Axiomatizing One-Variable Fragments of First-Order Logics

Inauguraldissertation
der Philosophisch-naturwissenschaftlichen Fakultät
der Universität Bern

vorgelegt von

Naomi Tokuda

von Eggiwil BE

Leiter der Arbeit:

Prof. Dr. George Metcalfe

Mathematisches Institut der Universität Bern

Axiomatizing One-Variable Fragments of First-Order Logics

Inauguraldissertation
der Philosophisch-naturwissenschaftlichen Fakultät
der Universität Bern

vorgelegt von

Naomi Tokuda
von Eggiwil BE

Leiter der Arbeit:
Prof. Dr. George Metcalfe
Mathematisches Institut der Universität Bern

Von der Philosophisch-naturwissenschaftlichen Fakultät angenommen.

Bern, 29.08.2024

Der Dekan
Prof. Dr. Jean-Louis Reymond

Contents

Introduction

This thesis studies the one-variable fragments of a broad family of first-order logics and (partially) tackles the challenge of providing a general approach to axiomatizing them. (Equational) consequence in one-variable first-order logics can be translated — by replacing any atom $P(x)$ with a propositional variable p and occurrences of the quantifiers $(\forall x)$ and $(\exists x)$ with the modalities $\Box$ and $\Diamond$, respectively — into consequence in a class of algebras with modalities. Therefore, the challenge of finding axiomatizations for one-variable fragments may be interpreted as the challenge of finding a (natural) equational basis for the corresponding classes of algebras. We show that in certain cases, this class can be defined by "$\mathcal{S}5$-like" equations. This thesis is based on the two papers [24] and [25].

Propositional logics deal with propositions, i.e., statements that hold some truth value, and ways of altering and combining them to form more (complicated) propositions. For example, in classical propositional logic, the statement 'this cat is cute' follows from the statement 'this cat is cute and all crocodiles are scary'. This argument can be formalized in classical propositional logic as follows:

$p \wedge q$	$p:$ this cat is cute
p	$q:$ all crocodiles are scary

Note that this argument holds independently of the meaning of p and q. Propositional logic considers the logical operations, such as 'and', 'or', 'not', and not the meaning of the individual parts of a statement, such as 'this', 'is', 'are', 'all'.

There are some quantified statements that can be formalized in propositional logic. For example,

All cats are cute.
Filou is a cat.
Hence, Filou is cute.

can be formalized in classical propositional logic as:

$A \to B$	$A : x$ is a cat
$C \to A$	$B : x$ is cute
Hence, $C \to B$	$C : x$ is Filou

However, this formalization is very clunky and the same approach does not work if we reason about more than one object at the same time. Thus, propositional logics can be very useful, but their expressive power is limited. Consider for example the following argument:

Every cat-owner wants to pet their cat.
Filou is my cat.
Therefore, I want to pet Filou.

This argument cannot be formalized in propositional logic; however, in first-order classical logic it can be formalized as:

$$
\begin{array}{rl}
(\forall x)(\forall y)((C(y) \wedge O(x,y)) \to P(x,y)) & \quad C(y) : y \text{ is a cat} \\
C(\text{Filou}) \wedge O(\text{Naomi, Filou}) & \quad O(x,y) : x \text{ owns } y \\
\text{Therefore, } P(\text{Naomi, Filou}) & \quad P(x,y) : x \text{ wants to pet } y
\end{array}
$$

First-order logics are much more expressive than their propositional counterparts. We can reason about objects and their relationships, we can quantify over these objects and we can argue about all or some of them without being specific. However this expressivity comes with a significant downside for many first-order logics, namely, the lack of decidability. We call a logic C *decidable*, if there is an algorithm to decide whether a given formula is valid and *undecidable*, if there is no such algorithm. Most of the first-order logics that are well studied are undecidable. Church [20] showed that first-order classical logic is undecidable and via Gödel's double negation translation (see, e.g., [87]) it was shown that any first-order intermediate logic — logics that are stronger than intuitionistic logic and weaker than classical logic — is also undecidable. Hilbert-style axiomatizations have been presented for first-order classical logic [41, 45] and some first-order intermediate logics, in particular, first-order intuitionistic logic [40].

Decidability and axiomatization results have been obtained for other (non-classical) logics, such as many-valued and substructural logics. *Many-valued logics* were first considered by Łukasiewicz in [51] where he considered a three-valued logic. This idea was expanded to n-valued logics and infinite-valued logic, the latter is nowadays called *Łukasiewicz logic*. The valid formulas for first-order Łukasiewicz logic are not recursively enumerable [78]. Gödel [35] introduced a family of finite-valued (intermediate) logics and Dummett [30] presented the infinite-valued version called *Gödel (or Gödel-Dummett) logic*. Gödel logic is an intermediate logic, which means that its first-order version is undecidable (as mentioned above). An axiomatization for first-order Gödel logic is given by Horn in [42]. *Substructural logics* can be considered as logics given by proof systems that lack some "structural" rules. Alternatively, substructural logics can be considered algebraically, as those logics that have a class of FL-*algebras* (or *pointed residuated lattices*) as an algebraic semantics. These logics have been studied extensively, see [31, 58, 73, 76]. The *Full Lambek calculus* FL and its extensions are prominent examples of substructural logics.

Whereas the function-free fragments of the first-order versions of FL, FL_e, FL_w, and FL_{ew} are decidable (see [44]), even propositional FL_c and FL_{ec} are undecidable (see [21] and [70], respectively). Hilbert-style axiomatizations for these substructural logics can be found in [26, 29].

One way to remedy this problem while maintaining some of the expressivity of first-order logics, is by restricting the signature or formulas and considering fragments of the first-order logics. There are many ways to do this. Fragments that have been considered in the literature include monadic fragments, fluted fragments, guarded fragments, prenex fragments and fragments that restrict the number of variables that occur in the formulas. The latter have been studied to some extent, although in order to obtain a decidable fragment, the maximum number of variables considered is quite small: For example the one-variable and two-variable fragments of first-order classical logic are decidable (see, e.g., [49] and [79, 80]), whereas its three-variable fragment is undecidable [81]. For first-order intuitionistic logic its one-variable fragment is decidable [11], but its two-variable fragment is already undecidable [47]. The one-variable fragments of first-order Gödel logic and first-order Łukasiewicz logic are decidable (see [13] and [77], respectively), but decidability for their two-variable fragments remain unknown. Note that the examples of one-, two-, and three-variable fragments given above are all equality-free logics.

In this thesis, we focus on one-variable fragments of first-order logics and the challenge of providing axiomatizations for them. The one-variable fragment of a first-order logic[1] is the restriction of the consequence relation to consequences in the logic constructed using one distinguished variable x, unary predicates, propositional operations, and the quantifiers $(\forall x)$ and $(\exists x)$. The statements "All cats are cute", "Filou is a cat", and "Filou is cute" can be formalized using one-variable formulas. Let us consider another example of an argument:

All cats either love going outside or love being petted.
There exists a cat that does not love going outside.
Hence, there exists a cat that loves being petted.

This argument can be formalized in first-order classical logic as follows:

$(\forall x)(C(x) \to (G(x) \vee P(x)))$	$C(x)$: x is a cat
$(\exists x)(C(x) \wedge \neg G(x))$	$G(x)$: x loves going outside
Hence, $(\exists x)(C(x) \wedge P(x))$	$P(x)$: x loves being petted

In particular, since only one variable is used, this is a formalization in the one-variable fragment of first-order classical logic. One-variable fragments may be reformulated as propositional modal logics, by replacing occurrences of an atom $P(x)$ with a propositional variable p, and occurrences of $(\forall x)$ and $(\exists x)$ with the modalities $\Box$ and $\Diamond$, respectively. Typically, this modal logic is

[1] In this thesis (first-order) logics are equated with consequence relations.

algebraizable, that is, it is sound and complete with respect to some suitable class of algebraic structures (in an algebraic signature containing $\Box$ and $\Diamond$). Therefore, these one-variable fragments may be studied via the corresponding class of algebraic structures using the tools of universal algebra. Using this translation, we can formulate the above argument in propositional modal logic as follows:

$\Box(c \to (g \vee p))$	c: x is a cat
$\Diamond(c \wedge \neg g)$	g: x loves going outside
Hence, $\Diamond(c \wedge p)$	p: x loves being petted

In Chapter 1 we define a semantics for a first-order logic based on a class of $\mathcal{L}$-lattices, i.e., algebras with a lattice-reduct, and see that this induces a semantics for its one-variable fragment. However, note that in general the task of finding an axiomatization of the one-variable fragment of a first-order logic is not trivial. A Hilbert-style axiomatization of a first-order logic does not (at least directly) yield a Hilbert-style axiomatization of (the modal counterpart of) its one-variable fragment, since derivations of one-variable formulas may introduce new variables. Nevertheless, axiomatizations for some one-variable first-order logics have been obtained. For example, the modal counterpart of the one-variable fragment of first-order classical logic is S5, first axiomatized by Wajsberg [91], and corresponds to the variety of monadic Boolean algebras introduced by Halmos [38]. The modal counterpart of the one-variable fragment of first-order intuitionistic logic is MIPC, as shown by Bull [11], and corresponds to the variety of monadic Heyting algebras introduced by Monteiro and Varsavsky [61]. Varieties of monadic Heyting algebras corresponding to the modal counterparts of the one-variable fragments of first-order intermediate logics have been investigated in [7, 8, 13, 72, 82, 83]. In particular, it was shown that the one-variable fragments of first-order Gödel logic and Corsi's first-order logic of linear frames correspond to the variety of monadic Gödel algebras [15] and the variety of monadic Heyting algebras satisfying the prelinearity axiom [14], respectively. The one-variable fragments of other first-order many-valued logics have also been studied, in particular, the modal counterparts of the one-variable fragments of first-order Łukasiewicz logic and Abelian logic correspond to monadic MV-algebras [17, 28, 77] and monadic Abelian ℓ-groups [59], respectively.

Despite of all these results, a general approach to axiomatizing this class of algebraic structures corresponding to the one-variable fragment of a first-order logic, has been lacking. In this thesis, we take a first step towards overcoming this challenge, by proving that the class of algebraic structures corresponding to the one-variable fragment of a first-order logic based on a variety of $\mathcal{L}$-lattices that has the superamalgamation property (corresponding to the Craig interpolation property in some cases) admits a (natural) axiomatization by "$\mathcal{S}$5-like" equations. To this end, we develop both algebraic and proof-theoretic approaches. We illustrate the main ideas of our methods via classes of $\mathsf{FL_e}$-*algebras* (or *commutative pointed residuated lattices*), that encompass the running examples in this thesis. Classes of $\mathsf{FL_e}$-algebras provide semantics for particular substructural logics. In the following, we give a more detailed outline of the thesis and of how this result is achieved.

Outline of the Thesis

In Chapter 1 we introduce the preliminary notions needed in this thesis. In particular, we introduce the logics that will be used as recurring examples. In Section 1.1 we first recall some basic definitions from universal algebra. We define $\mathcal{L}$-*lattices*, the main algebraic structures of this thesis, and $\mathsf{FL_e}$-*algebras*, particular $\mathcal{L}$-lattices that encompass the running examples used to illustrate the main concepts. In Section 1.2 we define first-order logics via a semantics based on classes of $\mathcal{L}$-lattices. We prove that if a class $\mathcal{K}$ of $\mathcal{L}$-lattices admits regular completions, then the first-order logics based on $\mathcal{K}$ and the class of complete members of $\mathcal{K}$ coincide. We conclude this section by presenting the first-order extensions of some of the logics introduced in Section 1.1. Section 1.3 is used to introduce some proof-theoretic notions. In particular, we define proof systems and present a Hilbert-style axiomatization for $\mathcal{FL}_e$, the variety of $\mathsf{FL_e}$-algebras, and a sequent calculus for $\mathcal{L}at$, the variety of lattices. We introduce the sequent calculus ∀CFL, a multiset version of the first-order Full Lambek calculus with exchange, and prove that it has cut elimination. We also consider the first-order versions of the running examples and discuss some proof systems that have been obtained for them. In Section 1.4 we define one-variable fragments of first-order logics via a restriction of the semantics for the first-order logics introduced in Section 1.2. Then we consider some axiomatizations that have been obtained for (the modal counterparts of) the one-variable fragments of the logics considered in Section 1.2.

In Chapter 2 we define modal extensions of $\mathcal{L}$-lattices and prove that in certain cases, they provide axiomatizations for the class of algebraic structures corresponding to the one-variable fragments of first-order logics defined over classes of $\mathcal{L}$-lattices. In Section 2.1 we define *m-$\mathcal{L}$-lattices*, extensions of $\mathcal{L}$-lattices with the unary operators $\Box$ and $\Diamond$, that satisfy "$\mathcal{S}5$-like" equations. For any class of $\mathcal{L}$-lattices $\mathcal{K}$, we denote by $m\mathcal{K}$ the class of m-$\mathcal{L}$-lattices with an $\mathcal{L}$-lattice reduct in $\mathcal{K}$. We consider some examples of one-variable fragments of first-order logics from the literature that correspond to classes of m-$\mathcal{L}$-lattices. In Section 2.2 we prove a one-to-one correspondence between m-$\mathcal{L}$-lattices and ordered pairs of $\mathcal{L}$-lattices and subalgebras that satisfy a relative completeness condition. This generalizes previous results in the literature (see, e.g., [7, 89]). In Section 2.3 we consider *functional* m-$\mathcal{L}$-lattices consisting of certain functions from a set W to an $\mathcal{L}$-lattice $\mathbf{A}$. We prove that the semantics of one-variable first-order logics can be identified with evaluations into functional m-$\mathcal{L}$-lattices. If $\mathcal{K}$ is a class of $\mathcal{L}$-lattices closed under taking subalgebras and direct powers, we obtain a correspondence between consequence in the one-variable first-order logic defined over $\mathcal{K}$ and consequence in the functional members of $m\mathcal{K}$. In Section 2.4 we achieve the main goal of the chapter, obtaining an axiomatization of the class of algebraic structures corresponding to the one-variable fragment of first-order logics defined over certain varieties of $\mathcal{L}$-lattices. We consider classes of $\mathcal{L}$-lattices, that have the *superamalgamation property*, an important algebraic property, studied for example in [54–56], that corresponds to the Craig interpolation property in the setting of $\mathsf{FL_e}$-algebras. We then prove a functional completeness theorem for such a class $\mathcal{K}$ that is closed under taking direct limits and subalgebras, showing that any member of $m\mathcal{K}$ is functional (generalizing a representation theorem of Bezhanishvili and Harding for monadic Heyting algebras [8]). This theorem together with the results from Section 2.3 yields Corollary 2.4.2:

If $\mathcal{V}$ is a variety of $\mathcal{L}$-lattices that has the superamalgamation property, then for any set $\Sigma \cup \{\varphi \approx \psi\}$ of one-variable equations,

$$\Sigma \models^{\forall 1}_{\mathcal{V}} \varphi \approx \psi \quad \Longleftrightarrow \quad \Sigma^* \models_{m\mathcal{V}} \varphi^* \approx \psi^*,$$

where $(-)^*$ denotes the (standard) translation from one-variable formulas to modal formulas that replaces atoms with propositional variables and quantifiers $(\forall x)$ and $(\exists x)$ with $\Box$ and $\Diamond$, respectively.

In Chapter 3 we provide an alternative proof-theoretic approach to proving Corollary 2.4.2 for the one-variable fragments of certain first-order substructural logics. In Section 3.1 we introduce the sequent calculus ∀1CFL. This sequent calculus is sound and complete with respect to consequence in the one-variable first-order logic defined over $\mathcal{FL}_e$, introduced in Section 1.4. Section 3.2 is used to prove an interpolation property for certain sequents derivable in ∀1CFL, in particular, for sequents that occur in the derivation of a one-variable sequent. In Section 3.3 we give an alternative proof of Corollary 2.4.2 for the variety $\mathcal{FL}_e$ using proof-theoretic methods. The key idea of this proof is to show (using the interpolation property) that additional variables in a derivation of a one-variable sequent can be eliminated. In Section 3.4 we apply the method from Section 3.3 to extensions of ∀1CFL with sets of simple rules that have exactly one premise. In particular, we prove Corollary 2.4.2 for the varieties $\mathcal{FL}_{ew}$ and $\mathcal{FL}_{ec}$.

In Chapter 4 we summarize the achievements of this thesis and consider cases of one-variable first-order logics that have been axiomatized in the literature, but are not covered by our methods. We also give an outlook on future avenues to continue this work and axiomatize an even broader family of one-variable first-order logics.

Chapter 1

The Logics

This chapter introduces the logics that are considered in this thesis. In Section 1.1 we introduce the algebraic structures that provide the algebraic semantics for the propositional versions of these logics as well as the semantics for the first-order versions defined in Section 1.2. In Section 1.3, we define some proof-theoretic notions and introduce a first-order version of the Full Lambek calculus with exchange. Finally, in Section 1.4, we introduce the one-variable fragments of the first-order logics defined in Section 1.2. We assume familiarity with basic notions of Universal Algebra as found in [12].

We begin by defining the formulas used in this thesis. Let $\mathcal{L}$ be an algebraic signature. The sets of propositional formulas are defined as follows:

1. The set $\mathrm{Fm}(\mathcal{L})$ of propositional formulas is built inductively using a countably infinite set of propositional variables $\{p_i\}_{i \in \mathbb{N}}$ and the operations in $\mathcal{L}$. The elements of $\mathrm{Fm}(\mathcal{L})$ are called *$\mathcal{L}$-formulas* and are usually denoted by $\alpha, \beta, \ldots$.

2. The set $\mathrm{Fm}_{\Box}(\mathcal{L})$ of modal propositional formulas is built inductively using a countably infinite set of propositional variables $\{p_i\}_{i \in \mathbb{N}}$, the operations in $\mathcal{L}$, and the unary operations symbols $\Box$ and $\Diamond$. The elements of $\mathrm{Fm}_{\Box}(\mathcal{L})$ are called *$\mathcal{L}_{\Box}$-formulas* and are usually denoted by $\alpha, \beta, \ldots$.

Note that the first-order formulas considered in this thesis are all function-free and equality-free. Hence, the sets of first-order formulas are defined as follows:

1. The set $\mathrm{Fm}_{\forall}(\mathcal{L})$ of first-order formulas is built inductively using the union over all $n \in \mathbb{N}$ of the countably infinite sets of n-ary predicates $\{P_{n,i}\}_{i \in \mathbb{N}}$, the countably infinite set Var of variables, the operations in $\mathcal{L}$, and quantifiers $(\forall x)$, $(\exists x)$ for any $x \in \mathrm{Var}$. The elements of $\mathrm{Fm}_{\forall}(\mathcal{L})$ are called *$\mathcal{L}_{\forall}$-formulas* and are usually denoted by $\varphi, \psi, \chi, \ldots$.

2. The set $\mathrm{Fm}^1_{\forall}(\mathcal{L})$ of one-variable formulas is built inductively using the countably infinite set of unary predicates $\{P_i\}_{i \in \mathbb{N}}$, a variable x, the operations in $\mathcal{L}$, and quantifiers $(\forall x)$ and $(\exists x)$. The elements of $\mathrm{Fm}^1_{\forall}(\mathcal{L})$ are called *one-variable $\mathcal{L}_{\forall}$-formulas* and are usually denoted by $\varphi, \psi, \chi, \ldots$.

3. The set $\mathrm{Fm}^{1+}_{\forall}(\mathcal{L})$ of extended one-variable formulas is built inductively using the countably infinite set of unary predicates $\{P_i\}_{i\in\mathbb{N}}$, the variables $\{x\} \cup \{x_i\}_{i\in\mathbb{N}}$ such that $\{x\} \cap \{x_i\}_{i\in\mathbb{N}} = \emptyset$, the operations in $\mathcal{L}$, and quantifiers $(\forall x)$, $(\exists x)$. The elements of $\mathrm{Fm}^{1+}_{\forall}(\mathcal{L})$ are called $\mathcal{L}^{+}_{\forall}$*-formulas* and are usually denoted by $\varphi, \psi, \chi, \ldots$.

We now give the definition of a logic (see, e.g., [43]) used in this thesis. Let Fm be one of the sets of formulas defined above. Let $\vdash \subseteq \mathcal{P}(\mathrm{Fm}) \times \mathrm{Fm}$ where a pair $\langle T, \alpha\rangle$ in $\vdash$ is denoted by $T \vdash \alpha$. We call $\vdash$ a *consequence relation* over Fm, if it satisfies the following properties for any $\alpha \in \mathrm{Fm}$ and $T, S \subseteq \mathrm{Fm}$:

1. if $\alpha \in T$, then $T \vdash \alpha$ *(reflexivity)*;
2. if $T \vdash \alpha$ and $T \subseteq S$, then $S \vdash \alpha$ *(monotonicity)*;
3. if $T \vdash \alpha$ and $S \vdash \beta$ for all $\beta \in T$, then $S \vdash \alpha$ *(transitivity)*.

A *logic* can be defined as the pair $\langle \mathrm{Fm}, \vdash\rangle$, where $\vdash$ is a consequence relation over Fm[1]. If the set Fm is clear from the context, we equate the logic $\langle \mathrm{Fm}, \vdash\rangle$ with the consequence relation $\vdash$.

1.1 Algebraic Semantics

In this section we recall some basic algebraic notions such as the formula algebra and consequence in a class of algebraic structures. We introduce $\mathcal{L}$-lattices, the algebraic structures that form the basis for the (algebraic) semantics for all the logics considered in this thesis. We conclude this section with the introduction of $\mathsf{FL_e}$-algebras, particular $\mathcal{L}$-lattices, and a list of examples of logics whose algebraic semantics are given by classes of $\mathsf{FL_e}$-algebras.

Let $\mathcal{L}$ be an arbitrary algebraic signature. Recall that we denote by $\mathrm{Fm}(\mathcal{L})$ the set of $\mathcal{L}$-*formulas* $\alpha, \beta, \ldots$ built inductively using a countably infinite set of propositional variables $\{p_i\}_{i\in\mathbb{N}}$ and the operations in $\mathcal{L}$. An ordered pair of $\mathcal{L}$-formulas $\alpha \approx \beta$ is called an $\mathrm{Fm}(\mathcal{L})$-*equation*.

An algebraic structure in the signature $\mathcal{L}$ is called an $\mathcal{L}$-*algebra*. Let us denote by $\mathcal{L}_n$ the set of n-ary operations of $\mathcal{L}$. Then the *formula algebra* of $\mathcal{L}$ is the $\mathcal{L}$-algebra

$$\mathbf{Fm}(\mathcal{L}) = \langle \mathrm{Fm}(\mathcal{L}), \{\star^{\mathbf{Fm}(\mathcal{L})} \mid n \in \mathbb{N}, \star \in \mathcal{L}_n\}\rangle,$$

where for each $n \in \mathbb{N}$, $\star \in \mathcal{L}_n$, and $\alpha_1, \ldots, \alpha_n \in \mathrm{Fm}(\mathcal{L})$,

$$\star^{\mathbf{Fm}(\mathcal{L})}(\alpha_1 \ldots, \alpha_n) = \star(\alpha_1, \ldots, \alpha_n).$$

[1]In the case where Fm is a set of propositional formulas, we can assume that $\vdash$ is also closed under substitutions.

A homomorphism from the formula algebra of $\mathcal{L}$ to an $\mathcal{L}$-algebra $\mathbf{A}$ is called an $\mathbf{A}$-*evaluation*. Let $\mathcal{K}$ be a class of $\mathcal{L}$-algebras. For a set of $\mathrm{Fm}(\mathcal{L})$-equations $\Sigma \cup \{\alpha \approx \beta\}$ we define

$$\Sigma \vDash_{\mathcal{K}} \alpha \approx \beta :\Longleftrightarrow \text{for every } \mathbf{A} \in \mathcal{K} \text{ and } \mathbf{A}\text{-evaluation } f,$$
$$f(\alpha') = f(\beta') \text{ for all } \alpha' \approx \beta' \in \Sigma \Longrightarrow f(\alpha) = f(\beta),$$

and say that $\alpha \approx \beta$ is a *consequence* of Σ in $\mathcal{K}$. For sets of $\mathrm{Fm}(\mathcal{L})$-equations Σ and Σ', we write $\Sigma \vDash_{\mathcal{K}} \Sigma'$ if $\Sigma \vDash_{\mathcal{K}} \varphi \approx \psi$ for all $\varphi \approx \psi \in \Sigma'$.

Remark 1.1.1. Note that $\vDash_{\mathcal{K}}$ is an *equational consequence relation* that satisfies the same properties for (sets of) equations instead of (sets of) formulas, as a consequence relation on formulas (i.e., reflexivity, monotonicity, transitivity), introduced at the beginning of Chapter 1.

Let $\mathcal{K}$ be a class of $\mathcal{L}$-algebras, then $\mathbb{H}(\mathcal{K})$, $\mathbb{S}(\mathcal{K})$, $\mathbb{P}(\mathcal{K})$, $\mathbb{I}(\mathcal{K})$, $\mathbb{P}_U(\mathcal{K})$, and $\mathbb{U}(\mathcal{K})$ denote the classes of homomorphic images of $\mathcal{L}$-algebras in $\mathcal{K}$, subalgebras of $\mathcal{L}$-algebras in $\mathcal{K}$, direct products of $\mathcal{L}$-algebras in $\mathcal{K}$, isomorphic copies of $\mathcal{L}$-algebras in $\mathcal{K}$, ultraproducts of $\mathcal{L}$-algebras in $\mathcal{K}$, and $\mathcal{L}$-algebras $\mathbf{A}$ such that every countably generated subalgebra of $\mathbf{A}$ belongs to $\mathcal{K}$, respectively.

A *variety* is a class $\mathcal{V}$ of algebraic structures in the same signature that is closed under taking homomorphic images, subalgebras and (direct) products. By Birkhoff's Theorem [10], a class $\mathcal{V}$ is a variety if and only if it is an equational class, that is, a class of algebraic structures defined by a set of equations.

Before we introduce $\mathcal{L}$-lattices, the main algebraic structures of this thesis, we recall the definition of a lattice. Let $\mathcal{L}_l$ be the algebraic signature consisting of the binary operations $\wedge$ and $\vee$. A *lattice* is an $\mathcal{L}_l$-algebra $\mathbf{L} = \langle L, \wedge, \vee \rangle$ such that $\wedge$ and $\vee$ are binary operations satisfying the following equations:

$$\begin{aligned}
&x \wedge y \approx y \wedge x, && x \vee y \approx y \vee x,\\
&x \wedge (y \wedge z) \approx (x \wedge y) \wedge z, && x \vee (y \vee z) \approx (x \vee y) \vee z,\\
&x \wedge x \approx x, && x \vee x \approx x,\\
&x \wedge (x \vee y) \approx x, && x \vee (x \wedge y) \approx x.
\end{aligned}$$

Alternatively, a lattice can be defined as a poset $\langle L, \leq \rangle$ — a nonempty set L with a reflexive, antisymmetric, transitive binary relation $\leq$ — such that for all $a, b \in L$

$$\inf\{a, b\} \qquad \text{and} \qquad \sup\{a, b\}$$

exist in L. The two definitions of lattices are equivalent. Starting with a lattice $\langle L, \wedge, \vee \rangle$, by setting

$$a \leq b \quad :\Longleftrightarrow \quad a \wedge b = a,$$

we obtain a lattice $\langle L, \leq \rangle$. Starting with a lattice $\langle L, \leq \rangle$ and defining

$$a \wedge b := \inf\{a, b\} \quad \text{and} \quad a \vee b := \sup\{a, b\},$$

we obtain a lattice $\langle L, \wedge, \vee\rangle$. The class of all lattices forms a variety, denoted by $\mathcal{L}at$.

Let $\mathcal{L}_l'$ be the algebraic signature extending $\mathcal{L}_l$ with the constants $\bot$ and $\top$. The $\mathcal{L}_l'$-algebra $\langle L, \wedge, \vee, \bot, \top\rangle$ is called a *bounded lattice* if $\langle L, \wedge, \vee\rangle$ is a lattice and for all $a \in L$,

$$\bot \leq a \leq \top.$$

A lattice or bounded lattice $\mathbf{L}$ is called *distributive*, if for any $a, b, c \in L$,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \quad \text{and} \quad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

Let $\mathcal{L}$ be an algebraic signature. Whenever $\mathcal{L}_2$ contains distinct operations $\wedge$ and $\vee$, we call $\mathcal{L}$ *lattice-oriented* and an $\mathcal{L}$-algebra $\mathbf{A}$ an *$\mathcal{L}$-lattice*, if the reduct $\langle A, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}\rangle$ is a lattice.

In particular, lattices are $\mathcal{L}_l$-lattices.

An $\mathcal{L}$-lattice $\mathbf{A}$ is called *complete*, if for all $X \subseteq A$,

$$\bigwedge X := \inf X \qquad \text{and} \qquad \bigvee X := \sup X$$

exist in A. For any class $\mathcal{K}$ of $\mathcal{L}$-lattices, we denote by $\mathcal{K}^c$ the class of all complete $\mathcal{L}$-lattices contained in $\mathcal{K}$.

Let $\mathbf{A}$ and $\mathbf{B}$ be $\mathcal{L}$-lattices, then $\mathbf{B}$ is called a *regular completion* of $\mathbf{A}$, if $\mathbf{B}$ is a complete $\mathcal{L}$-lattice and there exists an $\mathcal{L}$-lattice embedding $h\colon \mathbf{A} \to \mathbf{B}$ that preserves all existing meets and joins in $\mathbf{A}$, i.e., for any $X, Y \subseteq A$ such that the meet of X and the join of Y exist in A,

$$h(\bigwedge X) = \bigwedge h(X) \qquad \text{and} \qquad h(\bigvee Y) = \bigvee h(Y).$$

A class $\mathcal{K}$ of $\mathcal{L}$-lattices is said to *admit regular completions* if for any $\mathbf{A} \in \mathcal{K}$, there exists a regular completion of $\mathbf{A}$ in $\mathcal{K}$.

An $\mathcal{L}$-lattice $\mathbf{A}$ is called *totally ordered*, if for all $a, b \in A$,

$$a \leq b \qquad \text{or} \qquad b \leq a.$$

For a class $\mathcal{K}$ of $\mathcal{L}$-lattices, we denote by $\mathcal{K}_{\mathsf{to}}$ the class of all totally ordered members of $\mathcal{K}$.

At the beginning of this chapter, we introduced logics as consequence relations. We now define what it means for a class of $\mathcal{L}$-lattices to be an algebraic semantics for a logic. Let $\langle \mathrm{Fm}(\mathcal{L}), \vdash\rangle$ be a logic, then we call a class $\mathcal{K}$ of $\mathcal{L}$-lattices an *algebraic semantics for* $\vdash$ if there exists a set of equations $\tau(x)$ in one variable such that for all sets of $\mathcal{L}$-formulas $S \cup \{\varphi\}$ the following is satisfied:

$$S \vdash \varphi \quad \Longleftrightarrow \quad \tau(S) \vDash_{\mathcal{K}} \tau(\varphi),$$

where $\tau(\varphi)$ denotes the set of $\mathrm{Fm}(\mathcal{L})$-equations obtained by substituting x with φ in all equations in $\tau(x)$, and $\tau(S)$ is the union of the sets $\tau(\psi)$ for all $\psi \in S$.

Note that an algebraic semantics for a logic $\vdash$ is not unique. In particular, we have the following relationships between consequence in varieties and consequence in classes that generate them in different ways (see, e.g., [12]). Let $\mathcal{V}$ be a variety of $\mathcal{L}$-lattices. If $\mathcal{V}$ is generated as a variety by $\mathcal{K} \subseteq \mathcal{V}$, i.e., $\mathcal{V} = \mathbb{HSP}(\mathcal{K})$, then for any $\mathrm{Fm}(\mathcal{L})$-equation ε,

$$\models_{\mathcal{V}} \varepsilon \qquad \Longleftrightarrow \qquad \models_{\mathcal{K}} \varepsilon.$$

If $\mathcal{V}$ is generated as a quasivariety by $\mathcal{K} \subseteq \mathcal{V}$, i.e., $\mathcal{V} = \mathbb{ISPP}_U(\mathcal{K})$, then for any finite set $\Sigma \cup \{\varepsilon\}$ of $\mathrm{Fm}(\mathcal{L})$-equations,

$$\Sigma \models_{\mathcal{V}} \varepsilon \qquad \Longleftrightarrow \qquad \Sigma \models_{\mathcal{K}} \varepsilon.$$

If $\mathcal{V}$ is generated as a generalized quasivariety by $\mathcal{K} \subseteq \mathcal{V}$, i.e., $\mathcal{V} = \mathbb{ISPU}(\mathcal{K})$ (see [62]), then for any set $\Sigma \cup \{\varepsilon\}$ of $\mathrm{Fm}(\mathcal{L})$-equations,

$$\Sigma \models_{\mathcal{V}} \varepsilon \qquad \Longleftrightarrow \qquad \Sigma \models_{\mathcal{K}} \varepsilon.$$

Thus, if $\mathcal{K}$ generates $\mathcal{V}$ as a generalized quasivariety, $\mathcal{K}$ and $\mathcal{V}$ provide *equivalent* algebraic semantics for the same logic.

Let $\mathcal{L}_s$ be the signature consisting of the binary operations $\wedge, \vee, \cdot$, and $\to$ and the constants f and e. An $\mathsf{FL_e}$*-algebra* (see, e.g., [31, 58]) is an $\mathcal{L}_s$-lattice $\langle A, \wedge, \vee, \cdot, \to, \mathrm{f}, \mathrm{e}\rangle$ such that $\langle A, \cdot, \mathrm{e}\rangle$ is a commutative monoid and $\to$ is the residual of $\cdot$, that is, for any $a, b, c \in A$,

$$a \cdot b \le c \qquad \Longleftrightarrow \qquad a \le b \to c.$$

$\mathsf{FL_e}$-algebras are also referred to as *commutative pointed residuated lattices* and the class of all $\mathsf{FL_e}$-algebras can be defined by the equations defining lattices (given above), the equations defining commutative monoids, i.e.,

$$\begin{aligned} &x \cdot (y \cdot z) \approx (x \cdot y) \cdot z, \\ &x \cdot y \approx y \cdot x, \\ &x \cdot \mathrm{e} \approx x, \end{aligned}$$

and the following (in)equations

$$\begin{aligned} &x \cdot (y \vee z) \approx x \cdot y \vee x \cdot z, \\ &x \to y \le x \to (y \vee z), \\ &x \cdot (x \to y) \le y \le x \to (x \cdot y). \end{aligned}$$

Therefore, the class of all $\mathsf{FL_e}$-algebras forms a variety denoted by $\mathcal{FL}_e$, which provides an algebraic semantics for the Full Lambek Calculus with exchange $\mathrm{FL_e}$, or alternatively, the propositional version of the calculus $\forall$CFL introduced in Section 1.3. Varieties of FL_e-algebras provide algebraic semantics for substructural logics. For example, $\mathcal{FL}_{ew}$ and $\mathcal{FL}_{ec}$ are the varieties of $\mathsf{FL_e}$-algebras that satisfy the inequations

$$\mathrm{f} \le x \le \mathrm{e} \qquad \text{and} \qquad x \le x \cdot x,$$

respectively. $\mathcal{FL}_{ew}$ and $\mathcal{FL}_{ec}$ provide algebraic semantics for the Full Lambek calculus with exchange and weakening and the Full Lambek calculus with exchange and contraction, respectively. First-order versions of these calculi are introduced in Section 1.3 and in Chapter 3 the one-variable fragments of $\mathrm{FL_e}$, $\mathrm{FL_{ew}}$, and $\mathrm{FL_{ec}}$ are studied.

