Density Property of Stein Manifolds and Holomorphic Matrix Factorization

Inauguraldissertation
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vorgelegt von

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Der Dekan
Prof. Dr. Jean-Louis Reymond

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Abstract

We present results on the density property of Stein manifolds and on factorization of holomorphic matrices. A Stein manifold with the density property has an infinite dimensional group of holomorphic automorphisms.

We generalize a criterion for the density property. As an application, we find new examples of Stein manifolds with the density property.

We also work with the symplectic density property and the Hamiltonian density property. We establish these properties for the Calogero–Moser space of n particles and describe its group of holomorphic symplectic automorphisms. This gives a new class of Stein manifolds with the symplectic density property besides even dimensional Euclidean spaces.

The real Calogero–Moser space $\mathcal{C}_n^{\mathbb{R}}$ is a noncompact, totally real submanifold of the complex Calogero–Moser space $\mathcal{C}_n$. We prove that every symplectic diffeomorphism of $\mathcal{C}_n^{\mathbb{R}}$ smoothly isotopic to the identity can be approximated in the fine Whitney topology – the strongest in this context – by holomorphic symplectic automorphisms of $\mathcal{C}_n$ that preserve $\mathcal{C}_n^{\mathbb{R}}$. A key ingredient in our proof is a refined version of the symplectic density property of $\mathcal{C}_n$.

In holomorphic matrix factorization we factor a matrix, which has holomorphic functions on a Stein space as entries, into a product of specific matrices, e.g. unitriangular matrices or exponentials. In the final part we analyze bounds for the number of factors in some holomorphic matrix factorizations.

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Contents

Notations

$\mathrm{Aut}(X)$	The group of holomorphic automorphisms of X
$\mathrm{Aut}_\omega(X)$	The group of holomorphic symplectic automorphisms of (X, ω)
$\mathrm{Aut}_{\mathrm{alg}}(X)$	The group of algebraic automorphisms of X
$\mathbb{C}[X]$	The algebra of regular functions on X
$\mathrm{IVF}(X)$	The set of $\mathbb{C}$-complete holomorphic vector fields on X
$\mathrm{GL}_n(\mathbb{C})$	The general linear group over $\mathbb{C}$
$\mathrm{Lie}(X)$	The Lie subalgebra of $\mathrm{VF}(X)$ generated by the set of $\mathbb{C}$-complete holomorphic vector fields on X
$\mathrm{Lie}_{\mathrm{alg}}(X)$	The Lie subalgebra of $\mathrm{VF}_{\mathrm{alg}}(X)$ generated by the set of $\mathbb{C}$-complete algebraic vector fields on X
$\mathcal{J}^k(X)$	The kth jet space of X
$\mathcal{O}_{X,x}$	The ring of germs of holomorphic functions at a point $x \in X$
$\mathrm{VF}(X)$	The Lie algebra of all holomorphic vector fields on X
$\mathrm{VF}_{\mathrm{alg}}(X)$	The Lie algebra of all algebraic vector fields on X
$\mathrm{M}_n(\mathbb{C})$	The complex vector space of square matrices of size n
$\mathbb{N}$	The natural numbers, $1, 2, \ldots$.
$\mathrm{PGL}_n(\mathbb{C})$	The projective linear group over $\mathbb{C}$
$\mathrm{SL}_n(\mathbb{C})$	The special linear group over $\mathbb{C}$
$\mathrm{Sp}_{2n}(\mathbb{C})$	The symplectic linear group over $\mathbb{C}$
X	A Stein manifold or a smooth complex affine algebraic variety

Chapter 1

Introduction

A Stein manifold X (which is a complex manifold admitting a lot of holomorphic functions, cf. Definition 2.1) is said to have the *density property*, if the Lie subalgebra generated by $\mathbb{C}$-complete (cf. Definition 2.14) holomorphic vector fields on X is dense in the Lie algebra of all holomorphic vector fields on X, with respect to the compact-open topology. For approximation of local biholomorphic mappings by global holomorphic automorphisms, the density property plays a role comparable to cutoff functions for smooth objects. In particular, it says that the holomorphic automorphism group is very large (cf. Proposition 2.29) and provides great flexibility in constructing holomorphic automorphisms (cf. Chapter 5). We give a historical overview for the density property in Section 2.3.

A criterion for the density property
In 2008, Kaliman and Kutzschebauch [KK08a] worked out a strategy to find a sufficient criterion whether a given Stein manifold X has the density property. The idea is based on finding sufficiently many so-called *compatible pairs* of $\mathbb{C}$-complete holomorphic vector fields. They also established the algebraic density property for Danielewski surfaces [KK08b] despite the lack of compatible pairs in this particular case.

The first topic of this thesis, presented in Chapter 3, is a generalization of the notion of the compatible pair and the Kaliman–Kutzschebauch criterion. Let $\theta_1, \ldots, \theta_n$ ($n \geq 2$) be $\mathbb{C}$-complete holomorphic vector fields on X. Then the ordered n-tuple $(\theta_1, \ldots, \theta_n)$ is called a *compatible n-tuple* (of holomorphic vector fields on X) if the following two conditions are satisfied:

(1) There exists a non-zero ideal $I \subset \mathcal{O}(X)$ in the ring of holomorphic functions on X such that

$$I \subset \overline{\operatorname{span}\left(\prod_{i=1}^{n} \ker \theta_i\right)}$$

(2) There is a graph (G, π, ε), where

(i) G is a rooted directed tree with orientation towards the root

(ii) $\pi\colon \operatorname{Vert}(G) \to \{\theta_1, \ldots, \theta_n\}$ is a bijection with $\pi(\text{root}) = \theta_1$

(iii) $\varepsilon\colon \mathrm{Edge}(G) \to \mathcal{O}(X)$ is a mapping with

$$\varepsilon(v,w) \in \left(\ker \pi(v)^2 \setminus \ker \pi(v)\right) \cap \ker \pi(w)$$

for all $(v,w) \in \mathrm{Edge}(G)$.

Note that this definition actually generalizes the notion of a compatible pair, since the latter corresponds to the special case $n = 2$.

The above-mentioned Danielewski surfaces do not admit a compatible pair, but they admit a compatible 3-tuple according to our new definition. As we will see below, this enables us to give a conciser proof of the density property for Danielewski surfaces. Moreover, we have discovered another family of manifolds, which we call Gromov–Vaserstein fibers, for which we do not know whether there exist compatible pairs, but which turns out to have the density property thanks to the existence of a compatible 3-tuple.

Let $\mathrm{Aut}(X)$ denote the group of holomorphic automorphisms on X. Then the first main result can be formulated as follows.

Theorem 1 (Generalized Kaliman–Kutzschebauch criterion) Let X be a homogeneous Stein manifold with respect to $\mathrm{Aut}(X)$. Assume there exist $N = \dim(X)$ compatible n-tuples $\{(\theta_{1,i},\ldots,\theta_{n,i})\}_{i=1}^{N}$ of holomorphic vector fields on X such that the vectors $(\theta_{1,1})_x,\ldots,(\theta_{1,N})_x$ span the tangent space T_xX at some point $x \in X$. Then X has the density property.

In practice it could be difficult to find $\dim(X)$ compatible n-tuples. The next result transforms this condition into another form for an application of Theorem 1.

Theorem 2 Let X be a Stein manifold and $(\theta_1,\ldots,\theta_n)$ a compatible n-tuple of holomorphic vector fields on X. Assume that there exist $\mathbb{C}$-complete holomorphic vector fields $V_1,\ldots,V_N$ on X, $N \geq \dim(X)$, which span the tangent bundle $\mathrm{T}X$. If there exist functions $f_i \in \ker V_i$ such that $\mathrm{d}_x f_i(\theta_1) \neq 0$ for some point $x \in X$ and $i = 1,\ldots,N$, then X has the density property.

The reason for us to develop this new criterion is to prove the density property for a certain class of affine algebraic manifolds, naturally arising in the solution to the factorization problems of holomorphic matrices into elementary factors, namely the solutions to the Gromov–Vaserstein problems for the special linear groups by Ivarsson and Kutzschebauch [IK12a] and for the symplectic groups by Schott [Sch25]. These manifolds are the fibers of certain polynomial mappings $P\colon \mathbb{C}^m \to \mathbb{C}^k$, we call them Gromov–Vaserstein fibers. They share the property that each smooth fiber $P^{-1}(y)$ is biholomorphic to a product $\mathcal{G} \times \mathbb{C}^M$ for some holomorphically flexible Stein manifold $\mathcal{G}$ and some natural number M. In [IK12a] and [Sch25] it is shown that $\mathcal{G}$ is elliptic in the sense of Gromov and thus an Oka manifold. For the notion of Oka manifolds see [For17, Chapter 5]. Our main goal is to show that $\mathcal{G}$ has the density property which is much stronger than being Oka. We give an introduction with more details on Gromov–Vaserstein problem in Section 2.2.

