $Q'(V) \subseteq P(V)$ for all V and $Q'(\varphi) = P(\varphi)|_{Q'(V)}$ for all $\varphi \in \mathrm{Hom}(V, W)$. We call P *irreducible* if it contains no nonzero proper subfunctor. ♢

Definition 1.1.6. Let $P, Q : \mathbf{Vec} \to \mathbf{Vec}$ be functors. Notions like $P \oplus Q$, $P \otimes Q$ and, in case $Q \subseteq P$ is a subfunctor, P/Q are defined elementwise in the obvious way (e.g. $(P \oplus Q)(V) := P(V) \oplus Q(V)$, and similarly for morphisms). ♢

Definition 1.1.7. Let $P : \mathbf{Vec} \to \mathbf{Vec}$ be any functor. A functor $Q : \mathbf{Vec} \to \mathbf{Vec}$ is called a *quotient* of P if there exists a surjective natural transformation $\alpha : P \to Q$ (i.e. α_V is surjective for every V), or, equivalently, Q is isomorphic to a polynomial functor P/Q', where $Q' \subseteq P$ is a subfunctor. We call Q a *subquotient* of P if it is a quotient of a subfunctor of P. ♢

Even in general characteristic, the class of subquotients of P is closed under taking quotients and subfunctors (i.e. a subfunctor of a subquotient is still a subquotient).

The following characterisation follows from [FS97, Lemma 3.4], but we will give a more elementary proof here.

Proposition 1.1.8. *Let $P : \mathbf{Vec} \to \mathbf{Vec}$ be a functor. The following are equivalent:*

(1) P is a polynomial functor of degree at most d.

(2) P is a subquotient of a functor of the form of $\bigotimes_{e=0}^{d} (T^{\otimes e})^{\oplus n_e}$.

In other words, the set of polynomial functors is the smallest set of functors that contains the constant functor $V \mapsto K^1$ (which is equal to $T^{\otimes 0}$) and the identity functor (i.e. $T^{\otimes 1}$), and is closed under taking direct sums, tensor products and subquotients.

We need the following lemma for the proof:

Lemma 1.1.9. *Let P be a polynomial functor of degree d. Then P is generated by $P(U)$ where $U := K^d$, i.e. for every $V \in \mathbf{Vec}$*

$$P(V) = \operatorname{span}\{P(\varphi)p | p \in P(U), \varphi : U \to V\}$$

Proof.

- Let $V \in \mathbf{Vec}$, and let $\psi : V \to V$ be an arbitrary endomorphism. Choose coordinates $V = K^m$ so we can write ψ as a matrix $(\psi_{ij})_{ij}$. By definition of P, we can write

$$P(\psi) = \sum_{\substack{\alpha \in \mathbb{Z}^{m \times m} \\ |\alpha| \leq d}} A_\alpha \psi^\alpha = \sum_{\substack{\alpha \in \mathbb{Z}^{m \times m} \\ |\alpha| \leq d}} A_\alpha \prod_{i,j=1}^{m} \psi_{ij}^{\alpha_{ij}},\ A_\alpha \in \operatorname{End}(P(V))$$

- Note that for every α with $|\alpha| \leq d$, we have that $\alpha_{ij} > 0$ for at most d different index-pairs (i, j), and in particular, the sets

$$I_\alpha := \{i \in [m] : \exists j \in [m] \text{ s.t. } \alpha_{ij} > 0\}$$
$$J_\alpha := \{j \in [m] : \exists i \in [m] \text{ s.t. } \alpha_{ij} > 0\}$$

  have cardinality at most d. For any $I, J \subseteq [m]$ let $\psi_{I,J}$ be ψ where all i-th rows with $i \notin I$ and all j-th columns with $j \notin J$ are replaced by zeroes. We note that

$$P(\psi_{I,J}) = \sum_{\substack{\alpha \in \mathbb{Z}^{m \times m} \\ |\alpha| \leq d, I_\alpha \subseteq I, J_\alpha \subseteq J}} A_\alpha \psi^\alpha$$

  By a simple induction on $|I| + |J|$, one can show that

$$\sum_{\substack{\alpha \in \mathbb{Z}^{m \times m} \\ |\alpha| \leq d, I_\alpha = I, J_\alpha = J}} A_\alpha \psi^\alpha$$

  is a linear combination of maps $P(\psi_{I',J'})$, where $I' \subseteq I$, $J' \subseteq J$. Hence, also $P(\psi)$, which can be written as

$$P(\psi) = \sum_{\substack{I, J \subseteq [m] \\ |I|, |J| \leq d}} \sum_{\substack{\alpha \in \mathbb{Z}^{m \times m} \\ |\alpha| \leq d, I_\alpha = I, J_\alpha = J}} A_\alpha \psi^\alpha$$

is a linear combination of maps $P(\psi_{I,J})$ with $|I|, |J| \leq d$, say

$$P(\psi) = \sum_{\substack{I,J\subseteq[m]\\|I|,|J|\leq d}} c_{IJ} P(\psi_{I,J}).$$

- Note that $\psi_{I,J}$ is a map from a $|J|$-dimensional subspace of V to an $|I|$-dimensional subspace of V. Since $|I|, |J| \leq d = \dim(U)$ we can write

  $$\psi_{I,J} = \iota_I \circ \tilde{\psi}_{I,J} \circ \pi_J$$

  where $\pi_J : V \to U$, $\tilde{\psi}_{I,J} : U \to U$ and $\iota_I : U \to V$.

- Finally, let $v \in P(V)$. Taking $\psi = \mathrm{id}_V$, we can express v as an element in $\mathrm{span}\{P(\varphi)p | p \in P(U), \varphi : U \to V\}$ by

  $$v = P(\mathrm{id}_V)v = \sum_{\substack{I,J\subseteq[m]\\|I|,|J|\leq d}} c_{IJ} P((\mathrm{id}_V)_{I,J})v = \sum_{\substack{I,J\subseteq[m]\\|I|,|J|\leq d}} c_{IJ} P(\iota_I)(\underbrace{P(\tilde{\psi}_{I,J} \circ \pi_J)v}_{\in P(U)})$$

□

Proof of Proposition 1.1.8. Recall that we want to prove that a functor of vector spaces is, (1), a polynomial functor if and only if it is, (2) a subquotient of a direct sum of tensor products.

"(2) ⇒ (1)":

- Let P be a subquotient of a polynomial functor $S = \bigoplus_{e=0}^{d} (T^{\otimes e})^{\oplus n_e}$, i.e. there exists a subfunctor $R \subseteq S$ such that P is a quotient of R. Let $\varphi : V \to W$ be any linear map. We want to show that $P(\varphi)$ is polynomial of degree at most d in φ.

- We know that $S(\varphi)$ is polynomial of degree $\leq d$, and hence so is $R(\varphi)$. Since P is a quotient of R, there exists a surjective natural transformation $\pi : R \to P$. Let $\iota : P(V) \to R(V)$ such that $\pi_V \circ \iota = \mathrm{id}_{P(V)}$ (note that we choose ι specific to V, it is a priori not clear that there exists a natural transformation from P to R that gives the identity when combined with π). Then

  $$P(\varphi) = P(\varphi) \circ \pi_V \circ \iota = \pi_W \circ R(\varphi) \circ \iota$$

  Since ι and π_W are linear, and $R(\varphi)$ is polynomial of degree $\leq d$, we get that $P(\varphi)$ is polynomial of degree $\leq d$.

"(1) ⇒ (2)":

- Let P be a polynomial functor of degree $\leq d$. By Lemma 1.1.9

$$\begin{aligned} Q(V) \otimes P(U) &\to P(V) \\ \psi \otimes p &\mapsto \psi(p) \end{aligned}$$

  is surjective, where

$$Q(V) = \operatorname{span}\{P(\varphi) | \varphi \in \operatorname{Hom}(U, V)\} \subseteq \operatorname{Hom}(P(U), P(V))$$

  (note that Q is a polynomial functor since it is a subfunctor of the polynomial functor $V \mapsto \operatorname{Hom}(P(U), P(V)), \alpha \mapsto (\varphi \mapsto P(\alpha) \circ \varphi)$).

- We have shown that P is a quotient of $Q \otimes P(U)$, so it now suffices to show that Q is a subquotient of a direct sum of tensor powers (then so is $Q \otimes P(U)$, which just consists of $\dim(P(U))$ copies of Q).

- Consider the map $P_{UV} : \operatorname{Hom}(U, V) \to Q(V) \subseteq \operatorname{Hom}(P(U), P(V))$. By definition of P, this is a polynomial map of degree $\leq d$, so we can see it as an element in

$$\left(\bigoplus_{e=0}^{d} S^e(\operatorname{Hom}(U, V)^*) \right) \otimes Q(V) \cong \left(\bigoplus_{e=0}^{d} \Gamma^e(\operatorname{Hom}(U, V)) \right)^* \otimes Q(V)$$

  where Γ^e denotes the subfunctor of symmetric tensors of $T^{\otimes e}$. Hence, P_{UV} can be identified with a linear map

$$\left(\bigoplus_{e=0}^{d} \Gamma^e(\operatorname{Hom}(U, V)) \right) \to Q(V)$$

- So, Q is a quotient of

$$\left(\bigoplus_{e=0}^{d} \Gamma^e(\operatorname{Hom}(U, V)) \right) \cong \left(\bigoplus_{e=0}^{d} \Gamma^e(U^* \otimes V) \right) \cong \left(\bigoplus_{e=0}^{d} \Gamma^e \left(V^{\oplus \dim U^*} \right) \right)$$

  which is itself a subfunctor of a direct sum of tensor products, hence Q is a subquotient thereof. □

Remark 1.1.10. Let P be a nonzero polynomial functor. Then P is irreducible, if and only if for each $U \in \mathbf{Vec}$, $P(U)$ is zero or an irreducible $\operatorname{End}(U)$-module. For the "if", suppose that P is not irreducible, i.e. it contains a proper nonzero subfunctor Q, then also for U of high enough dimension, $Q(U)$ is a proper submodule of $P(U)$. For the "only if", suppose that M is a proper submodule of $P(U)$. Then the functor Q given by

$$Q(V) := \{p \in P(V) \mid \forall \varphi : V \to U,\, P(\varphi)p \in M\}$$

would define a nonzero proper subfunctor. ♢

Remark 1.1.11. It is well-known that in characteristic 0 tensor powers $T^{\otimes e}$ can be decomposed as a direct sum of Schur functors S^λ, where λ is a partition of e (see e.g. [FH91]). Note that for every $V \in \mathbf{Vec}$, $S^\lambda(V)$ is an irreducible $\mathrm{GL}(V)$-representation (if it is nonzero), hence in particular an irreducible $\mathrm{End}(V)$-module, so by the previous remark, S^λ is an irreducible polynomial functor.

So, Proposition 1.1.8 implies that every polynomial functor is isomorphic to a direct sum of Schur functors. This also implies that a polynomial functor Q is a subfunctor of some polynomial functor P, if and only if Q is a quotient of P (if and only if Q is a subquotient of P). ♢

1.1.3 Gradings

Let P be a polynomial functor of degree d, so by the previous remark $P = \bigoplus_{e=0}^{d} \bigoplus_{|\lambda|=e} (S^\lambda)^{\oplus n_\lambda}$. Let $P_e := \bigoplus_{|\lambda|=e} (S^\lambda)^{\oplus n_\lambda}$. We call P_e the *degree-e-part* of P. We can also define it without using the characterisation by

$$P_e(V) := \{p \in P(V) \mid \text{ for all } t \in K,\, P(t \cdot \mathrm{id}_V)p = t^e \cdot p\}.$$

Note that $P = P_0 \oplus P_1 \oplus \ldots \oplus P_d$, and each P_e is a subquotient of some $(T^{\otimes e})^{\oplus m_e}$. We call P_0 the *constant part* of P, and it can be identified with $P(0)$, i.e. P evaluated at the zero-space. We call P *pure* if P_0 is the zero-space, and we call $P_1 \oplus \ldots \oplus P_d$ the *pure part* of P. This is also denoted as $P_{\geq 1}$. Terms like $P_{\leq e}$ or $P_{>e}$ are defined accordingly in the obvious way. We call P *homogeneous of degree d* if $P = P_d$.

1.1.4 An Order on Polynomial Functors

Definition 1.1.12. We call a polynomial functor Q smaller than a polynomial functor P, written $Q < P$, if the two are not isomorphic, and for the largest e such that Q_e is not isomorphic to P_e, Q_e is a quotient of P_e. ♢

Writing these largest nonisomorphic parts Q_e and P_e as sums of Schur functors, i.e.

$$Q_e = \bigoplus_{\lambda:|\lambda|=e} (S^\lambda)^{\oplus m_\lambda}, \quad P_e = \bigoplus_{\lambda:|\lambda|=e} (S^\lambda)^{\oplus n_\lambda}$$

then Q is smaller than P if and only if $m_\lambda \leq n_\lambda$ for all partitions λ of e (where the inequality is strict for at least one such λ). This also demonstrates that this order on polynomial functors is a well-founded order (i.e. there are no infinite strictly decreasing chains).

1.2 Subsets and Polynomial Transformations

1.2.1 Subsets

Definition 1.2.1. Let P be a polynomial functor over K. A *subset* of P, $X \subseteq P$, consists of a subset $X(V) \subseteq P(V)$ for each $V \in \mathbf{Vec}$, such that for all $\varphi \in \mathrm{Hom}(V, W)$ and $v \in X(V)$ we have $P(\varphi)(v) \in X(W)$. We call X a *closed* subset, if for every $V \in \mathbf{Vec}$, $X(V)$ is closed (i.e. the zero-locus of a finite collection of polynomials). ♢

Example 1.2.2. Let $P = T^{\otimes d}$, $r \in \mathbb{N}$ fixed. Then, $X \subseteq P$ given by

$$X(V) = \{A \in P(V) : \mathrm{rk}\, A \leq r\}$$

is a subset. If, e.g., $d = 2$, this is a closed subset. ♢

Example 1.2.3. Let P be any polynomial functor, and A any subset (in the usual set-theoretic sense) of P_0 (recall that P_0 is a constant functor, so we can identify it with a vector space). Then

$$X(V) := \{(a, b) \in P(V) = P_0(V) \oplus P_{\geq 1}(V) \mid a \in A\}$$

is a subset, usually denoted by $A \times P_{\geq 1}$. It is a closed subset if and only if A is closed. We use the notation with $\times$ instead of $\oplus$ purely for aesthetic reasons. We will often consider sets of the form $A \times Q$, where A is a (finite-dimensional) affine variety, and Q is a pure polynomial functor. These can be implicitly seen as subsets of $K^n \oplus Q$, where n is big enough such that there is an embedding of A into K^n. ♢

Proposition 1.2.4. *If $X \subseteq P$ is a subset, then the closure $\overline{X}$ given by $\overline{X}(V) := \overline{X(V)}$ is also a (closed) subset.*

Proof. Consider the set

$$\{(\varphi, p) \in \mathrm{Hom}(V, W) \times P(V) | P(\varphi)p \in \overline{X(W)}\}$$

This set is closed as a preimage of a closed set, and it contains $\mathrm{Hom}(V, W) \times X(V)$. Hence, it also contains the closure $\mathrm{Hom}(V, W) \times \overline{X(V)}$, which proves that for any linear map $\varphi : V \to W$ we have that $P(\varphi)(\overline{X(V)}) \subseteq \overline{X(W)}$.

□

Definition 1.2.5. A closed subset X is called reducible, if it is empty or there exist closed subsets $X_1, X_2 \subsetneq X$ such that $X = X_1 \cup X_2$, and irreducible if it is not reducible. ♢

It is straightforward to check that X is irreducible if and only if for every $V \in \mathbf{Vec}$, $X(V)$ is irreducible. Note that there is an unavoidable terminology clash for the word irreducible, as it is used both in representation theory and algebraic geometry. For example, if $Q \subseteq P$ is a subfunctor, then Q is in particular an irreducible closed subset of P, but it might not be irreducible as a polynomial functor.

1.2.2 Polynomial Transformations

We now define the functorial equivalent of a polynomial map, so called polynomial transformations. The definition is the same as the definition for natural transformations, except that the word "linear" is replaced by the word "polynomial".

Definition 1.2.6. A polynomial transformation $\alpha : Q \to P$ consists of a polynomial map $\alpha_V : Q(V) \to P(V)$ for each $V \in \mathbf{Vec}$, such that for all $\varphi \in \mathrm{Hom}(V, W)$ the following diagram commutes:

$$\begin{array}{ccc} Q(V) & \xrightarrow{\alpha_V} & P(V) \\ \downarrow{\scriptstyle Q(\varphi)} & & \downarrow{\scriptstyle P(\varphi)} \\ Q(W) & \xrightarrow{\alpha_W} & P(W) \end{array}$$

♢

We will often consider polynomial transformations from sets of the form $A \times Q$, where A is an affine variety and Q is pure. These can simply be interpreted as restrictions of polynomial transformations as defined above.

Note that the image $X(V) := \mathrm{im}(\alpha_V)$ of any polynomial transformation is a subset. We revisit the example from the introduction:

Example 1.2.7. Let $Q = T^{\oplus r \cdot d}$, $P = T^{\otimes d}$. Then $\alpha : Q \to P$ given by

$$\begin{aligned} \alpha_V : Q(V) &\to P(V) \\ (v_{11}, \ldots, v_{1d}, \ldots, v_{r1}, \ldots, v_{rd}) &\mapsto \sum_{i=1}^{r} v_{i1} \otimes \ldots \otimes v_{id} \end{aligned}$$

is a polynomial transformation, and its image is the subset from Example 1.2.2. ♢

We make a few observations on the structure of polynomial transformations:

Remark 1.2.8. Note that the diagram in Definition 1.2.6 in particular commutes if φ is a multiple of the identity, i.e. $\varphi = t \cdot \mathrm{id}$ with $t \in K$. Say $Q = Q_e$ is a homogeneous polynomial functor of degree e, $P = P_d$ is a homogeneous polynomial functor of degree d, and $\alpha : Q \to P$ is a polynomial transformation. Then, for $q \in Q(V)$:

$$\alpha_V(Q(t \cdot \mathrm{id}_V)q) = \alpha_V(t^e q)$$

is equal to

$$P(t \cdot \mathrm{id}_V)\alpha_V(q) = t^d \alpha_V(q).$$

So, unless α is the zero-transformation, e must divide d, and α_V is a homogeneous polynomial of degree d/e, if $e \neq 0$ (this needs K to be an infinite field). In particular, if $d = e \neq 0$, then α is linear. Note that the only linear transformations from

$$Q = \bigoplus_{\lambda:|\lambda|=e} (S^\lambda)^{\oplus m_\lambda} \text{ to } P = \bigoplus_{\lambda:|\lambda|=e=d} (S^\lambda)^{\oplus n_\lambda}$$

are of the form

$$\alpha_V((q_{\lambda i})_{|\lambda|=e, 1\leq i\leq m_\lambda}) = (p_{\lambda j})_{|\lambda|=d=e, 1\leq j\leq n_\lambda}$$

where

$$p_{\lambda j} = \sum_{i=1}^{m_\lambda} A_{\lambda ij} q_{\lambda i},\ A_{\lambda ij} \in K.$$

If $e = 0$, we get that $\alpha_V(q) = t^d \alpha_V(q)$ so d has to be equal to 0, but α need not be linear. ♢

Remark 1.2.9. Now let $P = P_d$ still be homogeneous, but $\alpha : B \times Q \to P$, where Q is any pure polynomial functor and B is an affine variety. Write $Q = Q_{<d} \oplus Q_d \oplus Q_{>d}$. Then, by a similar argument as above, we can write α as

$$\alpha_V(b, q_{<d}, q_d, q_{>d}) = \alpha_{1,V}(b, q_{<d}) + \alpha_{2,V}(b, q_d)$$

where α_2 is of the same form as the linear transformation in the previous remark, except that the coefficients $A_{\lambda ij}$ are of the form $f_{\lambda ij}(b)$, where $f_{\lambda ij} \in K[B]$. ♢

Remark 1.2.10. We will also use the following observation: If $\alpha : B \times Q \to A \times P$ is a polynomial transformation (where A, B are affine varieties, P, Q are pure), then we can split α into two transformations

$$\alpha^{(0)} : B \to A \text{ and } \alpha^{(1)} : B \times Q \to P$$

such that

$$\alpha_V(b, q) = (\alpha^{(0)}(b), \alpha_V^{(1)}(b, q))$$

since the "A"-part of $\alpha_V(b, q)$ only depends on b and not on q. ♢

1.3 Important Results

We collect a few results about polynomial functors, which will be helpful for us later.

1.3.1 Topological Noetherianity

The following result by Draisma is fundamental to the theory of polynomial functors:

Theorem 1.3.1 ([Dra19]). *Let P be a polynomial functor over any infinite field. Any descending chain of closed subsets of P*

$$P \supseteq X_1 \supseteq X_2 \supseteq X_3 \supseteq \dots$$

stabilizes, i.e. there exists $N \in \mathbb{N}$ such that $X_N = X_{N+1} = X_{N+2} = \dots$.

This theorem implies that for any closed subset, there exists $U \in \mathbf{Vec}$ such that X is completely determined by its instance $X(U)$:

Corollary 1.3.2. *For each closed subset $X \subseteq P$ there exists a vector space $U \in \mathbf{Vec}$ with the property that for all $V \in \mathbf{Vec}$ we have*

$$X(V) = \{v \in P(V) | \forall \varphi : V \to U, P(\varphi)v \in X(U)\}$$

We say that the equations of X pull back from the equations of $X(U)$, or X is determined by $X(U)$, or, simply, X is determined by U.

Proof. For every $n \in \mathbb{N}$, consider the closed subset

$$X_n(V) := \{v \in P(V) | \forall \varphi : V \to K^n, P(\varphi)v \in X(K^n)\}$$

These are indeed closed subsets: It is straightforward to see that they are subsets, and $X_n(V)$ is closed for every n and V, because it is an intersection of the closed sets $P(\varphi)^{-1}(X(K^n))$. Now $X_1 \supseteq X_2 \supseteq \dots$ is a descending chain, so by Theorem 1.3.1 it stabilizes at, say, X_N.

We claim that $X_N = X$ and hence the corollary holds for $U = K^N$. Let $V \in \mathbf{Vec}$ and identify V with K^n, where $n = \dim(V)$. The inclusion $X(K^n) \subseteq X_N(K^n)$ is immediate. If $n < N$, then also $X(K^n) = X_n(K^n) \supseteq X_N(K^n)$. If $n \geq N$, then $X(K^n) = X_n(K^n) = X_N(K^n)$. □

Remark 1.3.3. Noetherianity also implies that a closed subset is the union of finitely many closed irreducible components (i.e. inclusion-wise maximal irreducible subsets). ♢

1.3.2 Shifting

Definition 1.3.4. Any fixed $U \in \mathbf{Vec}$ defines a polynomial functor Sh_U: $V \mapsto U \oplus V$, $\varphi \mapsto \mathrm{id}_U \oplus \varphi$. If P is a polynomial functor, then $\mathrm{Sh}_U P := P \circ \mathrm{Sh}_U$ is also a polynomial functor, called the *shift over U* of P. We also write $\mathrm{Sh}_U X := X \circ \mathrm{Sh}_U$ for subsets $X \subseteq P$, and, for polynomial transformations $\alpha : Q \to P$, $\mathrm{Sh}_U \alpha := \alpha_{U \oplus V} : \mathrm{Sh}_U Q \to \mathrm{Sh}_U P$. ♢

The concept of shifting is useful due to the following theorem, called the Shift Theorem:

Theorem 1.3.5 ([BDES21], Theorem 5.1.). *Let $X \subseteq P$ a closed subset that is not of the form $\widetilde{X} \times P_d$ (where P_d is the highest-degree part of P). Then there exist a vector space U and a nonzero polynomial $h \in K[P(U)]$, such that*

$$\mathrm{Sh}_U(X)[1/h] := \{p \in X(U \oplus V) : h(p) \neq 0\}$$

(where h is regarded as a polynomial on $P(U \oplus V)$ via the map $P(\pi_U) : P(U \oplus V) \to P(U)$, where π_U is the standard projection), is isomorphic to $B \times P'$, where B is an affine variety, and P' is a pure polynomial functor with $P' < P_{\geq 1}$.

(Note that the reference [BDES21] is written in the language of GL-varieties. A full translation to our language is given in [Bik20, Section 1.3.], and we will also give a small introduction below in Section 2.4.2.)

This theorem will allow us to use induction on the order of polynomial functors, by identifying big subsets with subsets in smaller polynomial functors.

The following lemma is used in the proof of the Shift Theorem, and will also be useful for us. Just like a univariate polynomial can be shifted over a constant, and then its leading term does not change, a polynomial functor can be shifted over a constant vector space, and its top-degree part does not change:

Lemma 1.3.6 ([Dra19], Lemma 14)**.** *For any polynomial functor P of degree d, $\mathrm{Sh}_U P$ is a polynomial functor of degree d, and their degree-d-parts are canonically isomorphic.*

Proof Sketch. The canonical isomorphisms are the respective restrictions of $P(\iota_V) : P(V) \to P(U \oplus V)$, where $\iota_V : V \to U \oplus V, v \mapsto (0, v)$ and $P(\pi_V) : P(U \oplus V) \to P(V)$, where $\pi_V : U \oplus V \to V, (u, v) \mapsto v$. □

1.3.3 Unirationality

We have seen that images of polynomial transformations are always subsets. The following unirationality theorem says that for closed subsets a sort of converse is true:

Theorem 1.3.7. *Let P be a polynomial functor, and $X \subseteq P$ a closed subset that is not of the form $A \times P_{\geq 1}$ for some affine variety A. Then there exist finitely many polynomial transformations $\alpha^{(j)} : B_j \times Q_j \to P$ (with B_j irreducible and closed, $Q_j < P_{\geq 1}$) such that $X = \bigcup_j \mathrm{im}(\alpha^{(j)})$.*

Proof. See either Theorem 4.2.5. in [Bik20] or, for a more precise statement but in the language of GL-Varieties, Proposition 5.6. in [BDES21]. □

Note that we certainly have to allow for B_j to be affine varieties, and not full affine spaces, otherwise this theorem would imply that all affine varieties are unirational, which is well-known to be wrong.