Remark 1.1.2. FL-*algebras*, i.e., *pointed residuated lattices* (see, e.g., [31, 58]) are also examples of $\mathcal{L}'_s$-lattices for the signature $\mathcal{L}'_s$ consisting of the binary operations $\wedge, \vee, \cdot, \backslash$, and $/$ and the constants f and e. The variety $\mathcal{FL}$ of FL-algebras forms an algebraic semantics for the Full Lambek calculus and $\mathcal{FL}_e$ is term-equivalent to the subvariety of $\mathcal{FL}$ defined by $x \cdot y \approx y \cdot x$ (identify $\backslash$ and $/$). Hence, FL-algebras fit in this framework of $\mathcal{L}$-lattices, however, we use FL_e-algebras as the basis for the examples in this thesis, since the main theorems do not apply in the case of $\mathcal{FL}$.

Let $\mathbf{A}$ be an $\mathcal{L}_s$-lattice and consider for each $X \subseteq A$,

$$X^u := \{a \in A \mid a \geq x \text{ for all } x \in X\},$$
$$X^l := \{a \in A \mid a \leq x \text{ for all } x \in X\}.$$

Then the *(Dedekind) MacNeille completion* of $\mathbf{A}$ (see, e.g., [31, 58]) is the $\mathcal{L}_s$-lattice

$$\mathcal{N}(\mathbf{A}) := \langle \mathcal{P}(A)^{ul}, \cap, \cup_{\mathcal{N}}, \cdot_{\mathcal{N}}, \to_{\mathcal{N}}, \mathrm{e}_{\mathcal{N}}, \mathrm{f}_{\mathcal{N}} \rangle,$$

where

$$\mathcal{P}(A)^{ul} := \{X \in \mathcal{P}(A) \mid (X^u)^l = X\}$$

and for all $X, Y \in \mathcal{P}(A)^{ul}$,

$$X \cup_{\mathcal{N}} Y := (X \cup Y)^{ul}, \qquad X \cdot_{\mathcal{N}} Y := \{x \cdot y \mid x \in X, y \in Y\}^{ul},$$
$$\mathrm{e}_{\mathcal{N}} := \{\mathrm{e}\}^l, \qquad X \to_{\mathcal{N}} Y := \{z \mid x \cdot z \in Y \text{ for all } x \in X\},$$
$$\mathrm{f}_{\mathcal{N}} := \{\mathrm{f}\}^l.$$

For any $\mathsf{FL_e}$-algebra $\mathbf{A}$, the $\mathcal{L}_s$-lattice $\mathcal{N}(\mathbf{A})$ is an $\mathsf{FL_e}$-algebra and

$$l \colon \mathbf{A} \to \mathcal{N}(\mathbf{A}); \; x \mapsto \{x\}^l$$

is an embedding that preserves all existing meets and joins, making $\mathcal{N}(\mathbf{A})$ a regular completion of $\mathbf{A}$. Thus, any class of $\mathsf{FL_e}$-algebras that is closed under MacNeille completions admits regular completions.

For any class $\mathcal{K}$ of $\mathsf{FL_e}$-algebras we can define a *consequence relation on formulas*[2] corresponding to $\vDash_{\mathcal{K}}$. Let $T \cup \{\alpha\}$ be a set of $\mathcal{L}_s$-formulas, then

$$T \vDash_{\mathcal{K}} \alpha \quad :\Longleftrightarrow \quad \{\mathrm{e} \leq \beta \mid \beta \in T\} \vDash_{\mathcal{K}} \mathrm{e} \leq \alpha.$$

[2]Indeed, such a consequence relation on formulas can be defined for any class $\mathcal{K}$ of $\mathcal{L}$-lattices such that any member of $\mathcal{K}$ has an $\mathsf{FL_e}$-algebra reduct. We will denote both consequence relations by $\vDash_{\mathcal{K}}$.

We can also go from a consequence of formulas to a consequence of equations. Let $\Sigma \cup \{\alpha \approx \beta\}$ be a set of $\mathrm{Fm}(\mathcal{L}_s)$-equations, then

$$\Sigma \vDash_{\mathcal{K}} \alpha \approx \beta \quad \iff \quad \{\alpha' \leftrightarrow \beta' \mid \alpha' \approx \beta' \in \Sigma\} \vDash_{\mathcal{K}} \alpha \leftrightarrow \beta,$$

where $\alpha \leftrightarrow \beta := (\alpha \to \beta) \wedge (\beta \to \alpha)$. That is, we may view every class $\mathcal{K}$ of $\mathsf{FL_e}$-algebras as determining a (substructural) propositional logic where $\mathcal{K}$ provides an algebraic semantics for it (via the set of equations $\tau(x) := \{\mathrm{e} \leq x\}$).

Now we present the main running examples of $\mathcal{L}$-lattices that provide algebraic semantics for the logics considered in this thesis.

Example 1.1.3. The following examples can all be defined as special classes of $\mathsf{FL_e}$-algebras. Let $\mathcal{L}_i$ be the signature $\mathcal{L}_s$ without $\cdot$ and f, e replaced with $0, 1$, respectively. The lattice-oriented signature $\mathcal{L}_b$ consists of the binary operations $\wedge, \vee$, the unary operation $\neg$ and the constants $0, 1$ and the lattice-oriented signature $\mathcal{L}_a$ contains the binary operations $\wedge, \vee$, and $+$, the unary operation $-$, and the constant 0.

- An $\mathcal{L}_i$-lattice $\mathbf{H} = \langle H, \wedge, \vee, \to, 0, 1\rangle$ is called a *Heyting algebra* whenever $\langle H, \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice and $\to$ is the residual of $\wedge$. The variety of Heyting algebras is denoted by $\mathcal{HA}$ and provides an algebraic semantics for *intuitionistic propositional logic* IPC (see e.g. [18]). Heyting algebras are term-equivalent to $\mathsf{FL_e}$-algebras satisfying

  $$x \cdot y \approx x \wedge y \qquad \text{and} \qquad \mathrm{f} \leq x \leq \mathrm{e},$$

  where we identify 0 and f, and 1 and e. Thus, the variety $\mathcal{HA}$ is term-equivalent to $\mathcal{FL}_{ewc} = \mathcal{FL}_{ew} \cap \mathcal{FL}_{ec}$. Varieties of Heyting algebras provide algebraic semantics for intermediate logics. See [9] for an extended survey of propositional intermediate logics and their different semantics.

- An $\mathcal{L}_b$-lattice $\mathbf{B} = \langle B, \wedge, \vee, \neg, 0, 1\rangle$ is called a *Boolean algebra* if the algebra $\langle B, \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice satisfying

  $$x \wedge \neg x \approx 0 \qquad \text{and} \qquad x \vee \neg x \approx 1.$$

  The class of all Boolean algebras forms the variety $\mathcal{BA}$ that is term-equivalent to the subvariety of $\mathcal{HA}$ defined by

  $$(x \to 0) \to 0 \approx x.$$

  The variety $\mathcal{BA}$ provides an algebraic semantics for *classical logic* CPC. The *standard Boolean algebra* is the 2-*element Boolean algebra* $\mathbf{2} = \langle \{0, 1\}, \wedge, \vee, \neg, 0, 1\rangle$, where

  $$a \wedge b = \min\{a, b\}, \quad a \vee b = \max\{a, b\}, \quad \neg 0 = 1, \quad \neg 1 = 0.$$

  The $\mathcal{L}_b$-algebra $\mathbf{2}$ generates $\mathcal{BA}$ as a generalized quasivariety and for any set of $\mathcal{L}_b$-formulas $T \cup \{\alpha\}$,

  $$T \vDash_{\mathcal{BA}} \alpha \qquad \iff \qquad T \vDash_{\mathbf{2}} \alpha.$$

- A Heyting algebra $\mathbf{G}$ is called a *Gödel algebra*, if it satisfies the equation

$$(x \to y) \vee (y \to x) \approx \mathrm{e}.$$

The *standard Gödel algebra* $\mathbf{G}$ is the Gödel algebra $\langle [0,1], \wedge, \vee, \to, 0, 1\rangle$, where for any $a, b \in [0,1]$, $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$, and

$$a \to b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise.} \end{cases}$$

We denote the variety of Gödel algebras by $\mathcal{GA}$, which provides an algebraic semantics for the *(infinite-valued) Gödel (or Gödel-Dummett) logic* LC (see, e.g., [3]), which was first presented by Dummett [30].

Subalgebras of $\mathbf{G}$ provide algebraic semantics for what are known as *Gödel logics*, a family of many-valued logics. An infinite family of finite-valued Gödel logics was introduced by Gödel to show that there are infinitely many logics between IPC and CPC [35]. The standard Gödel algebra $\mathbf{G}$ as well as any infinite subalgebra of $\mathbf{G}$ provide an alternative algebraic semantics for LC.

There are countably infinitely many different (propositional) Gödel logics and exactly one infinite-valued one.

- An MV-*algebra* (introduced by Chang in [19]) is an algebraic structure $\mathbf{A} = \langle A, \oplus, \neg, 0\rangle$, in the algebraic signature $\mathcal{L}_{mv}$ containing the binary operation $\oplus$, the unary operation $\neg$, and the constant 0, satisfying

$$\begin{array}{ll} x \oplus (y \oplus z) \approx (x \oplus y) \oplus z, & \neg\neg x \approx x, \\ x \oplus y \approx y \oplus x, & x \oplus \neg 0 \approx \neg 0, \\ x \oplus 0 \approx x, & \neg(\neg x \oplus y) \oplus y \approx \neg(\neg y \oplus x) \oplus x. \end{array}$$

The class of all MV-algebras forms the variety $\mathcal{MV}$ and is term-equivalent to the variety of $\mathsf{FL_e}$-algebras satisfying

$$\mathrm{f} \leq x \leq \mathrm{e} \quad \text{and} \quad (x \to y) \to y \approx x \vee y.$$

The variety $\mathcal{MV}$ provides an algebraic semantics for *Łukasiewicz logic* Ł (see e.g., [63], [23]), introduced by Łukasiewicz in [53]. The *standard* MV-*algebra* is the MV-algebra $\mathbf{Ł} = \langle [0,1], \oplus, \neg, 0\rangle$, where

$$a \oplus b \coloneqq \min\{a + b, 1\} \quad \text{and} \quad \neg a \coloneqq 1 - a.$$

The variety $\mathcal{MV}$ is generated as a quasivariety by $\mathbf{Ł}$ and for any finite set $T \cup \{\alpha\}$ of $\mathcal{L}_{mv}$-formulas,

$$T \models_{\mathcal{MV}} \alpha \qquad \iff \qquad T \models_{\mathbf{Ł}} \alpha.$$

- An $\mathcal{L}_a$-lattice $\mathbf{A} = \langle A, \wedge, \vee, +, -, 0\rangle$ is called a *lattice-ordered abelian group* (*abelian ℓ-group* for short), if $\langle A, +, -, 0\rangle$ is an abelian group and for all $a, b, c \in A$,
$$a \leq b \quad \Longrightarrow \quad a + c \leq b + c.$$
The class of all abelian ℓ-groups forms a variety, denoted by $\mathcal{LG}$, that is term-equivalent to the variety of $\mathsf{FL_e}$-algebras satisfying
$$(x \to \mathrm{e}) \cdot x \approx \mathrm{e} \qquad \text{and} \qquad \mathrm{f} \approx \mathrm{e}.$$
The variety $\mathcal{LG}$ provides an algebraic semantics for *Abelian logic*, introduced independently in [60] and [16]. We define the algebra $\mathbf{R} = \langle \mathbb{R}, \wedge, \vee, +, -, 0\rangle$ where $\langle \mathbb{R}, +, -, 0\rangle$ is the usual additive group of the reals and for all $a, b \in \mathbb{R}$,
$$a \wedge b \coloneqq \min\{a, b\} \qquad \text{and} \qquad a \vee b \coloneqq \max\{a, b\}.$$
The $\mathcal{L}_a$-lattice $\mathbf{R}$ generates $\mathcal{LG}$ as a quasivariety and for any finite set of $\mathcal{L}_a$-formulas $T \cup \{\alpha\}$,
$$T \models_{\mathcal{LG}} \alpha \qquad \Longleftrightarrow \qquad T \models_{\mathbf{R}} \alpha.$$

The final example in this section does not fully fit in the previous framework, since these $\mathcal{L}$-lattices are not special $\mathsf{FL_e}$-algebras, but they can be viewed as extensions of Boolean algebras with a particular unary operation $\Box$.

Example 1.1.4. Let $\mathcal{L}_m$ be the signature $\mathcal{L}_b$ together with a unary operation $\Box$. An $\mathcal{L}_m$-lattice $\mathbf{M} = \langle M, \wedge, \vee, \neg, 0, 1, \Box\rangle$ where $\langle M, \wedge, \vee, \neg, 0, 1\rangle$ is a Boolean algebra is called a *modal algebra* if it satisfies
$$\Box 1 \approx 1 \qquad \text{and} \qquad \Box(x \wedge y) \approx \Box x \wedge \Box y.$$
The class of all modal algebras, denoted by $\mathcal{MA}$ forms a variety and provides an algebraic semantics for the *modal logic* K. Subvarieties of $\mathcal{MA}$ provide algebraic semantics for well-known modal logics, e.g., the subvariety $\mathcal{S}4$ of $\mathcal{MA}$ defined by
$$\Box x \leq x \qquad \text{and} \qquad \Box x \leq \Box\Box x$$
provides an algebraic semantics for S4 and the subvariety $\mathcal{S}5$ of $\mathcal{S}4$ defined by
$$\Box\Diamond x \approx \Diamond x$$
provides an algebraic semantics for S5. The modal algebras in $\mathcal{S}5$ correspond to the *monadic Boolean algebras* studied in [38]. See [18] for a study of modal logic.

1.2 First-Order Logics

We now introduce a broad family of first-order logics defined algebraically based on classes of $\mathcal{L}$-lattices. Note that this is only one way to define first-order logics. First-order logics can also be defined via Kripke semantics (see, e.g., [48]), proof systems (introduced in Section 1.3), etc. In particular, (first-order) intermediate logics and (first-order) substructural logics are frequently defined via Kripke semantics and sequent calculi, respectively.

First-order logics can be defined over an arbitrary first-order language with formulas built using the propositional operations from the algebraic signature $\mathcal{L}$ (see, e.g., [26, Section 7.1]). However, we can restrict our attention to a fixed (generic) first-order language here. We consider the set of predicate symbols $\mathcal{P}$, the union of the countably infinite sets of n-ary predicates $\{P_{n,i}\}_{i\in\mathbb{N}}$, for each $n \in \mathbb{N}$, and the countably infinite set Var of variables, usually denoted by $x, y, z, x_1, x_2, \ldots$. Recall that $\mathrm{Fm}_\forall(\mathcal{L})$, the set of $\mathcal{L}_\forall$-*formulas*, denoted by $\varphi, \psi, \chi, \ldots$, is defined inductively as follows:

1. Let $x_1, \ldots, x_n$ be variables, then $P_{n,i}(x_1, \ldots, x_n)$ is an (atomic) $\mathcal{L}_\forall$-formula for all $n, i \in \mathbb{N}$.

2. Let $\star \in \mathcal{L}_n$ and let $\varphi_1, \ldots, \varphi_n$ be $\mathcal{L}_\forall$-formulas, then $\star(\varphi_1, \ldots, \varphi_n)$ is an $\mathcal{L}_\forall$-formula.

3. Let x be a variable and φ an $\mathcal{L}_\forall$-formula, then $(\forall x)\varphi$ and $(\exists x)\varphi$ are $\mathcal{L}_\forall$-formulas.

Remark 1.2.1. Note that a first-order language may contain function symbols, which means that in the inductive definition of first-order formulas above, the arguments of atomic first-order formulas are arbitrary terms. The terms considered in this thesis consist only of the variables in Var. We will be focusing on the function-free, equality-free fragments of these first-order logics, but we will refer to these fragments as the first-order logics throughout this thesis.

An $\mathrm{Fm}_\forall(\mathcal{L})$-*equation* is an ordered pair of $\mathcal{L}_\forall$-formulas φ, ψ and is usually written as $\varphi \approx \psi$.[3] We now define a semantics where $\mathcal{L}_\forall$-formulas are evaluated in $\mathcal{L}$-lattices. Let S be a non-empty set. An S-*valuation* is a map v from Var to S and for $a \in S$ and $x \in \mathrm{Var}$, we denote by $v_{x=a}$ the S-valuation defined by

$$v_{x=a}(y) = \begin{cases} v(y) & \text{if } y \neq x \\ a & \text{if } y = x. \end{cases}$$

Let $\mathbf{A}$ be an $\mathcal{L}$-lattice and let $\mathcal{I}(P_{n,i})$ be a map from S^n to A for each $n \in \mathbb{N}$ and $i \in \mathbb{N}$. Then we call $\mathfrak{S} = \langle S, \mathcal{I} \rangle$ an $\mathbf{A}_\forall$-*structure*, if the following values

[3] Let us emphasize that an $\mathrm{Fm}_\forall(\mathcal{L})$-equation $\varphi \approx \psi$ is a primitive syntactic object that relates two formulas and not terms. In some settings (e.g., first-order substructural logics), $\varphi \approx \psi$ can be replaced by a formula such as $\varphi \leftrightarrow \psi$ and semantical consequence can be defined between formulas, but this is not always the case.

are defined for any S-valuation v

$$\begin{aligned}
\|P_{n,i}\|_v^{\mathfrak{S}} &= \mathcal{I}(P_{n,i})(v(x_1), \ldots, v(x_n)) && n, i \in \mathbb{N} \\
\|(\forall x)\varphi\|_v^{\mathfrak{S}} &= \bigwedge\{\|\varphi\|_{v_{x=a}}^{\mathfrak{S}} \mid a \in S\} \\
\|(\exists x)\varphi\|_v^{\mathfrak{S}} &= \bigvee\{\|\varphi\|_{v_{x=a}}^{\mathfrak{S}} \mid a \in S\} \\
\|\star(\varphi_1, \ldots, \varphi_m)\|_v^{\mathfrak{S}} &= \star^{\mathbf{A}}(\|\varphi_1\|_v^{\mathfrak{S}}, \ldots, \|\varphi_m\|_v^{\mathfrak{S}}) && m \in \mathbb{N}, \star \in \mathcal{L}_m,
\end{aligned}$$

where we set $\|(\forall x)\varphi\|_v^{\mathfrak{S}}$ and $\|(\exists x)\varphi\|_v^{\mathfrak{S}}$ to be undefined if the respective (possibly infinite) meet and join do not exist in $\mathbf{A}$ and set $\|\star(\varphi_1, \ldots, \varphi_n)\|_v^{\mathfrak{S}}$ to be undefined if $\|\varphi_i\|_v^{\mathfrak{S}}$ is undefined for some $i \in \{1, \ldots, m\}$.

We say that an $\mathrm{Fm}_\forall(\mathcal{L})$-equation $\varphi \approx \psi$ is *valid* in an $\mathbf{A}_\forall$-structure $\mathfrak{S}$, denoted by $\mathfrak{S} \models \varphi \approx \psi$, if $\|\varphi\|_v^{\mathfrak{S}} = \|\psi\|_v^{\mathfrak{S}}$ for any S-valuation v. If $\mathfrak{S} \models \varphi \approx \psi$ for any $\mathbf{A}_\forall$-structure $\mathfrak{S}$, we write $\mathbf{A} \models \varphi \approx \psi$ and say that $\varphi \approx \psi$ is valid in $\mathbf{A}$, and if $\mathbf{A} \models \varphi \approx \psi$ for any $\mathcal{L}$-lattice $\mathbf{A}$ in a class $\mathcal{K}$ of $\mathcal{L}$-lattices, we write $\mathcal{K} \models \varphi \approx \psi$ and say that $\varphi \approx \psi$ is valid in $\mathcal{K}$.

These definitions can also be extended to sets of $\mathrm{Fm}_\forall(\mathcal{L})$-equations. Let Σ be a set of $\mathrm{Fm}_\forall(\mathcal{L})$-equations. Then we write $\mathfrak{S} \models \Sigma$, $\mathbf{A} \models \Sigma$, and $\mathcal{K} \models \Sigma$ if for all $\varphi \approx \psi \in \Sigma$, $\mathfrak{S} \models \varphi \approx \psi$, $\mathbf{A} \models \varphi \approx \psi$, and $\mathcal{K} \models \varphi \approx \psi$, respectively.

For any set $\Sigma \cup \{\varphi \approx \psi\}$ of $\mathrm{Fm}_\forall(\mathcal{L})$-equations and class of $\mathcal{L}$-structures $\mathcal{K}$ we write $\Sigma \vDash_{\mathcal{K}}^{\forall} \varphi \approx \psi$ and say that $\varphi \approx \psi$ is a *(sentential*[4]*) semantical consequence* of Σ in $\mathcal{K}$, if for any $\mathcal{L}$-lattice $\mathbf{A} \in \mathcal{K}$ and $\mathbf{A}_\forall$-structure $\mathfrak{S}$,

$$\mathfrak{S} \models \varphi' \approx \psi', \text{ for all } \varphi' \approx \psi' \in \Sigma \quad \Longrightarrow \quad \mathfrak{S} \models \varphi \approx \psi.$$

Note that $\vDash_{\mathcal{K}}^{\forall}$ is an equational consequence relation[5] in the sense of Remark 1.1.1, which we call the *first-order logic based on* $\mathcal{K}$.

In certain cases, we can restrict our attention to the complete members $\mathcal{K}^c$ of $\mathcal{K}$.

Proposition 1.2.2. *Let $\mathcal{K}$ be a class of $\mathcal{L}$-lattices that admits regular completions. Then for any set of* $\mathrm{Fm}_\forall(\mathcal{L})$*-equations* $\Sigma \cup \{\varphi \approx \psi\}$,

$$\Sigma \vDash_{\mathcal{K}}^{\forall} \varphi \approx \psi \quad \Longleftrightarrow \quad \Sigma \vDash_{\mathcal{K}^c}^{\forall} \varphi \approx \psi.$$

Proof. The left-to-right direction is clear, since $\mathcal{K}^c$ is a subclass of $\mathcal{K}$. For the right-to-left direction suppose that $\Sigma \nvDash_{\mathcal{K}}^{\forall} \varphi \approx \psi$. Then there exists $\mathbf{A} \in \mathcal{K}$ and an $\mathbf{A}_\forall$-structure $\langle S, \mathcal{I}\rangle$ such that for all $\varphi' \approx \psi' \in \Sigma$

$$\|\varphi'\|_v^{\mathfrak{S}} = \|\psi'\|_v^{\mathfrak{S}},$$

[4]Note that we use the qualifier "sentential" to distinguish between consequences of a set of equations of propositional formulas in $\mathcal{K}$ (introduced in Section 1.1) and consequences of a set of equations of first-order formulas in $\mathcal{K}$.

[5]Whenever it is clear from the context which consequence relation we mean, we call both $\vDash_{\mathcal{K}}$ and $\vDash_{\mathcal{K}}^{\forall}$ equational consequence relations.

for all S-valuations v, but for some S-valuation v'

$$\|\varphi\|^{\mathfrak{S}}_{v'} \neq \|\psi\|^{\mathfrak{S}}_{v'}.$$

By assumption, there exists a regular completion $\mathbf{B}$ of $\mathbf{A}$ in $\mathcal{K}^c$ together with an $\mathcal{L}$-lattice embedding $h\colon \mathbf{A} \to \mathbf{B}$ that preserves all existing meets and joins in $\mathbf{A}$. Since $\mathfrak{S}$ is an $\mathbf{A}_\forall$-structure and h preserves all existing meets and joins, taking $\mathfrak{S}^h = \langle S, \mathcal{I}^h \rangle$ with

$$\mathcal{I}^h(P_{n,i}) \coloneqq h \circ \mathcal{I}(P_{n,i})$$

for all $n, i \in I$, yields a $\mathbf{B}_\forall$-structure, that satisfies

$$\|\chi\|^{\mathfrak{S}^h}_{v} = h \circ \|\chi\|^{\mathfrak{S}}_{v}$$

for all $\chi \in \mathrm{Fm}_\forall(\mathcal{L})$ and S-valuations v. Then we obtain

$$\|\varphi'\|^{\mathfrak{S}^h}_{v} = h \circ \|\varphi'\|^{\mathfrak{S}}_{v} = h \circ \|\psi'\|^{\mathfrak{S}}_{v} = \|\psi'\|^{\mathfrak{S}^h}_{v}$$

for all $\varphi' \approx \psi' \in \Sigma$ and S-valuations v, as well as

$$\|\varphi\|^{\mathfrak{S}^h}_{v'} = h \circ \|\psi\|^{\mathfrak{S}}_{v'} \neq h \circ \|\psi\|^{\mathfrak{S}}_{v'} = \|\psi\|^{\mathfrak{S}^h}_{v'}.$$

Therefore, $\Sigma \not\models^{\forall}_{\mathcal{K}^c} \varphi \approx \psi$. □

The following example serves as an illustration of the definitions given above.

Example 1.2.3. We consider the signature $\mathcal{L}_b$ and the consequence relation $\models^{\forall}_{\mathbf{2}}$ between sets of $\mathrm{Fm}_\forall(\mathcal{L}_b)$-equations and an $\mathrm{Fm}_\forall(\mathcal{L}_b)$-equation, where $\mathbf{2}$ is the two-element Boolean algebra. Since $\mathbf{2}$ is a finite structure, the infinite meets and joins considered above always exist and any $\mathfrak{S} = \langle S, \mathcal{I} \rangle$ is a $\mathbf{2}_\forall$-structure. For an example, let us consider $S = \mathbb{N}$ and $\mathcal{I}$ such that for the unary predicate P and the binary predicate Q,

$$\mathcal{I}(P)\colon \mathbb{N} \to A; \quad k \mapsto \begin{cases} 1 & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd;} \end{cases}$$

$$\mathcal{I}(Q)\colon \mathbb{N}^2 \to A; \quad \langle k, l \rangle \mapsto \begin{cases} 1 & \text{if } k < l, \\ 0 & \text{otherwise.} \end{cases}$$

Then for the valuation $v(x) = 1$, $v(y) = 2$ some of the values are:

$$\begin{aligned}
\|P(x)\|^{\mathfrak{S}}_{v} &= \mathcal{I}(P)(v(x)) = \mathcal{I}(P)(1) = 0, \\
\|Q(x,y)\|^{\mathfrak{S}}_{v} &= \mathcal{I}(Q)(v(x), v(y)) = \mathcal{I}(Q)(1,2) = 1, \\
\|(\forall x)P(x)\|^{\mathfrak{S}}_{v} &= \bigwedge\{\|P(x)\|^{\mathfrak{S}}_{v_{x=k}} \mid k \in \mathbb{N}\} = \bigwedge\{0,1\} = 0, \\
\|(\exists x)P(x)\|^{\mathfrak{S}}_{v} &= \bigvee\{\|P(x)\|^{\mathfrak{S}}_{v_{x=k}} \mid k \in \mathbb{N}\} = \bigvee\{0,1\} = 1, \\
\|(\forall x)(\exists y)Q(x,y)\|^{\mathfrak{S}}_{v} &= \bigwedge\{\bigvee\{\|Q(x,y)\|^{\mathfrak{S}}_{v_{x=k,y=l}} \mid l \in \mathbb{N}\} \mid k \in \mathbb{N}\} \\
&= \bigwedge\{\bigvee\{\mathcal{I}(Q)(k,l) \mid l \in \mathbb{N}\} \mid k \in \mathbb{N}\} \\
&= 1.
\end{aligned}$$

Thus we get $\mathfrak{S} \models (\exists x)P(x) \approx (\forall x)(\exists y)Q(x,y)$.

For any first-order logic based on a class $\mathcal{K}$ of $\mathsf{FL_e}$-algebras, $\vDash^{\forall}_{\mathcal{K}}$, we can define a consequence relation on formulas the same way as in Section 1.1. For any set of $\mathcal{L}_{s\forall}$-formulas $T \cup \{\varphi\}$,

$$T \vDash^{\forall}_{\mathcal{K}} \varphi \quad \iff \quad \{\mathrm{e} \leq \psi \mid \psi \in T\} \vDash^{\forall}_{\mathcal{K}} \mathrm{e} \leq \varphi.$$

That is, the first-order logic based on $\mathcal{K}$ determines a (first-order) logic in the sense of the definition at the beginning of Chapter 1. In the examples below, we mean the consequence relation on formulas instead of the equational consequence relation whenever we write $\vDash^{\forall}_{\mathcal{K}}$.

Example 1.2.4. We consider the first-order versions of the logics introduced in Example 1.1.3.

- First-order Classical logic $\vDash^{\forall}_{\mathbf{2}}$ corresponds to $\vDash^{\forall}_{\mathcal{BA}}$ and, since $\mathcal{BA}$ admits regular completions, also to $\vDash^{\forall}_{\mathcal{BA}^c}$.

- First-order intuitionistic logic was first studied by Heyting [40]. As remarked at the beginning of this section, intuitionistic logic as well as its first-order version are usually defined via a Kripke semantics. However, we can give an equivalent definition of first-order intuitionistic logic via the semantics defined above. First-order intuitionistic logic corresponds to $\vDash^{\forall}_{\mathcal{HA}}$ and also to $\vDash^{\forall}_{\mathcal{HA}^c}$, since $\mathcal{HA}$ admits regular completions.

- First-order intermediate logics have been studied by Umezawa [90] and Ono [67]. In particular, the first-order version of infinite-valued Gödel logic $\vDash^{\forall}_{\mathbf{G}}$ was first investigated by Horn [42]. He showed that $\vDash^{\forall}_{\mathbf{G}}$ corresponds to $\vDash^{\forall}_{\mathcal{HA}_{\mathsf{to}}}$, where $\mathcal{HA}_{\mathsf{to}}$ is the class of totally ordered Heyting algebras. As mentioned in Section 1.1, there is only one infinite-valued propositional Gödel logic, however in [4], the authors show that there are (countably) infinitely many different infinite-valued first-order Gödel logics. See [3, 4] for a study of first-order Gödel logics.

  The consequence relation $\vDash^{\forall}_{\mathcal{GA}}$ corresponds to Corsi's first-order logic of linear frames introduced in [27].

- First-order substructural logics can be defined as consequence relations on formulas $\vDash^{\forall}_{\mathcal{V}}$, for varieties $\mathcal{V}$ of $\mathsf{FL_e}$-algebras. A broad family of varieties of $\mathsf{FL_e}$-algebras are closed under MacNeille completions and therefore, for any such variety $\mathcal{V}$, the consequence relations $\vDash^{\forall}_{\mathcal{V}}$ and $\vDash^{\forall}_{\mathcal{V}^c}$ coincide.

- First-order Łukasiewicz logic corresponds to $\vDash^{\forall}_{\mathbf{Ł}}$, where $\mathbf{Ł}$ is the standard MV-algebra defined in Example 1.1.3. The first-order logic based on $\mathcal{MV}_{\mathsf{to}}$, $\vDash^{\forall}_{\mathcal{MV}_{\mathsf{to}}}$ (studied by Hájek [37]) does not correspond to first-order Łukasiewicz logic.

  First-order Abelian logic can be considered as $\vDash^{\forall}_{\mathbf{R}}$. The one-variable fragment of $\vDash^{\forall}_{\mathbf{R}}$ is studied in [59, 89].

1.3 Proof Theory

In this section, we recall some basic proof-theoretic notions (found, e.g., in [58]) and present sequent calculi that correspond to some of the first-order logics introduced in Section 1.2. In particular, in Figure 1.4, we present ∀CFL, a multiset version of the *first-order Full Lambek Calculus with exchange*. We consider extensions of ∀CFL with structural rules. Finally, we give a proof of a fundamental property of these sequent calculi, that is, cut elimination.

Suppose that A is a set of structures, e.g., equations, formulas, or sequents. An ordered pair $\langle\{S_1,\ldots,S_n\},S\rangle$ consisting of a finite (possibly empty) set $\{S_1,\ldots,S_n\} \subseteq A$ and an element $S \in A$ is called an *inference* for A. The structures $S_1,\ldots,S_n$ are called *premises* and the structure S is called the *conclusion* of the inference. An inference is usually denoted by

$$\frac{S_1 \quad \ldots \quad S_n}{S}$$

A *rule* for A, usually denoted by r, is a set of inferences for A that are referred to as instances of r. Typically, rules are defined schematically using metavariables to denote arbitrary members of A or to construct the members of A. Schematically defined rules with no premises are called *axioms*. An ordered pair $\langle A,\mathcal{R}\rangle$, where A is a set of structures and $\mathcal{R}$ a set of rules for A, is called a *proof system*.

Let $\mathrm{C} = \langle A,\mathcal{R}\rangle$ be a proof system and $X \cup \{S\} \subseteq A$. We define a *derivation* d of S from X in C, denoted by $d; X \vdash_{\mathrm{C}} S$, to be a finite tree of members of A such that

1. the root is S;
2. each node S' is an element in X, or has child nodes $S_1,\ldots,S_n$ for $n \in \mathbb{N}$ such that

   $$\frac{S_1 \quad \ldots \quad S_n}{S'}$$

   is an instance of a rule in $\mathcal{R}$.

We say that S is *derivable* from X in C, if there exists a derivation d of S from X in C and we write $X \vdash_{\mathrm{C}} S$. For $X = \emptyset$, we just say that d is a derivation of S in C, or S is derivable in C, and write $d \vdash_C S$ and $\vdash_C S$, respectively. The *height* $\mathrm{ht}(d)$ of a derivation d is the maximum length of the branches of d.

If the set of structures is the set of (propositional or first-order) formulas Fm for some algebraic signature, the proof system is called a *Hilbert-style axiomatization*. Usually Hilbert-style axiomatizations have many axioms and only a few rules, and derivations are often given as a list of formulas.

Let C be a Hilbert-style axiomatization. Then $\vdash_C$ is a consequence relation over Fm, that is, C determines a logic $\vdash_C$. We call C a *Hilbert-style axiomatization for the logic* $\vdash$, if for any set of formulas $T \cup \{\varphi\}$,

$$T \vdash \varphi \quad \iff \quad T \vdash_C \varphi,$$

and we say that C *axiomatizes* the logic $\vdash$.

For a first-order formula $\varphi \in \mathrm{Fm}_\forall(\mathcal{L})$, we define the *complexity* of φ, denoted by $\mathrm{cp}(\varphi)$, inductively as follows:

1. If φ is an atomic formula, then $\mathrm{cp}(\varphi) = 0$.
2. If $\varphi = \star(\varphi_1, \ldots, \varphi_n)$ for $\star \in \mathcal{L}_n$ and formulas $\varphi_1, \ldots, \varphi_n$, then $\mathrm{cp}(\varphi) = \mathrm{cp}(\varphi_1) + \cdots + \mathrm{cp}(\varphi_n) + 1$.
3. If φ is $(\forall x)\psi$ or $(\exists x)\psi$ for a formula ψ, then $\mathrm{cp}(\varphi) = \mathrm{cp}(\psi) + 1$.