Note that it is much easier to prove that the Gromov–Vaserstein fibers $\mathcal{G} \times \mathbb{C}^M$ have the density property for $M \geq 1$ than it is for $M = 0$. In fact, an application of the original Kaliman–Kutzschebauch criterion shows that products of holomorphically flexible manifolds with affine spaces have the density property. Since this result is known in the literature only in a very restricted form (Varolin [Var01] proved that $G \times \mathbb{C}$ has the density property when G is a Stein complex Lie group) we include it here.

Recall that a complex manifold X is called *holomorphically flexible* at a point $x \in X$ if the values at x of $\mathbb{C}$-complete holomorphic vector fields on X span the tangent space $T_x X$. The manifold X is *flexible* if it is flexible at every point $x \in X$.

Theorem 3 Let X be a holomorphically flexible Stein manifold. Then $X \times \mathbb{C}$ has the density property.

The reader should compare this theorem to the result of Ugolini and Winkelmann [UW23], where they prove that the total space of a line bundle $\pi \colon E \to X$ over a Stein manifold X with the density property has again the density property, if there exists a so-called π-incompatible holomorphic automorphism of E.

As mentioned above, the manifolds $\mathcal{G}$ from the Gromov–Vaserstein fiber $\mathcal{G} \times \mathbb{C}^M$ are our new examples of manifolds with the density property. We shall encounter an interesting special case, where $\mathcal{G}$ is given by

$$\mathcal{G} = \left\{ (z_2, z_3, w_1, w_2, w_3) \in \mathbb{C}^5 : \begin{pmatrix} w_1 & w_2 \\ w_2 & w_3 \end{pmatrix} \begin{pmatrix} z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right\}$$

with $(b_1, b_2) \in \mathbb{C}^2 \setminus \{(0,0)\}$. We do not know whether there exist compatible pairs on $\mathcal{G}$. However, there is a compatible 3-tuple and, moreover, there are $\mathbb{C}$-complete holomorphic vector fields V_1, V_2 and V_3 on $\mathcal{G}$ which satisfy the conditions of Theorem 2. Hence $\mathcal{G}$ has the density property.

The symplectic density property

The main subject of Chapter 4 and Chapter 5 is the classical *Calogero–Moser space*, which has an origin in mathematical physics as an integrable dynamical system. Consider the phase space of n identical point-like particles on the real line, pairwise interacting with a repulsive, inverse square potential. This potential prevents any two particles having the same location. On the way to understand the connection between the Calogero–Moser system and the Kadomtsev–Petviashvili system, Wilson [Wil98] considered this n-particle problem on a complex line, where collisions will take place for most initial conditions after finite time. Following an earlier geometric construction by Kazhdan, Kostant and Sternberg [KKS78], Wilson introduced the following complex space which contains a dense subset where the particles have distinct locations (cf. Lemma 4.1).

Definition 1.1 Let $\widehat{\mathcal{C}}_n$ be the subvariety of $\mathrm{M}_n(\mathbb{C}) \times \mathrm{M}_n(\mathbb{C})$ given by

$$\operatorname{rank}([X, Y] + \mathrm{id}) = 1$$

where $(X, Y) \in \mathrm{M}_n(\mathbb{C}) \times \mathrm{M}_n(\mathbb{C})$. The group $\mathrm{GL}_n(\mathbb{C})$ acts on $\widehat{\mathcal{C}}_n$ by simultaneous conjugation in both factors:

$$g \cdot (X, Y) = (gXg^{-1}, gYg^{-1})$$

for $g \in \mathrm{GL}_n(\mathbb{C})$. We define the *Calogero–Moser space* $\mathcal{C}_n$ of n particles to be the GIT-quotient $\widehat{\mathcal{C}}_n // \mathrm{GL}_n(\mathbb{C})$.

It is easy to see that the group action is well defined, since it leaves the condition $\mathrm{rank}([X,Y] + \mathrm{id}) = 1$ invariant.

For $n = 1$ the matrices are just complex numbers, and we obtain $\mathcal{C}_1 = \mathbb{C}^2$. In higher dimensions, the structure becomes more complicated. The Calogero–Moser space $\mathcal{C}_n$ is a smooth complex-affine variety of dimension $2n$, see Wilson [Wil98, Section 1]. It has also been shown by Popov [Pop14, Section 13, Remark 5] that $\mathcal{C}_n$ is a rational variety.

The Calogero–Moser space is equipped with a holomorphic symplectic form and in fact carries a hyperkähler structure. Indeed, the Calogero–Moser space with its symplectic form is obtained as symplectic reduction of the preimage of a coadjoint orbit under a moment map, see Section 4.1 for more details.

We will make use of the following well-known complete flow maps.

Definition 1.2 The *Calogero–Moser flows* on $\mathcal{C}_n$ are defined as follows:

$$(X,Y) \mapsto (X, Y + tX^k)$$
$$(X,Y) \mapsto (X + tY^k, Y)$$

where $k \in \mathbb{N}_0$ and $t \in \mathbb{C}$.

It is easy to see that these are well-defined maps on $\mathcal{C}_n$ since they are invariant under conjugation and leave the commutator $[X,Y]$ invariant, and that they are algebraic isomorphisms for each $t \in \mathbb{C}$. Moreover, they leave the symplectic form invariant, see Section 4.1.

It has been established by Berest and Wilson [BW00] that the subgroup of the automorphism group generated by the Calogero–Moser flows acts transitively on the Calogero–Moser space $\mathcal{C}_n$. This was later improved to 2-transitivity by Berest, Eshmatov and Eshmatov [BEE16] and finally to m-transitivity for any $m \in \mathbb{N}$ (sometimes called *infinite transitivity*) by Kuyumzhiyan [Kuy20]. Therefore it is natural to consider properties of the holomorphic symplectic automorphism group of $\mathcal{C}_n$.

Definition 1.3 A complex manifold endowed with a holomorphic symplectic form has *the symplectic density property* if the Lie algebra generated by $\mathbb{C}$-complete holomorphic symplectic vector fields is dense in the Lie algebra of holomorphic symplectic vector fields in the compact-open topology.

For the Hamiltonian density property, replace ”symplectic” by ”Hamiltonian” in Definition 1.3 or see Section 2.4.

The density property for the Calogero–Moser spaces has been established by Andrist [And21]. However, for the symplectic or Hamiltonian density property, none of the known methods from Andersén–Lempert theory can be applied due to the lack of a module structure over the holomorphic functions for the symplectic/Hamiltonian vector fields: A powerful method for establishing the density property is the use of compatible pairs [KK08a] or compatible tuples [And+23] for which the module structure of the holomorphic vector fields over the holomorphic functions is essential.

The main goal of Chapter 4 is to establish the symplectic and the Hamiltonian density properties for the Calogero–Moser spaces, which is an entirely new class of examples with a large group of holomorphic symplectic automorphisms.

Theorem 1.4 *The Calogero–Moser space $\mathcal{C}_n$, $n \in \mathbb{N}$, has the Hamiltonian density property.*

In fact, we will show that the Hamiltonian density property can be established using only two complete holomorphic Hamiltonian vector fields that correspond to the Hamiltonian functions $\operatorname{tr} Y + \operatorname{tr} X^3$ and $(\operatorname{tr} X)^2 + \operatorname{tr} Y^2$. While establishing the Hamiltonian density property already turns out to be a highly non-trivial endeavor, it is even more surprising that it can be achieved with only finitely many generators. For the density property, this has been established before by Andrist for the case of $\mathbb{C}^n$ [And19] and for $\mathrm{SL}_2(\mathbb{C})$ as well as $xy = z^2$ [And23] with finitely many complete polynomial vector fields generating the Lie algebra of all polynomial vector fields.

Since every holomorphic symplectic vector field on $\mathcal{C}_n$ is Hamiltonian, see Lemma 4.3, the preceding theorem implies the following.

Theorem 1.5 *For any $n \in \mathbb{N}$ the Calogero–Moser space $\mathcal{C}_n$ has the symplectic density property.*

Using the symplectic version of Andersén–Lempert theory in Section 2.4, we can then establish the following description of the automorphism group.

