Asymptotic Phenomena through the Lens of Topological Noetherianity

Inauguraldissertation
der Philosophisch-naturwissenschaftlichen Fakultät
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vorgelegt von
Nafie Tairi
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Leiter der Arbeit:
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Von der Philosophisch-naturwissenschaftlichen Fakultät angenommen.

Bern, 04.09.2024

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Introduction

The guiding themes of this thesis are *universality* and *topological Noetherianity*. The core of both of these topics lies within *asymptotic algebra*. But what exactly is asymptotic algebra? In a very simplified way, in asymptotic algebra one tries to understand problems in a "large" setting (e.g. large dimension) or even in "*all* large" settings by showing that they can be completely determined by problems in a "small" setting. More precisely, we would like to have

- *Objects:* A collection of geometric or algebraic objects X_i for $i \in I$,
- *Structure:* with a reasonable structure on them, and
- *Relations:* some way to relate the X_i, e.g., morphisms between them.

Then one possible question in asymptotic algebra could be:

Can we characterize all objects $(X_i)_{i \in I}$ only using finitely many X_i?

Or more precisely, are there finitely many $i_1, \ldots, i_{n_0} \in I$ such that every X_i (where $i \in I$) can be rebuild from the finite collection $X_{i_1}, \ldots, X_{i_{n_0}}$? If this is indeed the case, i.e., if every X_i is completely determined by the finite collection, then the objects $X_{i_1}, \ldots, X_{i_{n_0}}$ are in some sense *universal* and therefore we will refer to this phenomenon as *universality*.

An illustrative example comes from the work of Kasman et al. in [KRPS08]. But before we can discuss their result, we first need to recall some facts about *Grassmannians*. Grassmannians, denoted by $\mathrm{Gr}(k, n)$, are geometric spaces representing all k-dimensional subspaces of an n-dimensional vector space V. The *Plücker embedding* is a way to embed the Grassmannian into the projective space $\mathbb{P}(\bigwedge^k V)$. In this way, the Plücker embedding provides a representation of the Grassmannian as a projective variety. The defining homogeneous equations of this variety are called *Plücker relations*. The smallest nontrivial Grassmannian is $\mathrm{Gr}(2, 4)$, which can be defined using only one Plücker relation, called the *Klein quadric*. In 2005, Kasman et al. proved an interesting insight: they showed that every Grassmannian $\mathrm{Gr}(k, n)$ can be set-theoretically defined purely by the Plücker relations obtained from pulling back the Klein quadric. Relating this back to our discussion of asymptotic algebra and the concept of universality, we recognize that the Klein quadric is *universal*, as it completely determines all other Grassmannians.

An essential part of this thesis is to understand how this result extends if we consider vector spaces equipped with an additional structure and special subspaces related to those structures. Specifically, we will consider *quadratic spaces* and *symplectic spaces* as well as their *maximal isotropic* subspaces. In each setting we will consider the analogues of Grassmannians and prove similar universality results.

A vast generalization of the result of Kasman et al. was obtained in 2014 by Draisma-Eggermont in [DE18] by considering what they call *Plücker varieties*. A Plücker variety is a family of varieties in exterior powers of vector spaces that, like the Grassmannian, is functorial in the vector space and behaves well under duals. While the approach of Kasman et al. is very concrete in nature, Draisma-Eggermont embark on a more abstract path by studying the *topological Noetherianity* of a certain limit space up to a certain group. Since topological Noetherianity represents the second central theme in this thesis we will now give its precise definition.

Definition (Topological G-Noetherianty)**.** Let X be a topological space equipped with the action $G \curvearrowright X$ of a group G by homeomorphisms. Then X is *topologically G-Noetherian* if every descending chain

$$X \supseteq X_1 \supseteq X_2 \supseteq X_3 \supseteq \cdots$$

of G-stable closed subsets stabilizes, i.e., if there exists $m_0 \in \mathbb{N}$ such that $X_m = X_{m+1}$ for all $m \geq m_0$. Here a subset $Y \subseteq X$ is called *G-stable* if $gY \subseteq Y$ for all $g \in G$.

In this thesis we will encounter certain infinite dimensional spaces and establish topological Noetherianty results for them up to an infinite dimensional group that is the limit of certain finite dimensional classical group. These results will be crucial for developing a theory similar to the theory of Plücker varieties due to Draisma-Eggermont.

After this initial overview of the thematic framework, we will now give a detailed chapter-by-chapter outline of the content of this thesis.

Universal Equations for Maximal Isotropic Grassmannians

The main goal of Chapter 1 is to establish an analogue of the universality result by Kasman et al. for isotropic Grassmannians in quadratic spaces. A quadratic space V is a vector space equipped with a non-degenerate symmetric bilinear form. In this context, we will consider the so-called isotropic Grassmannians $\mathrm{Gr}_{\mathrm{iso}}(k, V)$, which consist of the k-dimensional subspaces in V where the bilinear form is identically zero. Similar to the ordinary Grassmannian, the Plücker embedding maps the isotropic Grassmannian into the projective space $\mathbb{P}(\bigwedge^k V)$. For our universality result, we will consider the isotropic Grassmann cone $\widehat{\mathrm{Gr}}_{\mathrm{iso}}(k, V) \subseteq \bigwedge^k V$ and maps between exterior powers preserving the isotropic Grassmann cones, which we call IGCP-maps. Our focus will be on the isotropic subspaces of V that have the maximal possible dimension. Then, a consequence of our main result will be the following:

Theorem A. *Let V be a qaudratic space over an algebraically closed field $\mathbb{K}$ with* $\mathrm{Char}(\mathbb{K}) \neq 2$. *Then the maximal isotropic Grassmannian* $\mathrm{Gr}_{\mathrm{iso}}(\lfloor \frac{\dim V}{2} \rfloor, V)$ *in its Plücker embedding can be defined set-theoretically by pulling back the defining equations of*

- $\mathrm{Gr}_{\mathrm{iso}}(3, 7)$ *if V is odd-dimensional*
- $\mathrm{Gr}_{\mathrm{iso}}(4, 8)$ *if V is even-dimensional*

along all IGCP maps to $\bigwedge^3 \mathbb{K}^7$ resp. $\bigwedge^4 \mathbb{K}^8$.

Since the ideals of $\mathrm{Gr}_{\mathrm{iso}}(3,7)$ and $\mathrm{Gr}_{\mathrm{iso}}(4,8)$, and indeed of any isotropic Grassmannian, are generated by finitely many quadrics, Theorem A implies a universal bound on the ranks of the quadrics needed to set-theoretically define any maximal isotropic Grassmannian. Notably, this bound is precisely four.

Theorem A is a direct consequence of the main result in Chapter 1, namely, Theorem 1.3.1, where we consider only a specific family of IGCP maps. For this particular family, the isotropic Grassmannians $\mathrm{Gr}_{\mathrm{iso}}(3,7)$ and $\mathrm{Gr}_{\mathrm{iso}}(4,8)$ in Theorem A can *not* be replaced by isotropic Grasmannians of smaller dimension. Indeed, we have explicit counterexamples showing that Theorem 1.3.1 cannot be improved. However, it may be possible to reduce the dimension in Theorem A by considering additional IGCP maps.

Chapter 1 is organized as follows: In Section 1.1, we recall facts about the ordinary Grassmannian and state the universality result of Kasman et al. in [KRPS08]. Section 1.2 provides the necessary background on quadratic spaces and introduces isotropic Grassmann cones and IGCP maps. In Section 1.3, we state and prove Theorem 1.3.1, which implies Theorem A. A key ingredient in our proof is Proposition 1.3.7, which characterizes forms in $\widehat{\mathrm{Gr}}_{\mathrm{iso}}(\lfloor \frac{\dim V}{2} \rfloor, V)$. The counterexamples showing that Theorem 1.3.1 is optimal are presented in Section 1.4. Finally, in Section 1.5, we show that any maximal isotropic Grassmannian is defined set-theoretically by quadrics of rank at most four.

As we mentioned above, we are also interested in a universality result for Lagrangian Grassmannians in symplectic spaces. We will see, however, that the proofs in Chapter 1 do not apply directly for symplectic spaces. Most importantly, Theorem 1.3.1 is wrong in the symplectic setting, which we show using an explicit counterexample. Therefore, a central goal is to establish an appropriate setting in which universality for the Lagrangian Grassmannian holds. This will be discussed in Chapter 3.

Topological Noetherianity of the Infinite Half-Spin Representations

As mentioned earlier, Draisma-Eggermont obtained a generalization of the universality result due to Kasman et al. by studying the topological Noetherianty of a particular limit space. Now that we have established an analogous universality result for maximal isotropic Grassmannians in quadratic spaces, the following question arises: Can we prove topological Noetherianity of a certain limit space that implies Theorem A? This would indeed follow if we could prove that the projective limit $\varprojlim_n \bigwedge^n \mathbb{K}^{2n}$ is topologically SO-Noetherian. Unfortunately, despite much effort, we were not able to show this.

However, we succeeded in proving a topological Noetherianity result by considering a different embedding of the istropic Grassmannian, namely the spinor embedding. Unlike the Plücker embedding, the spinor embedding maps each of the two irreducible components of the isotropic Grassmannian into the projective space of the half-spin representations, which are irreducible representations of the spin group. Thus, the relevant limit space we will consider is the projective limit of the half-spin representations. More precisely, we consider a countable-dimensional vector space $V_\infty = \bigcup_n V_n$ with basis $e_1, f_1, e_2, f_2, e_3, f_3, \ldots$ and a bilinear form given by $(e_i|e_j) = (f_i|f_j) = 0$ and $(e_i|f_j) = \delta_{ij}$. Moreover, we will construct a direct limit $\mathrm{Spin}(V_\infty)$ of all spin groups $\mathrm{Spin}(2n)$ and define the direct limit $\bigwedge_\infty^+ E_\infty$ of all even half-spin representations. This space has as its basis all formal infinite products

$$e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \wedge \cdots$$

where $\{i_1 < i_2 < \ldots\}$ is a cofinite subset of the positive integers. The group $\mathrm{Spin}(V_\infty)$ acts naturally on this space, and hence on its dual $(\bigwedge_\infty^+ E_\infty)^*$, which we regard as the spectrum of the symmetric algebra on $\bigwedge_\infty^+ E_\infty$. The main theorem of Chapter 2 is as follows.

Theorem B. *The scheme $(\bigwedge_\infty^+ E_\infty)^*$ is topologically $\mathrm{Spin}(V_\infty)$-Noetherian, i.e., every descending chain*

$$\left(\bigwedge\nolimits_\infty^+ E_\infty\right)^* \supseteq X_1 \supseteq X_2 \supseteq X_3 \supseteq \cdots$$

of $\mathrm{Spin}(V_\infty)$-stable reduced closed subschemes stabilises.

The main result in [Nek20] is an exact analogue of Theorem B for the dual infinite wedge, acted upon by the infinite general linear group. Even though we now have much better tools available to study these kinds of questions than we had at the time of [DE18], notably the topological Noetherianity of polynomial functors [Dra19] and their generalisation to algebraic representations [ES22], spin representations are much more intricate than polynomial functors. Therefore, part of Chapter 2 will be devoted to establishing the precise relationship between the infinite half-spin representation and algebraic representations of the infinite general linear group, so as to use those tools.

Theorem B fits in a general program that asks for which sequences of representations of increasing groups one can expect Noetherianity results. This seems to be an extremely delicate question. Indeed, while Theorem B establishes Noetherianity of the dual infinite *half*-spin representation, we do not know whether the dual infinite spin representation is $\mathrm{Spin}(V_\infty)$-Noetherian; see Remark 2.3.9. Similarly, we do not know whether a suitable inverse limit of exterior powers $\bigwedge^n V_n$ is $\mathrm{SO}(V_\infty)$-Noetherian, and there are many more natural sequences of representations for which we do not yet have satisfactory results.

A main application of Theorem B is that we obtain a theory similar to the one for Plücker varieties due to Draisma-Eggermont in [DE18]. Namely, we will introduce the notion of *half-spin varieties*. Roughly speaking, a half-spin variety is a rule X that assigns to each $n \in \mathbb{N}$ a $\mathrm{Spin}(2n)$-stable closed reduced subscheme X_n of the even half-spin representation $\bigwedge^+ E_n$ satisfying some additional axioms. The maximal isotropic Grassmann cone over the spinor embedding represents the simplest example of such a variety. However, the half-spin varieties that we introduce go far beyond this. Indeed, this class of varieties is preserved under linear operations such as joins and tangential varieties, and under finite unions and arbitrary intersections. In particular, the secant variety of a half-spin variety will again be a half-spin variety.

One example how we can use Theorem B in the context of half-spin varieties is that every half-spin variety is completely determined by data coming from a single finite level.

Theorem C. *Let X be a half-spin variety. There exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ it holds that*

$$X_n = V\big(\mathrm{rad}(\mathrm{Spin}(2n) \cdot I_{n_0})\big),$$

where $\mathrm{rad}(\mathrm{Spin}(2n) \cdot I_{n_0})$ is the radical ideal generated by the $\mathrm{Spin}(2n)$-orbits of the ideal I_{n_0} defining $X_{n_0} \subseteq \bigwedge^+ E_{n_0}$.

Consequently, any variety obtained from several copies of the maximal isotropic Grassmannian by the operations mentioned above is defined by equations of some degree bounded independently of n. We stress, though, that these results are of a purely topological/set-theoretic nature. In the case where X is the isotropic Grassmann cone over the spinor embedding, we can use the Cartan map and Theorem A to show that n_0 can be taken to be equal to 4.

In the context of secant varieties, we point out the work by Sam on Veronese varieties: the k-th secant variety of the d-th Veronese embedding of $\mathbb{P}(\mathbb{K}^n)$ is defined *ideal-theoretically* by finitely many types of equations, independently of n, and in particular in bounded degree [Sam17a]. Furthermore, a similar statement holds for the p-th syzygies for any fixed p [Sam17b]. Similar results for ordinary Grassmannians were established by Laudone in [Lau18]. It would be very interesting to know whether their techniques apply to secant varieties of the maximal isotropic Grassmannian in its spinor embedding. Theorem C gives a weaker set-theoretic statement, but for a more general class of varieties.

After establishing Noetherianity, it would be natural to try to study additional geometric properties of $\mathrm{Spin}(V_\infty)$-stable subvarieties of the dual infinite half-spin representation. Perhaps there is a theory there analogous to the theory of GL-varietes [BDES23a, BDES23b]. However, we are currently quite far from any such deeper understanding!

Chapter 2 is organized as follows: In Section 2.1, we first recall the construction of the (finite-dimensional) half-spin representations. We mostly do this in a coordinate-free manner, only choosing, as one must, a maximal isotropic subspace of an orthogonal space for the construction. But for the construction of the infinite half-spin representation, we will need explicit formulas, and these are derived in Section 2.1, as well. We describe the spinor embedding of the maximal isotropic Grassmannian in the projectivised half-spin representation in Section 2.2. Then, we define suitable contraction and multiplication maps, which we show preserve the cones over these isotropic Grassmannians. Finally, we use these maps to construct the infinite-dimensional half-spin representations. Section 2.3 is devoted to the proof of Theorem B. Then, in Section 2.4, we state and prove the main results about half-spin varieties. Finally, in Section 2.5 we prove the universality of the isotropic Grassmannian of 4-spaces in an 8-dimensional space.

Noetherianity and Universality for Lagrangian Plücker Varieties

As mentioned at the beginning, the main goals of this thesis are to establish universality and topological Noetherianity results for quadratic and symplectic spaces. Whereas the results in Chapter 1 and Chapter 2 were about quadratic spaces, we now turn to symplectic spaces. A symplectic space V is a vector space equipped with a non-degenerate skew-symmetric bilinear form. In contrast to quadratic spaces, symplectic spaces always have even dimension $2n$. Maximal isotropic subspaces of a symplectic space are called Lagrangian subspaces. Similar to quadratic spaces, in this setting we will also consider Lagrangian Grassmann cones and Lagrangian Grassmann cone preserving (LGCP) maps. We mentioned in the description of Chapter 1 that the proof of the universality result in quadratic spaces does *not* directly apply to symplectic spaces, as there are explicit counterexamples to Theorem 1.3.1. Therefore, to establish a universality result for the Lagrangian Grassmannian, we need to find an appropriate setting. More precisely, we want to find a subspace of the exterior power that still contains the Lagrangian Grassmann cone but does *not* contain the counterexamples. A good candidate for this is the n-th fundamental representation of the symplectic group $\mathrm{Sp}(2n)$, which we denote by

$\ker_n$. The choice of this notation will become clear later.

The following statement is the first main result of Chapter 3.

Theorem D. *Let* V *be a symplectic space of dimension* $2n$ *over a field* $\mathbb{K}$ *of* $\mathrm{Char}(\mathbb{K}) = 0$. *Then, the Lagrangian Grassman cone* $\widehat{\mathrm{Gr}}_{\mathrm{L}}(V) \subseteq \ker_n$ *can set-theoretically be defined by pulling back the defining equations of* $\widehat{\mathrm{Gr}}_{\mathrm{L}}(\mathbb{K}^4)$ *along all LGCP maps to* $\ker_2 \subseteq \bigwedge^2 \mathbb{K}^4$.

Actually, Theorem D can be generalized by introducing *Lagrangian Plücker varieties*. Similar to half-spin varieties, a Lagrangian Plücker variety X is a collection of $\mathrm{Sp}(2n)$-stable closed subsets $X_n \subseteq \ker_n$. For these, we will show the following universality property:

Theorem E. *Let* X *be a Lagrangian Plücker variety. Then there exists* $n_0 \in \mathbb{N}$ *such that for all* $n \geq n_0$ *we have*

$$X_n = \left\{ \eta \in \ker_n \;\middle|\; f(g \cdot \eta) = 0 \quad \forall g \in \mathrm{Sp}(2n),\ \forall f \in I_{n_0} \right\},$$

where I_{n_0} *is the ideal of polynomials defining* $X_{n_0} \subseteq \ker_{n_0}$.

We will obtain Theorem E as a consequence of the Noetherianity result that we establish in the second part of Chapter 3. For this, we will consider the countable dimensional vector space $\ker_\infty$, which is the direct limit $\varinjlim_n \ker_n$ along suitable multiplication maps. The infinite symplectic group $\mathrm{Sp}(V_\infty)$, which is the direct limit of all symplectic groups $\mathrm{Sp}(2n)$, naturally acts on $\ker_\infty$. In fact, $\ker_\infty$ is an irreducible $\mathrm{Sp}(V_\infty)$-representation. There is an induced action of $\mathrm{Sp}(V_\infty)$ on the dual space $(\ker_\infty)^*$, which we see as the spectrum of the symmetric algebra $\mathrm{Sym}(\ker_\infty)$ endowed with the Zariski topology. We can now state the second main theorem of Chapter 3.

Theorem F. *The dual* $(\ker_\infty)^*$ *of the irreducible* $\mathrm{Sp}(V_\infty)$*-representation* $\ker_\infty$ *is topologically* $\mathrm{Sp}(V_\infty)$*-Noetherian.*

After reading the summary of Chapter 3, the reader might be inclined to think that the proofs in this chapter are direct adaptations of the results in Chapter 1 and Chapter 2. Even though the overall proof strategies remain the same, the technical details of both results required considerable modifications. In Chapter 1 and Chapter 2, we often used explicit calculations by working in a canonical basis. However, for the relevant spaces considered in Chapter 3, there was no obvious choice for a basis. So one of the main challenges was to find a useful coordinate-independent description of these spaces. Working in a basis in Chapter 1 and Chapter 2 was beneficial, as it often made our approach to the proofs straightforward. However, the calculations quickly became cumbersome, hiding the main ideas. In the coordinate-free approach of Chapter 3, it was more difficult to find the correct strategies, but once we had them, the essential ideas became more transparent.

Chapter 3 is organized as follows: In Section 3.1, we collect the necessary background information about general and symplectic vector spaces. Section 3.2 introduces the new setting for Theorem D. In Section 3.3, we present the main auxiliary results. The complete proof of Theorem D is presented in Section 3.4. We then state Theorem F and outline its proof strategy in Section 3.5. Section 3.6 and Section 3.7 contain the necessary preparation for the proof of Theorem F, which will then be given in Section 3.8. Finally, Section 3.9 presents the applications to Lagrangian Plücker varieties.

Final Thoughts

The results of this thesis naturally raise the question of whether there might be a more general universality and Noetherianity result, with our work serving as an example of such general principles. This seems plausible given the results on ordinary Grassmannians and Plücker Varieties in [DE18], maximal isotropic Grassmannians and Spin Varieties in [ST24, CDE+24], and Lagrangian Grassmannians and Lagrangian Plücker varieties in Chapter 3.

However, such a broad generalization seems currently to be out of reach. For example, in [CDE+24], we do not yet know how to prove the statement for the *full spin* representations, focusing only on the half spin representations. The details matter significantly, and despite using similar ideas across the different settings, we had to address the specific properties unique to each case. Therefore, a more abstract and general statement requires a deeper understanding.

Sources of the Material

A majority of the results in this thesis have been achieved through collaboration with fellow researchers:

- Chapter 1 is based on the article [ST24], joint with Tim Seynnaeve;
- Chapter 2 is based on the paper [CDE+24], joint with Jan Draisma, Rob Eggermont, Christopher Chiu and Tim Seynnaeve;
- Chapter 3 is independent work that has not yet appeared yet in print.

This thesis is not meant to be completely self-contained. Even though most preliminaries are explained in each chapter, we assume some familiarity with basic concepts from algebraic geometry, representation theory and the algebra of alternating tensors; all the needed standard definitions and results can be found in, e.g., [Pro07, FH91, Lee12].

How to Read this Thesis

Each chapter is self-contained and the chapters can be read in a any order. The chapters are arranged chronologically based on the appearance of the articles. While reading the thesis linearly is straightforward, as, for example, Chapter 2 refers to the results in Chapter 1 and Chapter 3 refers to the results in both Chapter 1 and Chapter 2, we recommend starting with Chapter 3. This chapter explores similar ideas and methods as used in the preceding ones but presents them in a more general context, without choosing a specific basis or working in coordinates, as often done in Chapter 1 or Chapter 2. We think that this makes it easier for the reader to grasp the key ideas and maintain an overview, which can then help understand the technical and coordinate-heavy parts in the previous chapters. Therefore, the recommended order is Chapter 3, followed by Chapter 1, and Chapter 2.

Chapter 1

Universal Equations for Maximal Isotropic Grassmannians

1.1 The Ordinary Grassmannian

Let V be a finite-dimensional vector space over any field $\mathbb{K}$, and $k \leq \dim V$. The *Grassmann cone* is defined as

$$\widehat{\mathrm{Gr}}(k,V) := \left\{ v_1 \wedge \cdots \wedge v_k \in \bigwedge^k V \;\middle|\; v_1, \ldots, v_k \in V \right\},$$

where $\bigwedge^k V$ is the k-th exterior power of V. For $\omega = v_1 \wedge \cdots \wedge v_k \in \widehat{\mathrm{Gr}}(k,V) \setminus \{0\}$, we will denote the corresponding subspace $\mathrm{span}\{v_1, \ldots, v_k\} \subseteq V$ as L_ω. If $\dim V = n$, we will sometimes write $\widehat{\mathrm{Gr}}(k,n)$ instead of $\widehat{\mathrm{Gr}}(k,V)$. Choosing a basis $e_1, \ldots, e_n$ of V induces coordinates $\{x_I \mid I \subset \{1, \ldots, n\}, |I| = k\}$ on $\bigwedge^k V$, which are known as the *Plücker coordinates*. The Grassmann cone is a subvariety of $\bigwedge^k V$, and its defining equations are quadrics referred to as the *Plücker relations* [Sha94, (1.24)]. In the case where $k = 2$ and $V = \mathbb{K}^4$, there exists only one Plücker relation, called the Klein quadric

$$P_{2,4} = x_{1,2}x_{3,4} - x_{1,3}x_{2,4} + x_{1,4}x_{2,3}.$$

The *Grassmannian* is the projectivization of the Grassmann cone:

$$\mathrm{Gr}(k,V) := \mathbb{P}\left(\widehat{\mathrm{Gr}}(k,V)\right) = \left(\widehat{\mathrm{Gr}}(k,V) \setminus \{0\}\right)/\mathbb{K}^* \subseteq \mathbb{P}\Big(\bigwedge^k V\Big).$$

It is a projective variety whose defining equations are the Plücker relations.

Notation 1.1.1. From now on, for vectors $v_1, \ldots, v_k \in V$, we will write $\langle v_1, \cdots, v_k \rangle$ to denote $\mathrm{span}\{v_1, \ldots, v_k\}$.

Definition 1.1.2 (GCP map)**.** A linear map $\varphi : \bigwedge^k V \to \bigwedge^q W$ is *Grassmann cone preserving* (GCP) if

$$\varphi\left(\widehat{\mathrm{Gr}}(k,V)\right) \subseteq \widehat{\mathrm{Gr}}(q,W).$$

Example 1.1.3. We give two examples of GCP maps.

1. If $f : V \to W$ is a linear map, the induced map $\bigwedge^k f : \bigwedge^k V \to \bigwedge^k W$ is Grassmann cone preserving.

2. For $\beta \in V^*$ the *contraction map*

$$i_\beta : \bigwedge^k V \to \bigwedge^{k-1} \ker \beta \subseteq \bigwedge^{k-1} V$$

defined as

$$v_1 \wedge \cdots \wedge v_k \mapsto \sum_{j=1}^{k} (-1)^{j-1} \beta(v_i) v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_k$$

is GCP. Below we give a coordinate-independent description of i_β.

Proof. Let $v_1 \wedge \cdots \wedge v_k \in \widehat{\mathrm{Gr}}(k, V) \setminus \{0\}$. Note that the vectors $v_1, \ldots, v_k$ are linearly independent. We will distinguish two cases. In the first case, assume that $\langle v_1, \ldots, v_k \rangle \subseteq \ker \beta$. Then, $i_\beta(v_1 \wedge \cdots \wedge v_k) = 0 \in \widehat{\mathrm{Gr}}(k-1, \ker \beta)$. Now consider the case that $\ker \beta \cap \langle v_1, \ldots, v_k \rangle$ has dimension $k-1$. After possibly replacing $v_1, \ldots, v_k$ with some $v_1', \ldots, v_k'$ such that $v_1' \wedge \cdots \wedge v_k' = v_1 \wedge \cdots \wedge v_k$, we can assume that $v_1 \notin \ker \beta$, but $v_2, \ldots, v_k \in \ker \beta$. Then, $i_\beta(v_1 \wedge \cdots \wedge v_k) = \beta(v_1) v_2 \wedge \cdots \wedge v_k$ is contained in $\widehat{\mathrm{Gr}}(k-1, \ker \beta)$. This proof also shows that i_β takes values in $\bigwedge^{k-1} \ker \beta$. □

The contraction map i_β can also be described coordinate-independently. Recall that there is a natural isomorphism $\bigwedge^k V \cong \mathrm{Alt}^k(V^*)$, where $\mathrm{Alt}^k(V^*)$ is the space of alternating multilinear maps $V^* \times \cdots \times V^* \to \mathbb{K}$. Under this identification, i_β agrees with the map

$$\mathrm{Alt}^k(V^*) \to \mathrm{Alt}^{k-1}(V^*), \quad \omega \mapsto \omega(\beta, \cdot, \ldots, \cdot).$$

Next, we recall the universality result by Kasman et al.

Theorem 1.1.4 ([KRPS08, Theorem 3.4])**.** *Let $\omega \in \bigwedge^k V$. Then $\omega \in \widehat{\mathrm{Gr}}(k, V)$ if and only if every GCP map to $\bigwedge^2 \mathbb{K}^4$ maps ω to $\widehat{\mathrm{Gr}}(2, 4)$.*

In fact, Kasman et al. show that the GCP maps can be chosen from an explicit finite collection. We can rephrase Theorem 1.1.4 in terms of the Klein quadric, as follows.

Corollary 1.1.5. *Any Grassmannian is set-theoretically defined by pullbacks of the Klein quadric $P_{2,4}$ along all GCP maps to $\bigwedge^2 \mathbb{K}^4$, i.e.,*

$$\widehat{\mathrm{Gr}}(k, V) = \left\{ \omega \in \bigwedge^k V \;\middle|\; P_{2,4}(\varphi(\omega)) = 0 \quad \forall \varphi \in GCP\left(\bigwedge^k V, \bigwedge^2 \mathbb{K}^4\right) \right\}.$$

1.2 Quadratic Spaces and the Isotropic Grassmannian

Throughout the remainder of this chapter, we will work in a field $\mathbb{K}$ of characteristic not 2. In this section, we will introduce quadratic spaces and isotropic Grassmannians, and establish several essential lemmas that we will later use to prove our main theorem.

1.2.1 Quadratic Spaces

In this subsection we introduce quadratic spaces. The material is fairly standard, for a reference see [Art57, Chapter 3]. For Lemma 1.2.6 we did not find a proof in the literature, so we opted to give a proof here.

A *quadratic space* refers to a vector space V equipped with a quadratic form, or equivalently, a symmetric bilinear form $(\cdot|\cdot)$. We always assume that the bilinear form is non-degenerate. A vector $v \in V$ is considered *isotropic* if $(v|v) = 0$. The set of all isotropic vectors in V is denoted by V_{iso}. The *orthogonal complement* of a subspace $L \subseteq V$ is defined as the space $L^\perp := \{v \in V \mid (v|u) = 0 \ \forall u \in L\}$. We call a subspace $L \subseteq V$ *isotropic* if $L \subseteq L^\perp$, i.e., if $(u|v) = 0$ for all $u, v \in L$. By polarization, using $\mathrm{Char}(\mathbb{K}) \neq 2$, this is equivalent to $L \subseteq V_{\text{iso}}$. If L is isotropic but any proper superset $L' \supsetneq L$ is not isotropic, we refer to L as *maximal isotropic*.

Definition 1.2.1 (Hyperbolic basis)**.** We call a collection of vectors $e_1, e_{-1}, \dots, e_k, e_{-k}$ in V *hyperbolic* if $(e_i|e_{-i}) = 1$ for $i = 1, \dots, k$, and $(e_i|e_j) = 0$ if $i \neq -j$. Note that the e_i are necessarily linearly independent. If $2k = \dim V$, then we call $e_1, e_{-1}, \dots, e_k, e_{-k}$ a *hyperbolic basis* of V.

Theorem 1.2.2. *Let L be an isotropic subspace of V, and $e_1, \dots, e_k$ a basis of L. Then we can find vectors $e_{-1}, \dots, e_{-k} \in V \setminus L$ such that $e_1, e_{-1}, \dots, e_k, e_{-k}$ forms a hyperbolic collection of vectors.*

Proof. This is [Art57, Theorem 3.8] in the case where $U = L$ is isotropic. □

Theorem 1.2.3 (See [Art57, Theorem 3.10])**.** *All maximal isotropic subspaces of V have the same dimension, which is referred to as the* Witt index *of V.*

Note that by Theorem 1.2.2, the Witt index can be at most $\lfloor \frac{\dim V}{2} \rfloor$. Moreover, this upper bound is attained when $\mathbb{K}$ is algebraically closed, regardless of the chosen non-degenerate quadratic form.

Convention 1.2.4. From this point onward, we make the assumption that V has maximal Witt index $\lfloor \frac{\dim V}{2} \rfloor$. We will denote this Witt index by p.

Remark 1.2.5. If $\dim V = 2p$ is even, then by Theorem 1.2.2, V has a hyperbolic basis. Note that then a subspace L is maximal isotropic if and only if $L = L^\perp$.

If $\dim V = 2p + 1$ is odd, then V has a basis

$$e_1, e_{-1}, \dots, e_p, e_{-p}, e_0 \tag{1.2.1}$$

such that $e_1, e_{-1}, \dots, e_p, e_{-p}$ is hyperbolic and $(e_0|e_i) = 0$ for all $i \neq 0$. We will call $e_1, e_{-1}, \dots, e_p, e_{-p}, e_0$ hyperbolic as well. Note that $(e_0|e_0) \neq 0$ by non-degeneracy. If $\mathbb{K}$ is algebraically closed we can rescale e_0 such that $(e_0|e_0) = 1$; in general we will write $c_0 := \frac{1}{2}(e_0|e_0)$. Note that we can also find a basis of V consisting of isotropic vectors, for instance by replacing e_0 by $e_0 + e_1 - c_0 e_{-1}$ in (1.2.1).

The following lemma will be used several times in the proof of our main theorem (to be precise: in Claim 1.3.8, Claim 1.3.10 and Claim 1.3.11).

Lemma 1.2.6. *Let $W_1, W_2 \subseteq V$ be maximal isotropic subspaces. Then for any choice of decomposition*

$$W_1 = (W_1 \cap W_2) \oplus U_1 \quad \text{and} \quad W_2 = (W_1 \cap W_2) \oplus U_2$$

the isomorphism $V \to V^*, v \mapsto (v|\cdot)$ restricts to an isomorphism $U_1 \to U_2^*$.

In particular, there exists a hyperbolic basis $e_1, e_{-1}, \ldots, e_p, e_{-p}, (e_0)$ of V, such that

$$W_1 = \langle e_1, \ldots, e_q, e_{q+1}, \ldots, e_p \rangle \quad \text{and} \quad W_2 = \langle e_1, \ldots, e_q, e_{-(q+1)}, \ldots, e_{-p} \rangle,$$

where $q = \dim(W_1 \cap W_2)$.

Proof. Note that U_1 and U_2 have the same dimension because the maximal isotropic subspaces W_1 and W_2 have the same dimension. Thus it suffices to show that the map $U_1 \to U_2^*$ is injective. Arguing by contradiction, assume there is some $u_1 \in U_1 \setminus \{0\}$ such that $(u_1|u_2) = 0$ for all $u_2 \in U_2$. Then it also holds that $(u_1|w_2) = 0$ for all $w_2 \in (W_1 \cap W_2) \oplus U_2 = W_2$ because $(W_1 \cap W_2) \oplus \langle u_1 \rangle \subseteq W_1$ is isotropic. Since W_2 and $u_1 \in U_1 \subseteq W_1$ are isotropic, this implies that also $W_2 \oplus \langle u_1 \rangle$ is isotropic. But W_2 is strictly contained in $W_2 \oplus \langle u_1 \rangle$ because $u_1 \in U_1 \setminus \{0\}$, contradicting the fact that W_2 is maximal isotropic.

To see how the first statement implies the second one, choose a basis $e_1, \ldots, e_p$ of W_1 such that $e_1, \ldots, e_q$ forms a basis for $W_1 \cap W_2$. Let $U_1 = \langle e_{q+1}, \ldots, e_p \rangle$ and choose some U_2 such that $W_2 = (W_1 \cap W_2) \oplus U_2$. It follows from the first part that there are unique $e_{-(q+1)}, \ldots, e_{-p} \in U_2$ such that $(e_i|e_{-j}) = \delta_{ij}$ for $i, j = q+1, \ldots, p$. Since W_1 and W_2 are isotropic, it holds that $W_1 \cap W_2 \subseteq U_1^\perp \cap U_2^\perp$, and $W_1 \cap W_2$ is a maximal isotropic subspace of $U_1^\perp \cap U_2^\perp$ by reasons of dimension. So there exist $e_{-1}, \ldots, e_{-q}, (e_0) \in U_1^\perp \cap U_2^\perp$ such that $e_1, e_{-1}, \ldots, e_q, e_{-q}, (e_0)$ forms a hyperbolic basis of $U_1^\perp \cap U_2^\perp$. This completes the proof. □

1.2.2 The Isotropic Grassmann Cone

We now introduce the isotropic Grassmann cone. We continue to work in a quadratic space V of dimension either $2p$ or $2p+1$ satisfying Convention 1.2.4.

Definition 1.2.7 (Isotropic Grassmann cone)**.** For $k \leq p$, the *isotropic Grassmann cone* is defined as

$$\widehat{\mathrm{Gr}}_{\mathrm{iso}}(k, V) := \left\{ v_1 \wedge \cdots \wedge v_k \in \bigwedge^k V \;\middle|\; (v_i|v_j) = 0 \text{ for all } 1 \leq i, j \leq k \right\}.$$

If $k = p$, then we call it the *maximal isotropic Grassmann cone.*

Note that $\omega \in \widehat{\mathrm{Gr}}(k, V) \setminus \{0\}$ lies in $\widehat{\mathrm{Gr}}_{\mathrm{iso}}(k, V)$ if and only if $L_\omega \subseteq V$ is isotropic.

Definition 1.2.8 (IGCP map)**.** A linear map $\Phi : \bigwedge^k V \to \bigwedge^q W$ is *isotropic Grassmann cone preserving* (IGCP) if

$$\Phi\left(\widehat{\mathrm{Gr}}_{\mathrm{iso}}(k, V)\right) \subseteq \widehat{\mathrm{Gr}}_{\mathrm{iso}}(q, W).$$

.

We will only need one explicit family of IGCP maps; they are the analogue of the GCP maps from Example 1.1.3. Let $v \in V_{\mathrm{iso}}$ be a nonzero isotropic vector. Define $V_v := v^{\perp}/\langle v\rangle$ (note that $\langle v\rangle \subseteq v^{\perp}$ because v is isotropic). It is easy to see that

$$(\bar{v}_1|\bar{v}_2)_{V_v} := (v_1|v_2)_V,$$

where $\bar{v}_i \in V_v$ denotes the equivalence class of $v_i \in v^{\perp}$ in V_v, is a well-defined non-degenerate bilinear form on V_v (i.e., the formula is independent of the choice of representatives $v_1, v_2 \in v^{\perp}$). Moreover, $(V_v, (\cdot|\cdot)_{V_v})$ again has maximal Witt index. We denote by π_v the projection $v^{\perp} \twoheadrightarrow V_v$.

Definition 1.2.9. For $v \in V_{\mathrm{iso}} \setminus \{0\}$ we define the linear map

$$\Phi_v : \bigwedge^k V \to \bigwedge^{k-1} V_v$$

as the following composition

$$\bigwedge^k V \xrightarrow{\varphi_v} \bigwedge^{k-1} v^{\perp} \xrightarrow{\bigwedge^{k-1}\pi_v} \bigwedge^{k-1} V_v,$$

where $\varphi_v := i_{(v|\cdot)}$ is the contraction map introduced in Example 1.1.3. Explicitly, this map is given by

$$\Phi_v(v_1 \wedge \cdots \wedge v_k) = \sum_{j=1}^{k} (-1)^{j-1}(v|v_j)\bar{v}_1 \wedge \cdots \wedge \widehat{v}_j \wedge \cdots \wedge \bar{v}_k. \tag{1.2.2}$$

Since Φ_v is a composition of two GCP maps, it is itself GCP. By the same proof as Example 1.1.3, one readily sees that Φ_v is in fact IGCP. More explicitly, the following holds.

Lemma 1.2.10. *For $\omega \in \widehat{\mathrm{Gr}}_{\mathrm{iso}}(k, V)$,*

1. *if $v \in L_\omega^{\perp}$, then $\Phi_v(\omega) = 0$,*
2. *if $v \notin L_\omega^{\perp}$, then $\Phi_v(\omega) \neq 0$, and*

$$L_{\Phi_v(\omega)} = (L_\omega \cap v^{\perp})/\langle v\rangle.$$

Proof. If $v \in L_\omega^{\perp}$, then $\langle v, w\rangle = 0$ for all $w \in L_\omega$. Therefore, $\varphi_v(\omega) = 0$ and hence also $\Phi_v(\omega) = 0$. This proves the first statement. For the second statement, suppose $v \notin L_\omega^{\perp}$ and choose a basis where $\omega = v_1 \wedge \cdots \wedge v_k$, $L_\omega = \langle v_1, \ldots, v_k\rangle$ and $L_\omega \cap v^{\perp} = \langle v_1, \ldots, v_{k-1}\rangle$. By evaluating $\Phi_v(\omega)$ using (1.2.2), we obtain the result. □

We now give a more coordinate-independent description of Φ_v. The bilinear form on V induces an isomorphism $V \xrightarrow{\cong} V^*, v \mapsto (v|\cdot)$. Together with the natural isomorphism $\bigwedge^k V^* \cong \mathrm{Alt}^k V$, this yields an isomorphism $\flat : \bigwedge^k V \xrightarrow{\cong} \mathrm{Alt}^k V$. Then Φ_v is the composition

$$\bigwedge^k V \xrightarrow{\flat} \mathrm{Alt}^k V \xrightarrow{\Phi_v^\flat} \mathrm{Alt}^{k-1} V_v \xrightarrow{\flat^{-1}} \bigwedge^{k-1} V_v,$$

where the middle map $\Phi_v^\flat$ is given by the formula

$$\Phi_v^\flat(\omega^\flat)(\bar{v}_1, \ldots, \bar{v}_{k-1}) = \omega^\flat(v, v_1, \ldots, v_{k-1}) \tag{1.2.3}$$

with $v_1, \ldots, v_{k-1} \in V^{\perp}$. Note that since $\omega^\flat$ is alternating, this does *not* depend on a choice of representatives $v_i \in V^{\perp}$ for $\bar{v}_i \in V_v$.

1.2.3 Two Lemmas About IGCP Maps

We finish this section by proving two lemmas that will play a central role throughout the proof of our main theorem. The first lemma states that no nonzero ω are annihilated by all IGCP maps:

Lemma 1.2.11. *Let $\omega \in \bigwedge^k V$ with $0 < k < \dim V$. If $\Phi_v(\omega) = 0$ for all $v \in V_{\text{iso}} \setminus \{0\}$, then $\omega = 0$.*

Proof. If $\Phi_v(\omega) = 0$ for all $v \in V_{\text{iso}} \setminus \{0\}$, then $\omega^\flat(v, v_2, \dots, v_k) = 0$ for all $v \in V_{\text{iso}} \setminus \{0\}$ and $v_2, \dots, v_k \in v^\perp$ due to (1.2.3). But then by Proposition 1.2.12 below, $\omega^\flat(w_1, \dots, w_k) = 0$ for all $w_1, \dots, w_k \in V$. So $\omega^\flat = 0$, and hence $\omega = 0$. This completes the proof. $\square$

Proposition 1.2.12. *If $0 < k < \dim V$, then the set*

$$\left\{ v \wedge v_2 \wedge \cdots \wedge v_k \in \bigwedge^k V \,\middle|\, v \in V_{\text{iso}} \setminus \{0\} \text{ and } v_2, \dots, v_k \in v^\perp \right\}$$

spans $\bigwedge^k V$.

Proof. Let $\mathcal{S}$ be the span of the given set in $\bigwedge^k V$. We choose a hyperbolic basis $e_1, e_{-1}, \dots, e_p, e_{-p}, (e_0)$ for V. It suffices to show that each pure wedge $e_{i_1} \wedge \cdots \wedge e_{i_k}$ is in $\mathcal{S}$. If there exists $j \neq 0$ such that $\#(\{j, -j\} \cap \{i_1, \dots, i_k\}) = 1$, then clearly $e_{i_1} \wedge \cdots \wedge e_{i_k} \in \mathcal{S}$. So we only need to show that $e_{j_1} \wedge e_{-j_1} \wedge \cdots \wedge e_{j_m} \wedge e_{-j_m} \in \mathcal{S}$ when $k = 2m$, or $e_{j_1} \wedge e_{-j_1} \wedge \cdots \wedge e_{j_m} \wedge e_{-j_m} \wedge e_0 \in \mathcal{S}$ when $k = 2m + 1$, where $j_1, ..., j_m \in \{1, ..., p\}$.

If $m < p$, we choose an index $j_0 \in \{1, \dots, p\} \setminus \{j_1, \dots, j_m\}$. We then define η as $\eta = e_{j_2} \wedge e_{-j_2} \wedge \cdots \wedge e_{j_m} \wedge e_{-j_m}$ if k is even, and $\eta = e_{j_2} \wedge e_{-j_2} \wedge \cdots \wedge e_{j_m} \wedge e_{-j_m} \wedge e_0$ if k is odd. Based on the definition of j_0 and $\mathcal{S}$, we have $(e_{j_0} + e_{j_1}) \wedge (e_{-j_0} - e_{-j_1}) \wedge \eta \in \mathcal{S}$. Expanding this expression, we obtain:

$$(e_{j_0} + e_{j_1}) \wedge (e_{-j_0} - e_{-j_1}) \wedge \eta = (e_{j_0} \wedge e_{-j_0} - e_{j_1} \wedge e_{-j_1}) \wedge \eta + (\text{terms in } \mathcal{S}).$$

Therefore, we conclude that

$$(e_{j_0} \wedge e_{-j_0} - e_{j_1} \wedge e_{-j_1}) \wedge \eta \in \mathcal{S}. \tag{1.2.4}$$

Similarly, by considering $(e_{j_0} + e_{-j_1}) \wedge (e_{-j_0} - e_{j_1}) \wedge \eta \in \mathcal{S}$, we can deduce

$$(e_{j_0} \wedge e_{-j_0} - e_{-j_1} \wedge e_{j_1}) \wedge \eta \in \mathcal{S}. \tag{1.2.5}$$

By subtracting (1.2.4) from (1.2.5) and using the anti-symmetry of the wedge product $\wedge$, we obtain that $2e_{j_1} \wedge e_{-j_1} \wedge \eta \in \mathcal{S}$. Given that $\text{Char}(\mathbb{K}) \neq 2$, this implies that $e_{j_1} \wedge e_{-j_1} \wedge \cdots \wedge e_{j_m} \wedge e_{-j_m} (\wedge e_0) \in \mathcal{S}$.

We still need to consider the case where $m = p$; i.e., we still need to show that $e_1 \wedge e_{-1} \wedge \cdots \wedge e_p \wedge e_{-p} \in \mathcal{S}$ if $\dim V = 2p+1$. For this we write $\eta = e_2 \wedge e_{-2} \wedge \cdots \wedge e_p \wedge e_{-p}$ as before, and note that

$$2c_0 e_1 \wedge e_{-1} \wedge \eta = \big((e_0 + e_1 - c_0 e_{-1}) \wedge (e_1 + c_0 e_{-1}) - e_0 \wedge e_1 - c_0 e_0 \wedge e_{-1}\big) \wedge \eta \in \mathcal{S},$$

where $c_0 = \frac{1}{2}(e_0|e_0)$. $\square$

The second lemma is a more technical variant of Lemma 1.2.11. We will use it to prove Claims 1.3.12 and 1.3.13 in the proof of the main theorem.

Lemma 1.2.13. *Assume $p \geq 2$ and $0 < k < \dim V$, and let $\omega \in \bigwedge^k V$ be nonzero. Let W and W' be maximal isotropic subspaces of V with $\dim(W \cap W') = p-1$, and suppose that $\Phi_v(\omega) = 0$ for every isotropic $v \in W \cup W'$.*

- *If $k > p$, then ω is of the form $\alpha \wedge \omega'$, where α lies in the one-dimensional space $\bigwedge^{p+1}(W + W')$.*
- *If $k \leq p$ then $\omega \in \bigwedge^k (W^\perp \cap W'^\perp)$.*

Proof. We choose a hyperbolic basis of the vector space V such that $W = \langle e_1, e_2, \ldots, e_p \rangle$ and $W' = \langle e_{-1}, e_2, \ldots, e_p \rangle$. For $\{i_1, \ldots, i_\ell\} \subset \{1, -1, \ldots, p, -p, (0)\}$, we will then write

$$V_{\widehat{i_1}, \ldots, \widehat{i_\ell}} \quad \text{for } \langle e_i \mid i \notin \{i_1, \ldots, i_\ell\} \rangle \subseteq V.$$

We prove by induction on $i = 2, \ldots, p+1$ that

$$\omega = e_1 \wedge e_{-1} \wedge e_2 \wedge \cdots \wedge e_{i-1} \wedge \omega'_i + \omega''_i, \tag{1.2.6}$$

with

$$\omega'_i \in \bigwedge^{k-i} V_{\widehat{1}, \widehat{2}, \widehat{3}, \ldots, \widehat{i-1}, -\widehat{1}} \quad \text{and} \quad \omega''_i \in \bigwedge^k V_{\widehat{1}, -\widehat{1}, -\widehat{2}, \ldots, -\widehat{i+1}},$$

and we put the first summand equal to zero if $i > k$.

First, let us show that (1.2.6) holds for $i = 2$. Indeed we can write

$$\omega = e_1 \wedge e_{-1} \wedge \omega'_2 + e_1 \wedge \alpha + e_{-1} \wedge \beta + \omega''_2$$

with

$$\omega'_2 \in \bigwedge^{k-2} V_{\widehat{1}, -\widehat{1}}, \qquad \alpha, \beta \in \bigwedge^{k-1} V_{\widehat{1}, -\widehat{1}}, \qquad \omega''_2 \in \bigwedge^k V_{\widehat{1}, -\widehat{1}}.$$

By assumption we have

$$0 = \Phi_{e_1}(\omega) = \Phi_{e_1}(e_{-1} \wedge \alpha),$$
$$0 = \Phi_{e_{-1}}(\omega) = \Phi_{e_{-1}}(e_1 \wedge \beta),$$

hence $\alpha = \beta = 0$.

Next we assume (1.2.6) for some i, and want to show it for $i+1$. We can write

$$\omega'_i = e_i \wedge \omega'_{i+1} + e_{-i} \wedge \alpha' + \beta'$$
$$\omega''_i = \omega''_{i+1} + e_i \wedge e_{-i} \wedge \alpha'' + e_{-i} \wedge \beta'',$$

where

$$\omega'_{i+1} \in \bigwedge^{k-i-1} V_{\widehat{1}, \widehat{2}, \ldots, \widehat{i}, -\widehat{1}}, \qquad \omega''_{i+1} \in \bigwedge^k V_{\widehat{1}, -\widehat{1}, -\widehat{2}, \ldots, -\widehat{i}},$$
$$\alpha' \in \bigwedge^{k-i-1} V_{\widehat{1}, \widehat{2}, \ldots, \widehat{i}, -\widehat{1}, -\widehat{i}}, \qquad \alpha'' \in \bigwedge^{k-2} V_{\widehat{1}, \widehat{i}, -\widehat{1}, -\widehat{2}, \ldots, -\widehat{i}},$$
$$\beta' \in \bigwedge^{k-i} V_{\widehat{1}, \widehat{2}, \ldots, \widehat{i}, -\widehat{1}, -\widehat{i}}, \qquad \beta'' \in \bigwedge^{k-1} V_{\widehat{1}, \widehat{i}, -\widehat{1}, -\widehat{2}, \ldots, -\widehat{i}}.$$

We compute $0 = \Phi_{e_i}(\omega) = \bar{e}_1 \wedge \bar{e}_{-1} \wedge \bar{e}_2 \wedge \cdots \wedge \bar{e}_{i-1} \wedge \overline{\alpha'} + \overline{\beta''}$, so we can conclude that $\alpha' = \beta'' = 0$.

Next we compute

$$\begin{aligned}\Phi_{e_i - e_1}(\omega) =& \Phi_{e_i - e_1}(e_1 \wedge e_{-1} \wedge e_2 \wedge \cdots \wedge e_{i-1} \wedge e_i \wedge \omega'_{i+1}) \\ &+ \Phi_{e_i - e_1}(e_1 \wedge e_{-1} \wedge e_2 \wedge \cdots \wedge e_{i-1} \wedge \beta') \\ &+ \Phi_{e_i - e_1}(\omega''_{i+1}) \\ &+ \Phi_{e_i - e_1}(e_i \wedge e_{-i} \wedge \alpha'').\end{aligned}$$

The first and third summand are zero by Lemma 1.2.10. So we get

$$\Phi_{e_i - e_1}(\omega) = \bar{e}_1 \wedge \bar{e}_2 \wedge \cdots \wedge \bar{e}_{i-1} \wedge \overline{\beta'} - \bar{e}_i \wedge \overline{\alpha''},$$

so $e_2 \wedge \cdots \wedge e_{i-1} \wedge \beta' = \alpha''$. If we do the analogous computation for $\Phi_{e_i - e_{-1}}(\omega)$ we find that $e_2 \wedge \cdots \wedge e_{i-1} \wedge \beta' = -\alpha''$. So $\beta' = \alpha'' = 0$, and we get

$$\omega = e_1 \wedge e_{-1} \wedge e_2 \wedge \cdots \wedge e_i \wedge \omega'_{i+1} + \omega''_{i+1},$$

which is exactly (1.2.6) for $i+1$ instead of i.