1.3.4 The Embedding Theorem

The Embedding Theorem is the main lemma for the theorems in the previous three paragraphs. It says the following:

Theorem 1.3.8. *Let $X \subseteq P$ a closed subset, R an irreducible subfunctor of P_d (where P_d is the highest-degree part of P), $U \in \mathbf{Vec}$, $f \in K[P(U)]$ an equation for $X(U)$, and h a partial derivative of f with respect to some coordinate in $R(U)$. Then the projection map*

$$\pi : \mathrm{Sh}_U P \to (\mathrm{Sh}_U P)/R$$

restricts to an isomorphism from $\mathrm{Sh}_U X[1/h]$ to a closed subset of $((\mathrm{Sh}_U P)/R)[1/h]$.

Proof. This theorem is implicitly proven in [Dra19], and explained more thoroughly in [BDES21, Theorem 4.1.]. □

This is the theorem that we will adapt to the finite field case in Chapter 4.

Example 1.3.9. Let $P = R = S^2$, X the subset of rank 1 symmetric matrices, $U = K^2$, $f = x_{11}x_{22} - x_{12}^2 \in K[P(U)] = K[x_{11}, x_{12}, x_{22}]$ and $h = x_{11} = \partial f/\partial x_{22}$. Note that, writing $V = K^n$,

$$\mathrm{Sh}_U P(K^n) = \left\{ \begin{pmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1,n+2} \\ x_{12} & x_{22} & x_{23} & & \\ x_{13} & x_{23} & x_{33} & & \\ \vdots & & & \ddots & \\ x_{1,n+2} & & & & x_{n+2,n+2} \end{pmatrix} \right\} \cong S^2(U) \oplus (T(K^n))^{\oplus 2} \oplus S^2(K^n).$$

The Embedding Theorem says that the projection map which deletes the $S^2(V)$-part,

$$\pi : \mathrm{Sh}_U P \to (\mathrm{Sh}_U P)/R \cong S^2(U) \oplus T^{\oplus 2}$$

restricts to an isomorphism from $\mathrm{Sh}_U X[1/h]$ (where $\mathrm{Sh}_U X[1/h](K^n)$ is the set of symmetric rank 1 matrices with $x_{11} \neq 0$) to a closed subset of $((\mathrm{Sh}_U P)/R)[1/h]$. This is easy to see in this example, because we can find an inverse: we can recover the elements x_{ij} $(i, j > 2)$ by $x_{ij} = x_{1i}x_{1j}/x_{11}$, since $x_{11} \neq 0$. ♢

Chapter 2

Implicitisation and Parameterisation in Polynomial Functors

2.1 Introduction

In this chapter, K is an algebraically closed field of characteristic 0.

2.1.1 Implicitisation

Let P, Q be polynomial functors and $\alpha : Q \to P$ be a polynomial transformation. It is immediate that $\mathrm{im}(\alpha)$ is a subset of P (given by $\mathrm{im}(\alpha)(V) := \alpha_V(Q(V))$), by Proposition 1.2.4, $X := \overline{\mathrm{im}(\alpha)}$ is a (closed) subset of P, and by Corollary 1.3.2, there exists a vector space U such that the equations for $X(U)$ pull back to the set-theoretic defining equations for $X(V)$ for all V.

The main goal of this chapter is to find an algorithm **implicitise** that computes this vector space U. Note that even though the input, the polynomial transformation α, consists of infinitely many polynomial maps, it actually can be given by finite information: Since by Lemma 1.1.9, the polynomial functor Q is generated by its instance $Q(K^d)$, where d is the degree of Q, α is completely determined by its instance α_{K^d}.

The naive approach to finding this algorithm is to calculate, starting at say $n = 0$, the equations for $X(K^n)$ with classical methods, then consider the closed subset

$$X_n(V) := \{v \in P(V) | \forall \varphi : V \to K^n, P(\varphi)v \in X(K^n)\},$$

check if $X_n = X$, and if they are not equal, do the same with $n+1$. This is actually precisely what we are going to do, the problem is that there seems to be no easy way to check whether $X_n = X$. We can always calculate equations of X_n for arbitrarily large n, but it is difficult to find a stopping criterion that tells us whether the equations for X_n already fully describe

X or not.

Example 2.1.1. Let $P = T^{\otimes 2}$ and X the closed subset of matrices of rank at most a fixed integer r (this is the image of the polynomial transformation from Example 1.2.7 with $d = 2$). Then, for $n \leq r$, $X(K^n) = P(K^n)$, so $X_n = P$. However, $X(K^{r+1})$ is a proper subset of $P(K^{r+1})$, and it turns out that $X_{r+1} = X$, which in this example can be verified with simple linear algebra, but this is difficult to generalize. ♢

2.1.2 Parameterisation

It turns out that to find this stopping criterion, it will be helpful to give an algorithm to the inverse problem first: Let $X \subseteq P$ be a closed subset, given by the equations for $X(U)$ for some $U \in \mathbf{Vec}$. By Theorem 1.3.7, X can be covered by the images of finitely many polynomial transformations $\alpha^{(j)} : B_j \times Q_j \to P$ (which we actually can pack into one transformation $\alpha : B \times Q \to P$). We will give an algorithm **parameterise** that computes this transformation.

2.1.3 The Stopping Criterion

Back to our algorithm **implicitise**: We were left off to check whether a subset X_n is equal to the subset $X = \overline{\mathrm{im}(\alpha)}$. To do this, we parameterise X_n with **parameterise**, i.e. we find a polynomial transformation $\alpha' : B' \times Q' \to P$ whose image is exactly X_n. Now the problem of determining whether $X_n = X$ is the same as the problem of determining whether the image of α' is equal to the closure of the image of α. The inclusion $\mathrm{im}(\alpha') \supseteq \overline{\mathrm{im}(\alpha)}$ is clear, so we just have to decide on the inclusion "$\subseteq$".

To check this, we pass to infinite dimensions and use a result from [BDES23] that says that this happens if and only if a suitably generic point of $\mathrm{im}(\alpha'_\infty)$ can be reached as the limit of a curve in $\mathrm{im}(\alpha_\infty)$. We will show that this curve, which lives in infinite-dimensional space, can be represented in finite terms, and searched for, on a computer.

If such a curve does not exist, i.e., if $\mathrm{im}(\alpha')$ is not contained in $\overline{\mathrm{im}(\alpha)}$, then this is because we do not see all the equations of X in X_n. In this case, the search for a curve does not terminate. So for our algorithm **implicitise** to terminate, it is necessary to run the search for witness curves in parallel to the search for equations: in each step, n is increased by 1, new equations are computed, and a new curve search is started. We model this behaviour by running the algorithm on countably many parallel processors. Of course, standard results in the theory of computation imply that this algorithm can then also run on an ordinary Turing machine (see e.g. [Ord]).

2.1.4 Computability Issues

We mostly ignore the technicalities of making sure that the inputs fed to our algorithms can actually be handled by a computer. Since all our inputs can be described with finitely many elements of the field K, we can think of them as living in a finitely generated field extension of $\mathbb{Q}$. This kind of field is computable, and there exists an algorithm for factoring polynomials over the field (see [GP08, Appendix B]).

2.2 Parameterisation

2.2.1 The Result

Notation 2.2.1. Recall from Corollary 1.3.2 that a closed subset X of a polynomial functor P is completely determined by an instance $X(U) \subseteq P(U)$ for some vector space U, and $X(U)$, being Zariski closed, is of the form $\{p \in P(U) | f_1(p) = \ldots = f_k(p) = 0\}$ for some polynomials $f_i \in K[P(U)]$. We write

$$X = \mathcal{V}_P(U; f_1, \ldots, f_k)$$

which is given by

$$X(V) = \{p \in P(V) | \forall \varphi : V \to U, i = 1 \ldots k, f_i(P(\varphi)p) = 0\}$$

Of course, if X is a closed subset of some $A \times P$, where A is an affine variety, and P is a pure polynomial functor, we accordingly write $X = \mathcal{V}_{A \times P}(U; f_1, \ldots, f_k)$

The goal of this section is to prove that the unirationality result, Theorem 1.3.7, can be made algorithmic:

Theorem 2.2.2. *There exists an algorithm* **parameterise** *that, on input an affine variety A, a pure polynomial functor P, a finite dimensional vector space U over K, and elements $f_i \in K[P(U)]$ for $i = 1, \ldots, k$, computes $(B; Q; \alpha)$, where B is an affine variety, Q is a pure polynomial functor, and $\alpha : B \times Q \to A \times P$ is a polynomial transformation such that $\mathrm{im}(\alpha) = \mathcal{V}_{A \times P}(U; f_1, \ldots, f_k) =: X$, i.e. for every $V \in \mathbf{Vec}$, $\alpha_V(B \times Q(V)) = X(V)$.*

We consider X as a subset of $A \times P$ instead of just a (possibly non-pure) polynomial functor P to make the recursion in the algorithm easier to explain.

Remark 2.2.3. Note that Theorem 1.3.7 only says that X can be covered by finitely many (i.e. possibly more than one) images of such maps $\alpha^{(i)} : B^{(i)} \times Q^{(i)} \to A \times P$, but it is convenient to just pack these finitely many maps into one big map $(\bigsqcup B^{(i)}) \times (\bigoplus Q^{(i)}) \to A \times P$. The price we have to pay for this is that we no longer can assume that B is irreducible. ♢

2.2.2 Smearing Out Equations

Before proving the theorem, we present an algorithm that computes the equations for any single instance $X(V)$.

Proposition 2.2.4. *There exists an algorithm* **smear** *that on the same input as* **parameterise** *plus a finite-dimensional vector space V, outputs generators in $K[A \times P(V)]$ of the radical ideal of $X(V)$.*

Proof. The algorithm $\mathbf{smear}(A; P; U; f_1, \ldots, f_k; V)$ proceeds as follows: Choose identifications $U = K^m$ and $V = K^n$, and construct the generic matrix

$$Z := \begin{pmatrix} z_{11} & \ldots & z_{1n} \\ \vdots & \ddots & \vdots \\ z_{m1} & \ldots & z_{mn} \end{pmatrix} \in K[(z_{ij})_{ij}]^{mn} \cong K[(z_{ij})_{ij}] \otimes \mathrm{Hom}(V, U)$$

and the generic vector $y = (y_1, \ldots, y_{\dim P(V)})^\top \in K[(y_i)_i] \otimes P(V)$. Then compute $P(Z)y \in K[(z_{ij})_{ij}, (y_i)_i] \otimes P(U)$, substitute $P(Z)y$ into the f_i, and expand as a polynomial in the z_{ij} with coefficients in $K[A][y_1, \ldots, y_{\dim P(V)}] \cong K[A \times P(V)]$. Finally, return generators of the radical of the ideal generated by all these coefficients. The correctness of this algorithm follows from the fact that these coefficients span the same space as the polynomials $f_i \circ P(\varphi) \in K[A \times P(V)]$ where φ runs through all linear maps from V to U.

□

2.2.3 The Parameterisation Algorithm

The algorithm **parameterise** is recursive and proceeds as follows; the algorithmic part is written in normal font, text that will be used in the analysis in italic. The proofs of termination and correctness are below.

(1) If $P = 0$, then compute the variety $B \subseteq A$ defined by $f_1, \ldots, f_k$ and return $(B; 0; B \hookrightarrow A)$, and exit.

(2) Decompose $P = P' \oplus R$ where R is an irreducible subfunctor of the top-degree part of P.

Writing $P = \bigoplus_{i=1}^m S^{\lambda_i}$, R will be equal to some S^{λ_i}, where $|\lambda_i|$ is maximal. Let $x_1, \ldots, x_n$ be a basis of $R(U)^*$. We regard elements in $K[A \times P(U)] \cong K[A \times P'(U)] \otimes K[R(U)] \cong K[A \times P'(U)][x_1, \ldots, x_n]$ as polynomials in $x_1, \ldots, x_n$ with coefficients in $K[A \times P'(U)]$.

(3) Compute

$$(f'_1, \ldots, f'_r) := \mathbf{smear}(A; P; U; f_1, \ldots, f_k; U)$$

and a Gröbner basis $(g_1, \ldots, g_l)$ of the elimination ideal in $K[A \times P'(U)]$ obtained by eliminating all x_i from $f'_1, \ldots, f'_r$.

The elements $f'_1, \ldots, f'_r$ generate the radical ideal of $X(U)$ in $P'(U) \oplus R(U)$. Let $X'(V)$ be the image of $X(V)$ under projection $A \times P(V) \to A \times P'(V)$, we will see that X' is a closed subset of $A \times P'$. Then $g_1, \ldots, g_l$ generate the radical ideal of $X'(U)$ by construction; and, as we will see below in the proof of correctness, we have $X' = \mathcal{V}_{A \times P'}(U; g_1, \ldots, g_l)$.

(4) For each $i = 1 \ldots k$ let $\tilde{f}_i$ be f_i, where every coefficent is replaced by its normal form modulo $g_1, \ldots, g_l$.

This has the effect that all coefficients that vanish identically on $X'(U)$ are set to zero.

(5) If all $\tilde{f}_i$ are zero, then compute

$$(B'; Q'; \alpha') := \textbf{parameterise}(A; P'; U; g_1, \ldots, g_l),$$

output $(B', Q' \oplus R, \alpha' \times \mathrm{id}_R)$, and exit.

(6) Pick the minimal i for which $\tilde{f}_i$ is nonzero and let x_j be a variable that appears in some monomial in $\tilde{f}_i$ with a nonzero coefficient.

There cannot be degree-0 elements among the $\tilde{f}_i$, because these would lie in the elimination ideal and hence have been reduced to zero.

(7) Compute the partial derivative $h := \partial \tilde{f}_i / \partial x_j \in K[A \times P'(U)][x_1, \ldots, x_n]$.

By construction, this h is nonzero and its coefficients, which are a subset of the coefficients of $\tilde{f}_i$ up to some positive integer scalars, do not lie in the ideal generated by $g_1, \ldots, g_l$.

(8) Compute

$$(B'; Q'; \alpha') := \textbf{parameterise}(A; P; U; h, f_1, \ldots, f_k).$$

(9) Compute $\tilde{P} := (\mathrm{Sh}_U P)/R$ via [FH91, Exercise 6.11], let P'' be the pure part of $\tilde{P}$, and compute $A'' := (A \times P(U))[1/h]$

So we have $(A \times \mathrm{Sh}_U P)[1/h] = A'' \times (P'' \oplus R)$.

(10) Compute

$$(f''_1, \ldots, f''_s) := \textbf{smear}(A; P; U; f_1, \ldots, f_k; U \oplus U)$$

and replace each f''_i by its image in $K[A'' \times (P''(U) \oplus R(U))]$ under the map $K[A \times P(U \oplus U)] \to K[A'' \times (P''(U) \oplus R(U))]$ dual to the inclusion

$$A'' \times (P''(U) \oplus R(U)) \cong (A \times \mathrm{Sh}_U P)(U)[1/h] \hookrightarrow A \times P(U \oplus U).$$

The elements $f''_1, \ldots, f''_s$ generate the radical ideal of $(\mathrm{Sh}_U X)(U)[1/h]$ in $A'' \times (P''(U) \oplus R(U))$. As we will see in the proof of correctness, we have

$$(\mathrm{Sh}_U X)[1/h] = \mathcal{V}_{A'' \times (P'' \oplus R)}(U; f''_1, \ldots, f''_s).$$

(11) Using Buchberger's algorithm to eliminate the coordinates on $R(U)$ from $f''_1, \ldots, f''_s$, compute equations $g''_1, \ldots, g''_t$ for the projection of the affine variety $(\mathrm{Sh}_U X)(U)[1/h]$ into $A'' \times P''(U)$.

Recall from Theorem 1.3.8 that the projection $(A'' \times (P'' \oplus R)) \to A'' \times P''$ restricts to a closed embedding from $(\mathrm{Sh}_U X)[1/h]$ to a closed subset X'' of the latter space. We will see below that $X'' = \mathcal{V}_{A'' \times P''}(U; g''_1, \ldots, g''_t)$.

(12) Compute the inverse $\iota : X'' \to (\mathrm{Sh}_U X)[1/h]$.

By Lemma 1.1.9, this inverse is uniquely determined by its instance ι_V with $\dim V = \deg(P)$.

(13) Compute

$$(B''; Q''; \alpha'') := \textbf{parameterise}(A''; P''; U; g''_1, \ldots, g''_t),$$

output

$$(B' \sqcup B'', Q' \oplus Q'', \alpha' \sqcup (\pi \circ \iota \circ \alpha'')),$$

with $\pi : A \times (\mathrm{Sh}_U P) \to A \times P$ given by $\pi_V = \mathrm{id}_A \times P(0_{U,0} \oplus \mathrm{id}_V)$, and exit.

Here α' is regarded as a map $B' \times (Q' \oplus Q'') \to A \times P$ that ignores the argument from Q'', and similarly α'' ignores the component in Q'.

2.2.4 Termination of parameterise

Proof of termination of **parameterise**. Assume for a contradiction that there exists an input on which **parameterise** does not terminate. This would mean that there is an infinite chain of recursive calls to itself.

In the recursive calls in steps (5) and (13), the polynomial functors P' and P'', respectively, are smaller than P in the order from section 1.1.4, whereas P remains the same in the call in step (8). Since the order on polynomial functors is well-founded the infinite chain consists, apart from a finite initial segment, entirely of consecutive calls in step (8).

Now note that, after each such call, $X'(U)$ either remains constant or becomes smaller. As long as it remains constant, i.e. the list $(g_1, \ldots, g_l)$ remains constant, the degree in $x_1, \ldots, x_n$ of the first equation keeps dropping. Hence after finitely many steps, $X'(U)$ becomes strictly smaller. It follows that $X'(U)$ becomes smaller infinitely often, which contradicts Noetherianity of the affine variety $A \times P'(U)$. □

2.2.5 Correctness of parameterise

Proof of correctness of **parameterise**. We now prove that the output of the algorithm is correct.

- If the algorithm exits in step (1), this is immediate.

- If it exits in step (5), then by Lemma 2.2.5 below, $X = X' \times R$, and X' is defined by $g_1, \ldots, g_l$. So, X is parameterised by $\alpha' \times \mathrm{id}_R$ for a parameterisation α' of X'.

- Finally, assume that the algorithm exits in step (13). We need to show that the union of the images of α' and $\pi \circ \iota \circ \alpha''$ equals X.

- First consider α', which is computed in step (8). The closed subset $Y := \mathcal{V}_{A\times P}(U; h, f_1, \ldots, f_k)$ parameterised by α' is clearly contained in X, so $\alpha' : B' \times Q' \to A \times P$ has its image contained in X.

- Next we argue that $\mathcal{V}_{A''\times(P''\oplus R)}(U; f_1'', \ldots, f_s'')$ is precisely $(\mathrm{Sh}_U X)[1/h]$. By construction, $f_1'', \ldots, f_s''$ generate the ideal of $(\mathrm{Sh}_U X)[1/h](U)$ in $A \times P(U \oplus U)[1/h]$. This shows "$\supseteq$".

- To show "$\subseteq$", i.e. that $(\mathrm{Sh}_U X)[1/h]$ is actually determined by U, let V be any finite-dimensional K-vector space and $(a, p) \in (A \times P(U \oplus V))[1/h] \setminus X(U \oplus V)$. We need to show that there exists a linear map $\varphi : V \to U$, such that $(a, P(\mathrm{id}_U \oplus \varphi)p) \notin X(U \oplus U)$.

- Since X is determined by U, we know that there exists a linear map $\psi : U \oplus V \to U$ such that $(a, P(\psi)p) \notin X(U)$. Let

$$\varphi := \psi|_V : V \to U, \varphi(v) = \psi(0, v)$$

  Then

$$\ker(\mathrm{id}_U \oplus \varphi : U \oplus V \to U \oplus U) = \{0\} \oplus \ker(\varphi) \subseteq \ker(\psi)$$

  and hence the linear map ψ factors as $\psi' \circ (\mathrm{id}_U \oplus \varphi)$ for some $\psi' : U \oplus U \to U$. But this implies that $(a, P(\mathrm{id}_U \oplus \varphi)p) \notin X(U \oplus U)$, because otherwise, by the subset property of X, $(a, P(\psi)p) \in X(U)$, which is a contradiction. This concludes the proof that $\mathcal{V}_{A''\times(P''\oplus R)}(U; f_1'', \ldots, f_s'') = (\mathrm{Sh}_U X)[1/h]$.

- Next, by Theorem 1.3.8, the projection $A'' \times (P'' \oplus R) \to A'' \times P''$ restricts to an isomorphism from $(\mathrm{Sh}_U X)[1/h]$ to a closed subset of $A'' \times P''$, and by Lemma 2.2.6 below, the equations of X'' can be pulled back from the equations of $X''(U)$, so we have indeed $X'' = \mathcal{V}_{A''\times P''}(U; g_1'', \ldots, g_t'')$.

- Finally, consider step (13). A straightforward calculation shows that setting $\pi_V := \mathrm{id}_A \times P(0_{U,0} \oplus \mathrm{id}_V)$ does indeed yield a polynomial transformation $A \times \mathrm{Sh}_U P \to A \times P$ that maps $\mathrm{Sh}_U X$ into X. We need to show that if $\alpha'' : B'' \times Q'' \to A'' \times P''$ is a transformation parameterising X'', then $\pi \circ \iota \circ \alpha''$ is a transformation $B'' \times Q'' \to A \times P$ whose image contains all points in X that are not in the subset Y parameterised by α'.

- So let $(a, p) \in X(V) \setminus Y(V)$. Then there exists a linear map $\varphi : V \to U$ such that $h(a, P(\varphi)p) \neq 0$. Let

$$\begin{aligned} \psi : V &\to U \oplus V \\ v &\mapsto (\varphi(v), v). \end{aligned}$$

Then, since $(\mathrm{id}_U \oplus 0_{V,0}) \circ \psi = \varphi$, the point $p' := P(\psi)p \in X(U \oplus V)$ satisfies

$$h(a, p') = h(a, P(\mathrm{id}_U \oplus 0_{V,0})p') = h(a, P(\varphi)p) \neq 0$$

i.e., p' lies in $(\mathrm{Sh}_U X)[1/h]$, which is the image of $\iota \circ \alpha''$. Moreover, since $(0_{U,0} \oplus \mathrm{id}_V) \circ \psi = \mathrm{id}_V$, we have $\pi_V(p') = p$. Hence (a, p) lies in the image of $\pi \circ \iota \circ \alpha''$. This concludes the proof of correctness of **parameterise**.

□

Lemma 2.2.5. *Let P be a pure polynomial functor and $X \subseteq A \times P$ a closed subset. Assume that X is defined by its equations in $K[A \times P(U)]$, let R be a subfunctor of P and set $P' := P/R$. Define X' as the image of X under the projection $\mathrm{id}_A \times \pi : A \times P \to A \times P'$. Assume that the closure of $X'(U)$ is the zero locus of the same polynomials as $X(U)$ (in particular, these polynomials do not contain any variables from $R^*(U)$, so they can be seen as both elements of $K[A \times P'(U)]$ and $K[A \times P(U)]$). Then, X' is actually a closed subset of $A \times P'$, $X = X' \times R$, and X' is defined by its equations in $K[A \times P'(U)]$.*

Proof. By assumption $X(U) = \overline{X'(U)} \times R(U)$, and hence $X(U) = X'(U) \times R(U)$ and $X'(U)$ is closed.

We now show that for all $V \in \mathbf{Vec}$, $X(V) = X'(V) \times R(V)$: Let $(a, p) \in A \times P(V)$, such that $(a, \pi_V(p)) \in X'(V)$. Then for all $\varphi : V \to U$ we have

$$(a, \pi_U(P(\varphi)(p))) = (a, P'(\varphi)(\pi_V(p))) \in X'(U)$$

and hence, $(a, P(\varphi)(p)) \in X(U)$. But since X is defined by its equations in $K[A \times P(U)]$, we have $(a, p) \in X(V)$. This proves $X = X' \times R$.

To show that X' is closed and determined by its instance $X'(U)$, suppose that $(a, p') \in A \times P'(V)$ such that $(a, P'(\varphi)p') \in X'(U)$ for all $\varphi : V \to U$. Pick any $p \in P(V)$ with $\pi_V(p) = p'$. Then the same computation as above shows that $(a, p) \in X(V)$, hence also $(a, p') \in X'(V)$. This shows that $X'(V) = \bigcap_{\varphi: V \to U} P'(\varphi)^{-1}(X'(U))$, so it is closed as an intersection of closed sets, and X' is defined by its equations in $K[A \times P'(U)]$. □

Lemma 2.2.6. *Let P be a pure polynomial functor, R a subfunctor, A an affine variety. Set $Y := A \times P$, $Y' := A \times (P/R)$. Let X be a closed subset of Y such that the projection $\pi : Y \to Y'$ restricts to an isomorphism from X to a closed subset X' of Y'. Let U be a vector space such that X is defined by its equations in $K[Y(U)]$. Then, X' is defined by its equations in $K[Y'(U)]$.*

Proof. By [Bik20, Proposition 1.3.22] there exists a polynomial transformation $\tilde{\psi} : Y' \to Y$ that extends $(\pi|_X)^{-1} : X' \to X$. Now set

$$\psi : Y' = A \times P' \to Y = A \times (P' \oplus R)$$
$$\psi_V(a, p') = (a, p', \pi_{Y(V) \to R(V)} \circ \tilde{\psi}_V(a, p'))$$

where $\pi_{Y(V) \to R(V)}$ is the standard projection. The following two properties apply to ψ:

1. $\pi \circ \psi = \mathrm{id}_{Y'}$
2. $\psi|_{X'} = (\pi|_X)^{-1}$

We claim that we are done now: Let $y' \in Y'(V)$, such that for every linear map $\varphi : V \to U$, $Y'(\varphi)(y') \in X'(U)$, and hence, by property (2), $\psi_U(Y'(\varphi)(y')) \in X(U)$. Since ψ is a morphism, we get

$$\psi_U(Y'(\varphi)(y')) = Y(\varphi)(\psi_V(y')) \in X(U).$$

Since X is defined by its equations in $K[Y(U)]$, we get that $\psi_V(y') \in X(V)$, and hence, with property (1) of ψ, $\pi_V(\psi_V(y')) = y' \in X'(V)$.