We call an occurrence of a variable x in a first-order formula *bound*, if it is in the scope of a quantifier $(\forall x)$ or $(\exists x)$ and *free*, otherwise. A formula in which every occurrence of a variable is bound is called a *sentence*.

Let x, y be variables and $\varphi \in \mathrm{Fm}_\forall(\mathcal{L})$. We define that y *is free for* x *in* φ inductively as follows:

1. y is free for x in any atomic formula.
2. If $\varphi = \star(\varphi_1, \ldots, \varphi_n)$ for $\star \in \mathcal{L}_n$, then y is free for x in φ, if y is free for x in φ_i for all $i \in \{1, \ldots, n\}$.
3. If $\varphi = (\forall z)\psi$ or $\varphi = (\exists z)\psi$, then y is free for x in φ, if
   (a) either $y \neq z$ and y is free for x in ψ
   (b) or x does not occur freely in φ.

Example 1.3.1. The variety of $\mathsf{FL_e}$-algebras $\mathcal{FL}_e$ provides an algebraic semantics for the logic determined by the Hilbert-style axiomatization HCFL (see, e.g., [31]) given in Figure 1.1. Then the translation given in the previous section of the defining equations of $\mathcal{FL}_e$ from Section 1.1 are all derivable in HCFL. We give the derivation of $(\alpha \to \beta) \to (\alpha \to (\beta \vee \gamma))$, the formula corresponding to the inequation $x \to y \leq x \to (y \vee z)$, as an example. The formulas in 1. and 2. are instances of axioms.

1.	$\beta \to (\beta \vee \gamma)$	
2.	$(\beta \to (\beta \vee \gamma)) \to ((\alpha \to \beta) \to (\alpha \to (\beta \vee \gamma)))$	
3.	$(\alpha \to \beta) \to (\alpha \to (\beta \vee \gamma))$	(mp) with 1., 2.

We obtain a Hilbert-style axiomatization $\forall$HCFL for $\models^\forall_{\mathcal{FL}_e}$, the substructural first-order logic based on $\mathcal{FL}_e$, by adding the rule and axioms from Figure 1.2 to HCFL (see, e.g., [29]).

$$
\begin{array}{ll}
\alpha \to \alpha & (\alpha \wedge \beta) \to \alpha \\
(\alpha \to \beta) \to ((\gamma \to \alpha) \to (\gamma \to \beta)) & (\alpha \wedge \beta) \to \beta \\
(\alpha \to (\beta \to \gamma)) \to (\beta \to (\alpha \to \gamma)) & \alpha \to (\alpha \vee \beta) \\
((\alpha \to \beta) \wedge (\alpha \to \gamma)) \to ((\alpha \to (\beta \wedge \gamma)) & \beta \to (\alpha \vee \beta) \\
((\alpha \to \gamma) \wedge (\beta \to \gamma)) \to ((\alpha \vee \beta) \to \gamma)) & \mathrm{e} \\
((\alpha \wedge \mathrm{e}) \cdot (\beta \wedge \mathrm{e})) \to (\alpha \wedge \beta) & \mathrm{e} \to (\alpha \to \alpha) \\
(\beta \to (\alpha \to \gamma)) \to (\alpha \cdot \beta \to \gamma) & \beta \to (\alpha \to \alpha \cdot \beta)
\end{array}
$$

$$
\frac{\alpha \quad \alpha \to \beta}{\beta}\ (\mathrm{mp}) \qquad \frac{\alpha}{\alpha \wedge \mathrm{e}}\ (\mathrm{adj}_u)
$$

Figure 1.1: The Proof System HCFL

$$
\begin{array}{ll}
(\forall x)\varphi(x) \to \varphi(y) & y \text{ is free for } x \text{ in } \varphi \\
\varphi(y) \to (\exists x)\varphi(x) & y \text{ is free for } x \text{ in } \varphi \\
(\forall x)(\psi \to \varphi) \to (\psi \to (\forall x)\varphi) & x \text{ does not occur freely in } \psi \\
(\forall x)(\varphi \to \psi) \to ((\exists x)\varphi \to \psi) & x \text{ does not occur freely in } \psi
\end{array}
$$

$$
\frac{\varphi}{(\forall x)\varphi}\ (\mathrm{gen})
$$

Figure 1.2: Additional axioms and rule for ∀HCFL

In [26], the authors show that for certain first-order logics given by a consequence relation on formulas, a suitable Hilbert-style axiomatization that extends the Hilbert-style axiomatization of the respective propositional logic can be obtained. In particular, this method can be applied to first-order logics given by a consequence relation $\models^{\forall}_{\mathcal{V}}$ based on a variety $\mathcal{V}$ of FL_e-algebras [26, Section 7.5]. Hence, we obtain an axiomatization for the first-order logic given by this consequence relation on formulas.

Example 1.3.2. In this example we give some references of Hilbert-style axiomatizations for some of the logics considered in the previous sections:

- A first system for first-order classical logic was given by Hilbert and Ackermann in 1928 in [41], where they pose the question of completeness for their system but do not answer it. It was Gödel in [34] that proved completeness of their system. A Hilbert-style axiomatization is also given by Kleene in [45].

- Heyting [40] was the first to study and give a Hilbert-style axiomatization for first-order intuitionistic logic. Kleene [45] also gave a Hilbert-style axiomatization for first-order intuitionistic logic, denoted by IQC.

- First-order Gödel logic was first axiomatized by Horn in [42]. He proved that adding the *prelinearity* and *constant domain* axioms,

$$(\varphi \to \psi) \vee (\psi \to \varphi) \quad \text{and} \quad (\forall x)(\varphi \vee (\forall x)\psi) \to ((\forall x)\varphi \vee (\forall x)\psi),$$

  to IQC, yields a Hilbert-style axiomatization for first-order Gödel logic $\models^{\forall}_{\mathbf{G}}$. Takeuti and Titani [85] provided a different axiomatization.

- Hilbert-style axiomatizations for some propositional substructural logics have been given in [1] and can be extended to Hilbert-style axiomatizations for the corresponding first-order substructural logics using the method in [26, Section 7.5]. Hilbert-style axiomatizations for the first-order logics $\models^{\forall}_{\mathcal{FL}}$ and $\models^{\forall}_{\mathcal{FL}_e}$ are given in [31].

- First-order Łukasiewicz logic was shown to not be recursively enumerable (see [78]). In contrast, the first-order logic based on $\mathcal{MV}_{\mathsf{to}}$ was axiomatized by Hájek in [37].

Let us now introduce the main proof systems that are used in this thesis, i.e., sequent calculi. In the scope of this thesis, a *sequent*[6] is an ordered pair of (finite) multisets of formulas $\langle \Gamma, \Delta \rangle$ such that Δ contains at most one formula, denoted by $\Gamma \Rightarrow \Delta$. If the formulas in Γ and Δ are in Fm, where Fm is one of the sets of formulas defined at the beginning of this chapter, then we call $\Gamma \Rightarrow \Delta$ an Fm-*sequent*. We denote by Γ, Π the multiset union of the multisets Γ and Π, moreover $\Rightarrow \Delta$ and $\Gamma \Rightarrow$ denote the sequents $\langle \emptyset, \Delta \rangle$ and $\langle \Gamma, \emptyset \rangle$, respectively. Suppose that S is a set of sequents and $\mathcal{R}$ is a set of rules for S, then the proof system $\mathrm{C} = \langle S, \mathcal{R} \rangle$ is called a *sequent calculus*.

In this section we give explicit sequent calculi only for a small number of logics. However, sequent calculi can be given for a large family of logics. The first sequent calculi were introduced by Gentzen in 1935 in [32], where he introduced LK and LJ, sequent calculi (where sequents are ordered pairs of sequences of formulas) for first-order classical and intuitionistic logic, respectively. LK is a multi-conclusion sequent calculus, whereas LJ is a single-conclusion sequent calculus.

Example 1.3.3. We present a basic example of a sequent calculus corresponding to consequence in the variety $\mathcal{L}at$ of lattices. Let us consider the signature $\mathcal{L}_l$. The structures we consider in order to obtain a proof system for lattices are unary sequents of $\mathcal{L}_l$-formulas, that is, ordered pairs of the form $\langle \alpha, \beta \rangle$ with $\alpha, \beta \in \mathrm{Fm}(\mathcal{L}_l)$, written $\alpha \Rightarrow \beta$[7]. The sequent calculus Lat consists of all unary $\mathrm{Fm}(\mathcal{L}_l)$-sequents together with the rules shown in Figure 1.3. The proof sys-

[6] Indeed, sequents of this form are called *single-conclusion*, whereas in some settings *multi-conclusion* versions are studied. Note also that in the literature, sequents are frequently considered to be ordered pairs of sequences of formulas instead of multisets.

[7] Here we consider a subset of all sequents of $\mathcal{L}_l$-formulas, namely the set of all unary sequents.

Identity Axioms

$$\frac{}{\alpha \Rightarrow \alpha}\ (\textsc{id})$$

Cut Rule

$$\frac{\alpha \Rightarrow \gamma \quad \gamma \Rightarrow \beta}{\alpha \Rightarrow \beta}\ (\textsc{cut})$$

Operation Rules

$$\frac{\alpha_1 \Rightarrow \beta}{\alpha_1 \wedge \alpha_2 \Rightarrow \beta}\ (\wedge\Rightarrow)_1 \qquad \frac{\alpha \Rightarrow \beta_1}{\alpha \Rightarrow \beta_1 \vee \beta_2}\ (\Rightarrow\vee)_1$$

$$\frac{\alpha_2 \Rightarrow \beta}{\alpha_1 \wedge \alpha_2 \Rightarrow \beta}\ (\wedge\Rightarrow)_2 \qquad \frac{\alpha \Rightarrow \beta_2}{\alpha \Rightarrow \beta_1 \vee \beta_2}\ (\Rightarrow\vee)_2$$

$$\frac{\alpha \Rightarrow \beta_1 \quad \alpha \Rightarrow \beta_2}{\alpha \Rightarrow \beta_1 \wedge \beta_2}\ (\Rightarrow\wedge) \qquad \frac{\alpha_1 \Rightarrow \beta \quad \alpha_2 \Rightarrow \beta}{\alpha_1 \vee \alpha_2 \Rightarrow \beta}\ (\vee\Rightarrow)$$

Figure 1.3: The Proof System Lat

tem Lat is sound and complete with respect to equational consequence in $\mathcal{L}at$ (cf. [58]), that is, for any set of unary $\mathrm{Fm}(\mathcal{L}_l)$-sequents Γ and $\alpha, \beta \in \mathrm{Fm}(\mathcal{L}_l)$,

$$\Gamma \vdash_{\mathrm{Lat}} \alpha \Rightarrow \beta \qquad \Longleftrightarrow \qquad \{\gamma \leq \delta \mid \gamma \Rightarrow \delta \in \Gamma\} \vDash_{\mathcal{L}at} \alpha \leq \beta.$$

We give an example of a derivation in Lat, proving that one inequation of the distributivity property for $\mathcal{L}at$ holds:

$$\frac{\dfrac{\dfrac{\dfrac{}{\alpha \Rightarrow \alpha}\,(\textsc{id})}{\alpha \wedge \beta \Rightarrow \alpha}\,(\wedge\Rightarrow)_1 \quad \dfrac{\dfrac{\dfrac{}{\beta \Rightarrow \beta}\,(\textsc{id})}{\beta \Rightarrow \beta \vee \gamma}\,(\Rightarrow\vee)_1}{\alpha \wedge \beta \Rightarrow \beta \vee \gamma}\,(\wedge\Rightarrow)_2}{\alpha \wedge \beta \Rightarrow \alpha \wedge (\beta \vee \gamma)}\,(\Rightarrow\wedge) \quad \dfrac{\dfrac{\dfrac{}{\alpha \Rightarrow \alpha}\,(\textsc{id})}{\alpha \wedge \gamma \Rightarrow \alpha}\,(\wedge\Rightarrow)_1 \quad \dfrac{\dfrac{\dfrac{}{\gamma \Rightarrow \gamma}\,(\textsc{id})}{\gamma \Rightarrow \beta \vee \gamma}\,(\Rightarrow\vee)_2}{\alpha \wedge \gamma \Rightarrow \beta \vee \gamma}\,(\wedge\Rightarrow)_2}{\alpha \wedge \gamma \Rightarrow \alpha \wedge (\beta \vee \gamma)}\,(\Rightarrow\wedge)}{(\alpha \wedge \beta) \vee (\alpha \wedge \gamma) \Rightarrow \alpha \wedge (\beta \vee \gamma)}\,(\vee\Rightarrow)$$

Hence, by soundness and completeness,

$$\vDash_{\mathcal{L}at} (\alpha \wedge \beta) \vee (\alpha \wedge \gamma) \leq \alpha \wedge (\beta \vee \gamma).$$

Let $\{x_i\}_{i \in \mathbb{N}}$ and $\{y_i\}_{i \in \mathbb{N}}$ be two sets of variables such that

$$\{x_i\}_{i \in \mathbb{N}} \cap \{y_i\}_{i \in \mathbb{N}} = \emptyset.$$

We now introduce $\forall$CFL, a multiset sequent calculus version for the *first-*

order Full Lambek calculus with exchange[8], where the sequents $\Gamma \Rightarrow \Delta$ consist of multisets Γ, Δ in $\mathrm{Fm}_\forall(\mathcal{L}_s)^* \subseteq \mathrm{Fm}_\forall(\mathcal{L}_s)$ and $\mathrm{Fm}_\forall(\mathcal{L}_s)^*$ is the set of $\mathcal{L}_{s\forall}$-formulas φ such that any bound variable in φ is in $\{x_i\}_{i\in\mathbb{N}}$ and any free variable in φ is in $\{y_i\}_{i\in\mathbb{N}}$.

The calculus $\forall$CFL is given in Figure 1.4, where the rules $(\Rightarrow\forall)$ and $(\exists\Rightarrow)$ satisfy the *eigenvariable condition* (marked by $*$ in Figure 1.4), i.e., the variable y in $\{y_i\}_{i\in\mathbb{N}}$ does not occur in the conclusion of the rule. Note that u in the premise of the rules $(\forall\Rightarrow)$ and $(\Rightarrow\exists)$ is a variable in $\{y_i\}_{i\in\mathbb{N}}$, since $\mathcal{L}_s$ does not contain any function symbols and the only terms are variables.

The *cut rule*

$$\frac{\Gamma \Rightarrow \varphi \qquad \Pi, \varphi \Rightarrow \Delta}{\Gamma, \Pi \Rightarrow \Delta}\ (\textsc{cut})$$

corresponds to the transitivity of $\leq$, i.e., the quasi-equation

$$x \leq y \ \&\ y \leq z \quad \Longrightarrow \quad x \leq z.$$

The cut rule is useful to establish soundness and completeness of $\forall$CFL with respect to consequence in the variety of all $\mathsf{FL_e}$-algebras $\mathcal{FL}_e$, but it is problematic when trying to establish the derivability of an arbitrary formula. We will address this problem later in this section. The formula φ of the cut rule is called the *cut-formula* of that specific instance of (CUT).

Let r be a rule in a sequent calculus. A formula that occurs in a premise and the conclusion of r in the same form is called *non-principal* and the multiset of all non-principal formulas of r is called the *context* of r. The formula(s) not in the context of r are called the *principal formula(s)* of r. For example in the cut rule of the sequent calculus $\forall$CFL, the context is Γ, Π, Δ and φ is the principal formula.

Rules are called *structural*, if they only manipulate the structure of sequents without referring to particular formulas.

We proceed by giving examples of some common structural rules:

1. The *weakening rules*

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \Pi \Rightarrow \Delta}\ (\textsc{wl}) \qquad\qquad \frac{\Pi \Rightarrow}{\Gamma, \Pi \Rightarrow \Delta}\ (\textsc{wr})$$

   allow us to add (multisets of) formulas on the left and a formula on the right. These rules correspond to the equations

$$x \leq \mathrm{e} \qquad \text{and} \qquad \mathrm{f} \leq x.$$

[8]Note here that this is not the usual Full Lambek calculus with exchange, since sequents are defined by using multisets of formulas instead of sequences of formulas. As "," is read as multiplication on the left we build commutativity $x \cdot y \approx y \cdot x$ into each rule of the Full Lambek calculus individually by considering multisets instead of sequences.

Axioms

$$\frac{}{\varphi \Rightarrow \varphi}\ (\textsc{id}) \qquad \frac{}{\mathrm{f} \Rightarrow}\ (\mathrm{f}\Rightarrow) \qquad \frac{}{\Rightarrow \mathrm{e}}\ (\Rightarrow \mathrm{e})$$

Cut Rule

$$\frac{\Gamma \Rightarrow \varphi \quad \Pi, \varphi \Rightarrow \Delta}{\Gamma, \Pi \Rightarrow \Delta}\ (\textsc{cut})$$

Operation Rules

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \mathrm{e} \Rightarrow \Delta}\ (\mathrm{e}\Rightarrow) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \mathrm{f}}\ (\Rightarrow \mathrm{f})$$

$$\frac{\Gamma_1 \Rightarrow \varphi \quad \Gamma_2, \psi \Rightarrow \Delta}{\Gamma_1, \Gamma_2, \varphi \to \psi \Rightarrow \Delta}\ (\to\Rightarrow) \qquad \frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi}\ (\Rightarrow\to)$$

$$\frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \cdot \psi \Rightarrow \Delta}\ (\cdot\Rightarrow) \qquad \frac{\Gamma_1 \Rightarrow \varphi \quad \Gamma_2 \Rightarrow \psi}{\Gamma_1, \Gamma_2 \Rightarrow \varphi \cdot \psi}\ (\Rightarrow\cdot)$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}\ (\wedge\Rightarrow)_1 \qquad \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \vee \psi}\ (\Rightarrow\vee)_1$$

$$\frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}\ (\wedge\Rightarrow)_2 \qquad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \vee \psi}\ (\Rightarrow\vee)_2$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta}\ (\vee\Rightarrow) \qquad \frac{\Gamma \Rightarrow \varphi \quad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \wedge \psi}\ (\Rightarrow\wedge)$$

$$\frac{\Gamma, \varphi(u) \Rightarrow \Delta}{\Gamma, (\forall x)\varphi(x) \Rightarrow \Delta}\ (\forall\Rightarrow) \qquad \frac{\Gamma \Rightarrow \psi(y)}{\Gamma \Rightarrow (\forall x)\psi(x)}\ (\Rightarrow\forall)_*$$

$$\frac{\Gamma, \varphi(y) \Rightarrow \Delta}{\Gamma, (\exists x)\varphi(x) \Rightarrow \Delta}\ (\exists\Rightarrow)_* \qquad \frac{\Gamma \Rightarrow \psi(u)}{\Gamma \Rightarrow (\exists x)\psi(x)}\ (\Rightarrow\exists)$$

Figure 1.4: The Sequent Calculus $\forall$CFL

2. The *contraction rule*[9]

$$\frac{\Gamma, \Pi, \Pi \Rightarrow \Delta}{\Gamma, \Pi \Rightarrow \Delta}\ (\textsc{c})$$

allows us to combine consecutive copies of (multisets of) formulas on the left and it corresponds to the "square-increasing law",

$$x \leq x \cdot x.$$

3. The *mingle rule*

$$\frac{\Gamma, \Pi_1 \Rightarrow \Delta \qquad \Gamma, \Pi_2 \Rightarrow \Delta}{\Gamma, \Pi_1, \Pi_2 \Rightarrow \Delta}\ (\textsc{mingle})$$

corresponds to the "square-decreasing" law,

$$x \cdot x \leq x.$$

4. For any $k \in \mathbb{N}^{>2}$, the *k-contraction rule*

$$\frac{\Gamma, \Pi^k \Rightarrow \Delta}{\Gamma, \Pi \Rightarrow \Delta}\ (\textsc{k-contr})$$

corresponds to the equation

$$x \leq x^k.$$

Note that the 2-contraction rule and the contraction rule coincide.

We denote the sequent calculi that extend $\forall$CFL by adding the rules (wl) and (wr), or (c) by $\forall\mathrm{CFL}_{\mathrm{w}}$ and $\forall\mathrm{CFL}_{\mathrm{c}}$, respectively. $\forall$CFL extended with the three rules (wl), (wr), and (c), denoted by $\forall\mathrm{CFL}_{\mathrm{wc}}$[10], provides an alternative sequent calculus for first-order intuitionistic logic.

A large family of substructural logics[11] is obtained by removing structural rules from sequent calculi for (fragments of) classical and intuitionistic logics, where the operations may be split whenever necessary. More generally, substructural logics can be considered as logics given by proof systems that lack some structural rules. Alternatively, substructural logics can be given as those logics whose algebraic semantics are given by classes of residuated lattices (seen

[9]In the context of a single-conclusion sequent calculus, this is indeed "the contraction rule". However, in a multi-conclusion sequent calculus, we also consider the version of the rule that combines consecutive copies of (multisets of) formulas on the right.

[10]Note that in the presence of (wl), (wr), and (c) the operations $\wedge$ and $\cdot$ coincide.

[11]A term coined by Došen during the conference on "logics with restricted structural rules" in Tübingen in 1990.

in Sections 1.1 and 1.2 for classes of $\mathsf{FL_e}$-algebras). In 1958 Lambek [50] introduced a sequent calculus without any structural rules that was first applied in the field of linguistics and is now called the *Lambek calculus*. Girard [33] developed a sequent calculus for *linear logic* that corresponds (with a different syntax) to the propositional part of LK with the contraction and the weakening rules removed. A semantical study of intuitionistic propositional logics without the contraction rule was developed by Ono and Komori in 1985 [71], where the authors show that these logics admit cut elimination and are sound and complete with respect to a semantics based on SO-monoids. Ono and Komori both begin to extend this work on propositional substructural logics to some of the respective first-order versions in [66] and [46]. The sequence-version sequent calculus FL obtained from ∀CFL by splitting the operation $\to$ into $\backslash$ and $/$ and adding rules for these symbols was first called the *Full Lambek calculus* by Ono in [68].

We define for $n \in \mathbb{N}^{>0}$ and $\varphi_1, \dots, \varphi_n, \psi \in \mathrm{Fm}_\forall(\mathcal{L}_s)$,

$$\prod(\varphi_1, \dots, \varphi_n) \coloneqq \varphi_1 \cdots \varphi_n, \qquad \prod() \coloneqq \mathrm{e},$$
$$\sum(\psi) \coloneqq \psi, \qquad \sum() \coloneqq \mathrm{f}.$$

Then we obtain the following soundness and completeness results:

Theorem 1.3.4 (cf. [46,71])**.** *Let* C *be* $\forall\mathrm{CFL}, \forall\mathrm{CFL_w}, \forall\mathrm{CFL_c}$*, or* $\forall\mathrm{CFL_{wc}}$ *and let* $\mathcal{V}$ *be* $\mathcal{FL}_e, \mathcal{FL}_{ew}, \mathcal{FL}_{ec}$*, or* $\mathcal{FL}_{ewc}$*, respectively. For any sequent* $\Gamma \Rightarrow \Delta$ *in* $\mathrm{Fm}_\forall(\mathcal{L}_s)^*$,

$$\vdash_{\mathrm{C}} \Gamma \Rightarrow \Delta \quad \Longleftrightarrow \quad \models^\forall_{\mathcal{V}} \prod \Gamma \le \sum \Delta.$$

Remark 1.3.5. Take C and $\mathcal{V}$ to be the same as in the previous theorem. Let Γ and Δ be multisets in $\mathrm{Fm}_\forall(\mathcal{L}_s)$. Then there are multisets Γ' and Δ' in $\mathrm{Fm}_\forall(\mathcal{L}_s)^*$ (obtained by substituting the bound and free variables with variables in $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$, respectively) such that

$$\models^\forall_{\mathcal{V}} \prod \Gamma \le \sum \Delta \quad \Longleftrightarrow \quad \models^\forall_{\mathcal{V}} \prod \Gamma' \le \sum \Delta'.$$

Using Theorem 1.3.4, we can prove

$$\vdash_{\mathrm{C}} \Gamma' \Rightarrow \Delta' \quad \Longleftrightarrow \quad \models^\forall_{\mathcal{V}} \prod \Gamma \le \sum \Delta,$$

and we obtain soundness and completeness of the sequent calculus C with respect to (sentential) semantical consequence in the variety $\mathcal{V}$.

Proof systems such as sequent calculi are usually built to satisfy certain properties. In particular, sequent calculi where every rule has the *subformula property* — every formula occurring in the premise(s) of an instance of that rule is the subformula of a formula occurring in the conclusion — are very useful. For example, every rule in Lat except for the cut rule has the subformula property. Starting with any unary sequent and applying the rules except (CUT) backwards, we terminate with a unary sequent consisting only of propositional variables. Thus, derivations are much easier dealt with in this setting. In $\forall$CFL, most of the rules have the subformula property, only the cut rule and the quantifier rules do not. However, the quantifier rules do have a version of the subformula property, where the premises of an instance of these rules are subformulas of the conclusion with some of the variables substituted by others. This means that as well as for Lat, starting with a sequent in $\mathrm{Fm}_\forall(\mathcal{L}_s)^*$ and applying the rules from $\forall$CFL except for (CUT) backwards, we terminate with a sequent consisting only of atomic formulas. Thus, we would like to be able to consider derivations without the use of the cut rule. Let $\forall\mathrm{CFL}^\circ$ denote $\forall$CFL without (CUT). We can show that $\forall$CFL has *cut elimination*, that is,

Theorem 1.3.6 (cf. [46, 71])**.** *For any $\mathrm{Fm}_\forall(\mathcal{L}_s)^*$-sequent S,*

$$\vdash_{\forall\mathrm{CFL}} S \qquad \Longleftrightarrow \qquad \vdash_{\forall\mathrm{CFL}^\circ} S.$$

Proof. It suffices to prove that any sequent $\Gamma \Rightarrow \Delta$ with a derivation d of the form

$$\dfrac{\begin{array}{c} d_1 \\ \vdots \\ \Gamma \Rightarrow \varphi \end{array} \qquad \begin{array}{c} d_2 \\ \vdots \\ \varphi, \Pi \Rightarrow \Delta \end{array}}{\Gamma, \Pi \Rightarrow \Delta}\ (\textsc{cut})$$

where d_1 and d_2 are cut-free derivations, can be derived without using the (CUT)-rule. We prove this by induction on the lexicographically ordered pair $\langle \mathrm{cp}(\varphi), \mathrm{ht}(d_1) + \mathrm{ht}(d_2)\rangle$. If both $\Gamma \Rightarrow \varphi$ and $\varphi, \Pi \Rightarrow \Delta$ are instances of an axiom, then they must both be an instance of (ID) (all other cases are not possible) and $\Gamma, \Pi \Rightarrow \Delta$ is again an instance of (ID).

For the induction step, there are three cases. Either the cut-formula φ is non-principal in the last rule applied in d_1, φ is non-principal in the last rule applied in d_2, or φ is principal in the last rules applied in both d_1 and d_2.

For the first case, suppose φ is non-principal in r, the last rule applied in d_1, and r is not ($\exists\Rightarrow$). Then we apply the (CUT)-rule to the premise(s) and $\varphi, \Pi \Rightarrow \Delta$. Applying the induction hypothesis to the derivation(s) (of lower height than d_1) and the rule again yields the desired cut-free derivation. Suppose the last rule applied in d_1 is ($\exists\Rightarrow$), then the premise is of the form $\Gamma', \psi(y) \Rightarrow \varphi$ for some variable y that does not occur in $\Gamma', (\exists x)\psi(x) \Rightarrow \varphi$. If y does not occur in $\varphi, \Pi \Rightarrow \Delta$, then we can again apply (CUT) to $\Gamma', \psi(y) \Rightarrow \varphi$ and $\varphi, \Pi \Rightarrow \Delta$, apply the induction hypothesis and ($\exists\Rightarrow$) to obtain the desired derivation. If y occurs in $\varphi, \Pi \Rightarrow \Delta$, then we take a variable y' in $\{y_i\}_{i\in\mathbb{N}}$ that does not occur in either sequent and consider the derivation d_1' of $\Gamma', \psi(y) \Rightarrow \varphi$. We denote by $d_1'(y')$, the derivation d_1' with all occurrences of y substituted by y'. Then $d_1'(y')$ is a derivation of $\Gamma', \psi(y') \Rightarrow \Delta$ with $\mathrm{ht}(d_1'(y')) < \mathrm{ht}(d_1)$ and we obtain the derivation

$$\dfrac{\begin{matrix} d_1'(y') \\ \vdots \\ \Gamma', \psi(y') \Rightarrow \varphi \end{matrix} \qquad \begin{matrix} d_2 \\ \vdots \\ \varphi, \Pi \Rightarrow \Delta \end{matrix}}{\Gamma', \psi(y'), \Pi \Rightarrow \Delta} \text{(CUT)}$$

By the induction hypothesis, there is a cut-free derivation of $\Gamma', \psi(y'), \Pi \Rightarrow \Delta$ and we can apply $(\exists\Rightarrow)$ to obtain a cut-free derivation of $\Gamma, \Pi \Rightarrow \Delta$. The cases where φ is non-principal in the last rule applied in d_2 are very similar.

Now we consider the cases where φ is principal in the last rules applied in both d_1 and d_2. Suppose that $\varphi = \varphi_1 \to \varphi_2$ and d is

$$\dfrac{\dfrac{\begin{matrix} d_{11} \\ \vdots \\ \Gamma, \varphi_1 \Rightarrow \varphi_2 \end{matrix}}{\Gamma \Rightarrow \varphi_1 \to \varphi_2} (\Rightarrow\to) \qquad \dfrac{\begin{matrix} d_{21} \\ \vdots \\ \Pi_1 \Rightarrow \varphi_1 \end{matrix} \qquad \begin{matrix} d_{22} \\ \vdots \\ \varphi_2, \Pi_2 \Rightarrow \Delta \end{matrix}}{\varphi_1 \to \varphi_2, \Pi_1, \Pi_2 \Rightarrow \Delta} (\to\Rightarrow)}{\Gamma, \Pi_1, \Pi_2 \Rightarrow \Delta} \text{(CUT)}$$

We obtain a derivation of the form

$$\dfrac{\begin{matrix} d_{21} \\ \vdots \\ \Pi_1 \Rightarrow \varphi_1 \end{matrix} \qquad \begin{matrix} d_{11} \\ \vdots \\ \Gamma, \varphi_1 \Rightarrow \varphi_2 \end{matrix}}{\Gamma, \Pi_1 \Rightarrow \varphi_2} \text{(CUT)}$$

Since $\text{cp}(\varphi_1) < \text{cp}(\varphi)$, an application of the induction hypothesis yields a cut-free derivation d_3 of $\Gamma, \Pi_1 \Rightarrow \varphi_2$ and we can consider the derivation of the form

$$\dfrac{\begin{matrix} d_3 \\ \vdots \\ \Gamma, \Pi_1 \Rightarrow \varphi_2 \end{matrix} \qquad \begin{matrix} d_{22} \\ \vdots \\ \varphi_2, \Pi_2 \Rightarrow \Delta \end{matrix}}{\Gamma, \Pi_1, \Pi_2 \Rightarrow \Delta} \text{(CUT)}$$

Since $\text{cp}(\varphi_2) < \text{cp}(\varphi)$, we can apply the induction hypothesis and obtain a cut-free derivation of $\Gamma, \Pi_1, \Pi_2 \Rightarrow \Delta$. All other propositional cases can be proved very similarly.

Suppose now that $\varphi = (\forall x)\psi(x)$ and d is

$$\dfrac{\dfrac{\begin{matrix} d_1' \\ \vdots \\ \Gamma \Rightarrow \psi(y) \end{matrix}}{\Gamma \Rightarrow (\forall x)\psi(x)} (\Rightarrow\forall) \qquad \dfrac{\begin{matrix} d_2' \\ \vdots \\ \psi(u), \Pi \Rightarrow \Delta \end{matrix}}{(\forall x)\psi(x), \Pi \Rightarrow \Delta} (\forall\Rightarrow)}{\Gamma, \Pi \Rightarrow \Delta} \text{(CUT)}$$

where y is a variable that does not occur (freely) in $\Gamma \Rightarrow (\forall x)\psi(x)$. Let $y' \in \{y_i\}_{i\in\mathbb{N}}$ be a variable that does not occur in $\Gamma \Rightarrow \psi(y)$ and $\psi(u), \Pi \Rightarrow \Delta$. Then $d_1'(y')$ denotes d_1' with all free occurrences of y substituted by y', and $d_2'(y')$, $\Pi(y')$ and $\Delta(y')$ denote d_2', Π and Δ with all free occurrences of u substituted by y'. By the eigenvariable condition, y does not occur in Γ and $\Gamma(y') = \Gamma$. Thus, $d_1'(y')$ is a derivation of $\Gamma \Rightarrow \psi(y')$ and $d_2'(y')$ is a derivation of $\psi(y'), \Pi(y') \Rightarrow \Delta(y')$ and we obtain

$$\dfrac{\begin{array}{c} d_1'(y') \\ \vdots \\ \Gamma \Rightarrow \psi(y') \end{array} \qquad \begin{array}{c} d_2'(y') \\ \vdots \\ \psi(y'), \Pi(y') \Rightarrow \Delta(y') \end{array}}{\Gamma, \Pi(y') \Rightarrow \Delta(y')} \ (\textsc{cut})$$

Since $\mathrm{cp}(\psi(y')) < \mathrm{cp}((\forall x)\psi(x))$, an application of the induction hypothesis yields a cut-free derivation d_3 of $\Gamma, \Pi(y') \Rightarrow \Delta(y')$. Let $d_3(u)$ denote d_3 with all free occurrences of y' substituted by u, then $d_3(u)$ is a cut-free derivation of $\Gamma, \Pi \Rightarrow \Delta$. The case where $\varphi = (\exists x)\psi(x)$ is principal in the last rule applied in both d_1 and d_2 is very similar. □

This method of cut elimination extends to a large family of both propositional and first-order logics, see for example the textbooks [57, 84, 88]. Cut elimination was first established by Gentzen in [32] for LK and LJ, sequent calculi for first-order classical and intuitionistic logic, respectively. Cut elimination has been proven for a number of substructural logics, in particular, cut elimination was shown for FL, $\mathrm{FL_e}$, $\mathrm{FL_w}$, $\mathrm{FL_{ew}}$, $\mathrm{FL_{ec}}$, and $\mathrm{FL_{ewc}}$ (see [70]), which also established decidability in these logics. The failure of cut elimination was shown for $\mathrm{FL_c}$. Ono and Kiriyama [44, 69] proved cut elimination and decidability for all the first-order extensions of the propositional substructural logics mentioned above except for the first-order version of $\mathrm{FL_{ec}}$, which was shown to have cut elimination but to be undecidable. Girard [33] also showed that Linear logic has cut elimination. Restall [76] provided conditions that guarantee cut elimination for a wide range of sequent calculi for substructural logics. Ciabattoni and Terui [22, 86] give a sufficient condition for single-conclusion sequent calculi to have cut elimination.