Finally, note that the case $i = p+1$ is exactly what we want. Indeed we have

$$\omega = e_1 \wedge e_{-1} \wedge e_2 \wedge \cdots \wedge e_p \wedge \omega' + \omega'',$$

with $\omega' \in \bigwedge^{k-p-1} V_{\hat{1},\hat{2},\ldots,\hat{p},-\hat{1}}$ and $\omega'' \in \bigwedge^k V_{\hat{1},\hat{2},\ldots,\hat{p},-\hat{1}} = \bigwedge^k (W^\perp \cap W'^\perp)$. However, if $k \leq p$, the first summand is zero, and if $k > p$ the second summand is zero since $\dim V_{\hat{1},\hat{2},\ldots,\hat{p},-\hat{1}} = \dim V - p - 1 \leq p < k$. □

1.3 Universality for Maximal Isotropic Grassmannians

1.3.1 Statement and Consequences of the Main Result

For this entire section, let V be a quadratic space of maximal Witt index $p = \lfloor \frac{\dim V}{2} \rfloor$ over a field $\mathbb{K}$ of characteristic not 2.

Theorem 1.3.1 (Main Result). *Assume* $\dim V > 8$ *and let* $\omega \in \bigwedge^p V$. *If for every isotropic vector* $v \in V_{\text{iso}}$, *the image of* v *under the isotropic Grassmann cone preserving map* Φ_v *lies in* $\widehat{\text{Gr}}_{\text{iso}}(p-1, V_v)$, *then* ω *itself lies in* $\widehat{\text{Gr}}_{\text{iso}}(p, V)$.

From this we easily deduce the following.

Corollary 1.3.2 (Universality). *For any* $\omega \in \bigwedge^p V$, *it holds that* $\omega \in \widehat{\text{Gr}}_{\text{iso}}(p, V)$ *if and only if*

- *every IGCP map to* $\bigwedge^3 \mathbb{K}^7$ *maps* ω *to* $\widehat{\text{Gr}}_{\text{iso}}(3, 7)$, *if* $\dim V = 2p+1$,
- *every IGCP map to* $\bigwedge^4 \mathbb{K}^8$ *maps* ω *to* $\widehat{\text{Gr}}_{\text{iso}}(4, 8)$, *if* $\dim V = 2p$.

Proof of Corollary 1.3.2 assuming Theorem 1.3.1. One direction follows directly from the definition of an IGCP map. To prove the other direction, we consider three cases depending on the dimension of V.

If $\dim V > 8$, we can repeatedly apply Theorem 1.3.1 to obtain the desired result. If $\dim V = 8$ or $\dim V = 7$, we can just apply the assumption to the identity map (which is trivially preserves the isotropic Grassmann cone). For $\dim V < 7$, we observe that the map $\varphi : \bigwedge^p V \to \bigwedge^{p+1} (V \oplus \langle e_{p+1}, e_{-p-1}\rangle)$, which sends ω to $\omega \wedge e_{p+1}$, has the property that ω lies in the isotropic Grassmann cone if and only if $\varphi(\omega)$ lies in the isotropic Grassmann cone. By applying these maps iteratively until we reach $\bigwedge^3 \mathbb{K}^7$ or $\bigwedge^4 \mathbb{K}^8$, we complete the proof. □

Similar to [KRPS08, Theorem 4.1], we obtain a statement about the ranks of quadrics defining the isotropic Grassmann cone, where we use the fact that $\widehat{\mathrm{Gr}}_{\mathrm{iso}}(p, 2p)$ has two irreducible components [Har92, Theorem 22.14].

Corollary 1.3.3. *The isotropic Grassmannian $\widehat{\mathrm{Gr}}_{\mathrm{iso}}(p, 2p+1)$ in its Plücker embedding can be defined by quadrics of rank at most 4. Furthermore, both irreducible components of $\widehat{\mathrm{Gr}}_{\mathrm{iso}}(p, 2p)$ can be defined by linear equations and quadrics of rank at most 4.*

Proof. By Corollary 1.1.5 it suffices to show the statement is true for $\widehat{\mathrm{Gr}}_{\mathrm{iso}}(3, 7)$ and $\widehat{\mathrm{Gr}}_{\mathrm{iso}}(4, 8)$. This can be done by an explicit calculation, see Section 1.5. □

Remark 1.3.4. The statement in Corollary 1.3.3 can also be deduced using the Cartan embedding, see Section 1.5.

Remark 1.3.5. A natural question arises: is there a similar result if we replace the symmetric form with a skew-symmetric form, focusing on Lagrangian Grassmannians? The answer, in the case of considering only the Lagrangian Grassmann cone preserving (LGCP) maps Φ_v for $v \in V$, defined as in Definition 1.2.9, is no.

To illustrate this, consider an 8-dimensional vector space V with basis $e_1, \ldots, e_{-4}$ and skew-symmetric form given by $(e_i|e_{-i}) = 1$ and $(e_{-i}|e_i) = -1$ for $i \geq 1$, and all other pairings equal to 0.

Now, consider the 2-form $\alpha = e_1 \wedge e_{-1} + e_2 \wedge e_{-2} + e_3 \wedge e_{-3} + e_4 \wedge e_{-4}$, and define

$$\omega = \alpha \wedge \alpha = 2 \sum_{1 \leq i < j \leq 4} e_i \wedge e_{-i} \wedge e_j \wedge e_{-j} \in \bigwedge^4 V.$$

It can be observed that ω does not lie in the Grassmann cone since $\omega \wedge \omega$ is a nonzero multiple of $e_1 \wedge e_{-1} \wedge e_2 \wedge e_{-2} \wedge e_3 \wedge e_{-3} \wedge e_4 \wedge e_{-4}$. However, upon explicit computation, it can be seen that every LGCP map Φ_v maps ω to zero, and thus it lies in the Lagrangian Grassmann cone.

This example can be generalized to any space of dimension $4m$ by considering the m-form $\omega = \alpha^{\wedge m} \in \bigwedge^{2m} V$. Therefore, we have a counterexample to the analogue of Theorem 1.3.1 (and even to the analogue of Lemma 1.2.11). However, it is not yet a counterexample to the analogue of Corollary 1.3.2, as there might be additional LGCP maps that could be considered.

1.3.2 Structure of the Proof

The aim of this subsection is twofold. First, we aim to prove Proposition 1.3.7, which will serve as the key ingredient in proving Theorem 1.3.1. Secondly, we will give an outline of the proof of Theorem 1.3.1 to make it more accessible, as it involves some technical aspects.

We assume $p \geq 2$. Note that we can always decompose V as

$$V = V' \oplus \langle e_p, e_{-p} \rangle, \tag{1.3.1}$$

where e_p, e_{-p} is a collection of hyperbolic vectors, and $V' := \langle e_p, e_{-p} \rangle^{\perp}$ which again has maximal Witt index. For the remaining part of this section, we will be working with this fixed decomposition. Any $\omega \in \bigwedge^p V$ can be uniquely written as

$$\omega = \omega_1 \wedge e_p \wedge e_{-p} + \omega_2 \wedge e_p + \omega_3 \wedge e_{-p} + \omega_4 \tag{1.3.2}$$

where $\omega_1 \in \bigwedge^{p-2} V'$, $\omega_2, \omega_3 \in \bigwedge^{p-1} V'$ and $\omega_4 \in \bigwedge^p V'$. The following observation shows that for $v \in V'$, a decomposition of ω maps to a decomposition of $\Phi_v(\omega)$.

Observation 1.3.6. *Let ω be as in (1.3.2). Then for any $v \in V'$ we have*

$$\Phi_v(\omega) =: \omega' = \omega_1' \wedge \bar{e}_p \wedge \bar{e}_{-p} + \omega_2' \wedge \bar{e}_p + \omega_3' \wedge \bar{e}_{-p} + \omega_4', \tag{1.3.3}$$

where $\omega_i' = \Phi_v(\omega_i)$.

Next, we give conditions for ω to be in the isotropic Grassmann cone.

Proposition 1.3.7. *Suppose we have written $\omega \in \bigwedge^p V$ in the form given by (1.3.2). Assume $\omega \in \widehat{\mathrm{Gr}}_{\mathrm{iso}}(p, V)$, then one of the following holds:*

1. *$\omega_1 = \omega_3 = \omega_4 = 0$ and $\omega_2 \in \widehat{\mathrm{Gr}}_{\mathrm{iso}}(p-1, V')$,*

2. *$\omega_1 = \omega_2 = \omega_4 = 0$ and $\omega_3 \in \widehat{\mathrm{Gr}}_{\mathrm{iso}}(p-1, V')$,*

3. *$\omega_1 = 0$, and ω_2, ω_3, ω_4 are nonzero. Then*
   - *$\omega_2, \omega_3 \in \widehat{\mathrm{Gr}}_{\mathrm{iso}}(p-1, V')$, $\omega_4 \in \widehat{\mathrm{Gr}}(p, V')$,*
   - *$L_{\omega_2} = L_{\omega_3} \subseteq L_{\omega_4}$.*

   This case only occurs if $\dim V$ is odd.

4. *ω_1, ω_2, ω_3, ω_4 are all nonzero. Then*
   - *$\omega_1 \in \widehat{\mathrm{Gr}}_{\mathrm{iso}}(p-2, V')$, $\omega_2, \omega_3 \in \widehat{\mathrm{Gr}}_{\mathrm{iso}}(p-1, V')$, $\omega_4 \in \widehat{\mathrm{Gr}}(p, V')$,*
   - *$L_{\omega_2} \cap L_{\omega_3} = L_{\omega_1}$ and $L_{\omega_2} + L_{\omega_3} = L_{\omega_4}$.*

Proof. We define $L' := L_\omega \cap V'$. Note that $p - 2 \leq \dim L' \leq p - 1$, where the second inequality holds since L' is an isotropic subspace of V'. We proceed by considering cases based on $\dim L'$. More precisely, we will show that (1), (2) or (3) hold if $\dim L' = p - 1$, and that (4) holds if $\dim L' = p - 2$.

Case 1. If $\dim L' = p - 1$, then L' is a maximal isotropic subspace of V'. Since L' has codimension one in L_ω, there exists a vector $v \in L_\omega$ such that $L_\omega = L' + \langle v \rangle$. Since L_ω is isotropic, we have $L_\omega \subseteq L_\omega^\perp \subseteq L'^\perp$. Therefore, $v \in L'^\perp$. We can write $v = w + v'$, where $w \in \langle e_p, e_{-p}\rangle$ and $v' \in V'$. Moreover, note that $w \in L'^\perp$, and therefore, $v' = v - w \in L'^\perp$. If $\dim V$ is even, then $L' = L'^\perp \cap V'$, hence $v' \in L'$. Consequently, we have

$$L_\omega = L' + \langle w + v'\rangle = L' + \langle w\rangle.$$

Since ω is isotropic, the vector w is also isotropic. Thus, we can conclude that either

$$w \in \langle e_p\rangle \quad \text{or} \quad w \in \langle e_{-p}\rangle.$$

So we conclude

$$L_\omega = L' + \langle e_p\rangle \quad \text{or} \quad L_\omega = L' + \langle e_{-p}\rangle,$$

and therefore

$$\omega = \omega_2 \wedge e_p \quad \text{or} \quad \omega = \omega_3 \wedge e_{-p},$$

where $\omega_2, \omega_3 \in \widehat{\mathrm{Gr}}_{\mathrm{iso}}(p-1, V')$.

If $\dim V$ is odd, there is a possibility that $v' \notin L'$. Nevertheless, we still have

$$L_\omega = L' + \langle w + v'\rangle.$$

Writing $w = \lambda e_p + \mu e_{-p}$, we obtain that $2\lambda\mu + (v'|v') = 0$. Since L' is maximal isotropic in V', the vector v' cannot be isotropic. Hence, we have $\lambda \neq 0 \neq \mu$. Consequently, we can write

$$\omega = \omega' \wedge (\lambda e_p + \mu e_{-p} + v'),$$

where $L_{\omega'} = L'$. By doing so, we have expressed ω in the form (1.3.2), with $\omega_1 = 0$, $\omega_2 = \lambda\omega'$, $\omega_3 = \mu\omega'$, and $\omega_4 = \omega' \wedge v'$. One can verify that this proves all the claims in (3).

Case 2. If $\dim L' = p-2$, we can write $L_\omega = \langle e_p + u, e_{-p} + v\rangle \oplus L'$ for some $u, v \in V'$. We choose $v_1, \dots, v_{p-2}$ as a basis of L' and express ω as

$$\begin{aligned}
\omega &= v_1 \wedge \cdots \wedge v_{p-2} \wedge (e_p + u) \wedge (e_{-p} + v) \\
&=: \omega_1 \wedge (e_p + u) \wedge (e_{-p} + v) \\
&= \omega_1 \wedge e_p \wedge e_{-p} - \omega_1 \wedge v \wedge e_p + \omega_1 \wedge u \wedge e_{-p} + \omega_1 \wedge u \wedge v \\
&=: \omega_1 \wedge e_p \wedge e_{-p} + \omega_2 \wedge e_p + \omega_3 \wedge e_{-p} + \omega_4.
\end{aligned}$$

To show that all ω_i are nonzero, we need to show that $v_1, v_2, \dots, v_{p-2}, u, v$ are linearly independent. We already know that $v_1, \dots, v_{p-2}$ are linearly independent since they form a basis of L'. Furthermore, v is also linearly independent from $v_1, \dots, v_{p-2}$; otherwise $e_{-p} \in L_\omega$, but this would imply $(e_p + u|e_{-p}) = 0$. Hence, we need to show that u is linearly independent from $v_1, \dots, v_{p-2}, v$. Assuming $u = \lambda v + v'$, where $v' \in L'$, we obtain

$$\omega = v_1 \wedge \cdots \wedge v_{p-2} \wedge (e_p + \lambda v) \wedge (e_{-p} + v),$$

where the vectors $v_1, \dots, v_{p-2}, e_p + \lambda v, e_{-p} + v$ are all isotropic. In particular, the pairing $(v|v) = 0$. However, this implies $(e_p + \lambda v|e_{-p} + v) = 1$, contradicting the isotropy of L_ω. Hence, the vectors $v_1, v_2, \dots, v_{p-2}, u, v$ are linearly independent, implying that all ω_i are nonzero. Note that by definition all ω_i belong to the corresponding Grassmann cone. Furthermore, since u and v are isotropic and $(v_i|v_j) = 0$, $(u|v_i) = 0$, $(v|v_i) = 0$ for all i, j, we can conclude that ω_1, ω_2 and ω_3 are isotropic. This proves the first statement in (4). The second statement follows from the definition and linear independence of $v_1, v_2, \dots, v_{p-2}, u, v$. □

For the proof of Theorem 1.3.1, we will fix $\omega \in \bigwedge^p V$ satisfying the assumption. We decompose ω as in (1.3.2). Then ω has one of the following zero patterns:

	ω_1	ω_2	ω_3	ω_4	ω_1	ω_2	ω_3	ω_4	
(0)	0	0	0	0	$*$	0	0	0	(8)
(1)	0	0	0	$*$	$*$	0	0	$*$	(9)
(2)	0	0	$*$	0	$*$	0	$*$	0	(10)
(3)	0	0	$*$	$*$	$*$	0	$*$	$*$	(11)
(4)	0	$*$	0	0	$*$	$*$	0	0	(12)
(5)	0	$*$	0	$*$	$*$	$*$	0	$*$	(13)
(6)	0	$*$	$*$	0	$*$	$*$	$*$	0	(14)
(7)	0	$*$	$*$	$*$	$*$	$*$	$*$	$*$	(15)

The proof splits into the Claims 1.3.8–1.3.14 which are based on the different zero patterns. First, we show that zero patterns (0), (1), (3), (5), (6), (8)-(14) and (7) (if V is even-dimensional) are not possible:

- Claim 1.3.8 shows that the only possible zero patterns are (2), (4), (7) and (15), with (7) only occurring when V has odd dimension.

Note that the highlighted zero patterns align with the cases in Proposition 1.3.7. We proceed by proving that Theorem 1.3.1 is true if $\omega \in \bigwedge^p V$ has one of the highlighted zero patterns as follows:

- Claim 1.3.9 proves that Theorem 1.3.1 is true if ω has zero pattern (2) or (4).
- Claim 1.3.10 and Claim 1.3.11 show that if ω has zero pattern (7) or (15), there are three possibilities for the dimension of the intersection $L_{\omega_2} \cap L_{\omega_3}$:
  (a) $\dim(L_{\omega_2} \cap L_{\omega_3}) = p - 2$ (when $\dim V$ is even);
  (b) $\dim(L_{\omega_2} \cap L_{\omega_3}) = p - 2$ (when $\dim V$ is odd);
  (c) $\dim(L_{\omega_2} \cap L_{\omega_3}) = p - 1$ (when $\dim V$ is odd).
- Claim 1.3.12 shows that Theorem 1.3.1 holds for case (a).
- Claim 1.3.13 shows that Theorem 1.3.1 holds for case (b).
- Claim 1.3.14 shows that Theorem 1.3.1 holds for case (c).

1.3.3 Proof of the Main Theorem

We will now prove Theorem 1.3.1 following the strategy we just explained. Throughout this section, let $\omega \neq 0$ satisfy the assumption of Theorem 1.3.1. Trivially, ω cannot have zero pattern (0).

Claim 1.3.8. *ω cannot have zero pattern (1), (3), (5), (6), or (8)-(14). If $\dim V$ is even it also cannot have zero pattern (7).*

Proof.

Step 1. If $\omega_1 \neq 0$, then ω_2, ω_3 and ω_4 are also nonzero. In other words, ω cannot have zero patterns (8)–(14).

Proof. If $\omega_1 \neq 0$, according to Lemma 1.2.11, there exists $v \in V'_{iso}$ such that $\Phi_v(\omega_1) \neq 0$. Therefore, applying case (4) of Proposition 1.3.7 to

$$\Phi_v(\omega) = \Phi_v(\omega_1) \wedge e_p \wedge e_{-p} + \Phi_v(\omega_2) \wedge e_p + \Phi_v(\omega_3) \wedge e_{-p} + \Phi_v(\omega_4)$$

we can conclude that $\Phi_v(\omega_2)$, $\Phi_v(\omega_3)$ and $\Phi_v(\omega_4)$ are nonzero. This implies that ω_2, ω_3 and ω_4 are nonzero as well. □

Step 2. If $\omega_4 \neq 0$, then either ω has zero pattern (15), or $\dim V$ is odd and ω has zero pattern (7). In other words, ω cannot have zero patterns (1), (3), (5), and also not (7) if $\dim V$ is even.

Proof. As before, by Lemma 1.2.11 there exists $v \in V'_{\mathrm{iso}}$ such that $\Phi_v(\omega_4) \neq 0$. The result follows by applying Proposition 1.3.7 to $\Phi_v(\omega)$ as before. □

Step 3. ω cannot have zero pattern (6).

Proof. Assume ω has zero pattern (6). Our goal is to find a vector $v \in V'_{\mathrm{iso}}$ such that $\Phi_v(\omega_2) \neq 0 \neq \Phi_v(\omega_3)$. Then $\omega' := \Phi_v(\omega)$ also has the property that $\omega'_1 = 0 = \omega'_4$ but $\omega'_2 \neq 0 \neq \omega'_3$. So by Proposition 1.3.7 ω' is not in $\widehat{\mathrm{Gr}}_{\mathrm{iso}}(p-1, V_v)$, which is a contradiction with the assumption of Theorem 1.3.1. We consider two cases:

Case 1. Assume $L_{\omega_2} + L_{\omega_3} \subsetneq V'$. This case holds if $\dim V$ is odd, and also if $\dim V$ is even except when $L_{\omega_2} \cap L_{\omega_3} = 0$. Since V is spanned by isotropic vectors, we can find an isotropic vector v that does not lie in the linear subspace $L_{\omega_2} + L_{\omega_3}$. Then we have the desired property that $\Phi_v(\omega_2) \neq 0 \neq \Phi_v(\omega_3)$ by Lemma 1.2.10.

Case 2. Assume $L_{\omega_2} + L_{\omega_3} = V'$. In this case, $\dim V$ is even and $L_{\omega_2} \cap L_{\omega_3} = 0$. By Lemma 1.2.6 we can choose a hyperbolic basis such that

$$\omega_2 = \alpha e_1 \wedge \cdots \wedge e_{p-1} \quad \text{and} \quad \omega_3 = \beta e_{-1} \wedge \cdots \wedge e_{-p+1}.$$

Taking $v := e_1 + e_{-2}$, we have $\Phi_v(\omega_2) \neq 0 \neq \Phi_v(\omega_3)$, satisfying the desired property. □

We now have considered all cases, and the proof of Claim 1.3.8 is complete. □

We now know that ω has one of the highlighted zero patterns. Next, we prove that Theorem 1.3.1 holds if ω has zero pattern (2) or (4).

Claim 1.3.9. *Theorem 1.3.1 is true if ω has zero pattern (2) or (4).*

Proof. Let ω have zero pattern (2). Then $\omega = \omega_3 \wedge e_{-p}$. For $v = e_p$, by (1.2.2), we have $\Phi_{e_p}(\omega) = \pm\omega_3$, which by assumption lies in $\widehat{\mathrm{Gr}}_{\mathrm{iso}}(p-1, V_v) = \widehat{\mathrm{Gr}}_{\mathrm{iso}}(p-1, V')$ and therefore also $\omega \in \widehat{\mathrm{Gr}}_{\mathrm{iso}}(p, V)$. If ω has zero pattern (4) we proceed analogously, using $v = e_{-p}$. □

For the rest of the proof, we assume that ω has zero pattern (15) or (7); where (7) can only occur if $\dim V$ is odd.

Claim 1.3.10. *The intersection $L_{\omega_2} \cap L_{\omega_3}$ is nonzero.*

Proof. Assume by contradiction that $L_{\omega_2} \cap L_{\omega_3} = 0$. Then by Lemma 1.2.6 we can find a hyperbolic basis of V' such that

$$\omega_2 = \alpha e_1 \wedge \cdots \wedge e_{p-1} \quad \text{and} \quad \omega_3 = \beta e_{-1} \wedge \cdots \wedge e_{-p+1}.$$

If $\dim V$ is even we take $v = e_1 + e_{-2}$. By Observation 1.3.6 we get

$$\Phi_v(\omega) =: \omega' = \omega_1' \wedge \bar{e}_p \wedge \bar{e}_{-p} + \omega_2' \wedge \bar{e}_p + \omega_3' \wedge \bar{e}_{-p} + \omega_4'.$$

Note that in the quotient space $\bigwedge^{p-1}(e_1 + e_{-2})^\perp / \langle e_1 + e_{-2} \rangle$, ω_2' and ω_3' have only the basis vector $\bar{e}_1 = \bar{e}_{-2}$ in common, thus

$$\dim(L_{\omega_2'} \cap L_{\omega_3'}) = 1.$$

Since $\omega' \in \widehat{\mathrm{Gr}}_{\mathrm{iso}}(p-1, V_v)$, we can conclude by Proposition 1.3.7 that the intersection $L_{\omega_2'} \cap L_{\omega_3'} = L_{\omega_1'}$, in particular

$$\dim(L_{\omega_2'} \cap L_{\omega_3'}) = p - 3.$$

This contradicts our assumption $\dim V > 8$, which for $\dim V$ even implies $p > 4$.

If $\dim V$ is odd we take $v = e_0 + e_{-1} - c_0 e_1$, where $c_0 = \frac{1}{2}(e_0|e_0)$. Then we find

$$\omega_2' = \alpha \bar{e}_2 \wedge \cdots \wedge \bar{e}_{p-1} \quad \text{and} \quad \omega_3' = -c_0 \beta \bar{e}_{-2} \wedge \cdots \wedge \bar{e}_{-p+1}.$$

Note that also in the quotient space $\bigwedge^{p-1}(e_0 + e_{-1} - c_0 e_1)^\perp / \langle e_0 + e_{-1} - c_0 e_1 \rangle$, we have that

$$\dim(L_{\omega_2'} \cap L_{\omega_3'}) = 0.$$

But by inspecting cases (3) and (4) of Proposition 1.3.7, we see that

$$\dim(L_{\omega_2'} \cap L_{\omega_3'}) \in \{p-2, p-3\}.$$

This is again a contradiction because, if $\dim V$ is odd, our assumption that $\dim V > 8$ implies $p > 3$. □

We will use this result and distinguish if the dimension of V is odd or even.

Claim 1.3.11. *One of the following holds.*

(a) $\dim V$ *is even, and* $\dim(L_{\omega_2} \cap L_{\omega_3}) = p - 2$,

(b) $\dim V$ *is odd, and* $\dim(L_{\omega_2} \cap L_{\omega_3}) = p - 2$,

(c) $\dim V$ *is odd, and* $\dim(L_{\omega_2} \cap L_{\omega_3}) = p - 1$ *and thus* $L_{\omega_2} = L_{\omega_3}$.

Proof. We write $q := \dim(L_{\omega_2} \cap L_{\omega_3}) > 0$. By Lemma 1.2.6 we can find a hyperbolic basis of V' such that

$$\omega_2 = e_1 \wedge \cdots \wedge e_q \wedge \widetilde{\omega}_2 \quad \text{and} \quad \omega_3 = e_1 \wedge \cdots \wedge e_q \wedge \widetilde{\omega}_3,$$

where $\widetilde{\omega}_2, \widetilde{\omega}_3 \in \widehat{\mathrm{Gr}}_{\mathrm{iso}}(p-q-1, \widetilde{V})$ and $\widetilde{V} = \langle e_{q+1}, e_{-q-1}, \ldots, e_{p-1}, e_{-p+1}\rangle$. Now if we choose v as any negative indexed basis vector e_{-i} and write $w' = \Phi_v(\omega)$, by Observation 1.3.6 we get

$$\omega_2' = \pm \bar{e}_1 \wedge \cdots \wedge \bar{e}_{i-1} \wedge \widehat{e}_i \wedge \bar{e}_{i+1} \wedge \cdots \wedge \bar{e}_q \wedge \widetilde{\omega}_2,$$
$$\omega_3' = \pm \bar{e}_1 \wedge \cdots \wedge \bar{e}_{i-1} \wedge \widehat{e}_i \wedge \bar{e}_{i+1} \wedge \cdots \wedge \bar{e}_q \wedge \widetilde{\omega}_3.$$

Therefore we have $L_{\omega_2'} \cap L_{\omega_3'} = \langle \bar{e}_1, \ldots, \bar{e}_{i-1}, \widehat{e}_i, \bar{e}_{i+1}, \ldots, \bar{e}_q\rangle$ and it holds that

$$\dim(L_{\omega_2'} \cap L_{\omega_3'}) = q - 1.$$

If $\dim V$ is even, then

$$\dim(L_{\omega_2'} \cap L_{\omega_3'}) = p - 3,$$

hence we conclude $q = p - 2$, as desired. Similarly, if $\dim V$ is odd, then

$$\dim(L_{\omega_2'} \cap L_{\omega_3'}) \in \{p-2, p-3\}.$$

We conclude $q \in \{p-2, p-1\}$. In other words, either $q = p-2$, or $L_{\omega_2} = L_{\omega_3}$. □

We will finish the proof by a case analysis of the cases in Claim 1.3.11. For the first two cases we need Lemma 1.2.13.

Claim 1.3.12. *Theorem 1.3.1 holds in case (a).*

Proof. Observe that for every vector $v \in L_{\omega_2} \cup L_{\omega_3}$, either $\Phi_v(\omega_2)$ or $\Phi_v(\omega_3)$ is zero. According to Observation 1.3.6 and Proposition 1.3.7 and consequently the possible zero patterns, this implies that $\Phi_v(\omega_1)$ and $\Phi_v(\omega_4)$ are also zero. Therefore, applying Lemma 1.2.13 with V' yields that $\omega_1 \in \bigwedge^{(p-1)-1}(L_{\omega_2}^{\perp} \cap L_{\omega_3}^{\perp}) = \bigwedge^{(p-1)-1}(L_{\omega_2} \cap L_{\omega_3})$ and $\omega_4 \in \bigwedge^{(p-1)+1}(L_{\omega_2} + L_{\omega_3})$. After choosing a hyperbolic basis for V' such that $L_{\omega_2} = \langle e_1, e_2, \ldots, e_{p-1}\rangle$ and $L_{\omega_3} = \langle e_{-1}, e_2, \ldots, e_{p-1}\rangle$, and defining $W := \langle e_1, e_{-1}, e_p, e_{-p}\rangle$, we can write

$$\omega = e_2 \wedge \cdots \wedge e_{p-1} \wedge \eta \tag{1.3.4}$$

for some $\eta \in \bigwedge^2 W$. Note that $L_{\omega_2} \cap L_{\omega_3} = \langle e_2, \ldots e_{p-1}\rangle$ is isotropic and orthogonal to W. Next, we choose $v = e_{-p+1}$. Consequently, V_v is isomorphic to $V_v'' \oplus W$, where $V'' = \langle e_2, e_{-2}, \ldots, e_{p-1}, e_{-p+1}\rangle$. Using (1.3.4) we can write

$$\Phi_v(\omega) = \pm \bar{e}_2 \wedge \cdots \wedge \bar{e}_{p-2} \wedge \eta.$$

By assumption we have $\Phi_v(\omega) \in \widehat{\mathrm{Gr}}_{\mathrm{iso}}(p-1, V_v)$. So we find that $\eta \in \widehat{\mathrm{Gr}}_{\mathrm{iso}}(2, W)$, which in turn implies $\omega \in \widehat{\mathrm{Gr}}_{\mathrm{iso}}(p, V)$. □

Claim 1.3.13. *Theorem 1.3.1 holds in case (b).*

Proof. We start by choosing a hyperbolic basis of V' where $L_{\omega_2} = \langle e_1, e_2, \ldots, e_{p-1}\rangle$ and $L_{\omega_3} = \langle e_{-1}, e_2, \ldots, e_{p-1}\rangle$. Applying Lemma 1.2.13 to ω_1 and ω_4 we get

$$\omega_1 \in \bigwedge^{p-2}(L_{\omega_2}^{\perp} \cap L_{\omega_3}^{\perp}) \quad \text{and} \quad \omega_4 = \nu e_1 \wedge e_{-1} \wedge e_2 \wedge \cdots \wedge e_{p-1}$$

for some $\nu \in \mathbb{K}^*$. So we can write

$$\omega = \left(\mu_0 e_2 \wedge \cdots \wedge e_{p-1} + e_0 \wedge \sum_{i=2}^{p-1} \mu_i e_2 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_{p-1}\right) \wedge e_p \wedge e_{-p}$$
$$+ \alpha e_1 \wedge e_2 \wedge \cdots \wedge e_{p-1} \wedge e_p + \beta e_{-1} \wedge e_2 \wedge \cdots \wedge e_{p-1} \wedge e_{-p}$$
$$+ \nu e_1 \wedge e_{-1} \wedge e_2 \wedge \cdots \wedge e_{p-1}.$$

Picking $v = e_{-2}$ yields

$$\Phi_v(\omega) = \left(\mu_0 e_3 \wedge \cdots \wedge e_{p-1} - e_0 \wedge \sum_{i=3}^{p-1} \mu_i e_3 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_{p-1}\right) \wedge e_p \wedge e_{-p}$$
$$- \alpha e_1 \wedge e_3 \wedge \cdots \wedge e_{p-1} \wedge e_p - \beta e_{-1} \wedge e_3 \wedge \cdots \wedge e_{p-1} \wedge e_{-p}$$
$$+ \nu e_1 \wedge e_{-1} \wedge e_3 \wedge \cdots \wedge e_{p-1}.$$

By assumption $\omega' = \Phi_v(\omega) \in \widehat{\mathrm{Gr}}_{\mathrm{iso}}(p-1, V_v)$. Thus, by Proposition 1.3.7 one of the cases (1) – (4) holds. Clearly, case (1) and (2), and since $L_{\omega'_2} \neq L_{\omega'_3}$, also case (3), are not possible. Thus, case (4) holds, which implies that $\omega'_1 \in \widehat{\mathrm{Gr}}_{\mathrm{iso}}(p-3, V'_v)$ and $L_{\omega'_1} = L_{\omega'_2} \cap L_{\omega'_3}$. In coordinates, this means that $\mu_i = 0$ for $i = 3, \dots, p-1$, and that $\mu_0 \neq 0$. The same argument[1] with $v = e_{-3}$ shows that also $\mu_2 = 0$. We now have written ω as in (1.3.4), and can proceed exactly as in Claim 1.3.12. □

Claim 1.3.14. *Theorem 1.3.1 holds in case (c).*

Proof. The proof is divided into several steps:

- Step 1 shows that $\omega_1 = 0$.
- Steps 2-4 show that $\omega_4 = e_1 \wedge \cdots \wedge e_{p-1} \wedge u$ for some $u \in V'$.
  - Step 2 shows that we can write:

    $$\omega_4 = e_1 \wedge \cdots \wedge e_{p-1} \wedge u + \sum_{j_1,\dots,j_\ell} \mu_J e_{j_1} \wedge e_{-j_1} \wedge \cdots \wedge e_{j_\ell} \wedge e_{-j_\ell} (\wedge e_0),$$

    where we write $p = 2\ell$ or $p = 2\ell + 1$, and the factor $\wedge e_0$ only appears in the latter case.
  - Step 3 shows that all μ_J are equal and thus:

    $$\omega_4 = e_1 \wedge \cdots \wedge e_{p-1} \wedge u + \mu \sum_{j_1,\dots,j_\ell} e_{j_1} \wedge e_{-j_1} \wedge \cdots \wedge e_{j_\ell} \wedge e_{-j_\ell} (\wedge e_0).$$

  - Step 4 shows that $\mu = 0$.
- Step 5 then concludes that $\omega \in \widehat{\mathrm{Gr}}_{\mathrm{iso}}(p, V)$.

[1] here we use $p \geq 4$

Step 1. $\omega_1 = 0$

Proof. Note that for every $v \in V'_{\mathrm{iso}}$, if we consider $\omega' := \Phi_v(\omega) \in \widehat{\mathrm{Gr}}_{\mathrm{iso}}(p-1, V_v)$, we have $L_{\omega'_2} = L_{\omega'_3}$. This implies that either $\omega' = 0$, or ω' is in case (3) of Proposition 1.3.7. In both cases we conclude $\omega'_1 = 0$. So we proved $\Phi_v(\omega_1) = 0$ for each $v \in V'_{\mathrm{iso}}$, which by Lemma 1.2.11 implies that $\omega_1 = 0$. □

Now we choose a hyperbolic basis of V' such that $L_{\omega_2} = L_{\omega_3} = \langle e_1, \dots, e_p \rangle$. For any $v \in V_{\mathrm{iso}}$, we can apply Proposition 1.3.7 to $\Phi_v(\omega)$ and find:

1. If $v \in L_{\omega_2}$, we have $\Phi_v(\omega_4) = 0$.
2. If $v \in V_{\mathrm{iso}} \setminus L_{\omega_2}$, we have $\Phi_v(\omega_4) \in \widehat{\mathrm{Gr}}(p-1, V_v)$ with $L_{\omega_2} \subset L_{\Phi_v(\omega_4)}$.

Step 2. ω_4 is of the form

$$e_1 \wedge \cdots \wedge e_{p-1} \wedge u + \sum_{j_1,\dots,j_\ell} \mu_J e_{j_1} \wedge e_{-j_1} \wedge \cdots \wedge e_{j_\ell} \wedge e_{-j_\ell} (\wedge e_0),$$

where we write $p = 2\ell$ or $p = 2\ell + 1$, and the factor $\wedge e_0$ only appears in the latter case.

Proof. We will write

$$\omega_4 = \sum_{i_1,\dots,i_p} \lambda_{i_1,\dots,i_p} e_{i_1} \wedge \cdots \wedge e_{i_p},$$

where we always order the indices as follows: $1, -1, 2, -2, \dots, p-1, -p+1, 0$. We will abbreviate $\lambda_{i_1,\dots,i_p}$ to λ_I, where $I = \{i_1, \dots, i_p\} \subset \{1, -1, 2, -2, \dots, p-1, -p+1, 0\}$. If we choose $v = e_i$, then (1) tells us that

$$0 = \Phi_v(\omega_4) = \sum_{-i \in I, i \notin I} \pm \lambda_I \bar{e}_{I \setminus \{-i\}} \in \bigwedge^{p-1} e_i^{\perp} / \langle e_i \rangle,$$

where the occurring vectors $\bar{e}_{I \setminus \{-i\}}$ are linearly independent. So if $-i \in I$ but $i \notin I$ then $\lambda_I = 0$. On the other hand, if we choose $v = e_{-i}$, then (2) tells us that

$$\Phi_v(\omega_4) = \sum_{-i \notin I, i \in I} \pm \lambda_I \bar{e}_{I \setminus \{i\}} \in \bigwedge^{p-1} e_{-i}^{\perp} / \langle e_{-i} \rangle$$

is of the form

$$\bar{e}_1 \wedge \cdots \wedge \widehat{\bar{e}_i} \wedge \cdots \wedge \bar{e}_{p-1} \wedge u$$

for some $u \in V'$. So if $i \in I$ but $-i \notin I$, then $\lambda_I = 0$, unless $\{1, 2, \dots, p-1\} \subset I$. Together with the above, this implies the claim. □

Step 3. All μ_I are equal, so we can write

$$\omega_4 = e_1 \wedge \cdots \wedge e_{p-1} \wedge u + \mu \sum_{j_1,\dots,j_\ell} e_{j_1} \wedge e_{-j_1} \wedge \cdots \wedge e_{j_\ell} \wedge e_{-j_\ell} (\wedge e_0).$$

Proof. Take $v = e_i - e_j$ with i, j positive. Then $\Phi_v(\omega_1) = 0$ and $\Phi_v(\omega_2) = \Phi_v(\omega_3) = 0$, hence by Proposition 1.3.7 we get $\Phi_v(\omega_4) = 0$. But

$$\begin{aligned}\varphi_v(\omega_4) = &-\sum_{i\in J}\mu_J e_{j_1}\wedge e_{-j_1}\wedge\cdots\wedge e_i\wedge\widehat{e}_{-i}\wedge\cdots\wedge e_{j_\ell}\wedge e_{-j_\ell}(\wedge e_0)\\ &+\sum_{j\in J}\mu_J e_{j_1}\wedge e_{-j_1}\wedge\cdots\wedge e_j\wedge\widehat{e}_{-j}\wedge\cdots\wedge e_{j_\ell}\wedge e_{-j_\ell}(\wedge e_0).\end{aligned}$$

After projecting to $\bigwedge^{p-1}(e_i - e_j)^\perp/\langle e_i - e_j\rangle$ we get

$$\begin{aligned}0 = \Phi_v(\omega_4) = &-\sum_{i\in J, j\notin J}\mu_J\bar{e}_{j_1}\wedge\bar{e}_{-j_1}\wedge\cdots\wedge\bar{e}_i\wedge\widehat{e}_{-i}\wedge\cdots\wedge\bar{e}_{j_\ell}\wedge\bar{e}_{-j_\ell}(\wedge\bar{e}_0)\\ &+\sum_{j\in J, i\notin J}\mu_J\bar{e}_{j_1}\wedge\bar{e}_{-j_1}\wedge\cdots\wedge\bar{e}_j\wedge\widehat{e}_{-j}\wedge\cdots\wedge\bar{e}_{j_\ell}\wedge\bar{e}_{-j_\ell}(\wedge\bar{e}_0)\\ =&\bar{e}_i\wedge\sum_{i,j\notin J'}(-\mu_{J'\cup\{i\}}+\mu_{J'\cup\{j\}})\bar{e}_{j'_1}\wedge\bar{e}_{-j'_1}\wedge\cdots\wedge\bar{e}_{j'_{\ell-1}}\wedge\bar{e}_{-j'_{\ell-1}}(\wedge\bar{e}_0).\end{aligned}$$

So we find that $\mu_{J'\cup\{i\}} = \mu_{J'\cup\{j\}}$ for every $J' \subset \{1, \ldots, \widehat{i}, \ldots, \widehat{j}, \ldots, p-1\}$. Letting i and j vary yields the result. □

Step 4. $\mu_4 = 0$

Proof. Finally, take $v = e_0 + e_{-1} - c_0 e_1$, where as before $c_0 = \frac{1}{2}(e_0|e_0)$. If $p = 2\ell$ we can write

$$\Phi_v(\omega_4) = \bar{e}_2\wedge\cdots\wedge\bar{e}_{p-1}\wedge\widetilde{u} + \mu\sum_{J\ni 1}(\bar{e}_{-1}+c_0\bar{e}_1)\wedge\bar{e}_{j_2}\wedge\bar{e}_{-j_2}\wedge\cdots\wedge\bar{e}_{j_\ell}\wedge\bar{e}_{-j_\ell}$$

and if $p = 2\ell + 1$ we have

$$\begin{aligned}\Phi_v(\omega_4) =&\bar{e}_2\wedge\cdots\wedge\bar{e}_{p-1}\wedge\widetilde{u} + \mu\sum_{J\ni 1}(\bar{e}_{-1}+c_0\bar{e}_1)\wedge\bar{e}_{j_2}\wedge\bar{e}_{-j_2}\wedge\cdots\wedge\bar{e}_{j_\ell}\wedge\bar{e}_{-j_\ell}\wedge\bar{e}_0\\ &+2c_0\mu\sum_J\bar{e}_{j_1}\wedge\bar{e}_{-j_1}\wedge\cdots\wedge\bar{e}_{j_\ell}\wedge\bar{e}_{-j_\ell}.\end{aligned}$$

In both cases we have $L_{\Phi_v(\omega_4)} \supset \langle\bar{e}_2, \ldots, \bar{e}_{p-1}\rangle$ by (2) from which we conclude $\mu = 0$. □

Step 5. $\omega \in \widehat{\mathrm{Gr}}_{\mathrm{iso}}(p, V)$

Proof. Since $\omega_1 = 0$ and $\omega_4 = e_1\wedge\cdots\wedge e_{p-1}\wedge u$ for some $u \in V'$, we can write

$$\omega = e_1\wedge\cdots\wedge e_{p-1}\wedge u'$$

for some $u' \in \langle e_0, e_p, e_{-1}, \ldots, e_{-p+1}, e_{-p}\rangle$. Choose $v = e_{-1}$, then we have

$$\Phi_v(\omega) = \bar{e}_2\wedge\cdots\wedge\bar{e}_{p-1}\wedge\overline{u'} \in \widehat{\mathrm{Gr}}_{\mathrm{iso}}(p-1, V_v)$$

hence $(u'|u') = 0$ and $(e_j|u') = 0$ for all $j = 2, \ldots, p-1$. Replacing $v = e_{-1}$ with $v = e_{-2}$ yields that also $(e_1|u') = 0$, hence $\omega \in \widehat{\mathrm{Gr}}_{\mathrm{iso}}(p, V)$. □

This proves Claim 1.3.14. □

1.4 Counterexamples in Small Dimensions

In Theorem 1.3.1, we assumed that $\dim V > 8$. In this section, we will show that this assumption is actually necessary. In both $\bigwedge^3 \mathbb{K}^7$ and $\bigwedge^4 \mathbb{K}^8$, we will give a p-form ω that does not lie in the isotropic Grassmannian, but which maps to the isotropic Grassmannian upon applying any IGCP map Φ_v. For case of simplicity, we assume the underlying field $\mathbb{K}$ is either $\mathbb{C}$ or $\mathbb{R}$.

1.4.1 Counterexample in Dimension 7

Let V be a 7-dimensional $\mathbb{K}$-vector space with a fixed basis $e_0, e_1, e_2, e_3, e_{-1}, e_{-2}, e_{-3}$, and a quadratic form given by the matrix

$$J = \left(\begin{array}{c|c|c} -\frac{1}{2} & 0 & 0 \\ \hline 0 & 0 & I_3 \\ \hline 0 & I_3 & 0 \end{array}\right),$$

where I_3 is the (3×3)-identity matrix. Choose

$$\omega_7 := e_1 \wedge e_2 \wedge e_3 + e_{-1} \wedge e_{-2} \wedge e_{-3} + e_0 \wedge (e_1 \wedge e_{-1} + e_2 \wedge e_{-2} + e_3 \wedge e_{-3}).$$

One verifies that $\omega_7 \notin \widehat{\mathrm{Gr}}(3, V)$, so in particular $\omega_7 \notin \widehat{\mathrm{Gr}}_{\mathrm{iso}}(3, V)$. In Claim 1.4.2 below we will show that every Φ_v maps ω_7 to the isotropic Grassmann cone $\widehat{\mathrm{Gr}}_{\mathrm{iso}}(2, 5)$. One could verify this by a direct computation for an arbitrary isotropic vector v. However, we will exploit the fact that ω_7 is sufficiently symmetric (Claim 1.4.1), so it suffices to do the computation for one fixed $v \in V_{\mathrm{iso}}$.

Consider the algebraic group

$$\begin{aligned} \mathrm{SO}(V) &= \{\phi \in \mathrm{SL}(V) \mid (\phi(x)|\phi(y)) = (x|y) \quad \forall x, y \in V\} \\ &= \left\{A \in \mathrm{SL}(7, \mathbb{K}) \mid A^T J A = J\right\} \end{aligned}$$

and its subgroup

$$G = \mathrm{stab}(\omega_7) = \{\phi \in \mathrm{SO}(V) \mid \phi \cdot \omega_7 = \omega_7\}.$$

Claim 1.4.1. *The action of G on V_{iso} is transitive.*

Proof. Take any $v_0 \in V_{\mathrm{iso}}$. We want to show that its orbit $G \cdot v_0$ has dimension six. Then $G \cdot v_0$ is a full-dimensional subvariety of the irreducible 6-dimensional variety V_{iso}, and hence is equal to V_{iso}. For this we use the formula

$$\dim(G \cdot v_0) = \dim G - \dim(\mathrm{stab}_G(v_0)),$$

where $\mathrm{stab}_G(v_0) = \{\phi \in G \mid \phi \cdot v_0 = v_0\}$ is the stabilizer. We will compute both terms $\dim G$ and $\dim(\mathrm{stab}_G(v_0))$ by switching to Lie algebras.

The Lie algebra of $\mathrm{SO}(V)$ is given by

$$\begin{aligned} \mathfrak{so}(V) &= \left\{X \in \mathfrak{sl}(7, \mathbb{K}) \mid X^T J + JX = 0\right\} \\ &= \left\{ \left(\begin{array}{c|c|c} 0 & -2y^T & -2x^T \\ \hline x & a & b \\ \hline y & c & -a^T \end{array}\right) \middle| \; x, y \in \mathbb{K}^3, a, b, c \in \mathbb{K}^{3\times 3}, b + b^T = c + c^T = 0 \right\}. \end{aligned}$$

We introduce the following notation.

$$\text{For } x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ we write } l_x := \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}.$$

We can compute the Lie algebra $\mathfrak{g} \subset \mathfrak{so}(V)$ of G as follows:

$$\begin{aligned} \mathfrak{g} &= \{X \in \mathfrak{so}(V) \mid X \cdot \omega_7 = 0\} \\ &= \left\{ \left(\begin{array}{c|c|c} 0 & -2y^T & -2x^T \\ \hline x & a & l_y \\ \hline y & l_x & -a^T \end{array} \right) \middle| \; x, y \in \mathbb{K}^3, a \in \mathfrak{sl}(3, \mathbb{K}) \right\}. \end{aligned}$$

Observe that $\dim \mathfrak{g} = 3 + 3 + 8 = 14$ and since $\dim G = \dim \mathfrak{g}$, G has also dimension 14. For the stabilizer, if we take $v_0 = e_{-3} \in V_{\text{iso}}$, we see that

$$\text{stab}_{\mathfrak{g}}(v_0) = \{X \in \mathfrak{g} \mid X \cdot e_{-3} = 0\}$$

is the set of matrices in $\mathfrak{g}$ whose final column is zero, which has dimension 8. So we get $\dim(G \cdot v_0) = 14 - 8 = 6 = \dim V_{\text{iso}}$, as desired. □

Claim 1.4.2. *For every $v \in V_{\text{iso}}$, it holds that $\Phi_v(\omega_7) \in \widehat{\text{Gr}}_{\text{iso}}(2, V_v)$.*

Proof. By the previous claim, it suffices to prove the claim for one fixed $v_0 \in V_{\text{iso}}$. Indeed, then any $v \in V_{\text{iso}}$ is of the form $g \cdot v_0$ for some $g \in \text{stab}(\omega_7)$, and we get

$$\Phi_v(\omega_7) = \Phi_{g \cdot v_0}(g \cdot \omega_7) = \bar{g} \cdot \Phi_{v_0}(\omega_7) \in Gr_{\text{iso}}(2, V_v),$$

where $\bar{g} : V_{v_0} \to V_v$ is the isometry induced by g. So we take $v_0 = e_{-3}$, and readily compute

$$\Phi_{e_{-3}}(\omega_7) = \bar{e}_1 \wedge \bar{e}_2 \in \widehat{\text{Gr}}_{\text{iso}}\left(2, V_{e_{-3}}\right).$$ □

In summary, this shows how Theorem 1.3.1 fails for the isotropic Grassmannian $\widehat{\text{Gr}}_{\text{iso}}(3, 7)$: by Claim 1.4.2 ω_7 satisfies the assumption but is itself not in $\widehat{\text{Gr}}_{\text{iso}}(3, 7)$. In particular, this means that $\widehat{\text{Gr}}_{\text{iso}}(3, 7)$ cannot be defined by pulling back the equations of $\widehat{\text{Gr}}_{\text{iso}}(2, 5)$ along IGCP maps of the form Φ_v. We originally constructed our counterexample by analyzing where our proof fails if $\dim V = 7$. However, it turned out, that ω_7 is interesting also from different points of view, which we will discuss in the following remarks.

Remark 1.4.3. In 1900, Engel [Eng00] showed that if ω is a generic 3-form on $\mathbb{C}^7$, its symmetry group is isomorphic to the exceptional group G_2, and that such a 3-form gives rise to a bilinear form β_ω. If we choose coordinates such that ω agrees with our form ω_7, then this group G_2 is precisely the stabilizer G we computed in Claim 1.4.1, and β_ω is up to scaling equal to our bilinear form given by J. For more about G_2, we refer the reader to [Fon18].

Remark 1.4.4. Alternatively we can construct ω_7 as the triple product on the split octonions. Here we will follow the notation from [BH14]. Recall that the space $\mathbb{H}$ of *quaternions* is the 4-dimensional real vector space with basis $\{1, i, j, k\}$, equipped with a bilinear associative product specified by Hamilton's formula

$$i^2 = j^2 = k^2 = ijk = -1.$$

The *conjugate* of a quaternion $x = a + bi + cj + dk$ is given by $\overline{x} = a - bi - cj - dk$. We also have a quadratic form given by $Q_{\mathbb{H}(x)} := x\overline{x} = \overline{x}x = a^2 + b^2 + c^2 + d^2$. The space of *split octonions* is the vector space $\mathbb{O}_s := \mathbb{H} \oplus \mathbb{H}$ with a bilinear (but nonassociative) product given by

$$(a, b)(c, d) := (ac + d\bar{b}, \bar{a}d + cb).$$

The conjugate of an octonion (a, b) is given by $\overline{(a, b)} = (\bar{a}, -b)$. Additionally, we define a quadratic form $Q_{\mathbb{O}_s}$, of signature $(4, 4)$, by $Q_{\mathbb{O}_s}(x) = x\overline{x} = \overline{x}x$; or equivalently $Q_{\mathbb{O}_s}((a, b)) = Q_{\mathbb{H}}(a) - Q_{\mathbb{H}}(b)$. We will write

$$\begin{array}{llll} e_0 := (1, 0) & e_1 := (i, 0) & e_2 := (j, 0) & e_3 := (k, 0) \\ e_4 := (0, 1) & e_5 := (0, i) & e_6 := (0, j) & e_7 := (0, k). \end{array}$$

Let $\mathbb{O}_{\mathrm{Im}} = \{x \in \mathbb{O}_s \mid \bar{x} = -x\} = \langle e_1, \ldots e_7 \rangle$ denote the *imaginary split octonions.* On $\mathbb{O}_{\mathrm{Im}}$ we can define a cross product given by the commutator:

$$x \times y := \frac{1}{2}(xy - yx),$$

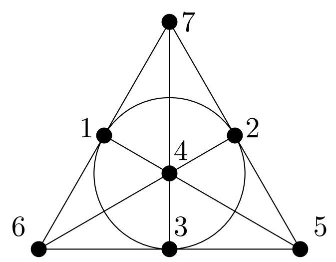
and a triple product $T : \mathbb{O}_{\mathrm{Im}} \times \mathbb{O}_{\mathrm{Im}} \times \mathbb{O}_{\mathrm{Im}} \to \mathbb{R}$, given by

$$T(x, y, z) := (x, y \times z),$$

where $(\cdot, \cdot)$ is the bilinear form coming from $Q_{\mathbb{O}_s}$. This triple product is an alternating trilinear form, and hence can be identified with an element $\omega \in \bigwedge^3 V^*$, where $V = \mathbb{O}_{\mathrm{Im}}$. Explicitly, writing $e_i^* \in V^*$ for the dual vector to e_i, we have

$$\begin{aligned} \omega = & e_1^* \wedge e_2^* \wedge e_3^* + e_1^* \wedge e_4^* \wedge e_5^* + e_1^* \wedge e_6^* \wedge e_7^* + e_2^* \wedge e_4^* \wedge e_6^* \\ & - e_2^* \wedge e_5^* \wedge e_7^* + e_3^* \wedge e_4^* \wedge e_7^* + e_3^* \wedge e_5^* \wedge e_6^*. \end{aligned}$$

Note that the terms in ω correspond to the lines in the Fano plane:



This ω agrees with ω_7 up to a change of basis. Explicitly, if we substitute

$$\begin{array}{llll} e_0 \mapsto \dfrac{e_4^*}{\sqrt{2}}, & e_1 \mapsto \dfrac{e_1^* + e_5^*}{\sqrt{2}}, & e_2 \mapsto \dfrac{e_2^* + e_6^*}{\sqrt{2}}, & e_3 \mapsto \dfrac{e_3^* + e_7^*}{\sqrt{2}}, \\ & e_{-1} \mapsto \dfrac{e_1^* - e_5^*}{\sqrt{2}}, & e_{-2} \mapsto \dfrac{e_2^* - e_6^*}{\sqrt{2}}, & e_{-3} \mapsto \dfrac{e_3^* - e_7^*}{\sqrt{2}} \end{array}$$

into ω_7, we recover ω (up to scaling).