Theorem 1.6 *The identity component of the group of holomorphic symplectic automorphisms of $\mathcal{C}_n$, $n \in \mathbb{N}$, is the closure (in the topology of uniform convergence on compacts) of the group generated by the following algebraic symplectic automorphisms:*

$$(X, Y) \mapsto (X + t\,\mathrm{id}, Y - 3X^2 t - 3Xt^2 - \mathrm{id}\, t^3)$$
$$(X, Y) \mapsto \big(X + 2Yt + f(t)\,\mathrm{id}, Y + f'(t)\,\mathrm{id}\big)$$

where $t \in \mathbb{C}$ and

$$f(t) = \frac{1}{n}\big(\cos(2\sqrt{n}t) - 1\big)\operatorname{tr} X + \frac{1}{n^{3/2}}(\sin(2\sqrt{n}t) - 2t\sqrt{n})\operatorname{tr} Y.$$

Remark 1.7 Alternatively, we can also use three algebraic families to generate a group whose closure is the identity component of the group of holomorphic symplectic automorphisms of $\mathcal{C}_n$:

$$(X, Y) \mapsto (X + t\,\mathrm{id}, Y - 3X^2 t - 3Xt^2 - \mathrm{id}\, t^3)$$
$$(X, Y) \mapsto (X + 2tY, Y)$$
$$(X, Y) \mapsto (X, Y - 2t\operatorname{tr}(X)\,\mathrm{id}).$$

Actually, it is also possible to use the following four simple families that are linear in time t:

$$\begin{aligned}
(X,Y) &\mapsto (X + t\,\mathrm{id}, Y) \\
(X,Y) &\mapsto (X, Y - 3X^2 t) \\
(X,Y) &\mapsto (X + 2tY, Y) \\
(X,Y) &\mapsto (X, Y - 2t\,\mathrm{tr}(X)\,\mathrm{id}).
\end{aligned}$$

Whether $\mathrm{Aut}_\omega(\mathcal{C}_n)$ has more than one connected component is still an open question.

Corollary 1.8 *The group generated by the holomorphic symplectic automorphisms in the preceding theorem acts m-transitively on $\mathcal{C}_n$ for any $m \in \mathbb{N}$.*

Proof This follows from the symplectic density property and Corollary 2.47. □

Holomorphic approximation of symplectic diffeomorphisms

In Chapter 5 we study approximation for symplectic diffeomorphisms from the real Calogero–Moser space $\mathcal{C}_n^{\mathbb{R}}$ onto itself by holomorphic symplectic automorphisms on the complex Calogero–Moser space $\mathcal{C}_n$, $n \in \mathbb{N}$.

The real Calogero–Moser space $\mathcal{C}_n^{\mathbb{R}}$ was constructed by Kazhdan, Kostant and Sternberg [KKS78] as a real symplectic reduction (cf. Definition 5.4) and two decades later the complex Calogero–Moser space $\mathcal{C}_n$ was introduced by Wilson [Wil98] in the complex setting as follows.

Let $\mathcal{M}$ be the direct sum $\mathrm{M}_n(\mathbb{C}) \oplus \mathrm{M}_n(\mathbb{C}) \oplus \mathbb{C}^n \oplus (\mathbb{C}^n)^*$, where $\mathrm{M}_n(\mathbb{C})$ is the $\mathbb{C}$-vector space of square matrices of size n. The space $\mathcal{M}$ can be endowed with the holomorphic symplectic form $\omega = \mathrm{tr}(dX \wedge dY + dv \wedge dw)$. Moreover, the general linear group $\mathrm{GL}_n(\mathbb{C})$ acts on $\mathcal{M}$ as

$$g \cdot (X, Y, v, w) = (gXg^{-1}, gYg^{-1}, gv, wg^{-1}),\ g \in \mathrm{GL}_n(\mathbb{C})$$

which preserves the symplectic form. Thus this action induces a complex moment map

$$\mu\colon \mathcal{M} \to \mathrm{M}_n(\mathbb{C}),\ (X, Y, v, w) \mapsto [X, Y] + vw$$

which is equivariant with respect to the above action on $\mathcal{M}$ and the coadjoint action on $\mathrm{M}_n(\mathbb{C}) \cong \mathfrak{gl}_n^*$. Take the coadjoint invariant point $iI_n \in \mathrm{M}_n(\mathbb{C})$. By Wilson [Wil98], the group action is free on the preimage $\mu^{-1}(iI_n)$.

Definition 1.9 The *Calogero–Moser space* $\mathcal{C}_n$ is the complex symplectic reduction $\mu^{-1}(iI_n) / \mathrm{GL}_n(\mathbb{C})$.

Remark 1.10 For Carleman approximation, we use a different rank condition from Definition 1.1. The subvarieties of $\mathrm{M}_n(\mathbb{C}) \oplus \mathrm{M}_n(\mathbb{C})$ defined by these two rank conditions are isomorphic to each other, and so are their quotients. For the sake of simplicity, we abuse the same notation for these Calogero–Moser spaces. Only as explicit comparisons are called for (later in Chapter 5), more precise notations will be introduced.

Let $\tau\colon \mathcal{C}_n \to \mathcal{C}_n$ be the antiholomorphic involution given by

$$(X, Y, v, w) \mapsto (X^*, Y^*, iw^*, iv^*).$$

Here, the star stands for complex conjugate transpose. Then the fixed-point set of τ turns out to be real symplectomorphic to the real Calogero–Moser space $\mathcal{C}_n^{\mathbb{R}}$. In other words, the complex space $\mathcal{C}_n$ is a symplectic complexification of $\mathcal{C}_n^{\mathbb{R}}$, see Definition 5.1.

Theorem 1.11 *The complex Calogero–Moser space $(\mathcal{C}_n, \omega, \tau)$ is a symplectic complexification of the real Calogero–Moser space $(\mathcal{C}_n^{\mathbb{R}}, \omega_{\mathbb{R}})$.*

The proof can be found in Section 5.5. The real form $\mathcal{C}_n^{\mathbb{R}}$ is simply connected, see Lemma 5.5. Hence we can even approximate any symplectic diffeomorphism from $\mathcal{C}_n^{\mathbb{R}}$ onto itself which is smoothly isotopic to the identity in Theorem 5.20 by a holomorphic symplectic automorphism of $\mathcal{C}_n$.

Our approximation is what complex analysts call the Carleman approximation. It is an approximation of real objects by holomorphic ones in the fine (Whitney) topology. Torsten Carleman was the first to obtain such a result. He proved that smooth functions on the real line can be approximated by holomorphic functions on the complex plane in the fine topology [Car27]. For the following definition we fix a norm $\|\cdot\|_{C^k}$ on the jet-space $\mathcal{J}^k(M)$, cf. Section 5.1.

Definition 1.12 Let $(M_{\mathbb{R}}, \omega_{\mathbb{R}})$ be a smooth symplectic manifold and (M, ω, τ) a symplectic complexification of $(M_{\mathbb{R}}, \omega_{\mathbb{R}})$. For a natural number k, we say that (M, ω, τ) admits *Hamiltonian C^k-Carleman approximation*, if for any Hamiltonian diffeomorphism φ of $M_{\mathbb{R}}$ onto itself (cf. Section 5.1) and any positive continuous function ε on $M_{\mathbb{R}}$, there exists a holomorphic symplectic automorphism Φ of M such that $\Phi(M_{\mathbb{R}}) = M_{\mathbb{R}}$ and the estimate $\|\Phi - \varphi\|_{C^k(p)} < \varepsilon(p)$ holds for any $p \in M_{\mathbb{R}}$.

The main result of Chapter 5 is:

Theorem 1.13 *The Calogero–Moser space $(\mathcal{C}_n, \omega, \tau)$ admits Hamiltonian C^k-Carleman approximation for all $k \in \mathbb{N}_0$.*

For such an approximation to hold, the complex symplectic manifold M must admit a large group of holomorphic symplectic automorphisms. A key ingredient in establishing Carleman approximation for automorphisms is the Andersén–Lempert theory, which concerns the approximation of biholomorphic mappings between Runge domains by holomorphic symplectic automorphisms.

A precise way in Andersén-Lempert theory to express the abundance of holomorphic automorphisms is Varolin's density property [Var01]. We need its symplectic counterpart – the symplectic density property – introduced in Definition 1.3.

Known nontrivial examples of Stein symplectic manifolds satisfying the symplectic density property include even-dimensional complex Euclidean space [For96] and the Calogero–Moser space [AH25]. In both cases, the vanishing of the first de Rham cohomology reduces the property to *the Hamiltonian density property*, where only Hamiltonian vector fields are considered. The Hamiltonian density property was also established for closed coadjoint orbits of complex Lie groups in [DW22]. This property allows approximation of real Hamiltonian vector fields on compact subsets by sums of holomorphic Hamiltonian vector fields, whose real parts are tangent to a given totally real submanifold; see [DW22, Lemma 3.1].

There were only two results on Carleman approximation of smooth diffeomorphisms on noncompact totally real submanifolds. Recall that a C^1 real submanifold N of a complex manifold M is called totally real, if at each point p of N the tangent space T_pN contains no complex line. The first result is by Kutzschebauch and Wold [KW18] who proved Carleman approximation of diffeomorphisms from $\mathbb{R}^s$ onto itself by holomorphic automorphisms of $\mathbb{C}^n$ in the fine topology when $s < n$. Their proof relies on Andersén–Lempert theory and Carleman approximation by entire functions. This result generalized earlier works on approximation by holomorphic automorphisms on compact totally real submanifolds, including Forstnerič–Rosay [FR93], Forstnerič [For94] and Forstnerič–Løw–Øvrelid [FLØ01]. In particular, [FLØ01] studied the approximation of certain diffeomorphisms between compact totally real submanifolds by holomorphic ω-automorphisms of $\mathbb{C}^{2n}$, where ω denotes either the standard holomorphic volume form or the holomorphic symplectic form.