For certain logics sequent calculi have not (yet) been found, but there are other more general proof systems that have been introduced for these logics. First-order Łukasiewicz logic was shown to not be recursively enumerable by Scarpellini [78]. Nonetheless, proof systems, containing some rule with infinitely many premises, have been obtained for it (see, e.g., [2, 5, 6, 37, 39]). Sequent calculi are not a convenient framework to deal with fuzzy logics, however, there are very elegant hypersequent calculi for these logics (see, e.g., [57]).

1.4 One-Variable Fragments

In Section 1.2, we introduced full first-order logics via a semantics based on classes of $\mathcal{L}$-lattices. In this section, we restrict these semantics to one variable and obtain semantics of the one-variable fragments of first-order logics based on $\mathcal{L}$-lattices. We then present some axiomatizations of such fragments that are known.

Although we can restrict certain semantics of a first-order logic to semantics for its one-variable fragment, a Hilbert-style axiomatization of a first-order logic does not directly yield a Hilbert-style axiomatization of the respective one-variable fragment, since derivations of one-variable formulas may involve additional variables. In this section, we give the standard translation functions between one-variable first-order formulas and modal formulas, where $(\forall x)$ and $(\exists x)$ are interpreted as $\Box$ and $\Diamond$, and vice versa. Therefore, we can consider a class of algebraic structures $\mathcal{K}$ such that consequence in the one-variable fragment corresponds to consequence in $\mathcal{K}$ (under this translation) and we may interpret the challenge of finding an axiomatization of the one-variable fragment of a first-order logic as finding an equational basis for $\mathcal{K}$. In the case of substructural logics, using the translations from Section 1.1, these axiomatizations of the corresponding varieties can be translated into Hilbert-style axiomatizations.

In order to define the one-variable fragments of the first-order logics introduced in Section 1.2, it suffices here to restrict our attention to the one-variable setting and a fixed (generic) predicate language. Thus, the following definitions will be analogous to the definitions in Section 1.2. Recall that $\mathrm{Fm}^1_\forall(\mathcal{L})$ denotes the set of *one-variable* $\mathcal{L}_\forall$*-formulas* $\varphi, \psi, \chi, \ldots$ built inductively using a countably infinite set of unary predicates $\{P_i\}_{i\in\mathbb{N}}$, a distinguished variable x, operations in $\mathcal{L}$, and quantifiers $(\forall x)$ and $(\exists x)$. We call an ordered pair of one-variable $\mathcal{L}_\forall$-formulas $\varphi, \psi \in \mathrm{Fm}^1_\forall(\mathcal{L})$, written $\varphi \approx \psi$, an $\mathrm{Fm}^1_\forall(\mathcal{L})$*-equation*.

Now let $\mathbf{A}$ be any $\mathcal{L}$-lattice, let S be a non-empty set, and let $\mathcal{I}(P_i)$ be a map from S to A for each $i \in \mathbb{N}$. We call the ordered pair $\mathfrak{S} = \langle S, \mathcal{I}\rangle$ an $\mathbf{A}$-*structure* if the following values are defined for any $u \in S$:

$$\begin{aligned} \|P_i(x)\|^{\mathfrak{S}}_u &= \mathcal{I}(P_i)(u) && i \in \mathbb{N} \\ \|(\forall x)\varphi\|^{\mathfrak{S}}_u &= \bigwedge \{ \|\varphi\|^{\mathfrak{S}}_v \mid v \in S\} \\ \|(\exists x)\varphi\|^{\mathfrak{S}}_u &= \bigvee \{ \|\varphi\|^{\mathfrak{S}}_v \mid v \in S\} \\ \|\star(\varphi_1, \ldots, \varphi_n)\|^{\mathfrak{S}}_u &= \star^{\mathbf{A}}(\|\varphi_1\|^{\mathfrak{S}}_u, \ldots, \|\varphi_n\|^{\mathfrak{S}}_u) && n \in \mathbb{N}, \star \in \mathcal{L}_n \end{aligned}$$

where we set $\|(\forall x)\varphi\|^{\mathfrak{S}}_u$ and $\|(\exists x)\varphi\|^{\mathfrak{S}}_u$ to be undefined if the respective meet and join do not exist in $\mathbf{A}$ and set $\|\star(\varphi_1, \ldots, \varphi_n)\|^{\mathfrak{S}}_u$ to be undefined if $\|\varphi_1\|^{\mathfrak{S}}_u$ is undefined for some $i \in \{1, \ldots, n\}$. Note that for each $\varphi \in \mathrm{Fm}^1_\forall(\mathcal{L})$, we can define a map from S to A as follows:

$$\|\varphi\|^{\mathfrak{S}} : S \to A; \quad u \mapsto \|\varphi\|^{\mathfrak{S}}_u .$$

Thus, $\langle S, \mathcal{I}\rangle$ is an $\mathbf{A}$-structure if and only if the partial map $\|\cdot\|^{\mathfrak{S}} : \mathrm{Fm}^1_\forall(\mathcal{L}) \to A^S$ is total. In the following, we work with the map $\|\cdot\|^{\mathfrak{S}}$ instead of the particular values.

If $\mathbf{A}$ is *complete*, then $\mathfrak{S} = \langle S, \mathcal{I}\rangle$ is always an $\mathbf{A}$-structure; otherwise, whether or not the partial map $\|\cdot\|^{\mathfrak{S}}$ is total depends on $\mathcal{I}$. E.g., for $\mathbf{A} = \langle \mathbb{N}, \min, \max\rangle$ and $S = \mathbb{N}$, if $\mathcal{I}(P_0)(n) \coloneqq n$, for all $n \in \mathbb{N}$, then $\|(\exists x)P_0(x)\|^{\mathfrak{S}}$

is undefined, but if $\mathcal{I}(P_i)(n) \leq K$ for all $i \in \mathbb{N}$ and $n \in S$, for some fixed $K \in \mathbb{N}$, then $\mathfrak{S}$ is an $\mathbf{A}$-structure.

We say that an $\mathrm{Fm}^1_\forall(\mathcal{L})$-equation $\varphi \approx \psi$ is *valid* in an $\mathbf{A}$-structure $\mathfrak{S}$, and write $\mathfrak{S} \models \varphi \approx \psi$, if $\|\varphi\|^{\mathfrak{S}} = \|\psi\|^{\mathfrak{S}}$. More generally, consider any class of $\mathcal{L}$-lattices $\mathcal{K}$. We say that an $\mathrm{Fm}^1_\forall(\mathcal{L})$-equation $\varphi \approx \psi$ is a *(sentential) semantical consequence* of a set of $\mathrm{Fm}^1_\forall(\mathcal{L})$-equations Σ in $\mathcal{K}$, and write $\Sigma \models^{\forall 1}_{\mathcal{K}} \varphi \approx \psi$, if for any $\mathbf{A} \in \mathcal{K}$ and $\mathbf{A}$-structure $\mathfrak{S}$,

$$\mathfrak{S} \models \varphi' \approx \psi', \text{ for all } \varphi' \approx \psi' \in \Sigma \implies \mathfrak{S} \models \varphi \approx \psi.$$

Similarly to the full first-order case, in certain cases, we can restrict our attention to the complete members of $\mathcal{K}$. We obtain the following corollary of Proposition 1.2.2:

Corollary 1.4.1. *Let $\mathcal{K}$ be a class of $\mathcal{L}$-lattices that admits regular completions. Then for any set of $\mathrm{Fm}^1_\forall(\mathcal{L})$-equations $\Sigma \cup \{\varphi \approx \psi\}$,*

$$\Sigma \models^{\forall 1}_{\mathcal{K}} \varphi \approx \psi \iff \Sigma \models^{\forall 1}_{\mathcal{K}^c} \varphi \approx \psi.$$

Similarly as in Section 1.2, we can define a consequence relation on formulas, whenever we consider a one-variable first-order logic based on a class $\mathcal{K}$ of $\mathsf{FL_e}$-algebras. We use $\models^{\forall 1}_{\mathcal{K}}$ to denote both consequence relations.

The following two theorems follow from [26].

Theorem 1.4.2 (Compactness)**.** *Let $\mathcal{V}$ be a variety of $\mathsf{FL_e}$-algebras. Then for any set of $\mathrm{Fm}^1_\forall(\mathcal{L}_s)$-equations $\Sigma \cup \{\varphi \approx \psi\}$,*

$$\Sigma \models^{\forall 1}_{\mathcal{V}} \varphi \approx \psi \iff \Sigma' \models^{\forall 1}_{\mathcal{V}} \varphi \approx \psi \text{ for some finite } \Sigma' \subseteq \Sigma.$$

Theorem 1.4.3 (Local Deduction Theorem, see also [57])**.** *Let $\mathcal{V}$ be a variety of $\mathsf{FL_e}$-algebras. Then for any set of one-variable $\mathcal{L}_{s\forall}$-formulas $T \cup \{\psi\}$ and any one-variable $\mathcal{L}_{s\forall}$-sentence φ,*

$$T \cup \{\varphi\} \models^{\forall 1}_{\mathcal{V}} \psi \iff T \models^{\forall 1}_{\mathcal{V}} (\varphi \wedge \mathrm{e})^n \to \psi \text{ for some } n \in \mathbb{N}^{>0}.$$

Recall that $\mathrm{Fm}_\Box(\mathcal{L})$ denotes the set of (propositional) $\mathcal{L}_\Box$-*formulas* $\alpha, \beta, \ldots$ built inductively using a countably infinite set of propositional variables $\{p_i\}_{i \in \mathbb{N}}$, the operations in $\mathcal{L}$, and the unary operations $\Box$ and $\Diamond$.

The (standard) translation functions $(-)^*$ and $(-)^\circ$ between $\mathrm{Fm}^1_\forall(\mathcal{L})$ and $\mathrm{Fm}_\Box(\mathcal{L})$ are defined inductively by

$$\begin{aligned}
(P_i(x))^* &= p_i & p_i^\circ &= P_i(x) & i \in \mathbb{N}\\
(\star(\varphi_1, \ldots, \varphi_n))^* &= \star(\varphi_1^*, \ldots, \varphi_n^*) & (\star(\alpha_1, \ldots, \alpha_n))^\circ &= \star(\alpha_1^\circ, \ldots, \alpha_n^\circ) & \star \in \mathcal{L}_n\\
((\forall x)\varphi)^* &= \Box\varphi^* & (\Box\alpha)^\circ &= (\forall x)\alpha^\circ\\
((\exists x)\varphi)^* &= \Diamond\varphi^* & (\Diamond\alpha)^\circ &= (\exists x)\alpha^\circ,
\end{aligned}$$

and lift in the obvious way to (sets of) $\mathrm{Fm}^1_\forall(\mathcal{L})$-equations and $\mathrm{Fm}_\Box(\mathcal{L})$-equations. Clearly, $(\varphi^*)^\circ = \varphi$ for any $\varphi \in \mathrm{Fm}^1_\forall(\mathcal{L})$ and $(\alpha^\circ)^* = \alpha$ for any $\alpha \in \mathrm{Fm}_\Box(\mathcal{L})$, and we may therefore switch between first-order and modal notations as convenient.

To axiomatize consequence in the one-variable first-order logic based on a class of $\mathcal{L}$-lattices $\mathcal{K}$, it therefore suffices to find a (natural) axiomatization of a class $\mathcal{C}$ of algebras in the signature of $\mathcal{L}$ expanded with $\Box, \Diamond$ such that $\models_{\mathcal{K}}^{\forall 1}$ corresponds to equational consequence in $\mathcal{C}$. Our goal in this thesis is to provide a (natural) axiomatization of a *variety* $\mathcal{V}$ such that for any set of $\mathrm{Fm}_{\forall}^{1}(\mathcal{L})$-equations $\Sigma \cup \{\varphi \approx \psi\}$,

$$\Sigma \models_{\mathcal{K}}^{\forall 1} \varphi \approx \psi \quad \Longleftrightarrow \quad \Sigma^* \models_{\mathcal{V}} \varphi^* \approx \psi^*.$$

Example 1.4.4. We now continue the running examples of this chapter by considering the one-variable fragments of the first-order logics introduced in Example 1.2.4.

- The first axiomatization of the one-variable fragment of first-order classical logic was given by Wajsberg in [91]. He proved that the one-variable fragment of first-order classical logic corresponds to the modal logic S5 (introduced in Section 1.1), which in particular shows, that

  $$\Sigma \models_{\mathbf{2}}^{\forall 1} \varphi \approx \psi \quad \Longleftrightarrow \quad \Sigma^* \models_{\mathcal{S}5} \varphi^* \approx \psi^*,$$

  for any set $\Sigma \cup \{\varphi \approx \psi\}$ of $\mathrm{Fm}_{\forall}^{1}(\mathcal{L}_b)$-equations.

- In [74], Prior axiomatized the S5-like modal logic MIPC as a modal extension of the intuitionistic logic IPC. In [61] it was shown that the variety $\mathcal{V}$ of monadic Heyting algebras provides an algebraic semantics for MIPC, where a *monadic Heyting algebra* as presented in [7][12] is an algebraic structure $\mathbf{H} = \langle H, \wedge, \vee, \rightarrow, 0, 1, \Box, \Diamond \rangle$ such that $\langle H, \wedge, \vee, \rightarrow, 0, 1 \rangle$ is a Heyting algebra, $\Box$ and $\Diamond$ are unary operations, and for all $a, b \in H$,

  $$\begin{array}{ll} \Box a \leq a, & a \leq \Diamond a, \\ \Box(a \wedge b) = \Box a \wedge \Box b, & \Diamond(a \vee b) = \Diamond a \vee \Diamond b, \\ \Box 1 = 1, & \Diamond 0 = 0, \\ \Box \Diamond a = \Diamond a, & \Diamond \Box a = \Box a, \\ \Diamond(\Diamond a \wedge b) = \Diamond a \wedge \Diamond b. & \end{array}$$

  Bull showed in [11] that the one-variable fragment of first-order intuitionistic logic can be axiomatized by MIPC, and thus, for any set $\Sigma \cup \{\varphi \approx \psi\}$ of $\mathrm{Fm}_{\forall}^{1}(\mathcal{L}_i)$-equations,

  $$\Sigma \models_{\mathcal{HA}}^{\forall 1} \varphi \approx \psi \quad \Longleftrightarrow \quad \Sigma^* \models_{\mathcal{V}} \varphi^* \approx \psi^*.$$

- Ono and Suzuki (see [64, 65, 67, 72, 82, 83]) found a continuum of logics over MIPC that correspond to the one-variable fragments of first-order intermediate logics. Bezhanishvili [7] further studied logics extending MIPC and showed that not all of those logics correspond to a one-variable fragment of a first-order intermediate logic.

[12] Bezhanishvili considers the unary operators $\forall$ and $\exists$ instead of $\Box$ and $\Diamond$.

- The one-variable fragment of first-order Gödel logic, $\models_{\mathbf{G}}^{\forall 1}$, was axiomatized in [15] by Caicedo and Rodríguez. They showed that for any set $\Sigma \cup \{\varphi \approx \psi\}$ of $\mathrm{Fm}_{\forall}^{1}(\mathcal{L}_i)$-equations,

$$\Sigma \models_{\mathbf{G}}^{\forall 1} \varphi \approx \psi \quad \Longleftrightarrow \quad \Sigma^* \models_{\mathcal{V}} \varphi^* \approx \psi^*,$$

  where $\mathcal{V}$ is the variety of monadic Gödel algebras, i.e., the variety of monadic Heyting algebras satisfying the prelinearity and the constant domain axioms:

$$(x \to y) \vee (y \to x) \approx \mathrm{e} \quad \text{and} \quad \Box(\Box x \vee y) \approx \Box x \vee \Box y.$$

  The one-variable fragment of (Corsi's) first-order logic of linear frames, $\models_{\mathcal{GA}}^{\forall 1}$, corresponds to the variety of monadic Heyting algebras defined by the prelinearity axiom [14].

- Interestingly, despite the fact that first-order Łukasiewicz logic is not recursively enumerable, the one-variable-fragment of first-order Łukasiewicz logic $\models_{\mathbf{Ł}}^{\forall 1}$ was axiomatized by Rutledge in [77], proving that it corresponds to the variety $\mathcal{MMV}$ of monadic MV-algebras. A *monadic* MV*-algebra* is an algebraic structure $\mathbf{M} = \langle M, \oplus, \neg, 0, \Diamond \rangle$ where $\langle M, \oplus, \neg, 0\rangle$ is an MV-algebra and $\Diamond$ is a unary operation satisfying for all $a, b \in M$,

$$\begin{array}{ll} a \leq \Diamond a, & \Diamond(a \vee b) = \Diamond b \vee \Diamond b, \\ \Diamond\neg\Diamond a = \neg\Diamond a, & \Diamond(\Diamond a \oplus \Diamond b) = \Diamond a \oplus \Diamond b, \\ \Diamond(a \otimes a) = \Diamond a \otimes \Diamond a, & \Diamond(a \oplus a) = \Diamond a \oplus \Diamond a, \end{array}$$

  where $a \otimes b \coloneqq \neg(\neg a \oplus \neg b)$ and $\Box a \coloneqq \neg\Diamond\neg a$.

- The one-variable fragment of first-order Abelian logic corresponds to the variety of monadic Abelian ℓ-groups [59]. A *monadic abelian ℓ-group* is an algebraic structure $\mathbf{G} = \langle G, \wedge, \vee, +, -, 0, \Box\rangle$ where $\langle G, \wedge, \vee, +, -, 0\rangle$ is an abelian ℓ-group, $\Box$ is a unary operation, and for all $a, b \in G$

$$\begin{array}{ll} \Box(a+b) \leq \Box a + \Box b, & \Box(a \wedge b) = \Box a \wedge \Box b, \\ \Box a \leq a, & \Diamond(a \wedge \Diamond b) = \Diamond a \wedge \Diamond b, \\ \Diamond a = \Box\Diamond a, & \Box(a+b) = \Box a + \Box b, \end{array}$$

  where $\Diamond a \coloneqq -\Box - a$.

Chapter 2

Algebraic approach

In this chapter we introduce potential modal counterparts of the one-variable first-order logics introduced in Section 1.4. We prove that for a large family of one-variable first-order logics based on a class $\mathcal{K}$ of $\mathcal{L}$-lattices a variety $\mathcal{V}$ of these modal counterparts provides a (natural) axiomatization, that is, for any set of $\mathrm{Fm}^1_\forall(\mathcal{L})$-equations $\Sigma \cup \{\varphi \approx \psi\}$,

$$\Sigma \models^{\forall 1}_{\mathcal{K}} \varphi \approx \psi \quad \iff \quad \Sigma^* \models_{\mathcal{V}} \varphi^* \approx \psi^*.$$

In Section 2.1 we define *m-$\mathcal{L}$-lattices* to be modal structures that extend $\mathcal{L}$-lattices with the modalities $\Box$ and $\Diamond$, and satisfy "$\mathcal{S}5$-like" equations, denoting for a class $\mathcal{K}$ of $\mathcal{L}$-lattices, the class of m-$\mathcal{L}$-lattices with an $\mathcal{L}$-lattice reduct in $\mathcal{K}$, by $m\mathcal{K}$. In Section 2.2 we formulate and prove some properties of m-$\mathcal{L}$-lattices, in particular, a correspondence theorem between m-$\mathcal{L}$-lattices and ordered pairs of $\mathcal{L}$-lattices and their relatively complete subalgebras. In Section 2.3 we introduce functional m-$\mathcal{L}$-lattices and establish a relationship between consequence in the one-variable first-order logic based on $\mathcal{K}$ and consequence in the functional members of $m\mathcal{K}$. Finally, in Section 2.4, we prove a functional representation theorem for a class $m\mathcal{K}$ such that $\mathcal{K}$ is closed under taking direct limits and subalgebras, and has the superamalgamation property. In particular, we prove that for a variety $\mathcal{V}$ that has the superamalgamation property, the equations defining $m\mathcal{V}$ provide a (natural) method for axiomatizing the one-variable first-order logic defined over $\mathcal{V}$.

2.1 Modal Extensions of $\mathcal{L}$-Lattices

In this section we define m-$\mathcal{L}$-lattices and show that they encompass several examples of known algebraic counterparts of the one-variable fragments of first-order logics found in the literature.

As our basic modal structures, let us define an *m-lattice* to be any algebraic structure $\langle L, \wedge, \vee, \Box, \Diamond \rangle$ with lattice reduct $\langle L, \wedge, \vee \rangle$ that satisfies the following equations:

$$
\begin{array}{llll}
(\mathrm{L1}_\Box) & \Box x \wedge x \approx \Box x & (\mathrm{L1}_\Diamond) & \Diamond x \vee x \approx \Diamond x \\
(\mathrm{L2}_\Box) & \Box(x \wedge y) \approx \Box x \wedge \Box y & (\mathrm{L2}_\Diamond) & \Diamond(x \vee y) \approx \Diamond x \vee \Diamond y \\
(\mathrm{L3}_\Box) & \Box\Diamond x \approx \Diamond x & (\mathrm{L3}_\Diamond) & \Diamond\Box x \approx \Box x.
\end{array}
$$

Recall from Section 1.1, that $x \leq y$ stands for $x \wedge y \approx x$, and since $x \wedge y \approx x$ if and only if $x \vee y \approx y$, $x \leq y$ also stands for $x \vee y \approx y$. It is easily shown that every m-lattice also satisfies the following equations and quasi-equations:

$$
\begin{array}{llll}
(\mathrm{L4}_\Box) & \Box\Box x \approx \Box x & (\mathrm{L4}_\Diamond) & \Diamond\Diamond x \approx \Diamond x \\
(\mathrm{L5}_\Box) & x \leq y \implies \Box x \leq \Box y & (\mathrm{L5}_\Diamond) & x \leq y \implies \Diamond x \leq \Diamond y.
\end{array}
$$

Now let $\mathcal{L}$ be any fixed lattice-oriented signature. We define an *m-$\mathcal{L}$-lattice* to be any algebraic structure $\langle \mathbf{A}, \Box, \Diamond \rangle$ such that $\mathbf{A}$ is an $\mathcal{L}$-lattice, $\langle A, \wedge, \vee, \Box, \Diamond \rangle$ is an m-lattice, and the following equation is satisfied for each $n \in \mathbb{N}$ and $\star \in \mathcal{L}_n$:

$$(\star_\Box) \quad \Box(\star(\Box x_1, \ldots, \Box x_n)) \approx \star(\Box x_1, \ldots, \Box x_n).$$

Let us consider an m-$\mathcal{L}$-lattice $\langle \mathbf{A}, \Box, \Diamond \rangle$. Then for each $n \in \mathbb{N}$ and $\star \in \mathcal{L}_n$ we obtain for any $a_1, \ldots, a_n \in A$,

$$
\begin{array}{rcll}
\Diamond(\star(\Diamond a_1, \ldots, \Diamond a_n)) & = & \Diamond(\star(\Box\Diamond a_1, \ldots, \Box\Diamond a_n)) & (\mathrm{L3}_\Box) \\
& = & \Diamond\Box(\star(\Box\Diamond a_1, \ldots, \Box\Diamond a_n)) & (\star_\Box) \\
& = & \Box(\star(\Box\Diamond a_1, \ldots, \Box\Diamond a_n)) & (\mathrm{L3}_\Diamond) \\
& = & \star(\Box\Diamond a_1, \ldots, \Box\Diamond a_n) & (\star_\Box) \\
& = & \star(\Diamond a_1, \ldots, \Diamond a_n) & (\mathrm{L3}_\Box)
\end{array}
$$

and $\langle \mathbf{A}, \Box, \Diamond \rangle$ also satisfies the equation

$$(\star_\Diamond) \quad \Diamond(\star(\Diamond x_1, \ldots, \Diamond x_n)) \approx \star(\Diamond x_1, \ldots, \Diamond x_n).$$

Given a class $\mathcal{K}$ of $\mathcal{L}$-lattices, let $m\mathcal{K}$ denote the class of m-$\mathcal{L}$-lattices with an $\mathcal{L}$-lattice reduct in $\mathcal{K}$. Note that if $\mathcal{K}$ is a variety, then so is $m\mathcal{K}$.

Example 2.1.1. It is straightforward to show that the notion of an m-$\mathcal{L}_s$-lattice encompasses other algebraic structures considered in the literature. In particular, $m\mathcal{BA}$ is the variety $\mathcal{S}5$ introduced in Example 1.1.4 and corresponds to the variety of monadic Boolean algebras [38] and $m\mathcal{HA}$ is the variety of monadic Heyting algebras [61] defined in Example 1.4.4.

Moreover, if $\mathbf{A}$ is an $\mathsf{FL_e}$-algebra, then every m-$\mathcal{L}_s$-lattice $\langle \mathbf{A}, \Box, \Diamond \rangle$ satisfies the equations

$$(\mathrm{L6}_\Box) \quad \Box(x \to \Box y) \approx \Diamond x \to \Box y \qquad (\mathrm{L6}_\Diamond) \quad \Box(\Box x \to y) \approx \Box x \to \Box y,$$

and $m\mathcal{FL}_e$ is therefore the variety of monadic $\mathsf{FL_e}$-algebras introduced in [89]. Let us just check ($\mathrm{L6}_\Box$), the proof for ($\mathrm{L6}_\Diamond$) being very similar. First we show that for any $a, b, c, d \in A$ such that $a \leq b$, also $a \cdot c \leq b \cdot c$ and $b \to d \leq a \to d$.

Since $b \cdot c \leq b \cdot c$, using residuation, $b \leq c \to b \cdot c$ and hence also $a \leq c \to b \cdot c$. Another application of residuation yields $a \cdot c \leq b \cdot c$. Since also $b \to d \leq b \to d$, applying residuation, we obtain $b \cdot (b \to d) \leq d$. Using the previous argument, we obtain $a \cdot (b \to d) \leq b \cdot (b \to c)$ by taking c to be $b \to d$. Hence, $a \cdot (b \to d) \leq d$ and applying residuation yields $b \to d \leq a \to d$. By (L1$_\Diamond$), $a \leq \Diamond a$, and we obtain $\Diamond a \to \Box b \leq a \to \Box b$. Hence, using (L3$_\Box$), ($\to_\Box$), and (L5$_\Box$),

$$\begin{aligned} \Diamond a \to \Box b &= \Box\Diamond a \to \Box b \\ &= \Box(\Box\Diamond a \to \Box b) \\ &= \Box(\Diamond a \to \Box b) \\ &\leq \Box(a \to \Box b). \end{aligned}$$

Conversely, by using residuation twice,

$$\begin{aligned} \Box(a \to \Box b) \leq a \to \Box b \quad &\iff \quad a \cdot \Box(a \to \Box b) \leq \Box b \\ &\iff \quad a \leq \Box(a \to \Box b) \to \Box b. \end{aligned}$$

Since $\Box(a \to \Box b) \leq a \to \Box b$ by (L1$_\Box$), also $a \leq \Box(a \to \Box b) \to \Box b$. Hence, using (L5$_\Diamond$), (L3$_\Diamond$), and ($\to_\Diamond$),

$$\begin{aligned} \Diamond a &\leq \Diamond(\Box(a \to \Box b) \to \Box b) \\ &= \Diamond(\Diamond\Box(a \to \Box b) \to \Diamond\Box b) \\ &= \Diamond\Box(a \to \Box b) \to \Diamond\Box b \\ &= \Box(a \to \Box b) \to \Box b. \end{aligned}$$

By residuation again,

$$\begin{aligned} \Diamond a \leq \Box(a \to \Box b) \to \Box b \quad &\iff \quad \Diamond a \cdot \Box(a \to \Box b) \leq \Box b \\ &\iff \quad \Box(a \to \Box b) \leq \Diamond a \to \Box b. \end{aligned}$$

Thus, from $\Diamond a \leq \Box(a \to \Box b) \to \Box b$ we obtain $\Box(a \to \Box b) \leq \Diamond a \to \Box b$.

In the following examples, we consider some more of the one-variable first-order logics from Example 1.4.4.

Example 2.1.2. The variety $m\mathcal{GA}$ corresponds to the one-variable fragment of Corsi's first-order logic of linear frames [14], whereas the variety of monadic Gödel algebras — axiomatized relative to $m\mathcal{GA}$ by the constant domain axiom $\Box(\Box x \vee y) \approx \Box x \vee \Box y$ — corresponds to the one-variable fragment of first-order Gödel logic, the first-order logic of linear frames with a constant domain [15].

Example 2.1.3. Note that the variety axiomatized relative to $m\mathcal{MV}$ by the constant domain axiom does not satisfy

$$\Diamond x \cdot \Diamond x \approx \Diamond(x \cdot x)$$

and therefore properly contains the variety $\mathcal{MMV}$ of monadic MV-algebras studied in [17, 28, 77] and defined in Example 1.4.4. Consider, for example, the MV-algebra (in the signature of $\mathsf{FL_e}$-algebras)

$$\mathbf{Ł}_3 = \langle \{0, \tfrac{1}{2}, 1\}, \wedge, \vee, \cdot, \to, 0, 1 \rangle$$

with the usual order, where

$$a \cdot b := \max(0, a + b - 1) \quad \text{and} \quad a \to b := \min(1, 1 - a + b).$$

Let

$$\Box 0 = \Box \tfrac{1}{2} = \Diamond 0 = 0 \qquad \text{and} \qquad \Box 1 = \Diamond \tfrac{1}{2} = \Diamond 1 = 1.$$

Then $\langle \mathbf{Ł}_3, \Box, \Diamond \rangle \in m\mathcal{MV}$ satisfies the constant domain axiom, but

$$\Diamond \tfrac{1}{2} \cdot \Diamond \tfrac{1}{2} = 1 \cdot 1 = 1 \neq 0 = \Diamond 0 = \Diamond (\tfrac{1}{2} \cdot \tfrac{1}{2})$$

and $m\mathcal{MV}$ does not correspond to the one-variable fragment of first-order Łukasiewicz logic.

2.2 A General Correspondence Theorem

We now establish a correspondence theorem between m-$\mathcal{L}$-lattices and ordered pairs of $\mathcal{L}$-lattices and their relatively complete subalgebras, which provides a useful description of m-$\mathcal{L}$-lattices that generalizes results in the literature for varieties such as monadic Heyting algebras [7] and monadic $\mathsf{FL_e}$-algebras [89].

Lemma 2.2.1. *Let $\langle \mathbf{A}, \Box, \Diamond \rangle$ be any m-$\mathcal{L}$-lattice. Then $\Box A := \{\Box a \mid a \in A\}$ forms a subalgebra $\Box \mathbf{A}$ of $\mathbf{A}$, where $\Box A = \Diamond A := \{\Diamond a \mid a \in A\}$ and for any $a \in A$,*

$$\Box a = \max\{b \in \Box A \mid b \leq a\} \quad \textit{and} \quad \Diamond a = \min\{b \in \Box A \mid a \leq b\}.$$

Proof. Let $\star \in \mathcal{L}_n$ and $\Box a_1, \ldots, \Box a_n \in \Box A$. Then using ($\star_\Box$), we obtain

$$\star(\Box a_1, \ldots, \Box a_n) = \Box(\star(\Box a_1, \ldots, \Box a_n)) \in \Box A,$$

and hence, $\Box A$ forms a subalgebra of $\mathbf{A}$. Since

$$\Box a = \Diamond \Box a \in \Diamond A \quad \text{and} \quad \Diamond a = \Box \Diamond a \in \Box A,$$

by (L3$_\Box$) and (L3$_\Diamond$), also $\Box A = \Diamond A$. Now consider any $a \in A$. If $b \in \Box A$ satisfies $b \leq a$, then

$$b = \Box b \leq \Box a,$$

by (L4$_\Box$) and (L5$_\Box$). But $\Box a \leq a$, by (L1$_\Box$), so

$$\Box a = \max\{b \in \Box A \mid b \leq a\}.$$

Analogous reasoning establishes that $\Diamond a = \min\{b \in \Box A \mid a \leq b\}$. □

Let us call a sublattice $\mathbf{L}_0$ of a lattice $\mathbf{L}$ *relatively complete* if for any $a \in L$, the sets

$$\{b \in L_0 \mid b \leq a\} \quad \text{and} \quad \{b \in L_0 \mid a \leq b\}$$

contain a maximum and minimum, respectively. Equivalently, $\mathbf{L}_0$ is relatively complete if the inclusion map

$$f_0 \colon \langle L_0, \leq \rangle \hookrightarrow \langle L, \leq \rangle$$

has left and right adjoints, that is, if there exist order-preserving maps

$$\Box \colon L \to L_0 \qquad \text{and} \qquad \Diamond \colon L \to L_0$$

such that for all $a \in L$ and $b \in L_0$,

$$f_0(b) \leq a \iff b \leq \Box a \quad \text{and} \quad a \leq f_0(b) \iff \Diamond a \leq b.$$

Let us also say that a subalgebra $\mathbf{A}_0$ of an $\mathcal{L}$-lattice $\mathbf{A}$ is relatively complete if this property holds with respect to their lattice reducts. In particular, by Lemma 2.2.1, the subalgebra $\Box\mathbf{A}$ of $\mathbf{A}$ is relatively complete for any m-$\mathcal{L}$-lattice $\langle \mathbf{A}, \Box, \Diamond \rangle$. The following result establishes a converse.

Lemma 2.2.2. *Let $\mathbf{A}_0$ be a relatively complete subalgebra of an $\mathcal{L}$-lattice $\mathbf{A}$, and define for each $a \in A$*

$$\Box_0 a \coloneqq \max\{b \in A_0 \mid b \leq a\} \quad \textit{and} \quad \Diamond_0 a \coloneqq \min\{b \in A_0 \mid a \leq b\}.$$

Then $\langle \mathbf{A}, \Box_0, \Diamond_0 \rangle$ is an m-$\mathcal{L}$-lattice and $\Box_0 A = \Diamond_0 A = A_0$.

Proof. It is straightforward to check that $\langle A, \wedge, \vee, \Box_0, \Diamond_0 \rangle$ is an m-lattice; for example, it satisfies (L2$_\Box$), since for any $a_1, a_2 \in A$,

$$\begin{aligned}
\Box_0(a_1 \wedge a_2) &= \max\{b \in A_0 \mid b \leq a_1 \wedge a_2\} \\
&= \max\{b \in A_0 \mid b \leq a_1 \text{ and } b \leq a_2\} \\
&= \max\{b \in A_0 \mid b \leq a_1\} \wedge \max\{b \in A_0 \mid b \leq a_2\} \\
&= \Box_0 a_1 \wedge \Box_0 a_2.
\end{aligned}$$

Since $\mathbf{A}_0$ is a subalgebra of $\mathbf{A}$, clearly $\langle \mathbf{A}, \Box_0, \Diamond_0 \rangle$ also satisfies ($\star_\Box$). Hence $\langle \mathbf{A}, \Box_0, \Diamond_0 \rangle$ is an m-$\mathcal{L}$-lattice and $\Box_0 A = \Diamond_0 A = A_0$. $\Box$

Combining Lemmas 2.2.1 and 2.2.2 yields the following representation theorem for m-$\mathcal{L}$-lattices.