1.4.2 Counterexample in Dimension 8

Let V be an 8-dimensional vector space with basis $e_1, e_2, e_3, e_4, e_{-1}, e_{-2}, e_{-3}, e_{-4}$, and quadratic form given by the matrix

$$J = \left(\begin{array}{c|c} 0 & I_4 \\ \hline I_4 & 0 \end{array}\right).$$

Choose

$$\begin{aligned} \omega_8 :=& 2e_1 \wedge e_2 \wedge e_3 \wedge e_4 + 2e_{-1} \wedge e_{-2} \wedge e_{-3} \wedge e_{-4} \\ &+ e_1 \wedge e_2 \wedge e_{-1} \wedge e_{-2} + e_1 \wedge e_3 \wedge e_{-1} \wedge e_{-3} + e_1 \wedge e_4 \wedge e_{-1} \wedge e_{-4} \\ &+ e_2 \wedge e_3 \wedge e_{-2} \wedge e_{-3} + e_2 \wedge e_4 \wedge e_{-2} \wedge e_{-4} + e_3 \wedge e_4 \wedge e_{-3} \wedge e_{-4}. \end{aligned} \tag{1.4.1}$$

One can verify that $\omega_8 \notin \widehat{\mathrm{Gr}}(4, V)$, so in particular $\omega_8 \notin \widehat{\mathrm{Gr}}_{\mathrm{iso}}(4, V)$. As before, we consider the algebraic group

$$\begin{aligned} \mathrm{SO}(V) =& \{\phi \in \mathrm{SL}(V) \mid (\phi(x)|\phi(y)) = (x|y) \quad \forall x, y \in V\} \\ =& \{A \in \mathrm{SL}(8, \mathbb{K}) \mid A^T J A = J\} \end{aligned}$$

and its subgroup

$$G := \mathrm{stab}(\omega_8) = \{\phi \in \mathrm{SO}(V) \mid \phi \cdot \omega_8 = \omega_8\}.$$

Claim 1.4.5. *The action of G on V_{iso} is transitive.*

Proof. Take any $v_0 \in V_{\mathrm{iso}}$; we want to show that its orbit $G \cdot v_0$ has dimension equal to $\dim V_{\mathrm{iso}} = 7$. The Lie algebra of $\mathrm{SO}(V)$ is given by

$$\begin{aligned} \mathfrak{so}(V) =& \{X \in \mathfrak{sl}(8, \mathbb{K}) \mid X^T J + JX = 0\} \\ =& \left\{ \left(\begin{array}{c|c} a & b \\ \hline c & -a^T \end{array}\right) \middle| \; a, b, c \in \mathbb{K}^{4\times 4}, b + b^T = c + c^T = 0 \right\}. \end{aligned}$$

We introduce the following notation

$$\text{for } b = \begin{pmatrix} 0 & b_{12} & b_{13} & b_{14} \\ -b_{12} & 0 & b_{23} & b_{24} \\ -b_{13} & -b_{23} & 0 & b_{34} \\ -b_{14} & -b_{24} & -b_{34} & 0 \end{pmatrix} \text{ write } \widetilde{b} := \begin{pmatrix} 0 & -b_{34} & b_{24} & -b_{23} \\ b_{34} & 0 & -b_{14} & b_{13} \\ -b_{24} & b_{14} & 0 & -b_{12} \\ b_{23} & -b_{13} & b_{12} & 0 \end{pmatrix}.$$

We can compute the Lie algebra $\mathfrak{g} \subset \mathfrak{so}(V)$ of G as follows:

$$\begin{aligned} \mathfrak{g} =& \{X \in \mathfrak{so}(V) \mid X \cdot \omega_8 = 0\} \\ =& \left\{ \left(\begin{array}{c|c} a & b \\ \hline \widetilde{b} & -a^T \end{array}\right) \middle| \; a \in \mathfrak{sl}(4, \mathbb{K}), b + b^T = 0 \right\}. \end{aligned}$$

As before $\dim G = \dim \mathfrak{g} = 21$. For the stabilizer, if we take $v_0 = e_{-4} \in V_{\mathrm{iso}}$, we see that

$$\mathrm{stab}_{\mathfrak{g}}(v_0) = \{X \in \mathfrak{g} \mid X \cdot e_{-4} = 0\}$$

is the set of matrices in $\mathfrak{g}$ whose final column is zero, which has dimension 14. So $\dim(G \cdot v_0) = 21 - 14 = 7 = \dim V_{\mathrm{iso}}$, as desired. □

As before, we conclude the following claim.

Claim 1.4.6. *For every $v \in V_{\mathrm{iso}}$, it holds that $\Phi_v(\omega_8) \in \widehat{\mathrm{Gr}}_{\mathrm{iso}}(3, V_v)$.*

Proof. As in Claim 1.4.2, it suffices to prove the claim for one fixed $v \in V_{\mathrm{iso}}$. Taking $v = e_{-1}$, we compute that

$$\Phi_v(\omega_8) = 2\bar{e}_2 \wedge \bar{e}_3 \wedge \bar{e}_4 \in \widehat{\mathrm{Gr}}_{\mathrm{iso}}(3, V_{e_{-1}}). \qquad \square$$

In summary, this shows how Theorem 1.3.1 fails for $\widehat{\mathrm{Gr}}_{\mathrm{iso}}(4, 8)$. As before, this means that $\widehat{\mathrm{Gr}}_{\mathrm{iso}}(4, 8)$ cannot be defined by pulling back the equations of $\widehat{\mathrm{Gr}}_{\mathrm{iso}}(3, 6)$ along IGCP maps of the form Φ_v.

Remark 1.4.7. The Lie algebra $\mathfrak{g}$ defined above is in fact isomorphic to $\mathfrak{so}(7)$. An explicit isomorphism $\mathfrak{so}(7) \to \mathfrak{g}$ can be given by

$$\left(\begin{array}{c|ccc|ccc} 0 & -2y_1 & -2y_2 & -2y_3 & -2x_1 & -2x_2 & -2x_3 \\ \hline x_1 & a_{11} & a_{12} & a_{13} & 0 & b_{12} & b_{13} \\ x_2 & a_{21} & a_{22} & a_{23} & -b_{12} & 0 & b_{23} \\ x_3 & a_{31} & a_{32} & a_{33} & -b_{13} & -b_{23} & 0 \\ \hline y_1 & 0 & c_{12} & c_{13} & -a_{11} & -a_{21} & -a_{31} \\ y_2 & -c_{12} & 0 & c_{23} & -a_{12} & -a_{22} & -a_{32} \\ y_3 & -c_{13} & -c_{23} & 0 & -a_{13} & -a_{23} & -a_{33} \end{array}\right) \mapsto \left(\begin{array}{cccc|cccc} d_{11} & a_{12} & a_{13} & -c_{23} & 0 & -y_3 & y_2 & x_1 \\ a_{21} & d_{22} & a_{23} & c_{13} & y_3 & 0 & -y_1 & x_2 \\ a_{31} & a_{32} & d_{33} & -c_{12} & -y_2 & y_1 & 0 & x_3 \\ b_{23} & -b_{13} & b_{12} & d_{44} & -x_1 & -x_2 & -x_3 & 0 \\ \hline 0 & -x_3 & x_2 & y_1 & -d_{11} & -a_{21} & -a_{31} & -b_{23} \\ x_3 & 0 & -x_1 & y_2 & -a_{12} & -d_{22} & -a_{32} & b_{13} \\ -x_2 & x_1 & 0 & y_3 & -a_{13} & -a_{23} & -d_{33} & -b_{12} \\ -y_1 & -y_2 & -y_3 & 0 & c_{23} & -c_{13} & c_{12} & -d_{44} \end{array}\right),$$

where for the left hand side we used the notation from Section 1.4.1, and in the right hand side we have

$$d_{11} := \frac{a_{11} - a_{22} - a_{33}}{2},\ d_{22} := \frac{-a_{11} + a_{22} - a_{33}}{2},\ d_{33} := \frac{-a_{11} - a_{22} + a_{33}}{2},\ d_{44} := \frac{a_{11} + a_{22} + a_{33}}{2}.$$

1.5 Ranks of Defining Quadrics

In Section 1.5.1, we will finish the proof of Corollary 1.3.3, by verifying the following fact:

Claim 1.5.1. $\widehat{\mathrm{Gr}}_{\mathrm{iso}}(3, 7)$*, as well as both irreducible components of* $\widehat{\mathrm{Gr}}_{\mathrm{iso}}(4, 8)$*, can be set-theoretically defined by quadrics of rank at most* 4*.*

In Section 1.5.2, we explain how Corollary 1.3.3 can be deduced from the literature on isotropic Grassmannians, in particular the Cartan embedding.

1.5.1 Computational Approach

Our verification is based on an algorithm, which we implemented in Macaulay2 [GS]. We sketch the steps of the algorithm below. Let X be either $\widehat{\mathrm{Gr}}_{\mathrm{iso}}(3, 7)$, or one of the components of $\widehat{\mathrm{Gr}}_{\mathrm{iso}}(4, 8)$.

1. Compute the ideal I defining X by parametrizing an open subset and performing a Gröbner basis computation. The ideal I is generated by linear equations and quadrics.

2. Get rid of the linear equations by substituting variables.

3. View the space I_2 of quadrics in I as a representation of $\mathrm{SO}(V)$, and decompose it into weight spaces.

4. Find a highest weight vector $p \in I_2$ of minimal rank.

5. Compute the subrepresentation generated by p, using the lowering operators in $\mathfrak{so}(V)$.

6. If we generated all of I_2, we are done.

7. Otherwise, find the highest weight space we did not yet generate, let p be a quadric of minimal rank in it, and return to step (5).

By construction, the $\mathrm{SO}(V)$-orbits of the quadrics p we found give sufficiently many equations to define X. Since acting with $\mathrm{SO}(V)$ does not change the rank of a quadric, it follows that if each of our quadrics has rank at most 4, then X can be defined by quadrics of rank at most 4. For $\widehat{\mathrm{Gr}}_{\mathrm{iso}}(3,7)$, our algorithm returned the following quadrics:

$$
\begin{aligned}
&x_{0,1,2}^2 + 2x_{1,2,3}x_{1,2,-3},\\
&x_{0,1,2}(x_{1,2,-2} + x_{1,3,-3}) + 2x_{1,2,3}x_{0,1,-3},\\
&x_{0,1,3}x_{0,1,-3} + x_{0,1,2}x_{0,1,-2},\\
&x_{1,2,3}(x_{0,1,-1} + x_{0,2,-2} - x_{0,3,-3}) + x_{0,1,2}(x_{2,3,-2} + x_{1,3,-1}),\\
&x_{0,1,-1}^2 - (x_{0,2,-2} + x_{0,3,-3})^2 + 2(x_{1,3,-3} + x_{1,2,-2})(x_{3,-1,-3} + x_{2,-1,-2}).
\end{aligned}
$$

For one of the components of $\widehat{\mathrm{Gr}}_{\mathrm{iso}}(4,8)$, we found the following quadrics:

$$
\begin{aligned}
&x_{1,2,3,-3}^2 - x_{1,2,3,4}x_{1,2,-3,-4},\\
&2x_{1,2,3,-3}x_{1,3,4,-1} - x_{1,2,3,4}(x_{1,2,-1,-2} - x_{1,3,-1,-3} - x_{1,4,-1,-4}),\\
&(x_{1,4,-1,-4} + x_{2,4,-2,-4} - x_{3,4,-3,-4})^2 - 4x_{3,4,-1,-2}x_{1,2,-3,-4}.
\end{aligned}
$$

Since all quadrics listed above have rank at most 4, and since both components of $\widehat{\mathrm{Gr}}_{\mathrm{iso}}(4,8)$ are isomorphic, our verification is now complete.

1.5.2 Rank 4 Quadrics via the Cartan Embedding

In this section we will sketch an alternative proof that $\mathrm{Gr}_{\mathrm{iso}}(p, 2p+1)$ and the connected components of $\mathrm{Gr}_{\mathrm{iso}}(p, 2p)$, in their Plücker embedding, are defined by linear equations and quadrics of rank at most 4, using the Cartan embedding (sometimes called spinor embedding), cf. [Car81] or [HB21]. The proof follows by combining the following facts:

- The image of the Cartan embedding is defined by quadrics [Car81].
- The Plücker embedding factors as the Cartan embedding followed by a degree two Veronese embedding ([BHH21, Theorem 2.1] and [CP13, Theorem 1]).
- The image of a degree two Veronese embedding is defined by quadrics of rank 3 and 4.

The idea is that the Veronese embedding turns the quadratic equations of the Cartan embedding into linear equations, so the only quadrics we need are the ones coming from the Veronese embedding.

Chapter 2

Topological Noetherianity of the Infinite Half-Spin Representations

2.1 Finite Spin Representations and the Spin Group

In this section we collect some preliminaries on spin groups and their defining representations. Throughout we will assume that $\mathbb{K}$ is an algebraically closed field of characteristic zero. We follow [Man09] in our set-up; for more general references on spin groups and their representations see [LM89, Pro07].

2.1.1 The Clifford Algebra

Let V be a finite-dimensional vector space over $\mathbb{K}$ endowed with a quadratic form q. The *Clifford algebra* $\mathrm{Cl}(V, q)$ of V is the quotient of the tensor algebra $T(V) = \bigoplus_{d\geq 0} V^{\otimes d}$ by the two-sided ideal generated by all elements

$$v \otimes v - q(v) \cdot 1, \; v \in V. \tag{2.1.1}$$

This is also the two-sided ideal generated by

$$v \otimes w + w \otimes v - 2(v|w) \cdot 1, \; v, w \in V, \tag{2.1.2}$$

where $(\cdot|\cdot)$ denotes the bilinear form associated to q defined by

$$(v|w) := \frac{1}{2}\big(q(v + w) - q(v) - q(w)\big).$$

The Clifford algebra is a functor from the category of vector spaces equipped with a quadratic form to the category of (unital) associative algebras. That is, any linear map $\varphi\colon (V, q) \to (V', q')$ with $q'(\varphi(v)) = q(v)$ for all $v \in V$ induces a homomorphism of associative algebras $\mathrm{Cl}(\varphi)\colon\ \mathrm{Cl}(V, q) \to \mathrm{Cl}(V', q')$. If ϕ is an inclusion $V \subseteq V'$, then $\mathrm{Cl}(\varphi)$ is injective, and hence $\mathrm{Cl}(V, q)$ is a subalgebra of $\mathrm{Cl}(V', q')$.

The decomposition of $T(V)$ into the even part $T^+(V) := \bigoplus_{d \text{ even}} V^{\otimes d}$ and the odd part $T^-(V) := \bigoplus_{d \text{ odd}} V^{\otimes d}$ induces a decomposition $\mathrm{Cl}(V, q) = \mathrm{Cl}^+(V, q) \oplus \mathrm{Cl}^-(V, q)$, turning $\mathrm{Cl}(V, q)$ into a $\mathbb{Z}/2\mathbb{Z}$-graded associative algebra. Note that, via the commutator on $\mathrm{Cl}(V, q)$, the even Clifford algebra $\mathrm{Cl}^+(V, q)$ is a Lie subalgebra of $\mathrm{Cl}(V, q)$.

The anti-automorphism of $T(V)$ determined by $v_1 \otimes \cdots \otimes v_d \mapsto v_d \otimes \cdots \otimes v_1$ preserves the ideal in the definition of $\mathrm{Cl}(V, q)$ and therefore induces an anti-automorphism $x \mapsto x^*$ of $\mathrm{Cl}(V, q)$.

2.1.2 The Grassmann Algebra as a $\mathrm{Cl}(V)$-Module

From now on, we will write $\mathrm{Cl}(V)$ for $\mathrm{Cl}(V, q)$ when q is clear from the context. If $q = 0$, then $\mathrm{Cl}(V) = \bigwedge V$, the Grassmann algebra of V. If $E \subseteq V$ is an isotropic subspace, that is, a subspace for which $q|_E = 0$, then this fact allows us to identify $\bigwedge E$ with the subalgebra $\mathrm{Cl}(E)$ of $\mathrm{Cl}(V)$.

For general q, $\mathrm{Cl}(V)$ is not isomorphic as an algebra to $\bigwedge V$, but $\bigwedge V$ is naturally a $\mathrm{Cl}(V)$-module as follows. For $v \in V$ define $o(v) : \bigwedge V \to \bigwedge V$ (the "outer product") as the linear map

$$o(v)\omega := v \wedge \omega$$

and $\iota(v) : \bigwedge V \to \bigwedge V$ (the "inner product") as the linear map determined by

$$\iota(v) w_1 \wedge \cdots \wedge w_k := \sum_{i=1}^{k} (-1)^{i-1} (v|w_i) w_1 \wedge \cdots \wedge \widehat{w_i} \wedge \cdots \wedge w_k.$$

Here, and elsewhere in the paper, $\widehat{\cdot}$ indicates a factor that is left out. Now $v \mapsto \iota(v) + o(v)$ extends to an algebra homomorphism $\mathrm{Cl}(V) \to \mathrm{End}(\bigwedge V)$. To see this, it suffices to consider $v, w_1, \ldots, w_k \in V$ and verify

$$(\iota(v) + o(v))^2 w_1 \wedge \cdots \wedge w_k = (v|v) w_1 \wedge \cdots \wedge w_k.$$

We write $a \bullet \omega$ for the outcome of $a \in \mathrm{Cl}(V)$ acting on $\omega \in \bigwedge V$. Using induction on the degree of a product, the linear map $\mathrm{Cl}(V) \to \bigwedge V, a \mapsto a \bullet 1$ is easily seen to be an isomorphism of vector spaces. In particular, $\mathrm{Cl}(V)$ has dimension $2^{\dim V}$.

2.1.3 Embedding $\mathfrak{so}(V)$ into the Clifford Algebra

From now on, we assume that q is non-degenerate and write $\mathrm{SO}(V) = \mathrm{SO}(V, q)$ for the special orthogonal group of q. Its Lie algebra $\mathfrak{so}(V)$ consists of linear maps $V \to V$ that are skew-symmetric with respect to $(\cdot|\cdot)$, that is, those $A \in \mathrm{End}(V)$ such that $(Av|w) = -(v|Aw)$ for all $v, w \in V$. We have a unique linear map $\psi : \bigwedge^2 V \to \mathrm{Cl}^+(V)$ with $\psi(u \wedge v) = uv - vu$, and ψ is injective. A straightforward computation shows that the image L of ψ is closed under the commutator in $\mathrm{Cl}(V)$, hence a Lie subalgebra. We claim that L is isomorphic to $\mathfrak{so}(V)$. Indeed, for $u, v, w \in V$ we have

$$[\psi(u \wedge v), w] = [[u, v], w] = 4(v|w)u - 4(u|w)v.$$

We see, first, that $V \subseteq \mathrm{Cl}(V)$ is preserved under the adjoint action of L; and second, that L acts on V via skew-symmetric linear maps, so that L maps into $\mathfrak{so}(V)$. Since every map in $\mathfrak{so}(V)$ is a linear combination of the linear maps above, and considering that $\dim(L) = \dim(\mathfrak{so}(V))$, the map $L \to \mathfrak{so}(V)$ is an isomorphism. We will identify $\mathfrak{so}(V)$ with the Lie subalgebra $L \subseteq \mathrm{Cl}(V)$ via the inverse of this isomorphism, and we will identify $\bigwedge^2 V$ with $\mathfrak{so}(V)$ via the map $u \wedge v \mapsto (w \mapsto (v|w)u - (u|w)v)$. The concatenation of these identifications is the linear map $\frac{1}{4}\psi$.

2.1.4 The Half-Spin Representations

From now on, we assume that $\dim(V) = 2n$. We believe that all our results hold *mutatis mutandis* also in the odd-dimensional case, but we have not checked the details. A *maximal* isotropic subspace U of V is an isotropic subspace which is maximal with respect to inclusion. Since $\mathbb{K}$ is algebraically closed, q has maximal Witt index, so that every maximal isotropic subspace of V has dimension n.

The spin representation of $\mathfrak{so}(V)$ is constructed as follows. Let F be a maximal isotropic subspace of V and let $f_1, \ldots, f_n$ be a basis of F. Define $f := f_1 \cdots f_n \in \mathrm{Cl}(F)$; this element in $\mathrm{Cl}(F) = \bigwedge F$ is well-defined up to a scalar. Then the left ideal $\mathrm{Cl}(V) \cdot f$ is a left module for the associative algebra $\mathrm{Cl}(V)$, and hence for its Lie subalgebra $\mathfrak{so}(V)$. This ideal is called the *spin representation* of $\mathfrak{so}(V)$. As $\mathrm{Cl}(V)$ is $\mathbb{Z}/2\mathbb{Z}$-graded, the spin representation splits into a direct sum of two subrepresentations for $\mathrm{Cl}^+(V)$, and hence for $\mathfrak{so}(V) \subseteq \mathrm{Cl}^+(V)$, namely, $\mathrm{Cl}^+(V) \cdot f$ and $\mathrm{Cl}^-(V) \cdot f$. These representations are called the *half-spin representations* of $\mathfrak{so}(V)$.

2.1.5 Explicit Formulas

We will need more explicit formulas for the action of $\mathfrak{so}(V)$ on the half-spin representations. To this end, let E be another isotropic n-dimensional subspace of V such that $V = E \oplus F$. Then the map

$$\bigwedge E = \mathrm{Cl}(E) \to \mathrm{Cl}(V)f, \quad \omega \mapsto \omega f$$

is a linear isomorphism, and we use it to identify $\bigwedge E$ with the spin representation. We write $\rho : \mathfrak{so}(V) \to \mathrm{End}(\bigwedge E)$ for the corresponding representation. This representation decomposes as a direct sum of the half-spin representations $\rho_+ : \mathfrak{so}(V) \to \mathrm{End}(\bigwedge^+ E)$ and $\rho_- : \mathfrak{so}(V) \to \mathrm{End}(\bigwedge^- E)$, where $\bigwedge^+ E = \bigoplus_{d \text{ even}} \bigwedge^d E$ and $\bigwedge^- E = \bigoplus_{d \text{ odd}} \bigwedge^d E$.

In this model of the spin representation, the action of $v \in E \subseteq \mathrm{Cl}(V)$ on the spin representation $\bigwedge E$ is just the outer product on $\bigwedge E : o(v) : \bigwedge E \to \bigwedge E,\ \omega \mapsto v \wedge \omega$, while the action of $v \in F \subseteq \mathrm{Cl}(V)$ is twice the inner product on $\bigwedge E$:

$$2\iota(v) w_1 \wedge \cdots \wedge w_k = 2 \sum_{i=1}^{k} (-1)^{i-1} (v|w_i) w_1 \wedge \cdots \wedge \widehat{w_i} \wedge \cdots \wedge w_k.$$

The factor 2 and the alternating signs come from the following identity in $\mathrm{Cl}(V)$:

$$vv_i = 2(v|v_i) - v_i v \text{ for } v \in F \text{ and } v_i \in E.$$

For a general $v \in V$ we write $v = v' + v''$ with $v' \in E$, $v'' \in F$. Then the action of V on $\bigwedge E$ is given by

$$v \mapsto o(v') + 2\iota(v'').$$

We now compute the linear maps by means of which $\mathfrak{so}(V)$ acts on $\bigwedge E$. To this end, recall that a pair $e, f \in V$ is called *hyperbolic* if e, f are isotropic and $(e|f) = 1$. Given the basis $f_1, \ldots, f_n$ of F, there is a unique basis $e_1, \ldots, e_n$ of E so that $(e_i|f_j) = \delta_{ij}$; then $e_1, \ldots, e_n, f_1, \ldots, f_n$ is called a *hyperbolic basis* of V. Now the element $e_i \wedge e_j \in \mathfrak{so}(V)$ acts on $\bigwedge E \simeq \mathrm{Cl}(V)f$ via the linear map

$$\frac{1}{4}\Big(o(e_i)o(e_j) - o(e_j)o(e_i)\Big) = \frac{1}{2} o(e_i)o(e_j);$$

the element $f_i \wedge f_j$ acts via the linear map

$$\frac{1}{4}\Big(4\iota(f_i)\iota(f_j) - 4\iota(f_j)\iota(f_i)\Big) = 2\iota(f_i)\iota(f_j);$$

and the element $e_i \wedge f_j$ acts via the linear map

$$\frac{1}{4}\Big(o(e_i)2\iota(f_j) - 2\iota(f_j)o(e_i)\Big) = \frac{1}{2}\Big(o(e_i)\iota(f_j) - \iota(f_j)o(e_i)\Big).$$

In particular, $\omega_0 := e_1 \wedge \cdots \wedge e_n \in \bigwedge E$ is mapped to 0 by all elements $e_i \wedge e_j$ and all elements $e_i \wedge f_j$ with $i \neq j$, and it is mapped to $\frac{1}{2}\omega_0$ by all $e_i \wedge f_i$.

2.1.6 Highest Weights of the Half-Spin Representations

Remember, for instance from [Jac62, Chapter IV, pages 140–141], that in the basis $e_1, \ldots, e_n, f_1, \ldots, f_n$, matrices in $\mathfrak{so}(V)$ have the form

$$\begin{bmatrix} A & B \\ C & -A^T \end{bmatrix} \text{ with } B^T = -B, \text{ and } C^T = -C.$$

Here the (e_i, e_j)-entry of A is the coefficient of $e_i \wedge f_j$, the (e_i, f_j)-entry of B is the coefficient of $e_i \wedge e_j$, and the (f_i, e_j)-entry of C is the coefficient of $f_i \wedge f_j$.

The diagonal matrices $e_i \wedge f_i$ span a Cartan subalgebra of $\mathfrak{so}(V)$ with standard (Chevalley) basis consisting of $h_i := e_i \wedge f_i - e_{i+1} \wedge f_{i+1}$ for $i = 1, \ldots, n-1$ and $h_n := e_{n-1} \wedge f_{n-1} + e_n \wedge f_n$ (this last element is forgotten in the basis of the Cartan algebra on [Jac62, page 140]).

Now $(e_i \wedge e_j)\omega_0 = (e_i \wedge f_j)\omega_0 = 0$ for all $i \neq j$. Furthermore, the elements $h_1, \ldots, h_{n-1}$ map ω_0 to 0, while h_n maps ω_0 to ω_0. Thus the Borel subalgebra maps the line $\mathbb{K}\omega_0$ into itself and ω_0 is a highest weight vector of the *fundamental weight* $\lambda_0 := (0, \ldots, 0, 1)$ in the standard basis. Summarising, $\omega_0 \in \bigwedge E$ generates a copy of the irreducible $\mathfrak{so}(V)$-module V_{λ_0} with highest weight λ_0. Clearly, the $\mathfrak{so}(V)$-module generated by ω_0 is contained in $\bigwedge^+ E$ if n is even, and contained in $\bigwedge^- E$ when n is odd. One can also show that both half-spin representations are irreducible, hence one of them is a copy of V_{λ_0}. For the other half-spin representation, consider the element

$$\omega_1 := e_1 \wedge \cdots \wedge e_{n-1} \in \bigwedge E.$$

This element is mapped to zero by $e_i \wedge e_j$ for all $i \neq j$ and by $e_i \wedge f_j$ for all $i < j$. It is further mapped to 0 by $h_1, \ldots, h_{n-2}, h_n$, and to ω_1 by h_{n-1}. For example, we have

$$\begin{aligned} h_n\omega_1 &= \frac{1}{2}\Big(o(e_{n-1})\iota(f_{n-1}) - \iota(f_{n-1})o(e_{n-1}) + o(e_n)\iota(f_n) - \iota(f_n)o(e_n)\Big)e_1 \wedge \cdots \wedge e_{n-1} \\ &= \frac{1}{2}(1 - 0 + 0 - 1)\omega_1 = 0, \text{ and similarly} \\ h_{n-1}\omega_1 &= \frac{1}{2}(1 - 0 - 0 + 1)\omega_1 = \omega_1. \end{aligned}$$

Hence ω_1 generates a copy of V_{λ_1}, the irreducible $\mathfrak{so}(V)$-module with the highest weight vector $\lambda_1 := (0, \ldots, 0, 1, 0)$; this is the other half-spin representation.

2.1.7 The Spin Group

Let $\rho : \mathfrak{so}(V) \to \mathrm{End}(\bigwedge E)$ be the spin representation. We can then define the spin group $\mathrm{Spin}(V)$ as the subgroup of $\mathrm{GL}(\bigwedge E)$ generated by the one-parameter subgroups $t \mapsto \exp(t\rho(X))$ where X runs over the root vectors $e_i \wedge e_j, f_i \wedge f_j$ and $e_i \wedge f_j$ with $i \neq j$. Note that $\rho(X)$ is nilpotent for each of these root vectors, so that $t \mapsto \exp(t\rho(X))$ is an *algebraic* group homomorphism $\mathbb{K} \to \mathrm{GL}(\bigwedge E)$. It is a standard fact that the subgroup generated by irreducible curves through the identity in an algebraic group is itself a connected algebraic group; see [Bor91, Proposition 2.2]. So $\mathrm{Spin}(V)$ is a connected algebraic group, and one verifies that its Lie algebra is isomorphic to the Lie algebra generated by the root vectors X, i.e., to $\mathfrak{so}(V)$.

By construction, the (half-)spin representations $\bigwedge E$, $\bigwedge^+ E$ and $\bigwedge^- E$ are representations of $\mathrm{Spin}(V)$. We use the same notation for these representations as we did for the corresponding Lie algebra representations:

$$\rho : \mathrm{Spin}(V) \to \mathrm{GL}\left(\bigwedge E\right),\ \rho_+ :\ \mathrm{Spin}(V) \to \mathrm{GL}\left(\bigwedge^+ E\right),$$

and

$$\rho_- :\ \mathrm{Spin}(V) \to \mathrm{GL}\left(\bigwedge^- E\right).$$

Remark 2.1.1. The algebraic group $\mathrm{Spin}(V)$ is usually constructed as a subgroup of the unit group $\mathrm{Cl}^*(V)$ as follows: consider first

$$\Gamma(V) = \{x \in \mathrm{Cl}^*(V) \mid xVx^{-1} = V\},$$

sometimes called the Clifford group. Then $\mathrm{Spin}(V)$ is the subgroup of $\Gamma(V)$ of elements of *spinor norm* 1; that is, $xx^* = 1$, where x^* denotes the involution defined in Section 2.1.1. In this model of the spin group, one can easily observe that it admits a $2:1$ covering $\mathrm{Spin}(V) \to \mathrm{SO}(V)$, namely, the restriction of the homomorphism $\Gamma(V) \to \mathrm{O}(V)$ given by associating to $x \in \Gamma(V)$ the orthogonal transformation $w \mapsto xwx^{-1}$. For more details see [Pro07]. Since our later computations involve the Lie algebra $\mathfrak{so}(V)$ only, the definition of $\mathrm{Spin}(V)$ above suffices for our purposes.

The half-spin representations are *not* representations of the group $\mathrm{SO}(V)$; this can be checked, e.g., by showing that the highest weights λ_0 and λ_1 are not in the weight lattice of $\mathrm{SO}(V)$.

2.1.8 Two Actions of $\mathfrak{gl}(E)$ on $\bigwedge E$

The definition of the (half-)spin representation(s) of $\mathfrak{so}(V)$ and $\mathrm{Spin}(V)$ as $\mathrm{Cl}^{(\pm)}(V)f$ involves only the quadratic form q and the choice of a maximal isotropic space $F \subseteq V$. Consequently, any linear automorphism of V that preserves q and maps F into itself also acts on $\mathrm{Cl}^{(\pm)}(V)f$. These linear automorphisms form the stabiliser of F in $\mathrm{SO}(V)$, which is the parabolic subgroup whose Lie algebra consists of the matrices in $\mathrm{SO}(V)$ that are block lower triangular in the basis $e_1, \ldots, e_n, f_1, \ldots, f_n$. So, while $\mathrm{SO}(V)$ does not act naturally on the (half-)spin representation(s), this stabiliser does.

In particular, in our model $\bigwedge^{(\pm)} E$ of the (half-)spin representation(s), the group $\mathrm{GL}(E)$, embedded into $\mathrm{SO}(V)$ as the subgroup of block diagonal matrices

$$\begin{bmatrix} a & 0 \\ 0 & -a^T \end{bmatrix}$$

acts on $\bigwedge E$ in the natural manner. We stress that this is *not* the action obtained by integrating the action of $\mathfrak{gl}(E) \subseteq \mathfrak{so}(V)$ on $\bigwedge E$ *regarded as the spin representation.* Indeed, the standard action of $e_i \wedge f_j \in \mathfrak{gl}(E)$ on $\omega := e_{i_1} \wedge \cdots \wedge e_{i_k} \in \bigwedge^k E$ yields

$$\sum_{l=1}^{k} e_{i_1} \wedge \cdots \wedge e_i(f_j|e_{i_l}) \wedge \cdots \wedge e_{i_k} = \begin{cases} 0 \text{ if } j \notin \{i_1, \dots, i_k\} \\ (-1)^{l-1} e_i \wedge e_{i_1} \wedge \cdots \wedge \widehat{e_{i_l}} \wedge \cdots \wedge e_{i_k} \text{ if } j = i_l. \end{cases}$$

On the other hand, in the spin representation the action is given by the linear map $\frac{1}{2}\Big(o(e_i)\iota(f_j) - \iota(f_j)o(e_i)\Big)$. If $j \neq i$ and $j \notin \{i_1, \dots, i_k\}$, then

$$o(e_i)\iota(f_j)\omega = \iota(f_j)o(e_i)\omega = 0.$$

If $j \neq i$ and $j = i_l$, then

$$o(e_i)\iota(f_j)\omega = (-1)^{l-1} e_i \wedge e_{i_1} \wedge \cdots \wedge \widehat{e_{i_l}} \wedge \cdots \wedge e_{i_k} = -\iota(f_j)o(e_i)\omega.$$

We conclude that for $i \neq j$, the action of $e_i \wedge f_j$ is the same in both representations. However, if $i = j$, then

$$\frac{1}{2}\Big(o(e_i)\iota(f_i) - \iota(f_i)o(e_i)\Big)\omega = \begin{cases} -\frac{1}{2}\omega \text{ if } i \notin \{i_1, \dots, i_k\}, \text{ and} \\ \frac{1}{2}(-1)^{l-1} e_i \wedge e_{i_1} \wedge \cdots \wedge \widehat{e_{i_l}} \wedge \cdots \wedge e_{i_k} = \frac{1}{2}\omega \text{ if } i = i_l. \end{cases}$$

We conclude that if $\tilde{\rho} : \mathfrak{gl}(E) \to \mathrm{End}\,(\bigwedge E)$ is the standard representation of $\mathfrak{gl}(E)$, then the restriction of the spin representation $\rho : \mathfrak{so}(V) \to \mathrm{End}(\bigwedge E)$ to $\mathfrak{gl}(E)$ *as a subalgebra of* $\mathfrak{so}(V)$ satisfies

$$\rho(A) = \tilde{\rho}(A) - \frac{1}{2}\,\mathrm{tr}(A)\,\mathrm{Id}_{\bigwedge E}\,. \tag{2.1.3}$$

At the group level, this is to be understood as follows. The pre-image of the subgroup $\mathrm{GL}(E) \subseteq \mathrm{SO}(V)$ in $\mathrm{Spin}(V)$ is isomorphic to the connected algebraic group

$$H := \Big\{(g, t) \in \mathrm{GL}(E) \times K^* \;\Big|\; \det(g) = t^2\Big\}$$

for which $(g, t) \mapsto g$ is a $2 : 1$ cover of $\mathrm{GL}(E)$, and the restriction of ρ to H satisfies $\rho(g, t) = \tilde{\rho}(g) \cdot t^{-1}$, a "twist of the standard representation by the inverse square root of the determinant".

2.2 The Isotropic Grassmannian and Infinite Spin Representations

2.2.1 The Isotropic Grassmannian in its Spinor Embedding

As before, let V be a $2n$-dimensional vector space over $\mathbb{K}$ endowed with a non-degenerate quadratic form. The (maximal) *isotropic Grassmannian* $\mathrm{Gr}_{\mathrm{iso}}(V, q)$ parametrizes all maximal isotropic subspaces of V. It has two connected components, denoted $\mathrm{Gr}^{+}_{\mathrm{iso}}(V)$ and $\mathrm{Gr}^{-}_{\mathrm{iso}}(V)$. The goal of this subsection is to introduce the isotropic Grassmann cone, which is an affine cone over $\mathrm{Gr}_{\mathrm{iso}}(V, q)$ in the spin representation.

Fix a maximal isotropic subspace $F \subseteq V$ and as before set $f \coloneqq f_1 \cdots f_n \in \mathrm{Cl}(V)$, where $f_1, \ldots, f_n$ is any basis of F. Now let $H \subseteq V$ be another maximal isotropic space. Then we claim that the space

$$S_H \coloneqq \Big\{\omega \in \mathrm{Cl}(V)f \;\Big|\; v \cdot \omega = 0 \text{ for all } v \in H\Big\} \subseteq \mathrm{Cl}(V)f \tag{2.2.1}$$

is 1-dimensional. Indeed, we may find a hyperbolic basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ of V such that $f_1, \ldots, f_k$ span $H \cap F$, $f_1, \ldots, f_n$ span F, and $e_{k+1}, \ldots, e_n, f_1, \ldots, f_k$ span H. We call this hyperbolic basis *adapted to H and F*. Then the element

$$\omega_H \coloneqq e_{k+1} \cdots e_n f_1 \cdots f_k f_{k+1} \cdots f_n \in \mathrm{Cl}(V)f$$

lies in S_H since $e_i \omega_H = f_j \omega_H = 0$ for all $i > k$ and $j \leq k$. Conversely, if $\mu \in S_H$, then write

$$\mu = \sum_{l=0}^{n} \sum_{i_1 < \ldots < i_l} c_{\{i_1, \ldots, i_l\}} e_{i_1} \cdots e_{i_l} f.$$

If $c_I \neq 0$ for some I with $I \not\supseteq \{k+1, \ldots, n\}$, then for any $j \in \{k+1, \ldots, n\} \setminus I$ we find that $e_j \mu \neq 0$. So all I with $c_I \neq 0$ contain $\{k+1, \ldots, n\}$. If some I with $c_I \neq 0$ further contains an $i \leq k$, then $f_i \mu$ is nonzero. Hence S_H is spanned by ω_H, as claimed. In what follows, by slight abuse of notation, we will write ω_H for any nonzero vector in S_H.

The space H can be uniquely recovered from ω_H via

$$H = \Big\{v \in V \;\Big|\; v \cdot \omega_H = 0\Big\}.$$

Indeed, we have already seen $\subseteq$. For the converse, observe that the vectors $e_i \omega_H, f_j \omega_H$ with $i \leq k$ and $j > k$ are linearly independent.

The map that sends $H \in \mathrm{Gr}_{\mathrm{iso}}(V, q)$ to the projective point representing it, i.e.,

$$H \mapsto [\omega_H] \in \mathbb{P}(\mathrm{Cl}(V)f),$$

is therefore injective, and it is called the *spinor embedding* of the isotropic Grassmannian (see [Man09]). The *isotropic Grassmann cone* is defined as

$$\widehat{\mathrm{Gr}}_{\mathrm{iso}}(V, q) \coloneqq \bigcup_H \langle \omega_H \rangle \subseteq \mathrm{Cl}(V)f,$$

where the union is the taken over all maximal isotropic subspaces $H \subseteq V$. We denote by $\widehat{\mathrm{Gr}}^{\pm}_{\mathrm{iso}}(V, q) \coloneqq \widehat{\mathrm{Gr}}_{\mathrm{iso}}(V, q) \cap \mathrm{Cl}^{\pm}(V)f$ the cones over the connected components of the isotropic Grassmannian in its spinor embedding.

2.2.2 Contraction with an Isotropic Vector

Let $e \in V$ be a nonzero isotropic vector. Then $V_e \coloneqq e^{\perp}/\langle e \rangle$ is equipped with a natural non-degenerate quadratic form, and there is a rational map $\mathrm{Gr}_{\mathrm{iso}}(V) \to \mathrm{Gr}_{\mathrm{iso}}(V_e)$ that maps an n-dimensional isotropic space H to the image in V_e of the $(n-1)$-dimensional isotropic space $H \cap e^{\perp}$ (this is defined if $e \notin H$, which by maximality of H is equivalent to $H \not\subseteq e^\perp$). This map is the restriction to the isotorpic Grassmannian $\mathrm{Gr}_{\mathrm{iso}}(V)$ of the rational map $\mathbb{P}(\bigwedge^n V) \to \mathbb{P}(\bigwedge^{n-1} V_e)$ induced by the linear map ("contraction with e"):

$$c_e : \bigwedge^n V \to \bigwedge^{n-1} V_e, \quad v_1 \wedge \cdots \wedge v_n \mapsto \sum_{i=1}^{n} (-1)^{i-1} (e|v_i) \overline{v_1} \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge \overline{v_n}$$

where $\overline{v_i}$ is the image of v_i in $V/\langle e \rangle$. Note first that this map is the inner product $\iota(e)$ followed by a projection. Furthermore, *a priori*, the codomain of this map is the larger space $\bigwedge^{n-1}(V/\langle e \rangle)$, but one may choose $v_1, \ldots, v_n$ such that $(e|v_i) = 0$ for $i > 1$, and then it is evident that the image is indeed in $\bigwedge^{n-1} V_e$.

We want to construct a similar contraction map at the level of the spin representation. For reasons that will become clear in a moment, we restrict our attention first to a map between two half-spin representations, as follows. Assume that $e \notin F$, and choose a basis $f_1, \ldots, f_n$ of F such that $(e|f_i) = \delta_{in}$. As usual, write $f := f_1 \cdots f_n$, and write $\overline{f} := \overline{f}_1 \cdots \overline{f}_{n-1}$, so that $\mathrm{Cl}^+(V_e)\overline{f}$ is a half-spin representation of $\mathfrak{so}(V_e)$.

Then we define the map

$$\pi_e : \mathrm{Cl}^+(V)f \to \mathrm{Cl}^+(V_e)\overline{f}, \quad \pi_e(af) := \text{ the image of } \frac{1}{2}\Big((-1)^{n-1} eaf + afe\Big) \text{ in } \mathrm{Cl}(V_e)\overline{f},$$

where the implicit claim is that the expression on the right lies in $\mathrm{Cl}(e^\perp) f_1 \cdots f_{n-1}$, so that its image in $\mathrm{Cl}(V_e)\overline{f}$ is well defined (note that the projection $e^\perp \to V_e$ induces a homomorphism of Clifford algebras), and that this image lies in the left ideal generated by $\overline{f}$. To verify this claim, and to derive a more explicit formula for the map above, let $e_1, \ldots, e_n = e$ be a basis of an isotropic space E complementary to F. Then it suffices to consider the case where $a = e_{i_1} \cdots e_{i_k}$ for some $i_1 < \ldots < i_k$. We then have

$$\begin{aligned} eaf &= ee_{i_1} \cdots e_{i_k} f_1 \cdots f_n \\ &= \begin{cases} 0 \text{ if } i_k = n, \text{ and} \\ 2(-1)^{k+n-1} e_{i_1} \cdots e_{i_k} f_1 \cdots f_{n-1} + (-1)^{k+n} e_{i_1} \cdots e_{i_k} f_1 \cdots f_n e \text{ otherwise.} \end{cases} \end{aligned}$$

Multiplying by $(-1)^{n-1}$ and using that k is even, the latter expression becomes

$$2e_{i_1} \cdots e_{i_k} f_1 \cdots f_{n-1} - afe.$$

Hence we conclude that

$$\pi_e(e_{i_1} \cdots e_{i_k} f) = \begin{cases} 0 & \text{if } i_k = n, \text{ and} \\ \overline{e}_{i_1} \cdots \overline{e}_{i_k} \overline{f} & \text{otherwise.} \end{cases}$$

In short, in our models $\bigwedge^+ E$ and $\bigwedge^+(E/\langle e \rangle)$ for the half-spin representations of $\mathfrak{so}(V)$ and $\mathfrak{so}(V_e)$, π_e is just the reduction-mod-e map. We leave it to the reader to check that the reduction-mod-e map $\bigwedge^- E \to \bigwedge^-(E/\langle e \rangle)$ arises in a similar fashion from the map

$$\pi_e : \mathrm{Cl}(V)^- f \to \mathrm{Cl}(V_e)^- \overline{f}, \quad \pi_e(af) := \text{ the image of } \frac{1}{2}\Big((-1)^n eaf + afe\Big) \text{ in } \mathrm{Cl}(V_e)\overline{f}.$$

We will informally call the maps π_e "contraction with e". Together they define a map on $\mathrm{Cl}(V)f$ which we also denote by π_e.

Proposition 2.2.1. *The contraction map* $\pi_e : \mathrm{Cl}(V)f \to \mathrm{Cl}(V_e)\overline{f}$ *is a homomorphism of* $\mathrm{Cl}(e^\perp)$*-representations.*

Proof. Let $v \in e^\perp$ and consider $a \in \mathrm{Cl}^-(V)$. Then $va \in \mathrm{Cl}^+(V)$ and hence $\pi_e(vaf)$ is the image in $\mathrm{Cl}(V_e)\overline{f}$ of

$$\frac{1}{2}\Big((-1)^{n-1}evaf + vafe\Big) = \frac{1}{2}\Big((-1)^n veaf + vafe\Big) = v\frac{1}{2}\Big((-1)^n eaf + afe\Big),$$

where we have used $(v|e) = 0$ in the first equality. The right-hand side clearly equals $\overline{v}$ times the image of $\pi_e(af)$ in $\mathrm{Cl}(V_e)\overline{f}$. □

2.2.3 Multiplying with an Isotropic Vector

In a sense dual to the contraction maps $c_e : \bigwedge^n V \to \bigwedge^{n-1} V_e$ are multiplication maps defined as follows. Let $e, h \in V$ be isotropic with $(e|h) = 1$; such a pair is called a *hyperbolic pair*. We then have $V = \langle e, h\rangle \oplus \langle e, h\rangle^\perp$, and the map from the second summand to $V_e = e^\perp/\langle e\rangle$ is an isometry. We use this isometry to identify V_e with the subspace $\langle e, h\rangle^\perp$ of V and write s_e for the corresponding inclusion map. Then we define

$$m_h : \bigwedge^{n-1} V_e \to \bigwedge^{n} V, \quad \overline{v}_1 \wedge \cdots \wedge \overline{v}_{n-1} \mapsto h \wedge \overline{v}_1 \wedge \cdots \wedge \overline{v}_{n-1},$$

which is just the outer product $o(h)$. The projectivisation of this map sends $\mathrm{Gr}_{\mathrm{iso}}(V_e)$ isomorphically to the closed subset of $\mathrm{Gr}_{\mathrm{iso}}(V)$ consisting of all H containing h. We further observe that

$$c_e \circ m_h = \mathrm{id}_{\bigwedge^{n-1} V_e}.$$

We define a corresponding multiplication map at the level of spin representations as follows: first, we assume that $h \in F$, and choose a basis $f_1, \ldots, f_n = h$ of F such that $(e|f_i) = \delta_{in}$. As usual, we set $f = f_1 \cdots f_n$ and $\overline{f} = \overline{f}_1 \cdots \overline{f}_{n-1}$. Then we define

$$\tau_h : \mathrm{Cl}(V_e)\overline{f} \to \mathrm{Cl}(V)f, \quad \tau_h(a\overline{f}) := a\overline{f}f_n = af.$$

Note that, for $a \in \mathrm{Cl}(V_e)$, we have

$$\pi_e\left(\tau_h(a\overline{f})\right) = \pi_e(af) = a\overline{f},$$

where the last identity can be seen verified in the model $\bigwedge E$ for the spin representation, where π_e is the reduction-mod-e map, and τ_h is just the inclusion $\bigwedge E/\langle e\rangle \to \bigwedge E$ corresponding to the inclusion $V_e \to V$. So $\pi_e \circ \tau_h = \mathrm{id}_{\mathrm{Cl}(V_e)\overline{f}}$. We will informally call τ_h the multiplication map with h.

Proposition 2.2.2. *The multiplication map* $\tau_h : \mathrm{Cl}(V_e)\overline{f} \to \mathrm{Cl}(V)f$ *is a homomorphism of* $\mathrm{Cl}(V_e)$*-representations, where* $\mathrm{Cl}(V_e)$ *is regarded a subalgebra of* $\mathrm{Cl}(V)$ *via the section* $s_e : V_e \to V$.

Proof. Let $v \in V_e$ and let $a \in \mathrm{Cl}(V_e)$. Then

$$\tau_h(va\overline{f}) = va\overline{f}f_n = vaf,$$

as desired. □

Corollary 2.2.3. *Both the mappings* $\pi_e : \mathrm{Cl}(V)f \to \mathrm{Cl}(V_e)\overline{f}$ *and* $\tau_h : \mathrm{Cl}(V_e)\overline{f} \to \mathrm{Cl}(V)f$ *are* $\mathrm{Spin}(V_e)$*-equivariant, where* $\mathrm{Spin}(V_e)$ *is regarded as a subgroup of* $\mathrm{Spin}(V)$ *via the orthogonal decomposition* $V = V_e \oplus \langle e, h\rangle$*.*

Proof. Proposition 2.2.1 and Proposition 2.2.2 imply that both maps are homomorphisms of $\mathfrak{so}(V_e)$-representations. Since $\mathrm{Spin}(V_e)$ is generated by one-parameter subgroups corresponding to nilpotent elements of $\mathfrak{so}(V_e)$, π_e and τ_h are $\mathrm{Spin}(V_e)$-equivariant. □

2.2.4 Properties of the Isotropic Grassmannian

The goal of this subsection is to collect properties of the isotropic Grassmann cone that will later motivate the definition of a *(half-)spin variety* (see Section 2.4). We fix a maximal isotropic subspace $F \subseteq V$ and a hyperbolic pair (e, h) with $h \in F$ and $e \notin F$ and identify $V_e = e^\perp/\langle e\rangle$ with the subspace $\langle e, h\rangle^\perp$ of V. We choose any basis $f_1, \ldots, f_n$ of F with $f_n = h$ and $(e|f_i) = 0$ for $i < n$ and write $f := f_1 \cdots f_n \in \mathrm{Cl}(V)$ and $\overline{f} := \overline{f_1} \cdots \overline{f_{n-1}} \in \mathrm{Cl}(V_e)$.

Proposition 2.2.4. *The isotropic Grassmann cone in* $\mathrm{Cl}(V)f$ *has the following properties:*

1. $\widehat{\mathrm{Gr}}_{\mathrm{iso}}(V) \subseteq \mathrm{Cl}(V)f$ *is Zariski closed and* $\mathrm{Spin}(V)$*-stable.*

2. *Let* $\pi_e : \mathrm{Cl}(V)f \to \mathrm{Cl}(V_e)\overline{f}$ *be the contraction defined in Section 2.2.2. Then for every maximal isotropic subspace* $H \subseteq V$ *we have*

$$\pi_e(S_H) \subseteq S_{H_e},$$

   where $H_e \subseteq V_e$ *is the image of* $e^\perp \cap H$ *in* V_e*.*

3. *Let* $\tau_h : \mathrm{Cl}(V_e)\overline{f} \to \mathrm{Cl}(V)f$ *be the map defined in Section 2.2.3. Then for every maximal isotropic* $H' \subseteq V_e$ *we have*

$$\tau_h(S_{H'}) = S_{H'\oplus\langle h\rangle}.$$

In particular, the contraction and multiplication map π_e *and* τ_h *preserve the isotropic Grassmann cones, i.e.,*

$$\pi_e\big(\widehat{\mathrm{Gr}}_{\mathrm{iso}}(V)\big) \subseteq \widehat{\mathrm{Gr}}_{\mathrm{iso}}(V_e) \quad \textit{and} \quad \tau_h\big(\widehat{\mathrm{Gr}}_{\mathrm{iso}}(V_e)\big) \subseteq \widehat{\mathrm{Gr}}_{\mathrm{iso}}(V).$$

Proof of Proposition 2.2.4.