□

2.3 Implicitisation

2.3.1 The Result

The goal of this section is to prove the following theorem:

Theorem 2.3.1. *There exists an algorithm* **implicitise** *that, on input finite-dimensional affine varieties A, B, pure polynomial functors P, Q, and a polynomial transformation $\alpha : B \times Q \to A \times P$, computes a $U \in \mathbf{Vec}$ and elements $f_1, \ldots, f_k \in K[A \times P(U)]$ such that $\mathcal{V}_{A\times P}(U; f_1, \ldots, f_k) = \overline{\mathrm{im}(\alpha)} =: X$, i.e. for every $V \in \mathbf{Vec}$, we have*

$$\overline{\alpha_V(B \times Q(V))} = \{(a, p) \in A \times P(V) | \forall i \, \forall \varphi : V \to U : f_i(a, P(\varphi)p) = 0\}$$

2.3.2 The Implicitisation Algorithm

Note that the only difficult part is finding the vector space U, or, writing $U = K^n$, finding the nonnegative integer n, such that X is determined by the instance $X(K^n)$. The idea of the algorithm **implicitise**(A, P, B, Q, α) is for $n = 0, 1, 2, 3, \ldots$ to calculate the equations for $X(K^n)$, use **parameterise** to calculate a parameterisation for the closed subset pulled back from $X(K^n)$ and then check with the below procedure **certify**, whether this closed subset is contained in (and therefore equal to) X.

Proposition 2.3.2. *There exists an algorithm that, on input finite-dimensional varieties A, B, B', pure polynomial functors P, Q, Q' and polynomial transformations $\alpha : B \times Q \to A \times P$ and $\alpha' : B' \times Q' \to A \times P$, has the following behaviour: if for all $V \in \mathbf{Vec}$ we have*

$$\alpha'_V(B' \times Q'(V)) \subseteq \overline{\alpha_V(B \times Q(V))}$$

then **certify**$(A, P, B, Q, \alpha, B', Q', \alpha')$ *terminates and returns "true". Otherwise, it does not terminate.*

The proof of Proposition 2.3.2 is deferred to Section 2.4. The fact that **certify** might not terminate means that we need to run **implicitise** on countably many parallel processors: the one where the original call is handled, plus countably many labelled $0, 1, 2, 3, \ldots$. We can now present the precise steps for **implicitise**(A, P, B, Q, α).

(1) Set $n := 0$.

(2) While no instance of **certify** has returned "true", perform steps (3) - (6):

(3) Set $U := K^n$ and, by classical elimination, compute defining equations $f_{n,1}, \ldots, f_{n,k_n} \in K[A \times P(U)]$ for the image closure of α_U
(*this is a finite-dimensional affine variety*).

(4) Compute $(B_n, Q_n, \alpha_n) :=$ **parameterise**$(A; P; U; f_{n,1}, \ldots, f_{n,k_n})$.

(5) On the n-th processor, start **certify**$(A, P, B, Q, \alpha, B_n, Q_n, \alpha_n)$.

(6) Set $n := n + 1$.

(7) if the m-th processor has returned "true", then return $(K^m; f_{m,1}, \ldots, f_{m,k_m})$.

One could imagine to reduce these infinitely many processors to one processor as follows: First, run steps (3) to (5) for $n = 0$ for 1 minute. Then run these steps for $n = 0$ and $n = 1$ for 2 minutes each, then for $n = 0, 1, 2$ for 3 minutes each, and so on and so on, until one of the **certify**-procedures terminates.

2.3.3 Correctness and Termination of implicitise

Proof of Theorem 2.3.1. By Corollary 1.3.2, there exists a value of n such that the tuple $(K^n; f_{n,1}, \ldots, f_{n,k_n})$ computed in iteration n is a correct output for **implicitise**. It then follows that α_n, which, by virtue of **parameterise**, parameterises the closed subset of $A \times P$ defined by $f_{n,1}, \ldots, f_{n,k_n}$, has its image contained in the closure of the image of α. Hence, by Proposition 2.3.2, the n-th call to **certify** terminates and returns "true". This shows that **implicitise** terminates.

Next, when it terminates with output $(K^m; f_{m,1}, \ldots, f_{m,k_m})$, then this is because the image of α_m, which equals the closed subset of $A \times P$ defined by $f_{m,1}, \ldots, f_{m,k_m}$, is contained in the image closure of α. Since, conversely, the image closure of α is contained in the closed subset defined by $f_{m,1}, \ldots, f_{m,k_m}$, the output is correct. □

2.4 Certifying Inclusion of Image Closures

2.4.1 An Instructive Example

Example 2.4.1. Let $\alpha : (S^1)^{\oplus 2} \to S^3$ be the morphism defined by $\alpha_V(u, v) = u^3 + v^3$. The image closure of α is the set of symmetric three-tensors of *border Waring rank* at most 2. On

the other hand, let $\beta : (S^1)^{\oplus 2} \to S^3$ be the morphism defined by $\beta_V(u,v) = 6u^2v$. Then we have

$$\beta_V(u,v) = 6u^2v = \lim_{t\to 0}[(t^2v + t^{-1}u)^3 + (t^2v - t^{-1}u)^3] = \lim_{t\to 0} \alpha_V(t^2v + t^{-1}u, t^2v - t^{-1}u)$$

and this implies that $\mathrm{im}(\beta) \subseteq \overline{\mathrm{im}(\alpha)}$. However, $\mathrm{im}(\beta) \not\subseteq \mathrm{im}(\alpha)$ as it is well known that the Waring rank of cubics of the form u^2v with u, v linearly independent vectors is equal to 3 ([CCG12]). ♢

This example shows a *certificate* for $\mathrm{im}(\beta) \subseteq \overline{\mathrm{im}(\alpha)}$, namely that β is a limit of a composition $\alpha \circ \gamma_t$ with

$$\gamma_{t,V} : (S^1)^{\oplus 2}(V) \to (S^1)^{\oplus 2}(V), (u,v) \mapsto (t^2v + t^{-1}u, t^2v - t^{-1}u).$$

Roughly speaking, whenever $\mathrm{im}(\beta)$ is contained in $\overline{\mathrm{im}(\alpha)}$, where β, α are morphisms into $A \times P$, with A an affine variety and P a pure polynomial functor, there is a certificate of this inclusion such as the one above - see below for the precise statement. However, we are not aware of any a priori lower bound on the (negative) exponents of t in such a certificate. This is why, in Proposition 2.3.2, the procedure **certify** does not terminate when no certificate exists.

2.4.2 An Excursion to Infinite Dimensions

We collect some material on GL-varieties. The results stated here appear in [BDES21, BDES23] or can directly be derived from results there.

Given an affine variety A and a pure polynomial functor P, we construct the inverse limit $\lim_{\leftarrow n}(A \times P(K^n)) = A \times P_\infty$, where the projections $P(K^{n+1}) \to P(K^n)$ are of the form $P(\pi)$ with π the standard projection $K^{n+1} \to K^n$. Rather than as a set of K-valued points, we will regard $A \times P_\infty$ as a reduced, affine K-scheme, namely, the spectrum of the ring $K[A] \otimes_K R$, where R is the symmetric algebra of the countable-dimensional vector space $\lim_{n\to\infty} P(K^n)^*$. The group $\mathrm{GL} := \bigcup_{n=0}^{\infty} \mathrm{GL}_n(K)$ acts on $A \times P_\infty$ by means of automorphisms and $A \times P_\infty$ is a GL-*variety* in the sense of [BDES21]. More generally, if X is a closed subset of a polynomial functor, then the inverse limit X_∞ of all $X(K^n)$ is a GL-variety. The association $X \mapsto X_\infty$ is an equivalence of categories with the category of affine GL-varieties, which sends a morphism $\alpha : X \to Y$ to a GL-equivariant morphism $\alpha_\infty : X_\infty \to Y_\infty$ of affine schemes over K.

Let $\alpha : B \times Q \to A \times P$ and $\alpha' : B' \times Q' \to A \times P$ be morphisms as in Proposition 2.3.2. Then the following two statements are equivalent:

(1) $\overline{\mathrm{im}(\alpha_\infty)} \supseteq \mathrm{im}(\alpha'_\infty)$ and

(2) $\overline{\mathrm{im}(\alpha_V)} \supseteq \mathrm{im}(\alpha'_V)$ for all $V \in \mathbf{Vec}$.

It is (2) which we want to certify in **certify**$(A, P, B, Q, \alpha, B', Q', \alpha')$. From now on, we assume that B, B' are irreducible - in the procedure **certify** we will reduce to this case.

Now [BDES23] contains the following useful criterion for (1): Let b' be the generic point of B' and let $q' \in Q'_\infty(K)$ be a point whose GL-orbit is dense in Q_∞ (such points exist, see [BDES21]). Let $x := \alpha'_\infty(b', q')$. This is an Ω-point of $A \times P_\infty =: X_\infty$, where $\Omega = K(B')$, and the GL-orbit of x is dense in $\mathrm{im}(\alpha'_\infty)$. Write $Y_\infty := B \times Q_\infty$.

Theorem 2.4.2. *We have* $\mathrm{im}(\alpha'_\infty) \subseteq \overline{\mathrm{im}(\alpha_\infty)}$ *if and only if there exists a finite-dimensional field extension* $\widetilde{\Omega}$ *of* Ω *and a bounded* $\widetilde{\Omega}((t))$*-point* $y(t)$ *of* Y_∞ *such that* $\lim_{t\to 0} \alpha_\infty(y(t)) = x$.

This follows from [BDES23, Theorem 6.6]. Here "bounded" means that the exponents of t in the countably many coordinates of the Q_∞-component of $y(t)$ are uniformly bounded from below. This ensures that $\alpha_\infty(y(t))$ is a well-defined $\widetilde{\Omega}((t))$-point of X_∞. The theorem says that we can choose $y(t)$ such that $\alpha_\infty(y(t))$ is in fact an $\widetilde{\Omega}[[t]]$-point of X_∞ and that setting t to zero yields x.

The procedure **certify** should certify the existence of $y(t)$. To this end, we will narrow down the space in which to search for $y(t)$ to an increasing chain of finite-dimensional varieties. First, we will show that $y(t)$ needs not use more of Q_∞ than can be obtained by applying morphisms to q'. To do so (see Proposition 2.4.7 below), we now introduce *systems of variables* in Schur functors.

2.4.3 Systems of variables in Schur functors

Fix a field extension Ω of K. For every nonempty partition λ, S^λ_∞ is an affine scheme over K whose Ω-valued points form an Ω-vector space of uncountable dimension. Let $V_\lambda \subseteq S^\lambda_\infty(\Omega)$ be the set of all points s for which there exist an integer k, partitions $\mu_1, \dots \mu_k$ with $0 < |\mu_i| < |\lambda|$, an Ω-valued point α of $\mathbf{Map}(S^{\mu_1} \oplus \cdots \oplus S^{\mu_k}, S^\lambda)$, and an Ω-valued point q of $S^{\mu_1}_\infty \oplus \cdots \oplus S^{\mu_k}_\infty$ such that $\alpha_\infty(q) = s$ (where $\mathbf{Map}(P', P)$ denotes the affine space of polynomial transformations from P' to P).

Example 2.4.3. If $\lambda = (2)$, then $S^\lambda_\infty(\Omega)$ is the space of infinite-by-infinite symmetric matrices with entries in Ω, and V_λ is the subspace of matrices of finite rank. ♢

Now, V_λ is a proper Ω-vector subspace of $S^\lambda_\infty(\Omega)$, and we choose any Ω-basis $(\xi_{\lambda,i})_{i\in I_\lambda}$ of a vector space complement to V_λ in $S^\lambda_\infty(\Omega)$, where I_λ is a (typically uncountable) index set. We call the $\xi_{\lambda,i}$ *variables*. We choose these variables for every λ and write ξ for the resulting uncountable tuple; ξ is what we call a *system of variables* (over Ω) for all Schur functors. If f is an Ω-valued point of $\mathbf{Map}(S^{\mu_1} \oplus \cdots \oplus S^{\mu_k}, S^\lambda)$ and we fix indices $i_1 \in I_{\mu_1}, \dots, i_k \in I_{\mu_k}$, then we will write $f(\xi)$ for $f(\xi_{\mu_1,i_1}, \dots, \xi_{\mu_k,i_k})$. This is slight abuse of notation, since it is not apparent from the formula $f(\xi)$ *which* indices were chosen, but the notation is compatible with the notation $f(x)$ for a polynomial that uses finitely many of an uncountable set of variables x.

Remark 2.4.4. We need to fix the extension Ω first and then choose the system of variables. It is not true that a system of variables chosen over K is also a system of variables over field extensions of K, as typically $S^\lambda_\infty(K) \otimes_K \Omega \neq S^\lambda_\infty(\Omega)$. ♢

The following proposition follows directly from the material in [BDES21].

Proposition 2.4.5. *Let $S = S^{\mu_1} \oplus \cdots \oplus S^{\mu_k}$ be a pure polynomial functor, and let $s = (s_1, \ldots, s_k) \in S_\infty(\Omega)$. Then the s_i can be chosen as part of a system of variables if and only if the* GL*-orbit of s is dense in $S_\infty(\Omega)$.*

The following theorem expresses that the variables in a system are independent and generate all vectors in all Schur functors.

Theorem 2.4.6. *Fix a field extension Ω of K and a system of variables ξ over Ω for all Schur functors. Then for every nonempty partition λ and any $p \in S^\lambda_\infty(\Omega)$, there exist partitions $\mu_1, \ldots, \mu_k$, an Ω-valued point f of $\mathbf{Map}(S^{\mu_1} \oplus \cdots \oplus S^{\mu_k}, S^\lambda)$ and variables ξ_{μ_j, i_j} for $j = 1, \ldots, k$ and $i_j \in I_{\mu_j}$ such that $p = f_\infty(\xi_{\mu_1,i_1}, \ldots, \xi_{\mu_k,i_k})$. Moreover, if f really depends on all ξ_{μ_j,i_j} in the sense that replacing one of them by zero changes the outcome, then, up to permutations of $\{1, \ldots, k\}$, the partitions μ_j, the variables ξ_{μ_j,i_j}, and f are unique.*

Proof. The existence of f follows by induction on $|\lambda|$: we may write p as

$$p = c_1 \xi_{\lambda,i_1} + c_2 \xi_{\lambda,i_2} + \cdots + c_l \xi_{\lambda,i_l} + \tilde{p}$$

with $\tilde{p} \in V_\lambda$ and (unique) $i_1, \ldots, i_l \in I_\lambda$ and $c_1, \ldots, c_l \in \Omega$. Now $\tilde{p} = \alpha_\infty(q)$, for a suitable α, where q is an Ω-valued point of $S^{\mu_1}_\infty \oplus \cdots \oplus S^{\mu_k}_\infty$ with $0 < |\mu_i| < |\lambda|$ for all i. By the induction hypothesis, the components $q_i, i = 1, \ldots, k$ of q are of the form $f_{i,\infty}(\xi)$, and then p equals

$$\alpha_\infty(f_{1,\infty}(\xi), \ldots, f_{k,\infty}(\xi)) + \sum_{i=1}^{l} c_i \xi_{\lambda,i},$$

so that $f := \alpha(f_1, \ldots, f_k) + \sum_{i=1}^{l} c_i \, \mathrm{id}_{S^\lambda}$ does the trick for the obvious choice of variables.

For uniqueness, it suffices to show that if f is a nonzero Ω-valued point of $\mathbf{Map}(S^{\mu_1} \oplus \ldots \oplus S^{\mu_k}, S^\lambda)$ and $\xi_{\mu_1,i_1}, \ldots, \xi_{\mu_k,i_k}$ are distinct variables, then $f_\infty(\xi) \neq 0$. Indeed, by Proposition 2.4.5, the GL-orbit of $(\xi_{\mu_1,i_1}, \ldots, \xi_{\mu_k,i_k})$ is dense, and f_∞ is GL-equivariant, so that $f_\infty(\xi) = 0$ implies that $f = 0$, which is a contradiction. □

2.4.4 Narrowing down the search for $y(t)$

In this section, we retain the notation from Section 2.4.2. The following diagram represents the situation:

$$\begin{array}{ccc} & & (b', q') \in B' \times Q'_\infty \\ & & \Big\downarrow{\scriptstyle \alpha'_\infty} \\ y(t) = (b(t), q(t)) \in (B \times Q_\infty)(\widetilde{\Omega}((t))) & \xrightarrow{\alpha_\infty} & x = (a, p) \in (A \times P_\infty)(\Omega) \end{array}$$

where $\Omega = K(B')$, b' is the generic point of B', q' is a K-point of Q'_∞ with dense GL-orbit, and we want to certify the existence of $y(t)$, defined over a finite extension $\widetilde{\Omega}$ of Ω, such that $\lim_{t\to 0} \alpha_\infty(y(t)) = x$.

Proposition 2.4.7. *If a $y(t)$ as in Theorem 2.4.2 exists, then it can be chosen of the form $(b(t), \gamma(t)_\infty(q'))$ with $b(t) \in B(\widetilde{\Omega}((t)))$ and $\gamma(t)$ a $\widetilde{\Omega}[t, t^{-1}]$-valued point of the finite-dimensional affine space $\mathbf{Map}(Q', Q)$.*

Proof.

- Write $y(t) = (b(t), q(t))$. First, terms in $q(t)$ of sufficiently high degree in t do not contribute to $\lim_{t\to 0} \alpha_\infty(b(t), q(t))$, so we may truncate $q(t)$ and assume that it is a finite sum $\sum_{d=m_1}^{m_2} t^d q_d$ with each $q_d \in Q_\infty(\widetilde{\Omega})$.

- Now write $Q' = S^{\lambda_1} \oplus \cdots \oplus S^{\lambda_k}$, where the λ_i are partitions. Accordingly, decompose $q' = (q'_1, \ldots, q'_k)$ with $q'_i \in S^{\lambda_i}_\infty$. Over the field extension $\widetilde{\Omega}$ of K, choose a system of variables ξ in such a manner that $q'_1, \ldots, q'_k$ are among these variables; this can be done by Proposition 2.4.5 because $\mathrm{GL}\cdot q'$ is dense in Q'_∞. Also, by Theorem 2.4.6, we have $q_d = f_{d,\infty}(\xi)$ for all d, where f_d is an (essentially unique) morphism into Q with coefficients in $\widetilde{\Omega}$.

- Recall from Remark 1.2.10 that α splits as $\alpha^{(0)} : B \to A$ and $\alpha^{(1)} : B \times Q \to P$, and similarly for α'. The limit $\lim_{t\to 0} \alpha^{(1)}_\infty(b(t), q(t))$ equals

$$g_\infty(q_{m_1}, \ldots, q_{m_2}) = g_\infty(f_{m_1,\infty}(\xi), \ldots, f_{m_2,\infty}(\xi))$$

  for some $\widetilde{\Omega}$-point g of $\mathbf{Map}(Q^{\oplus m_2 - m_1 + 1}, P)$. On the other hand, by the choice of $y(t)$, it equals $(\alpha')^{(1)}_\infty(q')$. In the latter expression, only the variables $q'_1, \ldots, q'_k$ appear. By the uniquess statement in Theorem 2.4.6, the same must apply to $g_\infty(f_{m_1,\infty}(\xi), \ldots, f_{m_2,\infty}(\xi))$.

- Therefore, replacing each q_d by $\tilde{q}_d := f_{d,\infty}(q', 0)$, where all variables not among the variables $q'_1, \ldots, q'_k$ are set to zero, yields a $\tilde{y}(t)$ with the same property as $y(t)$ that $\lim_{t\to 0} \alpha_\infty(\tilde{y}(t)) = x$. Now $\gamma(t) := \sum_d t^d f_d(\cdot, 0)$ is the desired $\widetilde{\Omega}[t, t^{-1}]$-valued point of $\mathbf{Map}(Q', Q)$.

□

Note that $(\alpha')^{(1)}$ can be regarded an Ω-point of $\mathbf{Map}(Q', P)$. Similarly, $\alpha^{(1)}(b(t), \cdot)$ can be regarded an $\widetilde{\Omega}((t))$-point of $\mathbf{Map}(Q, P)$.

Lemma 2.4.8. *A point $(b(t), \gamma(t)_\infty(Q'))$ as in Proposition 2.4.7 satisfies the property*

$$\lim_{t\to 0} \alpha_\infty(b(t), \gamma(t)_\infty(q')) = \alpha'(b', q') =: (a, p) \in (A \times P_\infty)(\Omega)$$

if and only if, first, $\lim_{t\to 0} \alpha^{(0)}(b(t)) = a$ and, second, the $\widetilde{\Omega}((t))$-point $\alpha^{(1)}(b(t), \cdot) \circ \gamma(t)$ of $\mathbf{Map}(Q', P)$ satisfies

$$\lim_{t\to 0} \alpha^{(1)}(b(t), \cdot) \circ \gamma(t) = (\alpha')^{(1)};$$

an equality of $\widetilde{\Omega}$-points in $\mathbf{Map}(Q', P)$.

Proof. The statement "if" is immediate, by substituting q'; and the statement "only if" follows from the fact that the GL-orbit of q' is dense in Q'_∞. □

2.4.5 Greenberg's approximation theorem

We have almost arrived at a countable chain of finite-dimensional varieties in which we can look for $y(t)$. The only problem is that the point $b(t) \in B(\widetilde{\Omega}((t)))$ does not yet have a finite representation. For concreteness, assume that B is given by a prime ideal $I = (f_1, \ldots, f_r)$ in $K[x_1, \ldots, x_m]$, A is embedded in K^n, and $\alpha^{(0)} : B \to A$ is the restriction of some polynomial map $\alpha^{(0)} : K^m \to K^n$.

Then $b(t)$ is an m-tuple in $\widetilde{\Omega}((t))^m$, and together with $\gamma(t)$ it is required to satisfy the following properties from Lemma 2.4.8:

(i) $f_i(b(t)) = 0$ for $i = 1, \ldots, r$;

(ii) $\lim_{t\to 0} \alpha^{(0)}(b(t)) = a$;

(iii) and $\lim_{t\to 0} \alpha^{(1)}(b(t), \cdot) \circ \gamma(t) = (\alpha')^{(1)}$.

Suppose that we fix a lower bound $-d_1$, with $d_1 \in \mathbb{Z}_{\geq 0}$, on the exponents of t appearing in $a(t)$ or in $\gamma(t)$. From the data of α and d_1, one can compute a bound $d_2 \in \mathbb{Z}_{\geq 0}$ such that the validity of (ii) and (iii) do not depend on the terms in $b(t)$ or $\gamma(t)$ with exponents $> d_2$. However, (i) does depend on all (typically infinitely many) terms of $b(t)$. Here Greenberg's approximation theorem comes to the rescue. As this theorem requires formal power series rather than Laurent series, we put $\tilde{b}(t) := t^{d_1} b(t)$. Accordingly, replace each f_i by $\tilde{f}_i := t^e f_i(t^{-d_1}x_1, \ldots, t^{-d_1}x_n)$ where e is large enough such that all coefficients of $\tilde{f}_i$ for all i are in $\widetilde{\Omega}[[t]]$. Note that $b(t)$ is a root of all f_i if and only if $\tilde{b}(t)$ is a root of all $\tilde{f}_i$.

Theorem 2.4.9 (Greenberg, [Gre66])**.** *There exist numbers $N_0 \geq 1, c \geq 1, s \geq 0$ such that for all $N \geq N_0$ and $\bar{b}(t) \in \widetilde{\Omega}[[t]]^n$ with $\tilde{f}_i(\bar{b}(t)) \equiv 0 \mod t^N$ for all $i = 1, \ldots, r$ there exists a $\tilde{b}(t) \in \widetilde{\Omega}[[t]]^n$ such that $\tilde{b}(t) \equiv \bar{b}(t) \mod t^{\lceil \frac{N}{c} \rceil - s}$ and moreover $f_i(\tilde{b}(t)) = 0$ for all i. Moreover, N_0, c, s can be computed from $\tilde{f}_1, \ldots, \tilde{f}_r$.*

In fact, the computability, which is crucial to our work, is only implicit in [Gre66]; it is made explicit in the overview paper [Ron18].

Corollary 2.4.10. *There exist nonnegative integers d_2, N_1, which can be computed from d_1 and $f_1, \ldots, f_r$, α, such that the following statements are equivalent:*

1. *A pair $b(t), \gamma(t)$ with properties (i)–(iii) exists that has no exponents of t smaller than $-d_1$;*
2. *A pair $b(t), \gamma(t)$ exists with all exponents of t in the interval $\{-d_1, \ldots, d_2\}$ that satisfies (ii) and (iii), and that satisfies (i) modulo t^{N_1}.*

Proof.