The second result is by Deng and Wold [DW22] who established Hamiltonian Carleman approximation for closed coadjoint orbits of complex Lie groups. Their approach not only yields holomorphic symplectic automorphisms preserving the real part but also addresses the case $\mathbb{R}^{2n}$ embedded in $\mathbb{C}^{2n}$ with the standard symplectic form $\omega = \sum_{j=1}^{n} dz_j \wedge dw_j$. The proofs of both results rely heavily on the density property of $\mathbb{C}^n$ or on the symplectic density property of $\mathbb{C}^{2n}$. Even the result in [DW22] on coadjoint orbits depends on the density property, as it is obtained by restricting the objects from the coadjoint representation to the coadjoint orbits.

Our result is the first one where the ambient complex space is much more complicated than affine space. A crucial component of our proof is the fact that the symplectic density property (see Definition 1.3) of the Calogero–Moser space is proven using complete holomorphic vector fields, whose real time flows preserve the real Calogero–Moser space; see Definition 5.10.

Factorization of holomorphic matrices

In the last chapter, Chapter 6, we deduce some algebraic properties for the group $\mathrm{Sp}_{2n}(\mathcal{O}(X))$ of holomorphic symplectic matrices on a Stein space X: Holomorphic factorization, exponential factorization, and Kazhdan's property (T). In holomorphic factorization we combine a result of Schott [Sch25] and tools from K-theory to give explicit bounds for the case where X is one-dimensional or two-dimensional. Then we use them to find bounds for exponential factorization. As a further application, we show that the elementary symplectic group $\mathrm{Ep}_{2n}(\mathcal{O}(X))$ admits Kazhdan's property (T). We give an introduction to the factorization for null-homotopic holomorphic mapping from a Stein space to $\mathrm{SL}_n(\mathbb{C})$ in Section 2.2.

Chapter 2

Preliminaries

2.1 Stein manifolds

First we set up a sketch of the stage following [For17]. We confine ourselves to the smooth case, although many classical results in this section are valid over Stein spaces.

For a compact set K in a complex manifold X, its $\mathcal{O}(X)$-(convex) hull is given by

$$\widehat{K} = \{p \in X : |f(p)| \leq \max_{x \in K} |f(x)| \text{ for all } f \in \mathcal{O}(X)\}.$$

K is called $\mathcal{O}(X)$-convex if $\widehat{K} = K$. When $X = \mathbb{C}^n$, an $\mathcal{O}(\mathbb{C}^n)$-convex compact K is said to be *polynomially convex*. A complex manifold X is *holomorphically convex*, if for any compact set $K \subset X$ the $\mathcal{O}(X)$-hull $\widehat{K}$ is compact.

Definition 2.1 A complex manifold M is called a *Stein manifold* if

(a) For any two distinct points $p, q \in M$, there is a holomorphic function $f \in \mathcal{O}(M)$ such that $f(p) \neq f(q)$.

(b) For every point $p \in M$ there exist functions $f_1, \ldots, f_n \in \mathcal{O}(M), n = \dim M$, whose differentials $\mathrm{d}f_1, \ldots, \mathrm{d}f_n$ are linearly independent at p.

(c) M is holomorphically convex.

The first two conditions convey that there are a lot of global holomorphic functions on a Stein manifold. Condition (c) implies that a Stein manifold M admits an exhaustion $K_1 \subset K_2 \subset \cdots \subset \cup_{j=1}^{\infty} K_j = M$ by compact $\mathcal{O}(M)$-convex subsets such that $K_j \subset \mathring{K}_{j+1}$ for every $j \in \mathbb{N}$.

Example 2.2 We mention the following classes of Stein manifolds

1. A closed complex submanifold of $\mathbb{C}^N$. In particular, smooth affine algebraic varieties.

2. Open Riemann surfaces.

3. The Cartesian product of two Stein manifolds.

4. The total space of a holomorphic vector bundle over a Stein base.

Runge's approximation theorem in one variable generalizes to the following.

Theorem 2.3 (Oka–Weil) *If X is a Stein manifold and K is a compact $\mathcal{O}(X)$-convex subset of X, then every holomorphic function in an open neighborhood of K can be approximated uniformly on K by functions in $\mathcal{O}(X)$.*

We turn this property into a definition.

Definition 2.4 A domain Ω in a complex manifold X is called *Runge in X* if every holomorphic function $f \in \mathcal{O}(\Omega)$ can be approximated uniformly on compact subsets in Ω by global functions in $\mathcal{O}(X)$.

More precisely, let d be a distance function on X induced by a Riemannian metric. Let $\varepsilon > 0$ and K a compact subset of Ω. For every $f \in \mathcal{O}(\Omega)$ on the Runge domain Ω, there exists a holomorphic function $g \in \mathcal{O}(X)$ such that $d(f(x), g(x)) < \varepsilon$ for every $x \in K$.

Let X be a complex manifold. Denote by $\mathcal{O}_{X,x}$ the ring of germs of holomorphic functions at a point $x \in X$. An *analytic sheaf* on X is a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, namely a sheaf whose stalk $\mathcal{F}_x$ over any point $x \in X$ is a module over the local ring $\mathcal{O}_{X,x}$. The sheaf $\mathcal{F}$ is *locally finitely generated* if for every point $x_0 \in X$ there exist an open neighborhood $U \subset X$ and finitely many sections $s_1, \ldots, s_k \in \mathcal{F}(U) = \Gamma(U, \mathcal{F})$ whose germs at any point $x \in U$ generate $\mathcal{F}_x$ as an $\mathcal{O}_{X,x}$-module. A simple example is the direct sum $\mathcal{O}_X^k$ of k copies of the structure sheaf $\mathcal{O}_X$ for any $k \in \mathbb{N}$, which is the sheaf of holomorphic sections of the trivial bundle $X \times \mathbb{C}^k \to X$.

An analytic sheaf is *coherent* if it is locally finitely generated and if for any set of local sections $s_1, \ldots, s_k \in \mathcal{F}(U)$ the corresponding sheaf of relations $\mathcal{R} = \mathcal{R}(s_1, \ldots, s_k)$ is also locally finitely generated. The sheaf $\mathcal{R}$ has stalks

$$\mathcal{R}_x = \{(g_{1,x}, \ldots, g_{k,x}) \in \mathcal{O}_{X,x}^k : \sum_{j=1}^{k} g_{j,x} s_{j,x} = 0\}, \quad x \in U$$

Thus an analytic sheaf $\mathcal{F}$ over X is coherent if and only if each point $x \in X$ has an open neighborhood $U \subset X$ such that for every analytic sheaf homomorphism

$$\beta \colon \mathcal{O}_U^k \to \mathcal{F}|_U, \quad (g_{1,x}, \ldots, g_{k,x}) \mapsto \sum_j g_{j,x} s_{j,x},$$

there exist $m \in \mathbb{N}$ and a homomorphism $\alpha \colon \mathcal{O}_U^m \to \mathcal{O}_U^k$ such that the following sequence is exact

$$\mathcal{O}_U^m \xrightarrow{\alpha} \mathcal{O}_U^k \xrightarrow{\beta} \mathcal{F}\big|_U \to 0 \tag{2.1}$$

Note that β maps the basis sections e_j of $\mathcal{O}_U^k$ onto the generators s_j of $\mathcal{F}|_U$ and $\ker \beta = \operatorname{Im} \alpha$ is the sheaf of relations.

Example 2.5 We list some examples of coherent analytic sheaves on X:

1. The structure sheaf $\mathcal{O}_X = \cup_{x \in X} \mathcal{O}_{X,x}$, that is the sheaf of germs of holomorphic functions

2. The structure sheaf $\mathcal{O}_A$ of a closed complex subvariety A in X. Recall that $\mathcal{O}_A$ is the quotient sheaf $\mathcal{O}_X/\mathcal{J}_A$, where $\mathcal{J}_A$ is the ideal sheaf $\cup_{x\in X}\mathcal{J}_{A,x}$ of A and $\mathcal{J}_{A,x}$ the ideal in $\mathcal{O}_{X,x}$ consisting of all holomorphic function germs at x vanishing on A.
3. A locally free analytic sheaf i.e. a sheaf of holomorphic sections of a holomorphic vector bundle, for example the tangent sheaf $\mathcal{T}_X$.
4. If $\beta\colon \mathcal{F}\to\mathcal{G}$ is a homomorphism of coherent analytic sheaves, then the kernel $\ker\beta$ and the image $\operatorname{Im}\beta$ are also coherent.