1. This is well known. Indeed, the isotropic Grassmann cone is the union of the cones over the two connected components, and these cones are the union of $\{0\}$ with the orbits of the highest weight vectors ω_0 and ω_1. These minimal orbits are always Zariski closed. For more detail see [Pro07, Theorem 1, p.428].

2. Let ω_H be a spanning element of S_H. Then for all $v \in e^\perp \cap H$ we have

$$\overline{v} \cdot \pi_e(\omega_H) = \pi_e(v \cdot \omega_H) = \pi_e(0) = 0,$$

   where the first equality follows from Proposition 2.2.1. Hence $\pi_e(\omega_H)$ lies in S_{H_e}.

3. Let $\omega_{H'}$ be a spanning element of $S_{H'}$. Then for all $v \in H'$ we have

$$v \cdot \tau_h(\omega_{H'}) = \tau_h(v \cdot \omega_{H'}) = \tau_h(0) = 0$$

where the first equality holds by Proposition 2.2.2. Furthermore, we have

$$h \cdot \tau_h(\omega_{H'}) = h \cdot \omega_{H'} f_n = 0,$$

where we used the definition of $\tau_h, h \perp V_e$ and $h = f_n$. Thus $\tau_h(\omega_{H'})$ lies in $S_{H' \oplus \langle h \rangle}$. The equality now follows from the fact that τ_h is injective. □

Remark 2.2.5. If $h \in H$, then $H = H_e \oplus \langle h \rangle$ and since $\pi_e \circ \tau_h$ is the identity on $\mathrm{Cl}(V_e)\overline{f}$ we find that

$$\pi_e(S_H) = \pi_e(\tau_h(S_{H_e})) = S_{H_e},$$

i.e., equality holds in (2) of Proposition 2.2.4. Later we will see that equality holds under the weaker condition that $e \notin H$, while $\pi_e(S_H) = \{0\}$ when $e \in H$. These statements can also be checked by direct computations, but some care is needed since for e, H, F in general position one cannot construct a hyperbolic basis adapted to H and F that moreover contains e.

2.2.5 The Dual of Contraction

Let $e \notin F \subseteq V$ be an isotropic vector. We want to compute the dual of the contraction map $\pi_e : \mathrm{Cl}(V)f \to \mathrm{Cl}(V_e)\overline{f}$; indeed, we claim that this is essentially the map

$$\psi_e : \mathrm{Cl}(V_e)\overline{f} \to \mathrm{Cl}(V)f$$

defined by its restriction $\mathrm{Cl}^{\pm}(V_e)\overline{f} \to \mathrm{Cl}^{\mp}(V)f$ as

$$\psi_e(\overline{b} \cdot \overline{f}_1 \cdots \overline{f_{n-1}}) := \pm e b f_1 \cdots f_n,$$

where the sign is $+$ on $\mathrm{Cl}^{+}(V_e)\overline{f}$ and $-$ on $\mathrm{Cl}^{-}(V_e)\overline{f}$. The reason for the "flip" of the choice of half-spin representation in the dual will become obvious below. Observe that ψ_e is well-defined and, given a basis $e_1, \ldots, e_n = e$ of an isotropic space complementary to F such that $e_1, \ldots, e_n, f_1, \ldots, f_n$ is a hyperbolic basis, maps $\overline{e_J}\overline{f}$ to $e_{J \cup \{n\}} f$.

Proposition 2.2.6. *The following diagram:*

$$\begin{array}{ccc} (\mathrm{Cl}(V_e)\overline{f})^* & \xrightarrow{\pi_e^*} & (\mathrm{Cl}(V)f)^* \\ \simeq \downarrow & & \downarrow \simeq \\ \mathrm{Cl}(V_e)\overline{f} & \xrightarrow[\psi_e]{} & \mathrm{Cl}(V)f \end{array}$$

can be made commuting via a $\mathrm{Spin}(V_e)$*-module isomorphism on the left vertical arrow and a* $\mathrm{Spin}(V)$*-module isomorphism on the right vertical arrow.*

Remark 2.2.7. The statement of Proposition 2.2.6 holds true when replacing $\mathrm{Cl}(V)f$ by either one of the two half-spin representations by considering the correct "flip". For example, if $n = \dim F$ is even, and $e_1, \ldots, e_n, f_1, \ldots, f_n$ is a hyperbolic basis as above, then in the $\bigwedge E$-model the correct grading is

$$\begin{array}{ccc} \left(\bigwedge^+ E_{n-1}\right)^* & \xrightarrow{\pi_{e_n}^*} & \left(\bigwedge^+ E_n\right)^* \\ \simeq\downarrow & & \downarrow\simeq \\ \bigwedge^- E_{n-1} & \xrightarrow[\psi_{e_n}]{} & \bigwedge^+ E_n. \end{array}$$

To prove Proposition 2.2.6 we will consider the bilinear form β on the spin representation $\mathrm{Cl}(V)f$ defined as in [Pro07] as follows: for $af, bf \in \mathrm{Cl}(V)f$ it turns out that $(af)^*bf = f^*a^*bf$, where $*$ denotes the anti-automorphism from Section 2.1.1, is a scalar multiple of f. The scalar is denoted $\beta(af, bf)$. We have the following properties:

Lemma 2.2.8 ([Pro07, p. 430])**.** *Let β be the bilinear form defined as above.*

1. *The form β is non-degenerate and* $\mathrm{Spin}(V)$*-invariant.*
2. *β is symmetric if $n \equiv 0, 1 \mod 4$, and it is skew-symmetric if $n \equiv 2, 3 \mod 4$.*
3. *The two half-spin representations are self-dual via β if n is even, and each is the dual of the other if n is odd.*

In the proof of Proposition 2.2.6 we will use a hyperbolic basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ with $e_n = e$. For a subset $I = \{i_1 < \ldots < i_k\} \subseteq [n]$ set $e_I := e_{i_1} \cdots e_{i_k} \in \mathrm{Cl}(E) \simeq \bigwedge E$, where E is the span of the e_i. We have seen in Section 2.1.5 that the spin representation has as a basis the elements $e_I f$ with I running through all subsets of $[n]$.

Proof of Proposition 2.2.6. Consider the bilinear forms β on $\mathrm{Cl}(V)f$ and β_e on $\mathrm{Cl}(V_e)\overline{f}$ as defined above. By Lemma 2.2.8 the spin representations $\mathrm{Cl}(V)f$ and $\mathrm{Cl}(V_e)\overline{f}$ are self-dual via β and β_e, respectively. Thus it suffices to prove, for $a \in \mathrm{Cl}(V)$ and $b \in \mathrm{Cl}(e^\perp)$, that

$$\beta_e\left(\pi_e(af), \overline{bf}\right) = \frac{(-1)^{n-1}}{2}\beta\left(af, \psi_e(\overline{bf})\right).$$

We may assume that $a = e_I$, $b = e_J$ with $I \subseteq [n]$, $J \subseteq [n-1]$.

In the $\bigwedge E$-model π_e is the mod-e map, and hence the left-hand side is zero if $n \in I$. If $n \notin I$, then the left-hand side equals the coefficient of $\overline{f}$ in $\overline{f}^* \overline{e_I}^* \overline{e_J}\overline{f}$. This is nonzero if and only if $[n-1]$ is the disjoint union of I and J, and then it is 2^{n-1} times a sign corresponding to the number of swaps needed to move the factors $\overline{f_i}$ of $\overline{f}^*$ to just before the corresponding factor $\overline{e_i}$ in either $\overline{e_I}^*$ or $\overline{e_J}$.

Apart from the factor $\frac{(-1)^{n-1}}{2}$, the right-hand side is the coefficient of f in $f^* e_I e_J e_n f$. This is nonzero if and only if $[n]$ is the disjoint union of the sets $\{n\}, J, I$, and in that case it is 2^n times a sign corresponding to the number of swaps needed to move the factors f_i of f^* to the corresponding factor e_i in either e_I or e_J or (in the case of f_n) to just before the factor e_n. The latter contributes $(-1)^{n-1}$, and apart from this factor the sign is the same as on the left-hand side. □

2.2.6 Two Infinite Spin Representations

Let V_∞ be the countable-dimensional vector space with basis $e_1, f_1, e_2, f_2, \ldots$, and equip V_∞ with the quadratic form for which this is a hyperbolic basis, i.e., $(e_i|e_j) = (f_i|f_j) = 0$ and $(e_i|f_j) = \delta_{ij}$ for all i, j. We write E_∞ and F_∞ for the subspaces of V_∞ spanned by the e_i and the f_i, respectively.

Let V_n be the subspace of V_∞ spanned by $e_1, f_1, e_2, f_2, \ldots, e_n, f_n$, with the restricted quadratic form. We further set $E_n := V_n \cap E_\infty$ and $F_n := V_n \cap F_\infty$. We define the *infinite spin group* as

$$\mathrm{Spin}(V_\infty) := \varinjlim_n \mathrm{Spin}(V_n)$$

where $\mathrm{Spin}(V_{n-1})$ is embedded into $\mathrm{Spin}(V_n)$ as the subgroup that fixes $\langle e_n, f_n \rangle$ element-wise. Similarly, we write $\mathrm{GL}(E_\infty) := \varinjlim_n \mathrm{GL}(E_n)$ and H for the preimage of $\mathrm{GL}(E_\infty)$ in $\mathrm{Spin}(V_\infty)$. We use the notation $\mathfrak{so}(V_\infty)$ and $\mathfrak{gl}(E_\infty)$ for the corresponding direct limits of the Lie algebras $\mathfrak{so}(V_n)$ and $\mathfrak{gl}(E_n)$. Here the direct limits are taken in the categories of abstract groups, and Lie algebras, respectively.

The previous paragraphs give rise to various $\mathrm{Spin}(V_{n-1})$-equivariant maps between the spin representations of $\mathrm{Spin}(V_{n-1})$ and $\mathrm{Spin}(V_n)$. First, contraction with e_n,

$$\pi_{e_n} : \mathrm{Cl}(V_n) f_1 \cdots f_n \to \mathrm{Cl}(V_{n-1}) f_1 \cdots f_{n-1},$$

and second, multiplication with f_n,

$$\tau_{f_n} : \mathrm{Cl}(V_{n-1}) f_1 \cdots f_{n-1} \to \mathrm{Cl}(V_n) f_1 \cdots f_n.$$

We have that these satisfy $\pi_{e_n} \circ \tau_{f_n} = \mathrm{id}$. Third, the map

$$\psi_{e_n} : \mathrm{Cl}(V_{n-1}) f_1 \cdots f_{n-1} \to \mathrm{Cl}(V_n) f_1 \cdots f_n$$

that is dual to π_{e_n} in the sense of Proposition 2.2.6.

Definition 2.2.9 (Direct and inverse spin representation)**.** The *direct (infinite) spin representation* is the direct limit of all spaces $\mathrm{Cl}(V_n) f_1 \cdots f_n$ along the maps ψ_{e_n}. The *inverse (infinite) spin representation* is the inverse limit of all spaces $\mathrm{Cl}(V_n) f_1 \cdots f_n$ along the maps π_{e_n}.

Since the maps ψ_{e_n} and π_{e_n} are $\mathrm{Spin}(V_{n-1})$-equivariant, both of these spaces are $\mathrm{Spin}(V_\infty)$-modules. As the dual of a direct limit is the inverse limit of the duals, and since the maps ψ_{e_n} and π_{e_n} are dual to each other by Proposition 2.2.6, the inverse spin representation is the dual space of the direct spin representation.

In our model $\bigwedge E_n$ of $\mathrm{Cl}(V_n) f_1 \cdots f_n$, the map ψ_{e_n} is just the right multiplication

$$\bigwedge E_{n-1} \to \bigwedge E_n,\ \omega \mapsto \omega \wedge e_n.$$

Hence the direct spin representation has as a basis formal infinite products

$$e_{i_1} \wedge e_{i_2} \wedge \ldots =: e_I$$

where $I = \{i_1 < i_2 < \ldots\}$ is a cofinite subset of $\mathbb{N}$. We will write $\bigwedge_\infty E_\infty$ for this countable-dimensional vector space. The action of the Lie algebra $\mathfrak{so}(V_\infty)$ of $\mathrm{Spin}(V_\infty)$ on this space is given via the explicit formulas from Section 2.1.5. In particular, the span of the e_I with $|\mathbb{N} \setminus I|$ even (respectively, odd) is a $\mathrm{Spin}(V_\infty)$-submodule, and $\bigwedge_\infty E_\infty$ is the direct sum of these (irreducible) modules.

Remark 2.2.10. The reader may wonder why we do not introduce the direct spin representation as the direct limit of all $\mathrm{Cl}(V)f_1 \cdots f_n$ along the maps τ_{f_n}. This would make the ordinary Grassmann algebra $\bigwedge E_\infty$ a model for the direct spin representation, instead of the slightly more complicated-looking space $\bigwedge_\infty E_\infty$. However, the maps dual to the τ_{f_n} correspond to contraction maps with $f_n \in F$, which we have not discussed and which interchange even and odd half-spin representations. We believe that our theorem below goes through for this different setting, as well, but we have not checked the details.

2.2.7 Four Infinite Half-Spin Representations

Keeping in mind that the maps ψ_{e_n} interchange the even and odd subrepresentations, we define the *direct (infinite) half-spin representations* $\bigwedge_\infty^\pm E_\infty$ to be the direct limit

$$\bigwedge\nolimits_\infty^\pm E_\infty = \varinjlim \left(\bigwedge\nolimits^\pm E_0 \to \bigwedge\nolimits^\mp E_1 \to \bigwedge\nolimits^\pm E_2 \to \bigwedge\nolimits^\mp E_3 \to \bigwedge\nolimits^\pm E_4 \to \cdots \right)$$

along the maps ψ_{e_n}. For the sake of readability we will abbreviate this by

$$\bigwedge\nolimits_\infty^\pm E_\infty = \varinjlim_n \bigwedge\nolimits^{\pm(-1)^n} E_n, \tag{2.2.2}$$

where $\pm(-1)^n$ denotes $\pm$ if n is even and $\mp$ if n is odd. In terms of the basis e_I introduced in Section 2.2.6, the half-spin representation $\bigwedge_\infty^+ E_\infty$ is spanned by all e_I with $|\mathbb{N} \setminus I|$ even, and $\bigwedge_\infty^- E_\infty$ by those with $|\mathbb{N} \setminus I|$ odd. The *inverse (infinite) half-spin representations* are defined as the duals of the direct (infinite) half-spin representations. Using the isomorphisms from Remark 2.2.7 we observe

$$\left(\bigwedge\nolimits_\infty^\pm E_\infty\right)^* = \varprojlim_n \left(\bigwedge\nolimits^{\pm(-1)^n} E_n\right)^* \simeq \varprojlim_n \bigwedge\nolimits^\pm E_n. \tag{2.2.3}$$

So the inverse (infinite) half-spin representations can be identified with the inverse limits of the half-spin representations $\bigwedge^\pm E_n$ along the projections π_{e_n}.

We can enrich the inverse spin representation to an affine scheme whose coordinate ring is the symmetric algebra on $\bigwedge_\infty E_\infty$, recalling the following remark.

Remark 2.2.11. Let $\mathbb{K}$ be any field (not necessarily algebraically closed) and W any $\mathbb{K}$-vector space (not necessarily finite dimensional). Then there are canonical identifications

$$W^* = \mathrm{Spec}\left(\mathrm{Sym}(W)\right)(\mathbb{K}) \subseteq \left\{\text{closed points in } \mathrm{Spec}\left(\mathrm{Sym}(W)\right)\right\}.$$

So $\mathrm{Spec}\big(\mathrm{Sym}(W)\big)$ can be seen as an enrichment of W^* to an affine scheme. If W is a linear representation for a group G, then G acts via $\mathbb{K}$-algebra automorphisms on $\mathrm{Sym}\, W$ and hence via $\mathbb{K}$-automorphisms on the affine scheme corresponding to W^*. For $W = \bigwedge_\infty^\pm E_\infty$, this construction extends the natural $\mathrm{Spin}(V_\infty)$-action on the vector space $\varprojlim_n \bigwedge^\pm E_n \simeq W^*$ to the corresponding affine scheme.

By abuse of notation, we will write $(\bigwedge_\infty E_\infty)^*$ also for the scheme itself, and similarly for the inverse half-spin representations $\left(\bigwedge_\infty^\pm E_\infty\right)^*$. Later we will also write $\bigwedge^\pm E_n$ for the affine scheme $\mathrm{Spec}\left(\mathrm{Sym}\left(\bigwedge^{\pm(-1)^n} E_n\right)\right)$ by identifying $\bigwedge^\pm E_n \cong \left(\bigwedge^{\pm(-1)^n} E_n\right)^*$ as in Equation (2.2.3).

2.3 Noetherianity of the Inverse Half-Spin Representations

In this section we prove our main theorem.

Theorem 2.3.1 (Noetherianty). *The inverse half-spin representation* $(\bigwedge_\infty^+ E_\infty)^*$ *is topologically Noetherian with respect to the action of* $\mathrm{Spin}(V_\infty)$*. That is, every descending chain*

$$\left(\bigwedge\nolimits_\infty^+ E_\infty\right)^* \supseteq X_1 \supseteq X_2 \supseteq \dots$$

of closed, reduced $\mathrm{Spin}(V_\infty)$*-stable subschemes stabilises, and the same holds for the other inverse half-spin representation.*

Recall that the action of $\mathrm{Spin}(V_\infty)$ on the inverse half-spin representation (as an affine scheme) is given by $\mathbb{K}$-automorphisms, as described in Remark 2.2.11. We write R for the symmetric algebra on the direct spin representation $\bigwedge_\infty E_\infty$, so the inverse spin representation is $\mathrm{Spec}(R)$. Similarly, we write $R^\pm$ for the symmetric algebras on the direct half-spin representations, so $R^\pm$ is the coordinate ring of $\varprojlim_n \bigwedge^\pm E_n$, respectively.

Let us briefly outline the proof strategy. We will proceed by induction on the minimal degree of an equation defining a closed subset X. Starting with such an equation p, we show that there exists a partial derivative $q := \frac{\partial p}{\partial e_I}$ such that the principal open $X[1/q]$ is topologically H_n-Noetherian, where H_n is the subgroup of $\mathrm{Spin}(V_\infty)$ defined below. For that we use that the H_n-action corresponds to a "twist" of the usual $\mathrm{GL}(E_\infty)$-action, as observed in Section 2.1.8 (for the exact formula see (2.1.3)); this allows us to apply the main result of [ES22]. Finally, for those points which are contained in the vanishing set of the $\mathrm{Spin}(V_\infty)$-orbit of q we can apply induction, as the minimal degree of a defining equation has been lowered by 1.

2.3.1 Shifting

Let G_n be the subgroup of G that fixes $e_1, \dots, e_n, f_1, \dots, f_n$ element-wise. Note that G_n is isomorphic to G; at the level of the Lie algebras the isomorphism from G to G_n is given by the map

$$\begin{bmatrix} A & B \\ C & -A^T \end{bmatrix} \mapsto \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & 0 & 0 \\ 0 & C & 0 & -A^T \end{bmatrix}$$

where the widths of the blocks are n, ∞, n, ∞, respectively. We write H_n for $H \cap G_n$, where $H \subseteq \mathrm{Spin}(V_\infty)$ is the subgroup corresponding to the subalgebra $\mathfrak{gl}(E_\infty) \subseteq \mathfrak{so}(V_\infty)$. Then H_n is the pre-image in $\mathrm{Spin}(V_\infty)$ of the subgroup $\mathrm{GL}(E_\infty)_n \subseteq \mathrm{GL}(E_\infty)$ of all g that fix $e_1, \dots, e_n$ element-wise and maps the span of the e_i with $i > n$ into itself. The Lie algebra of H_n and of $\mathrm{GL}(E_\infty)_n$ consists of the matrices above on the right with $B = C = 0$.

2.3.2 Acting with the General Linear Group on E

For every fixed $k \in \mathbb{Z}_{\geq 0}$, the Lie algebra $\mathfrak{gl}(E_\infty) \subseteq \mathfrak{so}(V_\infty)$ preserves the linear space

$$\left(\bigwedge\nolimits_\infty E_\infty\right)_k := \left\langle \{e_I : |\mathbb{N} \setminus I| = k\} \right\rangle,$$

and hence so does the corresponding subgroup $H \subseteq \mathrm{Spin}(V_\infty)$. We let $R_{\leq \ell} \subseteq R$ be the subalgebra generated by the spaces $(\bigwedge_\infty E_\infty)_k$ with $k \leq \ell$. Crucial in the proof of Theorem 2.3.1 is the following result.

Proposition 2.3.2. *For every choice of nonnegative integers ℓ and n, $\mathrm{Spec}(R_{\leq \ell})$ is topologically H_n-Noetherian, i.e., every descending chain*

$$\mathrm{Spec}(R_{\leq \ell}) \supseteq X_1 \supseteq X_2 \supseteq \dots$$

of H_n-stable closed and reduced subschemes stabilizes.

The key ingredient in the proof of Proposition 2.3.2 is the main result of [ES22]. In order to apply their result we need to do some preparatory work. We will start with the following lemma.

Lemma 2.3.3. *Every H_n-stable closed subscheme of $\mathrm{Spec}(R_{\leq \ell})$ is also stable under the group $\mathrm{GL}(E_\infty)_n$ acting in the natural manner on $\bigwedge_\infty E_\infty$ and its dual, and vice versa.*

Proof. Equation (2.1.3) implies that $\mathfrak{gl}(E_\infty) \subseteq \mathfrak{so}(V_\infty)$ acts on $\bigwedge_\infty E_\infty$ via

$$\rho(A) = \tilde{\rho}(A) - \frac{1}{2}\,\mathrm{tr}(A)\,\mathrm{id}_{\bigwedge_\infty E_\infty}$$

where $\tilde{\rho}$ is the standard representation of $\mathfrak{gl}(E_\infty)$ on $\bigwedge_\infty E_\infty$. An H_n-stable closed subscheme X of $\mathrm{Spec}(R_{\leq \ell})$ is given by an H_n-stable ideal I in the symmetric algebra $R_{\leq \ell}$. Such an I is then also stable under the action of the Lie algebra $\mathfrak{gl}(E_\infty)_n$ of H_n by derivations that act on variables in $\bigoplus_{k=0}^{\ell} (\bigwedge_\infty E_\infty)_k$ via ρ.

We claim that I is a homogeneous ideal. Indeed, for $f \in I$, choose $m > n$ such that all variables in f (which are basis elements e_I) contain the basis element e_m of E_∞. Let $A \in \mathfrak{gl}(E_\infty)_n$ be the diagonal matrix with 0's everywhere except a 1 on position (m, m). Then $\rho(A)$ maps each variable in f to $\frac{1}{2}$ times itself. Hence, by the Leibniz rule, $\rho(A)$ scales the homogeneous part of degree d in f by $\frac{d}{2}$. Since I is preserved by $\rho(A)$, it follows that I contains all homogeneous components of f, and hence I is a homogeneous ideal.

Now let $B \in \mathfrak{gl}(E_\infty)_n$ and $f \in I$ be arbitrary. By the previous paragraph we can assume f to be homogeneous of degree d, and we then have

$$\rho(B)f = \tilde{\rho}(B)f - \frac{d}{2}\,\mathrm{tr}(B)f,$$

and since I is $\rho(B)$-stable, we deduce $\tilde{\rho}(B)f \in I$. This completes the proof in one direction. The proof in the opposite direction is identical. □

Remark 2.3.4. Note that by the proof above, any $\mathrm{Spin}(V_\infty)$-stable closed subscheme X of $(\bigwedge_\infty E_\infty)^*$ is an affine cone.

Following [ES22] the *restricted dual* $(E_\infty)_*$ of E_∞ is defined as the union $\bigcup_{n \geq 1} (E_n)^*$. We will denote by $\varepsilon^1, \varepsilon^2, \dots$ the basis of $(E_\infty)_*$ that is dual to the canonical basis $e_1, e_2, \dots$ of E_∞ given by $\varepsilon^i(e_j) = \delta_{ij}$.

Lemma 2.3.5. *There is an $\mathrm{SL}(E_\infty)$-equivariant isomorphism*

$$\bigwedge\nolimits_\infty E_\infty \longrightarrow \bigwedge (E_\infty)_*,$$

which restricts to an isomorphism

$$\Big(\bigwedge\nolimits_\infty E_\infty\Big)_k \longrightarrow \bigwedge\nolimits^k (E_\infty)_*.$$

We will use this isomorphism to view $\bigwedge_\infty E_\infty$ as the restricted dual of the Grassmann algebra $\bigwedge E_\infty$. We stress, though, that this isomorphism is not $GL(E_\infty)$-equivariant.

Proof. We have a natural bilinear map

$$\bigwedge E_\infty \times \bigwedge\nolimits_\infty E_\infty \to \bigwedge\nolimits_\infty E_\infty, \quad (\omega, \omega') \mapsto \omega \wedge \omega'.$$

If $I \subseteq \mathbb{N}$ is finite and $J \subseteq \mathbb{N}$ is cofinite, then $e_I \wedge e_J$ is 0 if $I \cap J \neq \emptyset$ and $\pm e_{I\cup J}$ otherwise, where the sign is determined by the permutation required to order the sequence I, J. We then define a perfect pairing γ between the two spaces by

$$\gamma(\omega, \omega') := \text{the coefficient of } e_{\mathbb{N}} \text{ in } \omega \wedge \omega'.$$

The map $\Phi_\gamma : \bigwedge_\infty E_\infty \to \bigwedge(E_\infty)_*,\ \omega' \mapsto \gamma(\cdot, \omega')$ induced by γ is the isomorphism given by $e_I \mapsto \pm\varepsilon^{I^c}$, where $I^c \subseteq \mathbb{N}$ is the complement of I and $\varepsilon^J := \varepsilon^{j_1} \wedge \cdots \wedge \varepsilon^{j_k}$ for a finite set $J = \{j_1, \ldots, j_k\}$. Note that $\gamma(A \cdot \omega, A \cdot \omega') = \det(A)\gamma(\omega, \omega')$ for all $A \in \mathrm{GL}(E_\infty)$, and hence γ is $\mathrm{SL}(E_\infty)$-invariant. Therefore, the isomorphism Φ_γ is $\mathrm{SL}(E_\infty)$-equivariant. □

Lemma 2.3.6. *An ideal $I \subseteq \mathrm{Sym}(\bigwedge(E_\infty)_*)$ is $\mathrm{SL}(E_\infty)$-stable if and only if it is $\mathrm{GL}(E_\infty)$-stable. The same holds for $\mathrm{SL}(E_\infty)_n$ and $\mathrm{GL}(E_\infty)_n$.*

Proof. Assume that I is $\mathrm{SL}(E_\infty)$-stable. Let $f \in I$ and $A \in \mathrm{GL}(E_\infty)$ be arbitrary. Choose $m = m(f, A) \in \mathbb{N}$ large enough so that $f \in \mathrm{Sym}(\bigwedge(E_m)^*)$ and A is the image of some $A_m \in \mathrm{GL}(E_m)$. Define $A_{m+1} \in \mathrm{GL}(E_{m+1})$ as the map given by $A_{m+1}(e_i) = A_m(e_i)$ for $i \leq m$ and $A_{m+1}(e_{m+1}) = (\det(A_m))^{-1}(e_{m+1})$, and let A' be the image of A_{m+1} in $\mathrm{SL}(E_\infty)$. Then the action of A_m and A_{m+1} agree on $(E_m)^*$. Hence they also agree on $\mathrm{Sym}(\bigwedge(E_m)^*)$. So $A \cdot f = A' \cdot f \in I$ since I was assumed to be $\mathrm{SL}(E_\infty)$-stable and $A' \in \mathrm{SL}(E_\infty)$. As $f \in I$ and $A \in \mathrm{GL}(E_\infty)$ were arbitrary, this shows that I is $GL(E_\infty)$-stable. □

Proof of Proposition 2.3.2. First, we claim that $\mathrm{Spec}\left(\mathrm{Sym}\left(\bigoplus_{k=0}^{\ell} \bigwedge^k (E_\infty)_*\right)\right)$ is topologically $\mathrm{GL}(E_\infty)_n$-Noetherian. Indeed, the standard $\mathrm{GL}(E_\infty)$-representation of the space $\bigoplus_{k=0}^{\ell} \bigwedge^k (E_\infty)_*$ is an algebraic representation and this also remains true when we act with $\mathrm{GL}(E_\infty)$ via its isomorphism into $\mathrm{GL}(E_\infty)_n$. Hence, the claim follows from [ES22, Theorem 2]. Let $(X_i)_{i\in\mathbb{N}} \subseteq \mathrm{Spec}(R_{\leq\ell})$ be a descending chain of H_n-stable, closed, reduced subschemes. By Lemma 2.3.3 every X_i is also $\mathrm{GL}(E_\infty)_n$-stable. By Lemma 2.3.5 there is an $\mathrm{SL}(E_\infty)_n$-equivariant isomorphism $\mathrm{Spec}(R_{\leq\ell}) \cong \mathrm{Spec}\left(\mathrm{Sym}\left(\bigoplus_{k=0}^{\ell} \bigwedge^k (E_\infty)_*\right)\right)$. Let $X_i' \subseteq \mathrm{Spec}\left(\mathrm{Sym}\left(\bigoplus_{k=0}^{\ell} \bigwedge^k (E_\infty)_*\right)\right)$ be the closed, reduced, $\mathrm{SL}(E_\infty)$-stable subscheme corresponding to X_i under this isomorphism. Using Lemma 2.3.6 we see that the subschemes X_i' are also $\mathrm{GL}(E_\infty)_n$-stable. Therefore, the chain $(X_i')_{i\in\mathbb{N}}$ stabilizes by our first claim. Consequently, also the chain $(X_i)_{i\in\mathbb{N}}$ stabilizes. □

Before we come to the proof of Theorem 2.3.1, we first want to recall the action of $f_i \wedge f_j \in \mathfrak{so}(V_\infty)$ on $\bigwedge_\infty^+ E_\infty$ and its symmetric algebra R^+ in explicit terms. Recall from Section 2.2.6 that a basis for $\bigwedge_\infty^+ E_\infty$ is given by $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots$, where $I = \{i_1 < i_2 < \cdots\} \subseteq \mathbb{N}$ is cofinite and $|\mathbb{N} \setminus I|$ even. Then we have

$$(f_i \wedge f_j)e_I = \begin{cases} (-1)^{c_{i,j}(I)} e_{I\setminus\{i,j\}} & \text{if } i, j \in I, \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

where $c_{i,j}(I)$ depends on the position of i, j in I. (Note that there is no factor 4, since in our identification of $\bigwedge^2 V$ to the Lie subalgebra L of $\mathrm{Cl}(V)$ we had a factor $\frac{1}{4}$.) The corresponding action of $f_i \wedge f_j$ on polynomials in R^+ is as a derivation.

2.3.3 Proof of Theorem 2.3.1

Let $R^+ \subseteq R$ be the symmetric algebra on the direct half-spin representation $\bigwedge_\infty^+ E_\infty$, so that $\mathrm{Spec}(R^+)$ is the inverse half-spin representation $(\bigwedge_\infty^+ E_\infty)^*$. We will prove topological $\mathrm{Spin}(V_\infty)$-Noetherianity of $\mathrm{Spec}(R^+)$; the corresponding statement for $\mathrm{Spec}(R^-)$ is proved in exactly the same manner.

For a closed, reduced $\mathrm{Spin}(V_\infty)$-stable subscheme X of $\mathrm{Spec}(R^+)$, we denote by $\delta_X \in \{0, 1, 2, \ldots, \infty\}$ the lowest degree of a nonzero polynomial in the ideal $I(X) \subseteq R^+$ of X. Here we consider the natural grading on $R^+ = \mathrm{Sym}(\bigwedge_\infty^+ E_\infty)$, where the elements of $\bigwedge_\infty^+ E_\infty$ all have degree 1.

We proceed by induction on δ_X to show that X is topologically Noetherian; we may therefore assume that this is true for all Y with $\delta_Y < \delta_X$. We have $\delta_X = \infty$ if and only if $X = \mathrm{Spec}(R^+)$. Then a chain

$$\mathrm{Spec}(R^+) = X \supseteq X_1 \supseteq X_2 \supseteq \ldots$$

of $\mathrm{Spin}(V_\infty)$-closed subsets is either constant or else there exists an i with $\delta_{X_i} < \infty$. Hence it suffices to prove that X is Noetherian under the additional assumption that $\delta_X < \infty$. At the other extreme, if $\delta_X = 0$, then X is empty and there is nothing to prove. So we assume that $0 < \delta_X < \infty$ and that all Y with $\delta_Y < \delta_X$ are $\mathrm{Spin}(V_\infty)$-Noetherian.

Let $p \in R^+$ be a nonzero polynomial in the ideal of X of degree δ_X. By Remark 2.3.4, since X is a cone, p is in fact homogeneous of degree δ_X. Let e_I be a variable appearing in p such that $k := |I^c|$ is maximal among all variables in p; note that k is even. Then choose $n \geq k+2$ even such that all variables of p are contained in $\bigwedge^+ E_n$, i.e., they are of the form e_J with $J \supseteq \{n+1, n+2, \ldots\}$.

Now act on p with the element $f_{i_1} \wedge f_{i_2} \in \mathfrak{so}(V_\infty)$ with $i_1 < i_2$ the two smallest elements in I. Since X is $\mathrm{Spin}(V_\infty)$-stable, the result p_1 is again in the ideal of X. Furthermore, p_1 has the form

$$p_1 = \pm e_{I \setminus \{i_1, i_2\}} \cdot q + r_1$$

where $q = \frac{\partial p}{\partial e_I}$ contains only variables e_J with $|J^c| \leq k$ and where r_1 does not contain $e_{I \setminus \{i_1, i_2\}}$ but may contain other variables e_J with $|J^c| = k+2$ (namely, those with $i_1, i_2 \notin J$ for which $e_{J \cup \{i_1, i_2\}}$ appears in p).

If $n = k+2$, then $I \setminus \{i_1, i_2\} = \{n+1, n+2, \ldots\}$ and, since all variables e_J in p_1 satisfy $J \supseteq \{n+1, n+2, \ldots\}$, $e_{I \setminus \{i_1, i_2\}}$ is the only variable e_J in p_1 with $|J^c| = k+2$. If $n > k+2$, then we continue in the same manner, now acting with $f_{i_3} \wedge f_{i_4}$ on p_1, where $i_3 < i_4$ are the two smallest elements in $I \setminus \{i_1, i_2\}$. We write p_2 for the result, which is now of the form

$$p_2 = \pm e_{I \setminus \{i_1, i_2, i_3, i_4\}} \cdot q + r_2$$

where q is the same polynomial as before and r_2 does not contain the variable $e_{I \setminus \{i_1, i_2, i_3, i_4\}}$ but may contain other variables e_J with $|J^c| = k+4$.

Iterating this construction we find the polynomial

$$p_\ell = \pm e_{\{n+1,n+2,\ldots\}} \cdot q + r_\ell$$

in the ideal of X, where $\ell = (n-k)/2$, q is the same polynomial as before and r_ℓ only contains variables e_J with $|J^c| < n$. Let $Z := X[1/q]$ be the open subset of X where q is nonzero.

Lemma 2.3.7. *For every variable e_J with $|J^c| \geq n$, the ideal of Z in the localisation $R^+[1/q]$ contains a polynomial of the form $e_J - s/q^d$ for some $d \in \mathbb{Z}_{\geq 0}$ and some $s \in R^+_{\leq n-2}$.*

Proof. We proceed by induction on $|J^c| =: m$. By successively acting on p_ℓ with the elements $f_n \wedge f_{n+1}, f_{n+2} \wedge f_{n+3}, \ldots, f_{m-1} \wedge f_m$, we find the polynomial

$$\pm e_{\{m+1,m+2,\ldots\}} \cdot q + r$$

in the ideal of X, where r contains only variables e_L with $|L^c| < m$. Now act with elements of $\mathfrak{gl}(E_\infty)$ to obtain an element

$$\pm e_J \cdot q + \tilde{r}$$

where $\tilde{r}$ still contains only variables e_L with $|L^c| < m$. Inverting q, this can be used to express e_J in such variables e_L. By the induction hypothesis, all those e_L admit an expression, on Z, as a polynomial in $R^+_{\leq n-2}$ times a negative power of q. Then the same holds for e_J. □

Lemma 2.3.8. *The open subscheme $Z = X[1/q]$ is stable under the group H_n and H_n-Noetherian.*

Proof. By Lemma 2.3.3, X is stable under $\mathrm{GL}(E_\infty)_n$. The polynomial q is homogeneous and contains only variables e_J with $J \supseteq \{n+1, n+2, \ldots\}$. Every $g \in \mathrm{GL}(E_\infty)_n$ scales each such variable with $\det(g)$, and hence maps q to a scalar multiple of itself. We conclude that Z is stable under $\mathrm{GL}(E_\infty)_n$, hence by (a slight variant of) Lemma 2.3.3 also under H_n.

By Lemma 2.3.7, the projection dual to the inclusion $R^+_{\leq n-2}[1/q] \subseteq R^+[1/q]$ restricts on Z to a closed embedding, and this embedding is H_n-equivariant. By Proposition 2.3.2, the image of Z is H_n-Noetherian, hence so is Z itself. □

Proof of Theorem 2.3.1. Let

$$X \supseteq X_1 \supseteq \ldots$$

be a chain of reduced, $\mathrm{Spin}(V_\infty)$-stable closed subschemes. Let $Y \subseteq X$ be the reduced closed subscheme defined by the orbit $\mathrm{Spin}(V_\infty) \cdot q$. Since q has degree $\delta_X - 1$, we have $\delta_Y < \delta_X$ and hence Y is $\mathrm{Spin}(V_\infty)$-Noetherian by the induction hypothesis. It follows that the chain

$$Y \supseteq (Y \cap X_1)^{\mathrm{red}} \supseteq \ldots$$

is eventually stable. On the other hand, the chain

$$Z \supseteq (Z \cap X_1)^{\mathrm{red}} \supseteq \ldots$$

consists of reduced, H_n-stable closed subschemes of Z, hence it is eventually stable by Lemma 2.3.8.

Now pick a (not necessarily closed) point $P \in X_i$ for $i \gg 0$. If $P \in Y \cap X_i$, then $P \in Y \cap X_{i-1}$ by the first stabilisation. On the other hand, if $P \notin Y \cap X_i$, then there exists a $g \in \mathrm{Spin}(V_\infty)$ such that $gP \in Z$. Then gP lies in $X_i \cap Z$, which by the second stabilisation equals $X_{i-1} \cap Z$, hence $P = g^{-1}(gP)$ lies in X_{i-1}, as well. We conclude that the chain $(X_i)_i$ of closed, reduced subschemes of X stabilises. Hence the inverse half-spin representation $(\bigwedge_\infty^+ E_\infty)^*$ is topologically $\mathrm{Spin}(V_\infty)$-Noetherian. □

Remark 2.3.9. While the proof of Theorem 2.3.1 for the even half-spin case can be easily adapted to a proof for the odd half-spin case, we *do not know whether the spin representation* $(\bigwedge_\infty E_\infty)^*$ *itself is topologically* $\mathrm{Spin}(V_\infty)$*-Noetherian!* Also, despite much effort, we have not succeeded in proving that the inverse limit $\varprojlim_n \bigwedge^n V_n$ along the contraction maps c_{e_n} is topologically $\mathrm{SO}(V_\infty)$-Noetherian. Indeed, the situation is worse for this question: like the inverse spin representation, this limit is the dual of a countable-dimensional module that splits as a direct sum of two $\mathrm{SO}(V_\infty)$-modules, and here we do not even know whether the dual of *one* of these modules is topologically Noetherian!

2.4 Half-Spin Varieties and Applications

In this section we introduce the notion of half-spin varieties and reformulate our main result Theorem 2.3.1 in this language. We start by fixing the necessary data determining the half-spin representations of $\mathrm{Spin}(V)$.

Notation 2.4.1. As shorthand, we write $\mathcal{V} = (V, q, F) \in \mathcal{Q}$ to refer to a triple where

1. V is an even-dimensional vector space over $\mathbb{K}$,
2. q is a non-degenerate symmetric quadratic form on V, and
3. F is a maximal isotropic subspace of V.

An *isomorphism* $\mathcal{V} \to \mathcal{V}' = (V', q', F')$ of such triples is a linear bijection $\phi : V \to V'$ with $q'(\phi(v)) = q(v)$ and $\phi(F) = F'$.

Given a triple $\mathcal{V}$, we have half-spin representations $\mathrm{Cl}^{\pm}(V)f$, where $f = f_1 \cdots f_n$ with $f_1, \ldots, f_n$ a basis of F (recall that the left ideal $\mathrm{Cl}(V)f$ does not depend on this basis). Half-spin varieties are $\mathrm{Spin}(V)$-invariant subvarieties of these half-spin representations that are preserved by the contraction maps π_e from Section 2.2.2 and the multiplication maps τ_h from Section 2.2.3. The precise definition below is inspired by the that of a Plücker variety in [DE18]. It involves a uniform choice of either even or odd half-spin representations. For convenience of notation, we will only explicitly work with the even half-spin representations, but all further results are valid for the odd counterparts as well.

Definition 2.4.2 (Half-spin variety)**.** A *half-spin variety* is a rule X that assigns to each triple $\mathcal{V} = (V, q, F) \in \mathcal{Q}$ a closed, reduced subscheme $X(\mathcal{V}) \subseteq \mathrm{Cl}^+(V)f$ such that

1. $X(\mathcal{V})$ is $\mathrm{Spin}(V)$-stable;

2. for any isomorphism $\phi : \mathcal{V} \to \mathcal{V}'$, the map $\mathrm{Cl}^+(\phi)$ maps $X(\mathcal{V})$ into $X(\mathcal{V}')$;

3. for any isotropic $e \in V$ with $e \notin F$, if we set $V' := e^\perp/\langle e \rangle$, q' the induced form on V', F' the image of $F \cap e^\perp$ in V', and $\mathcal{V}' := (V', q', F')$, then the contraction map $\pi_e : \mathrm{Cl}^+(V)f \to \mathrm{Cl}^+(V')f'$ maps $X(\mathcal{V})$ into $X(\mathcal{V}')$; and

4. for any $\mathcal{V} = (V, q, F)$, if we denote by $\mathcal{V}'$ the triple $V' := V \oplus \langle e, h \rangle$, q' as the quadratic form that restricts to q on V, that makes the direct sum orthogonal, and e, h a hyperbolic basis, if we set $f' := f \cdot h$, then the map $\tau_h : \mathrm{Cl}^+(V)f \to \mathrm{Cl}^+(V')f'$ maps $X(\mathcal{V})$ into $X(\mathcal{V}')$.

Examples 2.4.3. The following are examples of half-spin varieties.

1. Trivially, $X(\mathcal{V}) := \mathrm{Cl}^+(V)f$, $X(\mathcal{V}) := \{0\}$ and $X(\mathcal{V}) := \emptyset$ define half-spin varieties.

2. By Proposition 2.2.4, the even component of the cone over the isotropic Grassmannian, $X(\mathcal{V}) := \widehat{\mathrm{Gr}}^+_{\mathrm{iso}}(V, q)$, is a half-spin variety.

3. For two half-spin varieties X and X' their *join* $X + X'$ defined by

$$(X + X')(\mathcal{V}) := \overline{\{x + x' \mid x \in X(\mathcal{V}), x' \in X'(\mathcal{V})\}}$$

is a half-spin variety.

4. The intersection of two half-spin varieties X and X' is a half-spin variety, which is defined by $(X \cap X')(\mathcal{V}) := X(\mathcal{V}) \cap X'(\mathcal{V})$.

Similar as in Section 2.2.6 we will use the following notation: for every $n \in \mathbb{N}$, we consider the vector space $V_n = \langle e_1, \ldots, e_n, f_1, \ldots, f_n \rangle$ together with the quadratic form q_n whose corresponding bilinear form $(\cdot|\cdot)$ satisfies

$$(e_i|e_j) = 0, \quad (f_i|f_j) = 0 \quad \text{and} \quad (e_i|f_j) = \delta_{ij}.$$

Furthermore, let $E_n = \langle e_1, \ldots, e_n \rangle$ and $F_n = \langle f_1, \ldots, f_n \rangle$; these are maximal isotropic subspaces of V_n. We will denote the associated tuple by $\mathcal{V}_n = (V_n, q_n, F_n)$.

Remark 2.4.4. A half-spin variety X is completely determined by the values $X(\mathcal{V}_n)$, that is, if X and X' are half-spin varieties such that $X(\mathcal{V}_n) = X'(\mathcal{V}_n)$ for all $n \in \mathbb{N}$, then $X(\mathcal{V}) = X'(\mathcal{V})$ for all tuples $\mathcal{V}$.

We now want to associate to each half-spin variety X an infinite-dimensional scheme X_∞ embedded inside the inverse half-spin representation $(\bigwedge^+_\infty E_\infty)^*$ as follows. Since $V_n = E_n \oplus F_n$, we can use the isomorphism from Section 2.1.5 to embed $X(\mathcal{V}_n)$ as a reduced subscheme of $\bigwedge^+ E_n$ (recall from Section 2.2.7 that we view $\bigwedge^+ E_n$ as the affine scheme with coordinate ring $\mathrm{Sym}(\bigwedge^{+(-1)^n} E_n)$). We abbreviate $X_n := X(\mathcal{V}_n) \subseteq \bigwedge^+ E_n$.

For $N \geq n$ let $\pi_{N,n} : \bigwedge^+ E_N \to \bigwedge^+ E_n$, resp. $\tau_{n,N} : \bigwedge^+ E_n \to \bigwedge^+ E_N$ be the maps induced by the canonical projection $E_N \to E_n$, resp. by the injection $E_n \hookrightarrow E_N$. Note that $\tau_{n,N}$ is a section of $\pi_{N,n}$. Recall that $(\bigwedge^+_\infty E_\infty)^* = \varprojlim_n \bigwedge^+ E_n$. We denote the structure maps by $\pi_{\infty,n} : (\bigwedge^+_\infty E_\infty)^* \to \bigwedge^+ E_n$ and by $\tau_{n,\infty} : \bigwedge^+ E_n \to (\bigwedge^+_\infty E_\infty)^*$ the inclusion maps induced by $\tau_{n,N}$.

From the definition of a half-spin variety it follows that

$$\pi_{N,n}(X_N) \subseteq X_n \quad \text{and} \quad \tau_{n,N}(X_n) \subseteq X_N. \tag{2.4.1}$$

Hence the inverse limit

$$X_\infty := \varprojlim_n X_n$$

is well-defined, and a closed, reduced, $\mathrm{Spin}(V_\infty)$-stable subscheme of $(\bigwedge_\infty^+ E_\infty)^*$. In order to see this, write $R_n := \mathrm{Sym}(\bigwedge^{+(-1)^n} E_n)$ and $R_\infty := \mathrm{Sym}(\bigwedge_\infty^+ E_\infty)$. Let $I_n \subseteq R_n$ be the radical ideal associated to $X_n \subseteq \mathrm{Spec}(R_n)$, i.e. $X_n = V(I_n) = \mathrm{Spec}(R_n/I_n)$. As $\mathrm{Spec}(\cdot)$ is a contravariant equivalence of categories, it holds that

$$X_\infty := \varprojlim_n X_n = \varprojlim_n \mathrm{Spec}(R_n/I_n) = \mathrm{Spec}\left(\varinjlim_n (R_n/I_n)\right).$$

So X_∞ corresponds to the ideal $I_\infty := \varinjlim_n I_n \subseteq R_\infty$. As all $I_n \subseteq R_n$ are radical, so is $I_\infty \subseteq R_\infty$ and therefore X_∞ is a reduced subscheme.

It follows from Equation (2.4.1) that

$$\pi_{\infty,n}(X_\infty) \subseteq X_n \quad \text{and} \quad \tau_{n,\infty}(X_n) \subseteq X_\infty. \tag{2.4.2}$$

Lemma 2.4.5. *The mapping*

$$X \mapsto X_\infty$$

is injective. That is, if X and X' are half-spin varieties such that $X_\infty = X'_\infty$, then $X = X'$, i.e. $X(\mathcal{V}) = X'(\mathcal{V})$ for all tuples $\mathcal{V}$.

Proof. Note that, for all $n \in \mathbb{N}$, it holds that

$$X_n = \pi_{\infty,n}(X_\infty).$$

Indeed, the inclusion $\supseteq$ is contained in Equation (2.4.2), and the other direction $\subseteq$ follows from the fact that $\tau_{n,\infty} : X_n \to X_\infty$ is a section of $\pi_{\infty,n}$. Hence, if $X_\infty = X'_\infty$, then

$$X_n = \pi_{\infty,n}(X_\infty) = \pi_{\infty,n}(X'_\infty) = X'_n.$$

By Remark 2.4.4 this shows that $X = X'$. □

For two half-spin varieties X and X', we will write $X \subseteq X'$ if $X(\mathcal{V}) \subseteq X'(\mathcal{V})$ for all $\mathcal{V} = (V, q, F)$. Theorem 2.3.1 then implies the following.

Theorem 2.4.6 (Noetherianity of half-spin varieties)**.** *Every descending chain of half-spin varieties*

$$X^{(0)} \supseteq X^{(1)} \supseteq X^{(2)} \supseteq X^{(3)} \supseteq \dots$$

stabilizes, that is, there exists $m_0 \in \mathbb{N}$ such that $X^{(m)} = X^{(m_0)}$ for all $m \geq m_0$.

Proof. Note that the mapping $X \mapsto X_\infty$ is order preserving, that is, if $X \subseteq X'$, then $X_\infty \subseteq X'_\infty$. Hence, a chain

$$X^{(0)} \supseteq X^{(1)} \supseteq X^{(2)} \supseteq X^{(3)} \supseteq \dots$$

of half-spin varieties induces a chain

$$X_\infty^{(0)} \supseteq X_\infty^{(1)} \supseteq X_\infty^{(2)} \supseteq X_\infty^{(3)} \supseteq \dots$$

of closed, reduced, $\mathrm{Spin}(V_\infty)$-stable subschemes in $(\bigwedge_\infty^+ E_\infty)^*$. By Theorem 2.3.1 we know that $(\bigwedge_\infty^+ E_\infty)^*$ is topologically $\mathrm{Spin}(V_\infty)$-Noetherian. Hence, the chain $X_\infty^{(m)}$ stabilizes. But then, by Lemma 2.4.5 also the chain of half-spin varieties $X^{(m)}$ stabilizes. This completes the proof. □

As a consequence we obtain the next results, which state how X_∞ is determined by the data coming from some finite level of X.

Theorem 2.4.7. *Let X be a half-spin variety. Then there exists $n_0 \in \mathbb{N}$ such that*

$$X_\infty = V\big(\operatorname{rad}(\mathrm{Spin}(V_\infty) \cdot I_{n_0})\big),$$

where $\operatorname{rad}(\mathrm{Spin}(V_\infty) \cdot I_{n_0}) \subseteq \mathrm{Sym}(\bigwedge_\infty^+ E_\infty)$ is the radical ideal that is generated by the $\mathrm{Spin}(V_\infty)$-orbits of the ideal $I_{n_0} \subseteq \mathrm{Sym}(\bigwedge^{+(-1)^{n_0}} E_{n_0})$ defining $X_{n_0} \subseteq \bigwedge^+ E_{n_0}$.

Proof. For each $n \in \mathbb{N}$ set $J_n := \operatorname{rad}(\mathrm{Spin}(V_\infty) \cdot I_n) \subseteq \mathrm{Sym}(\bigwedge_\infty^+ E_\infty)$. We denote by $I_\infty \subseteq \mathrm{Sym}(\bigwedge_\infty^+ E_\infty)$ the ideal associated to X_∞. This ideal is $\mathrm{Spin}(V_\infty)$-stable, radical and we have $I_\infty = \varinjlim_n I_n$. Therefore, $\bigcup_n J_n = I_\infty$.