- The implication (1) $\Rightarrow$ (2) holds for any choice of N_1 if d_2 is chosen large enough so that the terms of $b(t), \gamma(t)$ with degree $> d_2$ in t do not affect (ii), (iii), and do not contribute to the terms of degree $< N_1$ in $f_i(b(t))$ for any i.

- For the converse, first we compute $\tilde{f}_i$ and e as above; they depend on the choice of d_1. Then we compute N_0, c, s as in Greenberg's theorem. Compute $N_1 \geq N_0 - e$ such that terms in $b(t), \gamma(t)$ in which t has exponent at least $\lceil \frac{N_1+e}{c} \rceil - s - d_1$ do not affect properties (ii), (iii), and then compute d_2 as in the first paragraph.

- Given a pair $b(t), \gamma(t)$ as in (2), set $\bar{b}(t) := t^{d_1} b(t)$. Then, for each i,

$$\tilde{f}_i(\bar{b}(t)) = t^e f_i(b(t)) \equiv 0 \mod t^{N_1+e}.$$

  Then, since $N_1+e \geq N_0$, Greenberg's theorem yields $\tilde{b}(t) \in \widetilde{\Omega}[[t]]^n$ such that $\tilde{f}_i(\tilde{b}(t)) = 0$ for all i and such that

$$\tilde{b}(t) = \bar{b}(t) \mod t^{\lceil \frac{N_1+e}{c} \rceil - s}.$$

  Now set $b_1(t) := t^{-d_1}\tilde{b}(t)$, so that $f_i(b_1(t)) = 0$ for all i - this is property (i) - and

$$b_1(t) \equiv b(t) \mod t^{\lceil \frac{N_1+e}{c} \rceil - s - d_1}.$$

  Since the terms of $b(t)$ with exponent of degree at least $\lceil \frac{N_1+e}{c} \rceil - s - d_1$ do not affect (ii) and (iii), the pair $b_1(t), \gamma(t)$ also satisfy these conditions.

□

2.4.6 The procedure certify

To compute **certify**$(A, P, B, Q, \alpha, B', Q', \alpha')$, we proceed as follows. For convenience, we again assume that we have sufficiently many processors working in parallel.

1. If B and B' are not both irreducible, decompose B into irreducible components B_i and B' into irreducible components B'_j, and assign the computation of

$$\textbf{certify}(A, P, B_i, Q, \alpha|_{B_i \times Q}, B'_j, Q', \alpha'|_{B'_j \times Q'})$$

   for all i, j to distinct processors. As soon as for each j there exists at least one i such that the computation returns "true", return "true".

   So in what follows we may assume that B, B' are irreducible. They are given by prime ideals $I \subseteq K[x_1, \ldots, x_n]$ and $J \subseteq K[y_1, \ldots, y_m]$, respectively.

2. Let $f_1, \ldots, f_r$ be generators of I.

3. Compute $a := (\alpha')^{(0)}(b')$ where b' is the generic point of B'.

   So b' is just the m-tuple $(y_1 + J, \ldots, y_m + J) \in \Omega^m$, where Ω is the fraction field of $K[y_1, \ldots, y_m]/J$.

4. Compute the Ω-valued point $(\alpha')^{(1)}$ of $\mathbf{Map}(Q', P)$.

5. Construct a K-basis $\gamma_1, \ldots, \gamma_m$ of the vector space $\mathbf{Map}(Q', Q)$.

6. Set $d_1 := 0$, $r :=$"false".

7. While not r, perform the steps (8)–(10):

8. From α and d_1, compute the nonnegative integers N_1, d_2 from Corollary 2.4.10 and make the *Ansatz* $\gamma(t) = \sum_{i=1}^{m} c_i(t)\gamma_m$ where $c_i(t)$ is a linear combination of $t^{-d_1}, \ldots, t^{d_2}$ with coefficients to be determined in an extension of Ω; and the *Ansatz* $b(t) = (b_1(t), \ldots, b_n(t))$, where b_i is also a linear combination of $t^{-d_1}, \ldots, t^{d_2}$ with coefficients to be determined.

9. The desired properties of $(b(t), \gamma(t))$ from the second item of Corollary 2.4.10 translate into a system of polynomial equations for the $(m+n)\cdot(d_2+d_1+1)$ coefficients of the $c_i(t)$ and the $b_i(t)$. By a Gröbner basis computation, test whether a solution exists over an algebraic closure of Ω. If so, set $r :=$"true".

10. Set $d_1 := d_1 + 1$.

11. Return "true".

Proof of Proposition 2.3.2. The first step is justified by the observation that the image closure of α contains the image of α' if and only if for each j, the image of $\alpha'|_{B'_j \times Q'}$ is contained in the image closure of some $\alpha|_{B_i \times Q}$.

If the image closure of α contains the image of α', then by Theorem 2.4.2, Proposition 2.4.7, Lemma 2.4.8, and Corollary 2.4.10, the procedure **certify** terminates and returns "true". Otherwise, by the same results, the system of equations in step (9) does not have a solution, and the procedure does not terminate. □

Chapter 3

A Functorial Version of Chevalley's Theorem on Constructible Sets

3.1 Introduction

3.1.1 Images of Polynomial Transformations

Let $\alpha : B \times Q \to P$ be a polynomial transformation (where B is an affine variety and Q pure). One goal of this chapter is to show that $X := \mathrm{im}(\alpha)$ (without the closure) is completely determined by an instance $X(U)$ for some vector space U (see Theorem 3.4.1.(i) for the exact formulation).

Example 3.1.1. Let

$$\begin{aligned} \alpha : (K \setminus \{0\}) \times T &\to T^{\oplus 2} \\ \alpha_V(\lambda, v) &= (v, \lambda v) \end{aligned}$$

$X := \mathrm{im}(\alpha)$ consists of linearly dependent vectors where either both or none of them are 0. Note that for every V

$$X(V) = \{(u,v) \in V \oplus V | \forall \varphi : V \to K^1, T^{\oplus 2}(\varphi)(u,v) = (\varphi(u), \varphi(v)) \in X(K^1)\}$$

because, if $(u,v) \notin X(V)$, then either u, v are linearly independent or only one of u or v (say u) are 0, in both cases we can find a linear map φ, such that $\varphi(u) = 0$ and $\varphi(v) \neq 0$. ♢

Remark 3.1.2. We remark that in the above example, even though we can calculate all equations and inequations of $X(V)$ from the equations and inequations of $X(K^1)$ with quantifier elimination, we cannot really "see" them in $X(K^1)$, as for example, $X(K^1)$ does not fulfill any determinantal equation, but $X(V)$ does if $\dim V \geq 2$. ♢

We will use very similar methods as in the proof of topological Noetherianity in [Dra19]. We very much expect it to be possible to make the results from this chapter algorithmic, using the ideas from the previous chapter.

3.1.2 Constructible Subsets

Note that if K is algebraically closed, $X = \mathrm{im}(\alpha)$ is always a subset with the property that $X(V)$ is constructible for every V. One could ask, if any such subset X also has the property that it is determined by one instance $X(U)$. However, this is wrong, see Example 3.2.8.

We call a subset X constructible, if it does have both of these properties, i.e. $X(V)$ is constructible for every V, and it is determined by some $X(U)$ (see also Definition 3.2.1). Based on this definition, we are going to prove the analogue of Chevalley's Theorem, i.e. that the image of a constructible subset under a polynomial transformation is again constructible.

We can also talk about semialgebraic subsets, with the analogous definition, where the ground field is $\mathbb{R}$, and ask whether an analogue of Tarski-Seidenberg's Theorem holds. This question is still open, we neither have a proof nor a counterexample for this.

3.1.3 Parameterisation of Constructible Subsets

Recall that Theorem 1.3.7 says that every closed subset $X \subseteq P$ is the union of finitely many images of polynomial transformations $\alpha^{(j)} : B_j \times Q_j \to P$. We will generalize this theorem to the case where X is only a constructible subset. This reduces the proof of our Chevalley's Theorem to the case from Paragraph 3.1.1, where we only show that images of some $B \times Q$ are constructible.

This generalized parameterisation result is wrong for semialgebraic sets, see Remark 3.3.2, which is why we cannot use this idea for a potential proof of the Tarski-Seidenberg analogue.

3.1.4 Related Work

In [BDES21], a similar-looking version of Chevalley's Theorem is proven, but in the world of infinite-dimensional GL-varieties. A constructible subset of P_∞ (see 2.4.2 for the definition) is a subset that is given by finitely many equations and inequations, and that is invariant under the action of GL, i.e. the inductive limit of $(\mathrm{GL}_n)_n$. They prove that the image of a constructible set in this sense is again constructible. However, this result seems to be essentially different from ours, since attempts to derive our result from this have failed.

3.1.5 The Ground Field

By slight abuse of notation, we will use the letter $\mathbb{C}$ to denote any algebraically closed field of characteristic 0. Similarly, $\mathbb{R}$ denotes either the real numbers or any real closed field of characteristic 0. The letter K can refer to both $\mathbb{C}$ and $\mathbb{R}$.

We only state results over $\mathbb{C}$ and $\mathbb{R}$, but actually, it is true over any field of characteristic 0 that the image, X, of a closed subset under a polynomial transformation is determined by an

instance $X(U)$ (see Theorem 3.4.1.(ii)), as this does not use any properties of $\mathbb{R}$. However, the terminology is much nicer if we only talk about $\mathbb{C}$ and $\mathbb{R}$.

3.2 Constructible and Semialgebraic Subsets of Polynomial Functors

3.2.1 Definition

These are the main objects of this chapter:

Definition 3.2.1. Let P be a polynomial functor over $\mathbb{C}$. A subset $X \subseteq P$ is called

1. *pre-constructible*, if $X(V)$ is constructible for every $V \in \mathbf{Vec}$
2. *constructible*, if it is pre-constructible, and there exists $U \in \mathbf{Vec}$, such that for all $V \in \mathbf{Vec}$:

$$X(V) = \{v \in P(V) | \forall \varphi \in \mathrm{Hom}(V, U), P(\varphi)v \in X(U)\} \tag{3.1}$$

   We say that X is determined by U.

Replacing $\mathbb{C}$ by $\mathbb{R}$ and the word "constructible" by the word "semialgebraic" yields a definition for a (pre-)semialgebraic subset. ♢

Remark 3.2.2. It is straightforward to check that if X is determined by U, then it is also determined by any other vector space of dimension at least $\dim(U)$, in particular also by K^n for $n \geq \dim(U)$. ♢

Equation (3.1) is a finiteness condition that makes sure that all the information of $X(V)$, even if V is very big, is already contained in $X(U)$. Note that it is natural to ask for a finiteness condition when using the word "constructible" (or "semialgebraic"), since also the classical notion of a constructible set refers to a finite union of locally closed sets.

Also note that the inclusion "$\subseteq$" of equation (3.1) is true for all subsets. Hence, in order to check whether a pre-constructible subset $X \subseteq P$ is actually constructible, it suffices to show that for all $v \in P(V) \setminus X(V)$ there exists $\varphi \in \mathrm{Hom}(V, U)$, such that $P(\varphi)v \notin X(U)$.

3.2.2 Examples

In the following, we give some examples of constructible subsets over $\mathbb{C}$. They are also semialgebraic subsets, if you replace the ground field by $\mathbb{R}$.

Example 3.2.3. If X is a closed subset of P, i.e. it is a subset and $X(V)$ is Zariski-closed for every V, then X is a constructible subset by Corollary 1.3.2. For example:

1. $P = T^{\otimes 2}$ and $X(V) = \{A \in P(V) : \mathrm{rk}(A) \leq r\}$, i.e. matrices of rank at most some integer r.

2. $P = T^{\otimes d}$ and $X(V) = \{A \in P(V) : \mathrm{slicerank}(A) \leq r\}$, i.e. tensors of slice rank at most some integer r (see [TS16]).

3. $P = T^{\otimes 3}$ and $X(V) = \{A \in P(V) : \text{geometric rank}(A) \leq r\}$, i.e. tensors of geometric rank at most some integer r (see [KMZ20, Lemma 5.3.]).

♢

Example 3.2.4. Let $P = T^{\oplus d+1}$ (where d is fixed) and

$$X(V) = \{(v_0, v_1, \ldots, v_d) \in P(V) : v_0 \in \mathrm{span}(v_1, \ldots, v_d)\}$$

This is a constructible subset determined by $\mathbb{C}^1$: Let $(v_0, \ldots, v_d) \in P(V) \setminus X(V)$, i.e. $v_0 \notin \mathrm{span}(v_1, \ldots, v_d)$. Then we can find a linear map $\varphi : V \to \mathbb{C}^1$, such that $v_1, \ldots, v_d$ are in the kernel of φ, but not v_0. Note that the closure of X is not determined by $\mathbb{C}^1$ if $d \geq 1$. For example if $d = 1$, then the closure of X consists of pairs of linearly dependent vectors, and we need to consider $\overline{X}(\mathbb{C}^2)$ to see the equations for that. ♢

Example 3.2.5. Let $P = T^{\oplus d}$ and $([d] = \{1, 2, \ldots, d\}, \mathcal{I})$ be a matroid (see e.g. [Oxl06] for the definition of a matroid). Let

$$\widetilde{X}_{\mathcal{I}}(V) := \{(v_1, \ldots, v_d) \in P(V) : \forall I \in 2^{[d]}, (v_j)_{j \in I} \text{ linearly independent} \Leftrightarrow I \in \mathcal{I}\}$$

$$X_{\mathcal{I}}(V) := \bigcup_{g \in \mathrm{End}(V)} P(g)(\widetilde{X}_{\mathcal{I}}(V)).$$

An interesting example is $d = 3$, $\mathcal{I} = \{I \in 2^{[d]} : |I| \leq 2\}$. It turns out that

$$X_{\mathcal{I}} = \widetilde{X}_{\mathcal{I}} \cup \widetilde{X}_{\{\{1\},\{2\},\{3\},\emptyset\}} \cup \widetilde{X}_{\{\{1\},\{2\},\emptyset\}} \cup \widetilde{X}_{\{\{1\},\{3\},\emptyset\}} \cup \widetilde{X}_{\{\{2\},\{3\},\emptyset\}} \cup \widetilde{X}_{\{\emptyset\}}$$

(in particular, it does not include the sets $\widetilde{X}_{\{\{1\},\emptyset\}}$ or $\widetilde{X}_{\{\{1,2\},\{1\},\{2\},\emptyset\}}$).

Each such $X_{\mathcal{I}}$ is a constructible subset determined by $\mathbb{C}^d$, since for $(v_1, \ldots, v_d) \in P(V) \setminus X_{\mathcal{I}}(V)$, there exists a linear map $\varphi : V \to \mathbb{C}^d$, such that $\varphi|_{\mathrm{span}(v_1,\ldots,v_d)}$ is injective, so all linear independencies (and, trivially, all linear dependencies) in $(v_1, \ldots, v_d)$ are preserved, and hence $P(\varphi)(v_1, \ldots, v_d) \in P(\mathbb{C}^d) \setminus X_{\mathcal{I}}(\mathbb{C}^d)$. ♢

The following example also makes sense when replacing the number 3 by any other positive integer d, but we use the number 3 for ease of notation. It is also an illustration of how our theory of single-variable polynomial functors could be generalized to multivariable polynomial functors, which we allow multiple linear maps to act on.

Example 3.2.6. Let $P = T^{\otimes 3}$, $q \in \mathbb{N}$ fixed, and

$$X(V) := \{A \in P(V) : \mathrm{subrk}(A) \leq q\}$$

where the subrank of A, $\mathrm{subrk}(A)$, is the biggest integer q, such that there exist linear maps $\varphi_1, \varphi_2, \varphi_3 : V \to \mathbb{C}^q$ with

$$(\varphi_1 \otimes \varphi_2 \otimes \varphi_3)A = e_1^{\otimes 3} + \ldots + e_q^{\otimes 3}.$$

- We claim that X is a constructible subset of P. It is clear that X is a subset, and by quantifier elimination that every $X(V)$ is constructible, so X is pre-constructible.
- We claim that X is determined by $\mathbb{C}^{3(q+1)}$: Let $A \in P(V) \setminus X(V)$. Then there exist $\varphi_1, \varphi_2, \varphi_3 : V \to \mathbb{C}^{q+1}$ with $(\varphi_1 \otimes \varphi_2 \otimes \varphi_3)A = e_1^{\otimes 3} + \ldots + e_{q+1}^{\otimes 3}$.
- Let

  $$\Phi := \varphi_1 \oplus \varphi_2 \oplus \varphi_3 : V \to \mathbb{C}^{3(q+1)}$$

  Then $P(\Phi)A$ has subrank at least $q+1$, i.e. it does not lie in $X(\mathbb{C}^{3(q+1)})$, because

  $$(\pi_1 \otimes \pi_2 \otimes \pi_3)P(\Phi)A = e_1^{\otimes 3} + \ldots + e_{q+1}^{\otimes 3}$$

  where

  $$\pi_i : \mathbb{C}^{3(q+1)} \to \mathbb{C}^{q+1},\ (a_1, a_2, a_3) \mapsto a_i. \quad \diamondsuit$$

We can easily construct complicated constructible subsets, for example like this:

Example 3.2.7. Let $P = \mathbb{C} \times Q$, where Q is any pure polynomial functor, $X^{(0)}, X^{(1)}, \ldots, X^{(n)}$ constructible subsets of Q. Then

$$X = ((\mathbb{C} \setminus \{1, \ldots, n\}) \times X^{(0)}) \cup (\{1\} \times X^{(1)}) \cup \ldots \cup (\{n\} \times X^{(n)})$$

is a constructible subset. $\diamondsuit$

The following is an example of a pre-constructible subset that is not constructible:

Example 3.2.8. Let $P = \mathbb{C} \times T^{\otimes 2}$ and

$$X(V) = ((\mathbb{C} \setminus \mathbb{Z}_{\geq 0}) \times V^{\otimes 2}) \cup \bigcup_{m \in \mathbb{Z}_{\geq 0}} \{m\} \times \{A \in V^{\otimes 2} \mid rk(A) \leq m\}$$

Note that for every $n \in \mathbb{N}$

$$\begin{aligned} X(\mathbb{C}^n) =&((\mathbb{C}\setminus\mathbb{Z}_{\geq 0})\times\mathbb{C}^{n\times n})\cup\bigcup_{m\geq n}\{m\}\times\underbrace{\{A\in\mathbb{C}^{n\times n}\mid rk(A)\leq m\}}_{=\mathbb{C}^{n\times n}}\cup\\ &\bigcup_{m=0}^{n-1}\{m\}\times\{A\in\mathbb{C}^{n\times n}\mid rk(A)\leq m\}\\ =&((\mathbb{C}\setminus\{0,\dots,n-1\})\times\mathbb{C}^{n\times n})\cup\bigcup_{m=0}^{n-1}\{m\}\times\{A\in\mathbb{C}^{n\times n}\mid rk(A)\leq m\} \end{aligned}$$

is constructible. But for every $n\in\mathbb{N}$, the set

$$\{A\in P(V):\forall\varphi\in\mathrm{Hom}(V,\mathbb{C}^n),P(\varphi)(A)\in X(\mathbb{C}^n)\}$$

is equal to

$$((\mathbb{C}\setminus\{0,\dots,n-1\})\times V^{\otimes 2})\cup\bigcup_{m=0}^{n-1}\{m\}\times\{A\in V^{\otimes 2}\mid rk(A)\leq m\}$$

which is not the same as $X(V)$ if $\dim(V)>n$. ♢

Finally, an example of a semialgebraic set, with no equivalent in the complex world:

Example 3.2.9. $P=S^2$ (i.e. symmetric matrices), and $X(V)$ are the positive semi-definite elements in $P(V)$. This is a semialgebraic subset determined by $\mathbb{R}^1$, since for $A\in P(V)\setminus X(V)$, there exists $v\in V^*=\mathrm{Hom}(V,\mathbb{R}^1)$ such that $P(v)A=vAv^\top<0$, i.e. $P(v)A\notin X(\mathbb{R}^1)$. ♢

For the following example, we do not know whether it is semialgebraic:

Question 3.2.10. *For $P=S^{2d}$, is the subset X given by elements that can be written as sums of squares semialgebraic?*

3.2.3 Elementary Properties

We will later need the following easy Proposition. Also here, the word constructible can be replaced by the word semialgebraic (which would implicitly change the field from $\mathbb{C}$ to $\mathbb{R}$).

Proposition 3.2.11. *If X and Y are constructible subsets of a polynomial functor P, and $\alpha:Q\to P$ is a polynomial transformation then*

(i) The intersection $(X\cap Y)(V):=X(V)\cap Y(V)$ is a constructible subset.

(ii) The union $(X\cup Y)(V):=X(V)\cup Y(V)$ is a constructible subset.

(iii) The preimage $\alpha^{-1}(P)(V):=\alpha^{-1}(P(V))\subseteq Q(V)$ is a constructible subset.

Proof. Statements (i) and (iii) are completely straightforward, so we will only prove (ii): It is clear that $X \cup Y$ is pre-constructible. To prove that it is constructible, let U_1 and U_2 be the vector spaces that X resp. Y are determined by. We claim that $X \cup Y$ is determined by $U_1 \oplus U_2$.

Let $v \in P(V) \setminus (X \cup Y)(V)$. Then there exist $\varphi_1 : V \to U_1$, $\varphi_2 : V \to U_2$, such that $P(\varphi_1)(v) \notin X(U_1)$ and $P(\varphi_2)(v) \notin Y(U_2)$. Then $P(\varphi_1 \oplus \varphi_2)(v) \notin (X \cup Y)(U_1 \oplus U_2)$, because otherwise, denoting by π_{U_1} and π_{U_2} the corresponding projections from $U_1 \oplus U_2$ onto U_1 and U_2, $P(\pi_{U_1})P(\varphi_1 \oplus \varphi_2)(v) = P(\varphi_1)(v) \in X(U_1)$ or $P(\pi_{U_2})P(\varphi_1 \oplus \varphi_2)(v) = P(\varphi_2)(v) \in Y(U_2)$. □

3.3 Parameterisation of Constructible Subsets

3.3.1 Statement

Recall that Theorem 1.3.7 says that all closed subsets are images of some particularly nice sets under a polynomial transformation. The goal of this section is to prove that this is also true for constructible subsets. This will be an important ingredient for our main Theorem 3.4.1 but is also interesting in its own right.

Theorem 3.3.1 (Parameterisation of Constructible Subsets)**.** *Let P be a polynomial functor over $\mathbb{C}$ and $X \subseteq P$ a constructible subset. Then there exist finitely many polynomial transformations*

$$\alpha^{(i)} : A^{(i)} \times Q^{(i)} \to P$$

where $A^{(i)}$ are irreducible affine varieties, and $Q^{(i)}$ pure polynomial functors, such that

$$X = \bigcup_i \mathrm{im}(\alpha^{(i)}).$$

This theorem reduces the proof of our version of Chevalley's Theorem to showing that images of polynomial transformations on sets of the form $A \times Q$ as above are constructible.

Remark 3.3.2. The theorem is wrong for semialgebraic subsets. Let $P = S^2$ the symmetric matrices, and X its positive semidefinite elements (as in Example 3.2.9). Note that for every $V \in \mathbf{Vec}$, $X(V)$ has the same dimension as $S^2(V)$, namely $\binom{\dim(V)+1}{2}$, i.e. it is quadratic in $\dim(V)$. But if it was possible to cover X by images of polynomial transformations , then by Remark 1.2.8, it would have to be covered by images of transformations of the form

$$\alpha^{(i)} : A^{(i)} \times T^{\oplus d} \to P.$$

However, such a union of images can only have dimension linear in $\dim(V)$, which is a contradiction. ♢

3.3.2 Examples

Example 3.3.3. X as in Example 3.2.4 is the image of the polynomial transformation

$$\begin{aligned} \mathbb{C}^d \times V^{\oplus d} &\to V^{\oplus d+1} \\ (a_1, \ldots, a_d, v_1, \ldots, v_d) &\mapsto (a_1 v_1 + \ldots + a_d v_d, v_1, \ldots, v_d) \end{aligned}$$

♢

Example 3.3.4. Let $P(V) = S^2(V) \oplus S^2(V)$ (where $S^2(V)$ is thought of as degree-2-homogeneous-polynomials), and

$$X(V) = \{(f, g) \in P(V) : \forall a \in V^*,\ \ f(a) = 0 \Rightarrow g(a) = 0\}$$

This is a constructible subset determined by $\mathbb{C}^1$, because for $(f, g) \in P(V) \setminus X(V)$, there exists $a \in V^*$ such that $g(a) \neq 0$ and $f(a) = 0$, and hence $P(a)(f, g) \in P(\mathbb{C}^1) \setminus X(\mathbb{C}^1)$. It is also the union of the images of the following polynomial transformations:

$$\begin{aligned} \mathbb{C} \times S^2 &\to S^2 \oplus S^2 \\ (a, q) &\mapsto (q, a \cdot q) \end{aligned} \qquad \begin{aligned} S^1 \oplus S^1 &\to S^2 \oplus S^2 \\ (l, m) &\mapsto (l^2, lm) \end{aligned}$$

♢

3.3.3 Proof

The proof of Theorem 3.3.1 needs one more result from [BDDE] (this is also the part that requires the ground field to be algebraically closed):

Theorem 3.3.5. *Let P be a pure polynomial functor over $\mathbb{C}$ and $U \in$ **Vec**. Then there exists $V \in$ **Vec** and a dense open subset $\Sigma \subseteq P(V)$, such that for every $p \in \Sigma$ the map*

$$\begin{aligned} \mathrm{Hom}(V, U) &\to P(U) \\ \varphi &\mapsto P(\varphi)(p) \end{aligned}$$

is surjective.