Theorem 2.2.3. *Let $\mathcal{K}$ be any class of $\mathcal{L}$-lattices. Then there exists a one-to-one correspondence between the members of $\mathsf{m}\mathcal{K}$ and ordered pairs $\langle \mathbf{A}, \mathbf{A}_0 \rangle$ such that $\mathbf{A} \in \mathcal{K}$ and $\mathbf{A}_0$ is a relatively complete subalgebra of $\mathbf{A}$, implemented by the maps*

$$\langle \mathbf{A}, \Box, \Diamond \rangle \mapsto \langle \mathbf{A}, \Box\mathbf{A} \rangle \quad \textit{and} \quad \langle \mathbf{A}, \mathbf{A}_0 \rangle \mapsto \langle \mathbf{A}, \Box_0, \Diamond_0 \rangle.$$

2.3 Functional m-$\mathcal{L}$-Lattices

In this section we introduce functional m-$\mathcal{L}$-lattices and show that evaluations into functional m-$\mathcal{L}$-lattices can be identified with the semantics of one-variable first-order logics. This identification is then used to establish a relationship between consequence in one-variable first-order logics and consequence in a class of m-$\mathcal{L}$-lattices.

Given any $\mathcal{L}$-lattice $\mathbf{A}$ and set W, let $\mathbf{A}^W$ be the $\mathcal{L}$-lattice with universe A^W, where the operations are defined pointwise.

Proposition 2.3.1. *Let $\mathbf{A}$ be an $\mathcal{L}$-lattice, W a set, and $\mathbf{B}$ a subalgebra of $\mathbf{A}^W$ such that for each $f \in B$, the elements*

$$\bigwedge_{v\in W} f(v) \qquad \text{and} \qquad \bigvee_{v\in W} f(v)$$

exist in $\mathbf{A}$ and the following constant functions belong to B,

$$\Box f \colon W \to A;\ u \mapsto \bigwedge_{v\in W} f(v) \quad \text{and} \quad \Diamond f \colon W \to A;\ u \mapsto \bigvee_{v\in W} f(v).$$

Then $\langle \mathbf{B}, \Box, \Diamond \rangle$ is an m-$\mathcal{L}$-lattice. Moreover, if $\mathbf{A}$ belongs to a class $\mathcal{K}$ of $\mathcal{L}$-lattices closed under taking subalgebras and direct powers, then $\langle \mathbf{B}, \Box, \Diamond \rangle \in m\mathcal{K}$.

Proof. It is straightforward to check that $\langle B, \wedge, \vee, \Box, \Diamond \rangle$ satisfies (L1$_\Box$), (L2$_\Box$), (L1$_\Diamond$), and (L2$_\Diamond$). Let us just show that (L2$_\Diamond$) holds. For any $u \in W$,

$$\begin{aligned} \Diamond(f \vee g)(u) &= \bigvee_{v\in W} (f \vee g)(v) \\ &= \bigvee_{v\in W} f(v) \vee \bigvee_{v\in W} g(v) \\ &= \Diamond f(u) \vee \Diamond g(u). \end{aligned}$$

To confirm that $\langle \mathbf{B}, \Box, \Diamond \rangle$ is an m-$\mathcal{L}$-lattice — and therefore, if $\mathbf{A}$ belongs to a class $\mathcal{K}$ of $\mathcal{L}$-lattices closed under taking subalgebras and direct powers, a member of $m\mathcal{K}$ — observe that $\Box f$ and $\Diamond f$ are, by definition, constant functions for any $f \in B$. Hence $\langle B, \wedge, \vee, \Box, \Diamond \rangle$ clearly also satisfies (L3$_\Box$) and (L3$_\Diamond$). Moreover, for any $n \in \mathbb{N}$, $\star \in \mathcal{L}_n$, and $f_1, \ldots, f_n \in B$, the function $\star(\Box f_1, \ldots, \Box f_n)$ is constant and therefore equal to $\Box(\star(\Box f_1, \ldots, \Box f_n))$, so $\langle \mathbf{B}, \Box, \Diamond \rangle$ satisfies ($\star_\Box$). $\Box$

Let us call an m-$\mathcal{L}$-lattice $\langle \mathbf{B}, \Box, \Diamond \rangle$ $\langle \mathbf{A}, W \rangle$-*functional* if it is constructed as described in Proposition 2.3.1 for some $\mathcal{L}$-lattice $\mathbf{A}$ and set W. Given any class of $\mathcal{L}$-lattices $\mathcal{K}$, we call an m-$\mathcal{L}$-lattice $\mathcal{K}$-*functional* if it is isomorphic to an $\langle \mathbf{A}, W \rangle$-functional m-$\mathcal{L}$-lattice for some $\mathbf{A} \in \mathcal{K}$ and set W, omitting the prefix $\mathcal{K}$- if the class is clear from the context.

The following result identifies the semantics of one-variable first-order logics with evaluations into functional m-$\mathcal{L}$-lattices.

Proposition 2.3.2. *Let* $\mathbf{A}$ *be any* $\boldsymbol{\mathcal{L}}$*-lattice.*

(a) Let $\mathfrak{S} = \langle S, \mathcal{I} \rangle$ *be any* $\mathbf{A}$*-structure. Then* $B := \{\|\varphi\|^{\mathfrak{S}} \mid \varphi \in \mathrm{Fm}^1_\forall(\mathcal{L})\}$ *forms an* $\langle \mathbf{A}, S \rangle$*-functional m-*$\boldsymbol{\mathcal{L}}$*-lattice* $\mathbf{B}$ *and the* $\mathbf{B}$*-evaluation* $g^{\mathfrak{S}}$*, defined by setting* $g^{\mathfrak{S}}(p_i) := \mathcal{I}(P_i)$ *for each* $i \in \mathbb{N}$*, satisfies for all* $\varphi, \psi \in \mathrm{Fm}^1_\forall(\mathcal{L})$*,*

$$g^{\mathfrak{S}}(\varphi^*) = \|\varphi\|^{\mathfrak{S}} \quad \textit{and} \quad \mathfrak{S} \models \varphi \approx \psi \iff g^{\mathfrak{S}}(\varphi^*) = g^{\mathfrak{S}}(\psi^*).$$

(b) Let $\mathbf{B}$ *be any* $\langle \mathbf{A}, W \rangle$*-functional m-*$\boldsymbol{\mathcal{L}}$*-lattice for some set* W*, and let* g *be any* $\mathbf{B}$*-evaluation. Then* $\mathfrak{W} = \langle W, \mathcal{J} \rangle$*, where* $\mathcal{J}(P_i) := g(p_i)$ *for each* $i \in \mathbb{N}$*, is an* $\mathbf{A}$*-structure satisfying for all* $\varphi, \psi \in \mathrm{Fm}^1_\forall(\mathcal{L})$*,*

$$g(\varphi^*) = \|\varphi\|^{\mathfrak{W}} \quad \textit{and} \quad \mathfrak{W} \models \varphi \approx \psi \iff g(\varphi^*) = g(\psi^*).$$

Proof. (a) To show that $\mathbf{B}$ is $\langle \mathbf{A}, S \rangle$-functional, it suffices to observe that for any $\|\varphi\|^{\mathfrak{S}} \in B$, since $\mathfrak{S}$ is an $\mathbf{A}$-structure, the elements

$$\bigwedge\{\|\varphi\|^{\mathfrak{S}}(v) \mid v \in S\} \quad \text{and} \quad \bigvee\{\|\varphi\|^{\mathfrak{S}}(v) \mid v \in S\}$$

exist in $\mathbf{A}$ and correspond to the constant functions $\|(\forall x)\varphi\|^{\mathfrak{S}} \in B$ and $\|(\exists x)\varphi\|^{\mathfrak{S}} \in B$, respectively. The fact that $g^{\mathfrak{S}}(\varphi^*) = \|\varphi\|^{\mathfrak{S}}$ for all $\varphi \in \mathrm{Fm}^1_\forall(\mathcal{L})$, follows by an easy induction on the definition of φ, from which it follows also that for all $\varphi, \psi \in \mathrm{Fm}^1_\forall(\mathcal{L})$

$$\mathfrak{S} \models \varphi \approx \psi \iff g^{\mathfrak{S}}(\varphi^*) = g^{\mathfrak{S}}(\psi^*).$$

(b) Since $\mathbf{B}$ is $\langle \mathbf{A}, W \rangle$-functional, the elements

$$\bigwedge_{v \in W} f(v) \quad \text{and} \quad \bigvee_{v \in W} f(v)$$

exist in $\mathbf{A}$ for every $f \in B$. We prove that $g(\varphi^*) = \|\varphi\|^{\mathfrak{W}}$, by induction on the definition of φ, from which it follows immediately that $\mathfrak{W} = \langle W, \mathcal{J} \rangle$ is an $\mathbf{A}$-structure and

$$\mathfrak{W} \models \varphi \approx \psi \iff g(\varphi^*) = g(\psi^*)$$

for all $\varphi, \psi \in \mathrm{Fm}^1_\forall(\mathcal{L})$. In particular, for the case where $\varphi = (\forall x)\psi$, using the induction hypothesis for the second line,

$$\begin{aligned} \|(\forall x)\psi\|^{\mathfrak{W}}(u) &= \bigwedge\{\|\psi\|^{\mathfrak{W}}(v) \mid v \in W\} \\ &= \bigwedge\{g(\psi^*)(v) \mid v \in W\} \\ &= \Box g(\psi^*)(u) \\ &= g(((\forall x)\psi)^*)(u). \end{aligned}$$

The case where $\varphi = (\exists x)\psi$ is very similar. □

Let $\mathcal{K}$ be any class of $\mathcal{L}$-lattices and denote by $f\mathcal{K}$ the class of all $\mathcal{K}$-functional m-$\mathcal{L}$-lattices. Then, as a direct consequence of Proposition 2.3.2, for any set of $\mathrm{Fm}^1_\forall(\mathcal{L})$-equations $T \cup \{\varphi \approx \psi\}$,

$$\Sigma^* \models_{f\mathcal{K}} \varphi^* \approx \psi^* \quad \Longleftrightarrow \quad \Sigma \models^{\forall 1}_{\mathcal{K}} \varphi \approx \psi,$$

recalling from Section 1.4 that $(-)^*$ denotes the translation function from $\mathrm{Fm}^1_\forall(\mathcal{L})$ to $\mathrm{Fm}_\Box(\mathcal{L})$ and from (sets of) $\mathrm{Fm}^1_\forall(\mathcal{L})$-equations to $\mathrm{Fm}_\Box(\mathcal{L})$-equations. If $\mathcal{K}$ is closed under taking subalgebras and direct powers, then $f\mathcal{K} \subseteq m\mathcal{K}$, by Proposition 2.3.1, and we obtain the following relationship between consequence in the first-order logic based on $\mathcal{K}$ and consequence in the class $m\mathcal{K}$.

Corollary 2.3.3. *Let $\mathcal{K}$ be a class of $\mathcal{L}$-lattices closed under taking subalgebras and direct powers. Then for any set of $\mathrm{Fm}^1_\forall(\mathcal{L})$-equations $\Sigma \cup \{\varphi \approx \psi\}$,*

$$\Sigma^* \models_{m\mathcal{K}} \varphi^* \approx \psi^* \quad \Longrightarrow \quad \Sigma \models^{\forall 1}_{\mathcal{K}} \varphi \approx \psi.$$

Moreover, if every member of $m\mathcal{K}$ is $\mathcal{K}$-functional (i.e., $f\mathcal{K} = m\mathcal{K}$), then

$$\Sigma^* \models_{m\mathcal{K}} \varphi^* \approx \psi^* \quad \Longleftrightarrow \quad \Sigma \models^{\forall 1}_{\mathcal{K}} \varphi \approx \psi.$$

Let us remark that a stricter notion of a functional algebra for a class $\mathcal{K}$ of $\mathcal{L}$-lattices is considered in [8, 24] that coincides in our setting with the notion of being $\mathcal{K}^c$-*functional*, where $\mathcal{K}^c$ is the class of complete members of $\mathcal{K}$. That is, an m-$\mathcal{L}$-lattice $\langle \mathbf{B}, \Box, \Diamond \rangle$ is $\mathcal{K}^c$-functional if it is isomorphic to a subalgebra of $\langle \mathbf{A}^W, \Box, \Diamond \rangle$ for some complete $\mathcal{L}$-lattice $\mathbf{A} \in \mathcal{K}$ and set W, where $\Box$ and $\Diamond$ are defined as described in Proposition 2.3.1.

Adapting Proposition 2.3.2 slightly, we can formulate a stronger version of Corollary 2.3.3, which uses this notion of $\mathcal{K}^c$-functional m-$\mathcal{L}$-lattices.

Corollary 2.3.4. *Let $\mathcal{K}$ be a class of $\mathcal{L}$-lattices closed under taking subalgebras and direct powers. If every member of $m\mathcal{K}$ is $\mathcal{K}^c$-functional, then for any set of $\mathrm{Fm}^1_\forall(\mathcal{L})$-equations $\Sigma \cup \{\varphi \approx \psi\}$,*

$$\Sigma^* \models_{m\mathcal{K}} \varphi^* \approx \psi^* \quad \Longleftrightarrow \quad \Sigma \models^{\forall 1}_{\mathcal{K}^c} \varphi \approx \psi.$$

2.4 A Functional Representation Theorem

Adapting the proof of a similar result for Heyting Algebras [8, Theorem 3.6], we prove in this section that if a variety $\mathcal{V}$ of $\mathcal{L}$-lattices has the superamalgamation property, then every member of $m\mathcal{V}$ is $\mathcal{V}$-functional, and hence, by Corollary 2.3.3, consequence in the one-variable first-order logic based on $\mathcal{V}$ corresponds to consequence in $m\mathcal{V}$.

We first recall the necessary algebraic notions. Let $\mathcal{K}$ be a class of $\mathcal{L}$-lattices. A *V-formation* in $\mathcal{K}$ is a 5-tuple $\langle \mathbf{A}, \mathbf{B}_1, \mathbf{B}_2, f_1, f_2 \rangle$ consisting of $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{K}$ and embeddings $f_1 \colon \mathbf{A} \to \mathbf{B}_1$, $f_2 \colon \mathbf{A} \to \mathbf{B}_2$. An *amalgam* in $\mathcal{K}$ of a V-formation $\langle \mathbf{A}, \mathbf{B}_1, \mathbf{B}_2, f_1, f_2 \rangle$ in $\mathcal{K}$ is a triple $\langle \mathbf{C}, g_1, g_2 \rangle$ consisting of $\mathbf{C} \in \mathcal{K}$

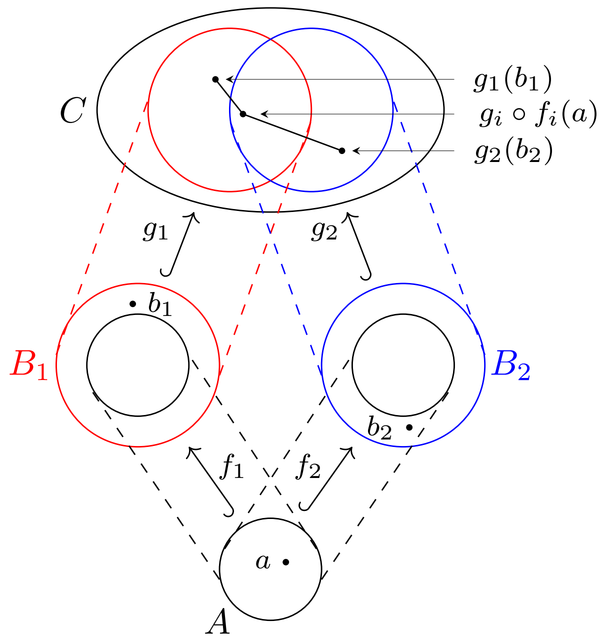


Figure 2.1: Depiction of the superamalgamation property

and embeddings $g_1 \colon \mathbf{B}_1 \to \mathbf{C}$, $g_2 \colon \mathbf{B}_2 \to \mathbf{C}$ such that $g_1 \circ f_1 = g_2 \circ f_2$; it is called a *superamalgam* if also for any $b_i \in B_i$, $b_j \in B_j$ and distinct $i, j \in \{1, 2\}$,

$$g_i(b_i) \leq g_j(b_j) \implies g_i(b_i) \leq g_i \circ f_i(a) = g_j \circ f_j(a) \leq g_j(b_j) \text{ for some } a \in A.$$

The class $\mathcal{K}$ is said to have the *superamalgamation property* if every V-formation in $\mathcal{K}$ has a superamalgam in $\mathcal{K}$. In Figure 2.1 we depict the superamalgam $\langle \mathbf{C}, g_1, g_2 \rangle$ of the V-formation $\langle \mathbf{A}, \mathbf{B_1}, \mathbf{B_2}, f_1, f_2 \rangle$.

Theorem 2.4.1. *Let $\mathcal{K}$ be a class of $\mathcal{L}$-lattices that is closed under taking direct limits and subalgebras, and has the superamalgamation property. Then every member of $m\mathcal{K}$ is functional.*

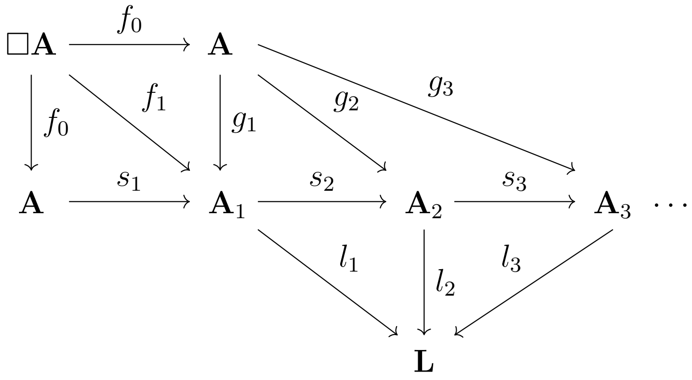
Proof. Consider any $\langle \mathbf{A}, \Box, \Diamond \rangle \in m\mathcal{K}$. Then $\mathbf{A} \in \mathcal{K}$ and, since $\mathcal{K}$ is closed under taking subalgebras, also $\Box\mathbf{A} \in \mathcal{K}$. We let $W := \mathbb{N}^{>0}$ and define inductively a sequence of $\mathcal{L}$-lattices $\langle \mathbf{A}_i \rangle_{i \in W}$ in $\mathcal{K}$ and sequences of $\mathcal{L}$-lattice embeddings

$$\langle f_i \colon \Box\mathbf{A} \to \mathbf{A}_i \rangle_{i \in W}, \quad \langle g_i \colon \mathbf{A} \to \mathbf{A}_i \rangle_{i \in W}, \quad \langle s_i \colon \mathbf{A}_{i-1} \to \mathbf{A}_i \rangle_{i \in W}.$$

Let $\mathbf{A}_0 := \mathbf{A}$ and let $f_0 \colon \Box\mathbf{A} \to \mathbf{A}$ be the inclusion map. For each $i \in W$, there exists inductively, by assumption, a superamalgam $\langle \mathbf{A}_i, s_i, g_i \rangle$ of the V-formation $\langle \Box\mathbf{A}, \mathbf{A}_{i-1}, \mathbf{A}, f_{i-1}, f_0 \rangle$, and we define also

$$f_i := s_i \circ f_{i-1} = g_i \circ f_0 = g_i|_{\Box A}. \tag{2.1}$$

Now let $\mathbf{L}$ be the direct limit of the system $\langle \langle \mathbf{A}_i, s_i \rangle \rangle_{i \in W}$ with an associated sequence of $\mathcal{L}$-lattice embeddings $\langle l_i \colon \mathbf{A}_i \to \mathbf{L} \rangle_{i \in W}$. Since $\mathcal{K}$ is closed under taking direct limits, $\mathbf{L}$ belongs to $\mathcal{K}$. The first two superamalgamation steps of this construction are depicted in the following diagram:



Since the operations of $\mathbf{L}^W$ are defined pointwise, $B := \{\langle l_i \circ g_i(a)\rangle_{i\in W} \mid a \in A\}$ is the universe of a subalgebra $\mathbf{B}$ of $\mathbf{L}^W$. We can also show that for each $a \in A$, the elements

$$\bigwedge_{j\in W} l_j \circ g_j(a) \quad \text{and} \quad \bigvee_{j\in W} l_j \circ g_j(a)$$

exist in L and hence that $\langle \mathbf{B}, \Box, \Diamond\rangle$, with $\Box$ and $\Diamond$ defined in Proposition 2.3.1, is an $\langle \mathbf{L}, W\rangle$-functional m-$\mathcal{L}$-lattice. Let $a \in A$ and fix some $i \in W$. It suffices to show that $l_i \circ g_i(\Box a)$ and $l_i \circ g_i(\Diamond a)$ are the greatest lower bound and least upper bound, respectively, of $S := \{l_j \circ g_j(a) \mid j \in W\}$. Observe first that for any $k \in W$,

$$\begin{aligned} l_k \circ g_k(\Box a) &= l_k \circ f_k(\Box a) \\ &= l_{k+1} \circ s_{k+1} \circ f_k(\Box a) \\ &= l_{k+1} \circ g_{k+1}(\Box a), \end{aligned}$$

where the first and last equations follow from (2.1) and the second follows from the fact that $\mathbf{L}$ is a direct limit. Hence for each $j \in W$,

$$\begin{aligned} l_i \circ g_i(\Box a) &= l_j \circ g_j(\Box a) \\ &\leq l_j \circ g_j(a). \end{aligned}$$

So $l_i \circ g_i(\Box a)$ is a lower bound of S. Now suppose that $c \in L$ is another lower bound of S. Since $\mathbf{L}$ is a direct limit, there exist $k \in W$ and $d \in A_k$ such that

$$\begin{aligned} l_{k+1} \circ s_{k+1}(d) &= l_k(d) \\ &= c \\ &\leq l_{k+1} \circ g_{k+1}(a). \end{aligned}$$

Since l_{k+1} is an embedding, $s_{k+1}(d) \leq g_{k+1}(a)$. Hence, since $\langle \mathbf{A}_{k+1}, s_{k+1}, g_{k+1} \rangle$ is a superamalgam of $\langle \Box \mathbf{A}, \mathbf{A}_k, \mathbf{A}, f_k, f_0 \rangle$, there exists $b \in \Box A$ such that

$$\begin{aligned} s_{k+1}(d) &\leq s_{k+1} \circ f_k(b) \\ &= g_{k+1} \circ f_0(b) \\ &\leq g_{k+1}(a). \end{aligned}$$

But s_{k+1} and g_{k+1} are embeddings and f_0 is the inclusion map, so

$$d \leq f_k(b) \qquad \text{and} \qquad b \leq a.$$

The latter inequality together with $b \in \Box A$, yields

$$b = \Box b \leq \Box a.$$

Hence also

$$f_k(b) \leq f_k(\Box a) = g_k(\Box a),$$

and, using the first inequality,

$$\begin{aligned} c &= l_k(d) \\ &\leq l_k \circ f_k(b) \\ &\leq l_k \circ g_k(\Box a) \\ &= l_i \circ g_i(\Box a). \end{aligned}$$

So

$$\bigwedge_{j \in W} l_j \circ g_j(a) = l_i \circ g_i(\Box a)$$

exists in L and the constant function $\langle l_i \circ g_i(\Box a) \rangle_{i \in W}$ belongs to B. Also, symmetrically,

$$\bigvee_{j \in W} l_j \circ g_j(a) = l_i \circ g_i(\Diamond a)$$

exists in L and the constant function $\langle l_i \circ g_i(\Diamond a) \rangle_{i \in W}$ belongs to B.

To show that $\langle \mathbf{A}, \Box, \Diamond \rangle$ is functional, it remains to prove that the following map is an isomorphism:

$$f \colon \langle \mathbf{A}, \Box, \Diamond \rangle \to \langle \mathbf{B}, \Box, \Diamond \rangle; \quad a \mapsto \langle l_i \circ g_i(a) \rangle_{i \in W}.$$

Since the operations of $\mathbf{L}^W$ are defined pointwise and l_i and g_i are $\mathcal{L}$-lattice embeddings for each $i \in W$, also f is an $\mathcal{L}$-lattice embedding. Clearly, it is onto, by the definition of B. Moreover, recalling that $l_i \circ g_i(\Box a) = \bigwedge_{j \in W} l_i \circ g_i(a)$ for each $a \in A$, it follows that

$$\begin{aligned} f(\Box a) &= \langle l_i \circ g_i(\Box a) \rangle_{i \in W} \\ &= \langle \bigwedge_{j \in W} l_j \circ g_j(a) \rangle_{i \in W} \\ &= \Box \langle l_i \circ g_i(a) \rangle_{i \in W} \\ &= \Box f(a), \end{aligned}$$

and, similarly, $f(\Diamond a) = \Diamond f(a)$. $\Box$

Combining Theorem 2.4.1 with Corollary 2.3.3 yields the following result.

Corollary 2.4.2. *If $\mathcal{V}$ is a variety of $\mathcal{L}$-lattices that has the superamalgamation property, then for any set $\Sigma \cup \{\varphi \approx \psi\}$ of $\mathrm{Fm}^1_\forall(\mathcal{L})$-equations,*

$$\Sigma \models^{\forall 1}_{\mathcal{V}} \varphi \approx \psi \quad \Longleftrightarrow \quad \Sigma^* \models_{m\mathcal{V}} \varphi^* \approx \psi^*.$$

Example 2.4.3. The variety of lattices has the superamalgamation property [36]. Hence, by Theorem 2.4.1, every m-lattice is functional, and consequence in the one-variable first-order lattice logic, understood as an equational consequence relation, corresponds to consequence in m-lattices.

Example 2.4.4. $\mathcal{FL}_e$, $\mathcal{FL}_{ew}$, and $\mathcal{FL}_{ec}$, and many other varieties of $\mathcal{FL}_e$-algebras have the superamalgamation property, which is equivalent in this setting to the Craig interpolation property for the associated substructural logic (see, e.g., [31]). Hence, for any such variety $\mathcal{V}$ — notably, for $\mathcal{V} \in \{\mathcal{FL}_e, \mathcal{FL}_{ew}, \mathcal{FL}_{ec}\}$ — every member of $m\mathcal{V}$ is functional, and consequence in the one-variable first-order substructural logic based on $\mathcal{V}$ corresponds to consequence in $m\mathcal{V}$.

Example 2.4.5. Maksimova [55] showed that there are exactly 7 varieties of Heyting algebras (e.g., $\mathcal{HA}$, $\mathcal{BA}$, and $\mathcal{GA}$) that have the (super-)amalgamation property. Therefore, for any variety $\mathcal{V}$ of these 7 varieties, every member of $m\mathcal{V}$ is functional, and consequence in the one-variable first-order intermediate logic based on $\mathcal{V}$ corresponds to consequence in $m\mathcal{V}$.

Example 2.4.6. A normal modal logic has the Craig interpolation property if and only if the associated variety of modal algebras — Boolean algebras with an operator — has the superamalgamation property [56]. Moreover, there exist infinitely many such logics [75], including well-known cases such as K, KT, K4, and S4. Hence our results yield axiomatizations for the one-variable fragments of infinitely many first-order logics based on varieties of modal algebras.

Remark 2.4.7. The one-variable fragments of first-order Gödel and first-order Łukasiewicz logic are based on the standard Gödel algebra **G** and the standard Łukasiewicz algebra **Ł**, and correspond to the varieties of monadic Gödel algebras and monadic MV-algebras, axiomatized relative to $m\mathcal{GA}$ and $m\mathcal{MV}$ by additional equations, respectively. Hence these fragments do not fit into the framework of this thesis. However, as mentioned in Example 2.4.5, the variety $\mathcal{GA}$ does have the superamalgamation property, but the first-order logic based on $\mathcal{GA}$ corresponds to Corsi's first-order logic of linear frames and our method provides an axiomatization of consequence in the one-variable first-order logic of linear frames.

Suppose finally that $\mathcal{K}$ is a class of $\mathcal{L}$-lattices that is not only closed under taking direct limits and subalgebras, and has the superamalgamation property, but also admits regular completions. In this case, we can adapt the proof of

Theorem 2.4.1 to show that every member of $\mathcal{K}$ is $\mathcal{K}^c$-functional, which — as noted at the end of Section 2.3 — corresponds to the stricter notion of a functional algebra considered in [8, 24].

Corollary 2.4.8. *Let $\mathcal{K}$ be a class of $\mathcal{L}$-lattices closed under taking direct limits and subalgebras, that has the superamalgamation property, and admits regular completions. Then every member of $m\mathcal{K}$ is $\mathcal{K}^c$-functional.*

Proof. Given some $\langle \mathbf{A}, \Box, \Diamond \rangle \in m\mathcal{K}$, the direct limit $\mathbf{L} \in \mathcal{K}$ constructed in the proof of Theorem 2.4.1 embeds into some $\bar{\mathbf{L}} \in \mathcal{K}^c$ and hence, reasoning as before, $\langle \mathbf{A}, \Box, \Diamond \rangle$ is isomorphic to a subalgebra of $\langle \bar{\mathbf{L}}^W, \Box, \Diamond \rangle$. □

Combining Corollary 2.3.4 and Corollary 2.4.8, we obtain the following result.

Corollary 2.4.9. *If $\mathcal{V}$ is a variety of $\mathcal{L}$-lattices that has the superamalgamation property and admits regular completions, then for any set $\Sigma \cup \{\varphi \approx \psi\}$ of $\mathrm{Fm}^1_\forall(\mathcal{L})$-equations,*

$$\Sigma \vDash^{\forall 1}_{\mathcal{V}^c} \varphi \approx \psi \quad \iff \quad \Sigma^* \vDash_{m\mathcal{V}} \varphi^* \approx \psi^*.$$

Example 2.4.10. In [8], Bezhanishvili and Harding proved that every monadic Heyting algebra is $\mathcal{HA}^c$-functional. Hence, by Corollary 2.3.4, the variety of monadic Heyting algebras provides an axiomatization for the one-variable fragment of first-order intuitionistic logic (since $\vDash^{\forall 1}_{\mathcal{HA}}$ and $\vDash^{\forall 1}_{\mathcal{HA}^c}$ coincide).

Chapter 3

Proof-Theoretic Approach

In this chapter, we describe an alternative proof-theoretic strategy for establishing completeness of axiomatizations for one-variable fragments of first-order logics. The key step is to prove that a derivation of a one-variable formula in a sequent calculus for the first-order logic can be transformed into a derivation that uses just one variable. To illustrate, we consider a first-order version of the full Lambek calculus with exchange, then extend the method to a broader family of first-order substructural logics.

In Section 3.1 we define a set of first-order formulas $\mathrm{Fm}^{1+}_{\forall}(\mathcal{L}_s)$ that extends $\mathrm{Fm}^{1}_{\forall}(\mathcal{L}_s)$, the one-variable $\mathcal{L}_{\forall}$-formulas given in Section 1.4, with a countably infinite set of variables that are distinct from x and are always free. We also introduce $\forall$1CFL, a cut-free sequent calculus satisfying for all $\mathrm{Fm}^{1}_{\forall}(\mathcal{L}_s)$-sequents $\Gamma \Rightarrow \Delta$,

$$\vdash_{\forall 1\mathrm{CFL}} \Gamma \Rightarrow \Delta \quad \Longleftrightarrow \quad \models^{\forall 1}_{\mathcal{FL}_e} \prod \Gamma \leq \sum \Delta,$$

where in a derivation of an $\mathrm{Fm}^{1}_{\forall}(\mathcal{L}_s)$-sequent extra (free) variables may be introduced. This explains our need for $\mathcal{L}^{+}_{s\forall}$-formulas and a sequent calculus $\forall$1CFL that operates on $\mathrm{Fm}^{1+}_{\forall}(\mathcal{L}_s)$-sequents. Section 3.2 is used to prove an interpolation property for derivations in $\forall$1CFL, finding for any derivable sequent $\Gamma(\bar{w}, y), \Pi(\bar{w}, z) \Rightarrow \Delta(\bar{w}, z)$ of a certain form an interpolant $\chi(\bar{w})$ such that the sequents $\Gamma(\bar{w}, y) \Rightarrow \chi(\bar{w})$ and $\Pi(\bar{w}, z), \chi(\bar{w}) \Rightarrow \Delta(\bar{w}, z)$ are derivable. In Section 3.3, we provide an alternative (proof-theoretic) proof of Corollary 2.4.2 for the variety $\mathcal{FL}_e$ using this interpolation property. In Section 3.4 we extend this proof for $\mathcal{FL}_e$ to varieties of $\mathsf{FL_e}$-algebras defined by equations of a certain simple form. In particular, we extend the proof to the varieties $\mathcal{FL}_{ew}$ and $\mathcal{FL}_{ec}$.

3.1 A Sequent Calculus for the One-Variable Fragment of $\forall$CFL

We begin this section by introducing $\mathrm{Fm}^{1+}_{\forall}(\mathcal{L}_s)$, a set of first-order formulas that use quantifiers $(\forall x)$ and $(\exists x)$ for a distinct variable x and a countably infinite set of variables $\{x_i\}_{i\in\mathbb{N}}$ that only occur freely. Then we present $\forall$1CFL, a sequent calculus based on $\mathcal{L}^+_{s\forall}$-formulas, that is sound and complete with respect to validity of equations in the one-variable first-order logic based on $\mathcal{FL}_e$.

The crucial feature of the first-order version of $\mathrm{FL_e}$ needed for our approach, proved in Section 1.3, is the fact that it can be presented as a *cut-free* sequent calculus with the standard rules for quantifiers. Any derivation of a one-variable formula φ in this calculus will therefore consist of sequents containing only subformulas of φ with some free occurrences of the variable x replaced by other variables. In particular, such a derivation will not introduce any new occurrences of quantifiers or bound variables, but may introduce free variables not occurring in φ via the rules for the universal quantifier on the right and the existential quantifier on the left. Hence, to reason about derivations of one-variable formulas, we may consider a fragment of the sequent calculus restricted to formulas that contain only unary predicates and one bound variable, but may contain further free variables.

More formally, let us recall from Chapter 1 that $\mathrm{Fm}^{1+}_{\forall}(\mathcal{L}_s)$ is the set of first-order formulas built inductively using unary predicates $\{P_i\}_{i\in\mathbb{N}}$, variables $\{x\}\cup\{x_i\}_{i\in\mathbb{N}}$, operations in $\mathcal{L}_s$, and quantifiers $(\forall x)$ and $(\exists x)$. The elements of $\mathrm{Fm}^{1+}_{\forall}(\mathcal{L}_s)$ are called *$\mathcal{L}^+_{s\forall}$-formulas*. Clearly, $\mathrm{Fm}^{1}_{\forall}(\mathcal{L}_s) \subseteq \mathrm{Fm}^{1+}_{\forall}(\mathcal{L}_s)$. We write $\varphi(\bar{w})$ to denote that the free variables of $\varphi \in \mathrm{Fm}^{1+}_{\forall}(\mathcal{L}_s)$ belong to a set $\bar{w}$, and indicate by $\varphi(\bar{w}, y)$ that y is not among the variables in $\bar{w}$.

In this chapter, *sequents* are ordered pairs of finite multisets of formulas Γ, Δ in $\mathrm{Fm}^{1+}_{\forall}(\mathcal{L}_s)$. We write $\Gamma(\bar{w})$ to denote that the free variables occurring in a finite multiset of formulas Γ belong to a set $\bar{w}$.