The two fundamental results in coherent sheaf theory are due to Cartan [Car53].

Theorem 2.6 *Let $\mathcal{F}$ be a coherent analytic sheaf on a Stein manifold X. Then*

(A) The stalk $\mathcal{F}_x$ of $\mathcal{F}$ at any point $x\in X$ is generated as an $\mathcal{O}_{X,x}$-module by global sections of the sheaf $\mathcal{F}$.

(B) $H^p(X,\mathcal{F})=0$ for all $p\in\mathbb{N}$.

The corresponding results hold for every coherent algebraic sheaf over a complex affine algebraic variety (Serre [Ser55]).

Corollary 2.7 *Every holomorphic function on a closed complex subvariety of a Stein manifold X extends to a holomorphic function on X.*

Proof Let A be a closed complex subvariety of X. We have the short exact sequence

$$0\longrightarrow\mathcal{J}_A\longrightarrow\mathcal{O}_X\longrightarrow\mathcal{O}_X/\mathcal{J}_A\longrightarrow 0$$

Then the ideal sheaf $\mathcal{J}_A$ is coherent and $H^1(X,\mathcal{J}_A)=0$ by Theorem B. The claim follows from the identification of $\mathcal{O}_A$ with the quotient sheaf $\mathcal{O}_X/\mathcal{J}_A$, and the exact cohomology sequence $\mathcal{O}(X)\to\mathcal{O}(A)\to H^1(X,\mathcal{J}_A)$. □

Corollary 2.8 *If $\mathcal{F}$ is a coherent analytic sheaf on a Stein manifold X and if $s_1,\ldots,s_k\in\mathcal{F}(X)$ generate each stalk $\mathcal{F}_x$ $(x\in X)$, then every section $s\in\mathcal{F}(X)$ is of the form $s=\sum g_is_i$ for some $g_i\in\mathcal{O}(X)$.*

Proof Consider the short exact sequence $0\to\mathcal{R}\to\mathcal{O}^k\xrightarrow{\beta}\mathcal{F}\to 0$ as in (2.1). Then the sheaf of relations $\mathcal{R}=\ker\beta$ is coherent and $H^1(X,\mathcal{R})=0$ by Theorem B. Now the exact cohomology sequence $\mathcal{O}^k(X)\xrightarrow{\beta}\mathcal{F}(X)\to H^1(X,\mathcal{R})$ yields the surjectivity of β. □

Theorem 2.9 (Oka–Weil for coherent analytic sheaves) *Let $\mathcal{F}$ be a coherent analytic sheaf on a Stein manifold X. If K is a compact $\mathcal{O}(X)$-convex set in X, then any section of $\mathcal{F}$ over an open neighborhood of K can be approximated uniformly on K by sections in $\mathcal{F}(X)$. More precisely, if sections $s_1,\ldots,s_m\in\mathcal{F}(X)$ generate every stalk $\mathcal{F}_x, x\in K$, then every section of $\mathcal{F}$ over an open neighborhood of K can be approximated uniformly on K by sections of the form $\sum g_js_j$ for some global functions $g_j\in\mathcal{O}(X)$.*

Proof Let s be a section of $\mathcal{F}$ over a relatively compact open Stein neighborhood $\Omega\subset X$ of K. Since a coherent analytic sheaf is locally finitely generated, there are sections $s_1,\ldots,s_m\in\mathcal{F}(X)$ which generate every stalk $\mathcal{F}_x$ for $x\in\Omega$. By Corollary 2.8 we can write $s=\sum h_is_i$ for some functions $h_i\in\mathcal{O}(\Omega)$. By the $\mathcal{O}(X)$-convexity of K we can approximate h_i by a global function $g_i\in\mathcal{O}(X)$. □

Theorem 2.10 (Parametric Oka–Weil on $\mathbb{C}^n$) *Let K be a compact polynomially convex set in $\mathbb{C}^n$ and let $U \subset \mathbb{C}^n$ be an open set containing K. Assume that P is a compact Hausdorff space and $f\colon P \times U \to \mathbb{C}$ is a continuous function such that $f_p = f(p,\cdot)\colon U \to \mathbb{C}$ is holomorphic for every $p \in P$. Given $\varepsilon > 0$ there exists a continuous function $F\colon P \times \mathbb{C}^n \to \mathbb{C}$ such that $F_p = F(p,\cdot)\colon \mathbb{C}^n \to \mathbb{C}$ is holomorphic for every $p \in P$ and*

$$\sup_{z \in K, p \in P} |F_p(z) - f_p(z)| < \varepsilon.$$

For a proof see [For17, §2.8].

2.1.1 Holomorphic vector fields

Let M be a complex manifold. Let us recall that a real vector field is a section of the real tangent bundle $\mathrm{T}M$, while a complex vector field is a section of the complexified tangent bundle $\mathrm{T}M \otimes_{\mathbb{R}} \mathbb{C}$. Through the almost complex structure J on $\mathrm{T}M$, the complex tangent bundle decomposes into the holomorphic and anti-holomorphic subbundles

$$\mathrm{T}M \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{T}^{1,0}M \oplus \mathrm{T}^{0,1}M.$$

We can identify the real tangent bundle TM and the holomorphic tangent bundle $\mathrm{T}^{1,0}M$ via the $\mathbb{R}$-linear isomorphism (see e.g. [For17, §1.6])

$$\alpha\colon \mathrm{T}M \hookrightarrow \mathrm{T}M \otimes_{\mathbb{R}} \mathbb{C} \to \mathrm{T}^{1,0}M,\ V \mapsto \frac{1}{2}(V - iJV)$$

which has the inverse $\alpha^{-1}(W) = 2\,\mathrm{Re}(W)$ for $W \in \mathrm{T}^{1,0}M$. We have the following commutative diagram

$$\begin{array}{ccc} \mathrm{T}M & \xrightarrow{J} & \mathrm{T}M \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\alpha} \\ \mathrm{T}^{1,0}M & \xrightarrow{i} & \mathrm{T}^{1,0}M \end{array}$$

A real vector field V on M is called *holomorphic* if $\alpha(V)$ is a holomorphic section of $\mathrm{T}^{1,0}M$.

Definition 2.11 A C^1 real submanifold N of a complex manifold M is called *totally real*, if at each point p of N the tangent space T_pN contains no complex line.

Definition 2.12 Let $\tau\colon M \to M$ be an antiholomorphic involution and N be the fixed point set of τ, which is a totally real submanifold. A holomorphic function $f \in \mathcal{O}(M)$ is called *τ-compatible* if $\tau^* f = \overline{f}$. A holomorphic vector field V on M is *τ-compatible* if $\tau_* V = \overline{V}$.

A τ-compatible function is real-valued on the fixed point set N. In the Hamiltonian setting τ-compatible functions and τ-compatible vector fields are closely related, cf. Lemma 5.3.

Lemma 2.13 *Let V be a holomorphic vector field on M with (N,τ) as above and $N \neq \emptyset$. The vector field V is τ-compatible if and only if $\alpha^{-1}(V_p) = 2\,\mathrm{Re}(V_p) \in \mathrm{T}_pN$ for all $p \in N$. In particular, the $\mathbb{R}$-flow of a τ-compatible vector field preserves the submanifold N.*

Proof Let x be a point in N. Consider the flow equation of V

$$V(x) = \frac{\partial}{\partial t}\Phi(t,x)\Big|_{t=0}$$

Under τ_* it becomes

$$\overline{V}(x) = \tau_* V(x) = \frac{\partial}{\partial t}\tau \circ \Phi(t,x)\Big|_{t=0}$$

Adding these two equations and writing the complex time as $t = u + iv$ yields

$$2\operatorname{Re} V(x) = \frac{\partial}{\partial t}(\mathrm{id} + \tau) \circ \Phi(t,x)\Big|_{t=0} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right)(\mathrm{id} + \tau) \circ \Phi(t,x)\Big|_{t=0}$$

When only flow in real time is taken into account

$$2\operatorname{Re} V(x) = \frac{1}{2}\frac{\partial}{\partial u}(\mathrm{id} + \tau) \circ \Phi(u,x)\Big|_{u=0}$$

On the other hand, $2\operatorname{Re} V(x) = \alpha^{-1}(V)$ is the corresponding smooth vector field for V. The flow equation for this smooth vector field in real time is

$$2\operatorname{Re} V(x) = \frac{\partial}{\partial u}\varphi(u,x)\Big|_{u=0}$$

where $\varphi(u,x) = \Phi(j(u),x)$ with the embedding $j\colon \mathbb{R} \to \mathbb{C}, t \mapsto (t,0)$. Therefore $\tau \circ \Phi(t,x) = \Phi(t,x)$ for $t \in \mathbb{R}$ and $x \in N$ as well as $\operatorname{Re} V(x) \in \mathrm{T}_x N$.