Since $(J_n)_{n\in\mathbb{N}}$ is an increasing chain of closed, $\mathrm{Spin}(V_\infty)$-stable, radical ideals, there exists $n_0 \in \mathbb{N}$ such that $J_n = J_{n_0}$ for all $n \geq n_0$ by Theorem 2.3.1. Consequently, $I_\infty = \bigcup_n J_n = J_{n_0}$ and hence $X_\infty = V(I_\infty) = V(J_{n_0})$. □

Corollary 2.4.8 (Universality for half-spin varieties)**.** *Let X be a half-spin variety. There exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ it holds that*

$$X_n = V(\operatorname{rad}(\mathrm{Spin}(V_n) \cdot I_{n_0})).$$

Proof. Take n_0 as in Theorem 2.4.7. Then the statement follows from that theorem and [Dra10, Lemma 2.1]. To apply that lemma, we must check condition (*) in that paper, namely, that for $q \geq n \geq n_0$ and $g \in \mathrm{Spin}(V_q)$ we can write

$$\pi_{q,n_0} \circ g \circ \tau_{n,q} = g'' \circ \tau_{m,n_0} \circ \pi_{n,m} \circ g'$$

for suitable $m \leq n_0$ and $g' \in \mathrm{Spin}(V_n)$ and $g'' \in \mathrm{Spin}(V_{n_0})$. In fact, since half-spin varieties are affine cones, it suffices that this identity holds up to a scalar factor. It also suffices to prove this for g in an open dense subset U of $\mathrm{Spin}(V_q)$, because the equations for X_{n_0} pulled back along the map on the left for $g \in U$ imply the equations for all g. We will prove this, with $m = n_0$, using the Cartan map in Lemma 2.5.6 below. □

2.5 Universality of $\widehat{\mathrm{Gr}}_{\mathrm{iso}}^+(4,8)$ and the Cartan Map

2.5.1 Statement

In Chapter 1 we saw that in even dimension, the isotropic Grassmannian in its Plücker embedding is set-theoretically defined by pulling back equations coming from $\widehat{\mathrm{Gr}}_{\mathrm{iso}}(4,8)$. Using the *Cartan map* we can translate this into a statement about the isotropic Grassmannian in its spinor embedding and prove the following result.

Theorem 2.5.1. *For all $n \geq 4$ we have*

$$\widehat{\mathrm{Gr}}^{+}_{\mathrm{iso}}(V_n) = V(\mathrm{rad}(\mathrm{Spin}(V_n) \cdot I_4)),$$

where I_4 is the ideal of polynomials defining $\widehat{\mathrm{Gr}}^{+}_{\mathrm{iso}}(V_4) \subseteq \mathrm{Cl}^{+}(V_4)f$.

In other words, the bound n_0 from Corollary 2.4.8 can be taken equal to 4 for the cone over the isotropic Grassmannian. We give the proof of Theorem 2.5.1 in Section 2.5.5 using properties of the Cartan map that will be established in the following sections.

2.5.2 Definition of the Cartan Map

When we consider $e_1 \wedge \cdots \wedge e_n$ as an element of the n-th exterior power $\bigwedge^n V$ of the standard representation V of $\mathfrak{so}(V)$, then it is a highest weight vector of weight $(0, \ldots, 0, 2) = 2\lambda_0$. Here, λ_0 is the fundamental weight introduced in Section 2.1.6 which is the highest weight of the half-spin representation $\mathrm{Cl}^{(-1)^n}(V)f$. Similarly, the element $e_1 \wedge e_2 \wedge \cdots \wedge e_{n-1} \wedge f_n \in \bigwedge^n V$ is a highest weight vector of weight $(0, \ldots, 0, 2, 0) = 2\lambda_1$, where λ_1 is the highest weight of the other half-spin representation. So $\bigwedge^n V$ contains copies of the irreducible representations $V_{2\lambda_0}, V_{2\lambda_1}$ of $\mathfrak{so}(V)$; in fact, it is well known to be the direct sum of these. To compare our results in this chapter about spin representations with the results from Chapter 1 about exterior powers, we will need the following considerations.

Consider any connected, reductive algebraic group G, with maximal torus T and Borel subgroup $B \supseteq T$. Let λ be a dominant weight of G, let V_λ be the corresponding irreducible representation, and let $v_\lambda \in V_\lambda$ be a nonzero highest-weight vector (which is unique up to scalar multiples). Then the symmetric square $S^2 V_\lambda$ contains a one-dimensional space of vectors of weight 2λ, spanned by $v_{2\lambda} := v_\lambda^2$. This vector is itself a highest-weight vector, and hence generates a copy of $V_{2\lambda}$; this is sometimes called the *Cartan component* of $S^2 V_\lambda$. By semisimplicity, there is a G-equivariant linear projection $\pi : S^2 V_\lambda \to V_{2\lambda}$ that restricts to the identity on $V_{2\lambda}$. The map

$$\widehat{\nu_2} : V_\lambda \to V_{2\lambda}, \quad v \mapsto \pi(v^2).$$

is a nonzero polynomial map, homogeneous of degree 2, and hence induces a rational map $\nu_2 : \mathbb{P}V_\lambda \to \mathbb{P}V_{2\lambda}$. Note that this is the composition of the quadratic Veronese embedding and the projection π. We will refer to ν_2 and to $\widehat{\nu_2}$ as the *Cartan map.*

Lemma 2.5.2. *The rational map ν_2 is a morphism and injective.*

We thank J.M. Landsberg for help with the following proof.

Proof. To show that ν_2 is a morphism, we need to show that $\pi(v^2)$ is nonzero whenever v is. Now the set Q of all $[v] \in \mathbb{P}V_\lambda$ for which $\pi(v^2)$ is zero is closed and B-stable. Hence, if $Q \neq \emptyset$, then by Borel's fixed point theorem, Q contains a B-fixed point. But the only B-fixed point in $\mathbb{P}V_\lambda$ is $[v_\lambda]$, and v_λ is mapped to the nonzero vector $v_{2\lambda}$. Hence $Q = \emptyset$.

Injectivity is similar but slightly more subtle. Assume that there exist distinct $[v], [w]$ with $\nu_2([v]) = \nu_2([w])$. Then $\{[v], [w]\}$ represents a point in the Hilbert scheme of two points in $\mathbb{P}V_\lambda$. Now the locus Q of points S in said Hilbert scheme such that $\nu_2(S)$ is a single reduced point is a closed subset of a projective scheme, hence Q contains a B-stable point S. This scheme S cannot consist of two distinct reduced points, since there is only one B-stable point. Therefore, the reduced subscheme of S is $\{[v_\lambda]\}$, and S represents the point $[v_\lambda]$ together with a nonzero tangent direction in $T_{[v_\lambda]}\mathbb{P}V_\lambda = V_\lambda/\mathbb{K}v_\lambda$, represented by $w \in V_\lambda$. Furthermore, B-stability of S implies that the B-module generated by w equals $\langle w, v_\lambda\rangle_{\mathbb{K}}$. That S lies in Q means that

$$\pi\left((v_\lambda + \epsilon w)^2\right) = v_{2\lambda} \mod \epsilon^2.$$

We find that $\pi(v_\lambda w) = 0$, so that the G-module generated by $v_\lambda w \in S^2V$ does not contain $V_{2\lambda}$. But since v_λ is (up to a scalar) fixed by B, the B-module generated by $v_\lambda w$ equals v_λ times the B-module gene rated by w, and hence contains $v_\lambda^2 = v_{2\lambda}$, a contradiction. □

Observe that ν_2 maps the unique closed orbit $G\cdot[v_\lambda]$ in $\mathbb{P}V_\lambda$ isomorphically to the unique closed orbit $G\cdot[v_{2\lambda}]$, both are isomorphic to G/P, where $P \supseteq B$ is the stabiliser of the line $\mathbb{K}v_\lambda$ and of the line $\mathbb{K}v_{2\lambda}$. In our setting above, where $G = \mathrm{Spin}(V)$ and $\lambda \in \{\lambda_0, \lambda_1\}$, the closed orbit $G\cdot[v_{2\lambda}]$ is one of the two connected components of the Grassmannian $\mathrm{Gr}_{\mathrm{iso}}(V)$ of n-dimensional isotropic subspaces of V, in its Plücker embedding; and the closed orbit in the projectivised half-spin representation $\mathbb{P}V_\lambda$ is the same component of the isotropic Grassmannian but now in its spinor embedding.

In what follows we will need a more explicit understanding both of the embedding of the isotropic Grassmannian in the projectivised (half-)spin representations and of the map $\widehat{\nu_2}$. These are treated in the next two paragraphs.

2.5.3 The Map $\widehat{\nu_2}$ from the Spin Representation to the Exterior Power

In Section 2.5.2 we argued the existence of $\mathrm{Spin}(V)$-equivariant quadratic maps from the half-spin representations to the two summands of $\bigwedge^n V$. In [Man09] these two maps are described jointly as

$$\widehat{\nu_2} : \mathrm{Cl}(V)f \to \bigwedge^n V, \quad af \mapsto \text{ the component in } \bigwedge^n V \text{ of } (afa^*) \bullet 1 \in \bigwedge V,$$

where $\bullet$ stands for the $\mathrm{Cl}(V)$-module structure of $\bigwedge V$ from Section 2.1.2 and a^* refers to the anti-automorphism of the Clifford algebra from Section 2.1.1.

Lemma 2.5.3. *The map $\widehat{\nu_2}$ maps the isotropic Grassmann cone in its spinor embedding to the isotropic Grassmann cone in its Plücker embedding, i.e.,*

$$\widehat{\nu_2}\big(\widehat{\mathrm{Gr}}_{\mathrm{iso}}(V)\big) = \widehat{\mathrm{Gr}}^{\mathrm{Pl}}_{\mathrm{iso}}(V),$$

where $\widehat{\mathrm{Gr}}^{\mathrm{Pl}}_{\mathrm{iso}}(V)$ is the isotropic Grassmann cone in its Plücker embedding (see Definition 1.2.7 in Chapter 1).

Proof. Let $H \subseteq V$ be a maximal isotropic subspace that intersects F in a k-dimensional space. Choose a hyperbolic basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ adapted to H and F, so that $H = \langle e_{k+1}, \ldots, e_n, f_1, \ldots, f_k\rangle$ is represented by the vector $\omega_H := e_{k+1}\cdots e_n f \in \widehat{\mathrm{Gr}}_{\mathrm{iso}}(V)$

where $f = f_1 \cdots f_n$; see Section 2.2.1. Set $a := e_{k+1} \cdots e_n$. Now

$$\begin{aligned} afa^* &= e_{k+1} \cdots e_n f_1 \cdots f_n e_n \cdots e_{k+1} \\ &= e_{k+1} \cdots e_n f_1 \cdots f_{n-1}(2 - e_n f_n) e_{n-1} \cdots e_{k+1} \\ &= 2e_{k+1} \cdots e_n f_1 \cdots f_{n-1} e_{n-1} \cdots e_{k+1} \\ &= \ldots \\ &= 2^{n-k} e_{k+1} \cdots e_n f_1 \cdots f_k \end{aligned}$$

where we have used the definition of $\mathrm{Cl}(V)$ (in the first step), the fact that the second copy of e_n is perpendicular to all elements before it and multiplies to zero with the first copy of e_n (in the second step), and have repeated this another $n - k - 1$ times in the last step. We now find that

$$(afa^*) \bullet 1 = 2^{n-k} e_{k+1} \wedge \cdots \wedge e_n \wedge f_1 \wedge \cdots \wedge f_k,$$

so that $(afa^*) \bullet 1$ lies in one of the two summands of $\bigwedge^n V$ and spans the line representing the space H in the Plücker embedding. This shows that $\widehat{\nu_2}$ maps the isotropic Grassmann cone in its spinor embedding to the isotropic Grassmann cone in its Plücker embedding, as desired. □

Remark 2.5.4. While the spin representation $\mathrm{Cl}(V)f$ depends only on the space F, since F determines f up to a scalar, which doesn't affect the left ideal $\mathrm{Cl}(V)f$, the map $\widehat{\nu_2}$ actually depends on f itself: for $\tilde{f} := tf$ with $t \in \mathbb{K}^*$, the map $\widehat{\nu_2}$ constructed from $\tilde{f}$ sends $af = (t^{-1}a)\tilde{f}$ to $t^{-1}a\tilde{f}t^{-1}a^* = t^{-1}afa^*$, so the new $\widehat{\nu_2}$ is t^{-1} times the old map.

2.5.4 Contraction and the Cartan Map Commute

Recall from Section 2.5.2 that we have quadratic maps $\widehat{\nu_2}$ from the half-spin representations to the two summands of $\bigwedge^n V$; together, these form a quadratic map $\widehat{\nu_2}$ which we discussed in Section 2.5.3. By abuse of terminology, we call this, too, the Cartan map. Given an isotropic vector $e \in V$ that is not in F, we write $\widehat{\nu_2}$ also for the Cartan map $\mathrm{Cl}(V_e)\overline{f} \to \bigwedge^{n-1} V_e$ (notation as in Section 2.2.2). Recall from Section 2.2.2 the contraction map $c_e : \bigwedge^n V \to \bigwedge^{n-1} V_e$ and its spin analogue $\pi_e : \mathrm{Cl}(V)f \to \mathrm{Cl}(V_e)\overline{f}$. Also, for a fixed $h = f_n \in F$ with $\langle e, h \rangle = 1$, recall from Section 2.2.3 the multiplication map $m_h : \bigwedge^{n-1} V_e \to \bigwedge^n V$ and its spin analogue $\tau_h : \mathrm{Cl}(V_e)\overline{f} \to \mathrm{Cl}(V)f$. The relations between these maps are as follows.

Proposition 2.5.5. *The following diagrams essentially commute:*

$$\begin{array}{ccc} \mathrm{Cl}(V)f & \xrightarrow{\pi_e} & \mathrm{Cl}(V_e)\overline{f} \\ {\scriptstyle\widehat{\nu_2}}\downarrow & & \downarrow{\scriptstyle\widehat{\nu_2}} \\ \bigwedge^n V & \xrightarrow[c_e]{} & \bigwedge^{n-1} V_e \end{array} \quad \textit{and} \quad \begin{array}{ccc} \mathrm{Cl}(V_e)\overline{f} & \xrightarrow{\tau_h} & \mathrm{Cl}(V)f \\ {\scriptstyle\widehat{\nu_2}}\downarrow & & \downarrow{\scriptstyle\widehat{\nu_2}} \\ \bigwedge^{n-1} V_e & \xrightarrow[m_h]{} & \bigwedge^n V. \end{array} \qquad (2.5.1)$$

More precisely, one can rescale the restrictions of c_e to the two $\mathfrak{so}(V)$-submodules of $\bigwedge^n V$ each by ± 1 in such a manner that the diagram commutes, and similarly for m_h.

Naturally, we could have chosen the scalars in the definition of c_e (or, using a square root of -1, in that of π_e) such that the diagram literally commutes. However, we have chosen the scalars such that c_e has the most natural formula and π_e, τ_h have the most natural formulas in our model $\bigwedge E$ for the spin representation.

Proof. We may choose a hyperbolic basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ of V such that $e = e_n$ and $f_1, \ldots, f_n$ is a basis of F. We write $f := f_1 \cdots f_n$ and $\overline{f} := \overline{f}_1 \cdots \overline{f}_{n-1}$.

Since the vertical maps are quadratic, it is not sufficient to show commutativity on a spanning set. We therefore consider

$$a := \sum_{I \subseteq [n]} c_I e_I$$

where, for $I = \{i_1 < \ldots < i_k\}$ we write $e_I := e_{i_1} \cdots e_{i_k}$. We then have

$$\pi_e(af) = \sum_{I: n \notin I} c_I \overline{e_I} \overline{f} =: \overline{a} \overline{f}$$

and

$$\widehat{\nu_2}(\overline{a}\overline{f}) = \text{the component in } \bigwedge^{n-1} V_e \text{ of } \sum_{I,J: n \notin I \cup J} (c_I c_J \overline{e_I} \overline{f} \overline{e_J}^*) \bullet 1 \in \bigwedge V_e.$$

Now note that, since $\overline{f}$ has $n-1$ factors, if I, J do not have the same parity, then acting with $\overline{e_I} \overline{f} \overline{e_J}^*$ on 1 yields a zero contribution in $\bigwedge^{n-1} V_e$. Hence the sum above may be split into two sums, one of which is

$$\text{the component in } \bigwedge^{n-1} V_e \text{ of } \sum_{I,J: |I|,|J| \text{ even},\, n \notin I \cup J} (c_I c_J \overline{e_I} \overline{f} \overline{e_J}^*) \bullet 1. \tag{2.5.2}$$

On the other hand, consider

$$\widehat{\nu_2}(af) = \text{the component in } \bigwedge^{n} V \text{ of} \sum_{I,J} (c_I c_J e_I f e_J^*) \bullet 1 \in \bigwedge V.$$

For the same reason as above, this splits into two sums, and we want to compare the following expression to (2.5.2):

$$c_e\left(\text{the component in } \bigwedge^{n} V \text{ of } \sum_{I,J: |I|,|J| \text{ even}} (c_I c_J e_I f e_J^*) \bullet 1\right). \tag{2.5.3}$$

Now recall that the action of $e = e_n \in V \subseteq \mathrm{Cl}(V)$ on $\bigwedge V$ is via $o(e) + \iota(e)$, while c_e is ι_e followed by projection to $\bigwedge^{n-1} V_e$. Hence to compute (2.5.3), we may as well compute the summands of

$$\text{the component in } \bigwedge^{n} V \text{ of } \sum_{I,J: |I|,|J| \text{ even}} (c_I c_J \cdot e \cdot e_I f e_J^*) \bullet 1$$

that do not contain a factor e. Terms with $n \in I$ do not contribute, because then $ee_I = 0$. Terms with $n \notin I$ but $n \in J$ do not contribute because when e gets contracted with f_n a factor e in e_J^* survives, and when e does not get contracted with f_n, we use $ee_J^* = 0$. So we may restrict attention to the terms with $n \notin I \cup J$. Let I, J correspond to such a term, that is, $|I|, |J|$ are even and $n \notin I \cup J$. Write $I = \{i_1 < \ldots < i_k\}$ and $J = \{j_1 < \ldots < j_l\}$. Then

$$\begin{aligned}(ee_I f e_J^*) \bullet 1 &= \Big((-1)^{n-1} e_I f_1 \cdots f_{n-1} e f_n e_J^*\Big) \bullet 1 \\ &= \Big((-1)^{n-1} e_I f_1 \cdots f_{n-1} e\Big) \bullet (f_n \wedge e_{j_l} \wedge \cdots \wedge e_{j_1}) \\ &= \Big((-1)^{n-1} e_I f_1 \cdots f_{n-1}\Big) \bullet (e_{j_l} \wedge \cdots \wedge e_{j_1} + e \wedge f_n \wedge e_{j_l} \wedge \cdots \wedge e_{j_1}).\end{aligned}$$

The second term in the last expression will contribute only terms with a factor e to the final result, and the former term contributes

$$\text{the component in } \textstyle\bigwedge^{n-1} V_e \text{ of } (-1)^{n-1}(\overline{e_I}\,\overline{f}\,\overline{e_J}^*) \bullet 1.$$

Comparing this with (2.5.2), we see that the diagram commutes on terms in $\mathrm{Cl}^+(V)f$ up to the factor $(-1)^{n-1}$. A similar computation shows that it commutes on terms in $\mathrm{Cl}^-(V)f$ up to a factor factor $(-1)^n$.

We now consider the second diagram, where V is split as the orthogonal direct sum $V_e \oplus \langle e, h\rangle$ with $e = e_n, h = f_n$. Consider $a \in \mathrm{Cl}(\langle e_1, \ldots, e_{n-1}\rangle)$. By the same argument as above, it suffices to consider the case where all summands of a in the basis e_I have indices I with $|I|$ of the same parity, say even. Then $\widehat{\nu}_2 \circ \tau_h$ in the diagram sends $a\overline{f}$ to the component in $\bigwedge^n V$ of $afa^* \bullet 1$. Since the summands e_I in a all have $n \notin I$, in $afa^* \bullet 1$ all summands have a factor f_n, and indeed

$$(afa^*) \bullet 1 = f_n \wedge (a\overline{f}a^* \bullet 1)$$

(when all terms in a have $|I|$ odd, we get a minus sign). The component in $\bigwedge^n V$ of this expression is the same as the one obtained via $m_h \circ \widehat{\nu}_2$. □

2.5.5 Proof of Theorem 2.5.1

In this section we use the Cartan map to prove Theorem 2.5.1, and finish the proof of Corollary 2.4.8 via a similar argument.

Proof of Theorem 2.5.1. For a quadratic space of dimension $2n$, we will denote the isotropic Grassmann cone over the Plücker embedding by $\widehat{\mathrm{Gr}}_{\mathrm{iso}}^{\mathrm{Pl}}(V) \subseteq \bigwedge^n V$. Given a maximal isotropic subspace $F \subseteq V$ with basis $f_1, \ldots, f_n$ and $f := f_1 \cdots f_n$, let $\widehat{\nu}_2 : \mathrm{Cl}^+(V)f \to \bigwedge^n V$ be the Cartan map defined in Section 2.5.3. For any isotropic $v \in V \setminus F$ the diagram

$$\begin{array}{ccc} \bigwedge^n V & \xrightarrow{c_v} & \bigwedge^{n-1} V_v \\ {\scriptstyle\widehat{\nu}_2}\uparrow & & \uparrow{\scriptstyle\widehat{\nu}_2} \\ \mathrm{Cl}(V)f & \xrightarrow{\pi_v} & \mathrm{Cl}(V_v)\overline{f} \end{array}$$

commutes up to scalar factor at the bottom by Proposition 2.5.5, where $V_v := v^\perp/\langle v\rangle$ and where $\overline{f}$ is the image of a product of a basis of $v^\perp \cap F_n$.

The proof of Corollary 1.3.2 in Chapter 1 shows that for $\omega \in \bigwedge^n V$ the following are equivalent:

1. $\omega \in \widehat{\mathrm{Gr}}^{\mathrm{Pl}}_{\mathrm{iso}}(V)$;

2. for any sequence $v_1 \in V$, $v_2 \in V_{v_1}$, $v_3 \in (V_{v_1})_{v_2}, \ldots, v_{n-4} \in (\cdots((V_{v_1})_{v_2})_{v_3}\cdots)_{v_{n-3}}$ of isotropic vectors, it holds that

$$C(\omega) \in \widehat{\mathrm{Gr}}^{\mathrm{Pl}}_{\mathrm{iso}}(W),$$

where we abbreviate $W := (\cdots((V_{v_1})_{v_2})_{v_3}\cdots)_{v_{n-4}}$ and $C : \bigwedge^n V \to \bigwedge^4 W$ is the composition $C := c_{v_{n-4}} \circ \cdots \circ c_{v_1}$ of the contraction maps c_{v_i} introduced in Section 2.2.2.

By slight abuse of notation, we also write $v_1, \ldots, v_{n-4}$ for preimages of these vectors in V. These span an $(n-4)$-dimensional isotropic subspace U of V (provided that each v_i chosen above in the successive quotients is nonzero), and W equals $U^\perp/U$. For any fixed ω, the condition that $C(\omega)$ lies in $\widehat{\mathrm{Gr}}^{\mathrm{Pl}}_{\mathrm{iso}}(W)$ is a closed condition on U, and hence it suffices to check that condition for U in a dense subset of the Grassmannian of isotropic $(n-4)$-dimensional subspaces of V. In particular, it suffices to check this when $U \cap F_n = \{0\}$.

Fix $n \geq 4$ and $x \in \mathrm{Cl}(V_n)f_1\cdots f_n$ such that $p(g\cdot x) = 0$ for all $g \in \mathrm{Spin}(V_n)$ and all $p \in I_4$. This means precisely that $\pi_{n,4}(g\cdot x) \in \widehat{\mathrm{Gr}}^{+}_{\mathrm{iso}}(V_4)$ for all $g \in \mathrm{Spin}(V_n)$. We need to show that $x \in \widehat{\mathrm{Gr}}^{+}_{\mathrm{iso}}(V_n)$. To this end, consider $\omega := \widehat{\nu}_2(x) \in \bigwedge^n V_n$. It suffices to show that $\omega \in \widehat{\mathrm{Gr}}^{\mathrm{Pl}}_{\mathrm{iso}}(V_n)$. Indeed, this follows from the fact that $\widehat{\nu}_2\big(\widehat{\mathrm{Gr}}^{+}_{\mathrm{iso}}(V)\big)$ is one of the two irreducible components of $\widehat{\mathrm{Gr}}^{\mathrm{Pl}}_{\mathrm{iso}}(V)$ (see Lemma 2.5.3) and because ν_2 is an injective morphism by Lemma 2.5.2. Let $v_1, v_2, \ldots, v_{n-4} \in V_n$ as above: linearly independent, and such that the span $U := \langle v_1, \ldots, v_{n-4}\rangle$ is an isotropic space that intersects F_n trivially. Let $C := c_{v_{n-4}} \circ \cdots \circ c_{v_1}$ be the composition of the associated contractions. We need to show that $C(\omega) \in \widehat{\mathrm{Gr}}^{\mathrm{Pl}}_{\mathrm{iso}}(W)$, where $W := U^\perp/U$.

Now $\widehat{\nu}_2\big(\widehat{\mathrm{Gr}}^{+}_{\mathrm{iso}}(W)\big) \subseteq \widehat{\mathrm{Gr}}^{\mathrm{Pl}}_{\mathrm{iso}}(W)$ by Lemma 2.5.3, and the diagram

$$\begin{array}{ccc} \bigwedge^n V_n & \xrightarrow{\quad C \quad} & \bigwedge^4 W \\ \widehat{\nu}_2 \big\uparrow & & \big\uparrow \widehat{\nu}_2 \\ \mathrm{Cl}(V_n)f & \xrightarrow{\pi_{v_{n-4}}\circ\cdots\circ\pi_{v_1}} & \mathrm{Cl}(W)\overline{f}, \end{array}$$

where $\overline{f}$ is the image of the product of a basis of $U^\perp \cap F_n$, commutes up to a scalar factor in the bottom map due to Proposition 2.5.5. Therefore, it suffices to check that $\pi_{v_{n-4}} \circ \cdots \circ \pi_{v_1}(x) \in \widehat{\mathrm{Gr}}^{+}_{\mathrm{iso}}(W)$. Now there exists an element $g \in \mathrm{Spin}(V_n)$ that maps F_n into itself (not with the identity!) and sends v_i to e_{n+1-i} for $i = 1, \ldots, n-4$. This induces an isometry $W := U^\perp/U \to (U')^\perp/U' = V_4 = \langle e_1, \ldots, e_4, f_1, \ldots, f_4\rangle$, where $U' := \langle e_5, \ldots, e_n\rangle$. This in turn induces a linear isomorphism (unique up to a scalar) $\mathrm{Cl}(W)\cdot\overline{f} \to \mathrm{Cl}(V_4)\cdot f_1\cdots f_4$ (where f on the left is the product of a basis of $F_n \cap U^\perp$) that maps $\widehat{\mathrm{Gr}}^{+}_{\mathrm{iso}}(W)$ onto $\widehat{\mathrm{Gr}}^{+}_{\mathrm{iso}}(V_4)$. Since, by assumption, $\pi_{n,4}(g\cdot x) = \pi_{e_5}\circ\cdots\circ\pi_{e_n}(g\cdot x)$ lies in the latter isotropic Grassmann cone, $\pi_{v_{n-4}} \circ \cdots \circ \pi_{v_1}(x)$ lies in the former. $\square$

Lemma 2.5.6. *Let $q \geq n \geq n_0$. Then for all g in some open dense subset of* $\mathrm{Spin}(V_q)$ *there exist $g' \in \mathrm{Spin}(V_n)$ and $g'' \in \mathrm{Spin}(V_{n_0})$ such that*

$$\pi_{q,n_0} \circ g \circ \tau_{n,q} = g'' \circ \pi_{n,n_0} \circ g'$$

holds up to a scalar factor.

Proof. The proof is similar to that above; we just give a sketch. Using the Cartan map, which is equivariant for the relevant spin groups, this lemma follows from a similar statement for the corresponding (halfs of) exterior power representations. Specifically, define

$$\begin{aligned} E &:= \langle e_{n_0+1}, \ldots, e_q \rangle \subseteq V_q, \\ E' &:= \langle e_{n_0+1}, \ldots, e_n \rangle \subseteq V_n, \text{ and} \\ F &:= \langle f_{n+1}, \ldots, f_q \rangle \subseteq V_q. \end{aligned}$$

Then the desired identity is

$$c_E \circ g \circ m_F = g'' \circ c_{E'} \circ g' \tag{2.5.4}$$

(up to a scalar), where

$$\begin{aligned} c_E &:= c_{e_{n_0+1}} \circ \cdots \circ c_{e_q} : \bigwedge^q V_q \to \bigwedge^{n_0} V_{n_0}, \\ c_{E'} &:= c_{n_0+1} \circ \cdots \circ c_{e_n} : \bigwedge^n V_n \to \bigwedge^{n_0} V_{n_0}, \text{ and} \\ m_F &:= m_{f_q} \circ \cdots \circ m_{f_{n+1}} : \bigwedge^n V_n \to \bigwedge^q V_q \end{aligned}$$

and the c_{e_i} and m_{f_j} are as defined in Section 2.2.2 and Section 2.2.3, respectively. Furthermore, since the exterior powers are representations of the special orthogonal groups, we may take g, g', g'' to be in $\mathrm{SO}(V_q), \mathrm{SO}(V_n), \mathrm{SO}(V_{n_0})$, respectively.

We investigate the effect of the map on the left on (a pure tensor in $\bigwedge^n V_n$ corresponding to) a maximal (i.e., n-dimensional) isotropic subspace W of V_n. First, W is extended to $W' := W \oplus F$, then g is applied to W', and the final contraction map sends gW' to the image in V_q/E of $(gW') \cap E^\perp$.

Instead of intersecting gW' with $E^\perp$, we may intersect $W' = W \oplus F$ with $(E'')^\perp$ where $E'' := g^{-1}E$, followed by the isometry $\overline{g} : (E'')^\perp/E'' \to E^\perp/E$ induced by g. Accordingly, one can verify that the map on the left-hand side of Equation (2.5.4) becomes (a scalar multiple of)

$$\overline{g} \circ c_{E''} \circ m_F$$

where $c_{E''} : \bigwedge^q V_q \to \bigwedge^{n_0} \left((E'')^\perp/E''\right)$ is the composition of contractions with a basis of E'', and where we write $\overline{g}$ also for the map that $\overline{g}$ induces from $\bigwedge^{n_0} \left((E'')^\perp/E''\right)$ to $\bigwedge^{n_0}(E^\perp/E)$.

Now consider the space $E'' \cap (V_n \oplus F) \subseteq V_q$. For g in an open dense subset of $\mathrm{SO}(V_q)$, this intersection has the expected dimension $(q - n_0) + (2n + q - n) - 2q = n - n_0$, and for g in an open dense subset of $\mathrm{SO}(V_q)$ we also have $(E'')^\perp \cap F = \{0\}$ (because $(E'')^\perp$ has codimension $q - n_0$, which is at least the dimension $q - n$ of F). We restrict ourselves to such g. Then in particular $E'' \cap F = \{0\}$ and therefore the projection $\widetilde{E} \subseteq V_n$ of

$E'' \cap (V_n \oplus F)$ along F has dimension $n - n_0$, as well. Note that $\widetilde{E}$ is isotropic because E'' is and because F is the radical of the bilinear form on $V_n \oplus F$.

Furthermore, the projection $V_n \oplus F \to V_n$ restricts to a linear isomorphism

$$(V_n \oplus F) \cap (E'')^\perp \to \widetilde{E}^\perp,$$

where the latter is the orthogonal complement of $\widetilde{E}$ inside V_n. This linear isomorphism induces an isometry

$$h_1 : \Big((V_n \oplus F) \cap (E'')^\perp\Big) \,/\, \big((V_n \oplus F) \cap E''\big) \to \widetilde{E}^\perp/\widetilde{E}$$

between spaces of dimension $2n_0$ equipped with a non-degenerate bilinear forms. On the other hand, the inclusion $V_n \oplus F \to V_q$ also induces an isometry

$$h_2 : \Big((V_n \oplus F) \cap (E'')^\perp\Big) \,/\, \big((V_n \oplus F) \cap E''\big) \to (E'')^\perp/E''.$$

Now a computation shows that, up to a scalar, we have

$$c_{E''} \circ m_F = h_2 \circ h_1^{-1} \circ c_{\widetilde{E}},$$

where $c_{\widetilde{E}} : \bigwedge^n V_n \to \bigwedge^{n_0}(\widetilde{E}^\perp/\widetilde{E})$ is a composition of contractions with a basis of $\widetilde{E}$. Now choose $g' \in \mathrm{SO}(V_n)$ such that $g'\widetilde{E} = E'$, so that we have

$$c_{E'} \circ g' = \overline{g'} \circ c_{\widetilde{E}},$$

where $\overline{g'}$ is the isometry $\widetilde{E}^\perp/\widetilde{E} \to (E')^\perp/E'$ induced by g'. We then conclude that

$$c_E \circ g \circ m_F = \overline{g} \circ h_2 \circ h_1^{-1} \circ (\overline{g'})^{-1} \circ c_{E'} \circ g'$$

and hence we are done if we set

$$g'' := \overline{g} \circ h_2 \circ h_1^{-1} \circ \left(\overline{g'}\right)^{-1} \in \mathrm{SO}\left((E')^\perp/E'\right) = \mathrm{SO}(V_{n_0}). \qquad \square$$

Chapter 3

Noetherianty and Universality for Lagrangian Plücker Varieties

3.1 Foundations

3.1.1 General Vector Spaces

Throughout this entire subsection V denotes a finite dimensional vector space over an arbitrary field $\mathbb{K}$.

Denote by $\mathrm{Alt}^k(V)$ the space of alternating multilinear maps $V \times \cdots \times V \to \mathbb{K}$. The following maps will be central in this article.

Definition 3.1.1 (Contraction maps)**.** For $\beta \in V^*$ the *contraction with* β, or *interior multiplication by* β, is the map

$$i_\beta : \bigwedge^k V \to \bigwedge^{k-1} \ker \beta \subseteq \bigwedge^{k-1} V$$

given by

$$v_1 \wedge \cdots \wedge v_k \mapsto \sum_{i=1}^{k} (-1)^{i-1} \beta(v_i) v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_k, \tag{3.1.1}$$

where $\widehat{\cdot}$ indicates that the factor is omitted. Similarly, for $\omega \in \mathrm{Alt}^2(V)$ the *contraction with* ω, or *interior multiplication by* ω, is the map

$$i_\omega : \bigwedge^k V \to \bigwedge^{k-2} V$$

sending $v_1 \wedge \cdots \wedge v_k$ to

$$\sum_{1 \le i < j \le k} (-1)^{i+j-1} \omega(v_i, v_j) v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge \widehat{v_j} \wedge \cdots \wedge v_k. \tag{3.1.2}$$

In the following sections, whenever we will consider symplectic vector spaces, we will exclusively consider i_ω with ω being the symplectic form on V.

Below we will give a more conceptual description of the contraction maps. To this avail, recall that the *evaluation pairing* is the non-degenerate pairing

$$(\cdot, \cdot)_{\mathrm{ev}} : \bigwedge^k V \times \bigwedge^k V^* \to \mathbb{K}$$

given by

$$(v_1 \wedge \cdots \wedge v_k, \beta^1 \wedge \cdots \wedge \beta^k)_{\text{ev}} := \det\left(\left(\beta^i(v_j)\right)_{1\le i,j\le k}\right). \tag{3.1.3}$$

Example 3.1.2. Let $e_1, \dots, e_n$ be a basis of V, and denote by $\varepsilon^1, \dots, \varepsilon^n \in V^*$ the dual basis given by $\varepsilon^j(e_i) = \delta_{ij}$. For any $1 \le i_1 < \ldots < i_k \le n$ and $1 \le j_1 < \ldots < j_k \le n$ it then holds

$$\left(e_{i_1} \wedge \cdots \wedge e_{i_k}, \varepsilon^{j_1} \wedge \cdots \wedge \varepsilon^{j_k}\right)_{\text{ev}} = \begin{cases} 1 & \text{if } i_1 = j_1, \dots, i_k = j_k \\ 0 & \text{otherwise.} \end{cases}$$

This also implies that the evaluation pairing is non-degenerate.

The evaluation pairing induces an isomorphism $\bigwedge^k(V^*) \cong (\bigwedge^k V)^*$ because it is non-degenerate. Moreover, by the universal property of $\bigwedge^k V$, there also is a canonical isomorphism $\text{Alt}^k(V) \cong (\bigwedge^k V)^*$ ([Lüc12, p. 183]). Composing these we get an isomorphism

$$\text{Alt}^k(V) \cong \bigwedge^k V^*. \tag{3.1.4}$$

Under this isomorphism, for all $\xi \in \text{Alt}^k(V) \cong \bigwedge^k(V^*)$ and $v_1, \dots, v_k \in V$ it holds

$$\xi(v_1, \dots, v_k) = (v_1 \wedge \cdots \wedge v_k, \xi)_{\text{ev}}, \tag{3.1.5}$$

where on the left hand side we think of ξ as an element of $\text{Alt}^k(V)$ and on the right hand side as an element of $\bigwedge^k(V^*)$.

Under the isomorphism $(\bigwedge^k V)^* \cong \bigwedge^k(V^*)$ induced from the evaluation pairing, the contraction maps i_β and i_ω (see Definition 3.1.1) are dual to the maps

$$\bigwedge^{k-1} V^* \to \bigwedge^k V^*,\ \xi \mapsto \beta \wedge \xi \quad \text{and} \quad \bigwedge^{k-2} V^* \to \bigwedge^k V^*,\ \xi \mapsto \omega \wedge \xi,$$

where we use the isomorphism (3.1.4) to think of $\omega \in \text{Alt}^2(V)$ as an element in $\bigwedge^2 V^*$. Namely, for all $\eta \in \bigwedge^k V$ and $\xi \in \bigwedge^{k-1} V^*$ it holds

$$(i_\beta(\eta), \xi)_{\text{ev}} = (\eta, \beta \wedge \xi)_{\text{ev}}, \tag{3.1.6}$$

and similarly

$$(i_\omega(\eta), \xi)_{\text{ev}} = (\eta, \omega \wedge \xi)_{\text{ev}} \tag{3.1.7}$$

for all $\eta \in \bigwedge^k V$ and $\xi \in \bigwedge^{k-2} V^*$ (see [FH91, p. 260, p. 476 and Exercise B.15(ii)]). In other words the maps i_β, resp. i_ω, can coordinate-independently be described as the maps dual to *wedging* with the respective form β or ω.

We now recall the definition of the (ordinary) Grassmann cone.

Definition 3.1.3 (Grassmann cone)**.** For $k \le \dim(V)$ the *Grassmann cone* $\widehat{\text{Gr}}(k, V)$ is defined as

$$\widehat{\text{Gr}}(k, V) := \left\{ v_1 \wedge \cdots \wedge v_k \in \bigwedge^k V \,\middle|\, v_1, \dots, v_k \in V \right\}.$$

For $\xi = v_1 \wedge \cdots \wedge v_k \in \widehat{\text{Gr}}(k, V) \setminus \{0\}$, we will denote the corresponding subspace $\langle v_1, \dots, v_k \rangle \subseteq V$ as L_ξ. Here we use the notation $\langle v_1, \dots, v_k \rangle$ to denote the span of the vectors $v_1, \dots, v_k$ in V.

Later (in the proof of Lemma 3.4.8) we will use the following result.

Lemma 3.1.4. *Suppose $\xi_1, \xi_2 \in \widehat{\mathrm{Gr}}(k, V) \setminus \{0\}$ are such that $\xi_1 + \xi_2 \in \widehat{\mathrm{Gr}}(k, V)$. Then $\dim(L_{\xi_1} \cap L_{\xi_2}) \geq k - 1$.*

Proof. We abbreviate $\xi = \xi_1 + \xi_2$ and $W = L_{\xi_1} + L_{\xi_2}$, so that $\xi \in \bigwedge^k W$ and $\xi \in \widehat{\mathrm{Gr}}(k, W)$ by assumption. If $\xi = 0$, then the lemma trivially holds. So we may assume $\xi \neq 0$. Consider the map $\psi_\xi : W \to \bigwedge^{k+1} W$, $w \mapsto w \wedge \xi$. Observe that

$$\dim\left(\ker(\psi_\xi)\right) \geq k \tag{3.1.8}$$

because $L_\xi \subseteq \ker(\psi_\xi)$ (L_ξ is defined since $\xi \in \widehat{\mathrm{Gr}}(k, W) \setminus \{0\}$). Arguing by contradiction, we assume that $\dim(L_{\xi_1} \cap L_{\xi_2}) = k - m$ for some $m \geq 2$. We fix a basis $\{w_{m+1}, \dots, w_k\}$ of $L_{\xi_1} \cap L_{\xi_2}$ and extend it to a basis of L_{η_1} and L_{η_2}, that is, we choose $u_1, \dots, u_m \in L_{\xi_1}$ and $u'_1, \dots, u'_m \in L_{\xi_2}$ such that

$$L_{\xi_1} = \langle u_1, \dots, u_m, w_{m+1}, \dots, w_k \rangle \quad \text{and} \quad L_{\xi_2} = \langle u'_1, \dots, u'_m, w_{m+1}, \dots, w_k \rangle.$$

Hence, after potentially rescaling u_1 and u'_1, it holds

$$\xi_1 = u_1 \wedge \cdots \wedge u_m \wedge w_{m+1} \wedge \cdots \wedge w_k \quad \text{and} \quad \xi_2 = u'_1 \wedge \cdots \wedge u'_m \wedge w_{m+1} \wedge \cdots \wedge w_k.$$

Therefore, keeping in mind that $\xi = \xi_1 + \xi_2$ and $m \geq 2$, we see that the images

$$\begin{aligned} \psi_\xi(u_i) &= u_i \wedge u'_1 \wedge \cdots \wedge u'_m \wedge w_{m+1} \wedge \cdots \wedge w_k \quad (1 \leq i \leq m) \\ \psi_\xi(u'_i) &= u'_i \wedge u_1 \wedge \cdots \wedge u_m \wedge w_{m+1} \wedge \cdots \wedge w_k \quad (1 \leq i \leq m) \end{aligned}$$

are linearly independent, so that $\dim(\mathrm{im}(\psi_\xi)) \geq 2m$. But then

$$\dim(\ker(\psi_\xi)) = \dim(W) - \dim(\mathrm{im}(\psi_\xi)) \leq (k + m) - 2m < k$$

due to the dimension formula. However, this contradicts (3.1.8). □

3.1.2 Symplectic Vector Spaces

From now on we consider a finite dimensional vector space V over a field $\mathbb{K}$ of $\mathrm{Char}(\mathbb{K}) = 0$ equipped with a *symplectic form* ω, i.e., a non-degenerate skew-symmetric bilinear form on V. It is well-known that such a symplectic space always has even dimension (see [Lee12, Proposition 22.7]).

We start by recalling some basic definitions. The *symplectic group* $\mathrm{Sp}(V)$ is the set of all automorphisms of V preserving the symplectic form, i.e.,

$$\mathrm{Sp}(V) = \left\{ A \in \mathrm{GL}(V) \;\middle|\; \omega(Av, Aw) = \omega(v, w) \text{ for all } v, w \in V \right\}$$

whose Lie algebra is

$$\mathfrak{sp}(V) = \left\{ L \in \mathrm{End}(V) \;\middle|\; \omega(Lv, w) + \omega(v, Lw) = 0 \text{ for all } v, w \in V \right\}.$$

The *orthogonal complement* of a subspace $L \subseteq V$ is

$$L^\perp := \left\{ v \in V \;\middle|\; \omega(v, u) = 0 \text{ for all } u \in L \right\}.$$

We call a subspace $L \subseteq V$ *isotropic* if $L \subseteq L^{\perp}$, i.e., if $\omega(u, v) = 0$ for all $u, v \in L$. Moreover, a subspace $L \subseteq V$ is called *Lagrangian* if $L = L^{\perp}$, or equivalently, if L is isotropic and $\dim(L) = \frac{1}{2}\dim(V)$.

Analogous to the definition of the (ordinary) Grassmann cone, we can now define the isotropic and Lagrangian Grassmann cone.

Definition 3.1.5 (Isotropic and Lagrangian Grassmann cone)**.** For $k \leq \frac{1}{2}\dim(V)$ the *isotropic Grassmann cone* $\widehat{\mathrm{Gr}}_{\mathrm{iso}}(k, V)$ is defined as

$$\widehat{\mathrm{Gr}}_{\mathrm{iso}}(k, V) := \left\{ v_1 \wedge \cdots \wedge v_k \in \bigwedge^k V \;\middle|\; \omega(v_i, v_j) = 0 \text{ for all } 1 \leq i, j \leq k \right\}.$$

The *Lagrangian Grassmann cone* $\widehat{\mathrm{Gr}}_{\mathrm{L}}(V)$ is defined as $\widehat{\mathrm{Gr}}_{\mathrm{iso}}(k, V)$ for $k = \frac{1}{2}\dim(V)$, i.e.,

$$\widehat{\mathrm{Gr}}_{\mathrm{L}}(V) := \widehat{\mathrm{Gr}}_{\mathrm{iso}}\left(\frac{1}{2}\dim(V), V\right).$$

Note that $\xi \in \widehat{\mathrm{Gr}}(k, V) \setminus \{0\}$ lies in $\widehat{\mathrm{Gr}}_{\mathrm{iso}}(k, V)$ if and only if $L_\xi \subseteq V$ is isotropic. Similarly, if $\dim(V) = 2n$, then $\xi \in \widehat{\mathrm{Gr}}(n, V) \setminus \{0\}$ is in $\widehat{\mathrm{Gr}}_{\mathrm{L}}(V)$ if and only if $L_\xi \subseteq V$ is Lagrangian.

Remark 3.1.6*.* The projectivization of $\widehat{\mathrm{Gr}}_{\mathrm{L}}(V)$ is the Lagrangian Grassmannian $\mathrm{Gr}_{\mathrm{L}}(V)$; it parametrizes Lagrangian subspaces of a symplectic space V.

Throughout the article we will frequently use the following terminology.

Definition 3.1.7 (Symplectic basis)**.** A basis $e_1, e_{-1}, \ldots, e_n, e_{-n}$ of V is called a *symplectic basis* if $\omega(e_i, e_{-i}) = 1$ for all $i \in \{1, \ldots, n\}$ and if for all $i, j \in \{\pm 1, \ldots, \pm n\}$ it holds $\omega(e_i, e_j) = 0$ whenever $i \neq -j$.

Note that, if $e_1, e_{-1}, \ldots, e_n, e_{-n}$ is a symplectic basis of V, then $\omega(e_{-i}, e_i) = -1$ for $1 \leq i \leq n$ since ω is skew-symmetric.

The following elementary lemma will be prove useful.

Lemma 3.1.8. *Let $L_1, L_2 \subseteq V$ be Lagrangian subspaces. Then for any choice of decomposition*

$$L_1 = (L_1 \cap L_2) \oplus U_1 \quad \text{and} \quad L_2 = (L_1 \cap L_2) \oplus U_2$$

the musical isomorphism $\flat$ (see (3.1.9)) restricts to an isomorphism $U_1 \to U_2^*$. In particular, there exists a symplectic basis $e_1, e_{-1}, \ldots, e_n, e_{-n}$ of V, such that

$$L_1 = \langle e_1, \ldots, e_q, e_{q+1}, \ldots, e_n \rangle \quad \text{and} \quad L_2 = \langle e_1, \ldots, e_q, e_{-(q+1)}, \ldots, e_{-n} \rangle,$$

where $q = \dim(L_1 \cap L_2)$.

Proof. This was proven in Lemma 1.2.6 in Chapter 1 for *quadratic* spaces, i.e., spaces equipped with a non-degenerate *symmetric* bilinear form. However, the proof also works for symplectic spaces without any modifications. ☐

Since the skew-symmetric form ω is non-degenerate, it induces *the musical isomorphism*

$$\flat : V \to V^*,\ v \mapsto v^\flat := \omega(v, \cdot\,). \tag{3.1.9}$$

This induces a map on the exterior powers, which by abuse of notation we still denote by $\flat$,

$$\flat : \bigwedge^k V \to \bigwedge^k V^*,\ v_1 \wedge \cdots \wedge v_k \mapsto v_1^\flat \wedge \cdots \wedge v_k^\flat. \tag{3.1.10}$$

We will denote the image of any $\eta \in \bigwedge^k V$ under this isomorphism by $\eta^\flat$.

Example 3.1.9. Let $e_1, e_{-1}, \ldots, e_n, e_{-n}$ be a symplectic basis of V, and denote the dual basis of V^* by $\varepsilon^1, \varepsilon^{-1}, \ldots, \varepsilon^n, \varepsilon^{-n}$. Then for $1 \leq i \leq n$ it holds

$$e_i^\flat = \varepsilon^{-i} \quad \text{and} \quad e_{-i}^\flat = -\varepsilon^i.$$

Hence $\xi_\omega := e_1 \wedge e_{-1} + \cdots + e_n \wedge e_{-n} \in \bigwedge^2 V$ satisfies

$$\xi_\omega^\flat = \varepsilon^1 \wedge \varepsilon^{-1} + \cdots + \varepsilon^n \wedge \varepsilon^{-n} = \omega.$$

Here we use the isomorphism $\mathrm{Alt}^2(V) \cong \bigwedge^2 V^*$ from (3.1.4) to interpret the symplectic form ω as an element of $\bigwedge^2 V^*$.

We want to point out a connection between the musical isomorphism and the evaluation map, as it will be important for us later on. Namely, for all $\eta, \xi \in \bigwedge^k V$ it holds

$$(\eta, \xi^\flat)_{\mathrm{ev}} = (-1)^k (\xi, \eta^\flat)_{\mathrm{ev}}. \tag{3.1.11}$$

This follows from the definition (3.1.3) of the evaluation pairing, and the fact that for all $v, w \in V$ it holds $v^\flat(w) = \omega(v, w) = -\omega(w, v) = -w^\flat(v)$ since ω is skew-symmetric.

Next we want to use the symplectic form to extend Definition 3.1.1 to also define the contraction with a vector $v \in V$.

Definition 3.1.10 (Contraction map)**.** For $v \in V$ the *contraction with* v is the map

$$\varphi_v : \bigwedge^k V \to \bigwedge^{k-1} v^\perp$$

defined by

$$v_1 \wedge \cdots \wedge v_k \mapsto \sum_{i=1}^k (-1)^{i-1} \omega(v, v_i) v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_k. \tag{3.1.12}$$

Note that $\varphi_v = i_\beta$ for $\beta = v^\flat$ (see Definition 3.1.1).

Let $v \in V$ be a nonzero vector. Define $V_v := v^\perp / \langle v \rangle$ (note that $\langle v \rangle \subseteq v^\perp$ because ω is skew-symmetric). It is easy to see that

$$\omega_v(\bar{v}_1, \bar{v}_2) := \omega(v_1, v_2), \tag{3.1.13}$$

where $\bar{v}_i \in V_v$ denotes the equivalence class of $v_i \in v^\perp$ in V_v, is a well-defined symplectic form on V_v (i.e., the formula is independent of the choice of representatives $v_1, v_2 \in v^\perp$). We denote by π_v the projection $v^\perp \twoheadrightarrow V_v$.

Definition 3.1.11. For any nonzero $v \in V$ we define the linear map

$$\Phi_v : \bigwedge^k V \to \bigwedge^{k-1} V_v \tag{3.1.14}$$

as the composition

$$\bigwedge^k V \xrightarrow{\varphi_v} \bigwedge^{k-1} v^\perp \xrightarrow{\bigwedge^{k-1} \pi_v} \bigwedge^{k-1} V_v,$$

where φ_v is the contraction map introduced in Definition 3.1.10. Explicitly, this map is given by

$$\Phi_v(v_1 \wedge \cdots \wedge v_k) = \sum_{j=1}^{k} (-1)^{j-1} \omega(v, v_j) \bar{v}_1 \wedge \cdots \wedge \widehat{v}_j \wedge \cdots \wedge \bar{v}_k. \tag{3.1.15}$$

The map Φ_v can alternatively be described as follows. Analogous to i_β (see (3.1.1)), any $v \in V$ defines an *interior multiplication* (also see [Lee12, p. 358])

$$i_v : \bigwedge^k V^* \to \bigwedge^{k-1} \ker(v),$$

where $\ker(v) := \{\alpha \in V^* \mid \alpha(v) = 0\} \subseteq V^*$. Choose a section $s_v : V_v \to v^\perp$ of π_v, and consider its dual map $s_v^* : V^* \to V_v^*$, $\alpha \mapsto \alpha \circ s_v$. Then the diagram

$$\begin{array}{ccccc} & & \overset{\Phi_v}{\curvearrowright} & & \\ \bigwedge^k V & \xrightarrow{\varphi_v} & \bigwedge^{k-1} v^\perp & \xrightarrow{\bigwedge^{k-1} \pi_v} & \bigwedge^{k-1} V_v \\ \flat_V \big\downarrow \cong & & \cong \big\downarrow \flat_V & & \cong \big\downarrow \flat_{V_v} \\ \bigwedge^k V^* & \xrightarrow{i_v} & \bigwedge^{k-1} \ker(v) & \xrightarrow{\bigwedge^{k-1} s_v^*} & \bigwedge^{k-1} V_v^* \end{array} \tag{3.1.16}$$

commutes (up to sign (-1)). Moreover, under the canonical isomorphisms from (3.1.4), the lower horizontal map $(\bigwedge^{k-1} s_v^*) \circ i_v$ agrees with the map

$$\mathrm{Alt}^k(V) \to \mathrm{Alt}^{k-1}(V_v),\ \xi \mapsto \xi(v, s_v(\cdot), \ldots, s_v(\cdot)). \tag{3.1.17}$$

So, up to the musical isomorphisms and the canonical isomorphisms from (3.1.4), (3.1.17) can be thought of as an coordinate-independent description for Φ_v (up to sign (-1)).