The sequent calculus $\forall$1CFL is displayed in Figure 3.1, where the quantifier rules are subject to the following side-conditions:

(i) if the conclusion of an application of $(\forall\Rightarrow)$ or $(\Rightarrow\exists)$ contains at least one free occurrence of a variable, then the variable u occurring in its premise also occurs freely in its conclusion[1];

(ii) the variable y occurring in the premise of an application of $(\Rightarrow\forall)$ or $(\exists\Rightarrow)$ does not occur freely in its conclusion.

Recall from Section 1.3 that if there exists a derivation d of a sequent $\Gamma \Rightarrow \Delta$ in a sequent calculus S, we write $d \vdash_{\mathrm{S}} \Gamma \Rightarrow \Delta$ or simply $\vdash_{\mathrm{S}} \Gamma \Rightarrow \Delta$.

The following relationship between derivability of $\mathrm{Fm}^{1}_{\forall}(\mathcal{L}_s)$-sequents in $\forall$1CFL and (first-order) validity of equations in the variety $\mathcal{FL}_e$ is a direct consequence of soundness and completeness and cut elimination for $\forall$CFL (Theorem 1.3.4 and Theorem 1.3.6).

[1]Note that in the literature, the variable u is often allowed to be an arbitrary term. However, by substituting certain variables, we can require u to be a variable that already occurs freely in the conclusion of the rule. Since this simplifies the proof of Lemma 3.2.1, we require our sequent calculus to satisfy condition (i).

Axioms

$$\frac{}{\varphi \Rightarrow \varphi}\ (\textsc{id}) \qquad \frac{}{\mathrm{f} \Rightarrow}\ (\mathrm{f}\Rightarrow) \qquad \frac{}{\Rightarrow \mathrm{e}}\ (\Rightarrow \mathrm{e})$$

Operation Rules

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \mathrm{e} \Rightarrow \Delta}\ (\mathrm{e}\Rightarrow) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \mathrm{f}}\ (\Rightarrow \mathrm{f})$$

$$\frac{\Gamma_1 \Rightarrow \varphi \quad \Gamma_2, \psi \Rightarrow \Delta}{\Gamma_1, \Gamma_2, \varphi \to \psi \Rightarrow \Delta}\ (\to\Rightarrow) \qquad \frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi}\ (\Rightarrow\to)$$

$$\frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \cdot \psi \Rightarrow \Delta}\ (\cdot\Rightarrow) \qquad \frac{\Gamma_1 \Rightarrow \varphi \quad \Gamma_2 \Rightarrow \psi}{\Gamma_1, \Gamma_2 \Rightarrow \varphi \cdot \psi}\ (\Rightarrow\cdot)$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}\ (\wedge\Rightarrow)_1 \qquad \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \vee \psi}\ (\Rightarrow\vee)_1$$

$$\frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}\ (\wedge\Rightarrow)_2 \qquad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \vee \psi}\ (\Rightarrow\vee)_2$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta}\ (\vee\Rightarrow) \qquad \frac{\Gamma \Rightarrow \varphi \quad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \wedge \psi}\ (\Rightarrow\wedge)$$

$$\frac{\Gamma, \varphi(u) \Rightarrow \Delta}{\Gamma, (\forall x)\varphi(x) \Rightarrow \Delta}\ (\forall\Rightarrow)_{(\mathrm{ii})} \qquad \frac{\Gamma \Rightarrow \psi(y)}{\Gamma \Rightarrow (\forall x)\psi(x)}\ (\Rightarrow\forall)_{(\mathrm{i})}$$

$$\frac{\Gamma, \varphi(y) \Rightarrow \Delta}{\Gamma, (\exists x)\varphi(x) \Rightarrow \Delta}\ (\exists\Rightarrow)_{(\mathrm{i})} \qquad \frac{\Gamma \Rightarrow \psi(u)}{\Gamma \Rightarrow (\exists x)\psi(x)}\ (\Rightarrow\exists)_{(\mathrm{ii})}$$

Figure 3.1: The Sequent Calculus ∀1CFL

Proposition 3.1.1 (cf. [46,71])**.** *For any sequent* $\Gamma \Rightarrow \Delta$ *containing formulas from* $\mathrm{Fm}^1_\forall(\mathcal{L}_s)$*,*

$$\vdash_{\forall 1\mathrm{CFL}} \Gamma \Rightarrow \Delta \quad \Longleftrightarrow \quad \models^{\forall 1}_{\mathcal{FL}_e} \prod \Gamma \leq \sum \Delta.$$

3.2 An Interpolation Property

We now establish an interpolation property for the calculus ∀1CFL. For any derivation d of a sequent in ∀1CFL, let $\mathrm{md}(d)$ denote the maximum number of applications of the rules $(\Rightarrow\forall)$ and $(\exists\Rightarrow)$ that occur on a branch of d. We prove by induction on the height of the derivation of a sequent $\Gamma(\bar{w}, y), \Pi(\bar{w}, z) \Rightarrow \Delta(\bar{w}, z)$ that there exists a formula $\chi(\bar{w})$ and derivations d_1 and d_2 of $\Gamma(\bar{w}, y) \Rightarrow \chi(\bar{w})$ and $\Pi(\bar{w}, z), \chi(\bar{w}) \Rightarrow \Delta(\bar{w}, z)$, respectively, such that $\mathrm{md}(d_1), \mathrm{md}(d_2) \leq \mathrm{md}(d)$. We prove this by constructing $\chi(\bar{w})$ according to the structure of the derivation of $\Gamma(\bar{w}, y), \Pi(\bar{w}, z) \Rightarrow \Delta(\bar{w}, z)$. Let us illustrate the idea of the proof by considering an example. Let

$$P(y), (\forall x)R(x), (\exists x)((\exists x)P(x) \to Q(x)) \Rightarrow (\exists x)Q(x) \cdot R(z)$$

be a sequent with the following derivation d

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(y) \Rightarrow P(y)}\,(\textsc{id})}{P(y) \Rightarrow (\exists x)P(x)}\,(\Rightarrow\exists) \qquad \dfrac{\dfrac{}{Q(w) \Rightarrow Q(w)}\,(\textsc{id})}{Q(w) \Rightarrow (\exists x)Q(x)}\,(\Rightarrow\exists)}{P(y), (\exists x)P(x) \to Q(w) \Rightarrow (\exists x)Q(x)}\,(\to\Rightarrow) \qquad \dfrac{\dfrac{}{R(z) \Rightarrow R(z)}\,(\textsc{id})}{(\forall x)R(x) \Rightarrow R(z)}\,(\forall\Rightarrow)}{P(y), (\forall x)R(x), (\exists x)P(x) \to Q(w) \Rightarrow (\exists x)Q(x) \cdot R(z)}\,(\Rightarrow\cdot)}{P(y), (\forall x)R(x), (\exists x)((\exists x)P(x) \to Q(x)) \Rightarrow (\exists x)Q(x) \cdot R(z)}\,(\exists\Rightarrow)$$

Following the structure of d, we now construct a formula χ (in this case it is even a sentence) and derivations d_1, d_2 such that

$$\begin{aligned} &d_1 \vdash_{\forall 1\mathrm{CFL}} P(y), (\forall x)R(x) \Rightarrow \chi, \\ &d_2 \vdash_{\forall 1\mathrm{CFL}} (\exists x)((\exists x)P(x) \to Q(x)), \chi \Rightarrow (\exists x)Q(x) \cdot R(z). \end{aligned}$$

We consider the premise of the last application of $(\exists\Rightarrow)$ in d, $P(y), (\exists x)P(x) \to Q(w), (\forall x)R(x) \Rightarrow (\exists x)Q(x) \cdot R(z)$ and try to find a formula $\chi_1(w)$ and derivations d_1', d_2' such that

$$\begin{aligned} &d_1' \vdash_{\forall 1\mathrm{CFL}} P(y), (\forall x)R(x) \Rightarrow \chi_1(w), \\ &d_2' \vdash_{\forall 1\mathrm{CFL}} (\exists x)P(x) \to Q(w), \chi_1(w) \Rightarrow (\exists x)Q(x) \cdot R(z). \end{aligned}$$

Next we consider the premises of $(\Rightarrow\cdot)$ in d,

$$P(y), (\exists x)P(x) \to Q(w) \Rightarrow (\exists x)Q(x) \quad \text{and} \quad (\forall x)R(x) \Rightarrow R(z)$$

and try to find formulas $\chi_2(w)$, χ_3 and derivations $d_{11}, d_{12}, d_{21}, d_{22}$ such that

$$\begin{aligned} &d_{11} \vdash_{\forall 1\mathrm{CFL}} P(y) \Rightarrow \chi_2(w), \qquad &&d_{12} \vdash_{\forall 1\mathrm{CFL}} (\exists x)P(x) \to Q(w), \chi_2(w) \Rightarrow (\exists x)Q(x), \\ &d_{21} \vdash_{\forall 1\mathrm{CFL}} (\forall x)R(x) \Rightarrow \chi_3, &&d_{22} \vdash_{\forall 1\mathrm{CFL}} \chi_3 \Rightarrow R(z). \end{aligned}$$

We can set $\chi_2(w) := ((\exists x)P(x) \to Q(w)) \to (\exists x)Q(x)$, since we are then able to obtain derivations d_{11} and d_{12} as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P(y) \Rightarrow P(y)}\,(\textsc{id})}{P(y) \Rightarrow (\exists x)P(x)}\,(\Rightarrow\exists) \qquad \dfrac{\dfrac{}{Q(w) \Rightarrow Q(w)}\,(\textsc{id})}{Q(w) \Rightarrow (\exists x)Q(w)}\,(\Rightarrow\exists)}{P(y), (\exists x)P(x) \to Q(w) \Rightarrow (\exists x)Q(x)}\,(\to\Rightarrow)}{P(y) \Rightarrow ((\exists x)P(x) \to Q(w)) \to (\exists x)Q(x)}\,(\Rightarrow\to)$$

$$\dfrac{\dfrac{}{(\exists x)P(x) \to Q(w) \Rightarrow (\exists x)P(x) \to Q(w)}\,(\textsc{id}) \qquad \dfrac{}{(\exists x)Q(x) \Rightarrow (\exists x)Q(x)}\,(\textsc{id})}{(\exists x)P(x) \to Q(w), \chi_2(w) \Rightarrow (\exists x)Q(x)}\,(\to\Rightarrow)$$

We can also set $\chi_3 \coloneqq (\forall x)R(x)$, since in that case d_{21} is an instance of (ID) and d_{22} is the an instance of (ID) together with an application of $(\forall\Rightarrow)$. Setting $\chi_1(w) = \chi_2(w) \cdot \chi_3$ then yields derivations d_1' and d_2' of the form

$$\dfrac{\begin{matrix} d_{11} \\ \vdots \\ P(y) \Rightarrow \chi_2(w) \end{matrix} \qquad \dfrac{}{(\forall x)R(x) \Rightarrow \chi_3}\,(\text{ID})}{P(y), (\forall x)R(x) \Rightarrow \chi_2(w) \cdot \chi_3}\,(\Rightarrow\cdot)$$

$$\dfrac{\dfrac{\begin{matrix} d_{12} \\ \vdots \\ (\exists x)P(x) \to Q(w), \chi_2(w) \Rightarrow (\exists x)Q(x) \end{matrix} \qquad \begin{matrix} d_{22} \\ \vdots \\ \chi_3 \Rightarrow R(z) \end{matrix}}{(\exists x)P(x) \to Q(w), \chi_2(w), \chi_3 \Rightarrow (\exists x)Q(x) \cdot R(z)}\,(\Rightarrow\cdot)}{(\exists x)P(x) \to Q(w), \chi_2(w) \cdot \chi_3 \Rightarrow (\exists x)Q(x) \cdot R(z)}\,(\cdot\Rightarrow)$$

Finally, we obtain the derivations d_1, d_2 by setting $\chi = (\forall x)\chi_1(x)$:

$$\dfrac{\begin{matrix} d_1' \\ \vdots \\ P(y), (\forall x)R(x) \Rightarrow \chi_1(w) \end{matrix}}{P(y), (\forall x)R(x) \Rightarrow (\forall x)\chi_1(x)}\,(\Rightarrow\forall)$$

$$\dfrac{\dfrac{\begin{matrix} d_2' \\ \vdots \\ (\exists x)P(x) \to Q(w), \chi_1(w) \Rightarrow (\exists x)Q(x) \cdot R(z) \end{matrix}}{(\exists x)P(x) \to Q(w), (\forall x)\chi_1(x) \Rightarrow (\exists x)Q(x) \cdot R(z)}\,(\forall\Rightarrow)}{(\exists x)((\exists x)P(x) \to Q(x)), (\forall x)\chi_1(x) \Rightarrow (\exists x)Q(x) \cdot R(z)}\,(\exists\Rightarrow)$$

Note that $\mathrm{md}(d) = 1$ and we found a formula

$$\chi = (\forall x)((((\exists x)P(x) \to Q(w)) \to (\exists x)Q(x)) \cdot (\forall x)R(x))$$

and derivations d_1 and d_2 satisfying $\mathrm{md}(d_1), \mathrm{md}(d_2) = 1 \leq \mathrm{md}(d)$ and

$$\begin{aligned} &d_1 \vdash_{\forall 1\mathrm{CFL}} P(y), (\forall x)R(x) \Rightarrow \chi, \\ &d_2 \vdash_{\forall 1\mathrm{CFL}} (\exists x)((\exists x)P(x) \to Q(x)), \chi \Rightarrow (\exists x)Q(x) \cdot R(z). \end{aligned}$$

Lemma 3.2.1. *Let $\Gamma(\bar{w}, y), \Pi(\bar{w}, z) \Rightarrow \Delta(\bar{w}, z)$ be any sequent such that $y \neq z$, $x \notin \bar{w} \cup \{y, z\}$, and no variable in $\bar{w} \cup \{y, z\}$ lies in the scope of a quantifier. If*

$$d \vdash_{\forall 1\mathrm{CFL}} \Gamma(\bar{w}, y), \Pi(\bar{w}, z) \Rightarrow \Delta(\bar{w}, z),$$

then there exist $\chi(\bar{w}) \in \mathrm{Fm}^{1+}_{\forall}(\mathcal{L}_s)$ and derivations d_1, d_2 in $\forall$1CFL *such that* $\mathrm{md}(d_1), \mathrm{md}(d_2) \leq \mathrm{md}(d)$ *and*

$$d_1 \vdash_{\forall 1\mathrm{CFL}} \Gamma(\bar{w}, y) \Rightarrow \chi(\bar{w}), \quad d_2 \vdash_{\forall 1\mathrm{CFL}} \Pi(\bar{w}, z), \chi(\bar{w}) \Rightarrow \Delta(\bar{w}, z).$$

Proof. By a straightforward inspection of the rules of $\forall$1CFL, no variable in $\bar{w} \cup \{y, z\}$ can lie in the scope of a quantifier in a sequent occurring in a derivation in $\forall$1CFL of $\Gamma(\bar{w}, y), \Pi(\bar{w}, z) \Rightarrow \Delta(\bar{w}, z)$. We prove the claim by induction on the height of d, considering in turn the last rule applied in the derivation.

Observe first that if y does not occur in Γ, we can define

$$\chi(\bar{w}) := \prod \Gamma$$

and obtain a derivation d_1 of $\Gamma(\bar{w}, y) \Rightarrow \chi(\bar{w})$, ending with repeated applications of $(\Rightarrow \cdot)$, $(\Rightarrow \mathrm{e})$, and (ID), and a derivation d_2 of $\Pi(\bar{w}, z), \chi(\bar{w}) \Rightarrow \Delta(\bar{w}, z)$ that extends d with repeated applications of $(\cdot \Rightarrow)$ and $(\mathrm{e} \Rightarrow)$, such that $\mathrm{md}(d_1) = 0$ and $\mathrm{md}(d_2) = \mathrm{md}(d)$.

Similarly, if z does not occur in Π, Δ, we can define

$$\chi(\bar{w}) := \prod \Pi \to \sum \Delta$$

and obtain a derivation d_1 of $\Gamma(\bar{w}, y) \Rightarrow \chi(\bar{w})$ that extends d with repeated applications of $(\cdot \Rightarrow)$, $(\mathrm{e} \Rightarrow)$, and $(\Rightarrow \mathrm{f})$, followed by an application of $(\Rightarrow \to)$, and a derivation d_2 of $\Pi(\bar{w}, z), \chi(\bar{w}) \Rightarrow \Delta(\bar{w}, z)$ ending with repeated applications of (ID), $(\Rightarrow \cdot)$, $(\Rightarrow \mathrm{e})$, and $(\mathrm{f} \Rightarrow)$, followed by an application of $(\to \Rightarrow)$, such that $\mathrm{md}(d_1) = \mathrm{md}(d)$ and $\mathrm{md}(d_2) = 0$.

For the base cases where d ends with (ID), $(\Rightarrow \mathrm{e})$, or $(\mathrm{f} \Rightarrow)$, either y does not occur in Γ or z does not occur in Π, Δ. For the remainder of the proof, let us assume that y occurs in Γ and z occurs in Π, Δ.

- $(\forall \Rightarrow)$: Suppose first that $\Gamma(\bar{w}, y)$ is $\Gamma'(\bar{w}, y), (\forall x)\varphi(x)$ and

  $$d' \vdash_{\forall 1\mathrm{CFL}} \Gamma'(\bar{w}, y), \varphi(u), \Pi(\bar{w}, z) \Rightarrow \Delta(\bar{w}, z),$$

  where $\mathrm{md}(d') = \mathrm{md}(d)$ and, using the assumption that no other variable lies in the scope of a quantifier, x is the only variable occurring in φ. Since y occurs in Γ and z occurs in Π, Δ, it follows from side-condition (i) for $(\forall \Rightarrow)$ that $u \in \bar{w} \cup \{y, z\}$. For the first subcase, suppose that $u \in \bar{w} \cup \{y\}$. An application of the induction hypothesis produces $\chi(\bar{w}) \in \mathrm{Fm}^{1+}_{\forall}(\mathcal{L}_s)$ and derivations d_1', d_2 such that $\mathrm{md}(d_1'), \mathrm{md}(d_2) \leq \mathrm{md}(d')$ and

  $$d_1' \vdash_{\forall 1\mathrm{CFL}} \Gamma'(\bar{w}, y), \varphi(u) \Rightarrow \chi(\bar{w}), \quad d_2 \vdash_{\forall 1\mathrm{CFL}} \Pi(\bar{w}, z), \chi(\bar{w}) \Rightarrow \Delta(\bar{w}, z).$$

  If u occurs in $\Gamma'(\bar{w}, y), \chi(\bar{w})$, then extending d_1' with an application of $(\forall \Rightarrow)$ yields a derivation d_1 such that $\mathrm{md}(d_1) = \mathrm{md}(d_1') \leq \mathrm{md}(d') = \mathrm{md}(d)$ and

  $$d_1 \vdash_{\forall 1\mathrm{CFL}} \Gamma'(\bar{w}, y), (\forall x)\varphi(x) \Rightarrow \chi(\bar{w}).$$

  Otherwise, by substituting u uniformly with y in d_1', we obtain a derivation of $\Gamma'(\bar{w}, y), \varphi(y) \Rightarrow \chi(\bar{w})$ and obtain d_1 as described previously.

For the second subcase, consider $u = z$. An application of the induction hypothesis produces $\chi'(\bar{w}) \in \mathrm{Fm}^{1+}_{\forall}(\mathcal{L}_s)$ and derivations d'_1, d'_2 such that $\mathrm{md}(d'_1), \mathrm{md}(d'_2) \leq \mathrm{md}(d')$ and

$$d'_1 \vdash_{\forall 1\mathrm{CFL}} \Gamma'(\bar{w}, y) \Rightarrow \chi'(\bar{w}), \quad d'_2 \vdash_{\forall 1\mathrm{CFL}} \varphi(z), \Pi(\bar{w}, z), \chi'(\bar{w}) \Rightarrow \Delta(\bar{w}, z).$$

Let $\chi(\bar{w}) \coloneqq \chi'(\bar{w}) \cdot (\forall x)\varphi(x)$. Combining an instance $(\forall x)\varphi(x) \Rightarrow (\forall x)\varphi(x)$ of (ID) with d'_1 and an application of $(\Rightarrow \cdot)$ yields a derivation d_1 such that $\mathrm{md}(d_1) = \mathrm{md}(d'_1) \leq \mathrm{md}(d') = \mathrm{md}(d)$ and

$$d_1 \vdash_{\forall 1\mathrm{CFL}} \Gamma'(\bar{w}, y), (\forall x)\varphi(x) \Rightarrow \chi(\bar{w}).$$

Also, d'_2 extended with applications of $(\forall \Rightarrow)$ and $(\cdot \Rightarrow)$ yields a derivation d_2 such that $\mathrm{md}(d_2) = \mathrm{md}(d'_2) \leq \mathrm{md}(d') = \mathrm{md}(d)$ and

$$d_2 \vdash_{\forall 1\mathrm{CFL}} \Pi(\bar{w}, z), \chi(\bar{w}) \Rightarrow \Delta(\bar{w}, z).$$

Suppose next that $\Pi(\bar{w}, z)$ is $\Pi'(\bar{w}, z), (\forall x)\varphi(x)$ and

$$d' \vdash_{\forall 1\mathrm{CFL}} \Gamma(\bar{w}, y), \Pi'(\bar{w}, z), \varphi(u) \Rightarrow \Delta(\bar{w}, z),$$

where $\mathrm{md}(d') = \mathrm{md}(d)$ and x is the only variable occurring in φ. Since y occurs in Γ and z occurs in Π, Δ, it follows from side-condition (i) for $(\forall \Rightarrow)$ that $u \in \bar{w} \cup \{y, z\}$. The case of $u \in \bar{w} \cup \{z\}$ is very similar to the first subcase above, so consider $u = y$. An application of the induction hypothesis produces $\chi'(\bar{w}) \in \mathrm{Fm}^{1+}_{\forall}(\mathcal{L}_s)$ and derivations d'_1, d'_2 such that $\mathrm{md}(d'_1), \mathrm{md}(d'_2) \leq \mathrm{md}(d')$ and

$$d'_1 \vdash_{\forall 1\mathrm{CFL}} \Gamma(\bar{w}, y), \varphi(y) \Rightarrow \chi'(\bar{w}), \quad d'_2 \vdash_{\forall 1\mathrm{CFL}} \Pi'(\bar{w}, z), \chi'(\bar{w}) \Rightarrow \Delta(\bar{w}, z).$$

Let $\chi(\bar{w}) \coloneqq (\forall x)\varphi(x) \to \chi'(\bar{w})$. Extending d'_1 with applications of $(\forall \Rightarrow)$ and $(\Rightarrow \to)$ yields a derivation d_1 such that $\mathrm{md}(d_1) = \mathrm{md}(d'_1) \leq \mathrm{md}(d') = \mathrm{md}(d)$ and

$$d_1 \vdash_{\forall 1\mathrm{CFL}} \Gamma(\bar{w}, y) \Rightarrow \chi(\bar{w}).$$

Also, d'_2 and an instance $(\forall x)\varphi(x) \Rightarrow (\forall x)\varphi(x)$ of (ID) combined with an application of $(\to \Rightarrow)$ yields a derivation d_2 such that $\mathrm{md}(d_2) = \mathrm{md}(d'_2) \leq \mathrm{md}(d') = \mathrm{md}(d)$ and

$$d_2 \vdash_{\forall 1\mathrm{CFL}} \Pi'(\bar{w}, z), (\forall x)\varphi(x), \chi(\bar{w}) \Rightarrow \Delta(\bar{w}, z).$$

- $(\Rightarrow \forall)$: Suppose that $\Delta(\bar{w}, z)$ is $(\forall x)\varphi(x)$ and for some variable u that does not occur freely in $\Gamma(\bar{w}, y), \Pi(\bar{w}, z) \Rightarrow (\forall x)\varphi(x)$,

$$d' \vdash_{\forall 1\mathrm{CFL}} \Gamma(\bar{w}, y), \Pi(\bar{w}, z) \Rightarrow \varphi(u),$$

where $\mathrm{md}(d') = \mathrm{md}(d) - 1$ and x is the only variable occurring in φ. An application of the induction hypothesis produces $\chi'(\bar{w}, u) \in \mathrm{Fm}^{1+}_{\forall}(\mathcal{L}_s)$ and derivations d'_1, d'_2 such that $\mathrm{md}(d'_1), \mathrm{md}(d'_2) \leq \mathrm{md}(d')$ and

$$d'_1 \vdash_{\forall 1\mathrm{CFL}} \Gamma(\bar{w}, y) \Rightarrow \chi'(\bar{w}, u), \quad d'_2 \vdash_{\forall 1\mathrm{CFL}} \Pi(\bar{w}, z), \chi'(\bar{w}, u) \Rightarrow \varphi(u).$$

Let $\chi(\bar{w}) \coloneqq (\forall x)\chi'(\bar{w}, x)$. Extending d'_1 with an application of $(\Rightarrow\forall)$ yields a derivation d_1 such that $\mathrm{md}(d_1) = \mathrm{md}(d'_1) + 1 \leq \mathrm{md}(d') + 1 = \mathrm{md}(d)$ and

$$d_1 \vdash_{\forall 1\mathrm{CFL}} \Gamma(\bar{w}, y) \Rightarrow \chi(\bar{w}).$$

Also, extending d'_2 with applications of $(\forall\Rightarrow)$ and $(\Rightarrow\forall)$ yield a derivation d_2 such that $\mathrm{md}(d_2) = \mathrm{md}(d'_2) + 1 \leq \mathrm{md}(d') + 1 = \mathrm{md}(d)$ and

$$d_2 \vdash_{\forall 1\mathrm{CFL}} \Pi(\bar{w}, z), \chi(\bar{w}) \Rightarrow (\forall x)\varphi(x).$$

- $(\Rightarrow\exists)$: Suppose that $\Delta(\bar{w}, z)$ is $(\exists x)\varphi(x)$ and

$$d' \vdash_{\forall 1\mathrm{CFL}} \Gamma(\bar{w}, y), \Pi(\bar{w}, z) \Rightarrow \varphi(u),$$

  where $\mathrm{md}(d') = \mathrm{md}(d)$ and x is the only variable occurring in φ. Since y occurs in Γ and z occurs in Π, Δ, it follows from side-condition (i) for $(\Rightarrow\exists)$ that $u \in \bar{w} \cup \{y, z\}$. For the first subcase, suppose that $u \in \bar{w} \cup \{z\}$. An application of the induction hypothesis produces $\chi(\bar{w}) \in \mathrm{Fm}^{1+}_{\forall}(\mathcal{L}_s)$ and derivations d_1, d'_2 such that $\mathrm{md}(d_1), \mathrm{md}(d'_2) \leq \mathrm{md}(d')$ and

$$d_1 \vdash_{\forall 1\mathrm{CFL}} \Gamma(\bar{w}, y) \Rightarrow \chi(\bar{w}), \quad d'_2 \vdash_{\forall 1\mathrm{CFL}} \Pi(\bar{w}, z), \chi(\bar{w}) \Rightarrow \varphi(u).$$

  If u occurs in $\Pi(\bar{w}, z), \chi(\bar{w})$, then extending d'_2 with an application of $(\Rightarrow\exists)$ yields a derivation d_2 such that $\mathrm{md}(d_2) = \mathrm{md}(d'_2) \leq \mathrm{md}(d')$ and

$$d_2 \vdash_{\forall 1\mathrm{CFL}} \Pi(\bar{w}, z), \chi(\bar{w}) \Rightarrow (\exists x)\varphi(x).$$

  Otherwise, by substituting u uniformly with z in d'_2, we obtain a derivation of $\Pi(\bar{w}, z), \chi(\bar{w}) \Rightarrow \varphi(z)$ and obtain d_2 as described previously.

  For the second subcase, consider $u = y$. An application of the induction hypothesis produces $\chi'(\bar{w}) \in \mathrm{Fm}^{1+}_{\forall}(\mathcal{L}_s)$ and derivations d'_1, d'_2 such that $\mathrm{md}(d'_1), \mathrm{md}(d'_2) \leq \mathrm{md}(d')$ and

$$d'_1 \vdash_{\forall 1\mathrm{CFL}} \Pi(\bar{w}, z) \Rightarrow \chi'(\bar{w}), \quad d'_2 \vdash_{\forall 1\mathrm{CFL}} \Gamma(\bar{w}, y), \chi'(\bar{w}) \Rightarrow \varphi(y).$$

  Let $\chi(\bar{w}) \coloneqq \chi'(\bar{w}) \to (\exists x)\varphi(x)$. Combining d'_2 with applications of $(\Rightarrow\exists)$ and $(\Rightarrow\to)$ yields a derivation d_1 such that $\mathrm{md}(d_1) = \mathrm{md}(d'_2) \leq \mathrm{md}(d') = \mathrm{md}(d)$ and

$$d_1 \vdash_{\forall 1\mathrm{CFL}} \Gamma(\bar{w}, y) \Rightarrow \chi(\bar{w}).$$

  Also, combining the instance $(\exists x)\varphi(x) \Rightarrow (\exists x)\varphi(x)$ of (ID) and d'_1 with $(\to\Rightarrow)$ yields a derivation d_2 such that $\mathrm{md}(d_2) = \mathrm{md}(d'_1) \leq \mathrm{md}(d') = \mathrm{md}(d)$ and

$$d_2 \vdash_{\forall 1\mathrm{CFL}} \Pi(\bar{w}, z), \chi(\bar{w}) \Rightarrow (\exists x)\varphi(x).$$

- $(\exists \Rightarrow)$: Suppose first that $\Gamma(\bar{w}, y)$ is $\Gamma'(\bar{w}, y), (\exists x)\varphi(x)$ and for some variable u that does not occur freely in $\Gamma(\bar{w}, y), \Pi(\bar{w}, z) \Rightarrow \Delta(\bar{w}, z)$,

$$d' \vdash_{\forall 1\text{CFL}} \Gamma'(\bar{w}, y), \varphi(u), \Pi(\bar{w}, z) \Rightarrow \Delta(\bar{w}, z),$$

where $\text{md}(d') = \text{md}(d) - 1$ and x is the only variable occurring in φ. An application of the induction hypothesis produces $\chi'(\bar{w}, u) \in \text{Fm}_{\forall}^{1+}(\mathcal{L}_s)$ and derivations d_1', d_2' such that $\text{md}(d_1'), \text{md}(d_2') \leq \text{md}(d')$ and

$$d_1' \vdash_{\forall 1\text{CFL}} \Gamma'(\bar{w}, y), \varphi(u) \Rightarrow \chi'(\bar{w}, u),$$
$$d_2' \vdash_{\forall 1\text{CFL}} \Pi(\bar{w}, z), \chi'(\bar{w}, u) \Rightarrow \Delta(\bar{w}, z).$$

Let $\chi(\bar{w}) \coloneqq (\exists x)\chi'(\bar{w}, x)$. Combining d_1' with applications of $(\Rightarrow\exists)$ and $(\exists\Rightarrow)$ yields a derivation d_1 such that $\text{md}(d_1) = \text{md}(d_1') + 1 \leq \text{md}(d') + 1 = \text{md}(d)$ and

$$d_1 \vdash_{\forall 1\text{CFL}} \Gamma'(\bar{w}, y), (\exists x)\varphi(x) \Rightarrow \chi(\bar{w}).$$

Also, extending d_2' with an application of $(\exists\Rightarrow)$ yields a derivation d_2 such that $\text{md}(d_2) = \text{md}(d_2') + 1 \leq \text{md}(d') + 1 = \text{md}(d)$ and

$$d_2 \vdash_{\forall 1\text{CFL}} \Pi(\bar{w}, z), \chi(\bar{w}) \Rightarrow \Delta(\bar{w}, z).$$

Now suppose that $\Pi(\bar{w}, z)$ is $\Pi'(\bar{w}, z), (\exists x)\varphi(x)$ and for some variable u that does not occur freely in $\Gamma(\bar{w}, y), \Pi(\bar{w}, z) \Rightarrow \Delta(\bar{w}, z)$,

$$d' \vdash_{\forall 1\text{CFL}} \Gamma(\bar{w}, y), \Pi'(\bar{w}, z), \varphi(u) \Rightarrow \Delta(\bar{w}, z),$$

where $\text{md}(d') = \text{md}(d) - 1$ and x is the only variable occurring in φ. An application of the induction hypothesis produces $\chi'(\bar{w}, u) \in \text{Fm}_{\forall}^{1+}(\mathcal{L}_s)$ and derivations d_1', d_2' such that $\text{md}(d_1'), \text{md}(d_2') \leq \text{md}(d')$ and

$$d_1' \vdash_{\forall 1\text{CFL}} \Gamma(\bar{w}, y) \Rightarrow \chi'(\bar{w}, u),$$
$$d_2' \vdash_{\forall 1\text{CFL}} \Pi'(\bar{w}, z), \varphi(u), \chi'(\bar{w}, u) \Rightarrow \Delta(\bar{w}, z).$$

Let $\chi(\bar{w}) \coloneqq (\forall x)\chi'(\bar{w}, x)$. The derivation d_1' together with an application of $(\Rightarrow\forall)$ yields a derivation d_1 such that $\text{md}(d_1) = \text{md}(d_1') + 1 \leq \text{md}(d') + 1 = \text{md}(d)$ and

$$d_1 \vdash_{\forall 1\text{CFL}} \Gamma(\bar{w}, y) \Rightarrow \chi(\bar{w}).$$

Also, d_2' together with applications of $(\forall\Rightarrow)$ and $(\exists\Rightarrow)$ yields a derivation d_2 such that $\text{md}(d_2) = \text{md}(d_2') + 1 \leq \text{md}(d') + 1 = \text{md}(d)$ and

$$d_2 \vdash_{\forall 1\text{CFL}} \Pi(\bar{w}, y), (\exists x)\varphi(x), \chi(\bar{w}) \Rightarrow \Delta(\bar{w}, z).$$

- (→⇒): Suppose for the first subcase that $\Gamma(\bar{w},y)$ is $\Gamma_1(\bar{w},y),\Gamma_2(\bar{w},y),\varphi(\bar{w},y)\to\psi(\bar{w},y)$ and $\Pi(\bar{w},z)$ is $\Pi_1(\bar{w},z),\Pi_2(\bar{w},z)$, and

$$\begin{aligned}
d_1' &\vdash_{\forall 1\mathrm{CFL}} \Gamma_1(\bar{w},y),\Pi_1(\bar{w},z)\Rightarrow\varphi(\bar{w},y),\\
d_2' &\vdash_{\forall 1\mathrm{CFL}} \Gamma_2(\bar{w},y),\psi(\bar{w},y),\Pi_2(\bar{w},z)\Rightarrow\Delta(\bar{w},z),
\end{aligned}$$

where $\mathrm{md}(d_1'),\mathrm{md}(d_2')\le\mathrm{md}(d)$. Two applications of the induction hypothesis produce $\chi_1(\bar{w}),\chi_2(\bar{w})\in\mathrm{Fm}_\forall^{1+}(\mathcal{L}_s)$ and derivations d_{11}', d_{12}', d_{21}', d_{22}' such that $\mathrm{md}(d_{11}'),\mathrm{md}(d_{12}')\le\mathrm{md}(d_1')$, $\mathrm{md}(d_{21}'),\mathrm{md}(d_{22}')\le\mathrm{md}(d_2')$, and

$$\begin{aligned}
d_{11}' &\vdash_{\forall 1\mathrm{CFL}} \Gamma_1(\bar{w},y),\chi_1(\bar{w})\Rightarrow\varphi(\bar{w},y),\\
d_{12}' &\vdash_{\forall 1\mathrm{CFL}} \Pi_1(\bar{w},z)\Rightarrow\chi_1(\bar{w}),\\
d_{21}' &\vdash_{\forall 1\mathrm{CFL}} \Gamma_2(\bar{w},y),\psi(\bar{w},y)\Rightarrow\chi_2(\bar{w}),\\
d_{22}' &\vdash_{\forall 1\mathrm{CFL}} \Pi_2(\bar{w},z),\chi_2(\bar{w})\Rightarrow\Delta(\bar{w},z).
\end{aligned}$$