Proof. The theorem follows directly from Corollary 2.5.4. in [BDDE] with V big enough, such that $\Sigma := P(V) \setminus (\bigcup_{i=1}^{k} \overline{\mathrm{im}\, \alpha_{i,V}})$ is dense (this is possible by a simple dimensionality argument). □

Proof of Theorem 3.3.1.

- Write $\overline{X} = X^{(1)} \cup \ldots \cup X^{(n)}$ where $X^{(i)}$ are the closed irreducible components of $\overline{X}$. To prove that X is parameterisable it suffices to prove that $X^{(i)} \cap X$ (which is again a constructible subset by Proposition 3.2.11.(i)) is parameterisable for every i. Hence, we can assume without loss of generality that $\overline{X}$ is irreducible.

- If $\overline{X}$ is not of the form $A \times P_{\geq 1}$ for some affine variety A, then by Theorem 1.3.7 there exist finitely many polynomial transformations $\beta^{(j)} : C_j \times Q_j \to P$ (with C_j irreducible affine varieties, $Q_j < P_{\geq 1}$) such that $\overline{X} = \bigcup_j \mathrm{im}(\beta^{(j)})$.

- By Proposition 3.2.11, (iii), $(\beta^{(j)})^{-1}(X)$ are constructible subsets. By induction on the order of polynomial functors each of them can be covered by finitely many maps $\gamma^{(ji)}$, and hence X is the union of the images of $\beta^{(j)} \circ \gamma^{(ji)}$.

- So assume that $\overline{X} = A \times P_{\geq 1}$, for some affine variety A. Note that A is irreducible, since $\overline{X}$ is irreducible. Our next goal is to find a dense open subset $B \subseteq A$ such that $B \times P_{\geq 1} \subseteq X$.

- Consider the set

$$\Omega := \{b \in A : \{b\} \times P_{\geq 1}(U) \subseteq X(U)\}$$

  (where U is the vector space that X is determined by). By quantifier elimination, Ω is constructible. We want to show that Ω is dense in A, so we can take B as an appropriate subset of Ω.

- By Theorem 3.3.5 there exists a vector space V and a dense open subset $\Sigma \subseteq P_{\geq 1}(V)$, such that for every $p \in \Sigma$ the map

$$\begin{aligned} \mathrm{Hom}(V, U) &\to P_{\geq 1}(U) \\ \varphi &\mapsto P_{\geq 1}(\varphi)(p) \end{aligned}$$

  is surjective. So in particular, if for some $p \in \Sigma$ and $b \in A$, (b, p) lies in $X(V)$, then b lies in Ω.

- So $X(V) \subseteq (\Omega \times P_{\geq 1}(V)) \cup ((A \setminus \Omega) \times (P_{\geq 1}(V) \setminus \Sigma))$. But since we assumed that $\overline{X} = A \times P_{\geq 1}$ (so in particular $\overline{X(V)} = A \times P_{\geq 1}(V)$), Ω must be dense in A.

- Hence there exists a subset $B \subseteq \Omega$ that is open (and dense) in A, and so $B \times P_{\geq 1} \subseteq X$. Since B is quasi-affine it can be written as a finite union of irreducible affine varieties, say B_i. Now, $B \times P_{\geq 1}$ can be covered with the images of identity maps on $B_i \times P_{\geq 1}$, and $((A \setminus B) \times P_{\geq 1}) \cap X$ can be covered by induction using Noetherianity of A.

□

3.4 Chevalley's and Weak Tarski-Seidenberg's Theorems

3.4.1 Statement

We now finally set out to prove our functorial version of Chevalley's Theorem, and a weaker version of Tarski-Seidenberg's Theorem:

Theorem 3.4.1.

(i) Let P, Q be polynomial functors over $\mathbb{C}$, $Y \subseteq Q$ a constructible subset, and $\alpha : Q \to P$ a polynomial transformation. Then, $X := \alpha(Y) \subseteq P$ is a constructible subset.

(ii) Let P, Q be polynomial functors over $\mathbb{R}$, $Y \subseteq Q$ a closed subset, and $\alpha : Q \to P$ a polynomial transformation. Then, $X := \alpha(Y) \subseteq P$ is a semialgebraic subset.

The statement reduces to the case $Y = A \times Q$ by Theorem 3.3.1 for the complex case and by Theorem 1.3.7 for the real case. We conjecture that statement (i) remains true when taking Y as semialgebraic, and not just closed, but our methods are insufficient to prove this.

We stress again, that statement (ii) is also true over any other field of characteristic 0, in the sense that X is determined by a particular vector space U, since the proof of this part does not use any particular properties of $\mathbb{R}$.

3.4.2 Proof

Remark 3.4.2. The proof of the second point of the theorem requires that Theorems 1.3.5 and 1.3.7 hold not only over $\mathbb{C}$, but also over $\mathbb{R}$ (not just as schemes but as $\mathbb{R}$-points). Even though the given sources do not explicitly state that this is the case, it is clear from the proofs that it is indeed the case. ♢

The proof uses similar methods as the proof of Theorem 1.3.1 (see [Dra19]) and also consists of a double induction. We will need the following lemma as a sort of base case:

Lemma 3.4.3. *Let $\alpha : A \times Q \to P = P_0 \oplus \ldots \oplus P_d$ (with Q a pure polynomial functor over $\mathbb{R}$ or $\mathbb{C}$, A affine irreducible, P_d not the zero-functor) a polynomial transformation, such that for $X := \mathrm{im}(\alpha)$ we have that $\overline{X}$ is of the form $\widetilde{X} \times P_d$ ($\widetilde{X} \subseteq P_{\leq d-1}$). Then there exists an open dense subset A' of A, such that $\alpha(A' \times Q)$ is of the form $X' \times P_d$ (with $X' \subseteq P_{\leq d-1}$).*

Example 3.4.4. Let $\alpha : A \times S^1 \oplus S^2 \to B \times S^1 \oplus S^2$ be a polynomial transformation. By [Bik20, Proposition 1.3.25], α is of the form

$$\alpha_V : (a, v, M) \mapsto (f_1(a), f_2(a)v, f_3(a)v^2 + f_4(a)M)$$

(where $f_1 : A \to B$ is a morphism, $f_2, f_3, f_4 \in K[A]$). Assume that A is irreducible, and $\overline{\mathrm{im}(\alpha)}$ is of the form $\widetilde{X} \times S^2$. This implies that f_4 is not the zero-polynomial, since the degree-2-part of the image has to be of dimension quadratic in $\dim V$, and the image of $f_3(a)v^2$ only has dimension linear in $\dim V$. Set $A' := A \setminus \mathcal{V}(f_4)$. Then,

$$\alpha(A' \times S^1 \oplus S^2) = ((f(A' \cap \mathcal{V}(f_2)) \times \{0\}) \cup (f(A' \setminus \mathcal{V}(f_2)) \times S^1)) \times S^2$$

which is of the required form. ♢

Proof of Lemma 3.4.3.

- Let $\pi : P \to P_d$ be the standard projection (this is a linear, and in particular polynomial, transformation), and consider $\pi \circ \alpha$. By the conditions in the Lemma, this map is dominant.
- Write $Q = Q_{<d} \oplus Q_d \oplus Q_{>d}$, and accordingly write elements of $A \times Q(V)$ as $(a, q_{<d}, q_d, q_{>d})$. Then, by Remark 1.2.9 we can write

  $$\pi_V \circ \alpha_V(a, q_{<d}, q_d, q_{>d}) = \alpha_{1,V}(a, q_{<d}) + \alpha_{2,V}(a, q_d)$$

  We claim that α_2 has to be dominant: Indeed, the image of $\alpha_{1,V}$ has dimension of order $O(\dim(V)^{d-1})$, and if α_2 were not dominant, its image would have codimension of order $O(\dim(V)^d)$.
- To further investigate what α_2 looks like, write

  $$Q_d = \bigoplus_{\lambda : |\lambda| = d} (S^\lambda)^{\oplus m_\lambda}, \quad P_d = \bigoplus_{\lambda : |\lambda| = d} (S^\lambda)^{\oplus n_\lambda}$$

  and

  $$\alpha_{2,V}(a, (q_{\lambda i})_{|\lambda|=d, 1 \le i \le m_\lambda}) = (p_{\lambda j})_{|\lambda|=d, 1 \le j \le n_\lambda}.$$

  So, α_2 is given by polynomials $f_{\lambda ij} \in K[A]$ by

  $$p_{\lambda j} = \sum_{i=1}^{m_\lambda} f_{\lambda ij}(a) q_{\lambda i}$$
- Since α_2 is dominant, for every partition λ, the matrix $(f_{\lambda ij}(a))_{ij}$ must be dominant (or, equivalently, surjective) for at least one $a \in A$. This implies that $m_\lambda \ge n_\lambda$, and that the variety

  $$B_\lambda := \{a \in A : (f_{\lambda ij}(a))_{ij} \text{ has not full rank}\}$$

  is a proper closed subvariety of A.

- Set $A' := A \setminus (\bigcup_{\lambda:|\lambda|=d} B_\lambda)$. This is open by definition, and using irreducibility of A, we conclude that it is dense. We claim that $\alpha(A' \times Q)$ is of the form $X' \times P_d$. Indeed, if $\alpha_V(a, q_{<d}, q_d, q_{>d}) = (p_{<d}, p_d)$ (with $a \in A'$) is in the image, then, since by construction $\alpha_{2,V}(a, \cdot)$ is surjective, we can modify q_d to reach any other point of the form $(p_{<d}, p'_d)$ with $p'_d \in P_d(V)$.

□

Proof of Theorem 3.4.1. Recall that we want prove that for $\alpha : Q \to P$ and $Y \subseteq Q$ constructible/closed, $\alpha(Y) =: X$ is constructible/semialgebraic. As usual, the word constructible in the proof can be replaced by the word semialgebraic (and the name Chevalley by the names Tarski-Seidenberg) for a proof of the second point of the theorem, except for the steps where the two cases are explicitly treated differently.

- It is clear that X is a subset of P, and, by classical Chevalley's Theorem, that $X(V)$ is constructible for every V. So we just have to show that X is determined by some vector space.

- By Theorem 3.3.1 in the complex case, and Theorem 1.3.7 in the real case, Y can be written as
$$Y = \bigcup_i \alpha^{(i)}(A^{(i)} \times R^{(i)})$$
where $\alpha^{(i)} : A^{(i)} \times R^{(i)} \to Q$ are finitely many polynomial transformations, $A^{(i)}$ are irreducible affine varieties, and $R^{(i)}$ are pure polynomial functors. So X is the union of the images of $\alpha \circ \alpha^{(i)}$, and since by Proposition 3.2.11, (ii) finite unions of constructible subsets are again constructible subsets, it is enough to prove the theorem when Y is of the form $Y = A \times Q$ (with A affine-irreducible and Q pure).

- The proof consists of a double induction: There is an outer induction hypothesis that assumes that all images of transformations $\alpha' : A' \times Q' \to P'$ are constructible, where A', Q' are arbitrary and $P' < P$. The inner induction hypothesis assumes that all images of maps $\alpha' : A' \times Q' \to P$ are constructible, where either $Q' < Q$, or $Q' \cong Q$ and $A' \subsetneq A$ (but with fixed codomain P).

- If X happens to be a subset of $P_0 \oplus \{(0, \dots, 0)\}$, then it is constructible by classical Chevalley's Theorem.

- If $\overline{X}$ is of the form $\widetilde{X} \times P_d$ as in the previous Lemma 3.4.3, then we can use the lemma to conclude that there exists an open, dense $A' \subseteq A$, such that $\alpha(A' \times Q)$ is of the form $X' \times P_d$. Now, $X' \times P_d$ is constructible if and only if X' is, but by Remark 1.2.9, X' can be identified with the image of α restricted to $A' \times Q_{\leq d-1}$, so it is constructible (using either the inner or the outer induction hypothesis). And $\alpha((A \setminus A') \times Q)$ is constructible by the inner induction hypothesis, hence X is constructible as the union of two constructible subsets.

- If $\overline{X}$ is of neither of the two above forms, then by the Shift Theorem (Theorem 1.3.5), there exist a vector space U, and a nonzero polynomial $h \in K[\overline{X}(U)]$, such that $\mathrm{Sh}_U(\overline{X})[1/h]$ is isomorphic to some $B \times P'$ via an isomorphism β, where B is an affine variety, and P' is a pure polynomial functor with $P' < P_{\geq 1}$.

- We take a step back and discuss the strategy for the rest of the proof: Let $p \in P(V) \setminus X(V)$. We need to show that there exists a vector space W, independent of p and V, such that there exists a linear map $\varphi : V \to W$ with $P(\varphi)(p) \notin X(W)$. If $p \notin \overline{X}(V)$ then we already know by Corollary 1.3.2 that there does exist such a vector space, say W'. If $p \in \overline{X}(V)$, then we consider two cases, namely

  $$p \in Z_1(V) := \{p \in \overline{X}(V) | \text{for all } \varphi : V \to U,\ h(P(\varphi)p) = 0\}$$

  and

  $$p \in Z_2(V) := \overline{X}(V) \setminus Z_1(V).$$

  The first case can be dealt with by the inner induction (again using unirationality), and for the second case we will use the Shift Theorem 1.3.5 and the outer induction, even though this will need a little more care, as Z_2 is typically not even a subset of P.

- We quickly do the first case: Note that $X \cap Z_1$ is the image of α restricted to $Y' := \alpha^{-1}(Z_1)$ which is a closed proper subset of $Y = A \times Q$. Then either $Y' = A' \times Q$ where $A' \subsetneq A$ and we can use the induction hypothesis directly to see that $X \cap Z_1 = \alpha(A' \times Q)$ is constructible. Or, Y' is not of this form, but then by Theorem 1.3.7, Y' is the union of finitely many images of maps $\alpha'^{(i)} : A'^{(i)} \times R'^{(i)} \to A \times Q$ with $R'^{(i)} < Q$, so $X \cap Z_1$, which is the union of all images $\alpha \circ \alpha'^{(i)}$, is constructible by the inner induction hypothesis. So there exists $W_1 \in \mathbf{Vec}$, such that for all $p \in Z_1(V) \setminus X(V)$, there exists $\varphi : V \to W_1$ with $P(\varphi)p \notin X(W_1)$.

- For the second case, consider first

  $$Z_2'(V) := \{p \in \overline{X(U \oplus V)} | h(P(\pi_V)p) \neq 0\} \subseteq \mathrm{Sh}_U(P)(V)$$

  (where $\pi_V : U \oplus V \to U$ is the standard projection). Note that h only contains variables from the degree-0-part of $\mathrm{Sh}_U P$. So $Y'(V) := \alpha_{U \oplus V}^{-1}(Z_2'(V)) \subseteq \mathrm{Sh}_U Q(V)$ is of the form $A' \times Q''$, where Q'' is the pure part of $\mathrm{Sh}_U Q$ and A' is an affine subvariety of $A \times Q(U)$. So we can use the outer induction hypothesis to conclude that $\beta_V \circ \alpha_{U \oplus V}(Y'(V)) = \beta_V(Z_2'(V) \cap X(U \oplus V))$ is a constructible subset of $B \times P'$ (since $P' < P_{\geq 1}$).

- So we get that there exists a vector space W_2, such that for all $p \in Z_2'(V) \setminus X(U \oplus V)$, there exists a linear map $\varphi : V \to W_2$ such that $P(\mathrm{id} \oplus \varphi)p \in Z_2'(W_2) \setminus X(U \oplus W_2)$.

- Now, finally, let $p \in Z_2(V) \setminus X(V)$. We first assume that $\dim(V) \geq \dim(U)$, so V is isomorphic to a vector space of the form $U \oplus V'$, and we can think of p as an element in $Z_2(U \oplus V') \setminus X(U \oplus V')$. By definition of Z_2 there exists $\psi : U \oplus V' \to U$ such that $h(P(\psi)p) \neq 0$. This is an open condition on ψ, so we can also assume that ψ has full rank. Hence, there exists $g \in \mathrm{GL}(U \oplus V')$ such that $\psi = \pi_V \circ g$, and therefore $P(g)p \in Z_2'(V') \setminus X(U \oplus V')$. Using now the map $\varphi : V' \to W_2$ from the previous bullet point we get

$$P((\mathrm{id}_U \oplus \varphi) \circ g)p \in Z_2'(W_2) \setminus X(U \oplus W_2) \subseteq P(U \oplus W_2) \setminus X(U \oplus W_2).$$

- So, if $\dim(V) \geq \dim(U)$ we are done, and if $\dim(V) < \dim(U)$ we can instead of $(\mathrm{id}_U \oplus \varphi) \circ g$ simply use an inclusion map $\iota : V \to U \oplus W_2$ such that $P(\iota)p \in P(U \oplus W_2) \setminus X(U \oplus W_2)$.

- Hence, X is determined by $K^{\min(\dim(W'),\dim(W_1),\dim(U \oplus W_2))}$. □

Chapter 4

Noetherianity in Polynomial Functors over Finite Fields

4.1 Introduction

4.1.1 The Main Result

The goal of this chapter is to prove a finite field analogue of Theorem 1.3.1 (topological Noetherianity of polynomial functors over infinite fields).

Let us first give the definition of a polynomial functor over a finite field K. The definition is essentially the same as over infinite fields, but let us still rewrite it precisely, to avoid any confusion that could result from the difference between polynomials and functions over finite fields:

Definition 4.1.1. A *polynomial functor* over the finite field K is a (covariant) functor $P : \mathbf{Vec} \to \mathbf{Vec}$ such that for any $U, V \in \mathbf{Vec}$ the map $P : \mathrm{Hom}(U, V) \to \mathrm{Hom}(P(U), P(V))$ can be written as a polynomial of degree at most some integer d that does not depend on U or V. The minimal such integer d is called the *degree* of P. ♢

Now we can state the main theorem of this chapter:

Theorem 4.1.2 (Noetherianity)**.** *Let P be a polynomial functor over the finite field K. Then any descending chain*

$$P \supseteq X_1 \supseteq X_2 \supseteq \dots$$

of subsets stabilises.

Note that we do not ask our subsets to be closed, because over finite fields, every subset is closed.

4.1.2 Idea of the Proof

As usual, the proof is a double induction, where the outer induction is on an order on polynomial functors, and we want to embed some big subset of a proper subset X of $\mathrm{Sh}_U P$ into $(\mathrm{Sh}_U P)/R$, where R is an irreducible subfunctor of the top-degree part of P, so hopefully $(\mathrm{Sh}_U P)/R$ is a smaller polynomial functor than P and we can use the induction hypothesis. However, adapting this idea to polynomial functors over finite fields is not straightforward. For instance, a polynomial functor over an infinite field has a $\mathbb{Z}_{\geq 0}$-grading, whereas a polynomial functor over the finite field K has a grading by $\{0, 1, \ldots, |K| - 1\}$, so a priori it is not even clear what we mean by top-degree part if the degree of P is bigger than $|K| - 1$.

Nevertheless, we show that P has a unique minimal subfunctor $P_{>d-1}$ the quotient by which has degree at most $d - 1$. We think of $P_{>d-1}$ as the top-degree part of P, and we will prove that any shift $\mathrm{Sh}_U P$ has the same top-degree part as P. So we can take an irreducible subfunctor R of $P_{>d-1}$, and assume that the Noetherianity statement holds for P/R and $\mathrm{Sh}_U P/R$ which will both be smaller polynomial functors than P.

Noetherianity of P/R implies that if $X_1 \supseteq X_2 \supseteq \ldots$ is a chain of subsets of P, then their projections $X_1' \supseteq X_2' \supseteq \ldots$ in P/R stabilise. Therefore, it suffices to prove Noetherianity for subsets $X \subseteq P$ that have a fixed projection $X' \subseteq P/R$. Then, to prove that any subset $X \subseteq P$ with projection X' is Noetherian, we think of each $X(V)$ as a Zariski-closed subset of $P(V)$, i.e., as given by polynomial equations in the finite vector space $P(V)$. The inner induction is on the minimal degree of an equation that vanishes identically on X but not on X'. Using *spreading operators* we show that from such an equation we can construct many equations of the same degree that are affine-linear in the R-direction. We establish an analogue of the Embedding Theorem which allows us to embed a certain subset of X into $\mathrm{Sh}_U P/R$, while on the complement of that subset a polynomial of strictly smaller degree vanishes. Both subsets can therefore be handled by induction.

4.2 Polynomial Functors over Finite Fields

From now on, K denotes a finite field with q elements. We now present the adaptations to our theory of polynomial functors to make it work over finite fields. All definitions from Chapter 1 that are not revised below can be directly adopted as the same definitions over finite fields.

4.2.1 Characterisation

The proof of Proposition 1.1.8, including Lemma 1.1.9, also holds over finite fields, i.e. a polynomial functor is a subquotient of a finite sum of tensor powers.

Remark 4.2.1. The requirement from Definition 4.1.1 that the degree of the maps $P(\varphi)$ must be universally bounded seems to be a lot stronger than in the case where K is an

infinite field. Recall from Remark 1.1.3 that our best example of a functor, where the $P(\varphi)$ are polynomial but of unbounded degree, was already quite complicated. However, for a polynomial functor P over a finite field, $P(\varphi)$ is always a polynomial map, since it is a map between finite sets, see the following example for such a functor that is not polynomial in our sense. ♢

Example 4.2.2. Consider the functor that sends V to the K-vector space KV with basis V and $\varphi : V \to W$ to the unique linear map $KV \to KW$ that sends the basis vector $v \in V$ to the basis vector $\varphi(v) \in W$. This is not a polynomial functor, because $\dim KV = |V| = |K|^{\dim V}$ is exponential in $\dim V$, so it cannot be a subquotient of a sum of tensor powers, because then $\dim KV$ would be bounded by a polynomial in $\dim V$. ♢

Note that in positive characteristic, we cannot decompose polynomial functors into Schur functors anymore, and it is not true that subfunctors are the same as quotients, see the following example:

Example 4.2.3. Let $P = T^{\otimes 2}$, and let σ be the linear map on $P(V)$ given by $\sigma(u \otimes v) = v \otimes u$. Consider

$$S^2(V) := P(V)/\{A - \sigma(A) | A \in P(V)\}$$
$$\Gamma^2(V) := \{A \in P(V) | A = \sigma(A)\}$$

S^2 is a quotient, and Γ^2 is a subfunctor of P, but they are not isomorphic if the characteristic of K is equal to 2. ♢

We make one more easy observation:

Proposition 4.2.4. *A polynomial functor P contains only finitely many subfunctors (seen as functors $Q \subseteq P$, where $Q(\varphi)$ is the restriction of $P(\varphi)$).*

Proof. Let d be the degree of P. By Lemma 1.1.9, P is generated, and hence completely determined, by its instance $P(K^d)$. Since any subfunctor $Q \subseteq P$ also has degree at most d, it is also determined by $Q(K^d) \subseteq P(K^d)$. But $P(K^d)$ is a finite set, so there exist only finitely many subsets $Q(K^d)$, and hence only finitely many subfunctors $Q \subseteq P$. □

4.2.2 Filtering by Degree

It is not true that we can write a polynomial functor P as a direct sum of polynomial functors P_e where P_e is homogeneous of degree e, as we did over infinite fields. We will see that there exists a *filtration* by degree. To see this, we first need a proposition:

Lemma 4.2.5. *For any polynomial functor P and any $e \in \mathbb{Z}_{\geq -1}$, there is a unique inclusionwise minimal subfunctor Q such that P/Q is a polynomial functor of degree at most e.*

Proof. Since by Proposition 4.2.4, P contains only finitely many subfunctors, it suffices to show that if Q_1, $Q_2 \subseteq P$ are subfunctors which have the property that P/Q_i $(i = 1, 2)$ is a polynomial functor of degree at most e, then so does $Q_1 \cap Q_2$. Consider the natural transformation

$$\begin{aligned} \iota : P/(Q_1 \cap Q_2) &\to P/Q_1 \oplus P/Q_2 \\ \iota_V(p) &:= (p + Q_1, p + Q_2) \end{aligned}$$

This is injective, so $P/(Q_1 \cap Q_2)$ is a subfunctor of $P/Q_1 \oplus P/Q_2$. But by assumption, P/Q_1, P/Q_2, and hence also $P/Q_1 \oplus P/Q_2$ have degree at most e, so $P/(Q_1 \cap Q_2)$ has degree at most e. □

Definition 4.2.6. Let P be a polynomial functor and let $e \in \mathbb{Z}_{\geq -1}$. The unique inclusionwise minimal subfunctor Q of P such that P/Q has degree $\leq e$ is denoted by $P_{>e}$. ♢

Example 4.2.7. Suppose that $\operatorname{char} K = 2$. Consider the polynomial functors S^2 and Γ^2 (see Example 4.2.3). Then S^2 has a subfunctor P that maps V to the space of squares of elements of V, and this is the only nontrivial subfunctor unequal to S^2 itself. The quotient S^2/P has degree 2, so $S^2_{>1} = S^2$. On the other hand, Γ has the subfunctor Q that assigns to V the set of skew-symmetric tensors in $V \otimes V$ - i.e., those in the linear span of tensors of the form $u \otimes v - v \otimes u$ as u, v range through V - and the quotient Γ/Q is isomorphic to P. Now if $K = \mathbb{F}_2$, then P has degree 1, so that $\Gamma_{>1} = Q$; while if $K \neq \mathbb{F}_2$, then P has degree 2, and therefore $\Gamma_{>1} = \Gamma$. ♢

We clearly have

$$P = P_{>-1} \supseteq P_{>0} \supseteq \cdots \supseteq P_{>d} = \{0\}$$

where $d = \deg(P)$. An alternative definition for $P_{>0}$ is

$$P_{>0}(V) = \{p \in P(V) \mid P(0 \cdot \mathrm{id}_V)p = 0\}.$$

We can still define the constant part

$$P_0(V) := \{p \in P(V) \mid P(0 \cdot \mathrm{id}_V)p = p\},$$

and this can also be identified with $P(0)$. It is true that $P = P_0 \oplus P_{>0}$ since

$$p \in P_0(V) \cap P_{>0}(V) \Rightarrow p \overset{p \in P_0(V)}{=} P(0 \cdot \mathrm{id}_V)p \overset{p \in P_{>0}(V)}{=} 0$$

and for every $p \in P(V)$, using that $0 \cdot \mathrm{id}_V$ is idempotent,

$$p = \underbrace{P(0 \cdot \mathrm{id}_V)p}_{\in P_0(V)} + \underbrace{(p - P(0 \cdot \mathrm{id}_V)p)}_{\in P_{>0}(V)}.$$

This also clarifies that our two definitions for $P_{>0}$ are equivalent.