Using this description we can characterize when $\Phi_v(\eta)$ is zero.

Lemma 3.1.12. *Let $v \in V$ be nonzero and $\eta \in \bigwedge^k V$. Then $\Phi_v(\eta) = 0$ if and only if*

$$\big(v \wedge v_2 \wedge \cdots \wedge v_k, \eta^\flat\big)_{\mathrm{ev}} = 0 \quad \textit{for all } v_2, \ldots, v_k \in v^\perp,$$

where $\eta^\flat \in \bigwedge^k V^*$ is the image of $\eta \in \bigwedge^k V$ under the induced musical isomorphism $\flat : \bigwedge^k V \to \bigwedge^k V^*$.

Proof. From (3.1.16) and (3.1.17) we see that $\Phi_v(\eta) = 0$ if and only if

$$\eta^\flat(v, s_v(\bar{v}_2), \ldots, s_v(\bar{v}_k)) = 0 \quad \text{for all } \bar{v}_2, \ldots, \bar{v}_k \in V_v.$$

Note that $v^\perp = s_v(V_v) \oplus \langle v \rangle$ because s_v is a section of $\pi_v : v^\perp \to v^\perp/\langle v \rangle = V_v$. Therefore, because $\eta^\flat$ is alternating, $\Phi_v(\eta) = 0$ holds if and only if

$$\eta^\flat(v, v_2, \ldots, v_k) \quad \text{for all } v_2, \ldots, v_k \in s_v(V_v) \oplus \langle v \rangle = v^\perp.$$

Keeping in mind (3.1.5), this completes the proof. $\square$

3.2 Counterexample and New Setting

3.2.1 Counterexample

A *quadratic space* is a vector space equipped with a non-degenerate *symmetric* bilinear form $(\cdot,\cdot)$, and a vector $v \in V$ is called *isotropic* if $(v,v) = 0$. In Chapter 1 the main result was the following.

Theorem 3.2.1 (Theorem 1.3.1 in Chapter 1). *Let V be a quadratic space over a field $\mathbb{K}$ with $\mathrm{Char}(\mathbb{K}) \neq 2$ and $\dim(V) > 8$. Consider $\eta \in \bigwedge^p$, where $p = \lfloor \frac{1}{2}\dim(V)\rfloor$. If $\Phi_v(\eta) \in \widehat{\mathrm{Gr}}_{\mathrm{iso}}(p-1, V_v)$ for every isotropic vector $v \in V_{\mathrm{iso}}$, then $\eta \in \widehat{\mathrm{Gr}}_{\mathrm{iso}}(p, V)$.*

This does *not* generalize directly to symplectic vector spaces. Indeed, the following example shows that there exist $\eta \in \bigwedge^n V$ such that $\Phi_v(\eta) = 0$ for all $v \in V$, but $\eta \notin \widehat{\mathrm{Gr}}_{\mathrm{L}}(V)$. Nonetheless, we will show in this chapter that the main result of Chapter 1 holds for the Lagrangian Grassmann cone if we consider a different setting.

Example 3.2.2 (Counterexample). Let V be a symplectic vector space of dimension $2n = 4m$. Denote by $\xi_\omega \in \bigwedge^2 V$ be the preimage of ω under the induced musical isomorphism $\flat : \bigwedge^2 V \to \bigwedge^2 V^*$ (see Example 3.1.9). Define

$$\eta_{\mathrm{ex}} = \xi_\omega^m = \xi_\omega \wedge \cdots \wedge \xi_\omega \in \bigwedge^{2m} V.$$

We claim that, for every $v \in V$,

$$\varphi_v(\eta_{\mathrm{ex}}) = -mv \wedge \xi_\omega^{m-1}, \tag{3.2.1}$$

where φ_v is the contraction we map introduced in Definition 3.1.10. Because the map $\Phi_v = \left(\bigwedge^{n-1} \pi_v\right) \circ \varphi_v$, where $\pi_v : v^\perp \to v^\perp/\langle v\rangle = V_v$ is the projection, this shows $\Phi_v(\eta_{\mathrm{ex}}) = 0$ for every $v \in V$. Moreover, one can easily check, for example by working in a hyperbolic basis (see Example 3.1.9), that $\eta_{\mathrm{ex}} \wedge \eta_{\mathrm{ex}} \neq 0$, and hence $\eta_{\mathrm{ex}} \notin \widehat{\mathrm{Gr}}(n, V)$.

So it remains to prove (3.2.1). Recall that $\varphi_v = i_\beta$ with $\beta = v^\flat$ (see Definition 3.1.1). We will use that for all $\xi_1 \in \bigwedge^k V$ and $\xi_2 \in \bigwedge^\ell V$ it holds (see [Lee12, Lemma 14.13(b)])

$$i_\beta(\xi_1 \wedge \xi_2) = i_\beta(\xi_1) \wedge \xi_2 + (-1)^k \xi_1 \wedge i_\beta(\xi_2). \tag{3.2.2}$$

Thus

$$\varphi_v(\eta_{\mathrm{ex}}) = i_{v^\flat}(\xi_\omega^m) = m i_{v^\flat}(\xi_\omega) \wedge \xi_\omega. \tag{3.2.3}$$

Keeping in mind that $\xi_\omega^\flat = \omega$, we obtain for all $w \in V$

$$(i_{v^\flat}(\xi_\omega), w^\flat)_{\mathrm{ev}} \overset{(3.1.6)}{=} (\xi_\omega, v^\flat \wedge w^\flat)_{\mathrm{ev}} \overset{(3.1.11)}{=} (v \wedge w, \xi_\omega^\flat)_{\mathrm{ev}} = (v \wedge w, \omega)_{\mathrm{ev}} \overset{(3.1.5)}{=} \omega(v,w),$$

and thus

$$(i_{v^\flat}(\xi_\omega), w^\flat)_{\mathrm{ev}} = -\omega(w,v) = -w^\flat(v) \overset{(3.1.5)}{=} -(v, w^\flat)_{\mathrm{ev}}$$

by the skew-symmetry of ω and the definition (3.1.9) of $w^\flat$. As $w \in W$ was arbitrary, this implies $i_{v^\flat}(\xi_\omega) = -v$. Therefore, (3.2.1) follows from (3.2.3).

3.2.2 New Setting

The goal of this subsection is to find, for each symplectic vector space V with $\dim(V) = 2n$, a subspace $U_V \subseteq \bigwedge^n V$ such that

(i) The counterexample η_{ex} is *not* contained in U_V;

(ii) The Lagrangian Grassmann cone $\widehat{\text{Gr}_{\text{L}}}(V)$ is a subset of U_V;

(iii) The map Φ_v introduced in (3.1.14) is well-defined on U_V, i.e, $\Phi_v(U_V) \subseteq U_{V_v}$.

We will prove that $\ker(i_\omega) \subseteq \bigwedge^n V$ satisfies all those conditions, where i_ω is the contraction with the symplectic from ω introduced in Definition 3.1.1.

Note that $\text{Sp}(V)$ naturally acts on $\bigwedge^n V$ by $A \cdot (v_1 \wedge \cdots \wedge v_n) = Av_1 \wedge \cdots \wedge Av_n$. Due to the definition of $\text{Sp}(V)$ and i_ω, we have $i_\omega(A\eta) = A \cdot i_\omega(\eta)$ for all $A \in \text{Sp}(V)$ and $\eta \in \bigwedge^n V$. So $A \cdot \eta \in \ker(i_\omega)$ if $\eta \in \ker(i_\omega)$, i.e., the action of $\text{Sp}(V)$ on $\bigwedge^n V$ restricts to an action on $\ker(i_\omega)$. This, in turn, induces an action of the Lie algebra $\mathfrak{sp}(V)$ on $\ker(i_\omega)$.

Even though we will not make use of it, we mention here the following result.

Theorem 3.2.3 (Theorem 17.5 in [FH91])**.** *Let (V, ω) be a symplectic vector space of dimension $2n$. Then, for every $1 \leq k \leq n$, $\ker(i_\omega) \subseteq \bigwedge^k V$ is the k-th irreducible fundamental representation of $\mathfrak{sp}(V)$.*

Equivalently, $\ker(i_\omega)$ is an irreducible $\text{Sp}(V)$-representation. This shows that $\ker(i_\omega)$ is also a natural space to consider.

Since the contraction i_ω is dual to the multiplication map $\omega\wedge\bullet : \bigwedge^{n-2} V^* \to \bigwedge^n V^*$ by (3.1.7), the kernel $\ker(i_\omega)$ can also be described as the orthogonal complement $\text{im}(\omega\wedge\bullet)^\perp$ of the image of the multiplication map.

We now check that $\ker(i_\omega)$ satisfies the conditions above.

Lemma 3.2.4. *The counterexample η_{ex} is not contained in* $\ker(i_\omega)$*.*

Proof. Note that, for all $\beta_1, \beta_2 \in V^*$ and $\xi \in \bigwedge^k V^*$, we have

$$i^*_{\beta_1\wedge\beta_2}(\xi) \overset{(3.1.7)}{=} \beta_1 \wedge \beta_2 \wedge \xi \overset{(3.1.6)}{=} (i^*_{\beta_1} \circ i^*_{\beta_2})(\xi) = (i_{\beta_2} \circ i_{\beta_1})^*(\xi),$$

and hence $i_{\beta_1\wedge\beta_2} = i_{\beta_2} \circ i_{\beta_1}$. Choose a hyperbolic basis $e_1, e_{-1}, \ldots, e_n, e_{-n}$ of V. Then $\omega = \sum_{i=1}^n e_i^\flat \wedge e_{-i}^\flat$ (see Example 3.1.9). Thus, as $\varphi_v = i_{v^\flat}$,

$$i_\omega = \sum_{i=1}^n \varphi_{e_{-i}} \circ \varphi_{e_i}. \tag{3.2.4}$$

From (3.2.1) we know

$$\varphi_{e_i}(\eta_{\text{ex}}) = -me_i \wedge \xi_\omega^{m-1}. \tag{3.2.5}$$

Moreover, from (3.2.2) and (3.2.1) (applied to ξ_ω^{m-1}), we obtain

$$\begin{aligned}
\varphi_{e_{-i}}\big(e_i \wedge \xi_\omega^{m-1}\big) &\overset{(3.2.2)}{=} \varphi_{e_{-i}}(e_i) \wedge \xi_\omega^{m-1} - e_i \wedge \varphi_{e_{-i}}(\xi_\omega^{m-1}) \\
&\overset{(3.2.1)}{=} -\xi_\omega^{m-1} - e_i \wedge \big(-(m-1)e_{-i} \wedge \xi_\omega^{m-2}\big) \\
&= -\xi_\omega^{m-1} + (m-1)e_i \wedge e_{-i} \wedge \xi_\omega^{m-2}.
\end{aligned}$$

Recall that $\xi_\omega = \sum_{i=1}^n e_i \wedge e_{-i}$ and $n = 2m$. So by summing over $i = 1, \dots, n$ we get

$$\sum_{i=1}^{n} \varphi_{e_{-i}}\big(e_i \wedge \xi_\omega^{m-1}\big) = -n\xi_\omega^{m-1} + (m-1)\xi_\omega^{m-1} = -(m+1)\xi_\omega^{m-1}.$$

Combining this with (3.2.4) and (3.2.5) yields

$$i_\omega(\eta_{\mathrm{ex}}) = m(m+1)\xi_\omega^{m-1}.$$

In particular, $i_\omega(\eta_{\mathrm{ex}}) \neq 0$, i.e., $\eta_{\mathrm{ex}} \notin \ker(i_\omega)$. ☐

Lemma 3.2.5. *The Lagrangian Grassmann cone $\widehat{\mathrm{Gr}_{\mathrm{L}}}(V)$ is a subset of* $\ker(i_\omega)$.

Proof. Let $\xi \in \widehat{\mathrm{Gr}_{\mathrm{L}}}(V)$ be arbitrary. By Definition 3.1.5 we can write $\xi = v_1 \wedge \cdots \wedge v_n$ for some $v_1, \dots, v_n \in V$ such that $\omega(v_i, v_j) = 0$ for all $1 \le i, j \le n$. But then the formula (3.1.2) for i_ω immediately yields $i_\omega(\xi) = 0$, i.e., $\xi \in \ker(i_\omega)$. ☐

Finally, we will show that Φ_v is well-defined on $\ker(i_\omega)$.

Lemma 3.2.6. *For every $v \in V$ we have $\Phi_v((\ker(i_\omega)) \subseteq \ker(i_{\omega_v})$, where ω_v is the symplectic form on V_v defined in (3.1.13).*

Proof. Since $\Phi_v = \big(\bigwedge^{n-1} \pi_v\big) \circ \varphi_v$, it suffices to show that the diagram

$$\begin{array}{ccccc} \bigwedge^n V & \xrightarrow{\varphi_v} & \bigwedge^{n-1} v^\perp & \xrightarrow{\bigwedge^{n-1}\pi_v} & \bigwedge^{n-1} V_v \\ {\scriptstyle i_\omega}\big\downarrow & & \big\downarrow{\scriptstyle i_\omega} & & \big\downarrow{\scriptstyle i_{\omega_v}} \\ \bigwedge^{n-2} V & \xrightarrow{\varphi_v} & \bigwedge^{n-3} v^\perp & \xrightarrow{\bigwedge^{k-1}\pi_v} & \bigwedge^{n-3} V_v \end{array}$$

commutes. To this avail, recall $\varphi_v = i_\beta$ for $\beta = v^\flat$ and that, by (3.1.6) and (3.1.7), the dual maps are given by $i^*_\beta(\xi) = \beta \wedge \xi$ and $i^*_\omega(\xi) = \omega \wedge \xi$. So it's easy to see that $i^*_\beta \circ i^*_\omega = i^*_\omega \circ i^*_\beta$, and thus the left square commutes. That the right square commutes follows immediately from (3.1.2) since $\omega(v_i, v_j) = \omega_v(\pi_v(v_i), \pi_v(v_j))$ for all $v_i, v_j \in v^\perp$ by the definition (3.1.13) of ω_v. ☐

3.3 Preliminary Results

In this section we prove the three main technical ingredients for the proof of the Main Theorem 3.4.1.

Proposition 3.3.1. *Let $\eta \in \ker(i_\omega) \subseteq \bigwedge^k V$ for some $2 \le k \le \dim(V)$ and assume $\Phi_v(\eta) = 0$ for all $v \in V$. Then $\eta = 0$.*

The general proof of Proposition 3.3.1 is a bit technical. However, in the case when $2 \le k \le \frac{1}{2}\dim(V)$, we can use Theorem 3.2.3 to give a simpler proof, which we include here for the convenience of the reader.

Proof of Proposition 3.3.1 when $2 \le k \le \frac{1}{2}\dim(V)$. Fix $2 \le k \le \frac{1}{2}\dim(V)$ and define

$$W := \left\{ \eta \in \ker(i_\omega) \subseteq \bigwedge^k V \;\middle|\; \Phi_v(\eta) = 0 \text{ for all } v \in V \right\}.$$

The statement of Proposition 3.3.1 is equivalent to $W = 0$. We will show that W is an $\mathrm{Sp}(V)$-invariant subspace of $\ker(i_\omega)$. Then, since $\ker(i_\omega)$ is an irreducible representation by Theorem 3.2.3, it follows that $W = 0$ or $W = \ker(i_\omega)$. But $W \neq \ker(i_\omega)$ because there clearly exist $\eta \in \ker(i_\omega)$ such that $\Phi_v(\eta) \neq 0$ for some $v \in V$.

It remains to show that W is an $\mathrm{Sp}(V)$-invariant. Because of Lemma 3.1.12 it suffices to show that for all $\eta \in \bigwedge^k V$, $v \in V$, $v_2, \ldots, v_k \in v^\perp$ and $g \in \mathrm{Sp}(V)$ it holds

$$\big(v \wedge v_2 \wedge \cdots \wedge v_k, (g\eta)^\flat\big)_{\mathrm{ev}} = \big(g^{-1}v \wedge g^{-1}v_2 \wedge \cdots \wedge g^{-1}v_k, \eta^\flat\big)_{\mathrm{ev}}. \tag{3.3.1}$$

Towards the proof of (3.3.1) we first observe that for all $u, w \in V$ we have

$$u^\flat(gw) = \omega(u, gw) = \omega(gg^{-1}u, gw) = \omega(g^{-1}u, w) = (g^{-1}u)^\flat(w),$$

where the third equality holds because $g \in \mathrm{Sp}(V)$ preserves ω. As a consequence, for all $\eta, \xi \in \bigwedge^k V$ it holds

$$\big(g\eta, \xi^\flat\big)_{\mathrm{ev}} = \big(\eta, (g^{-1}\xi)^\flat\big)_{\mathrm{ev}}. \tag{3.3.2}$$

Set $\xi := v \wedge v_2 \wedge \cdots \wedge v_k \in \bigwedge^k V$. Together with (3.1.11) we obtain

$$\big(\xi, (g\eta)^\flat\big)_{\mathrm{ev}} \overset{(3.1.11)}{=} (-1)^k \big(g\eta, \xi^\flat\big)_{\mathrm{ev}} \overset{(3.3.2)}{=} (-1)^k \big(\eta, (g^{-1}\xi)^\flat\big)_{\mathrm{ev}} \overset{(3.1.11)}{=} \big(g^{-1}\xi, \eta^\flat\big)_{\mathrm{ev}},$$

which, due to the definition of ξ, is equivalent to (3.3.1). This completes the proof. □

We now come to the general proof of Proposition 3.3.1.

Proof of Proposition 3.3.1. Let $\mathcal{S} \subseteq \bigwedge^k V$ be the span of $\mathcal{S}_1 \cup \mathcal{S}_2$ for the sets

$$\mathcal{S}_1 = \left\{ v \wedge v_2 \wedge \cdots \wedge v_k \in \bigwedge^k V \;\middle|\; v \in V \text{ and } v_2, \ldots, v_k \in v^\perp \right\}$$

and

$$\mathcal{S}_2 = \left\{ \xi_\omega \wedge v_3 \wedge \cdots \wedge v_k \in \bigwedge^k V \;\middle|\; v_3, \ldots, v_k \in V \right\},$$

where the 2-form $\xi_\omega \in \bigwedge^2 V$ is the preimage of ω under the induced musical isomorphism $\flat : \bigwedge^2 V \to \bigwedge^2 V^*$ (see Example 3.1.9).

We will prove Proposition 3.3.1 in two steps. First we will show that $(\xi, \eta^\flat)_{ev} = 0$ for all $\xi \in \mathcal{S}$. Then we will show that the $\mathcal{S} = \bigwedge^k V$. Since the evaluation pairing is non-degenerate, this will imply $\eta^\flat = 0$, which is equivalent to $\eta = 0$.

Step 1 ($\eta^\flat$ vanishes on $\mathcal{S}$): By assumption we have $\Phi_v(\eta) = 0$ for all $v \in V$. By Lemma 3.1.12 this is equivalent to $(v \wedge v_2 \wedge \cdots \wedge v_k, \eta^\flat)_{ev} = 0$ for all $v \in V$ and $v_2, \ldots, v_k \in v^\perp$. Thus, we have

$$\Phi_v(\eta) = 0 \quad \forall\, v \in V \iff (\xi, \eta^\flat)_{\mathrm{ev}} = 0 \quad \forall\, \xi \in \mathcal{S}_1. \tag{3.3.3}$$

Furthermore, $\eta \in \ker(i_\omega)$, meaning $i_\omega(\eta) = 0$, is equivalent to $(i_\omega(\eta), \beta)_{\mathrm{ev}} = 0$ for all $\beta \in \bigwedge^{k-2} V^*$ because the evaluation-pairing is non-degenerate. Moreover, using that i_ω

is dual to wedging with ω by (3.1.7), we get $\eta \in \ker(i_\omega)$ if and only if $(\eta, \omega \wedge \beta)_{\mathrm{ev}} = 0$ for all $\beta \in \bigwedge^{k-2} V^*$. Let $\xi = \xi_\omega \wedge v_3 \wedge \cdots \wedge v_k$. Then, $\xi^\flat = \omega \wedge v_3^\flat \wedge \cdots \wedge v_k^\flat$ since $\xi_\omega^\flat = \omega$. Therefore, by (3.1.11), we get

$$(\xi_\omega \wedge v_3 \wedge \cdots \wedge v_k, \eta^\flat)_{\mathrm{ev}} = (-1)^k (\eta, \omega \wedge v_3^\flat \wedge \cdots \wedge v_k^\flat)_{\mathrm{ev}} = 0.$$

So we obtain

$$\eta \in \ker(i_\omega) \iff (\xi, \eta^\flat)_{\mathrm{ev}} = 0 \quad \forall\, \xi \in \mathcal{S}_2. \tag{3.3.4}$$

Since $(\,\cdot\,, \eta^\flat)_{\mathrm{ev}} = 0$ is a linear condition, we conclude $(\xi, \eta^\flat)_{\mathrm{ev}} = 0$ for all $\xi \in \mathcal{S}$ by combining (3.3.3) and (3.3.4).

Step 2 ($\mathcal{S} = \bigwedge^k V$): We choose a symplectic basis $e_1, e_{-1}, \ldots, e_n, e_{-n}$ for V. It suffices to show that each pure wedge $e_{i_1} \wedge \cdots \wedge e_{i_k}$ is in $\mathcal{S}$.

If there exists $j \in \{\pm 1, \ldots, \pm n\}$ such that $j \in \{i_1, \ldots, i_k\}$ but $-j \notin \{i_1, \ldots, i_k\}$, then clearly $e_{i_1} \wedge \cdots \wedge e_{i_k} \in \mathcal{S}_1 \subseteq \mathcal{S}$ by the definition of $\mathcal{S}_1$. In particular, this always holds if k is odd. So we may from now on assume that $k = 2m$ is even.

It remains to show $e_{j_1} \wedge e_{-j_1} \wedge \cdots \wedge e_{j_m} \wedge e_{-j_m} \in \mathcal{S}$ for $j_1, \ldots, j_m \in \{1, \ldots, n\}$. Abbreviate $e_J = e_{j_2} \wedge e_{-j_2} \wedge \cdots \wedge e_{j_m} \wedge e_{-j_m}$. Define $R := \{1, \ldots, n\} \setminus \{j_1, \ldots, j_m\}$. Recall that $\xi_\omega = e_1 \wedge e_{-1} + \cdots + e_n \wedge e_{-n}$ (see Example 3.1.9). Note that by definition of R and $\mathcal{S}_2$ we have

$$\left(e_{j_1} \wedge e_{-j_1} + \sum_{r \in R} e_r \wedge e_{-r} \right) \wedge e_J = \xi_\omega \wedge e_J \in \mathcal{S}_2. \tag{3.3.5}$$

Moreover, by definition of R and $\mathcal{S}_1$, we also have

$$(e_r + e_{j_1}) \wedge (e_{-r} - e_{-j_1}) \wedge e_J \in \mathcal{S}_1$$

for all $r \in R$. Expanding this expression we obtain

$$(e_r + e_{j_1}) \wedge (e_{-r} - e_{-j_1}) \wedge e_J = (e_r \wedge e_{-r} - e_{j_1} \wedge e_{-j_1}) \wedge e_J + (\text{terms in } \mathcal{S}_2).$$

Thus, we conclude that $(e_r \wedge e_{-r} - e_{j_1} \wedge e_{-j_1}) \wedge e_J \in \mathcal{S}$ for all $r \in R$. Subtracting the sum $\sum_{r \in R} (e_r \wedge e_{-r} - e_{j_1} \wedge e_{-j_1}) \wedge e_J$ from (3.3.5) we get

$$(|R| + 1) e_{j_1} \wedge e_{-j_1} \wedge \cdots \wedge e_{j_m} \wedge e_{-j_m} \wedge e_J \in \mathcal{S}.$$

Since $\mathrm{Char}(\mathbb{K}) = 0$, we obtain $e_{j_1} \wedge e_{-j_1} \wedge \cdots \wedge e_{j_m} \wedge e_{-j_m} \in \mathcal{S}$. This completes the proof. □

We will also need the following more technical version of Proposition 3.3.1.

Proposition 3.3.2. *Let V be a symplectic space of with dimension $2n \geq 4$, $L, L' \subseteq V$ Lagrangian subspaces with $\dim(L \cap L') = n - 1$, and $\eta \in \bigwedge^k V$ such that $\Phi_v(\eta) = 0$ for all $v \in L \cup L'$.*

(i) If $k \leq n - 1$ and $\eta \in \ker(i_\omega)$, then $\eta \in \bigwedge^k (L \cap L')$.

(ii) If $k = n + 1$ and $i_\omega(\eta) \in \bigwedge^{n-1} (L \cap L')$, then $\eta \in \bigwedge^{n+1} (L + L')$.

Proof. By Lemma 3.1.8 there exists a symplectic basis $e_1, e_{-1}, \dots, e_n, e_{-n}$ for V such that

$$L = \langle e_1, \dots, e_n \rangle \quad \text{and} \quad L' = \langle e_{-1}, e_2 \dots, e_n \rangle.$$

We first prove the first statement (i). For this we let $\mathcal{S}$ be the span of $\mathcal{S}_1 \cup \mathcal{S}_2$ for the sets

$$\mathcal{S}_1 = \left\{ v \wedge v_2 \wedge \dots \wedge v_k \in \bigwedge^k V \;\middle|\; v \in L \cup L' \text{ and } v_2, \dots, v_k \in v^\perp \right\}$$

and

$$\mathcal{S}_2 = \left\{ \xi_\omega \wedge v_3 \wedge \dots \wedge v_k \in \bigwedge^k V \;\middle|\; v_i \in V \right\},$$

where, again, $\xi_\omega \in \bigwedge^2 V$ is the preimage of ω under the induced musical isomorphism $\flat : \bigwedge^2 V \to \bigwedge^2 V^*$ (see Example 3.1.9).

If $\Phi_v(\eta) = 0$ for every $v \in L \cup L'$ and $\eta \in \ker(i_\omega)$, then exactly as in Step 1 in the proof of Proposition 3.3.1 we deduce

$$(\xi, \eta^\flat)_{\text{ev}} = 0 \quad \text{for all } \xi \in \mathcal{S}. \tag{3.3.6}$$

Note that $e_2, \dots, e_n$ is a basis for $L \cap L'$. So, in terms of the basis e_J ($|J| = k$) for $\bigwedge^k V$, the desired conclusion $\eta \in \bigwedge^k (L \cap L')$ is equivalent to the fact that η is a linear combination of those e_J with $J \subseteq \{2, \dots, n\}$. Due to Example 3.1.9 this holds if and only if $\eta^\flat \in \bigwedge^k V^*$ is a linear combination of those ε^I with $I \subseteq \{-2, \dots, -n\}$, where $\varepsilon^1, \varepsilon^{-1}, \dots, \varepsilon^n, \varepsilon^{-n} \in V^*$ is the dual basis. In other words, the desired conclusion holds if and only if the ε^I-coefficient of $\eta^\flat$ is zero whenever I is *not* contained in $\{-2, \dots, -n\}$. By Example 3.1.2 the ε^I-coefficient of $\eta^\flat$ is $(e_I, \eta^\flat)_{\text{ev}}$. So $\eta \in \bigwedge^k (L \cap L')$ is equivalent to

$$(e_{i_1} \wedge \dots \wedge e_{i_k}, \eta^\flat)_{\text{ev}} = 0 \tag{3.3.7}$$

if at least one of the indices $i_1, \dots, i_k$ is equal to $-1, 1, 2, \dots n$. Therefore, combining Equation (3.3.6) and Equation (3.3.7) it suffices to show that

$$e_I \coloneqq e_{i_1} \wedge \dots \wedge e_{i_k} \in \mathcal{S} \quad \text{if} \quad \{i_1, \dots, i_k\} \cap \{-1, 1, 2, \dots, n\} \neq \varnothing.$$

We will prove this using a case analysis where we distinguish if e_I contains only singles, i.e., if $\pm i \in I$, then $\mp i \notin I$, or also pairs $e_j \wedge e_{-j}$.

Case 1 (The wedge e_I only contains singles): By assumption we know that at least one of the indices $i_1, \dots, i_k$ is contained in $\{-1, 1, 2, \dots, n\}$. We denote this index by j. Then, by choosing $v = e_j$, we obtain $e_I \in \mathcal{S}$ by definition of $\mathcal{S}_2$. This proves Case 1.

Case 2 (The wedge e_I contains at least one pair): We may assume that $e_{i_1} \wedge e_{-i_1}$ is the pair occurring in e_I, and write the wedge as$e_{i_1} \wedge e_{-i_1} \wedge e_{i_3} \wedge \dots \wedge e_{i_k}$ and set $e_J \coloneqq e_{i_3} \wedge \dots \wedge e_{i_k}$. Set

$$R \coloneqq \left\{ r \in \{1, \dots, \widehat{i_1}, \dots, p\} \;\middle|\; r, -r \notin \{i_3, \dots, i_k\} \right\}.$$

By the definition of R and $\mathcal{S}_1$ the terms

$$(e_r + e_{i_1}) \wedge (e_{-r} - e_{-i_1}) \wedge e_J \tag{3.3.8}$$

are contained in $\mathcal{S}$ for all $r \in R$. Expanding the term in 3.3.8 we obtain

$$(e_r + e_{i_1}) \wedge (e_{-r} - e_{-i_1}) \wedge e_J = (e_r \wedge e_{-r} - e_{i_1} \wedge e_{-i_1}) \wedge e_J + (\text{terms in } \mathcal{S}_1),$$

and hence we conclude that for all $r \in R$

$$(e_r \wedge e_{-r} - e_{i_1} \wedge e_{-i_1}) \wedge e_J \in \mathcal{S}. \tag{3.3.9}$$

Recall $\xi_\omega = \varepsilon^1 \wedge \varepsilon^{-1} + \cdots + \varepsilon^n \wedge \varepsilon^{-n}$ (see Example 3.1.9). Using the definition of R and $\mathcal{S}_2$, we obtain

$$\left(e_{i_1} \wedge e_{-i_1} + \sum_{r \in R} e_r \wedge e_{-r} \right) \wedge e_J = \xi_\omega \wedge e_J \in \mathcal{S}_2. \tag{3.3.10}$$

Therefore, subtracting $\sum_{r \in R}(e_r \wedge e_{-r} - e_{i_1} \wedge e_{-i_1}) \wedge e_J$ from (3.3.10) and keeping in mind (3.3.9), shows that

$$(|R| + 1) e_{i_1} \wedge e_{-i_1} \wedge e_J \in \mathcal{S}.$$

Since $\mathrm{Char}(\mathbb{K}) = 0$, also $e_{i_1} \wedge e_{-i_1} \wedge e_J \in \mathcal{S}$. This proves Case 2.

Next we will prove the second statement (ii). Let $\mathcal{S}$ be the span of $\mathcal{S}_1 \cup \mathcal{S}_2$ for the sets

$$\mathcal{S}_1 = \left\{ v \wedge v_2 \wedge \cdots \wedge v_{n+1} \in \bigwedge^{n+1} V \;\middle|\; v \in L \cup L' \text{ and } v_2, \ldots, v_{n+1} \in v^\perp \right\}$$

and

$$\mathcal{S}_2 = \left\{ \xi_\omega \wedge e_{j_3} \wedge \cdots \wedge e_{j_{n+1}} \in \bigwedge^{n+1} V \;\middle|\; \{j_3, \ldots, j_{n+1}\} \neq \{-2, \ldots, -n\} \right\}.$$

Similar to before, it suffices to show that $e_I = e_{i_1} \wedge \cdots \wedge e_{i_{n+1}}$ is contained in $\mathcal{S}$ whenever $I \neq \{1, -1, -2, \ldots, -n\}$. Fix such an I. Since $|I| = n + 1$, there exists $i \in \{1, \ldots, n\}$ such that $i, -i \in I$. So we can write $e_I = e_i \wedge e_{-i} \wedge e_J$ where $J = I \setminus \{i, -i\}$. Note that $J \neq \{-2, \ldots, -n\}$ since $I \neq \{1, -1, \ldots, -n\}$. Thus, $\xi_\omega \wedge e_J \in \mathcal{S}_2$. Now the arguments from Case 2 in the proof of (i) apply without any changes. $\square$

We choose a symplectic basis $e_1, e_{-1}, \ldots, e_n, e_{-n}$ of V and define V' as $\langle e_n, e_{-n} \rangle^\perp = \langle e_1, e_{-1}, \ldots, e_{n-1}, e_{-(n-1)} \rangle$. Any $\eta \in \bigwedge^k V$ can uniquely be written as

$$\eta = \eta_1 \wedge e_n \wedge e_{-n} + \eta_2 \wedge e_n + \eta_3 \wedge e_{-n} + \eta_4, \tag{3.3.11}$$

where $\eta_1 \in \bigwedge^{n-2} V', \eta_2, \eta_3 \in \bigwedge^{n-1} V'$ and $\eta_4 \in \bigwedge^n V'$. The next Proposition characterizes in terms of $\eta_1, \eta_2, \eta_3, \eta_4$ when η is in the Lagrangian Grassmann cone.

Proposition 3.3.3. *Suppose we have written $\eta \in \bigwedge^k V$ as in (3.3.11), and assume that $\eta \in \widehat{\mathrm{Gr}_{\mathrm{L}}}(V)$. Then one of the following holds:*

(a) It holds

$$\eta_1 = 0, \quad \eta_2, \eta_3 \in \widehat{\mathrm{Gr}_{\mathrm{L}}}(V') \quad \text{and} \quad \eta_4 = 0.$$

Moreover, η_2 and η_3 are multiples of each other, that is, either $\eta_2 = \lambda \eta_3$ or $\eta_3 = \lambda \eta_2$ for some $\lambda \in \mathbb{K}$.

(b) The $\eta_1, \eta_2, \eta_3, \eta_4$ are all nonzero. Moreover, it holds

$$\eta_1 \in \widehat{\mathrm{Gr}}_{\mathrm{iso}}(n-2, V'), \quad \eta_2, \eta_3 \in \widehat{\mathrm{Gr}}_{\mathrm{L}}(V'), \quad \eta_4 \in \widehat{\mathrm{Gr}}(n, V'),$$

$$L_{\eta_2} \cap L_{\eta_3} = L_{\eta_1} \quad \textit{and} \quad L_{\eta_2} + L_{\eta_3} = L_{\eta_4}.$$

Proof. We abbreviate $L' = L_\eta \cap V'$. Note that $\dim(L') \geq \dim(L_\eta) - 2 = n-2$ because $V' \subseteq V$ has codimension 2. Moreover, since $L' \subseteq V'$ is isotropic, we also know that $\dim(L') \leq \frac{\dim(V')}{2} = n-1$. Thus, $n-1 \geq \dim(L') \geq n-2$. We proceed by considering cases based on the dimension of L'. More precisely, we will show that the statement in (a) holds if $\dim(L') = n-1$, and the statement in (b) holds if $\dim(L') = n-2$.

Case 1 ($\dim(L') = n-1$*):* Then $L' \subseteq V'$ is Lagrangian. Moreover, $L' \subsetneq L_\eta$ has codimension one, and so there exists a vector $v \in L_\eta$ such that

$$L_\eta = L' \oplus \langle v \rangle. \tag{3.3.12}$$

Below we will show that in fact

$$\text{the exists a vector } w \in \langle e_n, e_{-n} \rangle \text{ such that } L_\eta = L' \oplus \langle w \rangle. \tag{3.3.13}$$

Then, if $v_1, \dots, v_{n-1}$ is a basis for L', it holds $L_\eta = L' \oplus \langle w \rangle = \langle v_1, \dots, v_{n-1}, w \rangle$. Therefore, as $w = ae_{-n} + be_n$ for some $a, b \in \mathbb{K}$, we can conclude (up to constants)

$$\eta = v_1 \wedge \cdots \wedge v_{n-1} \wedge w = (av_1 \wedge \cdots \wedge v_{n-1}) \wedge e_{-n} + (bv_1 \wedge v_{n-1}) \wedge e_n.$$

Hence $\eta_1, \eta_4 = 0, \eta_2 = bv_1 \wedge \cdots \wedge v_{n-1}$ and $\eta_3 = av_1 \wedge \cdots \wedge v_{n-1}$. This proves the statement in (a) of Proposition 3.3.3.

It remains to prove the claim in (3.3.13). We write $v = v' + w$ with $v' \in V'$ and $w \in \langle e_{-n}, e_n \rangle$. It suffices to show $v' \in L'$, because then

$$L_\eta = L' \oplus \langle v \rangle = L' \oplus \langle v' + w \rangle = L' \oplus \langle w \rangle.$$

Because $L' \subseteq V'$ is Lagrangian, it holds

$$L' = \{ u \in V' \mid \omega(u, \ell) = 0 \text{ for all } \ell \in L' \} = (L')^\perp \cap V',$$

where the second equality holds due to the definition of $(L')^\perp$. Therefore, since $v' \in V'$ by definition, it suffices to show $v' \in (L')^\perp$.

Keeping in mind that L_η is Lagrangian, (3.3.12) shows that $v \perp L'$, i.e., $v \in (L')^\perp$. Moreover, $w \perp L'$ because $w \in \langle e_n, e_{-n} \rangle$, $L' \subseteq V'$ and $\langle e_n, e_{-n} \rangle \perp V'$. Since $v' = v - w$, this shows that $v' \in (L')^\perp$, completing the proof (3.3.13).

Case 2 ($\dim(L') = n-2$*):* In this case we can write $L_\eta = \langle e_n + u, e_{-n} + v \rangle \oplus L'$ for some $u, v \in V'$. Choose a basis $v_1, \dots, v_{n-2}$ of L', and set $\eta_1 := v_1 \wedge \cdots \wedge v_{n-2}$. Then it holds (up to scalars)

$$\begin{aligned} \eta &= \eta_1 \wedge (e_n + u) \wedge (e_{-n} + v) \\ &= \eta_1 \wedge e_n \wedge e_{-n} - (\eta_1 \wedge v) \wedge e_n + (\eta_1 \wedge u) \wedge e_{-n} + \eta_1 \wedge u \wedge v, \end{aligned}$$

so that $\eta_2 = -\eta_1 \wedge v$, $\eta_3 = \eta_1 \wedge u$ and $\eta_4 = \eta_1 \wedge u \wedge v$. Thus, it will suffice to show that $v_1, v_2, \dots, v_{n-2}, u, v$ are linearly independent. Indeed, if they are independent, then

$\eta_1, \eta_2, \eta_3, \eta_4$ are all nonzero elements of the Grassmann cone and its easy to see that the identities $L_{\eta_2} \cap L_{\eta_3} = L_{\eta_1}$ and $L_{\eta_2} + L_{\eta_3} = L_{\eta_4}$ are true. As we already know that $L_{\eta_1} = L'$ is isotropic, in order to see that L_{η_2} and L_{η_3} are Lagrangian, it suffices to check $u, v \perp L'$. However, this follows from the fact that for all $\ell \in L'$ it holds

$$\omega(u, \ell) = \omega(e_n, \ell) + \omega(u, \ell) = \omega(e_n + u, \ell) = 0, \tag{3.3.14}$$

and similarly for v. Here the first equality holds because $L' \subseteq V' \perp e_n$ and the last one because $\langle e_n + u, e_{-n} + v\rangle \oplus L' = L_\eta$ is Lagrangian by assumption.

Arguing by contradiction we assume that $v_1, v_2, \ldots, v_{n-2}, u, v$ linearly dependent, so that we can write $v = \lambda u + v'$ for some $\lambda \in \mathbb{K}$ and $\ell \in L'$. But then

$$\begin{aligned}\omega(e_n + u, e_{-n} + v) &= \omega(e_n + u, e_{-n} + \lambda u + \ell)\\ &= \omega(e_n, e_{-n}) + \omega(e_n, \lambda u + \ell) + \omega(u, e_{-n}) + \lambda\omega(u, u) + \omega(u, \ell)\\ &= 1 + 0 + 0 + 0 + 0 = 1,\end{aligned}$$

contradicting the fact that $e_n + u, e_{-n} + v \in L_\eta$ and that L_η is Lagrangian. Here we used that $\omega(e_n, \lambda u + \ell) = 0 = \omega(u, e_{-n})$ since $\lambda u + \ell, u \in V' \perp e_n, e_{-n}$, $\omega(u, u) = 0$ by the anti-symmetry of ω, and $\omega(u, \ell) = 0$ by (3.3.14). This shows that $v_1, v_2, \ldots, v_{n-2}, u, v$ linearly independent, completing the proof of Case 2. □

3.4 Universality Result for the Lagrangian Grassmannian

In this section we prove the first main result of the article, which we state here again for the convenience of the reader.

Theorem 3.4.1. *Let (V, ω) be a symplectic vector space of dimension $2n \geq 6$, and let $\eta \in \ker(i_\omega) \subseteq \bigwedge^n V$. If $\Phi_v(\eta) \in \widehat{\mathrm{Gr}}_{\mathrm{L}}(V_v)$ for all $v \in V$, then $\eta \in \widehat{\mathrm{Gr}}_{\mathrm{L}}(V)$.*

Remark 3.4.2. The Main Theorem 3.4.1 is wrong if $\dim(V) = 4$. Indeed, if $\dim(V) = 4$, then V_v is a two dimensional symplectic space, and thus $\widehat{\mathrm{Gr}}_{\mathrm{L}}(V_v) = \bigwedge^1 V_v = V_v$. So $\Phi_v(\eta) \in \widehat{\mathrm{Gr}}_{\mathrm{L}}(V_v)$ for every $\eta \in \ker(i_\omega)$. But there clearly exist $\eta \in \ker(i_\omega) \subseteq \bigwedge^2 V$ that are not in $\widehat{\mathrm{Gr}}_{\mathrm{L}}(V)$.

If V, W are symplectic spaces of respective dimensions $2n$ and $2m$, a *Lagrangian Grassmann cone preserving map (LGCP map) from V to W* is a linear map

$$\Phi : \ker(i_{\omega_V}) \subseteq \bigwedge^n V \to \ker(i_{\omega_W}) \subseteq \bigwedge^m W$$

such that $\Phi\left(\widehat{\mathrm{Gr}}_L(V)\right) \subseteq \widehat{\mathrm{Gr}}_{\mathrm{L}}(W)$. For example, for any $v \in V$, the maps Φ_v are an LGCP map from V to V_v (see Example 1.1.3(2) in Chapter 1).

As an immediate corollary we get that the Lagrangian Grassmann cone can set-theoretically be defined by pullbacks of the Lagrangian Grassmann cone $\widehat{\mathrm{Gr}}_{\mathrm{L}}(\mathbb{K}^4)$ along all maps that preserve the Lagrangian Grassmann cone (compare with [KRPS08, Theorem 3.4]).

Corollary 3.4.3 (Universality for the Lagrangian Grassmannian)**.** *Let (V,ω) be a symplectic space of dimension $2n$ and $\eta \in \ker(i_\omega) \subseteq \bigwedge^n V$. Then the following are equivalent.*

1. $\eta \in \widehat{\mathrm{Gr}}_{\mathrm{L}}(V)$;
2. $\Phi(\eta) \in \widehat{\mathrm{Gr}}_{\mathrm{L}}(\mathbb{K}^4)$ *for every LGCP map from V to $\mathbb{K}^4$.*

Throughout this entire section we assume that $\eta \in \ker(i_\omega) \subseteq \bigwedge^n V$ satisfies the assumption in the Main Theorem 3.4.1. Moreover, we can without loss of generality assume that $\eta \neq 0$.

We will again write η as in (3.3.11). Namely, we choose a symplectic pair $e_n, e_{-n} \in V$ and write $V' = \langle e_n, e_{-n}\rangle^\perp$. Then η can uniquely be written as

$$\eta = \eta_1 \wedge e_n \wedge e_{-n} + \eta_2 \wedge e_n + \eta_3 \wedge e_{-n} + \eta_4, \tag{3.4.1}$$

where $\eta_1 \in \bigwedge^{n-2} V', \eta_2, \eta_3 \in \bigwedge^{n-1} V'$ and $\eta_4 \in \bigwedge^n V'$. Note that η then has one of the following zero patterns, where "$*$" stands for "nonzero":

	η_1	η_2	η_3	η_4	η_1	η_2	η_3	η_4	
(0)	0	0	0	0	$*$	0	0	0	(8)
(1)	0	0	0	$*$	$*$	0	0	$*$	(9)
(2)	0	0	$*$	0	$*$	0	$*$	0	(10)
(3)	0	0	$*$	$*$	$*$	0	$*$	$*$	(11)
(4)	0	$*$	0	0	$*$	$*$	0	0	(12)
(5)	0	$*$	0	$*$	$*$	$*$	0	$*$	(13)
(6)	0	$*$	$*$	0	$*$	$*$	$*$	0	(14)
(7)	0	$*$	$*$	$*$	$*$	$*$	$*$	$*$	(15)

We will split the proof of the Main Theorem 3.4.1 into different lemmas which are based on these zero patterns.

We will often apply Proposition 3.3.1 and Proposition 3.3.2 to η_1, η_2, η_3 or η_4. To check the assumptions in Proposition 3.3.1 and Proposition 3.3.2 we will make use of the following lemma.

Lemma 3.4.4. *For $\eta \in \bigwedge^n V$ consider the decomposition in (3.4.1). Then $\eta \in \ker(i_\omega)$ if and only if $\eta_1 \in \ker(i_\omega) \subseteq \bigwedge^{n-2} V'$, $\eta_2, \eta_3 \in \ker(i_\omega) \subseteq \bigwedge^{n-1} V'$ and $i_\omega(\eta_4) = -\eta_1$.*

Proof. Since $\eta_1 \in \bigwedge^{n-2} V'$ and $e_n, e_{-n} \perp V'$, it follows from the formula (3.1.2) for i_ω that

$$\begin{aligned} i_\omega(\eta_1 \wedge e_n \wedge e_{-n}) =& i_\omega(\eta_1) \wedge e_n \wedge e_{-n} + (-1)^{(n-1)+n-1}\omega(e_n, e_{-n})\eta_1 \\ =& i_\omega(\eta_1) \wedge e_n \wedge e_{-n} + \eta_1. \end{aligned}$$

Similarly, one checks $i_\omega(\eta_2 \wedge e_n) = i_\omega(\eta_2) \wedge e_n$ and $i_\omega(\eta_3 \wedge e_{-n}) = i_\omega(\eta_3) \wedge e_{-n}$. Hence

$$i_\omega(\eta) = i_\omega(\eta_1) \wedge e_n \wedge e_{-n} + i_\omega(\eta_2) \wedge e_n + i_\omega(\eta_3) \wedge e_{-n} + (i_\omega(\eta_4) + \eta_1).$$

From this the desired equivalence easily follows. □

Moreover, we will frequently apply Proposition 3.3.3 to $\Phi_v(\eta)$. To do so the following observation will be useful. For its formulation, note that for $v \in V'$ there is a canonical isomorphism $V_v \cong (V')_v \oplus \langle e_n, e_{-n}\rangle$.

Observation 3.4.5. *Let η be as in (3.4.1) and $v \in V'$. Then, under the canonical isomorphism $V_v = V'_v \oplus \langle e_n, e_{-n} \rangle$, it holds*

$$\Phi_v(\eta) = \Phi_v(\eta_1) \wedge e_n \wedge e_{-n} + \Phi_v(\eta_2) \wedge e_n + \Phi_v(\eta_3) \wedge e_{-n} + \Phi_v(\eta_4). \tag{3.4.2}$$

After these preliminary observations we can now start with the proof of Theorem 3.4.1. First, we will rule out all the non highlighted zero patterns.

Lemma 3.4.6. *η cannot have zero pattern* $(1), (3), (5), (7)$, *or* $(8) - (14)$.

Proof. We will first show that if $\eta_1 \neq 0$, then η_2, η_3 and η_4 are also nonzero. So η cannot have zero patterns $(8) - (14)$.

Let $\eta_1 \neq 0$. Since $\eta \in \ker(i_\omega)$, by Lemma 3.4.4 also $\eta_1 \in \ker(i_\omega) \subseteq \bigwedge^{n-2} V'$. But then according to Proposition 3.3.1, there exists $v \in V'$ such that $\Phi_v(\eta_1) \neq 0$. Therefore, applying Proposition 3.3.3(b) to

$$\Phi_v(\eta) = \Phi_v(\eta_1) \wedge e_n \wedge e_{-n} + \Phi_v(\eta_2) \wedge e_n + \Phi_v(\eta_3) \wedge e_{-n} + \Phi_v(\eta_4)$$

we can conclude that $\Phi_v(\eta_1), \Phi_v(\eta_2), \Phi_v(\eta_3)$ and $\Phi_v(\eta_4)$ are nonzero. This then implies that η_2, η_3 and η_4 are nonzero as well.

We may assume $\eta_1 = 0$ from now on. We will show that then $\eta_4 = 0$, that is, η can not have zero patterns $(1), (3), (5)$ or (7). Arguing by contradiction assume $\eta_4 \neq 0$. By Lemma 3.4.4 $i_\omega(\eta_4) = -\eta_1$. By assumption $\eta_1 = 0$ and therefore clearly $\eta_4 \in \ker(i_\omega)$. Similar as before, we may invoke Proposition 3.3.1 to find $v \in V'$ such that $\Phi_v(\eta_4) \neq 0$, and then apply Proposition 3.3.3 to $\Phi_v(\eta)$ to conclude that $\Phi_v(\eta_i) \neq 0$ for $i = 1, 2, 3, 4$. Therefore also $\eta_i \neq 0$ for $i = 1, 2, 3, 4$. But this contradicts the assumption $\eta_1 = 0$. □

So we have shown that η has one of the highlighted zero patterns. Next, we will prove the Main Theorem 3.4.1 if η has zero pattern (2) or (4).

Lemma 3.4.7. *If η has zero pattern (2) or (4), then $\eta \in \widehat{\mathrm{Gr}}_{\mathrm{L}}(V)$.*

Proof. Let η have zero pattern (2). Then, $\eta = \eta_3 \wedge e_{-n}$. Choose $v = e_n$ and note that then there is a canonical identification $V_v = V'$. Using the formula (3.1.15) for Φ_v, we obtain $\Phi_{e_n}(\eta) = (-1)^{n-1}\eta_3$, which by assumption lies in $\widehat{\mathrm{Gr}}_{\mathrm{L}}(V')$. Therefore, $\eta = \eta_3 \wedge e_{-n} \in \widehat{\mathrm{Gr}}_{\mathrm{L}}(V)$ since $e_{-n} \perp V'$. If η has zero pattern (4) we proceed analogously using $v = e_{-n}$. □

So we are left to show that the Main Theorem 3.4.1 holds if η has zero pattern (6) or (15). In particular, η_2 and η_3 are nonzero. We saw in the proof of Lemma 3.4.7 that choosing $v = e_{\pm n}$ implies that $\eta_2, \eta_3 \in \widehat{\mathrm{Gr}}_{\mathrm{L}}(V')$. So $L_{\eta_2}, L_{\eta_3} \subseteq V'$ are well-defined Lagrangian subspaces. In order to prove the Main Theorem 3.4.1, we will first show that $n - 2 \leq \dim(L_{\eta_2} \cap L_{\eta_3}) \leq n - 1$.

Lemma 3.4.8. *If η has zero pattern (6) or (15), then* $\dim(L_{\eta_2} \cap L_{\eta_3}) \geq n - 2$.