Let $\chi(\bar{w}):=\chi_1(\bar{w})\to\chi_2(\bar{w})$. Then d_{11}' and d_{21}', together with applications of (→⇒) and (⇒→), and d_{12}' and d_{22}', together with an application of (→⇒), yield derivations d_1 and d_2, respectively, such that $\mathrm{md}(d_1),\mathrm{md}(d_2)\le\mathrm{md}(d)$ and

$$\begin{aligned}
d_1 &\vdash_{\forall 1\mathrm{CFL}} \Gamma_1(\bar{w},y),\Gamma_2(\bar{w},y),\varphi(\bar{w},y)\to\psi(\bar{w},y)\Rightarrow\chi(\bar{w}),\\
d_2 &\vdash_{\forall 1\mathrm{CFL}} \Pi_1(\bar{w},z),\Pi_2(\bar{w},z),\chi(\bar{w})\Rightarrow\Delta(\bar{w},z).
\end{aligned}$$

For the second subcase, suppose that $\Gamma(\bar{w},y)$ is $\Gamma_1(\bar{w},y),\Gamma_2(\bar{w},y)$ and $\Pi(\bar{w},z)$ is $\Pi_1(\bar{w},z),\Pi_2(\bar{w},z),\varphi(\bar{w},z)\to\psi(\bar{w},z)$, and

$$\begin{aligned}
d_1' &\vdash_{\forall 1\mathrm{CFL}} \Gamma_1(\bar{w},y),\Pi_1(\bar{w},z)\Rightarrow\varphi(\bar{w},z),\\
d_2' &\vdash_{\forall 1\mathrm{CFL}} \Gamma_2(\bar{w},y),\Pi_2(\bar{w},z),\psi(\bar{w},z)\Rightarrow\Delta(\bar{w},z),
\end{aligned}$$

where $\mathrm{md}(d_1'),\mathrm{md}(d_2')\le\mathrm{md}(d)$. Two applications of the induction hypothesis produce $\chi_1(\bar{w}),\chi_2(\bar{w})\in\mathrm{Fm}_\forall^{1+}(\mathcal{L}_s)$ and derivations d_{11}', d_{12}', d_{21}', d_{22}' such that $\mathrm{md}(d_{11}'),\mathrm{md}(d_{12}')\le\mathrm{md}(d_1')$, $\mathrm{md}(d_{21}'),\mathrm{md}(d_{22}')\le\mathrm{md}(d_2')$, and

$$\begin{aligned}
d_{11}' &\vdash_{\forall 1\mathrm{CFL}} \Gamma_1(\bar{w},y)\Rightarrow\chi_1(\bar{w}),\\
d_{12}' &\vdash_{\forall 1\mathrm{CFL}} \Pi_1(\bar{w},z),\chi_1(\bar{w})\Rightarrow\varphi(\bar{w},z),\\
d_{21}' &\vdash_{\forall 1\mathrm{CFL}} \Gamma_2(\bar{w},y)\Rightarrow\chi_2(\bar{w}),\\
d_{22}' &\vdash_{\forall 1\mathrm{CFL}} \Pi_2(\bar{w},z),\psi(\bar{w},z),\chi_2(\bar{w})\Rightarrow\Delta(\bar{w},z).
\end{aligned}$$

Let $\chi(\bar{w}):=\chi_1(\bar{w})\cdot\chi_2(\bar{w})$. Then d_{11}' and d_{21}', together with an application of (⇒·), and d_{12}' and d_{22}', together with applications of (→⇒) and (·⇒), yield derivations d_1 and d_2, respectively, such that $\mathrm{md}(d_1),\mathrm{md}(d_2)\le\mathrm{md}(d)$ and

$$\begin{aligned}
d_1 &\vdash_{\forall 1\mathrm{CFL}} \Gamma_1(\bar{w},y),\Gamma_2(\bar{w},y)\Rightarrow\chi(\bar{w}),\\
d_2 &\vdash_{\forall 1\mathrm{CFL}} \Pi_1(\bar{w},z),\Pi_2(\bar{w},z),\varphi(\bar{w},z)\to\psi(\bar{w},z),\chi(\bar{w})\Rightarrow\Delta(\bar{w},z).
\end{aligned}$$

- ($\Rightarrow\to$): Suppose that $\Delta(\bar{w}, z)$ is $\varphi(\bar{w}, z) \to \psi(\bar{w}, z)$ and

$$d' \vdash_{\forall 1\text{CFL}} \Gamma(\bar{w}, y), \Pi(\bar{w}, z), \varphi(\bar{w}, z) \Rightarrow \psi(\bar{w}, z).$$

By the induction hypothesis, there exist $\chi(\bar{w}) \in \text{Fm}^{1+}_{\forall}(\mathcal{L}_s)$ and derivations d_1, d_2' such that

$$d_1 \vdash_{\forall 1\text{CFL}} \Gamma(\bar{w}, y) \Rightarrow \chi(\bar{w}),\ \ d_2' \vdash_{\forall 1\text{CFL}} \Pi(\bar{w}, z), \varphi(\bar{w}, z), \chi(\bar{w}) \Rightarrow \psi(\bar{w}, z).$$

The derivation d_2' with an application of ($\Rightarrow\to$) yields a derivation d_2 such that

$$d_2 \vdash_{\forall 1\text{CFL}} \Pi(\bar{w}, z), \chi(\bar{w}) \Rightarrow \varphi(\bar{w}, z) \to \psi(\bar{w}, z).$$

The constraints on $\text{md}(d_1)$ and $\text{md}(d_2)$ clearly hold.

- ($\vee\Rightarrow$): Suppose first that $\Gamma(\bar{w}, y)$ is $\Gamma'(\bar{w}, y), \varphi(\bar{w}, y) \vee \psi(\bar{w}, y)$ and

$$\begin{aligned} d_1' &\vdash_{\forall 1\text{CFL}} \Gamma'(\bar{w}, y), \varphi(\bar{w}, y), \Pi(\bar{w}, z) \Rightarrow \Delta(\bar{w}, z), \\ d_2' &\vdash_{\forall 1\text{CFL}} \Gamma'(\bar{w}, y), \psi(\bar{w}, y), \Pi(\bar{w}, z) \Rightarrow \Delta(\bar{w}, z). \end{aligned}$$

By the induction hypothesis, there exist $\chi_1(\bar{w}), \chi_2(\bar{w}) \in \text{Fm}^{1+}_{\forall}(\mathcal{L}_s)$ and derivations $d_{11}', d_{12}', d_{21}', d_{22}'$ such that

$$\begin{aligned} d_{11}' &\vdash_{\forall 1\text{CFL}} \Gamma'(\bar{w}, y), \varphi(\bar{w}, y) \Rightarrow \chi_1(\bar{w}), \\ d_{12}' &\vdash_{\forall 1\text{CFL}} \Pi(\bar{w}, z), \chi_1(\bar{w}) \Rightarrow \Delta(\bar{w}, z), \\ d_{21}' &\vdash_{\forall 1\text{CFL}} \Gamma'(\bar{w}, y), \psi(\bar{w}, y) \Rightarrow \chi_2(\bar{w}), \\ d_{22}' &\vdash_{\forall 1\text{CFL}} \Pi(\bar{w}, z), \chi_2(\bar{w}) \Rightarrow \Delta(\bar{w}, z). \end{aligned}$$

Define $\chi(\bar{w}) := \chi_1(\bar{w}) \vee \chi_2(\bar{w})$. The derivations d_{11}', d_{21}', together with applications of $(\Rightarrow\vee)_1$, $(\Rightarrow\vee)_2$, and ($\vee\Rightarrow$), yield a derivation d_1, and the derivations d_{12}', d_{22}', together with an application of ($\vee\Rightarrow$), yield a derivation d_2, satisfying

$$\begin{aligned} d_1 &\vdash_{\forall 1\text{CFL}} \Gamma'(\bar{w}, y), \varphi(\bar{w}, y) \vee \psi(\bar{w}, y) \Rightarrow \chi_1(\bar{w}) \vee \chi_2(\bar{w}), \\ d_2 &\vdash_{\forall 1\text{CFL}} \Pi(\bar{w}, z), \chi_1(\bar{w}) \vee \chi_2(\bar{w}) \Rightarrow \Delta(\bar{w}, z). \end{aligned}$$

Clearly, $\text{md}(d_1), \text{md}(d_2) \leq \text{md}(d)$.

Suppose now that $\Pi(\bar{w}, z)$ is $\Pi'(\bar{w}, z), \varphi(\bar{w}, z) \vee \psi(\bar{w}, z)$ and

$$\begin{aligned} d_1' &\vdash_{\forall 1\text{CFL}} \Gamma(\bar{w}, y), \Pi'(\bar{w}, z), \varphi(\bar{w}, z) \Rightarrow \Delta(\bar{w}, z), \\ d_2' &\vdash_{\forall 1\text{CFL}} \Gamma(\bar{w}, y), \Pi'(\bar{w}, z), \psi(\bar{w}, z) \Rightarrow \Delta(\bar{w}, z). \end{aligned}$$

By the induction hypothesis, there exist $\chi_1(\bar{w}), \chi_2(\bar{w}) \in \text{Fm}^{1+}_{\forall}(\mathcal{L}_s)$ and derivations $d_{11}', d_{12}', d_{21}', d_{22}'$ such that

$$\begin{aligned} d_{11}' &\vdash_{\forall 1\text{CFL}} \Gamma(\bar{w}, y) \Rightarrow \chi_1(\bar{w}), \\ d_{12}' &\vdash_{\forall 1\text{CFL}} \Pi'(\bar{w}, z), \varphi(\bar{w}, z), \chi_1(\bar{w}) \Rightarrow \Delta(\bar{w}, z), \\ d_{21}' &\vdash_{\forall 1\text{CFL}} \Gamma(\bar{w}, y) \Rightarrow \chi_2(\bar{w}), \\ d_{22}' &\vdash_{\forall 1\text{CFL}} \Pi'(\bar{w}, z), \psi(\bar{w}, z), \chi_2(\bar{w}) \Rightarrow \Delta(\bar{w}, z). \end{aligned}$$

Let $\chi(\bar{w}) \coloneqq \chi_1(\bar{w}) \wedge \chi_2(\bar{w})$. Then the derivations d'_{11}, d'_{21}, together with an application of $(\Rightarrow\wedge)$, and the derivations d'_{12}, d'_{22}, together with applications of $(\wedge\Rightarrow)_1$, $(\wedge\Rightarrow)_2$, and $(\vee\Rightarrow)$, yield derivations d_1 and d_2, respectively, such that

$$d_1 \vdash_{\forall 1\mathrm{CFL}} \Gamma(\bar{w}, y) \Rightarrow \chi_1(\bar{w}) \wedge \chi_2(\bar{w}),$$
$$d_2 \vdash_{\forall 1\mathrm{CFL}} \Pi'(\bar{w}, z), \varphi(\bar{w}, z) \vee \psi(\bar{w}, z), \chi_1(\bar{w}) \wedge \chi_2(\bar{w}) \Rightarrow \Delta(\bar{w}, z).$$

Clearly, again $\mathrm{md}(d_1), \mathrm{md}(d_2) \leq \mathrm{md}(d)$.

- $(\Rightarrow\vee)$: Suppose that $\Delta(\bar{w}, z)$ is $\varphi_1(\bar{w}, z) \vee \varphi_2(\bar{w}, z)$ and

  $$d' \vdash_{\forall 1\mathrm{CFL}} \Gamma(\bar{w}, y), \Pi(\bar{w}, z) \Rightarrow \varphi_i(\bar{w}, z).$$

  By the induction hypothesis, there exist $\chi(\bar{w}) \in \mathrm{Fm}^{1+}_{\forall}(\mathcal{L}_s)$ and derivations d_1, d'_2 such that

  $$d_1 \vdash_{\forall 1\mathrm{CFL}} \Gamma(\bar{w}, y) \Rightarrow \chi(\bar{w}), \quad d'_2 \vdash_{\forall 1\mathrm{CFL}} \Pi(\bar{w}, z), \chi(\bar{w}) \Rightarrow \varphi_i(\bar{w}, z).$$

  The derivation d'_2 together with an application of $(\Rightarrow\vee)_i$ yields a derivation d_2 such that

  $$d_2 \vdash_{\forall 1\mathrm{CFL}} \Pi(\bar{w}, z), \chi(\bar{w}) \Rightarrow \varphi_1(\bar{w}, z) \vee \varphi_2(\bar{w}, z).$$

  The constraints on $\mathrm{md}(d_1)$ and $\mathrm{md}(d_2)$ clearly hold.

- $(\Rightarrow\wedge)$: Suppose that $\Delta(\bar{w}, z)$ is $\varphi(\bar{w}, z) \wedge \psi(\bar{w}, z)$ and

  $$d'_1 \vdash_{\forall 1\mathrm{CFL}} \Gamma(\bar{w}, y), \Pi(\bar{w}, z) \Rightarrow \varphi(\bar{w}, z),$$
  $$d'_2 \vdash_{\forall 1\mathrm{CFL}} \Gamma(\bar{w}, y), \Pi(\bar{w}, z) \Rightarrow \psi(\bar{w}, z).$$

  By the induction hypothesis, there exist $\chi_1(\bar{w}), \chi_2(\bar{w}) \in \mathrm{Fm}^{1+}_{\forall}(\mathcal{L}_s)$ and derivations $d'_{11}, d'_{12}, d'_{21}, d'_{22}$ such that

  $$d'_{11} \vdash_{\forall 1\mathrm{CFL}} \Gamma(\bar{w}, y) \Rightarrow \chi_1(\bar{w}), \quad d'_{12} \vdash_{\forall 1\mathrm{CFL}} \Pi(\bar{w}, z), \chi_1(\bar{w}) \Rightarrow \varphi(\bar{w}, z),$$
  $$d'_{21} \vdash_{\forall 1\mathrm{CFL}} \Gamma(\bar{w}, y) \Rightarrow \chi_2(\bar{w}), \quad d'_{22} \vdash_{\forall 1\mathrm{CFL}} \Pi(\bar{w}, z), \chi_2(\bar{w}) \Rightarrow \psi(\bar{w}, z).$$

  Let $\chi(\bar{w}) \coloneqq \chi_1(\bar{w}) \wedge \chi_2(\bar{w})$. Then the derivations d'_{11}, d'_{21}, together with an application of $(\Rightarrow\wedge)$, and the derivations d'_{12}, d'_{22}, together with applications of $(\wedge\Rightarrow)_1$, $(\wedge\Rightarrow)_2$, and $(\Rightarrow\wedge)$, yield derivations d_1 and d_2, respectively, such that

  $$d_1 \vdash_{\forall 1\mathrm{CFL}} \Gamma(\bar{w}, y) \Rightarrow \chi_1(\bar{w}) \wedge \chi_2(\bar{w}),$$
  $$d_2 \vdash_{\forall 1\mathrm{CFL}} \Pi(\bar{w}, z), \chi_1(\bar{w}) \wedge \chi_2(\bar{w}) \Rightarrow \varphi(\bar{w}, z) \wedge \psi(\bar{w}, z).$$

  Clearly, the constraints on $\mathrm{md}(d_1)$ and $\mathrm{md}(d_2)$ are satisfied in this case.

- $(\wedge\Rightarrow)$: Suppose that $\Gamma(\bar{w}, y)$ is $\Gamma'(\bar{w}, y), \varphi_1(\bar{w}, y) \wedge \varphi_2(\bar{w}, y)$ and

$$d' \vdash_{\forall 1\mathrm{CFL}} \Gamma'(\bar{w}, y), \varphi_i(\bar{w}, y), \Pi(\bar{w}, z) \Rightarrow \Delta(\bar{w}, z).$$

  By the induction hypothesis, there exist $\chi(\bar{w}) \in \mathrm{Fm}^{1+}_{\forall}(\mathcal{L}_s)$ and derivations d'_1, d_2 such that $\mathrm{md}(d'_1), \mathrm{md}(d_2) \leq \mathrm{md}(d)$ and

$$d'_1 \vdash_{\forall 1\mathrm{CFL}} \Gamma'(\bar{w}, y), \varphi_i(\bar{w}, y) \Rightarrow \chi(\bar{w}),\ \ d_2 \vdash_{\forall 1\mathrm{CFL}} \Pi(\bar{w}, z), \chi(\bar{w}) \Rightarrow \Delta(\bar{w}, z).$$

  The derivation d'_1 and an application of $(\wedge\Rightarrow)_i$ yield a derivation d_1 satisfying $\mathrm{md}(d_1) = \mathrm{md}(d'_1)$ and

$$d_1 \vdash_{\forall 1\mathrm{CFL}} \Gamma'(\bar{w}, y), \varphi_1(\bar{w}, y) \wedge \varphi_2(\bar{w}, y) \Rightarrow \chi(\bar{w}).$$

  The case where $\Pi(\bar{w}, z)$ is $\Pi'(\bar{w}, z), \varphi_1(\bar{w}, z) \wedge \varphi_2(\bar{w}, z)$ is very similar.

- $(\Rightarrow\cdot)$: Suppose that $\Delta(\bar{w}, z)$ is $\varphi(\bar{w}, z)\cdot\psi(\bar{w}, z)$. Suppose also that $\Gamma(\bar{w}, y)$ is $\Gamma_1(\bar{w}, y), \Gamma_2(\bar{w}, y)$, and $\Pi(\bar{w}, z)$ is $\Pi_1(\bar{w}, z), \Pi_2(\bar{w}, z)$, and

$$d'_1 \vdash_{\forall 1\mathrm{CFL}} \Gamma_1(\bar{w}, y), \Pi_1(\bar{w}, z) \Rightarrow \varphi(\bar{w}, z),$$
$$d'_2 \vdash_{\forall 1\mathrm{CFL}} \Gamma_2(\bar{w}, y), \Pi_2(\bar{w}, z) \Rightarrow \psi(\bar{w}, z).$$

  By the induction hypothesis, there exist $\chi_1(\bar{w}), \chi_2(\bar{w}) \in \mathrm{Fm}^{1+}_{\forall}(\mathcal{L}_s)$ and derivations $d'_{11}, d'_{12}, d'_{21}, d'_{22}$ such that

$$d'_{11} \vdash_{\forall 1\mathrm{CFL}} \Gamma_1(\bar{w}, y) \Rightarrow \chi_1(\bar{w}), \quad d'_{12} \vdash_{\forall 1\mathrm{CFL}} \Pi_1(\bar{w}, z), \chi_1(\bar{w}) \Rightarrow \varphi(\bar{w}, z),$$
$$d'_{21} \vdash_{\forall 1\mathrm{CFL}} \Gamma_2(\bar{w}, y) \Rightarrow \chi_2(\bar{w}), \quad d'_{22} \vdash_{\forall 1\mathrm{CFL}} \Pi_2(\bar{w}, z), \chi_2(\bar{w}) \Rightarrow \psi(\bar{w}, z).$$

  Let $\chi(\bar{w}) \coloneqq \chi_1(\bar{w}) \cdot \chi_2(\bar{w})$. Then the derivations d'_{11}, d'_{21}, together with an application of $(\Rightarrow\cdot)$, and the derivations d'_{12}, d'_{22}, together with applications of $(\Rightarrow\cdot)$ and $(\cdot\Rightarrow)$, yield derivations d_1 and d_2, respectively, such that

$$d_1 \vdash_{\forall 1\mathrm{CFL}} \Gamma_1(\bar{w}, y), \Gamma_2(\bar{w}, y) \Rightarrow \chi_1(\bar{w}) \cdot \chi_2(\bar{w}),$$
$$d_2 \vdash_{\forall 1\mathrm{CFL}} \Pi_1(\bar{w}, z), \Pi_2(\bar{w}, z), \chi_1(\bar{w}) \cdot \chi_2(\bar{w}) \Rightarrow \varphi(\bar{w}, z) \cdot \psi(\bar{w}, z).$$

  The constraints on $\mathrm{md}(d_1)$ and $\mathrm{md}(d_2)$ clearly hold.

- $(\cdot\Rightarrow)$: Suppose that $\Gamma(\bar{w}, y)$ is $\Gamma'(\bar{w}, y), \varphi(\bar{w}, y) \cdot \psi(\bar{w}, y)$ and

$$d' \vdash_{\forall 1\mathrm{CFL}} \Gamma'(\bar{w}, y), \varphi(\bar{w}, y), \psi(\bar{w}, y), \Pi(\bar{w}, z) \Rightarrow \Delta(\bar{w}, z).$$

  By the induction hypothesis, there exist $\chi(\bar{w}) \in \mathrm{Fm}^{1+}_{\forall}(\mathcal{L}_s)$ and derivations d'_1, d_2 such that

$$d'_1 \vdash_{\forall 1\mathrm{CFL}} \Gamma'(\bar{w}, y), \varphi(\bar{w}, y), \psi(\bar{w}, y) \Rightarrow \chi(\bar{w}),$$
$$d_2 \vdash_{\forall 1\mathrm{CFL}} \Pi(\bar{w}, z), \chi(\bar{w}) \Rightarrow \Delta(\bar{w}, z).$$

  Then d'_1 and an application of $(\cdot\Rightarrow)$ yield a derivation d_1 such that

$$d_1 \vdash_{\forall 1\mathrm{CFL}} \Gamma'(\bar{w}, y), \varphi(\bar{w}, y) \cdot \psi(\bar{w}, y) \Rightarrow \chi(\bar{w}).$$

  The constraints on $\mathrm{md}(d_1)$ and $\mathrm{md}(d_2)$ clearly hold.

  The case that $\Pi(\bar{w}, z)$ is $\Pi'(\bar{w}, z), \varphi(\bar{w}, z) \cdot \psi(\bar{w}, z)$ is very similar. □

3.3 An Alternative Completeness Proof

In this section, we reprove, by proof-theoretic means, the special case of Corollary 2.4.2 for the variety $\mathcal{FL}_e$. The main idea is to prove for any sequent $\Gamma \Rightarrow \Delta$ of one-variable $\mathcal{L}_{s\forall}$-formulas

$$d \vdash_{\forall 1\mathrm{CFL}} \Gamma \Rightarrow \Delta \quad \Longrightarrow \quad \models_{m\mathcal{FL}_e} (\prod \Gamma)^* \leq (\sum \Delta)^*.$$

This proof is established by induction on the lexicographically ordered pair $\langle \mathrm{md}(d), \mathrm{ht}(d) \rangle$, where most cases are a quite straightforward application of the induction hypothesis and the equations defining $m\mathcal{FL}_e$. If the last rule applied in d is $(\Rightarrow\forall)$ or $(\exists\Rightarrow)$, then we introduce a new variable and we cannot apply the induction hypothesis. With an application of Lemma 3.2.1 we obtain sequents of one-variable $\mathcal{L}_{s\forall}$-formulas and we can finish the proof with applications of the induction hypothesis and the equations defining $m\mathcal{FL}_e$.

Theorem 3.3.1. *For any set $\Sigma \cup \{\varphi \approx \psi\}$ of $\mathrm{Fm}^1_\forall(\mathcal{L}_s)$-equations,*

$$\Sigma \models^{\forall 1}_{\mathcal{FL}_e} \varphi \approx \psi \quad \Longleftrightarrow \quad \Sigma^* \models_{m\mathcal{FL}_e} \varphi^* \approx \psi^*.$$

Proof. The right-to-left direction follows directly from Corollary 2.3.3. For the converse, note first that due to compactness and the local deduction theorem for $\models^{\forall 1}_{\mathcal{V}}$, stated in Section 1.4, we can restrict to the case where $\Sigma = \emptyset$. Hence, by Proposition 3.1.1, it suffices to prove for any sequent $\Gamma \Rightarrow \Delta$ of one-variable $\mathcal{L}_{s\forall}$-formulas

$$d \vdash_{\forall 1\mathrm{CFL}} \Gamma \Rightarrow \Delta \quad \Longrightarrow \quad \models_{m\mathcal{FL}_e} (\prod \Gamma)^* \leq (\sum \Delta)^*.$$

We proceed by induction on the lexicographically ordered pair $\langle \mathrm{md}(d), \mathrm{ht}(d) \rangle$. The base cases are clear and the cases for the last application of a rule in d except $(\Rightarrow\forall)$ and $(\exists\Rightarrow)$ all follow by applying the induction hypothesis and the equations defining $m\mathcal{FL}_e$. Just note that for each such application, the premise(s) contain only formulas from $\mathrm{Fm}^1_\forall(\mathcal{L}_s)$ with at least one fewer symbol. In particular, for $(\forall\Rightarrow)$ and $(\Rightarrow\exists)$, it can be assumed that the variable u occurring in the premise is x and the result follows using $(\mathrm{L1}_\Box)$ or $(\mathrm{L1}_\Diamond)$. Let us consider one case where d ends with a rule for one of the propositional operations. Suppose that the last rule applied in d is $(\Rightarrow\cdot)$ with premises $\Gamma_1 \Rightarrow \varphi$ and $\Gamma_2 \Rightarrow \psi$. Since $\Gamma_1 \Rightarrow \varphi$ and $\Gamma_2 \Rightarrow \psi$ are $\mathrm{Fm}^1_\forall(\mathcal{L}_s)$-sequents, an application of the induction hypothesis yields

$$\models_{m\mathcal{FL}_e} (\prod \Gamma_1)^* \leq \varphi^* \quad \text{and} \quad \models_{m\mathcal{FL}_e} (\prod \Gamma_2)^* \leq \psi^*.$$

Since $\cdot$ is order preserving in both arguments,

$$\models_{m\mathcal{FL}_e} (\prod \Gamma_1)^* \cdot (\prod \Gamma_2)^* \leq \varphi^* \cdot \psi^*.$$

By the definition of the translation function $(-)^*$ and $\prod$,

$$\models_{m\mathcal{FL}_e} (\prod \Gamma_1)^* \cdot (\prod \Gamma_2)^* \approx (\prod(\Gamma_1, \Gamma_2))^* \quad \text{and} \quad \models_{m\mathcal{FL}_e} \varphi^* \cdot \psi^* \approx (\varphi \cdot \psi)^*,$$

and therefore,

$$\models_{m\mathcal{FL}_e} (\prod(\Gamma_1, \Gamma_2))^* \leq (\varphi \cdot \psi)^*.$$

Suppose the last rule applied in d is $(\forall\Rightarrow)$. Then

$$d' \vdash_{\forall 1\mathrm{CFL}} \Gamma, \varphi(x) \Rightarrow \Delta,$$

and since $\Gamma, \varphi(x) \Rightarrow \Delta$ is an $\mathrm{Fm}^1_\forall(\mathcal{L}_s)$-sequent, an application of the induction hypothesis yields

$$\models_{m\mathcal{FL}_e} (\prod \Gamma)^* \cdot \varphi(x)^* \approx (\prod(\Gamma, \varphi(x)))^* \leq (\sum \Delta)^*.$$

Since $\models_{m\mathcal{FL}_e} ((\forall x)\varphi(x))^* \approx \Box\varphi(x)^* \leq \varphi(x)^*$ by $(\mathrm{L1}_\Box)$, also

$$\models_{m\mathcal{FL}_e} (\prod \Gamma)^* \cdot ((\forall x)\varphi(x))^* \leq (\prod \Gamma)^* \cdot \varphi(x)^*$$

and

$$\models_{m\mathcal{FL}_e} (\prod(\Gamma, (\forall x)\varphi(x)))^* \leq (\prod \Gamma)^* \cdot \varphi(x)^* \leq (\sum \Delta)^*.$$

Suppose now that the last rule applied in d is $(\Rightarrow\forall)$, where Δ is $(\forall x)\psi(x)$ and x may occur freely in Γ. Then

$$d' \vdash_{\forall 1\mathrm{CFL}} \Gamma \Rightarrow \psi(z)$$

with $\mathrm{md}(d') = \mathrm{md}(d) - 1$, where z is a variable distinct from x. We write $\Gamma(y)$ and $d'(y)$ to denote Γ and d' with all free occurrences of x replaced by y. Clearly,

$$d'(y) \vdash_{\forall 1\mathrm{CFL}} \Gamma(y) \Rightarrow \psi(z)$$

with $\mathrm{md}(d'(y)) = \mathrm{md}(d')$. Note also that no occurrence of y or z lies in the scope of a quantifier in $\Gamma(y) \Rightarrow \psi(z)$. Hence, by Lemma 3.2.1, there exist a sentence χ and derivations d_1, d_2 such that $\mathrm{md}(d_1), \mathrm{md}(d_2) \leq \mathrm{md}(d')$ and

$$d_1 \vdash_{\forall 1\mathrm{CFL}} \Gamma(y) \Rightarrow \chi, \quad d_2 \vdash_{\forall 1\mathrm{CFL}} \chi \Rightarrow \psi(z).$$

Since χ is a sentence and x does not occur freely in $\Gamma(y)$ or $\psi(z)$, we can assume that d_1 and d_2 do not contain any free occurrences of x, and, by substituting all occurrences of y in d_1, and z in d_2, with x, obtain derivations d_1' of $\Gamma \Rightarrow \chi$ and d_2' of $\chi \Rightarrow \psi(x)$ with $\mathrm{md}(d_1') = \mathrm{md}(d_1)$ and $\mathrm{md}(d_2') = \mathrm{md}(d_2)$. Hence, by the induction hypothesis twice,

$$\models_{m\mathcal{FL}_e} (\prod \Gamma)^* \leq \chi^* \quad \text{and} \quad \models_{m\mathcal{FL}_e} \chi^* \leq \psi(x)^*.$$

Since $((\forall x)\chi)^* = \Box\chi^*$ and χ is a sentence,

$$\models_{m\mathcal{FL}_e} \chi^* \approx ((\forall x)\chi)^*,$$

and hence the equations defining $m\mathcal{FL}_e$ yield also

$$\models_{m\mathcal{FL}_e} \chi^* \leq ((\forall x)\psi(x))^*.$$

So finally we obtain

$$\vDash_{m\mathcal{FL}_e} (\prod \Gamma)^* \leq ((\forall x)\psi(x))^*.$$

For the last case, suppose that the last rule applied in d is ($\exists\Rightarrow$), where Γ is $\Gamma', (\exists x)\psi(x)$ and x may occur freely in Γ' and Δ. Then

$$d' \vdash_{\forall 1\mathrm{CFL}} \Gamma', \psi(y) \Rightarrow \Delta$$

with $\mathrm{md}(d') = \mathrm{md}(d) - 1$, where y is a variable distinct from x. We write $\Gamma'(z)$, $\Delta(z)$, and $d'(z)$ to denote Γ', Δ, and d' with all free occurrences of x replaced by z. Clearly,

$$d'(z) \vdash_{\forall 1\mathrm{CFL}} \Gamma'(z), \psi(y) \Rightarrow \Delta(z)$$

with $\mathrm{md}(d'(z)) = \mathrm{md}(d')$. By Lemma 3.2.1, there exist a sentence χ and derivations d_1, d_2 such that $\mathrm{md}(d_1), \mathrm{md}(d_2) \leq \mathrm{md}(d')$ and

$$d_1 \vdash_{\forall 1\mathrm{CFL}} \psi(y) \Rightarrow \chi, \quad d_2 \vdash_{\forall 1\mathrm{CFL}} \Gamma'(z), \chi \Rightarrow \Delta(z).$$

Since χ is a sentence and x does not occur freely in $\psi(y)$, $\Gamma'(z)$, or $\Delta(z)$, we can assume that d_1 and d_2 do not contain any free occurrences of x, and, by substituting all occurrences of y in d_1, and z in d_2, with x, obtain derivations d_1' of $\psi(x) \Rightarrow \chi$ and d_2' of $\Gamma', \chi \Rightarrow \Delta$ with $\mathrm{md}(d_1') = \mathrm{md}(d_1)$ and $\mathrm{md}(d_2') = \mathrm{md}(d_2)$. Hence, by the induction hypothesis,

$$\vDash_{m\mathcal{FL}_e} \psi(x)^* \leq \chi^* \quad \text{and} \quad \vDash_{m\mathcal{FL}_e} (\prod(\Gamma', \chi))^* \leq (\sum \Delta)^*.$$

Since $((\exists x)\chi)^* = \Diamond \chi^*$ and χ is a sentence,

$$\vDash_{m\mathcal{FL}_e} \chi^* \approx ((\exists x)\chi)^*,$$

and hence the equations defining $m\mathcal{FL}_e$ yield also

$$\vDash_{m\mathcal{FL}_e} ((\exists x)\psi(x))^* \leq \chi^*.$$

Therefore, we finally obtain

$$\vDash_{m\mathcal{FL}_e} (\prod(\Gamma', (\exists x)\psi(x)))^* \leq (\sum \Delta)^*. \qquad \square$$

3.4 Extensions of the Calculus $\forall$1CFL

In this section, we extend the proof-theoretic strategy for proving Corollary 2.4.2 to varieties of $\mathsf{FL_e}$-algebras axiomatized relative to $\mathcal{FL}_e$ by equations of a certain simple form.

Given a variable x, let $x^0 := \mathrm{e}$ and $x^{k+1} := x \cdot x^k$, for each $k \in \mathbb{N}$, and given a multiset Π and $k \in \mathbb{N}$, let Π^k denote the multiset union of k copies of Π. We define sequent rules

$$r(x \leq x^k) := \frac{\Gamma, \Pi^k \Rightarrow \Delta}{\Gamma, \Pi \Rightarrow \Delta} \quad \text{and} \quad r(\mathrm{f} \leq x) := \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \Delta} .$$

Note that these rules correspond to simple structural rules with exactly one premise, that is, some of the structural rules introduced in Section 1.3. In particular, $r(x \leq x^0)$ is (WL), $r(\mathrm{f} \leq x)$ is (WR), $r(x \leq x^2)$ is (C), and $r(x \leq x^k)$ for $k > 2$ is (K-CONTR). The rule (MINGLE) is not covered, since it corresponds to the equation $x^2 \leq x$ and has two premises. We show at the end of this section why (even simple) rules with multiple premises may cause problems with our method.