Backtracking the argument above and using $\mathbb{R}$ totally real in $\mathbb{C}$ shows $\tau_* V = \overline{V}$ on N. Since $N = \operatorname{Fix}(\tau) \neq \emptyset$ has real dimension $n = \dim_{\mathbb{C}} M$ by [AH77, Proposition 1.3], it is totally real of maximal dimension in M. Then the holomorphicity of $V - \overline{\tau_* V}$ implies $\tau_* V = \overline{V}$ on M. □

By definition, N being a totally real submanifold means $\mathrm{T}_p N \cap J\mathrm{T}_p N = \{0\}$ for any $p \in N$. This together with $\alpha(JV) = i\alpha(V)$ from the above diagram implies the complex flow of a τ-compatible vector field does not leave the totally real submanifold N invariant.

Definition 2.14 A vector field V on a manifold M is $\mathbb{R}$-*complete* if its flow $\varphi_t(p)$ exists for all $t \in \mathbb{R}$ and $p \in M$. A holomorphic vector field V on a complex manifold M is called $\mathbb{C}$-*complete* if V and JV are both $\mathbb{R}$-complete.

For a holomorphic vector field, $\mathbb{C}$-completeness is equivalent to the existence of its flow for all complex times.

Lemma 2.15 *Let M be a complex manifold. Let V be a $\mathbb{C}$-complete vector field on M and $f \in \ker V, g \in \ker V^2 \setminus \ker V$. Then fV and gV are complete.*

Proof Let Φ_t denote the flow map of V. Then the flow map Ψ_t of fV is given by

$$\Psi_t(x) = \Phi_{tf(x)}(x)$$

since

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi_t(x) = f(x)V(\Psi_t(x)) = f(\Psi_t(x))V(\Psi_t(x))$$

where the last step is due to $f \in \ker V$. As for gV, the flow map is given by

$$\Phi_{\varepsilon(tV_x g)\cdot tg(x)}(x), \quad \varepsilon(\zeta) = \frac{e^{\zeta}-1}{\zeta},$$

see e.g. [AK18, Lemma 3.3]. □

2.2 Holomorphic matrix factorization

To understand an invertible square matrix (e.g. representing the linear part of an automorphism), it is natural to see if it can be written as a finite product of matrices of simpler forms. Among the many ways of factorization, one that arises canonically from Gauss elimination process is the unitriangular decomposition. It means that each factor is unipotent and either in upper or lower triangular form, with 1's on the diagonal, and in interchanging order.

In the simple case with complex matrix entries, any matrix in $\mathrm{SL}_n(\mathbb{C})$ can be written as a product of 4 unitriangular factors. The same question can be asked for $\mathrm{SL}_n(R)$, where R is any commutative ring. Such a decomposition does not always exist, as demonstrated by Cohn's counterexample [Coh66]: The matrix

$$\begin{pmatrix} 1 - z_1 z_2 & z_1^2 \\ -z_2^2 & 1 + z_1 z_2 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}[z_1, z_2])$$

cannot be written as a finite product of unitriangular matrices in $\mathrm{SL}_2(\mathbb{C}[z_1, z_2])$.

When R is the ring of continuous complex-valued functions on a normal Hausdorff topological space X, it was studied and solved partially by Thurston and Vaserstein [TV86] and later fully by Vaserstein in 1988.

Theorem 2.16 ([Vas88]) *For any natural number n and an integer $d \geq 0$ there is a natural number K such that for any finite-dimensional normal Hausdorff topological space X of dimension d and null-homotopic continuous mapping $f\colon X \to \mathrm{SL}_n(\mathbb{C})$, the mapping can be written as a finite product of no more than K unitriangular matrices in $\mathrm{SL}_n(\mathcal{C}(X))$.*

In his groundbreaking paper for modern Oka theory, Gromov [Gro89, §3.5.G] asked the following question for the ring of holomorphic functions on $\mathbb{C}^d$

> *Vaserstein Problem.* Does every holomorphic map $\mathbb{C}^d \to \mathrm{SL}_n(\mathbb{C})$ decompose into a finite product of holomorphic maps sending $\mathbb{C}^d$ into unipotent subgroups in $\mathrm{SL}_n(\mathbb{C})$?

This problem was positively answered by Ivarsson and Kutzschebauch [IK12a] by applying the modern Oka-Grauert-Gromov principle to certain stratified fibrations, which implies the existence of a holomorphic solution from Vaserstein's continuous solution.

First we need some handy notations. For an **odd** natural number K we denote elements in $\mathbb{C}^{n(n-1)/2}$ as follows

$$Z_K = (z_{K,21}, \dots, z_{K,kl}, \dots, z_{K,n(n-1)}), \quad k > l.$$

Then we set

$$M_K(Z_K) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ z_{K,21} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ z_{K,n1} & \cdots & z_{K,n(n-1)} & 1 \end{pmatrix}.$$

For **even** K we proceed similarly. We only consider the transposed version, that is, we write elements in $\mathbb{C}^{n(n-1)/2}$ as

$$Z_K = (z_{K,12}, \dots, z_{K,kl}, \dots, z_{K,(n-1)n}), \quad k < l,$$

and we set

$$M_K(Z_K) = \begin{pmatrix} 1 & z_{K,12} & \cdots & z_{K,1n} \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & z_{K,(n-1)n} \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Definition 2.17 We consider topological spaces that are Hausdorff and paracompact (thus normal). For a definition of *complex spaces*, we refer to Grauert and Remmert [GR84, §1].

A complex space X is called *reduced at* x, if the local ring $\mathcal{O}_{X,x}$ does not contain nonzero nilpotent elements. We call X a *reduced* complex space, if X is reduced at all points. A point $x \in X$ is called *regular*, if $\mathcal{O}_{X,x}$ is isomorphic to a $\mathbb{C}$-algebra of convergent power series. We call a complex space X *finite dimensional*, if its regular part as a complex manifold has finite dimension.

Furthermore, a second countable complex space X is called a *Stein space* if it satisfies (a), (c) in Definition 2.1 and

(b') Every local ring $\mathcal{O}_{X,x}$ is generated by functions in $\mathcal{O}(X)$.

Property (b') says that there is a holomorphic map $X \to \mathbb{C}^N$ which embeds a neighborhood of x as a local complex subvariety of $\mathbb{C}^N$.

Theorem 2.18 *Let X be a finite dimensional reduced Stein space and $f\colon X \to \mathrm{SL}_n(\mathbb{C})$ be a holomorphic mapping that is null-homotopic. Then there exist natural number K and holomorphic mappings $G_1, \dots, G_K\colon X \to \mathbb{C}^{n(n-1)/2}$ such that f can be written as a product of upper and lower unitriangular matrices*

$$f(x) = M_1(G_1(x)) \cdots M_K(G_K(x)).$$

Ivarsson and Kutzschebauch's idea for solving Gromov's Vaserstein problem goes as follows. Let $\Psi_K\colon (\mathbb{C}^{n(n-1)/2})^K \to \mathrm{SL}_n(\mathbb{C})$ be

$$\Psi_K(Z_1, \dots, Z_K) = M_1(Z_1)^{-1} \cdots M_K(Z_K)^{-1}.$$

The problem amounts to finding a holomorphic map

$$G = (G_1, \dots, G_K)\colon X \to (\mathbb{C}^{n(n-1)/2})^K$$

such that the diagram

$$\begin{tikzcd} & (\mathbb{C}^{n(n-1)/2})^K \arrow[d, "\Psi_K"] \\ X \arrow[ur, dashed, "G"] \arrow[r, "f"'] & \mathrm{SL}_n(\mathbb{C}) \end{tikzcd}$$

commutes. Vaserstein's result, Theorem 2.16, yields a continuous map such that the diagram is commutative.

To apply the Oka-Grauert-Gromov principle for sections of holomorphic submersions over X, one candidate submersion is the pullback of $\Psi_K\colon (\mathbb{C}^{n(n-1)/2})^K \to \mathrm{SL}_n(\mathbb{C})$. The map Ψ_K is not submersive at all points of $(\mathbb{C}^{n(n-1)/2})^K$ and becomes a surjective holomorphic submersion if a subset of $(\mathbb{C}^{n(n-1)/2})^K$ is removed. The difficulty is that the fibers of this submersion turn out to be very complicated. They chose instead the following candidate

$$\begin{tikzcd} & (\mathbb{C}^{n(n-1)/2})^K \arrow[d, "\pi_n \circ \Psi_K"] \\ X \arrow[ur, dashed] \arrow[r, "\pi_n \circ f"'] & \mathbb{C}^n \setminus \{0\} \end{tikzcd}$$

where $\pi_n\colon \mathrm{SL}_n(\mathbb{C}) \to \mathbb{C}^n \setminus \{0\}$ is the projection of a matrix to its last row. For $K \geq 2$, the map $\Phi_K = \pi_n \circ \Psi_K$ turns out to be submersive exactly on the complement of a singularity set S_K. The singularity appears when the entries in the last row of each lower triangular matrix and the entries in the last column of each upper triangular matrix are all 0, except for the Kth matrix where no conditions are placed [IK12a, Lemma 2.6]. After showing that the holomorphic submersions $\Phi_K\colon (\mathbb{C}^{n(n-1)/2})^K \setminus S_K \to \mathbb{C}^n \setminus \{0\}, K \geq 3$ admit stratified sprays, they applied the stratified Oka-Grauert-Gromov principle for sections of holomorphic submersions to the pullback of Φ_K under $\pi_n \circ f$ to obtain the following.