4.2.3 An Order on Polynomial Functors

The above filtration now allows us to define an order on polynomial functors over finite fields, in a similar way as we did over infinite fields:

Definition 4.2.8. We call a polynomial functor Q smaller than a polynomial functor P, written $Q < P$, if the two are not isomorphic, and for the largest e such that $Q_{>e}$ is not isomorphic to $P_{>e}$, $Q_{>e}$ is a quotient of $P_{>e}$. We also write $Q \leq P$, if either $Q < P$ or $Q \cong P$. ♢

Lemma 4.2.9. *The relation $\leq$ is a well-founded pre-order on polynomial functors, meaning it is reflexive, transitive, and every descending chain stabilizes.*

Proof.

- Reflexivity is immediate. To see transitivity, assume $R \leq Q \leq P$. If one of the inequalities is an isomorphism, it follows immediately that $R \leq P$. Suppose that they are both not isomorphisms. Let e be maximal such that $Q_{>e} \ncong P_{>e}$ and let e' be maximal such that $R_{>e'} \ncong Q_{>e'}$. If $e' = e$, then e is maximal such that $R_{>e} \ncong P_{>e}$ and $R_{>e}$ is a quotient of $P_{>e}$. If $e' < e$, then e is maximal such that $(Q_{>e} \cong) R_{>e} \ncong P_{>e}$, and $R_{>e}$ is a quotient of $P_{>e}$, and similarly if $e' > e$. In all cases, we find $R \leq P$, as desired.

- To see that $\leq$ is well-founded, suppose we had an infinite chain

$$P^{(1)} \geq P^{(2)} \geq \dots.$$

  Let d be the degree of $P^{(1)}$, so $P^{(1)}_{>d} = \{0\}$ and $P^{(1)}_{>d-1} \neq \{0\}$. Then, for every $n \in \mathbb{N}$, $P^{(n+1)}_{>d-1}$ is either isomorphic to or a quotient of $P^{(n)}_{>d-1}$, and since a quotient of a quotient is again a quotient, it is also a quotient of $P^{(1)}_{>d-1}$.

- By Proposition 4.2.4, $P^{(1)}_{>d-1}$ has only finitely many subfunctors, so it also has only finitely many quotients. Hence, the chain

$$P^{(1)}_{>d-1} \geq P^{(2)}_{>d-1} \geq \dots$$

  stabilizes, at say N_{d-1}.

- Now, for every $n \geq N_{d-1}$, we have that $P^{(n+1)}_{>d-2}$ is either isomorphic to or a quotient of $P^{(n)}_{>d-2}$, and by the same argument as above the chain

$$P^{(N_{d-1})}_{>d-2} \geq P^{(N_{d-1}+1)}_{>d-2} \geq \dots$$

  stabilizes. We conclude by induction that the chain

$$P^{(1)} = P^{(1)}_{>-1} \geq P^{(2)} = P^{(2)}_{>-1} \geq \dots$$

stabilizes.

□

4.2.4 Shifting

Recall that for a polynomial functor P and $U \in \mathbf{Vec}$, the shift over U of P is defined as $\mathrm{Sh}_U P(V) = P(U \oplus V)$. The goal of this section is to prove an analogue of Lemma 1.3.6, i.e. that the top-degree part of $\mathrm{Sh}_U P$ is isomorphic to the top-degree part of P.

We first need a lemma:

Lemma 4.2.10. *If $\alpha : P \to Q$ is a natural transformation, then for each e we have $\alpha(P_{>e}) \subseteq Q_{>e}$.*

Proof. We compute

$$P/\alpha^{-1}(Q_{>e}) \cong \mathrm{im}(\alpha)/(\mathrm{im}(\alpha) \cap Q_{>e}) \subseteq Q/Q_{>e}.$$

Since the latter polynomial functor has degree at most e, so does the first. The defining property of $P_{>e}$ then implies that $P_{>e} \subseteq \alpha^{-1}(Q_{>e})$. This is equivalent to the statement in the lemma. □

Let $\alpha : P \to \mathrm{Sh}_U P$ be the natural transformation given by $\alpha_V = P(\iota_V) : P(V) \to P(U \oplus V)$, where $\iota_V : V \to U \oplus V$ is the inclusion $v \mapsto (0, v)$. Indeed, that $(\alpha_V)_V$ is a natural transformation follows from the commutativity of the following diagram, for any $\varphi : V \to W$:

$$\begin{array}{ccc} P(V) & \xrightarrow{P(\iota_V)} & P(U \oplus V) \\ {\scriptstyle P(\varphi)}\downarrow & & \downarrow{\scriptstyle P(\mathrm{id}_U \oplus \varphi)} \\ P(W) & \xrightarrow[P(\iota_W)]{} & P(U \oplus W), \end{array}$$

which in turn follows from the fact that P is a functor and that

$$(\mathrm{id}_U \oplus \varphi) \circ \iota_V = \iota_W \circ \varphi = (v \mapsto (0, \varphi(v)))$$

Similarly, we have a natural transformation $\beta : \mathrm{Sh}_U P \to P$ defined by $\beta_V = P(\pi_V) : P(U \oplus V) \to P(V)$, where $\pi : U \oplus V \to V$ is the projection $(u, v) \mapsto v$. The relation $\pi \circ \iota = \mathrm{id}_V$ translates to $\beta \circ \alpha = \mathrm{id}_P$. This implies that $\mathrm{Sh}_U P$ is the direct sum of $\mathrm{im}(\alpha) \cong P$ and the polynomial functor $Q := \ker(\beta)$.

Lemma 4.2.11. *Assume that $\deg(P) = d \geq 0$. Then $(\mathrm{Sh}_U P)_{>d-1} \cong P_{>d-1}$.*

Proof. Using the notation α and β from above, we have $\alpha(P_{>d-1}) \subseteq (\mathrm{Sh}_U P)_{>d-1}$ and $\beta((\mathrm{Sh}_U P)_{>d-1}) \subseteq P_{>d-1}$ by Lemma 4.2.10. Combining these facts shows that α maps $P_{>d-1}$ injectively into $(\mathrm{Sh}_U P)_{>d-1}$. To argue that it also maps surjectively there, it suffices to show that $(\mathrm{Sh}_U P)/\alpha(P_{>d-1})$ has degree $\leq d-1$.

To see this, we recall that $\mathrm{Sh}_U P = \mathrm{im}(\alpha) \oplus Q$, where $Q = \ker(\beta)$. Accordingly,

$$(\mathrm{Sh}_U P)/\alpha(P_{>d-1}) \cong (\alpha(P)/\alpha(P_{>d-1})) \oplus Q.$$

Here, the first summand on the right is isomorphic to $P/P_{>d-1}$, hence of degree $\leq d-1$. So it suffices to show that Q has degree $\leq d-1$, as well. Consider a vector $q \in Q(V)$ and a linear map $\varphi \in \mathrm{Hom}(V, W)$. Then we have

$$\begin{aligned} Q(\varphi)(q) &= P(\mathrm{id}_U \oplus \varphi)(q) \\ &= P(\mathrm{id}_U \oplus \varphi)(q) - P(\mathrm{id}_U \oplus \varphi)(\alpha_V(\beta_V(q))) \\ &= (P(\mathrm{id}_U \oplus \varphi) - P(0_U \oplus \varphi))(q) \end{aligned}$$

where the second equality follows from $\beta_V(q) = 0$ and the last equality follows from the definition of α and β. Now, for ψ running through $\mathrm{Hom}(U \oplus V, U \oplus W)$, $P(\psi)$ can be described by a polynomial map of degree at most d. If in this map we substitute for ψ the maps $\mathrm{id}_U \oplus \varphi$ and $0_U \oplus \varphi$, respectively, we obtain the same degree-d parts in φ. Hence the map $\varphi \mapsto P(\mathrm{id}_U \oplus \varphi) - P(0_U \oplus \varphi)$ is given by a polynomial map of degree $\leq d-1$ in the entries of φ. This shows that Q has degree $\leq d-1$, as desired. □

Lemma 4.2.12. *Let P be a polynomial functor, let $e \in \mathbb{Z}_{\geq -1}$, and let R be a subfunctor of $P_{>e}$, and hence of P. Then $(P/R)_{>e} \cong P_{>e}/R$.*

Proof. By Lemma 4.2.10, the natural transformation $P \to P/R$ maps $P_{>e}$ into $(P/R)_{>e}$, and its kernel on $P_{>e}$ is R, so that $P_{>e}/R$ maps injectively into $(P/R)_{>e}$. To see that it also maps surjectively, we note that

$$(P/R)/(P_{>e}/R) \cong P/P_{>e}$$

has degree $\leq e$. Hence $P_{>e}/R$ contains $(P/R)_{>e}$ by definition of the latter functor. □

We will intensively use the following construction.

Example 4.2.13. Let $P \neq 0$ be a polynomial functor of degree $d \geq 0$, and let R be an irreducible subfunctor of $P_{>d-1}$. Let $U \in \mathbf{Vec}$ and set $Q := \mathrm{Sh}_U P$. By Lemma 4.2.11, R is also naturally a subfunctor of $Q_{>d-1}$, which in turn is a subfunctor of Q. By Lemma 4.2.12, we have $(Q/R)_{>d-1} \cong Q_{>d-1}/R$, which in turn is $\cong P_{>d-1}/R$, a quotient of $P_{>d-1}$. Since $P_{>e} = Q_{>e} = 0$ for $e \geq d$, we conclude that $Q/R < P$. ♢

4.2.5 The Coordinate Ring of a Polynomial Functor

Let V be any vector space over K. We will use as the coordinate ring of V not the ring of polynomials on V but the ring of functions:

Definition 4.2.14. Given $V \in \mathbf{Vec}$, we write $K[V]$ for the K-algebra of functions $V \to K$. This has a natural algebra filtration

$$\{0\} = K[V]_{\leq -1} \subseteq K[V]_{\leq 0} \subseteq K[V]_{\leq 1} \subseteq K[V]_{\leq 2} \subseteq \dots$$

where $K[V]_{\leq d}$ is the set of functions $f : V \to K$ that can be realized as a polynomial of degree at most d. ♢

$K[V]$ is the quotient of the symmetric algebra SV^* by the ideal generated by the polynomials $x^q - x$ as x runs through (a basis of) V^*. Note further that $K[V]$ is a finite-dimensional K-vector space, of dimension $q^{\dim(V)}$, the number of elements of V.

Definition 4.2.15. Given a basis $x_1, \dots, x_n$ of V^*, every element f of $K[V]$ has a unique representative polynomial in which all exponents of all variables are $\leq q - 1$; we will call this representative - which does depend on the choice of basis - the *reduced* polynomial representation for f relative to the choice of coordinates. ♢

The following lemma is immediate; the natural isomorphisms in it will be interpreted as equalities throughout the chapter.

Lemma 4.2.16. *For $V, W \in \mathbf{Vec}$ we have $K[V \times W] \cong K[V] \otimes K[W]$ via the K-linear map from right to left that sends $f \otimes g$ to the function $(v, w) \mapsto f(v)g(w)$; this is a K-algebra isomorphism. Similarly, the set of arbitrary maps $V \to W$ is canonically isomorphic to $K[V] \otimes W$ via the K-linear map from right to left that sends $f \otimes w$ to the function $v \mapsto f(v) \cdot w$.* □

Furthermore, we write $K[V]_0 = K$ for the sub-K-algebra of constant functions, and $K[V]_{>0}$ for the K-vector space spanned by all functions that vanish at zero.

We can now define the coordinate ring of a polynomial functor:

Definition 4.2.17. Let $P : \mathbf{Vec} \to \mathbf{Vec}$ be a polynomial functor. We define $K[P]$ as the contravariant functor from $\mathbf{Vec}$ to K-algebras that assigns to V the ring $K[P(V)]$ and to a linear map $\varphi : V \to W$ the pullback $P(\varphi)^\# : K[P(W)] \to K[P(V)]$. We call $K[P]$ the *coordinate ring of P*. ♢

Note that $P(\varphi)^\#$ is an algebra homomorphism; this is going to be of crucial importance in Section 4.4.2. The coordinate ring comes with a natural ring filtration:

$$\{0\} = K[P]_{\leq -1} \subseteq K[P]_{\leq 0} \subseteq K[P]_{\leq 1} \subseteq K[P]_{\leq 2} \subseteq \dots$$

where $K[P]_{\leq e}$ assigns to V the space $K[P(V)]_{\leq e}$.

Lemma 4.2.18. *If P is a polynomial functor of degree $\leq d$, then $V^* \mapsto K[P(V)]_{\leq e}$ is a polynomial functor of degree $\leq d \cdot e$.*

Proof. This functor assigns to a linear map $\varphi : V^* \to W^*$ the restriction of the pullback $P(\varphi^*)^\# : K[P(V)] \to K[P(W)]$ to $K[P(V)]_{\leq e}$. Since $P(\varphi^*)$ is a linear map, this pullback does indeed map $K[P(V)]_{\leq e}$ into $K[P(W)]_{\leq e}$, and it does so via a linear map that is polynomial of degree $\leq e$ in $P(\varphi^*)$, hence of degree $\leq d \cdot e$ in φ^*, which in turn depends linearly on φ. □

Example 4.2.19. Let $P = S^2$ and assume $|K| > 2$. Take $V = K^n$ with basis $e_1, \ldots, e_n$, so that $P(V)$ has basis $e_i e_j$ with $i \leq j$. For $k > l$ distinct, let $g_{kl}(s) \in \mathrm{End}(V)$ be the matrix with 1's on the diagonal, an s on position (k, l), and zeros elsewhere. We have

$$\begin{aligned} P(g_{kl}(s)) \sum_{i \leq j} a_{ij} e_i e_j &= \sum_{i \leq j} a_{ij} (g_{kl}(s) e_i)(g_{kl}(s) e_j) \\ &= \sum_{i \leq j} a_{ij} (e_i + \delta_{il} s e_k)(e_j + \delta_{jl} s e_k) \\ &= \sum_{i \leq j} a_{ij} (e_i e_j + s(\delta_{il} e_k e_j + \delta_{jl} e_i e_k) + s^2 \delta_{il} \delta_{jl} e_k^2) \\ &= \left(\sum_{i \leq j} a_{ij} e_i e_j \right) + s \left(\sum_{j \geq l} a_{lj} e_k e_j + \sum_{i \leq l} a_{il} e_i e_k \right) + s^2 a_{ll} e_k^2 \end{aligned}$$

Observe that by acting with g_{kl} on (linear combinations of) the basis vectors $e_i e_j$, indices l either remain the same or turn into indices k.
We now look at the dual. Let $\{x_{ij} \mid i \leq j\}$ be the basis of $P(V)^*$ dual to the given basis of $P(V)$. Then, for instance, for $l < i < k$ we have

$$P(g_{kl}(s))^\# x_{ik} = x_{ik} + s x_{li},$$

as can be seen by taking the coefficient of $e_i e_k$ in the expression above. We observe here that indices k either remain the same or turn into indices l. We can also write the above as

$$P(g_{lk}(s)^T)^\# x_{ik} = x_{ik} + s x_{li}.$$

Note that $P(g)^\#$ is contravariant in g, and hence $P(g^T)^\#$ is again covariant. This explains the V^* in Lemma 4.2.18. ♢

4.2.6 Subsets

Recall that a subset X of a polynomial functor P is the data of a set $X(V) \subseteq P(V)$ for every $V \in \mathbf{Vec}$, such that for all $\varphi : V \to W$, $P(\varphi)X(V) \subseteq X(W)$. Note that since K is a finite field, every subset is a closed subset (since every $X(V)$ is finite, and hence closed).

Definition 4.2.20. Given a subset $X \subseteq P$, we denote by $I_X(V) \subseteq K[P(V)]$ the ideal of all functions $P(V) \to K$ that vanish identically on $X(V)$. ♢

We stress that conversely, since X is automatically a closed subset, $X(V)$ is also the set of all common zeros of $I_X(V)$ in $P(V)$.

4.3 Weight Theory

4.3.1 Multiplicative Monoid Homomorphisms $K \to K$

A monoid homomorphism $(K, \cdot) \to (K, \cdot)$ is a map $\varphi : K \to K$ with $\varphi(1) = 1$ and $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in K$. In particular, φ restricts to a group homomorphism from the multiplicative group $K^\times = K \setminus \{0\}$ to itself. Since $K^\times$ is cyclic, say with generator g, the monoid homomorphism φ is uniquely determined by its values on g and on 0. Write $\varphi(g) = g^e$ for a unique exponent $e \in \{1, \dots, q-1\}$. If $e \neq q-1$, so that $\varphi(g) \neq 1$, then $\varphi(0)$ is forced to be 0, since otherwise $\varphi(g) \cdot \varphi(0)$ does not equal $\varphi(g \cdot 0) = \varphi(0)$. If $e = q-1$, then there are two possiblities for $\varphi(0)$, namely, $\varphi(0) = 1$ and $\varphi(0) = 0$. In the first case, we will denote φ by $c \mapsto c^0$, and in the second case, we denote by $c \mapsto c^{q-1}$. The following lemma is now straightforward.

Lemma 4.3.1. *The monoid of monoid homomorphisms $K \to K$ is isomorphic to the monoid $\{0, \dots, q-1\}$ with operation $i \oplus j$ defined by $i \oplus j = i+j$ if $i+j \leq q-1$ and $i \oplus j = i+j-(q-1)$ otherwise.*

Note that this monoid is not cancellative, since $0 \oplus j = (q-1) \oplus j$ for all $j \in \{1, \dots, q-1\}$. Still, it will be convenient to have a notation for subtracting elements in the following sense: for $i \in \{1, \dots, q-1\}$ and $j \in \{0, \dots, q-1\}$ we write $i \ominus j$ for the unique element in $\{1, \dots, q-1\}$ that equals $i - j$ modulo $q-1$.

4.3.2 Acting with Diagonal Matrices

Let P be a polynomial functor, and set $V = K^n$, so that we may identify $\mathrm{End}(V)$ with the space of $n \times n$-matrices, and let D_n be its submonoid of diagonal matrices.

Lemma 4.3.2. *We have*

$$P(V) = \bigoplus_{\chi : D_n \to K} P(V)_\chi$$

where χ runs over all monoid homomorphims $(D_n, \cdot) \to (K, \cdot)$ and where

$$P(V)_\chi := \{p \in P(V) | \forall \varphi \in D_n : P(\varphi)p = \chi(\varphi)p\}.$$

Proof. Each element $\varphi \in D_n$ satisfies $\varphi^q = \varphi$, and therefore also $P(\varphi)^q = P(\varphi^q) = P(\varphi)$. Consequently, $P(\varphi)$ is a root of the polynomial $h = T \cdot (T^{q-1} - 1) = \prod_{a \in K}(T - a) \in K[T]$. So $P(\varphi)$ is diagonalisable over K. Moreover, all elements of D_n commute, and therefore so do all elements of $P(D_n)$. Hence, the latter are all simultaneously diagonalisable. We therefore have

$$P(V) = \bigoplus_{\chi : D_n \to K} P(V)_\chi$$

where, a priori, χ runs through all maps $D_n \to K$. Now, if $P(V)_\chi \neq 0$, then it follows that $\chi(\mathrm{diag}(1, \dots, 1)) = 1$ and $\chi(\varphi\psi) = \chi(\varphi)\chi(\psi)$, i.e. χ is a monoid homomorphism $D_n \to K$. □

Note that monoid homomorphisms $\chi : D_n \to K$ can be naturally identified with n-tuples of monoid homomorphisms by

$$\begin{aligned} \{\chi : D_n \to K\} &\leftrightarrow \{(\chi_i : K \to K)_{i=1}^n\} \\ (\mathrm{diag}(t_1, \dots, t_n) \mapsto \chi_1(t_1) \cdots \chi_n(t_n)) &\leftarrow\!\shortmid (\chi_i : K \to K)_{i=1}^n \\ \chi &\mapsto (t \mapsto \chi(\mathrm{diag}(1, \dots, 1, \underset{\substack{\uparrow \\ \text{i-th place}}}{t}, 1, \dots, 1)))_{i=1}^n \end{aligned}$$

and hence, by Lemma 4.3.1, with elements of $\{0, \dots, q-1\}^n$. We write $\chi = (a_1, \dots, a_n)$ if $\chi(\mathrm{diag}(t_1, \dots, t_n)) = t_1^{a_1} \cdots t_n^{a_n}$ for all $(t_1, \dots, t_n) \in K^n$.
We use the word *weight* for monoid homomorphisms $\chi : D_n \to K$, and we call a vector in $P(V)_\chi$ a *weight vector* of weight χ.

We use the notation $\oplus$ also in this context: if $\chi, \mu \in \{0, \dots, q-1\}^n$ are weights, then $\chi \oplus \mu$ is their componentwise sum with respect to $\oplus$ (and accordingly for $\ominus$). Note that

$$(\chi \oplus \mu)(\mathrm{diag}(t_1, \dots, t_n)) = \chi(\mathrm{diag}(t_1, \dots, t_n)) \cdot \mu(\mathrm{diag}(t_1, \dots, t_n)).$$

Example 4.3.3. If U is the subspace of $V = K^n$ spanned by the first k basis vectors, then $P(U)$, regarded as a subspace of $P(V)$, is the direct sum of all $P(V)_\chi$ where χ runs over the weights in $\{0, \dots, q-1\}^k \times \{0\}^{n-k}$. In particular, the constant part of P is $P(0) = P(V)_{(0,\dots,0)}$. ♢

Lemma 4.3.4. *Let $\chi = (a_1, \dots, a_n) \in \{0, \dots, q-1\}^n$ be a weight such that $P(K^n)_\chi$ is nonzero. Then, $\sum_i a_i$ is at most $\deg(P)$.*

Proof. Choose a nonzero $p \in P(K^n)_\chi$. Then $P(\mathrm{diag}(t_1, \dots, t_n))p = t_1^{a_1} \cdots t_n^{a_n} p$, and we note that $t_1^{a_1} \cdots t_n^{a_n}$ is a reduced polynomial in $t_1, \dots, t_n$. On the other hand, $P(\mathrm{diag}(t_1, \dots, t_n))$ can be expressed as a reduced polynomial of degree at most $\deg(P)$ in $t_1, \dots, t_n$ with coefficients that are linear maps $P(K^n) \to P(K^n)$. Evaluating this at p yields a reduced polynomial of degree at most $\deg(P)$ in $t_1, \dots, t_n$ whose coefficients are elements of $P(K^n)$. But we already know which polynomial that is, namely $t_1^{a_1} \cdots t_n^{a_n} p$. Hence $\sum_i a_i \le \deg(P)$. □

4.3.3 Acting with Additive One-Parameter Subgroups

Let P be a polynomial functor, $n \in \mathbb{Z}_{\ge 2}$, and $i, j \in [n]$ distinct. Then we have a one-parameter subgroup

$$g_{ij} : (K, +) \to \mathrm{GL}_n(K),\ g_{ij}(s) := I + sE_{ij}$$

where E_{ij} is the matrix with zeroes everywhere except for a 1 in position (i, j). For $b = 0 \dots q-1$ we define the linear map $F_{ij}[b] : P(K^n) \to P(K^n)$ by

$$F_{ij}[b]p = \text{ the coefficient of } s^b \text{ in } P(g_{ij}(s))p,$$

where we write $P(g_{ij}(s))p$ as a reduced polynomial in s with coefficients in $P(K^n)$.

Lemma 4.3.5. *For any subfunctor Q of P, the linear space $Q(K^n)$ is stable under $F_{ij}[b]$.*

Proof. Let $p \in Q(K^n)$. Then for all $s \in K$ the element

$$P(g_{ij}(s))p = F_{ij}[0]p + sF_{ij}[1]p + \ldots + s^{q-1}F_{ij}[q-1]p$$

lies in $Q(K^n)$. The Vandermonde matrix $(s^b)_{s\in K, b\in\{0,\ldots,q-1\}}$ is invertible, and this implies that each of the $F_{ij}[b]p$ above are linear combinations of the $P(g_{ij}(s))p$, and therefore in $Q(K^n)$. □

Lemma 4.3.6. *Let $p \in P(K^n)$ be a weight vector of weight $a = (a_1, \ldots, a_n)$, let $b \in \{0, \ldots, q-1\}$, and set $\tilde{p} := F_{ij}[b]p$. Then we have the following.*

(1) if $b = 0$, then $\tilde{p} = p$;

(2) if $b \neq 0$ and $a_j = 0$, then $\tilde{p} = 0$;

(3) if $0 < a_j \neq b$, then $\tilde{p}$ is a weight vector of weight $a \ominus be_j \oplus be_i$;

(4) if $0 < a_j = b$, then $\tilde{p}$ is a sum of a weight vector of weight

$$a \ominus be_j \oplus be_i = (a_1, \ldots, a_i \oplus b, \ldots, q-1, \ldots, a_n)$$

and a weight vector of weight

$$a - be_j \oplus be_i = (a_1, \ldots, a_i \oplus b, \ldots, 0, \ldots, a_n)$$

Proof.