Proof. We choose $v = e_n + e_{-n}$ and consider $\varphi_v(\eta_1 \wedge e_n \wedge e_{-n})$ where $\eta_1 \in \bigwedge^{n-2} V'$. Using the formula (3.1.12) for φ_v we get

$$\begin{aligned}
\varphi_v(\eta_1 \wedge e_n \wedge e_{-n}) &= (-1)^{(n-1)-1}\omega(v, e_n)\eta_1 \wedge e_{-n} + (-1)^{n-1}\omega(v, e_{-n})\eta_1 \wedge e_n \\
&= (-1)^{n-1}\eta_1 \wedge (e_{-n} + e_n), \\
&= (-1)^{n-1}\eta_1 \wedge v
\end{aligned}$$

since the first $(n-2)$ summands in (3.1.12) are zero because $v \perp V'$. After projecting to V_v this will become zero, i.e., we get $\Phi_v(\eta_1 \wedge e_n \wedge e_{-n}) = 0$. Similarly, one can show $\Phi_v(\eta_2 \wedge e_n) = -(-1)^{n-1}\eta_2$, $\Phi_v(\eta_3 \wedge e_{-n}) = (-1)^{n-1}\eta_3$ and $\Phi_v(\eta_4) = 0$, where we implicitly use the canonical identification $V_v = V'$. So by assumption we get

$$\widehat{\mathrm{Gr}_{\mathrm{L}}}(V') \ni \Phi_v(\eta) = (-1)^n \eta_2 + (-1)^{n-1}\eta_3.$$

Thus, η_2, η_3 and $\eta_2 - \eta_3$ are all contained in $\widehat{\mathrm{Gr}_{\mathrm{L}}}(V')$. But then by Lemma 3.1.4 we have that

$$\dim(L_{\eta_2} \cap L_{\eta_3}) \geq (n-1) - 1 = n-2.$$

This completes the proof. □

We can now complete the proof of the Main Theorem 3.4.1.

Lemma 3.4.9. *If η has zero pattern (6) or (15), then $\eta \in \widehat{\mathrm{Gr}_{\mathrm{L}}}(V)$.*

Proof. Because of Lemma 3.4.8 we know $\dim(L_{\eta_2} \cap L_{\eta_3}) \in \{n-2, n-1\}$. We will make a case distinction. The proof will show that if $\dim(L_{\eta_2} \cap L_{\eta_3}) = n-1$, then η has zero pattern (6), and if $\dim(L_{\eta_2} \cap L_{\eta_3}) = n-2$, then η has zero pattern (15).

Case 1 ($\dim(L_{\eta_2} \cap L_{\eta_3}) = n-1$*):* Then η_2 and η_3 are multiples of each other. Let $v \in V'$ be arbitrary. Then, by Observation 3.4.5 we have

$$\Phi_v(\eta) = \Phi_v(\eta_1) \wedge e_n \wedge e_{-n} + \Phi_v(\eta_2) \wedge e_n + \Phi_v(\eta_3) \wedge e_{-n} + \Phi_v(\eta_4),$$

which by assumption lies in $\widehat{\mathrm{Gr}_{\mathrm{L}}}(V_v)$. Since $\Phi_v(\eta_2)$ and $\Phi_v(\eta_3)$ are multiples of each other, Proposition 3.3.3 shows that $\Phi_v(\eta_1) = 0$ and $\Phi_v(\eta_4) = 0$ for all $v \in V'$. As $\eta_1 \in \ker(i_\omega) \subseteq \bigwedge^{n-2} V'$ by Lemma 3.4.4, it follows from Proposition 3.3.1 that $\eta_1 = 0$. Consequently, $i_\omega(\eta_4) = -\eta_1 = 0$ due to Lemma 3.4.4, and thus also $\eta_4 = 0$ thanks to Proposition 3.3.1.

Since η_2 and η_3 are multiples of each other, and because $\eta_1 = 0$ and $\eta_4 = 0$, we can write $\eta = \eta_2 \wedge e_n + \eta_3 \wedge e_{-n} = \eta_2 \wedge (e_n + \lambda e_{-n})$ for some $\lambda \in \mathbb{K}$. Therefore, as $\eta_2 \in \widehat{\mathrm{Gr}_{\mathrm{L}}}(V')$ and $e_n + \lambda e_{-n} \perp V'$, we conclude $\eta \in \widehat{\mathrm{Gr}_{\mathrm{L}}}(V)$. This completes the proof of Case 1.

Case 2 ($\dim(L_{\eta_2} \cap L_{\eta_3}) = n-2$*):* Observe that for every $v \in L_{\eta_2} \cup L_{\eta_3}$ either $\Phi_v(\eta_2)$ or $\Phi_v(\eta_3)$ is zero. Thus applying Proposition 3.3.3 to

$$\Phi_v(\eta) = \Phi_v(\eta_1) \wedge e_n \wedge e_{-n} + \Phi_v(\eta_2) \wedge e_n + \Phi_v(\eta_3) \wedge e_{-n} + \Phi_v(\eta_4)$$

shows that $\Phi_v(\eta_1) = 0$ and $\Phi_v(\eta_4) = 0$ for all $v \in L_{\eta_2} \cup L_{\eta_3}$. Therefore, and due to Lemma 3.4.4, we can apply Proposition 3.3.2(i) to $\eta_1 \in \bigwedge^{(n-1)-1} V'$ to conclude that $\eta_1 \in \bigwedge^{(n-1)-1}(L_{\eta_2} \cap L_{\eta_3})$. Consequently, $i_\omega(\eta_4) = -\eta_1 \in \bigwedge^{(n-1)-1}(L_{\eta_2} \cap L_{\eta_3})$ by Lemma 3.4.4. Hence, $\eta_4 \in \bigwedge^{(n-1)+1}(L_{\eta_2} + L_{\eta_3})$ follows from Proposition 3.3.2(ii).

By Lemma 3.1.8 there exists a symplectic basis $e_1, e_{-1}, \ldots, e_{n-1}, e_{-(n-1)}$ of V' such that

$$L_{\eta_2} = \langle e_1, e_2, \ldots, e_{n-1} \rangle \quad \text{and} \quad L_{\eta_3} = \langle e_{-1}, e_2, \ldots, e_{n-1} \rangle.$$

Set $W := \langle e_1, e_{-1}, e_n, e_{-n} \rangle$. It follows from the above and the decomposition (3.4.1) that we can write

$$\eta = e_2 \wedge \cdots \wedge e_{n-1} \wedge \xi$$

for some $\xi \in \bigwedge^2 W$. Choosing $v = e_{-(n-1)}$ shows that

$$\widehat{\mathrm{Gr}}_{\mathrm{L}}(V_v) \ni \Phi_v(\eta) = e_2 \wedge \cdots \wedge e_{n-2} \wedge \xi,$$

where we implicitly use the canonical identification $V_v = \langle \{e_i, e_{-i}\}_{i \neq n-1} \rangle$ (this uses the assumption $n \geq 3$ from the Main Theorem 3.4.1). This implies $\xi \in \widehat{\mathrm{Gr}}_{\mathrm{L}}(W)$. Therefore, $\eta \in \widehat{\mathrm{Gr}}_{\mathrm{L}}(V)$ because $W \perp V'$. This completes the proof of Case 2. □

3.5 Topological Noetherianity of the Dual $(\ker_\infty)^*$

3.5.1 Statement of the Noetherianity Result

Let again $\mathbb{K}$ be a field of characteristic zero. For each $n \in \mathbb{N}$ we denote by V_n the symplectic vector space with a fixed symplectic basis $e_1, e_{-1}, \ldots, e_n, e_{-n}$. For ease of notation we will abbreviate the spaces $\ker(i_\omega)$ introduced in Section 3.2.2 by

$$\ker_n := \ker(i_\omega) \subseteq \bigwedge^n V_n.$$

Observe that the embedding

$$m_{e_{n+1}} : \bigwedge^n V_n \longrightarrow \bigwedge^{n+1} V_{n+1},\ \eta \mapsto \eta \wedge e_{n+1}$$

restricts to an embedding

$$\ker_n \hookrightarrow \ker_{n+1}.$$

Indeed, for all $\eta \in \ker_n$ we have $i_\omega(\eta \wedge e_{n+1}) = i_\omega(\eta) \wedge e_{n+1} = 0 \wedge e_{n+1} = 0$, i.e., $\eta \wedge e_{n+1} \in \ker_{n+1}$, since e_{n+1} is orthogonal to V_n. So we can define the direct limit

$$\begin{aligned} \ker_\infty &:= \varinjlim_n \ker_n \\ &:= \varinjlim_n \left(\ker_1 \xrightarrow{m_{e_2}} \ker_2 \xrightarrow{m_{e_3}} \ker_3 \xrightarrow{m_{e_4}} \cdots \right). \end{aligned}$$

Explicitly, this is the set of infinite wedges

$$\ker_\infty = \left\{ \eta_n \wedge e_{n+1} \wedge e_{n+2} \wedge \cdots \;\middle|\; \eta_n \in \ker_n \text{ and } n \in \mathbb{N} \right\}.$$

Next, we want to define the *infinite symplectic group* $\mathrm{Sp}(V_\infty)$. The obvious embedding $V_n \hookrightarrow V_{n+1}$ induces an inclusion $\mathrm{Sp}(V_n) \hookrightarrow \mathrm{Sp}(V_{n+1})$ by sending a matrix $A \in \mathrm{Sp}(V_n)$ to

$$A' = \begin{pmatrix} A & & \\ & 1 & \\ & & 1 \end{pmatrix} \in \mathrm{Sp}(V_{n+1}),$$

where the order of the basis on V_{n+1} is $e_1, e_{-1}, \ldots, e_{n+1}, e_{-(n+1)}$. Taking the direct limit over these inclusions we define

$$V_\infty := \varinjlim_n V_n$$

and

$$\mathrm{Sp}(V_\infty) := \varinjlim_n \mathrm{Sp}(V_n).$$

The actions of $\mathrm{Sp}(V_n)$ on $\ker_n$ introduced in Section 3.2.2 induce an action of $\mathrm{Sp}(V_\infty)$ on $\ker_\infty$, and hence also on the dual space $(\ker_\infty)^*$. Since every $\ker_n$ is $\mathrm{Sp}(V_n)$-irreducible (Theorem 3.2.3), $\ker_\infty$ is an irreducible $\mathrm{Sp}(V_\infty)$-representation.

For Theorem 3.5.1 below, it will be important to consider $(\ker_\infty)^*$ not just as a vector space, but as an affine scheme. For this we recall that for any field $\mathbb{K}$ (not necessarily algebraically closed) and any $\mathbb{K}$-vector space W (not necessarily finite dimensional) there are canonical identifications

$$W^* = \mathrm{Spec}\left(\mathrm{Sym}(W)\right)(\mathbb{K}) \subseteq \left\{\text{closed points in } \mathrm{Spec}\left(\mathrm{Sym}(W)\right)\right\}.$$

So $\mathrm{Spec}\left(\mathrm{Sym}(W)\right)$ can be seen as an enrichment of W^* to an affine scheme. From now on we will, for any vector space W, denote by W^* the affine scheme $\mathrm{Spec}(\mathrm{Sym}(W))$.

We can now finally state our Noetherianity result.

Theorem 3.5.1 (Noetherianity)**.** *The dual* $(\ker_\infty)^*$ *of the* $\mathrm{Sp}(V_\infty)$*-representation* $\ker_\infty$ *is topologically* $\mathrm{Sp}(V_\infty)$*-Noetherian. That is, every descending chain*

$$(\ker_\infty)^* \supseteq X_1 \supseteq X_2 \supseteq X_3 \supseteq \cdots$$

of $\mathrm{Sp}(V_\infty)$*-stable closed subsets stabilizes.*

Given that any closed subset X of an affine scheme corresponds uniquely to a reduced closed subscheme X_{red} whose underlying topological space is X (see [GW10, Proposition 3.52]), we could express Theorem 3.5.1 equivalently by stating that any chain of $\mathrm{Sp}(V_\infty)$-stable reduced closed subschemes in $(\ker_\infty)^*$ stabilizes.

3.5.2 Proof Strategy for the Noetherianity Result

In this subsection, we will outline the strategy we will use in order to prove Theorem 3.5.1. More details will be given in Section 3.8.

Let $X \subseteq (\ker_\infty)^*$ be a closed subset. We denote by $\delta_X \in \{0, 1, 2, \ldots, \infty\}$ the lowest degree of a nonzero polynomial in the radical ideal $I_X \subseteq \mathrm{Sym}(\ker_\infty)$ defining X. It will suffice to show that any X with $\delta_X < \infty$ is topologically $\mathrm{Sp}(V_\infty)$-Noetherian. We will proceed by induction on δ_X to show that this holds. So we assume $0 < \delta_X < \infty$ and that all $\mathrm{Sp}(V_\infty)$-stable closed subsets Y with $\delta_Y < \delta_X$ are $\mathrm{Sp}(V_\infty)$-Noetherian.

We choose a polynomial $p \in I_X$ with $\deg(p) = \delta_X$ and assume that a specific variable (which we will explain in more detail in Section 3.7.4) e_I is a variable of p. We then consider the formal partial derivative

$$q = \frac{\partial p}{\partial e_I}$$

and define

$$Y := V(\mathrm{Sp}(V_\infty) \cdot q) \quad \text{and} \quad Z := X[1/q],$$

where $X[1/q]$ is the open subset of X where q is nonzero. Since q has degree at most $\delta_X - 1$, we have $\delta_Y < \delta_X$ and hence the closed Y defined by the orbit $\mathrm{Sp}(V_\infty) \cdot q$ is $\mathrm{Sp}(V_\infty)$-Noetherian by the induction hypothesis.

The main effort to prove Theorem 3.5.1 will be to show that, for some $m \in \mathbb{N}$ large enough, Z is topologically $\mathrm{SL}(E_{-\infty})_m$-Noetherian, where the precise definition of the subgroups $\mathrm{SL}(E_{-\infty})_m \leqslant \mathrm{Sp}(V_\infty)$ will be given in Section 3.7.1.

Proposition 3.5.2. *For all large enough $m \in \mathbb{N}$ the open subset $Z := X[1/q] \subseteq X$ is $\mathrm{SL}(E_{-\infty})_m$-stable and topologically $\mathrm{SL}(E_{-\infty})_m$-Noetherian.*

To prove that X is topologically $\mathrm{Sp}(V_\infty)$-Noetherian take a chain

$$X \supseteq X_1 \supseteq X_2 \supseteq \cdots$$

of $\mathrm{Sp}(V_\infty)$-stable closed subsets in X. It will follow from the definition of Y and Z that

$$X_i = (Y \cap X_i) \cup \mathrm{Sp}(V_\infty) \cdot (Z \cap X_i).$$

Since Y and Z are both topologically Noetherian relative to suitable groups, the chains $(Y \cap X_i)_{i \in \mathbb{N}} \subseteq Y$ and $(Z \cap X_i)_{i \in \mathbb{N}} \subseteq Z$ will stabilize. Consequently, the chain $(X_i)_{i \in \mathbb{N}} \subseteq X$ stabilizes, showing that X is topologically $\mathrm{Sp}(V_\infty)$-Noetherian.

3.6 Preliminaries for the Proof of Proposition 3.5.2

In this section we collect some elementary preliminary results that will be needed in the forthcoming discussion.

3.6.1 An Equivariant Isomorphism

For any vector space W of dimension n the Hodge $*$-isomorphism identifies $\bigwedge^{n-k} W$ with $\bigwedge^k W^*$. We can combine this with the musical isomorphism $\flat$ as follows.

Consider V_n as introduced in Section 3.5, with E_n and E_{-n} denoting the Lagrangian subspaces $\langle e_1, \ldots, e_n \rangle$ and $\langle e_{-1}, \ldots, e_{-n} \rangle$ respectively. Note that the musical isomorphism $\flat : V \to V^*$ defined in (3.1.9) induces an isomorphism $\flat : \bigwedge^k V \to \bigwedge^k V^*$, which further restricts to an isomorphism $\flat : \bigwedge^k E_n \to \bigwedge^k (E_{-n})^*$ (see Example 3.1.9). That is, for each $\xi \in \bigwedge^k E_n$ we have $\xi^\flat \in \bigwedge^k (E_{-n})^*$. Note that the pairing

$$\bigwedge^k E_n \times \bigwedge^{n-k} E_n \xrightarrow{\wedge} \bigwedge^n E_n \cong \mathbb{K}$$

and the evaluation pairing $(\cdot,\cdot)_{\mathrm{ev}} : \bigwedge^k E_{-n} \times \bigwedge^k (E_{-n})^* \longrightarrow \mathbb{K}$ are both non-degenerate. Thus there exists, for each $0 \le k \le n$, an isomorphism

$$\phi : \bigwedge^{n-k} E_n \xrightarrow{\cong} \bigwedge^k E_{-n}$$

that for all $\eta \in \bigwedge^{n-k} E_n$ and $\xi \in \bigwedge^k E_n$ satisfies

$$\xi \wedge \eta = (\phi(\eta), \xi^\flat)_{\mathrm{ev}} \cdot e_{[n]}, \tag{3.6.1}$$

where $e_{[n]} := e_1 \wedge \cdots \wedge e_n \in \bigwedge^n E_n$ is the canonical generator.

Example 3.6.1. The isomorphism ϕ satisfies

$$\phi(e_{k+1} \wedge \cdots \wedge e_n) = e_{-1} \wedge \cdots \wedge e_{-k}.$$

More generally, for any $I \subseteq \{1, \ldots, n\}$ with $|I| = n-k$ we have $\phi(e_I) = \pm e_{-I^c}$ for an appropriate sign $\pm$. This can be checked using Example 3.1.2 and Example 3.1.9.

The following twisted equivariance property of ϕ will be central.

Lemma 3.6.2. *Let $A \in \mathrm{Sp}(V_n)$ be such that $A(E_n) \subseteq E_n$ and $A(E_{-n}) \subseteq E_{-n}$. Then for all $\eta \in \bigwedge^{n-k} E_n$ we have*

$$\phi(A\eta) = \det(A|_{E_n}) A\phi(\eta).$$

Proof. Since the evaluation-paring is non-degenerate, it suffices to show that for every $\xi \in \bigwedge^k E_n$ we have

$$(\phi(A\eta), \xi^\flat)_{\mathrm{ev}} = \det(A|_{E_n})(A\phi(\eta), \xi^\flat)_{\mathrm{ev}},$$

or equivalently,

$$(\phi(A\eta), \xi^\flat)_{\mathrm{ev}} \cdot e_{[n]} = \det(A|_{E_n})(A\phi(\eta), \xi^\flat)_{\mathrm{ev}} \cdot e_{[n]}, \tag{3.6.2}$$

where $e_{[n]} = e_1 \wedge \cdots \wedge e_n$. By the defining property (3.6.1) of ϕ we can write the left hand side of (3.6.2) as

$$(\phi(A\eta), \xi^\flat)_{\mathrm{ev}} \cdot e_{[n]} = \xi \wedge A\eta$$

Recall from (3.3.2) that for all $A \in \mathrm{Sp}(V_n)$ and $\eta, \xi \in \bigwedge^k V$ we have

$$(A\eta, \xi^\flat)_{\mathrm{ev}} = (\eta, (A^{-1}\xi)^\flat)_{\mathrm{ev}}. \tag{3.6.3}$$

Combining this with (3.6.1) and (3.6.3) we obtain for the right hand side of (3.6.2)

$$\begin{aligned} \det(A|_{E_n})(A\phi(\eta), \xi^\flat)_{\mathrm{ev}} \cdot e_{[n]} &\overset{(3.6.3)}{=} \det(A|_{E_n})(\phi(\eta), (A^{-1}\xi)^\flat)_{\mathrm{ev}} \cdot e_{[n]} \\ &\overset{(3.6.1)}{=} \det(A|_{E_n})\big(A^{-1}\xi \wedge \eta\big), \end{aligned}$$

By the definition of the determinant, and as $A(E_n) \subseteq E_n$, the action of A on $\bigwedge^n E_n$ is just multiplication with $\det(A|_{E_n})$. Hence

$$\det(A|_{E_n})\big(A^{-1}\xi \wedge \eta\big) = A \cdot \big(A^{-1}\xi \wedge \eta\big) = \xi \wedge A\eta.$$

Therefore, both sides of (3.6.2) equal $\xi \wedge A\eta$. This completes the proof. □

3.6.2 The Root System of $\mathfrak{sp}(V_n)$

In this subsection we recall some standard facts about the root system of $\mathfrak{sp}(V_n)$ and use this as an opportunity to fix notation. We refer the reader to [FH91, Section 16.1] for more details.

In the following we will write $E_{k,\ell}$ for the endomorphism of V_n mapping e_ℓ to e_k and all other basis vectors to zero. The elements

$$H_i := E_{i,i} - E_{-i,-i} \quad (1 \leq i \leq n)$$

span the Cartan subalgebra $\mathfrak{h}_n$ of $\mathfrak{sp}(V_n)$. We use the notation $L_1, \dots, L_n \in \mathfrak{h}_n^*$ for the dual basis.

The root system of $\mathfrak{sp}(V_n)$ is then as follows. For $1 \leq i, j \leq n$

$$X_{i,j} = E_{i,j} - E_{-j,-i} \in \mathfrak{sp}(V_n) \tag{3.6.4}$$

is a root vector for the root $L_i - L_j \in \mathfrak{h}_n^*$. Moreover, for $i \neq j$

$$Y_{i,j} = E_{i,-j} + E_{j,-i} \quad \text{and} \quad Z_{i,j} = E_{-i,j} + E_{-j,i}$$

are root vectors for the roots $L_i + L_j \in \mathfrak{h}_n^*$ and $-L_i - L_j \in \mathfrak{h}_n^*$, respectively. Finally,

$$U_i = E_{i,-i} \quad \text{and} \quad V_i = E_{-i,i} \tag{3.6.5}$$

are root vectors for the roots $2L_i \in \mathfrak{h}_n^*$ and $-2L_i \in \mathfrak{h}_n^*$, respectively.

We define a functional $\ell : \mathfrak{h}_n^* \to \mathbb{K}$ by $\ell(\sum_{i=1}^n a_i L_i) = c_1 a_1 + \dots + c_n a_n$, where $c_1 > \dots > c_n > 0$ is a fixed choice of constants (this is *not* the same as in [FH91]). Then the positive roots are

$$R^+ = \{L_i + L_j\}_{i \geq j} \cup \{L_i - L_j\}_{i > j}. \tag{3.6.6}$$

3.7 Z is Topologically Noetherian

The main goal of this section is to prove Proposition 3.5.2. However, this requires some preliminary work. The main ingredients for the proof are Proposition 3.7.2 and Lemma 3.7.5. For the sake of clarity we will split these two results and their proofs into separate subsections. Before we can come to these proofs, we however first have to make some definitions.

Throughout this entire section let V_n be as in Section 3.5 and denote by E_n and E_{-n} the Lagrangian subspaces $\langle e_1, \dots, e_n \rangle$ and $\langle e_{-1}, \dots, e_{-n} \rangle$.

3.7.1 The Groups $\mathrm{SL}(E_{-\infty})_m$

In this section we define the subgroups $\mathrm{SL}(E_{-\infty})_m$ of $\mathrm{Sp}(V_\infty)$ appearing in Proposition 3.5.2.

Let $E_{-\infty}$ be the direct limit along the canonical inclusions $E_{-n} \to E_{-(n+1)}$, i.e., $\varinjlim_n E_{-n}$. Analogous to the definition of $\mathrm{Sp}(V_\infty)$ we also define $\mathrm{GL}(E_{-\infty})$ as the direct limit $\varinjlim_n \mathrm{GL}(E_{-n})$ along the canonical inclusions. For each $m \in \mathbb{N}$ we define $GL(E_{-\infty})_m$ to be the image in $\mathrm{GL}(E_{-\infty})$ of the *shifting by m embedding*

$$\mathrm{GL}(E_{-\infty}) \to \mathrm{GL}(E_{-\infty}), A \mapsto \begin{pmatrix} 1_m & 0 \\ 0 & A \end{pmatrix},$$

where 1_m is the $(m \times m)$-identity matrix, and the 0's stand for the zero $(m \times \infty)$- and $(\infty \times m)$-matrices. The groups $\mathrm{SL}(E_{-\infty})$ and $\mathrm{SL}(E_{-\infty})_m$ are defined analogously.

We are left to explain how we see these groups as subgroups of $\mathrm{Sp}(V_\infty)$. For this fix $n \in \mathbb{N}$. The group $\mathrm{GL}(E_{-n})$ embedds into $\mathrm{Sp}(V_n)$ via

$$g \longmapsto \begin{pmatrix} g^{-t} & 0 \\ 0 & g \end{pmatrix} \tag{3.7.1}$$

where the order of the basis of V_n is $e_1, \dots, e_n, e_{-1}, \dots, e_{-n}$. This induces the embedding of Lie algebras $\mathfrak{gl}(E_{-n}) \hookrightarrow \mathfrak{sp}(V_n)$,

$$L \longmapsto \begin{pmatrix} -L^t & 0 \\ 0 & L \end{pmatrix}.$$

We will always identify $\mathrm{GL}(E_{-n})$ and $\mathfrak{gl}(E_{-n})$ with their images in $\mathrm{Sp}(V_n)$ and $\mathfrak{sp}(V_n)$. Note that the root vectors $X_{i,j}$ defined in (3.6.4) span $\mathfrak{gl}(E_{-n}) \subseteq \mathfrak{sp}(V_\infty)$.

We obtain embeddings $\mathrm{GL}(E_{-\infty}) \hookrightarrow \mathrm{Sp}(V_\infty)$ and $\mathfrak{gl}(E_{-\infty}) \hookrightarrow \mathfrak{sp}(V_\infty)$ by taking the direct limit over these embeddings. Again, we will always identify $\mathrm{GL}(E_{-\infty})$ and $\mathfrak{gl}(E_{-\infty})$ with their images under these embeddings. We also use this embedding to see $\mathrm{SL}(E_{-\infty})_m$ as subgroups of $\mathrm{Sp}(V_\infty)$.

3.7.2 The Spaces $(\ker_\infty)_{\leq k}$

Fix $n \in \mathbb{N}$. Since $V_n = E_n \oplus E_{-n}$ we have the decomposition

$$\bigwedge^n V_n = \bigoplus_{k=0}^{n} \left(\bigwedge^k E_{-n} \otimes \bigwedge^{n-k} E_n \right),$$

where $\bigwedge^k E_{-n} \otimes \bigwedge^{n-k} E_n$ is the span of all wedges $e_{i_1} \wedge \cdots \wedge e_{i_n}$ with exactly k of the indices $i_1, \dots, i_n$ being negative. For each $0 \leq k \leq n$ we define the subspace $(\ker_n)_k \subseteq \ker_n$ as the space of all $\eta \in \ker_n$ that can be written as a sum of wedges with k negative indexed vectors, i.e.,

$$(\ker_n)_k := (\ker_n) \cap \left(\bigwedge^k E_{-n} \otimes \bigwedge^{n-k} E_n \right).$$

Similarly, we define

$$(\ker_n)_{\leq k} := \bigoplus_{i=0}^{k} (\ker_n)_i.$$

Due to the definition (3.7.1) of the embedding $\mathrm{GL}(E_{-n}) \subseteq \mathrm{Sp}(V_n)$ all the spaces $(\ker_n)_k$ are $\mathrm{GL}(E_{-n})$-stable. In fact, we have the following, explaining the representation theoretic significance of these spaces.

Lemma 3.7.1. *The space* $\ker_n$ *decomposes as*

$$\ker_n = \bigoplus_{k=0}^{n} (\ker_n)_k,$$

and for all $0 \leq k \leq n$ *the space* $(\ker_n)_k$ *is an irreducible* $\mathfrak{sl}(E_{-n})$*-representation with highest weight vector* $e_{-1} \wedge \cdots \wedge e_{-k} \wedge e_{k+1} \wedge \cdots \wedge e_n$.

Proof. According to [FH91, p. 261], the decomposition of $\ker_n \subseteq \bigwedge^n V_n$ into irreducible representations of $\mathfrak{sl}_n$ is given by $\ker_n = \bigoplus_{b=0}^{n} W^{(n-b,b)}$, where $W^{(n-b,b)}$ is the irreducible $\mathfrak{sl}_n$-representation with highest weight vector (in the convention of [FH91], which is opposite to ours)

$$w^{(n-b,b)} := e_1 \wedge \cdots \wedge e_{n-b} \wedge e_{-(n-b+1)} \wedge \cdots \wedge e_{-n}.$$

Clearly, $w^{(n-b,b)} \in (\ker_n)_b$, and hence $W^{(n-b,b)} \subseteq (\ker_n)_b$ since $(\ker_n)_b$ is $\mathfrak{sl}_n$-stable. Since the subspaces $(\ker_n)_b$ are pairwise disjoint, this implies $W^{(n-b,b)} = (\ker_n)_b$. Therefore, $(\ker_n)_b$ is $\mathfrak{sl}(E_{-n})$-irreducible and $\ker_n = \bigoplus_{b=0}^{n}(\ker_n)_b$.

It remains to check that $e_{-1} \wedge \cdots \wedge e_{-k} \wedge e_{k+1} \wedge \cdots \wedge e_n$ is the highest weight vector of $(\ker_n)_k$. By (3.6.6) the positive root vectors of $\mathfrak{gl}(E_{-n}) \subseteq \mathfrak{sp}(V_n)$ are the $X_{i,j}$ defined in (3.6.4) with $i > j$. But one can easily check that, for all $1 \leq j < i \leq n$, the action of $X_{i,j}$ sends $e_{-1} \wedge \cdots \wedge e_{-k} \wedge e_{k+1} \wedge \cdots \wedge e_n$ to zero. This implies the desired result. □

The structure maps $m_{e_{n+1}} : \ker_n \to \ker_{n+1}$ map $(\ker_n)_k$ into $(\ker_{n+1})_k$. Thus we can take the direct limit and define

$$(\ker_\infty)_k := \varinjlim_n (\ker_n)_k.$$

Then every subspace $(\ker_\infty)_k$ is $\mathrm{GL}(E_{-\infty})$-stable and $\mathfrak{sl}(E_{-\infty})$-irreducible with highest weight vector $e_{-1} \wedge \cdots \wedge e_{-k} \wedge e_{k+1} \wedge e_{k+2} \wedge \cdots$. Moreover,

$$\ker_\infty = \bigoplus_{k=0}^{\infty} (\ker_\infty)_k.$$

Finally, we define

$$(\ker_\infty)_{\leq k} = \bigoplus_{i=0}^{k} (\ker_\infty)_i.$$

3.7.3 First Key Ingredient

The goal of this subsection is to prove the following proposition, which is the first key result towards the proof of Proposition 3.5.2.

Proposition 3.7.2. *The dual* $(\ker_\infty)^*_{\leq k}$ *is topologically* $\mathrm{SL}(E_{-\infty})_m$*-Noetherian for all* $k, m \in \mathbb{N}$.

The main technical ingredient of the proof will be Draisma's main result in [Dra19] about the topological Noetherianity of finite degree polynomial functors. But we have to do some preliminary work before we can apply it.

For each $n \in \mathbb{N}$ and $0 \leq k \leq n$ the spaces $(\ker_n)_k$ are by definition subspaces of $\bigwedge^k E_{-n} \otimes \bigwedge^{n-k} E_n$. Recall from Section 3.6.1 that we have, for every $0 \leq k \leq n$, an isomorphism

$$\bigwedge^{n-k} E_n \xrightarrow{\cong} \bigwedge^k E_{-n}.$$

From Lemma 3.6.2 and the definition (3.7.1) of the embedding $\mathrm{SL}(E_{-n}) \subseteq \mathrm{GL}(E_{-n})$ it follows that this isomorphism is $\mathrm{SL}(E_{-n})$-equivariant. Moreover, this isomorphism is compatible with the structure maps, i.e., the diagram

$$\begin{array}{ccc} \bigwedge^{n-k} E_n & \xrightarrow{\cong} & \bigwedge^k E_{-n} \\ {\scriptstyle m_{e_{n+1}}}\downarrow & & \downarrow \\ \bigwedge^{n+1-k} E_{n+1} & \xrightarrow{\cong} & \bigwedge^k E_{-(n+1)} \end{array}$$

commutes, where the horizontal map on the right is induced by the obvious inclusion $E_{-n} \subseteq E_{-(n+1)}$ (see Example 3.6.1). Therefore, we obtain $\mathrm{SL}(E_{-n})$-equivariant embeddings

$$(\ker_n)_k \hookrightarrow \bigwedge^k E_{-n} \otimes \bigwedge^k E_{-n}$$

that are compatible with the structure maps. Taking the direct limit we obtain an $\mathrm{SL}(E_{-\infty})$-equivariant embedding

$$(\ker_\infty)_k \hookrightarrow \bigwedge^k E_{-\infty} \otimes \bigwedge^k E_{-\infty}.$$

By taking the direct sum we also obtain an $\mathrm{SL}(E_{-\infty})$-equivariant embedding

$$\varphi : (\ker_\infty)_{\leq k} \hookrightarrow \bigoplus_{i=0}^{k} \bigwedge^i E_{-\infty} \otimes \bigwedge^i E_{-\infty} =: W_{-\infty}^{\leq k}. \tag{3.7.2}$$

It follows from Lemma 3.7.3(1) and Lemma 3.7.4 below that the dual U^* of the image $U := \mathrm{Im}(\varphi) \subseteq W_{-\infty}^{\leq k}$ is topologically Noetherian.

Lemma 3.7.3. *The following statements hold:*

1. *A subspace $U \subseteq W_{-\infty}^{\leq k}$ is $\mathrm{SL}(E_{-\infty})$-stable if and only if it is $\mathrm{GL}(E_{-\infty})$-stable.*

2. *An ideal $I \subseteq \mathrm{Sym}(W_{-\infty}^{\leq k})_m$ is $\mathrm{SL}(E_{-\infty})$-stable if and only if it is $\mathrm{GL}(E_{-\infty})$-stable.*

The analogous statements with $\mathrm{SL}(E_{-\infty})_m$ and $\mathrm{GL}(E_{-\infty})_m$ hold as well.

Proof. Since the proof of both statements are similar, we only prove the second statement. Assume that $I \subseteq \mathrm{Sym}(W_{-\infty}^{\leq k})$ is $\mathrm{SL}(E_{-\infty})$-stable. Let $f \in I$ and $A \in \mathrm{GL}(E_{-\infty})$ be arbitrary. Choose $n = n(f, A) \in \mathbb{N}$ large enough so that $f \in \mathrm{Sym}(\bigoplus_{i=0}^{k} \bigwedge^i E_{-n} \otimes \bigwedge^i E_{-n})$ and A is the image of some $A_{-n} \in \mathrm{GL}(E_{-n})$. Define $A_{-(n+1)} \in \mathrm{SL}(E_{-(n+1)})$ as the map given by $A_{-(n+1)}(e_i) = A_{-n}(e_i)$ for $-n \leq i \leq -1$ and $A_{-(n+1)}(e_{-(n+1)}) = (\det(A_{-n}))^{-1}(e_{-(n+1)})$, and let A' be the image of $A_{-(n+1)}$ in $\mathrm{SL}(E_{-\infty})$. Then the action of A_{-n} and $A_{-(n+1)}$ agree on E_{-n}. Hence they also agree on $\mathrm{Sym}(\bigoplus_{i=0}^{k} \bigwedge^i E_{-n} \otimes \bigwedge^i E_{-n})$. Therefore, $A \cdot f = A' \cdot f \in I$ because I was assumed to be $\mathrm{SL}(E_{-\infty})$-stable and $A' \in \mathrm{SL}(E_{-\infty})$. As $f \in I$ and $A \in \mathrm{GL}(E_{-\infty})$ were arbitrary, this shows that I is $\mathrm{GL}(E_{-\infty})$-stable. □

Lemma 3.7.4. *If $U \subseteq W_{-\infty}^{\leq k}$ is $\mathrm{GL}(E_{-\infty})_m$–stable then its dual U^* is topologically $\mathrm{GL}(E_{-\infty})_m$–Noetherian.*

Proof. We start by showing that $(W_{-\infty}^{\leq k})^*$ is topologically $\mathrm{GL}(E_{-\infty})_m$–Noetherian. For every $k, m \in \mathbb{N}$ consider the functor $F_{k,m} : \mathrm{Vec}_{\mathbb{K}} \to \mathrm{Vec}_{\mathbb{K}}$ defined by

$$F_{k,m}(V) = \bigoplus_{i=0}^{k} \left(\bigwedge^k (\mathbb{K}^m \oplus V) \otimes \bigwedge^k (\mathbb{K}^m \oplus V) \right).$$

This is a polynomial functor of finite degree. So by applying Draisma's Noetherianity result in [Dra19] we deduce that the dual of the direct limit $\varinjlim_n F_{k,m}(\mathbb{K}^n)$ is topologically GL_∞-Noetherian, where $\mathrm{GL}_\infty := \varinjlim_n \mathrm{GL}(\mathbb{K}^n)$. But, up to the canonical isomorphism

$\mathrm{GL}(E_{-\infty})_m \cong \mathrm{GL}(E_{-\infty})$, the action of $\mathrm{GL}(E_{-\infty})_m$ on $W_{-\infty}^{\leq k}$ is the same as the action of GL_∞ on $\varinjlim_n F_{k,m}(\mathbb{K}^n)$. Therefore, $(W_{-\infty}^{\leq k})^*$ is topologically $\mathrm{GL}(E_{-\infty})_m$-Noetherian.

The rest follows from the following general claim. Let G be a group and W a G-vector space such that W^* is topologically G-Noetherian. Then, for any G-stable subspace $U \subseteq W$, the dual U^* is also topologically G-Noetherian.

To prove this claim we first observe that the map $i_U^* : W^* \to U^*$ induced by the inclusion $i_U : U \hookrightarrow W$ is G-equivariant and surjective. Indeed, as U is a G-stable subspace, the inclusion i_U is clearly G-equivariant, and thus the induced map i_U^* is also G-equivariant. To prove surjectivity choose a projection $\pi : W \to U$ such that $\pi|_U = \mathrm{id}_U$, or equivalently, so that the diagram

$$\overbrace{U \xrightarrow{\;i_U\;} W \xrightarrow{\;\pi\;} U}^{\mathrm{id}_U}$$

commutes (π does *not* need to be G-equivariant). Applying the contravariant functor $\mathrm{Spec}(\mathrm{Sym}(\cdot))$ we get the commutative diagram

$$\overbrace{U^* \xleftarrow{\;i_U^*\;} W^* \xleftarrow{\;\pi^*\;} U^*}^{\mathrm{id}_{U^*}} \;,$$

which implies that i_U^* is surjective.

Now take a chain $(X_i)_{i\in\mathbb{N}} \subseteq U^*$ of G-stable closed subsets. For all $i \in \mathbb{N}$ we set $\tilde{X}_i := (i_U^*)^{-1}(X_i) \subseteq W^*$. Then $(\tilde{X}_i)_{i\in\mathbb{N}} \subseteq W^*$ is a chain of G-stable closed subsets because i_U^* is G-equivariant and continuous. As W^* is topologically G-Noetherian, we deduce that the chain $(\tilde{X}_i)_{i\in\mathbb{N}}$ stabilizes. But, due to the surjectivity of i_U^*, we have $X_i = i_U^*(\tilde{X}_i)$ for all $i \in \mathbb{N}$, and hence the chain $(X_i)_{i\in\mathbb{N}}$ also stabilizes. This proves that U^* is topologically G-Noetherian. □

We can now prove Proposition 3.7.2

Proof of Proposition 3.7.2. We define $U := \varphi((\ker_\infty)_{\leq k})$ where φ is the embedding mentioned in (3.7.2). Then, because φ is an $\mathrm{SL}(E_{-\infty})$-equivariant embedding, we have a $\mathrm{SL}(E_{-\infty})$-equivariant isomorphism $(\ker_\infty)_{\leq k} \cong U$ and hence also $((\ker_\infty)_{\leq k})^* \cong U^*$. In particular, $U \subseteq W_{-\infty}^{\leq k}$ is $\mathrm{SL}(E_{-\infty})$-stable, and therefore, by Lemma 3.7.3, it is also $\mathrm{GL}(E_{-\infty})$-stable. Lemma 3.7.4 then implies that the dual U^* is topologically $GL(E_{-\infty})_m$-Noetherian for every $m \in \mathbb{N}$.

Now take a chain $(X_i)_{i\in\mathbb{N}} \subseteq ((\ker_\infty)_{\leq k})^*$ of $\mathrm{SL}(E_{-\infty})_m$-stable closed subsets. Since there is an $\mathrm{SL}(E_{-\infty})$-equivariant isomorphism $((\ker_\infty)_{\leq k})^* \cong U^*$, this corresponds to a chain $(\tilde{X}_i)_{i\in\mathbb{N}} \subseteq U^*$ of $\mathrm{SL}(E_{-\infty})_m$-stable closed subsets. Note that by the second part of Lemma 3.7.3 every element of the chain is in fact $\mathrm{GL}(E_{-\infty})_m$-stable. Using that, by the above paragraph, U^* is topologically $\mathrm{GL}(E_{-\infty})_m$-Noetherian, we deduce that the chain $(\tilde{X}_i)_{i\in\mathbb{N}}$ stabilizes. Consequently, the chain $(X_i)_{i\in\mathbb{N}}$ stabilizes. As the chain $(X_i)_{i\in\mathbb{N}}$ was arbitrary, this proves that $((\ker_\infty)_{\leq k})^*$ is $\mathrm{SL}(E_{-\infty})_m$-Noetherian. □

3.7.4 Second Key Ingredient

In this subsection we prove the second key ingredient for the proof of Proposition 3.5.2. Before we can formulate it, we need to fix some notation.

For a $\mathrm{Sp}(V_\infty)$-stable closed subset $X \subseteq (\ker_\infty)^*$ let $\delta_X \in \{0, 1, 2, \ldots, \infty\}$ be the lowest degree of a nonzero polynomial in the radical ideal $I_X \subseteq \mathrm{Sym}(\ker_\infty)$.

Assume that X is such that $\delta_X < \infty$. Choose a (nonzero) polynomial $p \in I_X$ of minimal degree δ_X. Consider the smallest integer k such that $p \in \mathrm{Sym}((\ker_\infty)_{\leq k})$, and fix $n > k$ such that $p \in \mathrm{Sym}((\ker_n)_{\leq k})$. In words this means that the variables of the polynomial p have at most k negative-indexed vectors and every variable in p is of the form $\eta_n \wedge e_{n+1} \wedge e_{n+2} \wedge \cdots$ for some $\eta_n \in \ker_n$. Because $(\ker_\infty)_k$ is an irreducible $\mathfrak{sl}(E_{-\infty})$-representation with highest weight vector $e_{I_k} := e_{-1} \wedge \cdots \wedge e_{-k} \wedge e_{k+1} \wedge e_{k+2} \wedge \cdots$, and since I_X is $\mathfrak{sp}(V_\infty)$-stable, we may without loss of generality assume that p contains the variable e_{I_k}. Set $p_k := p$. We define $q \in \mathrm{Sym}(\ker_\infty)$ as the formal partial derivative

$$q = \frac{\partial p_k}{\partial e_{I_k}}.$$

Observe that for $m > n$ the action of $\mathrm{SL}(E_{-\infty})_m$ fixes q. Indeed, this follows from the fact that for $m > n$ the groups $\mathrm{SL}(E_{-\infty})_m$ act trivially on V_n and the fact that the action of $g \in \mathrm{SL}(E_{-\infty})_m$ on $e_{n+1} \wedge e_{n+2} \wedge \cdots$ is just multiplication with $\det(g^{-1}) = 1$.

Recall that $(\ker_\infty)_{\leq k}$ is $\mathrm{GL}(E_{-\infty})$-stable subspace of $\ker_\infty$. So, $\mathrm{Sym}((\ker_\infty)_{\leq k})[1/q]$ is an $\mathrm{SL}(E_{-\infty})_m$-stable subring of the localization ring $\mathrm{Sym}(\ker_\infty)[1/q]$. In particular, the map

$$i^* : (\ker_\infty)^*[1/q] \longrightarrow (\ker_\infty)^*_{\leq k}[1/q]$$

induced by the inclusion is $\mathrm{SL}(E_{-\infty})_m$-equivariant. Also, if we define $Z \subseteq (\ker_\infty)^*[1/q]$ as the open subset of X where q does not vanish, i.e.,

$$Z := X[1/q],$$

then Z is $\mathrm{SL}(E_{-\infty})_m$-stable.

We can now formulate the second key ingredient for the proof of Proposition 3.5.2. The first part just summarizes the above discussion. The key part is the second statement.

Lemma 3.7.5. *For every $m > n$ the map*

$$i^* : (\ker_\infty)^*[1/q] \longrightarrow (\ker_\infty)^*_{\leq k}[1/q]$$

induced by the inclusion is $\mathrm{SL}(E_{-\infty})_m$-equivariant. Moreover, the restriction of i^* to Z is a closed embedding.

Lemma 3.7.6. *There exists $p_n \in I_X$ of the form*

$$p_n = e_{I_n} q + r_n,$$

where $e_{I_n} = e_{-1} \wedge \cdots \wedge e_{-n} \wedge e_{n+1} \wedge \cdots$, $q = \frac{\partial p_k}{\partial e_{I_k}}$ and $r_n \in \mathrm{Sym}((\ker_\infty)_{\leq n-1})$.

Proof. During the proof we will abbreviate $e_{I_\ell} := e_{-1} \wedge \cdots \wedge e_{-\ell} \wedge e_{\ell+1} \wedge e_{\ell+2} \wedge \cdots$. By (3.6.5) the maps V_i sending e_i to e_{-i} (and all other basis vectors to zero) are contained in $\mathfrak{sp}(V_n)$. Note that

$$(V_n \circ \cdots \circ V_{k+1})(e_{I_k}) = e_{I_n},$$

where by abuse of notation we write $(V_n \circ \cdots \circ V_{k+1})(e_{I_k})$ for the successive action $V_n(V_{n-1}(\cdots(V_{k+1}(e_{I_k}))\cdots))$ of $\mathfrak{sp}(V_n)$ on e_{I_k}.

First, we act with V_{k+1} on p_k. Since X is assumed to be $\mathrm{Sp}(V_\infty)$-stable, the resulting polynomial p_{k+1} is again in the ideal I_X. Moreover, p_{k+1} has the form

$$p_{k+1} := V_{k+1}(p_k) = e_{I_{k+1}} q + r_{k+1},$$

where $q = \frac{\partial p_k}{\partial e_{I_k}}$ and $r_{k+1} \in \mathrm{Sym}((\ker_\infty)_{\leq k+1})$. To illustrate why p_{k+1} is of this form, we consider the following example.

Example 3.7.7. Let $e_J \in (\ker_\infty)_{\leq k}$ be a variable different from e_{I_k}, e.g.,

$$e_J = e_{-1} \wedge e_{-2} \wedge e_3 \wedge e_{-4} \wedge \cdots \wedge e_{-k} \wedge e_{k+1} \wedge e_{k+2} \wedge e_{-(k+3)} \wedge e_{k+4} \wedge \cdots .$$

Let p_k be the polynomial $p_k = (e_{I_k})^2 e_J$. Then, by the Leibniz-Rule, we get for any $L \in \mathfrak{sp}(V_n)$

$$\begin{aligned}
L(p_k) &= L((e_{I_k})^2 e_J) \\
&= L(e_{I_k}) e_{I_k} e_J + e_{I_k} L(e_{I_k}) e_J + e_{I_k} e_{I_k} L(e_J) \\
&= L((e_{I_k})\big(2 e_{I_k} e_J\big) + (e_{I_k})^2 L(e_J) \\
&= L(e_{I_k}) \frac{\partial p_k}{\partial e_{I_k}} + (e_{I_k})^2 L(e_J) \\
&=: L(e_{I_k}) q + r_{k+1}.
\end{aligned}$$

Note that for $L = V_{k+1}$ we have $L(e_{I_k}) = e_{I_{k+1}}$ and

$$L(e_J) = e_{-1} \wedge e_{-2} \wedge e_3 \wedge e_{-4} \wedge \cdots \wedge e_{-(k+1)} \wedge e_{k+2} \wedge e_{-(k+3)} \wedge e_{k+4} \wedge \cdots$$

Therefore, the remainder r_{k+1} can indeed still contain variables in $(\ker_\infty)_{k+1}$, i.e., variables that have exactly $k+1$ negative indexed vectors.

From this example we can also see that the only variables in r_{k+1} that are in $(\ker_\infty)_{k+1}$ are those of the form $V_{k+1}(\eta)$ for a variable $\eta \in (\ker_\infty)_k$ in p_k different from e_{I_k}.

Next, we act with V_{k+2} on p_{k+1}. The resulting polynomial p_{k+2} will again be in I_X since I_X is $\mathfrak{sp}(V_n)$-stable. We compute

$$\begin{aligned}
p_{k+2} &:= V_{k+2}(p_{k+1}) \\
&= V_{k+2}\big(e_{I_{k+1}} q + r_{k+1}\big) \\
&= V_{k+2}(e_{I_{k+1}}) q + \big(e_{I_{k+1}} V_{k+2}(q) + V_{k+2}(r_{k+1})\big) \\
&=: e_{I_{k+2}} q + r_{k+2}.
\end{aligned}$$

Observe that the remainder r_{k+2} is contained in $\mathrm{Sym}((\ker_\infty)_{\leq k+2})$. Indeed, we clearly have $e_{I_{k+1}} \in (\ker_\infty)_{k+1}$. Also $V_{k+2}(q) \in \mathrm{Sym}((\ker_\infty)_{\leq k+1})$ since $q = \frac{\partial p_k}{\partial e_{I_k}} \in \mathrm{Sym}((\ker_\infty)_k)$ by construction. Finally, $r_{k+1} \in (\ker_\infty)_{\leq k+1}$ implies $V_{k+2}(r_1) \in (\ker_\infty)_{\leq k+2}$.

Similarly as for r_{k+1}, the only variables of r_{k+2} that are in $(\ker_\infty)_{k+2}$ are those of the form $V_{k+2}(V_{k+1}(\eta))$, where η is a variable of p_k in $(\ker_\infty)_k$ different form e_{I_k}.

Iterating this construction we find a polynomial

$$p_n = e_{I_n} q + r_n \in I_X$$

where now r_n is not just in $\mathrm{Sym}((\ker_\infty)_{\leq n})$, but in fact r_n is in $\mathrm{Sym}((\ker_\infty)_{\leq n-1})$. Indeed, if r_n had a variable in $(\ker_\infty)_n$, then this variable would have to be of the form $(V_n \circ \cdots \circ V_{k+1})(\eta)$, where η is a variable of $p_k \in (\ker_\infty)_k$ different form e_{I_k}. Recall that by construction every variable in p_k is in $(\ker_n)_k \subseteq \bigwedge^k E_{-n} \otimes \bigwedge^{n-k} E_n$. But the composition

$$V_n \circ \cdots \circ V_{k+1} : \bigwedge^k E_{-n} \otimes \bigwedge^{n-k} E_n \longrightarrow \bigwedge^n E_{-n} \otimes \bigwedge^0 E_n$$

maps any variable e_J different from e_{I_k} to zero. In particular, any variable η in $(\ker_n)_k$ different from e_{I_k} gets mapped to zero. Therefore, r_n does *not* contain any variables in $(\ker_\infty)_n$, so that $r_n \in \mathrm{Sym}((\ker_\infty)_{\leq n-1})$. We again illustrate this by continuing the example from before.

Example 3.7.8. In the example above, the only variable of r_{k+1} that is in $(\ker_\infty)_{k+1}$ is

$$V_{k+1}(e_J) = e_{-1} \wedge e_{-2} \wedge e_3 \wedge e_{-4} \wedge \cdots \wedge e_{-(k+1)} \wedge e_{k+2} \wedge e_{-(k+3)} \wedge e_{k+4} \wedge \cdots .$$

Hence, after applying V_{k+2} the only variable of r_{k+2} in $(\ker_\infty)_{k+2}$ is

$$V_{k+2}(V_{k+1}(e_J)) = e_{-1} \wedge e_{-2} \wedge e_3 \wedge e_{-4} \wedge \cdots \wedge e_{-(k+1)} \wedge e_{-(k+2)} \wedge e_{-(k+3)} \wedge e_{k+4} \wedge \cdots .$$

So $r_{k+2} \in \mathrm{Sym}((\ker_\infty)_{\leq k+2})$. However, in the next step, when applying V_{k+3} the remainder r_{k+3} will be in $\mathrm{Sym}((\ker_\infty)_{\leq k+2})$ because the above variable gets send to zero by V_{k+3} as it does *not* contain e_{k+3}.

This completes the proof of Lemma 3.7.6. □

We will from now on abbreviate $R = \mathrm{Sym}(\ker_\infty)$ and $R_{\leq n} := \mathrm{Sym}((\ker_\infty)_{\leq n})$. Recall that $Z := X[1/q] \subseteq (\ker_\infty)^*[1/q]$. We denote by $I_Z \subseteq R[1/q]$ the radical ideal of Z.