Proposition 2.19 *([IK12a, Proposition 2.8]) Let X be a finite dimensional reduced Stein space and $f\colon X \to \mathrm{SL}_n(\mathbb{C})$ be a null-homotopic holomorphic map. Assume that there exists a natural number K and a continuous map $F\colon X \to (\mathbb{C}^{n(n-1)/2})^K \setminus S_K$ such that*

$$\begin{tikzcd} & (\mathbb{C}^{n(n-1)/2})^K \arrow[d, "\Phi_K"] \\ X \arrow[ur, dashed, "F"] \arrow[r, "\pi_n \circ f"'] & \mathbb{C}^n \setminus \{0\} \end{tikzcd}$$

commutes. Then there exists a holomorphic map $G\colon X \to (\mathbb{C}^{n(n-1)/2})^K \setminus S_K$ homotopic to F through continuous $F_t\colon X \to (\mathbb{C}^{n(n-1)/2})^K \setminus S_K$, such that the above diagram is commutative for all F_t.

This proposition was in turn used, via an induction on the size of matrix, to transfer Vaserstein's continuous solution to a holomorphic solution, for details see [IK12a].

For the symplectic group $\mathrm{Sp}_{2n}(\mathbb{C})$, Ivarsson, Kutzschebauch and Løw showed in [IKL20] the continuous factorization problem for symplectic matrices in $\mathrm{Sp}_{2n}(\mathbb{C})$ and in [IKL23] the holomorphic factorization problem for $\mathrm{Sp}_4(\mathbb{C})$. The general case, namely the holomorphic factorization problem for $\mathrm{Sp}_{2n}(\mathbb{C})$ was finished by Josua Schott [Sch25], see Theorem 6.4.

2.3 Density property

The density property arises as a tool in the study of holomorphic automorphisms of Stein manifolds.

In their seminal paper [RR88] Rosay and Rudin studied the holomorphic automorphism group of $\mathbb{C}^n$ and encountered many interesting phenomenons. In particular, $\mathbb{C}^n$ with $n \geq 2$ has a very large automorphism group.

Let $\mathrm{Aut}(\mathbb{C}^n)$ be the group of holomorphic automorphisms of $\mathbb{C}^n$, equipped with the topology of uniform convergence on compact subsets. Note that this topology is metrizable, e.g. we can take

$$d(\Phi, \Psi) = \sum_{k=1}^{\infty} 2^{-k} \frac{d_k(\Phi, \Psi)}{1 + d_k(\Phi, \Psi)} \tag{2.2}$$

for any $\Phi, \Psi \in \mathrm{Aut}(\mathbb{C}^n)$ and

$$d_k(\Phi, \Psi) = \sup_{|z| \leq k} (|\Phi(z) - \Psi(z)| + |\Phi^{-1}(z) - \Psi^{-1}(z)|)$$

gives a metric on the space of continuous functions on the closed ball with radius k. With this metric $\mathrm{Aut}(\mathbb{C}^n)$ becomes a complete metric space.

For $n = 1$, $\mathrm{Aut}(\mathbb{C})$ consists only of affine linear maps: $z \mapsto az + b$ $(a, b \in \mathbb{C}, a \neq 0)$. Let us consider $n \geq 2$. Let $\lambda \colon \mathbb{C}^n \to \mathbb{C}^k$ be a $\mathbb{C}$-linear map for some $k < n$ and $v \in \ker \lambda$ a vector in the kernel, $f \in \mathcal{O}(\mathbb{C}^k)$ an entire function on $\mathbb{C}^k$. For every $t \in \mathbb{C}$ and $z \in \mathbb{C}^n$ consider the following one-parameter group of holomorphic automorphisms of $\mathbb{C}^n$

$$\Phi_t(z) = z + t f(\lambda z) v. \tag{2.3}$$

Setting $t = 1$, we call an automorphism of type (2.3) a *shear*. Denote by $\mathcal{S}_1(n)$ the group generated by shears and by $\mathrm{Aut}_1(\mathbb{C}^n)$ the group of holomorphic automorphisms with Jacobian one.

In 1990 Erik Andersén [And90] discovered that $\mathcal{S}_1(n)$ is dense in $\mathrm{Aut}_1(\mathbb{C}^n)$, namely every holomorphic automorphism of $\mathbb{C}^n$ with Jacobian one can be approximated uniformly on compact subsets by compositions of polynomial shears.

Another type of automorphisms of $\mathbb{C}^n$ is given by

$$\Psi_t(z) = z + \frac{1}{|v|^2} (e^{t|v|^2 f(\lambda z)} - 1) \langle z, v \rangle v, \quad v \neq 0 \tag{2.4}$$

where $\langle z, v\rangle$ denotes the standard Hermitian scalar product on $\mathbb{C}^n$. We call such automophism an *overshear* and denote by $\mathcal{S}(n)$ the group generated by shears and overshears.

In 1992 Andersén and László Lempert [AL92] extended Andersén's result showing that $\mathcal{S}(n)$ is dense in $\mathrm{Aut}(\mathbb{C}^n)$, namely every holomorphic automorphism of $\mathbb{C}^n$ can be approximated uniformly on compact subsets by compositions of shears and overshears.

Remark 2.20 Andersén and Lempert also showed that $\mathcal{S}(n)$ is a proper subgroup of $\mathrm{Aut}(\mathbb{C}^n)$ for every $n > 1$ by constructing a holomorphic automorphism that is not a finite composition of shears and overshears. For example

$$(z_1, z_2) \mapsto (z_1 \,\mathrm{e}^{z_1 z_2}, z_2 \,\mathrm{e}^{-z_1 z_2})$$

is not a composition of shears and overshears. Further, they also showed that the group $\mathcal{S}(n)$ is meagre in $\mathrm{Aut}(\mathbb{C}^n)$.

The above one-parameter groups of holomorphic automorphisms of $\mathbb{C}^n$ have the following infinitesimal generators which are holomorphic vector fields on $\mathbb{C}^n$

$$\begin{aligned} V_z &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \Phi_t(z) = f(\lambda z) v = f(\lambda z) \sum_{k=1}^{n} v_k \frac{\partial}{\partial z_k} \\ W_z &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \Psi_t(z) = f(\lambda z)\langle z, v\rangle v. \end{aligned} \tag{2.5}$$

For example, when $v = e_n$ and $\lambda = \lambda_n \colon \mathbb{C}^n \to \mathbb{C}^{n-1}$ is the projection onto the first $n-1$ coordinates, we have

$$V_z = f(\lambda_n z)\frac{\partial}{\partial z_n}, \quad W_z = f(\lambda_n z) z_n \frac{\partial}{\partial z_n}.$$

In particular, the shear vector field V_z points in the nth direction and has a coefficient depending only on the first $n-1$ coordinates $z_1, \ldots, z_{n-1}$; while the coefficient of the overshear W_z is linear in z_n.

Andersén and Lempert also showed a Runge type approximation for holomorphic injections on star-shaped domains in $\mathbb{C}^n$ by holomorphic automorphisms. Their key technical observation is [AL92, Proposition 3.7]:

> *Let p be a polynomial of n variables. Then there exist a finite number of polynomials $q_1, \ldots, q_N$ of one variable, and linear forms $l_1, \ldots, l_N$ of n variables such that*
>
> $$p(z) = \sum_{k=1}^{N} q_k(l_k(z)).$$

Subsequently, Franc Forstnerič and Jean-Pierre Rosay [FR93; FR94] recast the above observation explicitly in terms of holomorphic vector fields on $\mathbb{C}^n$: Every polynomial vector field V on $\mathbb{C}^n$ for $n > 1$ is a finite sum of $\mathbb{R}$-complete polynomial vector fields $V_1, \ldots, V_N$ whose flows consist of shears and overshears. The flow of a $\mathbb{R}$-complete holomorphic vector field is a one-parameter group of holomorphic automorphisms, while the infinitesimal generator of a one-parameter group of automorphisms is a $\mathbb{R}$-complete holomorphic vector field, cf. Equation (2.3) and (2.5). Because the flow of the sum $V = V_1 + \cdots + V_N$ can be approximated by compositions of the flows of vector fields $V_1, \ldots, V_N$ (cf. Theorem 2.24), the flow of any holomorphic vector field on $\mathbb{C}^n$ can be approximated uniformly on compact subsets by compositions of shears and overshears.