- We write

  $$P(g_{ij}(s))p = p_0 + sp_1 + \ldots + s^{q-1}p_{q-1}.$$

  By setting s equal to zero we obtain $P(g_{ij}(0))p = P(\mathrm{id}_{K^n})p = p$ on the left-hand side, and p_0 on the right-hand side. This proves (1).

- If $a_j = 0$, then

  $$P(\mathrm{diag}(1, \ldots, 1, 0, 1, \ldots, 1))p = p$$

  where the 0 is on position j. Therefore

  $$P(g_{ij}(s))p = P(\underbrace{g_{ij}(s)\,\mathrm{diag}(1, \ldots, 1, 0, 1, \ldots, 1)}_{=\mathrm{diag}(1,\ldots,1,0,1,\ldots,1)})p = p$$

  does not depend on s and hence $F_{ij}[b]p = 0$, for $b \neq 0$.

- We now assume $a_j > 0$. We have $F_{ij}[b]p = p_b$. To determine the weight(s) appearing in p_b, we act on p_b with diagonal matrices. For $t = (t_1, \ldots, t_n) \in K^n$ and $t_j \neq 0$ we have

$$\mathrm{diag}(t_1, \ldots, t_n) \cdot g_{ij}(s) = g_{ij}(t_i s t_j^{-1}) \cdot \mathrm{diag}(t_1, \ldots, t_n)$$

and therefore

$$\begin{aligned}\sum_{d=0}^{q-1} s^d P(\mathrm{diag}(t_1, \ldots, t_n))p_d &= P(\mathrm{diag}(t_1, \ldots, t_n) \cdot g_{ij}(s))p \\ &= P(g_{ij}(t_i s t_j^{-1}) \cdot \mathrm{diag}(t_1, \ldots, t_n))p \\ &= t_1^{a_1} \cdots t_n^{a_n} \cdot P(g_{ij}(t_i s t_j - 1))p \\ &= t_1^{a_1} \cdots t_n^{a_n} \cdot \sum_{d=0}^{q-1} (t_i s t_j^{-1})^d p_d.\end{aligned}$$

Comparing coefficients of s^b, we find

$$P(\mathrm{diag}(t))p_b = t^{a - be_j + be_i} p_b$$

for all $t \in K^n$ with $t_j \neq 0$. Hence, if $a_j \neq b$, p_b is a weight vector of weight $a \ominus be_j \oplus be_i$, and if $a_j = b$, p_b is the sum of two weight vectors with weights $a \ominus be_j \oplus be_i$ and $a - be_j \oplus be_i$.

□

4.3.4 Spreading Out Weight

Retaining the notation from before, suppose we are given a nonzero weight vector $p \in P(K^n)$ of weight $(a_1, \ldots, a_n) \in \{0, \ldots, q-1\}^n$ and a $j \in [n]$ with $a_j > 0$. We construct vectors $\tilde{p} \in P(K^{n+1})$ by identifying p with $P(\iota)p$ where $\iota : K^n \to K^{n+1}$ is the embedding adding a 0 in the last position. Then p is a vector of weight $a = (a_1, \ldots, a_n, 0)$ in $P(K^{n+1})$, and we compute

$$\tilde{p} := F_{n+1,j}[b]p$$

for various b. The vector $\tilde{p}$ is guaranteed to be nonzero for at least two values of b, namely for $b = 0$ (in which case $\tilde{p} = p$), and as we now will see, for $b = a_j$. Indeed, in the latter case, by Lemma 4.3.6, $\tilde{p}$ is the sum of a weight vector $\tilde{p}_0$ of weight $a - a_j e_j + a_j e_{n+1}$ and a weight vector $\tilde{p}_1$ of weight $a + (q - 1 - a_j)e_j + a_j e_{n+1}$.

Lemma 4.3.7. *In the case where $b = a_j$, we have $\tilde{p}_0 = P((j, n+1))p$, where $(j, n+1)$ is short-hand for the permutation matrix corresponding to the transposition $(j, n+1)$.*

Proof. The vector $\tilde{p}_0$ is obtained by applying $P(\pi_j)$ to $\tilde{p}$, where π_j is the projection $K^{n+1} \to K^{n+1}$ that sets the j-th coordinate to zero. Furthermore, we have $P(\pi_{n+1})p = p$, where π_{n+1} sets the $(n+1)$st coordinate to zero. We can then compute $\tilde{p}_0$ as the coefficient of s^{a_j} in

$$\begin{aligned} P(\pi_j)P(g_{n+1,j}(s))p &= P(\pi_j g_{n+1,j}(s)\pi_{n+1})p \\ &= P((j,n+1))P(\mathrm{diag}(1,\dots,1,s,1,\dots,1,0))p \\ &= P((j,n+1))s^{a_j}p. \end{aligned}$$

□

If either $F_{n+1,j}[b]p \neq 0$ for some $b \neq 0, a_j$, or if $F_{n+1,j}[a_j]p \neq P((j,n+1))p$, then we find a new vector p' in the subfunctor of P generated by p whose weight has strictly more nonzero entries - we have *spread out the weight* of p.

Definition 4.3.8. A nonzero weight vector $p \in P(K^n) \subseteq P(K^{n+1})$ of weight $a \in \{0,\dots,q-1\}^n$ is called *maximally spread out*, if for all $j \in [n]$ with $a_j > 0$ we have

$$P(g_{n+1,j}(s))p = p + s^{a_j}P((j,n+1))p.$$

♢

Proposition 4.3.9. *For any nonzero polynomial functor P, there exist an n and a nonzero weight vector $p \in P(K^n)$ that is maximally spread out.*

Proof. Let $p \in P(K^m)$ be a nonzero weight vector. As long as p is not maximally spread out, by the above discussion we can replace p by a nonzero weight vector in $P(K^{m+1})$ whose weight has strictly more nonzero entries. But by Lemma 4.3.4, the number of nonzero entries is bounded from above by $\deg(P)$. Hence this process must terminate, with a maximally spread out vector. □

Example 4.3.10. It is not true that every polynomial functor is generated by its maximally spread out vectors. Consider, for instance, K of characteristic 2 and the functor Γ^2 that sends V to the space of symmetric tensors in $V \otimes V$. The weight vectors in $\Gamma^2(K^n)$ are of the forms $e_i \otimes e_i$ and $e_i \otimes e_j + e_j \otimes e_i \in \Gamma^2(K^n)$ with $i \neq j$. Only the latter are maximally spread out. But they generate the subfunctor of Γ^2 consisting of all skew-symmetric tensors in $T^{\otimes 2}$. ♢

4.3.5 The Prime Field Case

In this section we assume that q is prime, so that K is a prime field. We retain the notation from above.

Definition 4.3.11. Let $\iota : K^n \to K^{n+1}$ be the standard embedding and $F_{n+1,j} = F_{n+1,j}[1] : P(K^n) \to P(K^{n+1})$ be the operator that sends p to the coefficient of s^1 in $P(g_{n+1,j}(s) \circ \iota)(p)$. ♢

Lemma 4.3.12. *Assume that K is a prime field. Then the operator $F_{n+1,j} : P(K^n) \to P(K^{n+1})$ is injective on the direct sum of all weight spaces corresponding to weights $\chi = (a_1,\dots,a_n)$ with $a_j > 0$, and it is zero on the weight spaces corresponding to weights with $a_j = 0$.*

Proof. The last part follows immediately from Lemma 4.3.6; we now prove the first part. The operator $F_{n+1,j}$ maps the weight space of χ into that of $\chi \ominus e_j + e_{n+1}$ if $a_j > 1$, and into the sum of the weight spaces with weights $\chi - e_j + e_{n+1}$ and $\chi \ominus e_j + e_{n+1}$ if $a_j = 1$. Since these weights are distinct for distinct χ, it suffices to show that $F_{n+1,j}$ is injective on a single weight space, corresponding to the weight $(a_1, \dots, a_n)$, where $a_j > 0$. Let p be a nonzero vector in this weight space.

Define $\varphi : K^{n+1} \to K^n$ by

$$\varphi(c_1, \dots, c_{n+1}) := (c_1, \dots, c_j + c_{n+1}, \dots, c_n).$$

We then have

$$\varphi \circ g_{n+1,j}(s) \circ \iota = \mathrm{diag}(1, \dots, 1+s, \dots, 1)$$

and therefore

$$P(\varphi)P(g_{n+1,j}(s))P(\iota)p = (1+s)^{a_j} \cdot p.$$

The coefficient of s^1 in the latter expression is $a_j \cdot p$, which is nonzero since $a_j < q$ and q is prime. That coefficient is also equal to $P(\varphi)\tilde{p}$, where $\tilde{p} := F_{n+1,j}p$. Hence $\tilde{p} \neq 0$. □

By Lemma 4.3.12, if $\chi = (a_1, \dots, a_n)$ with $a_j > 1$, then $F_{n+1,j}$ maps $P(K^n)_\chi$ injectively into $P(K^{n+1})_{\chi'}$, where $\chi' = \chi - e_j + e_{n+1}$. On the other hand, if $a_j = 1$, then by Lemma 4.3.7, $F_{n+1,j}$ followed by the projection to the weight space of $\chi' = (a_1, \dots, 0, \dots, a_n, 1)$ agrees on $P(K^n)_\chi$ with the map $P((n+1, j))$, which of course we already knew is injective.

Example 4.3.13. Lemma 4.3.12 is wrong for non-prime fields. Indeed, take $K = \mathbb{F}_4$ and $P = S^2$. Consider the element $p := e_1^2 \in P(K^1)$ of weight (2). Now $P(g_{21}(s))p = (e_1 + se_2)^2 = e_1^2 + s^2 e_2^2$, and hence $F_{2,1}p = 0$. On the other hand, if $K = \mathbb{F}_2$, then $s^2 = s$, and $F_{2,1}p = e_2^2$. ♢

Remark 4.3.14. Note that, as a consequence of the lemma, if a weight vector p of weight $(a_1, \dots, a_n)$ is maximally spread out, then $a_j \in \{0, 1\}$ for all j, and moreover $F_{n+1,j}p = P((n+1, j))p$ for all j with $a_j = 1$. Indeed, if $a_j > 1$, then $F_{n+1,j}p$ is a weight vector of weight $(a_1, \dots, a_j - 1, \dots, a_n, 1)$, and if $a_j = 1$ but $F_{n+1,j}p \neq P((n+1, j))p$, then the left-hand side has a component of weight $(a_1, \dots, q-1, \dots, a_n, 1)$. In either case, p was not maximally spread out. ♢

Remark 4.3.15. For $\varphi \in \mathrm{Hom}(V, W)$, P a polynomial functor and $p \in P(V)$, we will sometimes just write φp instead of $P(\varphi)p$. For instance, we will do this for $\varphi = g_{n+1,j}(s)$. The advantage of this is that we do not need to make explicit in which polynomial functor we are computing. ♢

4.4 Proof of Noetherianity

Recall that our goal is to prove Theorem 4.1.2, i.e. that every descending chain of subsets in a polynomial functor stabilizes.

4.4.1 Reduction to the Prime Field Case

Proposition 4.4.1. *Suppose that Theorem 4.1.2 holds when K is a prime field. Then it also holds when K is an arbitrary finite field.*

Proof.

- Let F be the prime field of K and set $e := \dim_F K$. For an n-dimensional K-vector space U, we write U_F for the $e \cdot n$-dimensional F-vector space obtained by restricting the scalar multiplication on U from $K \times U \to U$ to $F \times U \to U$.

- Now let P be a polynomial functor over K. Define a polynomial functor P_F over F by setting, for a finite-dimensional F-vector space U, $P_F(U) := (P(K \otimes_F U))_F$, and sending an F-linear map $\varphi : U \to V$ to the map $P(\varphi)$, which is K-linear, and therefore also F-linear. It is easy to see from the definitions that P_F is polynomial of the same degree as P.

- For a subset X of P, we define a subset of P_F via $X_F(U) := X(K \otimes_F U)$. If $X_1 \supseteq X_2 \supseteq \dots$ is a chain of subsets in P, then $(X_1)_F \supseteq (X_2)_F \supseteq \dots$ is a chain of subsets of P_F. By assumption, the latter stabilises, say at $(X_{n_0})_F$. Then it follows that, for any $n \geq n_0$ and any m,

$$X_n(K^m) = X_n(K \otimes_F F^m) = (X_n)_F(F^m) = (X_{n_0})_F(F^m) = X_{n_0}(K^m)$$

  and this suffices to conclude that $X_n = X_{n_0}$

□

In view of Proposition 4.4.1, from now on we assume that K is a prime field.

An important reason for this assumption is that we can then use Lemma 4.3.12. We believe that the proof below can be adapted to arbitrary finite fields, and this might actually give more general results. In particular, in the proof below we will act with the operators $F_{n+1,j} = F_{n+1,j}[1]$; and in the general case we would have to work with the operators $F_{n+1,j}[b]$ for $b \in \{1, \dots, q-1\}$. But the reasoning below is already rather subtle, and we prefer not to make it more opaque by the additional technicalities coming from non-prime fields.

4.4.2 The Embedding Theorem

As in the infinite-field case, we will prove Noetherianity (Theorem 4.1.2) via an auxiliary result, namely an analogue of the Embedding Theorem 1.3.8. Let us recall the setting:

Let P be a polynomial functor of positive degree d and let R an irreducible subfunctor of $P_{>d-1}$. Let $\pi : P \to P/R =: P'$ be the projection. Dually, this gives rise to an embedding

$K[P/R] \subseteq K[P]$. For a fixed $V \in \mathbf{Vec}$, if we choose elements $y_1, \dots, y_n \in P(V)^*$ that map to a basis of $R(V)^*$, then we can write elements of $K[P(V)]$ as reduced polynomials in $y_1, \dots, y_n$ with coefficients that are elements of $K[P'(V)]$. We note, however, that R is typically not a direct summand of P. This implies, for instance, that when acting with $\mathrm{End}(V)$ on y_i, we typically do not stay within the linear span of the $y_1, \dots, y_n$ but also get terms that are linear functions in $K[P'(V)]$.

Let X be a subset of P, and let X' be the image of X in P/R, i.e. $X'(V) := \pi(X(V))$ (to simplify notation, we write π instead of π_V).

Now there are two possibilities:

1. $X = \pi^{-1}(X')$, i.e., $X(V) = \pi^{-1}(X'(V))$ for all V. In this case, I_X is generated by $I_{X'} \subseteq K[P'] \subseteq K[P]$.
2. there exists a space V and an element $f \in I_X(V)$ such that f does not lie in $K[P] \cdot I_{X'}$.

Theorem 4.4.2 (Embedding Theorem)**.** *Assume, as above, that K is a prime field. From any $f \in I_X(V) \setminus (K[P(V)] \cdot I_{X'}(V))$, we can construct a $U \in \mathbf{Vec}$ and a polynomial h in $K[P(U)]$ of degree strictly smaller than that of f, such that also h does not vanish identically on $\pi^{-1}(X'(U))$ and such that the projection $\mathrm{Sh}_U P \to (\mathrm{Sh}_U P)/R$ restricts to an injective map on $(\mathrm{Sh}_U X)[1/h]$.*

Here $(\mathrm{Sh}_U X)(V) = X(U \oplus V) \subseteq P(U \oplus V) = (\mathrm{Sh}_U P)(V)$ and $(\mathrm{Sh}_U X)[1/h]$ is the subset of $\mathrm{Sh}_U P$ consisting of points p where $h(p) \neq 0$. A warning here is that h may actually vanish identically on $X(U)$, in which case the conclusion is trivial because $(\mathrm{Sh}_U X)[1/h]$ is empty. But in our application to the Noetherianity theorem, this will be irrelevant.

4.4.3 Proof of Noetherianity from the Embedding Theorem

This proof is very similar to the proof of Noetherianity in [Dra19].

Proof of Theorem 4.1.2.

- Proceeding by induction on P along the partial order from Paragraph 4.2.3, we may assume that Noetherianity holds for every polynomial functor $Q < P$; we call this the *outer induction assumption*.
- Let d be the degree of P. If $d = 0$, then $P(V)$ is a fixed finite set, and clearly any chain of subsets stabilises. So we may assume that $d > 0$.
- Let R be an irreducible subfunctor in the subfunctor $P_{>d-1} \subseteq P$. Given a subset X of P, we write X' for its projection in $P' := P/R$.
- We define $\delta_X \in \{1, 2, \dots, \infty\}$ as the minimal degree of a polynomial in any $I_X(V) \setminus (K[P(V)] \cdot I_{X'}(V))$; this is ∞ if $I_X(V) = K[P(V)] \cdot I_{X'}(V)$ for every $V \in \mathbf{Vec}$.

- For X, Y subsets of P we write $Y < X$ if either $Y' \subsetneq X'$ or else $Y' = X'$ but $\delta_Y < \delta_X$. Since, by the outer induction assumption, P' is Noetherian, this is a well-founded partial order on subsets of P. To prove that a given subset $X \subseteq P$ is Noetherian, we may therefore assume that all subsets $Y \subseteq P$ with $Y < X$ are Noetherian; this is the inner induction hypothesis.

- If $\delta_X = \infty$, then any proper subset Y of X satisfies $Y < X$, so we are done. We are therefore left with the case where $\delta_X \in \mathbb{Z}_{\geq 1}$.

- Let $V \in \mathbf{Vec}$ such that there exists $f \in I_X(V) \setminus (K[P(V)] \cdot I_{X'}(V))$ of degree δ_X. By the Embedding Theorem, there exists $U \in \mathbf{Vec}$ and an element $h \in K[P(U)] \setminus I_X(U)$ of degree $< \delta_X$, such that $(\mathrm{Sh}_U X)[1/h] \to (\mathrm{Sh}_U P)/R$ is an injective map. Since $(\mathrm{Sh}_U P)/R < P$ (by Example 4.2.13), $(\mathrm{Sh}_U X)[1/h]$ is Noetherian by the outer induction hypothesis.

- Define Y as the subset of X defined by the vanishing of h, i.e.

$$Y(V) := \{p \in X(V) | \forall \varphi : V \to U : h(P(\varphi)p) = 0\}.$$

  Let $Y' \subseteq X'$ be the projection of Y in P/R. If $Y' \subsetneq X'$, then $Y < X$ and hence Y is Noetherian by the inner induction hypothesis. If $Y' = X'$, then $h \in I_Y(U) \setminus (K[P(U)] \cdot I_{Y'}(U)$, and hence $\delta_Y \leq \deg(h) < \delta_X$. So then, too, $Y < X$, and Y is Noetherian by the induction hypothesis.

- Now consider a chain

$$X \supseteq X_1 \supseteq X_2 \supseteq \dots$$

  of subsets. By the above two bullet points, from some point on both the chain $(X_i \cap Y)_i$ and the chain $((\mathrm{Sh}_U X_i)[1/h])_i$ have stabilised. We claim that then also the chain $(X_i)_i$ has stabilised.

- Indeed, take $p \in X_i(W)$. If $p \in X_i(W) \cap Y(W)$, then also $p \in X_{i+1}(W) \cap Y(W)$ by the first chain, and we are done. If not, then let $\varphi : W \to U$ be a linear map such that $h(P(\varphi)p) \neq 0$. Let $\iota : W \to U \oplus W$ be the embedding $w \mapsto (\varphi(w), w)$. Then we find that

$$\begin{aligned} P(\iota)p &\in X_i(U \oplus W)[1/h] = (\mathrm{Sh}_U X_i)(W)[1/h] \\ &= (\mathrm{Sh}_U X_{i+1})(W)[1/h] \subseteq X_{i+1}(U \oplus W). \end{aligned}$$

  Now if $\rho : U \oplus W \to W$ is the projection, then we find that $p = P(\rho)P(\iota)p \in P(\rho)X_{i+1}(U \oplus W) = X_{i+1}(W)$, as desired. □

4.4.4 Proof of the Embedding Theorem

Proof of Theorem 4.4.2.

- Recall that P has degree $d > 0$, $X \subseteq P$ is a subset, R is an irreducible subfunctor of $P_{>d-1}$, $\pi : P \to P/R$ is the projection, $X' = \pi(X)$, $X \neq \pi^{-1}(X')$, and $f \in I_X(V) \setminus (K[P(V)] \cdot I_{X'}(V))$. Assume that f has degree δ. Recall from Lemma 4.2.18 that $V^* \mapsto K[P(V)]_{\leq \delta}$ is a polynomial functor. Furthermore, this has subfunctors $V^* \mapsto I_X(V)_{\leq \delta}$ and $V^* \mapsto (K[P(V)] \cdot I_{X'}(V))_{\leq \delta}$.

- We may assume that $V = K^n$, and without loss of generality, f is a weight vector. We will act on f with elements $g_{n+1,j}(s)^\top$; see Example 4.2.19 for an explanation of the transpose. The part of degree b in s is then captured by the operator $F_{n+1,j}[b]$.

- After acting repeatedly with operators $F_{n+1,j}[b]$ (for increasing values of n and possibly j and observing that this does not increase the degree of f), we may assume that the image of $f \in K[P(K^n)]$ in the quotient functor

$$I_X(K^n)_{\leq \delta}/(K[P(K^n)] \cdot I_{X'}(K^n))_{\leq \delta}$$

  is maximally spread out (see Proposition 4.3.9). By Remark 4.3.14, we can pass to a coordinate subspace, such that the weight of the image of f, and hence also of f, is $(1, \ldots, 1)$. Moreover, it implies that if we split, for any $j \in \{1, \ldots, n\}$, $\tilde{f} := F_{n+1,j} f$ as $\tilde{f}_0 + \tilde{f}_1$ where $\tilde{f}_0$ has weight $(1, \ldots, 0, \ldots, 1, 1)$ and $\tilde{f}_1$ has weight $(1, \ldots, q-1, \ldots, 1, 1)$, then $\tilde{f}_1$ vanishes identically on $X'(K^{n+1})$, otherwise $\tilde{f}_1$ would be a more spread out polynomial that vanishes identically on X but not on X'.

- Choose a basis $\mathbf{x}$ of $P'(K^n)^* \subseteq P(K^n)^*$ consisting of weight vectors, and extend this to a basis $\mathbf{x}, \mathbf{y}$ of $P(K^n)^*$ of weight vectors. This means that $\mathbf{y}$ maps to a weight basis of $R(K^n)^*$. Relative to these choices, we can write f as a reduced polynomial

$$f = \sum_\alpha f_\alpha(\mathbf{x}) \mathbf{y}^\alpha \tag{4.1}$$

  for suitable exponent vectors α and nonzero functions $f_\alpha \in K[P'(K^n)]$. We choose this expression reduced relative to $I_{X'}(K^n)$ in the following sense: no nonempty subset of the terms of any f_α add up to a polynomial in $I_{X'}(K^n)$. This implies that no f_α is in the ideal of $I_{X'}(K^n)$, but the requirement is a bit stronger than that.

- Let y_0 be one of the elements in $\mathbf{y}$ that appears in f; we further choose y_0 such that the support in $\{1, \ldots, n\}$ is inclusion-wise minimal. Consider the expression (coarser than (4.1)):

$$f = f_0(\mathbf{x}, \mathbf{y} \setminus \{y_0\}) y_0^0 + \cdots + f_c(\mathbf{x}, \mathbf{y} \setminus \{y_0\}) y_0^c$$

  where f_e is a reduced polynomial in $\mathbf{x}$ and the variables in $\mathbf{y}$ except for y_0; and where $f_c \neq 0$ and $c \in \{1, \ldots, q-1\}$. Note that f_0 is a weight vector of the same weight as f. A priori, the coefficients f_e with $e > 0$ need not be weight vectors, since the weight monoid $(\{0, \ldots, q-1\}^n, \oplus)$ is not cancellative. However, all terms in f_e have the same weight up to identifying 0 and $q-1$, and upon adding e times the weight of y_0 to any of them (using the operation $\oplus$), one obtains the weight $(1, \ldots, 1)$ of f.

Lemma 4.4.3. *We have $c = 1$, f_1 is a weight vector, and after a permutation, f_1 has weight $(1^m, 0^{n-m})$, and y_0 has weight $(0^m, 1^{n-m})$.*

Proof.

- To prove the claim, let $j \in [n]$ be such that the weight $\chi = (a_1, \ldots, a_n)$ of y_0 has $a_j > 0$.
- We partition the variables $\mathbf{y}$ into three subsets: those whose weight has an entry 0 in position j are collected in the tuple $\mathbf{y}_0$; those with a 1 there in the tuple $\mathbf{y}_1$; and those with an entry > 1 there in the tuple $\mathbf{y}_{>1}$.
- We construct a weight basis of $P(K^{n+1})^*$ consisting of:
  - $\mathbf{x}$, $\mathbf{y}_0$, $\mathbf{y}_1$ and $\mathbf{y}_{>1}$;
  - the tuple $(n+1, j)\mathbf{y}_1$ obtained by applying $(n+1, j)$ to each variable in $\mathbf{y}_1$;
  - the tuple $F_{n+1,j}\mathbf{y}_{>1}$ obtained by applying $F_{n+1,j}$ to each variable in $\mathbf{y}_{>1}$ (recall that $F_{n+1,j}$ is short for $F_{n+1,j}[1]$);
  - weight elements that together with $\mathbf{x}$ form a basis of $P'(K^{n+1})^*$; and
  - weight elements that along with $\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_{>1}, (n+1, j)\mathbf{y}_1, F_{n+1,j}\mathbf{y}_{>1}$ project to a weight basis of $R(K^{n+1})^*$.