Let S be the set of equations $\{x \leq x^k \mid k \in \mathbb{N}\} \cup \{\mathrm{f} \leq x\}$. Given any $S' \subseteq S$, denote by $\mathcal{FL}_e + S'$ the variety of $\mathcal{FL}_e$-algebras axiomatized relative to $\mathcal{FL}_e$ by the equations in S', and by $\forall$1CFL $+ r(S')$ the sequent calculus $\forall$1CFL extended with the rules $r(\varepsilon)$ for each equation ε in S'. Then for any sequent $\Gamma \Rightarrow \Delta$ containing formulas from $\mathrm{Fm}^1_\forall(\mathcal{L}_s)$ (see, e.g., [46, 71]),

$$\vdash_{\forall\mathrm{1CFL+r(S')}} \Gamma \Rightarrow \Delta \quad \Longleftrightarrow \quad \models^{\forall 1}_{\mathcal{FL}_e + S'} \prod \Gamma \leq \sum \Delta.$$

We now formulate Lemma 3.2.1 for $\forall$1CFL $+ r(S')$ and extend the proof:

Lemma 3.4.1. *Let $\Gamma(\bar{w}, y), \Pi(\bar{w}, z) \Rightarrow \Delta(\bar{w}, z)$ be any sequent such that $y \neq z$, $x \notin \bar{w} \cup \{y, z\}$, and no variable in $\bar{w} \cup \{y, z\}$ lies in the scope of a quantifier. If*

$$d \vdash_{\forall\mathrm{1CFL+r(S')}} \Gamma(\bar{w}, y), \Pi(\bar{w}, z) \Rightarrow \Delta(\bar{w}, z),$$

then there exist $\chi(\bar{w}) \in \mathrm{Fm}^{1+}_\forall(\mathcal{L}_s)$ and derivations d_1, d_2 in $\forall$1CFL$+r(S')$ such that $\mathrm{md}(d_1), \mathrm{md}(d_2) \leq \mathrm{md}(d)$ and

$$d_1 \vdash_{\forall\mathrm{1CFL+r(S')}} \Gamma(\bar{w}, y) \Rightarrow \chi(\bar{w}), \quad d_2 \vdash_{\forall\mathrm{1CFL+r(S')}} \Pi(\bar{w}, z), \chi(\bar{w}) \Rightarrow \Delta(\bar{w}, z).$$

Proof. This proof is an extension of the proof for Lemma 3.2.1 and it is enough to just consider the additional cases for the rules in $r(S')$.

Suppose first that (WR) $= r(f \leq x) \in r(S')$ is the last rule applied in d and

$$d' \vdash_{\forall\mathrm{1CFL+r(S')}} \Gamma(\bar{w}, y), \Pi(\bar{w}, z) \Rightarrow \quad .$$

An application of the induction hypothesis yields $\chi(\bar{w}) \in \mathrm{Fm}^{1+}_\forall(\mathcal{L}_s)$ and derivations d_1, d'_2 satisfying $\mathrm{md}(d_1), \mathrm{md}(d'_2) \leq \mathrm{md}(d') = \mathrm{md}(d)$ and

$$d_1 \vdash_{\forall\mathrm{1CFL+r(S')}} \Gamma(\bar{w}, y) \Rightarrow \chi(\bar{w}), \quad d'_2 \vdash_{\forall\mathrm{1CFL+r(S')}} \Pi(\bar{w}, z), \chi(\bar{w}) \Rightarrow \quad .$$

Then d'_2 together with an application of (WR) yields a derivation d_2 with $\mathrm{md}(d_2) = \mathrm{md}(d'_2) \leq \mathrm{md}(d)$ and

$$d_2 \vdash_{\forall\mathrm{1CFL+r(S')}} \Pi(\bar{w}, z), \chi(\bar{w}) \Rightarrow \Delta(\bar{w}, z).$$

Suppose now that (WL) $= r(x \leq x^0) \in r(S')$ is the last rule applied in d and $\Gamma(\bar{w}, y)$ is $\Gamma_1(\bar{w}, y), \Gamma_2(\bar{w}, y)$ and $\Pi(\bar{w}, z)$ is $\Pi_1(\bar{w}, z), \Pi_2(\bar{w}, z)$, and

$$d' \vdash_{\forall 1\mathrm{CFL+r(S')}} \Gamma_1(\bar{w}, y), \Pi_1(\bar{w}, z) \Rightarrow \Delta(\bar{w}, z).$$

An application of the induction hypothesis yields $\chi(\bar{w}) \in \mathrm{Fm}^{1+}_{\forall}(\mathcal{L}_s)$ and derivations d_1', d_2' satisfying $\mathrm{md}(d_1'), \mathrm{md}(d_2') \leq \mathrm{md}(d') = \mathrm{md}(d)$ and

$$d_1' \vdash_{\forall 1\mathrm{CFL+r(S')}} \Gamma_1(\bar{w}, y) \Rightarrow \chi(\bar{w}), \quad d_2' \vdash_{\forall 1\mathrm{CFL+r(S')}} \Pi_1(\bar{w}, z), \chi(\bar{w}) \Rightarrow \Delta(\bar{w}, z).$$

Taking d_1' and d_2' together with an application of (WL) yields

$$d_1 \vdash_{\forall 1\mathrm{CFL+r(S')}} \Gamma_1(\bar{w}, y), \Gamma_2(\bar{w}, y) \Rightarrow \chi(\bar{w}),$$
$$d_2 \vdash_{\forall 1\mathrm{CFL+r(S')}} \Pi_1(\bar{w}, z), \Pi_2(\bar{w}, z), \chi(\bar{w}) \Rightarrow \Delta(\bar{w}, z),$$

where the derivations d_1 and d_2 satisfy $\mathrm{md}(d_1), \mathrm{md}(d_2) \leq \mathrm{md}(d)$.

Suppose finally that for some $k \in \mathbb{N}^{>0}$, $r(x \leq x^k) \in r(S')$ is the last rule in d and $\Gamma(\bar{w}, y)$ is $\Gamma_1(\bar{w}, y), \Gamma_2(\bar{w}, y)$ and $\Pi_1(\bar{w}, z), \Pi_2(\bar{w}, z)$, and

$$d' \vdash_{\forall 1\mathrm{CFL+r(S')}} \Gamma_1(\bar{w}, y), \Gamma_2(\bar{w}, y)^k, \Pi_1(\bar{w}, z), \Pi_2(\bar{w}, z)^k \Rightarrow \Delta(\bar{w}, z).$$

An application of the induction hypothesis yields $\chi(\bar{w}) \in \mathrm{Fm}^{1+}_{\forall}(\mathcal{L}_s)$ and derivations d_1', d_2' satisfying $\mathrm{md}(d_1'), \mathrm{md}(d_2') \leq \mathrm{md}(d') = \mathrm{md}(d)$ and

$$d_1' \vdash_{\forall 1\mathrm{CFL+r(S')}} \Gamma_1(\bar{w}, y), \Gamma_2(\bar{w}, y)^k \Rightarrow \chi(\bar{w}),$$
$$d_2' \vdash_{\forall 1\mathrm{CFL+r(S')}} \Pi_1(\bar{w}, z), \Pi_2(\bar{w}, z)^k, \chi(\bar{w}) \Rightarrow \Delta(\bar{w}, z).$$

The derivations d_1' and d_2' together with an application of $r(x \leq x^k)$ yields derivations d_1 and d_2 satisfying $\mathrm{md}(d_1), \mathrm{md}(d_2) \leq \mathrm{md}(d)$ and

$$d_1 \vdash_{\forall 1\mathrm{CFL+r(S')}} \Gamma_1(\bar{w}, y), \Gamma_2(\bar{w}, y) \Rightarrow \chi(\bar{w}),$$
$$d_2 \vdash_{\forall 1\mathrm{CFL+r(S')}} \Pi_1(\bar{w}, z), \Pi_2(\bar{w}, z), \chi(\bar{w}) \Rightarrow \Delta(\bar{w}, z). \qquad \square$$

Hence, following the proof of Theorem 3.3.1 yields the following more general result.

Theorem 3.4.2. *For any $S' \subseteq S$ and set $\Sigma \cup \{\varphi \approx \psi\}$ of $\mathrm{Fm}^1_{\forall}(\mathcal{L})$-equations,*

$$\Sigma \models^{\forall 1}_{\mathcal{FL}_e + S'} \varphi \approx \psi \quad \iff \quad \Sigma^* \models_{m\mathcal{FL}_e + S'} \varphi^* \approx \psi^*.$$

In particular, we obtain alternative proof-theoretic proofs of completeness for the axiomatizations of the one-variable fragments of the first-order extensions of $\mathrm{FL_{ew}}$, $\mathrm{FL_{ec}}$, and $\mathrm{FL_{ewc}}$ (intuitionistic logic).

As mentioned in the beginning of this section, we only consider extensions of ∀1CFL with simple rules that only have one premise. Let us consider ∀1CFL + {(MINGLE)} to illustrate why our method fails when we add rules with more than one premise to ∀1CFL. Suppose we have a derivation of the sequent

$$\Gamma(\bar{w}, y), \Gamma_1(\bar{w}, y), \Gamma_2(\bar{w}, y), \Pi(\bar{w}, z), \Pi_1(\bar{w}, z), \Pi_2(\bar{w}, z) \Rightarrow \Delta(\bar{w}, z)$$

of the form

$$\dfrac{\begin{array}{cc} d_1 & d_2 \\ \vdots & \vdots \\ \Gamma, \Gamma_1, \Pi, \Pi_1 \Rightarrow \Delta & \Gamma, \Gamma_2, \Pi, \Pi_2 \Rightarrow \Delta \end{array}}{\Gamma, \Gamma_1, \Gamma_2, \Pi, \Pi_1, \Pi_2 \Rightarrow \Delta} \ (\text{MINGLE})$$

where we leave out the free variables to save space. If we assume d_1 and d_2 are derivations that do not contain an application of (MINGLE), then by Lemma 3.2.1, we obtain formulas $\chi_1(\bar{w}), \chi_2(\bar{w})$ in $\text{Fm}_\forall^{1+}(\mathcal{L}_s)$ and derivations $d_{11}, d_{12}, d_{21}, d_{22}$ such that

$$\begin{array}{ll} d_{11} \vdash_{\forall 1\text{CFL}} \Gamma, \Gamma_1 \Rightarrow \chi_1, & d_{12} \vdash_{\forall 1\text{CFL}} \Pi, \Pi_1, \chi_1 \Rightarrow \Delta, \\ d_{21} \vdash_{\forall 1\text{CFL}} \Gamma, \Gamma_2 \Rightarrow \chi_2, & d_{22} \vdash_{\forall 1\text{CFL}} \Pi, \Pi_2, \chi_2 \Rightarrow \Delta. \end{array}$$

We would like to find $\chi(\bar{w}) \in \text{Fm}_\forall^{1+}(\mathcal{L}_s)$ constructed from $\chi_1(\bar{w})$ and $\chi_2(\bar{w})$, and derivations d'_1, d'_2 such that

$$d'_1 \vdash_{\forall 1\text{CFL}+\{(\text{MINGLE})\}} \Gamma, \Gamma_1, \Gamma_2 \Rightarrow \chi \quad \text{and} \quad d'_2 \vdash_{\forall 1\text{CFL}+\{(\text{MINGLE})\}} \Pi, \Pi_1, \Pi_2, \chi \Rightarrow \Delta.$$

However, the only rules with two premises where the contexts of the premises do not match are $(\to\Rightarrow)$, (MINGLE), and $(\Rightarrow\cdot)$. An application of $(\to\Rightarrow)$ can not work, since it would combine formulas where y occurs freely with formulas where z occurs freely. The rule (MINGLE) cannot be applied to the sequents $\Gamma, \Gamma_1 \Rightarrow \chi_1$ and $\Gamma, \Gamma_2 \Rightarrow \chi_2$, since χ_1 and χ_2 do not necessarily match. We could take d_{11} and d_{21} and apply $(\Rightarrow\vee)_1$ and $(\Rightarrow\vee)_2$, respectively. Then an application of (MINGLE) yields a derivation of $\Gamma, \Gamma_1, \Gamma_2 \Rightarrow \chi_1 \vee \chi_2$, but since we cannot apply $(\vee\Rightarrow)$ to $\Pi, \Pi_1, \chi_1 \Rightarrow \Delta$ and $\Pi, \Pi_2, \chi_2 \Rightarrow \Delta$ we cannot find a suitable derivation d'_2. Considering $(\Rightarrow\cdot)$ we can almost solve the problem. Consider the following derivations:

$$\dfrac{\begin{array}{cc} d_{11} & d_{21} \\ \vdots & \vdots \\ \Gamma, \Gamma_1 \Rightarrow \chi_1 & \Gamma, \Gamma_2 \Rightarrow \chi_2 \end{array}}{\Gamma, \Gamma, \Gamma_1, \Gamma_2 \Rightarrow \chi_1 \cdot \chi_2} \ (\Rightarrow\cdot)$$

$$\dfrac{\dfrac{\begin{array}{cc} d_{12} & d_{22} \\ \vdots & \vdots \\ \Pi, \Pi_1, \chi_1 \Rightarrow \Delta & \Pi, \Pi_2, \chi_2 \Rightarrow \Delta \end{array}}{\Pi, \Pi_1, \Pi_2, \chi_1, \chi_2 \Rightarrow \Delta} \ (\text{MINGLE})}{\Pi, \Pi_1, \Pi_2, \chi_1 \cdot \chi_2 \Rightarrow \Delta} \ (\cdot\Rightarrow)$$

The conclusion of the first derivation is the desired sequent with an additional Γ on the left-hand-side. There is no method to remove it in $\forall 1\text{CFL} + \{(\text{MINGLE})\}$, however, if we add (C), then we obtain

$$\dfrac{\dfrac{\begin{matrix} d_{11} \\ \vdots \\ \Gamma, \Gamma_1 \Rightarrow \chi_1 \end{matrix} \qquad \begin{matrix} d_{21} \\ \vdots \\ \Gamma, \Gamma_2 \Rightarrow \chi_2 \end{matrix}}{\Gamma, \Gamma, \Gamma_1, \Gamma_2 \Rightarrow \chi_1 \cdot \chi_2}\,(\Rightarrow\cdot)}{\Gamma, \Gamma_1, \Gamma_2 \Rightarrow \chi_1 \cdot \chi_2}\,(\textsc{c})$$

and $\chi(\bar{w}) \coloneqq \chi_1(\bar{w}) \cdot \chi_2(\bar{w})$ together with the above derivations are the desired formula and derivations, respectively.

Chapter 4

Concluding Remarks

In this thesis, we studied the one-variable fragments of a family of first-order logics based on classes of $\mathcal{L}$-lattices $\mathcal{K}$ and addressed the challenge of providing a (natural) axiomatization of the equational consequence relation $\vDash_{\mathcal{K}}^{\forall 1}$, or, equivalently, in algebraic terms, providing a (natural) axiomatization of the generalized quasivariety generated by the class of all $\langle \mathbf{A}, W \rangle$-functional m-$\mathcal{L}$-lattices, where $\mathbf{A} \in \mathcal{K}$ and W is a set.

In Chapter 2 we addressed this challenge algebraically for certain classes of $\mathcal{L}$-lattices. We defined m-$\mathcal{L}$-lattices to be $\mathcal{L}$-lattices expanded with $\Box$ and $\Diamond$ satisfying certain "$\mathcal{S}5$-like" equations, and for any class $\mathcal{K}$ of $\mathcal{L}$-lattices, we let $m\mathcal{K}$ denote the class of m-$\mathcal{L}$-lattices with an $\mathcal{L}$-lattice reduct in $\mathcal{K}$. We saw that a number of one-variable fragments of first-order logics defined over classes of $\mathcal{L}$-lattices that have already been axiomatized in the literature, correspond to some variety $m\mathcal{V}$ of m-$\mathcal{L}$-lattices. We proved a correspondence theorem between m-$\mathcal{L}$-lattices and pairs consisting of $\mathcal{L}$-lattices and their relatively complete subalgebras. We showed that whenever $\mathcal{K}$ is closed under taking subalgebras and direct powers, for $\mathbf{A} \in \mathcal{K}$ and a set W, any $\langle \mathbf{A}, W \rangle$-functional m-$\mathcal{L}$-lattice belongs to $m\mathcal{K}$. We identified the semantics of one-variable first-order logics with evaluations into functional m-$\mathcal{L}$-lattices, which allowed us to prove that if $\mathcal{K}$ is closed under taking subalgebras and direct powers, then consequence in $m\mathcal{K}$ implies consequence in the one-variable first-order logic defined over $\mathcal{K}$. We proved a functional completeness theorem, that is, if $\mathcal{K}$ is closed under taking direct limits and subalgebras, and has the superamalgamation property, then any member of $m\mathcal{K}$ is functional. With this functional completeness theorem, we obtained the correspondence of consequence in $m\mathcal{K}$ and consequence in the one-variable first-order logic based on $\mathcal{K}$ in certain cases, specifically, we proved that if $\mathcal{K}$ is a variety that has the superamalgamation property, then for any set $\Sigma \cup \{\varphi \approx \psi\}$ of $\mathrm{Fm}_{\forall}^{1}(\mathcal{L})$-equations (Corollary 2.4.2),

$$\Sigma \vDash_{\mathcal{K}}^{\forall 1} \varphi \approx \psi \quad \iff \quad \Sigma^* \vDash_{m\mathcal{K}} \varphi^* \approx \psi^*.$$

However, if $\mathcal{K}$ lacks the superamalgamation property or is not a variety, further axioms may be required.

In Chapter 3 we approached this challenge proof-theoretically and provided an alternative proof of Corollary 2.4.2 for certain substructural logics extending $\mathrm{FL_e}$ that have a cut-free sequent calculus. We introduced the cut-free sequent calculus ∀1CFL that is sound and complete with respect to the one-variable fragment of the first-order logic based on $\mathcal{FL}_e$. Derivations of $\mathrm{Fm}^1_\forall(\mathcal{L}_s)$-sequents in ∀1CFL do not introduce new quantifiers, but they may introduce additional variables. We proved an interpolation property for ∀1CFL showing that these additional variables can be eliminated from such a derivation. We extended this strategy to varieties of $\mathsf{FL_e}$-algebras axiomatized relative to $\mathcal{FL}_e$ by equations of a certain simple form, that correspond to simple structural rules with exactly one premise.

Let us conclude this thesis by mentioning some interesting directions for further research. We have obtained axiomatizations for consequence in the one-variable fragments of first-order logics based on varieties of $\mathcal{L}$-lattices that have the superamalgamation property. Hence, a potential generalization is to consider varieties of $\mathcal{L}$-lattices that have a weaker property, the *super generalized amalgamation property*. For substructural logics (even those without exchange) the super generalized amalgamation property corresponds to the Craig interpolation property [31]. Obtaining a functional completeness theorem for varieties of $\mathcal{L}$-lattices that have this property, would yield axiomatization results for an even larger family of one-variable fragments of first-order logics. In particular, such a result would yield an axiomatization for the one-variable fragment of the first-order version of the full Lambek Calculus FL. However, we conjecture that in these cases, completeness does not hold for consequences but only for valid equations. An alternative generalization is to extend our proof-theoretic approach from Chapter 3 to first-order versions of substructural logics like FL that have a cut-free sequent calculus, although considering the first-order version of FL would require us to lift this proof-theoretic strategy to sequent calculi where sequents are based on sequences of formulas instead of multisets of formulas.

Another interesting way of continuing this investigation of one-variable fragments of first-order logics and the problem of axiomatizing them is to consider classes that consist of the totally ordered members of a variety of $\mathcal{L}$-lattices, and hence forms a positive universal class. An $\mathcal{L}_s$-lattice is called *semilinear*, if it is isomorphic to a subdirect product of totally ordered $\mathsf{FL_e}$-algebras. Then for any variety $\mathcal{V}$ of semilinear $\mathsf{FL_e}$-algebras, we can show that

$$\models^{\forall 1}_{\mathcal{V}} (\exists x)\varphi \cdot (\exists x)\varphi \approx (\exists x)(\varphi \cdot \varphi).$$

In Example 2.1.3 we proved that whenever $\mathbf{Ł}_3 \in \mathcal{V}$, then

$$\not\models_{m\mathcal{V}} \Diamond x \cdot \Diamond x \approx \Diamond(x \cdot x),$$

and in this case $m\mathcal{V}$ does not correspond to the one-variable fragment of the first-order logic based on $\mathcal{V}$. In particular, if $\mathcal{V}$ is $\mathcal{MV}$ or the variety of all semilinear $\mathsf{FL_e}$-algebras, $m\mathcal{V}$ does not correspond to the one-variable fragment of the first-order logic based on $\mathcal{V}$. If we consider $\mathcal{V}_{\mathsf{to}}$, then additionally to

$$\vDash^{\forall 1}_{\mathcal{V}_{\mathsf{to}}} (\exists x)\varphi \cdot (\exists x)\varphi \approx (\exists x)(\varphi \cdot \varphi)$$

we can also show

$$\vDash^{\forall 1}_{\mathcal{V}_{\mathsf{to}}} (\forall x)(\varphi \vee \psi) \approx (\forall x)\varphi \vee \psi$$

where x does not occur in ψ. Although a general approach to obtaining axiomatizations of the one-variable fragments of the first-order logics based on $\mathcal{V}$ and $\mathcal{V}_{\mathsf{to}}$ is lacking, success for certain cases suggests a possible future line of investigation. Let us consider a specific case; the one-variable fragment of first-order Łukasiewicz logic can be defined over $\mathcal{MV}_{\mathsf{to}}$, the class of totally ordered $\mathcal{MV}$-algebras, which corresponds to the variety of monadic $\mathcal{MV}$-algebras [77]. The variety of monadic $\mathcal{MV}$-algebras [77] can be defined relative to $m\mathcal{MV}$ by

$$\Diamond x \cdot \Diamond x \approx \Diamond(x \cdot x) \qquad \text{and} \qquad \Box(\Box x \vee y) \approx \Box x \vee \Box y.$$

Interestingly, a proof that the one-variable fragment of first-order Łukasiewicz logic corresponds to the variety of monadic $\mathcal{MV}$-algebras is given in [17] using the fact that $\mathcal{MV}_{\mathsf{to}}$ has the amalgamation property (see also [59, 89] for related results). This suggests that the method of Chapter 2 might be adapted to one-variable fragments of first-order logics defined over classes of totally ordered algebras that have the amalgamation property.

Bibliography

[1] A. R. Anderson and N. D. Belnap, *Entailment*, Vol. 1, Princeton University Press, 1975.

[2] M. Baaz and G. Metcalfe, *Proof theory for first order Łukasiewicz logic*, Proceedings of tableaux 2007, 2007, pp. 24–28.

[3] M. Baaz and N. Preining, *Gödel-Dummett Logics*, Handbook of mathematical fuzzy logic, 2011, pp. 585–625.

[4] M. Baaz, N. Preining, and R. Zach, *First-order Gödel logics*, Ann. Pure Appl. Logic **147** (2007), 23–47.

[5] L. P. Belluce, *Further results on infinite valued predicate logic*, J. Symb. Log. **29** (1964), 69–78.

[6] L. P. Belluce and C. C. Chang, *A weak completeness theorem for infinite valued first order logic*, J. Symb. Log. **28** (1963), 43–50.

[7] G. Bezhanishvili, *Varieties of monadic Heyting algebras - part I*, Studia Logica **61** (1998), no. 3, 367–402.

[8] G. Bezhanishvili and J. Harding, *Functional monadic Heyting algebras*, Algebra Universalis **48** (2002), 1–10.

[9] G. Bezhanishvili and W. H. Holliday, *A semantic hierarchy for intuitionistic logic*, Indagationes Mathematicae **30** (2019), no. 3, 403–469.

[10] G. Birkhoff, *On the Structure of Abstract Algebras*, Mathematical Proceedings of the Camridge Philosophical Society **31** (1935), no. 4, 433–454.

[11] R.A. Bull, *MIPC as formalisation of an intuitionist concept of modality*, J. Symb. Log. **31** (1966), 609–616.

[12] S. Burris and H. P. Sankappanavar, *A Course in Universal Algebra*, Springer, 1981.

[13] X. Caicedo, G. Metcalfe, R. Rodríguez, and J. Rogger, *Decidability in order-based modal logics*, J. Comput. System Sci. **88** (2017), 53–74.

[14] X. Caicedo, G. Metcalfe, R. Rodríguez, and O. Tuyt, *One-variable fragments of intermediate logics over linear frames.*, Inform. and Comput. **287** (2022).

[15] X. Caicedo and R. Rodríguez, *Bi-modal Gödel logic over* $[0,1]$*-valued Kripke frames*, J. Logic Comput. **25** (2015), no. 1, 37–55.

[16] E. Casari, *Comparative logics and abelian* ℓ*-groups*, Logic colloquium '88, 1989, pp. 161–190.

[17] D. Castaño, C. Cimadamore, J.P.D. Varela, and L. Rueda, *Completeness for monadic fuzzy logics via functional algebras*, Fuzzy Sets and Systems **407** (2021), 161–174.

[18] A. Chagrov and M. Zakharyaschev, *Modal logic*, Oxford Univ. Press, 1996.

[19] C. C. Chang, *Algebraic analysis of many-valued logics*, Transactions of the American Mathematical Society **88** (1958), 467–490.

[20] A. Church, *A note on the Entscheidungsproblem*, J. Symb. Log. **1** (1936), 40–41.

[21] K. Chvalovský and R. Horčík, *Full lambek calculus with contraction is undecidable*, The Journal of Symbolic Logic **81** (2016), no. 2, 524–540.

[22] A. Ciabattoni and K. Terui, *Towards a semantic characterization of cut-elimination.*, Studia Logica **82** (2006), no. 1, 95–119.

[23] R. Cignoli, I. M. L. D'Ottaviano, and D. Mundici, *Algebraic foundations of many-valued reasoning*, Trends in Logic, vol. 7, Kluwer, Dordrecht, 1999.

[24] P. Cintula, G. Metcalfe, and N. Tokuda, *Algebraic semantics for one-variable lattice-valued logics*, Proceedings of AiML 2022, 2022, pp. 237–257.

[25] ______, *One-variable fragments of first-order logics*, The Bulletin of Symbolic Logic (2024), 1–23.

[26] P. Cintula and C. Noguera, *Logic and Implication: An Introduction to the General Algebraic Study of Non-classical Logics*, Trends in Logic, vol. 57, Springer, 2021.

[27] G. Corsi, *Completeness theorem for Dummett's* LC *quantified*, Studia Logica **51** (1992), 317–335.

[28] A. di Nola and R. Grigolia, *On monadic* MV*-algebras*, Ann. Pure Appl. Logic **128** (2004), no. 1-3, 125–139.

[29] K. Došen, *Modal translations in substructural logics*, Journal of Philosophical Logic **21** (1992), no. 3, 283–336.

[30] M. Dummett, *A propositional calculus with denumerable matrix*, J. Symb. Log. **24** (1959), 97–106.

[31] N. Galatos, P. Jipsen, T. Kowalski, and H. Ono, *Residuated lattices: An algebraic glimpse at substructural logics*, Elsevier, 2007.

[32] G. Gentzen, *Untersuchungen über das Logische Schliessen*, Math. Zeitschrift **39** (1935), 176,–210,405–431.

[33] J. Girard, *Linear logic*, Theoret. Comput. Sci. **50** (1987), 1–101.

[34] K. Gödel, *Die Vollständigkeit der Axiome des logischen Funktionenkalküls*, Monatshefte für Mathematik und Physik **37** (1930), 349–360.

[35] K. Gödel, *Zum intuitionistischen Aussagenkalkül*, Anzeiger der Akademie der Wissenschaften in Wien **69** (1932), 65–66.

[36] G. Grätzer, *General lattice theory*, 2nd ed., Birkhäuser, 1998.

[37] P. Hájek, *Metamathematics of fuzzy logic*, Kluwer, Dordrecht, 1998.

[38] P.R. Halmos, *Algebraic logic, I. Monadic Boolean algebras*, Compos. Math. **12** (1955), 217–249.

[39] L. S. Hay, *Axiomatization of the infinite-valued predicate calculus*, J. Symb. Log. **28** (1963), no. 1, 77–86.

[40] A. Heyting, *Die formalen Regeln der intuitionistischen Logik*, Sitzungsbericht PreuBische Akademie der Wissenschaften Berlin, physikalisch-mathematische Klasse II (1930), 42–56.

[41] D. Hilbert and W. Ackermann, *Grundzüge der theoretischen Logik*, Springer, Berlin, 1928.

[42] A. Horn, *Logic with truth values in a linearly ordered Heyting algebra*, J. Symb. Log. **34** (1969), no. 3, 395–409.

[43] R. Iemhoff, *Consequence relations and admissible rules*, Journal of Philosophical Logic **45** (2016), no. 3, 327–348.

[44] E. Kiriyama and H. Ono, *The contraction rule and decision problems for logics without structural rules*, Studia Logica **50** (1991), 299–319.

[45] S. C. Kleene, *Introduction to metamathematics*, P. Noordhoff N.V., Groningen, 1952.

[46] Y. Komori, *Predicate logics without the structural rules*, Studia Logica **45** (1986), no. 4, 393–104.

[47] R. Kontchakov, A. Kurucz, and M. Zakharyaschev, *Undecidability of first-order intuitionistic and modal logics with two variables*, Bulletin of Symbolic Logic **11** (2005), no. 3, 428–438.

[48] S.A. Kripke, *Semantical analysis of intuitionistic logic I*, Formal systems and recursive functions, 1965, pp. 92–130.

[49] R. E. Ladner, *The computational complexity of provability in systems of modal propositional logic*, SIAM J. Comput. **6** (1977), no. 3, 467–480.

[50] J. Lambek, *The mathematics of sentence structure*, American Mathematical Monthly **65** (1958), 154–170.

[51] J. Łukasiewicz, *O logice tròjwartościowej*, Ruch Filozoficzny **5** (1920), 169–171. Reprinted and translated in [52].

[52] ———, *Selected works*, North-Holland Publishing Company, 1970. Edited by L. Borowski.

[53] J. Łukasiewicz and A. Tarski, *Untersuchungen über den Aussagenkalkül*, Comptes Rendus des Séances de la Societé des Sciences et des Lettres de Varsovie, Classe III **23** (1930). Reprinted and translated in [52].

[54] Judit X. Madarász, *Interpolation and amalgamation; pushing the limits. part i*, Studia Logica: An International Journal for Symbolic Logic **61** (1998), no. 3, 311–345.

[55] L.L. Maksimova, *Craig's theorem in superintuitionistic logics and amalgamable varieties of pseudo-Boolean algebras*, Algebra i Logika **16** (1977), 643–681.

[56] ———, *Modal logics and varieties of modal algebras: The Beth properties, interpolation, and amalgamation*, Algebra i Logika **31** (1992), no. 2, 145–166.

[57] G. Metcalfe, N. Olivetti, and D. Gabbay, *Proof theory for fuzzy logics*, Applied Logic, vol. 36, Springer, 2008.

[58] G. Metcalfe, F. Paoli, and C. Tsinakis, *Residuated structures in algebra and logic*, Mathematical Surveys and Monographs, vol. 277, American Mathematical Society, 2023.

[59] G. Metcalfe and O. Tuyt, *A monadic logic of ordered abelian groups*, Proceedings of AiML 2020, 2020, pp. 441–457.

[60] R.K. Meyer and J.K. Slaney, *Abelian logic from A to Z*, Paraconsistent logic: Essays on the inconsistent, 1989, pp. 245–288.

[61] A. Monteiro and O. Varsavsky, *Algebras de Heyting monádicas*, Actas de las X Jornadas de la Unión Matemática Argentina, Bahía Blanca (1957), 52–62.

[62] T. Moraschini, *A gentle introduction to the leibniz hierarchy*, Janusz czelakowski on logical consequence, 2024, pp. 123–201.

[63] A. Di Nola and I. Leuştean, *Łukasiewicz Logic and MV-Algebras*, Handbook of mathematical fuzzy logic, 2011, pp. 469–583.

[64] H. Ono, *On some intuitionistic modal logics*, Publ. RIMS, Kyoto Univ. **13** (1977), 687–722.

[65] ______, *Model extension theorem and craig's interpolation theorem for intermediate predicate logics*, Reports on Mathematical Logic **15** (1983), 41–58.

[66] ______, *Semantical analysis of predicate logics without the contraction rule*, Studia Logica **44** (1985), no. 2, 187–196.

[67] ______, *Some problems in intermediate predicate logics*, Reports on Mathematical Logic **21** (1987), 55–67.

[68] ______, *Structural rules and a logical hierarchy*, Mathematical logic, 1990, pp. 95–104.

[69] ______, *Decidability and finite model property of substructural logics* (199512).

[70] ______, *Proof-theoretic methods in nonclassical logic – an introduction*, Theories of types and proofs, 1998, pp. 207–254.

[71] H. Ono and Y. Komori, *Logic without the contraction rule*, J. Symb. Log. **50** (1985), 169–201.

[72] H. Ono and N.-Y. Suzuki, *Relations between intuitionistic modal logics and intermediate predicate logics*, Rep. Math. Logic **22** (1988), 65–87.

[73] F. Paoli, *Substructural logics: A primer*, Kluwer Academic Publishers, Dordrecht, 2002.

[74] A. Prior, *Time and modality*, Clarendon Press, Oxford, 1957.

[75] W. Rautenberg, *Modal tableau calculi and interpolation*, J. Philos. Log. **12** (1983), no. 4, 403–423.

[76] G. Restall, *An introduction to substructural logics*, Routledge, London, 1999.

[77] J.D. Rutledge, *A preliminary investigation of the infinitely many-valued predicate calculus*, Ph.D. Thesis, 1959.

[78] B. Scarpellini, *Die Nichtaxiomatisierbarkeit des unendlichwertigen Prädikatenkalküls von Łukasiewicz*, J. Symb. Log. **27** (1962), no. 2, 159–170.

[79] D. Scott, *A decision method for validity of sentences in two variables*, Journal of Symbolic Logic **27** (1962), 377.

[80] K. Segerberg, *Two-dimensional modal logic*, Journal of Philosophical Logic **2** (1973), no. 1, 77–96.

[81] J. Surányi, *Zur Reduktion des Entscheidungsproblems des logischen Funktioskalküls*, Mathematikai és Fizikai Lapok **50** (1943), 51–74.

[82] N.-Y. Suzuki, *An algebraic approach to intuitionistic modal logics in connection with intermediate predicate logics*, Studia Logica **48** (1989), 141–155.

[83] ______, *Kripke bundles for intermediate predicate logics and Kripke frames for intuitionistic modal logics*, Studia Logica **49** (1990), no. 3, 289–306.

[84] G. Takeuti, *Proof theory*, Second, Studies in Logic and the Foundations of Mathematics, vol. 81, North Holland, Amsterdam, 1987.

[85] G. Takeuti and T. Titani, *Intuitionistic fuzzy logic and intuitionistic fuzzy set theory*, J. Symb. Log. **49** (1984), no. 3, 851–866.

[86] K. Terui, *Which structural rules admit cut elimination? — an algebraic criterion*, J. Symb. Log. **72** (2007), no. 3, 738–754.

[87] A. S. Troelstra, *Principles of intuitionism. lectures presented at the summer conference on intuitionism and proof theory at suny at buffalo, n.y*, Journal of Symbolic Logic **40** (1975), no. 3, 447–448.

[88] A. S. Troelstra and H. Schwichtenberg, *Basic proof theory*, 2nd ed., Cambridge University Press, 2000.

[89] O. Tuyt, *One-variable fragments of first-order many-valued logics*, Ph.D. Thesis, 2021.

[90] T. Umezawa, *On logics intermediate between intuitionistic and classical predicate logic*, Journal of Symbolic Logic **24** (1959), no. 2, 141–153.

[91] M. Wajsberg, *Ein erweiterter Klassenkalkül*, Monatshefte für Mathematik und Physik **40** (1933), 113–126.