Lemma 3.7.9. *The composition*

$$R_{\leq n}[1/q] \xrightarrow{i} R[1/q] \xrightarrow{\pi_{I_Z}} R[1/q]/I_Z$$

of the canonical inclusion i and the canonical projection π_{I_Z} is surjective, i.e., for all $a \in R[1/q]$ there exists $b \in R_{\leq n}[1/q]$ such that $a \equiv b \pmod{I_Z}$.

Proof. Note $R = \bigcup_{m \in \mathbb{N}} R_{\leq m}$. Therefore it suffices to prove that for all $m \geq n$ and all $a \in R_{\leq m}$ there exists $b \in R_{\leq n}$ and $k \in \mathbb{N}$ such that

$$a \equiv \frac{b}{q^k} \pmod{I_Z}.$$

We will prove this by induction on m. So fix $m > n$ and assume the statement holds for $m - 1$. Since $(\ker_\infty)_m$ is an $\mathfrak{sl}(E_{-\infty})$-irreducible representation with highest weight vector $e_{I_m} := e_{-1} \wedge \cdots \wedge e_{-m} \wedge e_{m+1} \wedge e_{m+2} \wedge \cdots$ (see Lemma 3.7.1), we get from [FH91, Observation 14.16] that there exists a basis $\{w_\alpha\}_{\alpha \in A_m}$ of $(\ker_\infty)_m$ such that for any basis element w_α there exist $L_1, \ldots, L_N \in \mathfrak{sl}(E_{-\infty})$ such that successively acting on e_{I_m} by $L_1, \ldots, L_N$ yields w_α, i.e.,

$$L_N(L_{N-1}(\cdots(L_1(e_{I_m}))\cdots) = w_\alpha.$$

Since $\{w_\alpha\}_{\alpha \in A_m}$ is a basis for $(\ker_\infty)_m$, it suffices to prove the above claim for $a = w_\alpha$.

From Lemma 3.7.6 we know that there exists $r_n \in R_{\leq n-1}$ such that

$$p_n = e_{I_n} q + r_n \in I_X.$$

Proceeding as in the proof of Lemma 3.7.6, by successively applying the root vectors $V_{n+1}, \ldots, V_m \in \mathfrak{sp}(V_n)$ defined in (3.6.5) we see that

$$p_m = e_{I_m} q + r_m \in I_X$$

for some $r_m \in R_{\leq m-1}$. Choose $L_1, \ldots, L_N \in \mathfrak{sl}(E_{-\infty})$ such that successively acting on e_{I_m} by $L_1, \ldots, L_N$ yields w_α. Recall that the rings $R_{\leq m-1}$ are $\mathfrak{gl}(E_{-\infty})$-stable. Thus, by successively applying $L_1, \ldots, L_N$ to p_m we get that

$$w_\alpha q + r'_m \in I_X$$

for some $r'_m \in R_{\leq m-1}$. Dividing by q in $R[1/q]$ yields

$$w_\alpha + \frac{r'_m}{q} \in \frac{1}{q} I_X \subseteq I_X[1/q] = I_Z,$$

i.e., $w_\alpha \equiv -\frac{r'_m}{q} \pmod{I_Z}$. Since $r'_m \in R_{\leq m-1}$, by the induction hypothesis there exists $b \in R_{\leq n}$ and $k \in \mathbb{N}$ such that

$$r'_m \equiv \frac{b}{q^k} \pmod{I_Z}.$$

Hence $w_\alpha \equiv \frac{-b}{q^{k+1}} \pmod{I_Z}$ with $-b \in R_{\leq n}$, completing the proof. □

We now use this to show that the induced map i^* when restricted to Z is a closed embedding.

Proof of Lemma 3.7.5. By Lemma 3.7.9 the composition $\pi_{I_Z} \circ i$ is surjective. Now set $J := \ker(\pi_{I_Z} \circ i)$. Then, by the first isomorphism theorem, there is an induced isomorphism $R_{\leq n}[1/q]/J \cong R[1/q]/I_Z$. Applying the contravariant functor $\operatorname{Spec}(\cdot)$ to

$$\begin{array}{ccc} R_{\leq n}[1/q] & \xrightarrow{\ i\ } & R[1/q] \\ \downarrow{\scriptstyle \pi_J} & & \downarrow{\scriptstyle \pi_{I_Z}} \\ R_{\leq n}[1/q]/J & \overset{\cong}{\dashrightarrow} & R[1/q]/I_Z \end{array}$$

yields the commutative diagram

$$\begin{array}{ccc} (\ker_\infty)^*_{\leq n}[1/q] & \xleftarrow{\ i^*\ } & (\ker_\infty)^*[1/q] \\ \uparrow & & \uparrow \\ \operatorname{Spec}(R_{\leq n}[1/q]/J) & \overset{\cong}{\dashleftarrow} & \operatorname{Spec}(R[1/q]/I_Z) \end{array} .$$

Recall that for any ring A and any ideal $\mathfrak{a} \subseteq A$ the map $\mathrm{Spec}(A/\mathfrak{a}) \to \mathrm{Spec}(A)$ induced by the projection $A \to A/\mathfrak{a}$ is a closed embedding with image $V(\mathfrak{a})$. Since $V(I_Z) = Z$ by definition of I_Z, this implies that the restriction of i^* to Z is a closed embedding. □

3.7.5 Poof of Proposition 3.5.2

We have established all necessary preliminary results for the proof of Proposition 3.5.2.

Proof of Proposition 3.5.2. With the same notation as in Section 3.7.4 fix some $m > n$. Let $(Z_i)_{i\in\mathbb{N}} \subseteq Z$ be a chain of $\mathrm{SL}(E_{-\infty})_m$-stable closed subsets. By Lemma 3.7.5 the restriction

$$i^*|_Z : Z \hookrightarrow (\ker_\infty)^*_{\leq n}[1/q]$$

is an $\mathrm{SL}(E_{-\infty})_m$-equivariant closed embedding. Thus $(Z_i')_{i\in\mathbb{N}} := \big((i^*|_Z)(Z_i)\big)_{i\in\mathbb{N}}$ is a chain of $\mathrm{SL}(E_{-\infty})_m$-stable closed subsets in $(\ker_\infty)^*_{\leq n}[1/q]$. By Proposition 3.7.2 the dual $(\ker_\infty)^*_{\leq n}$ is topologically $\mathrm{SL}(E_{-\infty})_m$-Noetherian. But then $(\ker_\infty)^*_{\leq n}[1/q]$ is also topologically $\mathrm{SL}(E_{-\infty})_m$-Noetherian. So we can conclude that the chain $(Z_i')_{i\in\mathbb{N}}$ stabilizes. But as $i^*|_Z$ is an embedding, this implies that the chain $(Z_i)_{i\in\mathbb{N}}$ itself stabilizes. This proves that Z is topologically $\mathrm{SL}(E_{-\infty})_m$-Noetherian. □

3.8 Proof of Theorem 3.5.1

After having established Proposition 3.5.2 we can now give the detailed proof of our Noetherianity result.

Proof of Theorem 3.5.1. As before, for any closed subset $X \subseteq (\ker_\infty)^*$ we denote by $\delta_X \in \{0, 1, 2, \ldots, \infty\}$ the lowest degree of a nonzero polynomial in the radical ideal $I_X \subseteq \mathrm{Sym}(\ker_\infty)$ of X.

Observe that $\delta_X = \infty$ if and only if $X = (\ker_\infty)^*$. So a chain

$$(\ker_\infty)^* \supseteq X_1 \supseteq X_2 \supseteq \cdots$$

of $\mathrm{Sp}(V_\infty)$-stable closed subsets is either constant or else there exists an $i \in \mathbb{N}$ with $\delta_{X_i} < \infty$. Therefore, it suffices to show that any $\mathrm{Sp}(V_\infty)$ closed subset $X \subseteq (\ker_\infty)^*$ with $\delta_X < \infty$ is topologically $\mathrm{Sp}(V_\infty)$-Noetherian. We will prove this by induction on δ_X.

For the base case $\delta_X = 0$, note that $\delta_X = 0$ if and only if $X = \emptyset$. So the base case trivially holds.

Now fix a $\mathrm{Sp}(V_\infty)$ closed subset $X \subseteq (\ker_\infty)^*$ with $0 < \delta_X < \infty$, and assume that all $\mathrm{Sp}(V_\infty)$ closed subset $Y \subseteq (\ker_\infty)^*$ with $\delta_Y < \delta_X$ are $\mathrm{Sp}(V_\infty)$-Noetherian.

Choose a nonzero polynomial $p \in I_X$ with $\deg(p) = \delta_X$ and define $q := \frac{\partial p}{\partial e_I}$ as at the beginning of Section 3.7.4. Set

$$Y := V(\mathrm{Sp}(V_\infty) \cdot q) \quad \text{and} \quad Z := X[1/q].$$

Then Y is a $\mathrm{Sp}(V_\infty)$-stable closed subset with $\delta_Y \leq \deg(q) \leq \deg(p) - 1 < \delta_X$, and so Y is topologically $\mathrm{Sp}(V_\infty)$-Noetherian by the induction hypothesis. By Proposition 3.5.2 there exists $m \in \mathbb{N}$ large enough such that Z is $\mathrm{SL}(E_{-\infty})_m$-stable and topologically $\mathrm{SL}(E_{-\infty})_m$-Noetherian.

Take a chain $(X_i)_{i\in\mathbb{N}} \subseteq X$ of $\mathrm{Sp}(V_\infty)$-stable closed subsets. Observe that for all $i \in \mathbb{N}$ we have

$$X_i = (Y \cap X_i) \cup \mathrm{Sp}(V_\infty) \cdot (Z \cap X_i). \tag{3.8.1}$$

Indeed, fix a point $\mathfrak{p} \in X_i$ (not necessarily closed). If $\mathfrak{p} \notin Y$, then, by the definition of Y and Z, there exists $g \in \mathrm{Sp}(V_\infty)$ such that $g \cdot \mathfrak{p} \in Z$, and hence $g \cdot \mathfrak{p} \in Z \cap X_i$ since X_i is $\mathrm{Sp}(V_\infty)$-stable.

Since Y and X_i are $\mathrm{Sp}(V_\infty)$-stable closed subsets, their intersections $(Y \cap X_i)_{i\in\mathbb{N}}$ are a chain of $\mathrm{Sp}(V_\infty)$-stable closed subsets in Y. As Y is topologically $\mathrm{Sp}(V_\infty)$-Noetherian, the chain $(Y \cap X_i)_{i\in\mathbb{N}}$ stabilizes.

Similarly, every $Z \cap X_i$ is $\mathrm{SL}(E_{-\infty})_m$-stable and a closed subset of Z (by the definition of the subspace topology), and so the chain $(Z \cap X_i)_{i\in\mathbb{N}}$ also stabilizes.

Because the chains $(Y \cap X_i)_{i\in\mathbb{N}}$ and $(Z \cap X_i)_{i\in\mathbb{N}}$ both stabilize, it follows from (3.8.1) that the chain $(X_i)_{i\in\mathbb{N}}$ itself stabilizes. As the chain $(X_i)_{i\in\mathbb{N}}$ was arbitrary, this proves that X is topologically $\mathrm{Sp}(V_\infty)$-Noetherian. □

3.9 Lagrangian Plücker Varieties and Applications

3.9.1 The Dual $(\ker_\infty)^*$ as a Projective Limit

Before we can come to the applications of our Noetherianity result (Theorem 3.5.1), we first have to find an alternative description of $(\ker_\infty)^*$ as a projective limit.

Recall that $\ker_\infty$ was defined as the direct limit $\ker_\infty := \varinjlim_n \ker_n$ along the multiplication maps

$$m_{e_n} : \ker_{n-1} \subseteq \bigwedge^{n-1} V_{n-1} \to \ker_n \subseteq \bigwedge^{n} V_n,\ \eta \mapsto \eta \wedge e_n.$$

Thus, as taking the dual turns direct limits into projective limits, we have

$$(\ker_\infty)^* \cong \varprojlim \left(\cdots \xrightarrow{m_{e_4}^*} (\ker_3)^* \xrightarrow{m_{e_3}^*} (\ker_2)^* \xrightarrow{m_{e_2}^*} (\ker_1)^* \right).$$

By Lemma 3.9.1 below, under the musical isomorphism, the duals $m_{e_i}^*$ of the multiplication maps correspond to the maps Φ_{e_i} defined in Definition 3.1.10. Therefore,

$$(\ker_\infty)^* \cong \varprojlim \left(\cdots \xrightarrow{\Phi_{e_4}} \ker_3 \xrightarrow{\Phi_{e_3}} \ker_2 \xrightarrow{\Phi_{e_2}} \ker_1 \right),$$

i.e., the dual $(\ker_\infty)^*$ can be identified with the projective limit $\varprojlim_n \ker_n$ along the maps $\Phi_{e_n} : \ker_n \to \ker_{n-1}$. We will use this interpretation of $(\ker_\infty)^*$ in Section 3.9.2 below.

It remains to show that contraction maps are dual to multiplication maps, which is contained in the next lemma. In its formulation, for a symplectic vector space V and a nonzero $v \in V$, we choose a section $s_v : V_v \to v^\perp$ of the projection $\pi_v : v^\perp \to v^\perp/\langle v\rangle = V_v$, and define the multiplication m_v as $v \wedge \bigwedge^{n-1} s_v$, i.e.,

$$m_v : \bigwedge^{n-1} V_v \to \bigwedge^{n} V,\ \bar{v}_1 \wedge \cdots \wedge \bar{v}_{n-1} \mapsto v \wedge s_v(\bar{v}_1) \wedge \cdots \wedge s_v(\bar{v}_{n-1}).$$

Using that s_v is an isometric embedding, one can easily check that m_v restricts to a map $m_v : \ker(i_{\omega_v}) \to \ker(i_\omega)$. Moreover, m_v is independent of the choice of section s_v.

Notice that, up to this point, multiplication has always been defined by wedging with a vector from the right. Moving forward, it will be more convenient to consider multiplication maps as wedging with a vector from the left. However, aside from a sign change, these operations are the same.

Lemma 3.9.1. *The restriction of the musical ismorphism*

$$\ker(i_\omega) \subseteq \bigwedge^n V \xrightarrow{\flat} \bigwedge^n (V^*) \cong \Big(\bigwedge^n V\Big)^* \xrightarrow{\text{restriction}} \big(\ker(i_\omega)\big)^*$$

is again an isomorphism, and the diagram

$$\begin{array}{ccc} \ker(i_\omega) & \xrightarrow{\Phi_v} & \ker(i_{\omega_v}) \\ {\scriptstyle \flat_V}\big\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\big\downarrow{\scriptstyle \flat_{V_v}} \\ \big(\ker(i_\omega)\big)^* & \xrightarrow{m_v^*} & \big(\ker(i_{\omega_v})\big)^* \end{array}$$

commutes (up to the sign -1), where Φ_v is the map defined in Definition 3.1.10.

Proof. Recall from (3.1.6) that $v \wedge \bullet$ is dual to i_v. Hence the dual m_v^* of $m_v = v \wedge \bigwedge^{n-1} s_v$ is the composition

$$\bigwedge^n V^* \xrightarrow{i_v} \bigwedge^{n-1} V^* \xrightarrow{\bigwedge^{n-1} s_v^*} \bigwedge^{n-1} V_v^*.$$

But by (3.1.16) the diagram

$$\begin{array}{ccccc} & & \overset{\Phi_v}{\dashrightarrow} & & \\ \bigwedge^n V & \xrightarrow{\varphi_v} & \bigwedge^{n-1} v^\perp & \xrightarrow{\bigwedge^{n-1} \pi_v} & \bigwedge^{n-1} V_v \\ {\scriptstyle \flat_V}\big\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\big\downarrow{\scriptstyle \flat_V} & & {\scriptstyle \cong}\big\downarrow{\scriptstyle \flat_{V_v}} \\ \bigwedge^n V^* & \xrightarrow{i_v} & \bigwedge^{n-1} \ker(v) & \xrightarrow{\bigwedge^{n-1} s_v^*} & \bigwedge^{k-1} V_v^* \end{array}$$

commutes (up to the sign -1). If we restrict to $\ker(i_\omega)$ we see that the diagram in Lemma 3.9.1 commutes (up to the sign -1).

It remains to show for each symplectic vector space V that the composition

$$\ker(i_\omega) \subseteq \bigwedge^n V \xrightarrow{\flat} \bigwedge^n (V^*) \cong \Big(\bigwedge^n V\Big)^* \xrightarrow{\text{restriction}} \big(\ker(i_\omega)\big)^*$$

is an isomorphism for $n = \frac{1}{2}\dim(V)$. By reasons of dimension, it suffices to show that this map is injective. This translates into showing that for all $\eta \in \ker(i_\omega)$ nonzero there exists $\xi \in \ker(i_\omega)$ such that $(\xi, \eta^\flat)_{\text{ev}} \neq 0$. Since $(\xi, \eta^\flat)_{\text{ev}} = (-1)^n (\eta, \xi^\flat)_{\text{ev}}$ by (3.1.11), this is equivalent to showing that for all nonzero $\eta \in \ker(i_\omega)$ there exists $\xi \in \ker(i_\omega)$ such that $(\eta, \xi^\flat)_{\text{ev}} \neq 0$, i.e., we have to show that

$$W := \Big\{\eta \in \ker(i_\omega) \,\Big|\, (\eta, \xi^\flat)_{\text{ev}} = 0 \text{ for all } \xi \in \ker(i_\omega)\Big\}$$

satisfies $W = 0$. Observe that W is $\mathrm{Sp}(V)$-stable. Indeed, by (3.3.2) we have for all $A \in \mathrm{Sp}(V)$, $\eta \in W$ and $\xi \in \ker(i_\omega)$

$$\big(A\eta, \xi^\flat\big)_{\text{ev}} \overset{(3.3.2)}{=} \big(\eta, (A^{-1}\xi)^\flat\big)_{\text{ev}} = 0,$$

showing that $A\eta \in W$. Since $\ker(i_\omega)$ is $\mathrm{Sp}(V)$-irreducible by Theorem 3.2.3, it follows that either $W = 0$ or $W = \ker(i_\omega)$. But $(e_{-1} \wedge \cdots \wedge e_{-n}, (e_1 \wedge \cdots \wedge e_n)^\flat)_{\mathrm{ev}} = 1$ due to Example 3.1.9 and Example 3.1.2, and hence $e_{-1} \wedge \cdots \wedge e_{-n} \notin W$. So $W \neq \ker(i_\omega)$, and therefore $W = 0$. This completes the proof. $\square$

In accordance with Lemma 3.9.1 we will also write $\ker_n$ for the affine scheme $(\ker_n)^* = \mathrm{Spec}(\mathrm{Sym}(\ker_n))$, so that $(\ker_\infty)^*$ is not just the projective limit $\varprojlim_n \ker_n$ when seen as a vector space, but also when we think of it as an affine scheme (as in Theorem 3.5.1).

3.9.2 Lagrangian Plücker Varieties

Our goal in this section is to introduce Lagrangian Plücker varieties and explain how Theorem 3.5.1 can be used to prove some important properties they have as we will see in Theorem 3.9.8 and Corollary 3.9.11.

We want to introduce the definition of Lagrangian Plücker Variety, which takes inspiration from the notion of Plücker variety in [DE18] or more specifically from the notion of a half-spin variety in Chapter 2. Similar to half-spin varieties we will consider linear maps $\ker(i_\omega) \subseteq \bigwedge^n V \to \ker(i_{\omega'}) \subseteq \bigwedge^{n'} V'$, where (V, ω) resp. (V', ω') are symplectic vector spaces of dimension $2n$ resp. $2n'$, that are compositions of maps of the following type:

- For any isometry $\varphi : V \to V'$ the induced map $\bigwedge^n \varphi : \bigwedge^n V \xrightarrow{\cong} \bigwedge^n V'$ restricts to an isomorphism
$$\bigwedge^n \varphi : \ker(i_\omega) \xrightarrow{\cong} \ker(i_{\omega'});$$
- For each nonzero $v \in V$ the contraction maps $\Phi_v : \bigwedge^n V \to \bigwedge^{n-1} V_v$ restrict to a map
$$\Phi_v : \ker(i_\omega) \to \ker(i_{\omega_v})$$
by Lemma 3.2.6;
- The multiplication maps
$$m_{e_{-n}} : \bigwedge^{n-1} V_{n-1} \to \bigwedge^n V_n, \eta \mapsto e_{-n} \wedge \eta$$
restrict to a map
$$m_{e_{-n}} : \ker_{n-1} \to \ker_n,$$
where V_n and $\ker_n$ are as in Section 3.5.

Observe that $m_{e_{-n}}$ is a section of Φ_{e_n}.

Definition 3.9.2 (Lagrangian Plücker variety)**.** A *Lagrangian Plücker variety* is a rule X that assigns to every finite dimensional symplectic vector space $V = (V, \omega)$ a Zariski closed $\mathrm{Sp}(V)$-stable subset
$$X(V) \subseteq \ker(i_\omega) \subseteq \bigwedge^n V,$$
where $n = \frac{1}{2}\dim(V)$, that is stable under the above type of maps, i.e., such that

1. for every isometry $\varphi : V \to V'$
$$\left(\bigwedge^n \varphi\right)\left(X(V)\right) = X(V');$$

2. for every $v \in V \setminus \{0\}$
$$\Phi_v\big(X(V)\big) \subseteq X(V_v);$$

3. for all $n \in \mathbb{N}$
$$m_{e_{-n}}\big(X(V_{n-1})\big) \subseteq X(V_n).$$

Equivalently, one could, as for Theorem 3.5.1, define $X(V)$ to be a reduced closed subscheme (instead of just a Zariski closed subset) with the given properties.

Note that, by definition, any $A \in \mathrm{Sp}(V)$ is an isometry $A : V \to V$. Thus, as $A \in \mathrm{Sp}(V)$ acts on $\bigwedge^n V$ via $\bigwedge^n A$, Definition 3.9.2(1) automatically implies that $X(V)$ is $\mathrm{Sp}(V)$-stable.

Examples 3.9.3. The following are examples of Lagrangian Plücker varieties.

1. Trivially, $X(V) := \ker(i_\omega)$, $X(V) := \{0\}$ and $X(V) := \emptyset$ define Lagrangian Plücker varieties.

2. For two Lagrangian Plücker varieties X and X' their *join* $X + X'$, which is defined by
$$(X + X')(V) := \overline{\{x + x' \mid x \in X(V),\ x' \in X'(V)\}},$$
is a Lagrangian Plücker variety.

3. For a Lagrangian Plücker variety X, the k-th secant variety $\mathrm{Sec}^k(X)$ of X defined by
$$\mathrm{Sec}^k(X)(V) := \overline{\{x_1 + \cdots + x_k \mid x_i \in X(V)\}}$$
is again a Lagrangian Plücker variety.

4. The intersection of two Lagrangian Plücker varieties X and X' is again a Lagrangian Plücker variety. We denote it as $(X \cap X')(V) := X(V) \cap X'(V)$.

Example 3.9.4. The Lagrangian Grassmann cone $X(V) := \widehat{\mathrm{Gr}}_{\mathrm{L}}(V)$ defined in Definition 3.1.5 is a Lagrangian Plücker variety.

Proof. It is well-known that through the Plücker embedding the (ordindary) Grassmannian $\mathrm{Gr}(n, V)$ is a projective variety in $\mathbb{P}(\bigwedge^n V)$. In particular, the (ordinary) Grassmann cone $\widehat{\mathrm{Gr}}(n, V) \subseteq \bigwedge^n V$ is Zariski closed. It easily follows from Definition 3.1.5 that $\widehat{\mathrm{Gr}}_{\mathrm{L}}(V) = \widehat{\mathrm{Gr}}(n, V) \cap \ker(i_\omega)$. Hence, the Lagrangian Grassmann cone $\widehat{\mathrm{Gr}}_{\mathrm{L}}(V) \subseteq \ker(i_\omega)$ is also Zariski closed. Keeping in mind that being $\mathrm{Sp}(V)$-stable follows automatically from Definition 3.9.2(1), it remains to check the properties (1)-(3) in Definition 3.9.2.

First, let $\varphi : V \to V'$ be an isometry and let $\xi \in \widehat{\mathrm{Gr}}_{\mathrm{L}}(V)$ be arbitrary. By Definition 3.1.5 we can write can write $\xi = v_1 \wedge \cdots \wedge v_n$ for some $v_1, \ldots, v_n \in V$ with $\omega(v_i, v_j) = 0$ for all $1 \leq i, j \leq n$. Observe that $\omega'(\varphi(v_i), \varphi(v_j)) = \omega(v_i, v_j) = 0$ for all $1 \leq i, j \leq n$ since φ is an isometry, and thus
$$\left(\bigwedge^n \varphi\right)(\xi) = \varphi(v_1) \wedge \cdots \wedge \varphi(v_n) \in \widehat{\mathrm{Gr}}_{\mathrm{L}}(V').$$

This shows $\left(\bigwedge^n \varphi\right)\left(\widehat{\mathrm{Gr}}_{\mathrm{L}}(V)\right) \subseteq \widehat{\mathrm{Gr}}_{\mathrm{L}}(V')$. Replacing φ by φ^{-1} also implies the other inclusion, thus proving equality.

Second, the same argument as in Example 1.1.3(2) in Chapter 1 shows that φ_v maps $\widehat{\mathrm{Gr}}_{\mathrm{L}}(V)$ into $\widehat{\mathrm{Gr}}_{\mathrm{iso}}(n-1, v^{\perp})$. Since $\Phi_v = \left(\bigwedge^{n-1} \pi_v\right) \circ \varphi_v$ by Definition 3.1.10, this shows that Φ_v maps $\widehat{\mathrm{Gr}}_{\mathrm{L}}(V)$ into $\widehat{\mathrm{Gr}}_{\mathrm{iso}}(n-1, V_v) = \widehat{\mathrm{Gr}}_{\mathrm{L}}(V_v)$.

Finally, let $\xi \in \widehat{\mathrm{Gr}}_{\mathrm{L}}(V_{n-1})$ be arbitrary. Again, we can write $\xi = v_1 \wedge \cdots \wedge v_{n-1}$ for some $v_1, \ldots, v_{n-1} \in V_{n-1}$ with $\omega(v_i, v_j) = 0$ for all $1 \leq i, j \leq n-1$. Then

$$m_{e_{-n}}(\xi) = e_{-n} \wedge \xi = e_{-n} \wedge v_1 \wedge \cdots \wedge v_{n-1} \in \widehat{\mathrm{Gr}}_{\mathrm{L}}(V_n)$$

because $e_{-n} \perp V_{n-1}$ and $\langle v_1, \ldots, v_{n-1} \rangle \subseteq V_{n-1}$. □

Since any symplectic vector space is isometric to some V_n (see Lemma 3.1.8), the following remark follows immediately from Definition 3.9.2(1).

Remark 3.9.5. A Lagrangian Plücker variety X is completely determined by the values $X(V_n)$, that is, if X and X' are Lagrangian Plücker varieties such that $X(V_n) = X'(V_n)$ for all $n \in \mathbb{N}$, then $X(V) = X'(V)$ for all V.

We will from now on abbreviate $X_n := X(V_n) \subseteq \ker_n$.

We want to associate to each Lagrangian Plücker variety X a closed $\mathrm{Sp}(V_\infty)$-stable subset $X_\infty \subseteq (\ker_\infty)^*$. Recall from Section 3.9.1 that $(\ker_\infty)^*$ can be identified with the projective limit $\varprojlim_n \ker_n$ along the contraction maps Φ_{e_n}, i.e.,

$$(\ker_\infty)^* \cong \varprojlim \left(\cdots \xrightarrow{\Phi_{e_4}} \ker_3 \xrightarrow{\Phi_{e_3}} \ker_2 \xrightarrow{\Phi_{e_2}} \ker_1 \right).$$

By Definition 3.9.2(2) each $\Phi_{e_n} : \ker_n \to \ker_{n-1}$ satisfies $\Phi_{e_n}(X_n) \subseteq X_{n-1}$. Thus the projective limit

$$X_\infty := \varprojlim \left(\cdots \xrightarrow{\Phi_{e_4}} X_3 \xrightarrow{\Phi_{e_3}} X_2 \xrightarrow{\Phi_{e_2}} X_1 \right) \subseteq (\ker_\infty)^*$$

is well defined. Moreover, $X_\infty \subseteq (\ker_\infty)^*$ is closed and $\mathrm{Sp}(V_\infty)$-stable since $X_n \subseteq \ker_n$ is closed and $\mathrm{Sp}(V_n)$-stable for all $n \in \mathbb{N}$. If one thinks of X_n as closed reduced subschemes of $\ker_n$, then also X_∞ is a closed reduced subscheme of $(\ker_\infty)^*$. Moreover, we have the following.

Remark 3.9.6. The closed subset $X_\infty \subseteq (\ker_\infty)^*$ is an affine cone.

Proof. Let $I_\infty \subseteq \mathrm{Sym}(\ker_\infty)$ be the radical ideal defining X_∞, and note that I_∞ is $\mathfrak{sp}(V_\infty)$-stable since X_∞ is $\mathrm{Sp}(V_\infty)$-stable.

We have to show that I_∞ is a homogeneous ideal. Let $f \in I_\infty$ be arbitrary. Choose $n \in \mathbb{N}$ large enough such that f is contained in $\mathrm{Sym}(\ker_n) \subseteq \mathrm{Sym}(\ker_\infty)$, i.e., so that any variable $\eta \in \ker_\infty$ of f is of the form $\eta_n \wedge e_{n+1} \wedge e_{n+2} \wedge \cdots$ for some $\eta_n \in \ker_n$. By Section 3.6.2 the endomorphism H_{n+1} of V_∞ mapping e_{n+1} to e_{n+1}, $e_{-(n+1)}$ to $-e_{-(n+1)}$ and all other basis vectors to zero is an element of $\mathfrak{sp}(V_\infty)$. The action of H_{n+1} on $\ker_\infty$ sends any variable of f to itself. So, by the Leibniz rule, the action of H_{n+1} multiplies the degree d homogeneous part of f by d. Since $H_{n+1} \cdot f \in I_\infty$, this implies that every homogeneous part of f is contained in I_∞. This completes the proof. □

For all $N > n$ we denote by $\pi_{N,n} : \ker_N \to \ker_n$ the composition $\Phi_{e_{n+1}} \circ \cdots \circ \Phi_{e_N}$ of the contraction maps. Similarly, $\tau_{n,N} : \ker_n \to \ker_N$ shall denote the composition $m_{e_{-N}} \circ \cdots \circ m_{e_{-(n+1)}}$. Note that $\tau_{n,N}$ is a section of $\pi_{N,n}$ because, for every $k \in \mathbb{N}$, $m_{e_{-k}}$ is a section of Φ_{e_k}. It follows from Definition 3.9.2(2),(3) that

$$\pi_{N,n}(X_N) \subseteq X_n \quad \text{and} \quad \tau_{n,N}(X_n) \subseteq X_N. \tag{3.9.1}$$

We also denote by $\pi_{\infty,n} : (\ker_\infty)^* \to \ker_n$ the structure maps of the projective limit $\varprojlim_n \ker_n$, and we write $\tau_{n,\infty} : \ker_n \to (\ker_\infty)^*$ for the map induced by the maps $\tau_{n,N}$. Then $\tau_{n,\infty}$ is a section of $\pi_{\infty,n}$. Moreover, it follows from Equation (3.9.1) that

$$\pi_{\infty,n}(X_\infty) \subseteq X_n \quad \text{and} \quad \tau_{n,\infty}(X_n) \subseteq X_\infty. \tag{3.9.2}$$

We are now in a position to prove the following useful lemma.

Lemma 3.9.7. *The mapping*

$$X \mapsto X_\infty$$

is injective, i.e., if X and X' are Lagrangian Plücker varieties such that $X_\infty = X'_\infty$, then $X = X'$.

Proof. Note that, for all $n \in \mathbb{N}$, we have

$$X_n = \pi_{\infty,n}(X_\infty).$$

Indeed, the inclusion $\supseteq$ is contained in (3.9.2). The other direction $\subseteq$ follows from the fact that $\tau_{n,\infty}$ is a section of $\pi_{\infty,n}$ and that $\tau_{n,\infty}(X_n) \subseteq X_\infty$ by (3.9.2).

Hence, if $X_\infty = X'_\infty$, then

$$X_n = \pi_{\infty,n}(X_\infty) = \pi_{\infty,n}(X'_\infty) = X'_n.$$

By Remark 3.9.5 this shows that $X = X'$. □

For two Lagrangian Plücker varieties X and X', we will write $X \subseteq X'$ if, for all symplectic vector spaces V, we have $X(V) \subseteq X'(V)$. Theorem 3.5.1 then implies the following.

Theorem 3.9.8 (Noetherianity of Lagrangian Plücker varieties)**.** *Every descending chain of Lagrangian Plücker varieties*

$$X^{(0)} \supseteq X^{(1)} \supseteq X^{(2)} \supseteq X^{(3)} \supseteq \cdots$$

stabilizes, that is, there exists $m_0 \in \mathbb{N}$ such that $X^{(m)} = X^{(m_0)}$ for all $m \geq m_0$.

Proof. Note that the mapping $X \mapsto X_\infty$ is order preserving, that is, if $X \subseteq X'$, then $X_\infty \subseteq X'_\infty$. Hence, a chain

$$X^{(0)} \supseteq X^{(1)} \supseteq X^{(2)} \supseteq X^{(3)} \supseteq \cdots$$

of Lagrangian Plücker varieties induces a chain

$$X^{(0)}_\infty \supseteq X^{(1)}_\infty \supseteq X^{(2)}_\infty \supseteq X^{(3)}_\infty \supseteq \ldots$$

of closed $\mathrm{Sp}(V_\infty)$-stable subsets in $(\ker_\infty)^*$. By Theorem 3.5.1 we know that $(\ker_\infty)^*$ is topologically $\mathrm{Sp}(V_\infty)$-Noetherian. Hence, the chain $\big(X^{(m)}_\infty\big)_{m\in\mathbb{N}}$ stabilizes. But then, by Lemma 3.9.7 also the chain of Lagrangian Plücker varieties $\big(X^{(m)}\big)_{m\in\mathbb{N}}$ stabilizes. This completes the proof. □

As a further consequence of Theorem 3.5.1 we obtain the next results, which states that X_∞ is determined by the data coming from some finite level of X.

Theorem 3.9.9. *Let X be a Lagrangian Plücker variety. Then there exists $n_0 \in \mathbb{N}$ such that*

$$X_\infty = V\left(\mathrm{Sp}(V_\infty) \cdot I_{n_0}\right),$$

where $\mathrm{Sp}(V_\infty) \cdot I_{n_0}$ are the $\mathrm{Sp}(V_\infty)$-orbits of the radical ideal $I_{n_0} \subseteq \mathrm{Sym}(\ker_{n_0})$ defining $X_{n_0} \subseteq \ker_{n_0}$.

Here, by a slight abuse of notation, we identify $I_{n_0} \subseteq \mathrm{Sym}(\ker_{n_0})$ with its image in $\mathrm{Sym}(\ker_\infty)$ under the embedding induced by the inclusion $\ker_{n_0} \to \ker_\infty$.

Remark 3.9.10. If we would think of a Lagrangian Plücker variety as a rule assigning to each symplectic vector space V a reduced closed subscheme $X(V)$, then in Theorem 3.9.9 we would have to replace $V(\mathrm{Sp}(V_\infty)\cdot I_{n_0})$ by $V\big(\mathrm{rad}(\mathrm{Sp}(V_\infty)\cdot I_{n_0})\big)$, where $\mathrm{rad}(\mathrm{Sp}(V_\infty)\cdot I_{n_0})$ is the radical ideal generated by $\mathrm{Sp}(V_\infty) \cdot I_{n_0}$.

Proof. For each $n \in \mathbb{N}$ set $J_n := \mathrm{rad}(\mathrm{Sp}(V_\infty) \cdot I_n) \subseteq \mathrm{Sym}(\ker_\infty)$. Note that as subsets of $(\ker_\infty)^*$ we have $V(J_n) = V(\mathrm{Sp}(V_\infty) \cdot I_n)$. Let $I_\infty = \varinjlim_n I_n \subseteq \mathrm{Sym}(\ker_\infty)$ be the radical ideal associated to X_∞. Then $\bigcup_n J_n = I_\infty$ because I_∞ is $\mathrm{Sp}(V_\infty)$-stable and radical. Since $(J_n)_{n\in\mathbb{N}}$ is an increasing chain of $\mathrm{Sp}(V_\infty)$-stable radical ideals, by Theorem 3.5.1 there exists $n_0 \in \mathbb{N}$ such that $J_n = J_{n_0}$ for all $n \geq n_0$. Therefore, $I_\infty = \bigcup_n J_n = J_{n_0}$, and hence $X_\infty = V(I_\infty) = V(J_{n_0}) = V(\mathrm{Sp}(V_\infty) \cdot I_{n_0})$. ☐

Using a result of Draisma [Dra10] , we will obtain the following corollary.

Corollary 3.9.11 (Universality for Lagrangian Plücker varieties)**.** *Let X be a Lagrangian Plücker variety. There exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ it holds that*

$$X_n = V(\mathrm{Sp}(V_n) \cdot I_{n_0}).$$

Before coming to the proof of Corollary 3.9.11, we first make some comments.

In Corollary 3.9.11 we make the same abuse of notation as in Theorem 3.9.9 and identify $I_{n_0} \subseteq \mathrm{Sym}(\ker_{n_0})$ with its image in $\mathrm{Sym}(\ker_n)$ under the inclusion of rings $\mathrm{Sym}(\ker_{n_0}) \to \mathrm{Sym}(\ker_n)$ induced by the inclusion $\ker_{n_0} \to \ker_n$. In other words, for a polynomial f defined on $\ker_{n_0}$ we denote the polynomial $f \circ \pi_{n,n_0}$ defined on $\ker_n$ still by f. Therefore, Corollary 3.9.11 can be reformulated as follows.

Remark 3.9.12. Let X be a Lagrangian Plücker variety. Then there exists $n_0 \in \mathbb{N}$ with the following property: For every $n \geq n_0$ and $x \in \ker_n \subseteq \bigwedge^n V_n$ we have

$$x \in X_n \iff \pi_{n,n_0}(A \cdot x) \in X_{n_0} \quad \text{for all } A \in \mathrm{Sp}(V_n).$$

Together with the following example, which is a consequence of Theorem 3.4.1, this explains how Corollary 3.9.11 can be thought of as a generalization of Theorem 3.4.1 to arbitrary Lagrangian Plücker varieties.

Example 3.9.13. For all $n \geq 2$ and $\eta \in \ker_n \subseteq \bigwedge^n V_n$ we have

$$\eta \in \widehat{\mathrm{Gr}}_{\mathrm{L}}(V_n) \iff \pi_{n,2}(A \cdot \eta) \in \widehat{\mathrm{Gr}}_{\mathrm{L}}(V_2) \text{ for all } A \in \mathrm{Sp}(V_n),$$

where $\pi_{n,2} = \Phi_{e_3} \circ \cdots \circ \Phi_{e_n}$.

Here we think of $\ker_n$ only as a vector space.

Proof. We prove this by induction. In the base case $n = 2$ there is nothing to prove. Now take $n > 2$ and assume that the result is true for $n-1$. If $\eta \in \widehat{\mathrm{Gr}}_{\mathrm{L}}(V_n)$, then $\pi_{n,2}(A \cdot \eta) \in \widehat{\mathrm{Gr}}_{\mathrm{L}}(V_2)$ for all $A \in \mathrm{Sp}(V_n)$ because $V \mapsto \widehat{\mathrm{Gr}}_{\mathrm{L}}(V)$ is a Lagrangian Plücker variety by Example 3.9.4. To prove the other direction, we argue by contradiction, i.e., we assume that there exists $\eta \in \ker_n$ such that $\pi_{n,2}(A \cdot \eta) \in \widehat{\mathrm{Gr}}_{\mathrm{L}}(V_2)$ for all $A \in \mathrm{Sp}(V_n)$ but $\eta \notin \widehat{\mathrm{Gr}}_{\mathrm{L}}(V_n)$.

Since $\eta \notin \widehat{\mathrm{Gr}}_{\mathrm{L}}(V_n)$ there exists, due to Theorem 3.4.1, a vector $v \in V_n \setminus \{0\}$ such that $\Phi_v(\eta) \notin \widehat{\mathrm{Gr}}_{\mathrm{L}}((V_n)_v)$. Choose some $A \in \mathrm{Sp}(V_n)$ such that $Av = e_n$. Note that $A(v^\perp) = e_n^\perp$ because A preserves the symplectic form ω. So A induces an isometry $\bar{A} : (V_n)_v \xrightarrow{\cong} (V_n)_{e_n} = V_{n-1}$. Using again that A preserves ω, one checks that the diagram

$$\begin{array}{ccccc}
\bigwedge^n V_n & \xrightarrow{\varphi_v} & \bigwedge^{n-1} v^\perp & \xrightarrow{\bigwedge^{n-1} \pi_v} & \bigwedge^{n-1} (V_n)_v \\
{\scriptstyle \bigwedge^n A} \big\downarrow {\scriptstyle \cong} & & {\scriptstyle \bigwedge^{n-1} A} \big\downarrow {\scriptstyle \cong} & & {\scriptstyle \cong} \big\downarrow {\scriptstyle \bigwedge^{n-1} \bar{A}} \\
\bigwedge^n V_n & \xrightarrow{\varphi_{e_n}} & \bigwedge^{n-1} e_n^\perp & \xrightarrow{\bigwedge^{n-1} \pi_{e_n}} & \bigwedge^{n-1} V_{n-1}
\end{array}$$

commutes. In particular, $\eta' := \Phi_{e_n}(A \cdot \eta) = \left(\bigwedge^{n-1} \bar{A}\right)\left(\Phi_v(\eta)\right) \notin \widehat{\mathrm{Gr}}_{\mathrm{L}}(V_{n-1})$. By induction hypothesis there exists $A' \in \mathrm{Sp}(V_{n-1})$ such that $\pi_{n-1,2}(A' \cdot \eta') \notin \widehat{\mathrm{Gr}}_{\mathrm{L}}(V_2)$. Note that, due to the definition of the embedding $\mathrm{Sp}(V_{n-1}) \subseteq \mathrm{Sp}(V_n)$, A' fixes e_n, and hence Φ_{e_n} is A'-equivariant. So $A' \cdot \eta' = \Phi_{e_n}(A'A \cdot \eta)$, and therefore

$$\pi_{n,2}(A'A \cdot \eta) = \pi_{n-1,2}\left(\Phi_{e_n}(A'A \cdot \eta)\right) = \pi_{n-1,2}(A' \cdot \eta') \notin \widehat{\mathrm{Gr}}_{\mathrm{L}}(V_2).$$

This contradicts our assumption on η and thus completes the proof. $\square$

We now come to the proof of Corollary 3.9.11.

Proof of Corollary 3.9.11. Let $n_0 \in \mathbb{N}$ be as in Theorem 3.9.9. Then the statement follows from Theorem 3.9.9 and [Dra10, Lemma 2.1]. To apply that lemma, we must check condition (*) in that paper, i.e., that for all $q \geq n \geq n_0$ and $g \in \mathrm{Sp}(V_q)$ we can write

$$\pi_{q,n_0} \circ g \circ \tau_{n,q} = g'' \circ \tau_{m,n_0} \circ \pi_{n,m} \circ g'$$

for suitable $m \leq n_0$ and $g' \in \mathrm{Sp}(V_n)$ and $g'' \in \mathrm{Sp}(V_{n_0})$. Since X_∞ is an affine cone by Remark 3.9.6, the proof of [Dra10, Lemma 2.1] shows that it suffices that this identity holds up to a scalar factor. It also suffices to prove this for g in an open dense subset U of $\mathrm{Sp}(V_q)$, because the equations for X_{n_0} pulled back along the map on the left for $g \in U$ imply the equations for all g. We will prove this, with $m = n_0$, using Lemma 3.9.14 below. $\square$

Lemma 3.9.14. *Let $g \geq n \geq n_0$. Then for all A in an open dense subset of $\mathrm{Sp}(V_q)$ there exist $A' \in \mathrm{Sp}(V_n)$ and $A'' \in \mathrm{Sp}(V_{n_0})$ such that*

$$\pi_{q,n_0} \circ A \circ \tau_{n,q} = c(g) A'' \circ \pi_{n,n_0} \circ A',$$

where $c(g)$ is a constant only depending on g.

A sketch of proof for the analogous statement for the spin group was given in Lemma 2.5.6 in Chapter 2. Nonetheless, we include a more detailed proof for the convenience of the reader.

Proof. Throughout this proof we will abbreviate

$$E := \langle e_{n_0+1}, \dots, e_q\rangle \subseteq V_q,\ E' := \langle e_{n_0+1}, \dots, e_n\rangle \subseteq V_n,\ F := \langle e_{-(n+1)}, \dots, e_{-q}\rangle \subseteq V_q.$$

We start by making two observations. First, the map $\pi_{q,n_0} = \Phi_{e_{n+1}} \circ \dots \circ \Phi_{e_q}$ is the same as the composition of

$$\bigwedge^q V_q \xrightarrow{\varphi_{e_q}} \bigwedge^{q-1} e_q^\perp \xrightarrow{\varphi_{e_{q-1}}} \bigwedge^{q-2} \langle e_{q-1}, e_q\rangle^\perp \xrightarrow{\varphi_{e_{q-2}}} \cdots \xrightarrow{\varphi_{e_{n_0+1}}} \bigwedge^{n_0} E^\perp,$$

and the map $\bigwedge^{n_0} E^\perp \to \bigwedge^{n_0} V_{n_0}$ induced by the projection $E^\perp \to E^\perp/E = V_{n_0}$ (here $(\cdot)^\perp$ stands for the orthogonal complement in V_q). Second, for any subspace $U = \langle u_1, \dots u_k\rangle$ of V_q the composition

$$\bigwedge^q V_q \xrightarrow{\varphi_{u_k}} \bigwedge^{q-1} u_k^\perp \xrightarrow{\varphi_{u_{k-1}}} \bigwedge^{q-2} \langle u_{k-1}, u_k\rangle^\perp \xrightarrow{\varphi_{u_{k-2}}} \cdots \xrightarrow{\varphi_{u_1}} \bigwedge^{q-k} U^\perp$$

is well-defined (i.e., independent of the choice of basis) up to a constant. Indeed, by (3.1.6) this composition is dual to $u_1 \wedge \dots \wedge u_k \wedge \bullet$, and for a different basis $u_1', \dots, u_k'$ of U the two k-forms $u_1 \wedge \dots \wedge u_k$ and $u_1' \wedge \dots \wedge u_k'$ agree up to a multiplicative constant. So for any subspace U the map $\varphi_U := \varphi_{u_1} \circ \dots \circ \varphi_{u_k}$ is well-defined (up to scalars). Furthermore, we will denote by $\Phi_U : \bigwedge^q V_q \to \bigwedge^{q-k} U^\perp/U$ the composition of φ_U and the map induced by the projection $V_q \to V_q/U$, e.g., $\Phi_E = \pi_{q,n_0}$.

For any $A \in \mathrm{Sp}(V_q)$ we will for ease of notation also abbreviate

$$E'' := A^{-1}E \subseteq V_q,$$

which is isotropic because E is isotropic. Then we have (up to scalars)

$$\pi_{q,n_0} \circ A = \bar{A} \circ \Phi_{E''}, \tag{3.9.3}$$

where $\bar{A} : (E'')^\perp/E'' \xrightarrow{\cong} E^\perp/E = V_{n_0}$ is an isometry induced by A.

Consider the subspace $E'' \cap (V_n \oplus F) \subseteq V_q$. For A in an open dense subset of $\mathrm{Sp}(V_q)$ this has the expected dimension

$$\begin{aligned} \dim\big(E'' \cap (V_n \oplus F)\big) &= \dim(E'') + \dim(V_n \oplus F) - \dim(V_q) \\ &= (q - n_0) + (2n + q - n) - 2q \\ &= n - n_0. \end{aligned} \tag{3.9.4}$$

Moreover, since $(E'')^\perp \subseteq V_q$ has codimension $\dim(E'') = q - n_0 \geq q - n = \dim(F)$, for A in an open dense subset of $\mathrm{Sp}(V_q)$ we also have

$$(E'')^\perp \cap F = 0. \tag{3.9.5}$$

We will from now on only consider $A \in \mathrm{Sp}(V_q)$ satisfying (3.9.4) and (3.9.5). Note that, as E'' is isotropic, (3.9.5) also implies $E'' \cap F = 0$. So the restriction of the projection $V_n \oplus F \to V_n$ to $E'' \cap (V_n \oplus F)$ has trivial kernel, and hence the image

$$\tilde{E} := \mathrm{Im}\big(E'' \cap (V_n \oplus F) \subseteq V_n \oplus F \xrightarrow{\text{mod } F} V_n\big) \subseteq V_n$$

also has dimension $n - n_0$ due to (3.9.4). Note that $\tilde{E}$ is also isotropic because the projection $V_n \oplus F \to V_n$ preserves the symplectic form. Again using that the projection preserves the symplectic form, we see that it restricts to an isomorphism

$$(E'')^\perp \cap (V_n \oplus F) \xrightarrow{\cong} \tilde{E}^\perp,$$

and hence induces an isometry

$$h_1 : \Big((E'')^\perp \cap (V_n \oplus F)\Big) / \big(E'' \cap (V_n \oplus F)\big) \xrightarrow{\cong} \tilde{E}^\perp / \tilde{E}$$

between symplectic vector spaces of dimension $2n_0$ (here $\tilde{E}^\perp \subseteq V_n$ denotes the orthogonal complement in V_n). Similarly, due to reasons of dimension, the inclusion $V_n \oplus F \subseteq V_q$ also induces an isometry

$$h_2 : \Big((E'')^\perp \cap (V_n \oplus F)\Big) / \big(E'' \cap (V_n \oplus F)\big) \xrightarrow{\cong} (E'')^\perp / E''.$$

One can check that the following diagram commutes (up to scalars):

$$\begin{array}{ccccccc}
 & & \bigwedge^q V_q & \xrightarrow{\varphi_{E''}} & \bigwedge^{n_0} (E'')^\perp & \xrightarrow{\text{mod } E''} & \bigwedge^{n_0} (E'')^\perp / E'' \\
 & \nearrow^{\tau_{n,q}} & \uparrow{\scriptstyle \text{incl.}} & & \uparrow{\scriptstyle \text{incl.}} & & \cong \uparrow \bigwedge^{n_0} h_2 \\
\bigwedge^n V_n & \xrightarrow{\tau_{n,q}} & \bigwedge^q (V_n \oplus F) & \xrightarrow{\varphi_{E''}} & \bigwedge^{n_0} (E'')^\perp \cap (V_n \oplus F) & \xrightarrow{\text{mod } E''} & \bigwedge^{n_0} \frac{(E'')^\perp \cap (V_n \oplus F)}{(E'') \cap (V_n \oplus F)} \\
 & & \searrow^{\varphi_{\tilde{E}}} & & \downarrow{\scriptstyle \text{mod } F} & & \cong \downarrow \bigwedge^{n_0} h_1 \\
 & & & & \bigwedge^{n_0} \tilde{E}^\perp & \xrightarrow{\text{mod } \tilde{E}} & \bigwedge^{n_0} \tilde{E}^\perp / \tilde{E}
\end{array}$$

So we have (up to scalars)

$$\Phi_{E''} \circ \tau_{n,q} = h_2 \circ h_1^{-1} \circ \Phi_{\tilde{E}},$$

where $\Phi_{E''} = (\text{mod } E'') \circ \varphi_{E''}$ and similarly for $\Phi_{\tilde{E}}$. Choose $A' \in \mathrm{Sp}(V_n)$ such that $A'(\tilde{E}) = E'$. Then

$$\overline{A'} \circ \Phi_{\tilde{E}} = \pi_{n,n_0} \circ A',$$

where $\overline{A'} : \tilde{E}^\perp / \tilde{E} \xrightarrow{\cong} (E')^\perp / E' = V_{n_0}$ is the isometry induced by A'. Together with (3.9.3) we thus obtain (up to scalars)

$$\pi_{q,n_0} \circ A \circ \tau_{n_0,q} = \overline{A} \circ h_2 \circ h_1^{-1} \circ (\overline{A'})^{-1} \circ \pi_{n,n_0} \circ A',$$

which is exactly the desired equality if we define

$$A'' := \overline{A} \circ h_2 \circ h_1^{-1} \circ (\overline{A'})^{-1} \in \mathrm{Sp}(V_{n_0}).$$

This completes the proof.

□

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