  The only non-obvious thing here is that the elements in $F_{n+1,j}\mathbf{y}_{>1}$ can be chosen as part of a set mapping to a basis of $R(K^{n+1})^*$ and this follows from Lemma 4.3.12.
- Either y_0 belongs to $\mathbf{y}_1$ or to $\mathbf{y}_{>1}$. In the first case, we define $y_1 := (n+1, j)y_0$, and in the second case we define $y_1 := F_{n+1,j}y_0$. In both cases, y_1 is the (nonzero) weight-$(a_1, \ldots, a_j - 1, \ldots, a_n, 1)$-component of $F_{n+1,j}y_0$ and one of the chosen variables (in the first case, this uses Lemma 4.3.7).
- Consider

$$g_{n+1,j}(s)^\top f = \sum_{e=0}^{c} (f_e(g_{n+1,j}(s)^\top \mathbf{x}, g_{n+1,j}(s)^\top (\mathbf{y} \setminus \{y_0\})))(g_{n+1,j}(s)^\top y_0)^e. \tag{4.2}$$

  From the c-th term we get a contribution $f_c \cdot c \cdot s \cdot y_0^{c-1} \cdot y_1$, which is nonzero because $c < q - 1$ and q is prime.
- Rewriting (4.2) as a reduced polynomial in s, y_0, y_1 with coefficients that are reduced polynomials in the remaining chosen variables in $P(K^{n+1})^*$, we claim that $f_c \cdot c$ is precisely the coefficient of $s \cdot y_0^{c-1} \cdot y_1$. Indeed, y_0, y_1 only appear in the terms $y_0 = F_{n+1,j}[0]y_0$ and $F_{n+1,j}[1]y_0$ from $g_{n+1,j}(s)^\top y_0$, and nowhere in $f_e(g_{n+1,j}(s)^\top \mathbf{x}, g_{n+1,j}(s)^\top (\mathbf{y} \setminus \{y_0\}))$ because:
  - $g_{n+1,j}(s)^\top$ maps the coordinates $\mathbf{x}$ into linear combinations of $\mathbf{x}$ and the further chosen variables in $P'(K^{n+1})^*$;

- y_0, y_1 do not appear in $F_{n+1,j}[b]\mathbf{y}$ for $b > 1$ for weight reasons: expressing the elements in the latter tuple on the basis of the chosen variables, all variables have weights with $b > 1$ at position $n+1$, while y_0, y_1 have a 0 and 1 there, respectively;
- y_0 does not appear in $F_{n+1,j}[1]y$ for any variable y in $\mathbf{y}$, again by comparing the weights in position $n+1$;
- y_1 is different from all variables $F_{n+1,j}[1]y$, where y ranges over the variables in $\mathbf{y}_{>1}$ (other than y_0, if y_0 is in $\mathbf{y}_{>1}$);
- y_1 is different from all variables $(n+1,j)y$, where y ranges over the variables in $\mathbf{y}_1$ (other than y_0, if y_0 is in $\mathbf{y}_1$); and
- y_1 does not appear in the weight component $y' = F_{n+1,j}y - (n+1,j)y$ of any variable y in $\mathbf{y}_1$. Indeed, if $(a'_1, \ldots, 1, \ldots, a'_n)$ is the weight of y, then y' has weight $(a'_1, \ldots, q-1, \ldots, a'_n, 1)$. But, as remarked earlier, the variable y_1 constructed from y_0 does not have a $q-1$ on position j in its weight.

- We conclude that, when writing $\tilde{f} = F_{n+1,j}f$ as a polynomial in the chosen variables, the terms divisible by $y_0^{c-1}y_1$ are precisely those in $c \cdot f_c \cdot y_0^{c-1} \cdot y_1$. Now in f_c, expanded as a reduced polynomial in $\mathbf{y} \setminus \{y_0\}$ with coefficients that are reduced polynomials in $\mathbf{x}$, consider any nonzero term $\ell(\mathbf{x})(\mathbf{y} \setminus \{y_0\})^\alpha$. By reducedness of f, no nonempty subset of the terms of $\ell(\mathbf{x})$ add up to a polynomial that vanishes identically on $X'(K^n)$.

- Group the terms in $\ell(\mathbf{x})$ into two parts: $\ell(\mathbf{x}) = \ell_0(\mathbf{x}) + \ell_1(\mathbf{x})$, in such a manner that $\ell_0(\mathbf{x}) \cdot (\mathbf{y} \setminus \{y_0\})^\alpha y_0^{c-1}y_1$ is the part of $\ell(\mathbf{x})(\mathbf{y} \setminus \{y_0\})^\alpha y_0^{c-1}y_1$ that has weight $(1, \ldots, 0, \ldots, 1, 1)$ and hence is part of $\tilde{f}_0$, and $\ell_1(\mathbf{x}) \cdot (\mathbf{y} \setminus \{y_0\})^\alpha y_0^{c-1}y_1$ has weight $(1, \ldots, q-1, \ldots, 1, 1)$ and hence is part of $\tilde{f}_1$. Since $\tilde{f}_1$ vanishes identically on $\pi^{-1}(X')$, we find that $\ell_1(\mathbf{x})$ does so too. Hence, since no nonempty set of terms in ℓ adds up to a polynomial that vanishes on $X'(K^n)$, we have $\ell_1(\mathbf{x}) = 0$ and $\ell(\mathbf{x}) = \ell_0(\mathbf{x})$. It follows that $\ell(\mathbf{x})(\mathbf{y} \setminus \{y_0\}^\alpha)y_0^{c-1}y_1$ is a weight vector of weight $(1, \ldots, 0, \ldots, 1, 1)$. Since the term $\ell(\mathbf{x})(\mathbf{y} \setminus \{y_0\})^\alpha$ in $f_c(\mathbf{x}, \mathbf{y} \setminus \{y_0\})$ was arbitrary, we find that $f_c y_0^{c-1}y_1$ is a weight vector of weight $(1, \ldots, 0, \ldots, 1, 1)$. Since the weight of y_0 has a positive entry on position j, we find that $c-1 = 0$ and all weights appearing in f_c have a 0 on position j.

- Now j was arbitrary in the support of the weight of y_0, so the weights appearing in f_c all have disjoint support from that of y_0. But the only way, in the weight monoid $\{0, 1, \ldots, q-1\}^n$, to obtain the weight $(1, \ldots, 1)$ as a $\oplus$-sum of two weights with disjoint supports is that, after a permutation, one weight is $(1^m, 0^{n-m})$ and the other weight is $(0^m, 1^{n-m})$. Hence f_c is a weight vector that, after that permutation, has the former weight, and then y_0 has the latter. □

- Now we have found that
$$f = f_0 + f_1 \cdot y_0$$
where f_1 does not vanish identically on $\pi^{-1}(X')$; f_1 has weight $(1^m, 0^{n-m})$, y_0 has weight $\chi = (0^m, 1^{n-m})$ and f_0 does not involve y_0. It might be though, that f_0 still

contains other variables y in $\mathbf{y}$ of the same weight $\chi = (0^m, 1^{n-m})$. Therefore, among the $\mathbf{y}$-variables, let $y_0 = \hat{y}_1, \hat{y}_2, \ldots, \hat{y}_N$ be those that have weight equal to χ; so N is the multiplicity of χ in $R(K^n)^*$. Then the above implies that

$$f = \hat{f}_1\hat{y}_1 + \cdots + \hat{f}_N\hat{y}_N + r \tag{4.3}$$

where each $\hat{f}_i$ has weight $(1^m, 0^{n-m})$ and where the $\mathbf{y}$-variables that appear in r have weight vectors with at least one nonzero entry in the first m positions (here we use that y_0 had a weight of minimal support). Note that $\hat{f}_1 = f_0$ does not vanish identically on $X'(K^n)$.

- Now set $U := K^m$, $W := K^{n-m}$, and $h := \hat{f}_1$. Note that $h \in K[P(U)]$, since its weight is $(1^m, 0^{n-m})$. Also, h has lower degree than f, as desired, and does not vanish identically on $\pi^{-1}(X'(U))$. In fact, all $\hat{f}_i$ are polynomials in $K[P(U)]$, the $\hat{y}_i$ map to coordinates on $R(W)$, and r is a polynomial in $K[(\mathrm{Sh}_U P)(W)/R(W)]$ because every $\mathbf{y}$-variable in r has at least one nonzero entry among the first m entries of its weight.

- We claim that $(\mathrm{Sh}_U X)[1/h] \to (\mathrm{Sh}_U P)/R$ is injective. We first show that this is the case when evaluating at $W = K^{n-m}$. Consider two points $p, p' \in (\mathrm{Sh}_U X)[1/h](W)$ with the same projection in $(\mathrm{Sh}_U P)(W)/R(W)$, so that $p - p' \in R(W)$. Then f vanishes at both p and p' and, in (4.3), we have $\hat{f}_i(p) = \hat{f}_i(p') =: c_i \in K$ for all i, as well as $r(p) = r(p')$. Then (4.3) shows that

$$c_1\hat{y}_1(p) + \cdots + c_N\hat{y}_N(p) = c_1\hat{y}_1(p') + \cdots + c_N\hat{y}_N(p').$$

  This can be expressed as $L(p - p') = 0$ for a linear form $L \in R(W)^*$ which is nonzero because $c_1 = h(p) = h(p') \neq 0$. Now act with an element $\psi \in \mathrm{End}(K^{n-m})$ on (4.3), and then substitute p and p'. This yields the identity $L(R(\varphi)(p - p')) = 0$. Hence we obtain a nonzero $\mathrm{End}(K^{n-m})$-submodule of linear forms in $R(K^{n-m})^*$ that are zero on $p - p'$. But since $R(K^{n-m})$, and hence $R(K^{n-m})^*$, are irreducible $\mathrm{End}(K^{n-m})$-modules by Remark 1.1.10, this means that $p - p' = 0$. The same argument applies when W is replaced by K^s for any s. This completes the proof of the Embedding Theorem. □

4.5 Applications

4.5.1 The Restriction Theorem

Definition 4.5.1. Let $A \in P(V)$ and $B \in P(W)$. Then B is called a *restriction* of A, if there exists a linear map $\varphi : V \to W$ such $P(\varphi)A = B$. This is also denoted $A \geq B$. ♢

The following is now an easy corollary from Theorem 4.1.2:

Corollary 4.5.2 (Restriction Theorem)**.** *Let P be a polynomial functor over a finite field K. For every $i \in \mathbb{N}$ let $V_i \in \mathbf{Vec}$ and $A_i \in P(V_i)$. Then there exists $i < j$ such that $A_j \geq A_i$.*

It is easy to generalize this theorem to multivariate polynomial functors, e.g. if every A_i is an element of some $V_i^{(1)} \otimes V_i^{(2)} \otimes V_i^{(3)}$, then there exist $i < j$ and $\varphi_k : V_j^{(k)} \to V_i^{(k)}$ such that $A_i = P(\varphi_1 \otimes \varphi_2 \otimes \varphi_3)A_j$ (compare also Example 3.2.6).

For the proof, the following notation will be convenient:

Notation 4.5.3. Let $A \in P(U)$. Then

$$X_{\not\geq A}(V) := \{B \in P(V) | \nexists \varphi : V \to U \text{ s.t. } P(\varphi)B = A\}$$

It is straightforward to check that $X_{\not\geq A}$ is a subset. We call A a *forbidden restriction* for $X_{\not\geq A}$.

Proof of Corollary 4.5.2. For every $k \in \mathbb{N}$ let X_k be the subset

$$X_k := \bigcap_{i=1}^{k} X_{\not\geq A_i}.$$

Now, $X_1 \supseteq X_2 \supseteq \ldots$ is a descending chain of subsets. By Theorem 4.1.2, this chain stabilizes, in particular, there exists $n \in \mathbb{N}$, such that $X_n = X_{n+1}$. Since $A_{n+1} \notin X_{n+1}(V_{n+1})$, then also $A_{n+1} \notin X_n(V_{n+1})$. But by definition of X_n this implies that there exist $i \leq n$ and $\varphi : V_{n+1} \to V_i$, such that $P(\varphi)(A_{n+1}) = A_i$. □

Unsurprisingly, the Restriction Theorem is wrong over infinite fields, after all, forbidden restrictions are typically not closed subsets of polynomial functors over infinite fields, which would be required for the descending chain to stabilize. A trivial counterexample would be the constant functor $V \mapsto K^1$, and A_i disjoint one-point-sets. The following counterexample is a little more interesting:

Example 4.5.4. Let K be an infinite field with $\operatorname{char} K \neq 2$, then consider the matrices

$$M_a := \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} \in T^{\otimes 2}(K^2)$$

for a ranging through K. Assume $M_a \geq M_b$, i.e. there exists $g : K^2 \to K^2$ such that $P(g)M_a = M_b$. Identifying g with its matrix, this translates to $gM_ag^\top = M_b$. Looking at the symmetric parts of M_a and M_b, we find that $gIg^T = I$, so g is an orthogonal matrix and M_a, M_b have the same characteristic polynomial. But the characteristic polynomial of M_a equals $(t-1)^2 + a^2$, so $M_a \geq M_b$ holds (if and) only if $a^2 = b^2$. Since $|K| = \infty$, we have found infinitely many matrices that are incomparable with respect to $\geq$. ♢

4.5.2 Restriction-Monotone Functions

Definition 4.5.5. A function f that assigns to any $A \in P(V)$ a real-number is called *restriction-monotone* if $A \geq B$ implies $f(A) \geq f(B)$. ♢

Most, if not all, well-known rank functions on tensor powers (like tensor rank, slice rank, subrank) are restriction-monotone. They all have values in $\mathbb{Z}$. The following is an example of a restriction-monotone function whose values (presumably) are not contained in $\mathbb{Z}$:

Example 4.5.6. Let $P = T^{\otimes d}$, and denote by rk the usual tensor rank, i.e. for $S \in V^{\otimes d}$, $\mathrm{rk}(S)$ is the minimal r such that S can be written as

$$S = \sum_{i=1}^{r} v_{i,1} \otimes \cdots \otimes v_{i,d}$$

for suitable vectors $v_{i,1}, \ldots, v_{i,d}$. The *asymptotic rank* of S is the limit

$$\lim_{t \to \infty} \sqrt[t]{\mathrm{rk}(S^{\boxtimes t})},$$

where the *vertical tensor product* $S^{\boxtimes t}$ is the d-way tensor in $(V^{\otimes t})^{\otimes d}$ obtained by tensoring t copies of S and grouping together the t copies of the first copy of V, the t copies of the second copy, and so on. ♢

Asymptotic ranks are interesting, because they relate to complexity theory. For instance, if ω is the exponent of matrix multiplication, then 2^ω is the asymptotic rank of the matrix multiplication tensor.

Theorem 4.1.2 allows us to make a statement on what the set of all possible asymptotic ranks looks like:

Corollary 4.5.7. *Let f be any restriction-monotone function on a polynomial functor over the finite field K. Then the set of values of f in $\mathbb{R}$ is a well-ordered set.*

Proof. If not, then there exist elements $A_1 \in P(V_1), A_2 \in P(V_2), \ldots$ on which f takes values $a_1 > a_2 > \ldots$. Let $X_{\leq a_i} \subseteq P$ be the subset given by

$$X_{\leq a_i}(V) = \{A \in P(V) | f(A) \leq a_i\}.$$

(we need that f is restriction monotone to conclude that this is a subset). Furthermore, since $A_i \in X_{\leq a_i}(V_i) \setminus X_{\leq a_{i+1}}(V_i)$, we have

$$X_{\leq a_1} \supsetneq X_{\leq a_2} \supsetneq \ldots$$

But this contradicts Theorem 4.1.2. □

For asymptotic tensor ranks, this result has since been improved (with completely different methods) in [BCL+23], where it is proven that the set of all possible asymptotic tensor ranks is discrete, so there are no accumulation points at all, while we could only show that there are no accumulation points "from above".

Another example of a restriction-monotone function with possibly non-integer values is the analytic rank from [Lov19].

4.6 Addendum: Countably Many Asymptotic Tensor Ranks over $\mathbb{C}$

We do not know whether the set of asymptotic tensor ranks over $\mathbb{C}$ is well-ordered. However, there is quite an easy argument that shows that this set is countable. We will present a slightly more general version of this result.

4.6.1 Tensor Invariants and Their Asymptotic Counterparts

Let $d \in \mathbb{Z}_{\geq 0}$. An *invariant* of d-way tensors is the data of a function

$$f = f_{V_1,\dots,V_d} : V_1 \otimes \cdots \otimes V_d \to \mathbb{R}_{\geq 0}$$

for every choice of finite-dimensional complex vector spaces $V_1, \dots, V_d$, satisfying the condition that whenever $\varphi_i : V_i \to W_i$ are linear isomorphisms for $i = 1, \dots, d$, we have

$$f_{V_1,\dots,V_d} = f_{W_1,\dots,W_d} \circ \varphi_1 \otimes \cdots \otimes \varphi_d.$$

Note that any restriction-monotone function is certainly an invariant.

We call f *algebraic* if, for every choice of $n_1, \dots, n_d \in \mathbb{Z}_{\geq 0}$, any $T \in \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_d}$ and any field automorphism σ of $\mathbb{C}$, we have $f(\sigma(T)) = f(T)$, where $\sigma(T)$ is the tensor obtained by applying σ to all entries of T.
The *asymptotic counterpart* $\widetilde{f}$ of an invariant of d-way tensors is another invariant of d-way tensors defined by

$$\widetilde{f}(T) := \lim_{n\to\infty} \sqrt[n]{f(T^{\boxtimes n})},$$

provided that this limit exists. As before, $T^{\boxtimes n}$ is the n-th vertical tensor product of $T \in V_1 \otimes \cdots \otimes V_d$, obtained by taking the n-th tensor power of T and regarding this as a d-way tensor in

$$(V_1^{\otimes n}) \otimes \cdots \otimes (V_d^{\otimes n}).$$

A sufficient condition for $\widetilde{f}$ to be defined is that f is sub-multiplicative, i.e., $f(T \boxtimes S) \leq f(T) \cdot f(S)$ for all S and T (this follows from Fekete's Lemma, see e.g. [Lan17]).

4.6.2 The Result

We will establish the following fundamental result, which, as we will see in Paragraph 4.6.3, in particular applies to asymptotic versions of all known versions of rank.

Theorem 4.6.1. *Let f be an algebraic invariant of d-way tensors, and assume that its asymptotic counterpart $\widetilde{f}$ exists. Then*

(1) also $\widetilde{f}$ is an algebraic invariant of d-way tensors; and

(2) the set of values of f, as $V_1, \dots, V_d$ run through all complex vector spaces and T runs through $V_1 \otimes \cdots \otimes V_d$, is at most countably infinite.

Corollary 4.6.2. *The set of asymptotic ranks, the set of asymptotic subranks, and the set of asymptotic geometric ranks of complex d-tensors is countable.*

Proof of the corollary. As we will see in Section 4.6.3, tensor rank, subrank, and geometric rank are all algebraic, and hence, by the first item of the theorem, so are their asymptotic counterparts $\widetilde{f}$. By the second item applied to $\widetilde{f}$, it takes on countably many values. □

4.6.3 Algebraicity of Tensor Ranks

Recall that the rank of $T \in V_1 \otimes \cdots \otimes V_d$ is the smallest r such that T can be written as

$$T = \sum_{i=1}^{r} v_{i,1} \otimes \cdots \otimes v_{i,d}.$$

Dually, the subrank of T is the largest r such that there exist linear maps $\varphi_i : V_i \to \mathbb{C}^r$ with

$$(\varphi_1 \otimes \cdots \otimes \varphi_d)(T) = \sum_{i=1}^{r} e_i \otimes \cdots \otimes e_i.$$

Finally, the geometric rank of T equals the codimension in $V_1^* \times \cdots \times V_{d-1}^*$ of the algebraic variety

$$X(T) = \{(x_1, \dots, x_{d-1}) \in V_1^* \times \cdots \times V_{d-1}^* \mid T(x_1, \dots, x_{d-1}, \cdot) \equiv 0\}.$$

Lemma 4.6.3. *Rank, subrank, and geometric rank are algebraic tensor invariants.*

Proof. For tensor rank, we note that applying a field automorphism σ to all vectors in a decomposition of T as a sum of r tensors product of vectors yields such a decomposition of σT. For subrank, we note that if

$$(A_1 \otimes \cdots \otimes A_d)T = \sum_{i=1}^{r} e_i^{\otimes d},$$

then

$$(\sigma A_1) \otimes \cdots \otimes (\sigma A_d))(\sigma T) = \sum_{i=1}^{r} e_i^{\otimes d}.$$

For geometric rank, we first observe that the action of $\mathrm{Aut}(\mathbb{C})$ on $\mathbb{C}$ and on each $\mathbb{C}^{n_i}$ naturally yields an action on the dual space $(\mathbb{C}^{n_i})^*$ given by $(\sigma x)(v) := \sigma(x(\sigma^{-1}v))$. With respect to this action we have

$$\begin{aligned}
\sigma X(T) &= \{(\sigma x_1, \dots, \sigma x_{d-1}) \mid \forall y \in V_d^* : T(x_1, \dots, x_{d-1}, y) = 0\} \\
&= \{(x_1, \dots, x_{d-1}) \mid \forall y \in V_d^* : T(\sigma^{-1}x_1, \dots, \sigma^{-1}x_{d-1}, \sigma^{-1}y) = 0\} \\
&= \{(x_1, \dots, x_{d-1}) \mid \forall y \in V_d^* : \sigma T(\sigma^{-1}x_1, \dots, \sigma^{-1}x_{d-1}, \sigma^{-1}y) = 0\} \\
&= X(\sigma T).
\end{aligned}$$

In the last step, we used that the natural action of $\mathrm{Aut}(\mathbb{C})$ on tensors agrees with the action of $\mathrm{Aut}(\mathbb{C})$ on multilinear forms given by $\sigma \circ T \circ (\sigma^{-1})^d$.

Since field automorphisms are continuous in the Zariski topology and dimension is a topological invariant, this implies that the geometric rank of T equals that of σT. □

4.6.4 Proof of the Theorem

Proof.

- We first show that if f is an algebraic tensor invariant, then so is $\tilde{f}$. To this end, let $T \in \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_d}$ and let σ be a field automorphism of $\mathbb{C}$. Set $r := \tilde{f}(T)$ and let $\varepsilon > 0$. Then there exists an n_0 such that for $n > n_0$ we have

$$\left| \sqrt[n]{f(T^{\boxtimes n})} - r \right| < \varepsilon.$$

  We then have

$$\left| \sqrt[n]{f((\sigma T)^{\boxtimes n})} - r \right| = \left| \sqrt[n]{f(\sigma(T^{\boxtimes n}))} - r \right| = \left| \sqrt[n]{f(T^{\boxtimes n})} - r \right| < \varepsilon,$$

  where the first step uses that the entries of $T^{\boxtimes n}$ are polynomial functions, defined over $\mathbb{Z}$, of the entries of T, and that these functions commute with the field automorphism σ; and the last step uses that f is algebraic.

- To see that the number of values of an algebraic tensor invariant is at most countable, we proceed as follows. As the tensor invariant f is invariant under linear isomorphisms, any d-way tensor has the same f-value as some tensor in a tensor product $\mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_d}$. Since the number of tuples $(n_1, \ldots, n_d) \in \mathbb{Z}_{\geq 0}$ is countable, it suffices to show that for fixed $n_1, \ldots, n_d$, f takes at most a countable number of values on $\mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_d}$.

- To this end, we claim that the value of f on a d-way tensor $T = (t_{i_1,\ldots,i_d})_{i_1,\ldots,i_d}$ depends only on the ideal

$$I(T) := \{h \in R := \mathbb{Q}[(x_{i_1,\ldots,i_d})_{i_1,\ldots,i_d}] : h(T) = 0\}$$

  of algebraic relations over $\mathbb{Q}$ among the entries of T. Indeed, let T' be another tensor with $I(T) = I(T')$, and let L, L' be the subfields of $\mathbb{C}$ generated by the entries of T and of T' respectively. Since L is isomorphic to the fraction field of the quotient $R/I(T)$, and similarly for L', there is a unique field isomorphism $\sigma : L \to L'$ that maps each entry $t_{i_1,\ldots,i_d}$ of T to the corresponding entry $t'_{i_1,\ldots,i_d}$ of T'.

- Any field isomorphism between finitely generated subfields of $\mathbb{C}$ extends to an automorphism of $\mathbb{C}$. (Indeed, the transcendence degrees of $\mathbb{C}$ over L and over L' are equal, so if we choose transcendence bases B, B' of $\mathbb{C}$ over L, L', respectively, then there exists a bijection $B \to B'$. Extend $\sigma : L \to L'$ to the unique isomorphism $L(B) \to L'(B')$ that

agrees with this bijection on B. And then apply [Yal66, Theorem 6] to see that this isomorphism extends to an isomorphism of the algebraic closure $\mathbb{C}$ of $L(B)$ into the algebraic closure of $L'(B')$, which is also $\mathbb{C}$.) Denote by σ an extension of σ to an automorphism of $\mathbb{C}$. Then $\sigma T = T'$ by construction, and therefore $f(T) = f(\sigma T) = f(T')$ since f is algebraic.

- Finally, $I(T)$ is an ideal in a polynomial ring over $\mathbb{Q}$ with a finite number of variables, and hence, by Hilbert's basis theorem, generated by a finite number of polynomials. Since the number of polynomials in said polynomial ring is countable, so is the number of finite subsets of that polynomial ring, and so is the number of ideals in it. We conclude that $I(T)$ takes only countably many values, hence, by the above, so does f. □

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Erklärung

gemäss RSL Phil.-nat. 18 Artikel 30

Vorname, Name: Andreas Blatter
Matrikelnummer: 14-127-799
Studiengang: Dissertation
Titel der Arbeit: Images of Tensor Product Polynomials
Leiter der Arbeit: Prof.dr.ir. Jan Draisma

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Bern, den 30.4.2024