Since then, the central stage of Andersén–Lempert theory shifts from the automorphism group $\mathrm{Aut}(\mathbb{C}^n)$ to the dynamics in $\mathbb{C}^n$. Taking this new focus, Forstnerič and Rosay generalized the approximation result to Runge domains in $\mathbb{C}^n$. It is essentially their interpretation that makes Andersén–Lempert theory an important tool in complex analysis.

Theorem 2.21 (Andersén–Lempert theorem) *Let Ω be a domain in $\mathbb{C}^n$ and $\Phi_t \colon \Omega \to \mathbb{C}^n (t \in [0,1])$ be a $\mathcal{C}^1$-isotopy of holomorphic injections such that Φ_0 is the identity map on Ω and $\Omega_t = \Phi_t(\Omega)$ is Runge in $\mathbb{C}^n$ for every t. Then Φ_1 can be approximated uniformly on compact subsets of Ω by elements of the group $\mathcal{S}(n)$.*

Proof Here we give a sketch: Consider the time dependent vector field V on the domain $\tilde{\Omega} = \{(t,z) : t \in [0,1], z \in \Omega_t\}$ given by

$$V(t,x) = V_t(x) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=t} \Phi_s(\Phi_t^{-1}(x)), \quad x \in \Omega_t.$$

Choose an integer $N \in \mathbb{N}$ and subdivide $[0,1]$ into subintervals $I_k = [t_k, t_{k+1}]$ of length $1/N$, where $t_k = k/N$ for $k = 0, 1, \ldots, N$. Take the locally constant vector field $\hat{V}_t$ which equals V_{t_k} on Ω_{t_k} for $t \in I_k$ and denote its flow by φ_t^k. The flows of $\hat{V}$ and of V differ on the interval I_k by a term of order $o(N^{-1})$. Thus as $N \to \infty$, the composition

$$\psi_{t,N} = \varphi_{t/N}^{N-1} \circ \varphi_{t/N}^{N-2} \circ \cdots \circ \varphi_{t/N}^{1} \circ \varphi_{t/N}^{0}$$

converges to Φ_t uniformly on compact subsets in Ω for all $t \in [0,1]$. Note that given a compact subset $K \subset \Omega$, for large enough N the orbit of K under the composition $\psi_{t,N}$ for $t \in I_k$ remains in Ω_{t_k}. Therefore it suffices to approximate the flow φ_t^k (chosen such that $\varphi_0^k(z) = z$ for $z \in \Omega_{t_k}$) of the time independent vector field V_{t_k} uniformly on compact subset $L \subset \Omega_{t_k}$, whose trajectory

$$\tilde{L} = \{\varphi_t^k(z) : z \in L,\ t \in [0, 1/N]\}$$

stays in Ω_{t_k}, by holomorphic automorphisms of $\mathbb{C}^n$. Then one makes use of the Runge property of Ω_{t_k} to approximate the holomorphic vector field V_{t_k} by a polynomial vector field on a compact subset in Ω_{t_k} which contains $\tilde{L}$ in its interior. Here the observation of Andersén–Lempert gives us a sum of $\mathbb{R}$-complete polynomial vector fields, and correspondingly the flow φ_t^k may be approximated by a finite composition of shears and overshears. See [For17, §4.9] for more details. $\square$

Remark 2.22 (i) The Runge condition in Theorem 2.21 is essential: If a biholomorphic map $\Phi \colon \Omega \to \Omega'$ between domains in $\mathbb{C}^n$ is a limit of automorphisms of $\mathbb{C}^n$, then Ω is Runge if and only if Ω' is.

(ii) Together with the fact that every holomorphic automorphism of $\mathbb{C}^n$ can be joined by a smooth path to the identity [For17, Lemma 4.9.4], the Forstnerič–Rosay result implies that the group $\mathcal{S}(n)$ generated by shears and overshears is dense in $\mathrm{Aut}(\mathbb{C}^n)$.

Next, we introduce the notion of algorithm which is useful for approximating the flow of Lie combinations of vector fields. To a vector field, an algorithm captures the linear part of its flow.

Definition 2.23 [AM78] Let V be a continuous vector field on a manifold M, and $A_t(x)$ a continuous map from an open set in $\mathbb{R}_{\geq 0} \times M$ containing $\{0\} \times M$ to M such that its t-derivative exists and is continuous. We say that A is *an algorithm for* V if for all $p \in M$

$$A_0(p) = p, \quad \left.\frac{\partial}{\partial t}\right|_{t=0} A(t,p) = V_p.$$

Theorem 2.24 *[AM78, Theorem 4.1.26] Let V be a Lipschitz continuous vector field with flow Φ_t on a manifold M. Let Ω be the fundamental domain of V and $\Omega_+ = \Omega \cap (\mathbb{R}_{\geq 0} \times M)$. If A is an algorithm for V, then for all $(t,p) \in \Omega_+$ the n-th iterate $(A_{t/n})^{\circ n}(p)$ of the map $A_{t/n}$ is defined for sufficiently large $n \in \mathbb{N}$ (depending on p and t), and*

$$\lim_{n\to\infty} (A_{t/n})^{\circ n}(p) = \Phi_t(p).$$

The convergence is uniform on compact subsets of Ω_+.

The idea of using composition of algorithms to approximate the flow map of a vector field is essentially the Euler method: First discretize in time, then integrate time independent vector fields over small intervals, and in the limit the deviation converges to zero.

Lemma 2.25 *[For17, Proposition 4.8.3] For vector fields V and W with flows Φ_t, Ψ_t, respectively.*

(i) The composition $\Phi_t \circ \Psi_t$ is an algorithm for $V + W$.

(ii) For $t > 0$, $\Psi_{-\sqrt{t}} \circ \Phi_{-\sqrt{t}} \circ \Psi_{\sqrt{t}} \circ \Phi_{\sqrt{t}}$ is an algorithm for $[V, W]$.

Proof (i) Let $x = (x_1, \dots, x_n)$ be a coordinate map around a point $x \in M$. The Taylor expansion of the flows is given by

$$\Phi_t(x) = x + tV(x) + O(t^2), \quad \Psi_t(x) = tW(x) + O(t^2)$$

which gives $\Phi_t \circ \Psi_t(x) = x + tV(x) + tW(x) + O(t^2)$.

(ii) Similarly use local Taylor expansion of composition of flows, cf. [For17, p.37].□

Combining the preceeding Theorem and Lemma, we have the transition from Lie-combination of $\mathbb{C}$-complete holomorphic vector fields to composition of holomorphic automorphisms.

Corollary 2.26 *[For17] Let $V_1, \dots, V_m$ be $\mathbb{R}$-complete holomorphic vector fields on a complex manifold M. Let V be a holomorphic vector field on M that is in the Lie algebra* *generated by* $V_1, \ldots, V_m$*. Assume that* K *is a compact subset in* M *and* $t_0 > 0$ *is such that the flow* $\Phi_t(p)$ *of* V *exists for all* $p \in K$ *and for all* $t \in [0, t_0]$*. Then* Φ_{t_0} *is a uniform limit on* K *of a sequence of compositions of time-forward maps of* $V_1, \ldots, V_m$*.*

The next major step in the development of Andersén–Lempert theory and the study of holomorphic automorphism groups was taken by Dror Varolin in his dissertation and in the papers [Var99; Var00; Var01]. His main observation is

> *The flow of any holomorphic vector field on a complex manifold* M*, which is a Lie combination of complete holomorphic vector fields, is a limit of holomorphic automorphisms of* M*.*

The notion of *density property* for complex manifolds conceptualizes the proof of the Andersén–Lempert theorem and enlarges the possible operations for complete holomorphic vector fields from linear combinations to Lie combinations. Let

$$\mathrm{IVF}(M) = \text{the set of } \mathbb{C}\text{-complete holomorphic vector fields on } M.$$

Definition 2.27 A complex manifold M has the *density property (DP)* if the Lie algebra $\mathrm{Lie}(M) = \mathrm{Lie}(\mathrm{IVF}(M))$ generated by $\mathrm{IVF}(M)$ is dense (in the topology of locally uniform convergence) in the Lie algebra of all holomorphic vector fields on M.

The reason for taking Lie combinations to define DP is that we can find algorithms for addition and Lie bracket, cf. Lemma 2.25. Varolin also gives a version of Andersén–Lempert theorem for Stein manifolds with DP [Var00], cf. Theorem 2.46.

Remarks 2.28 (i) DP is a precise tool to say that the holomorphic automorphism group of a Stein manifold is very big. Intuitively we can compare DP to cutoff functions in the smooth category, both fulfilling the role of globalizing local objects. Of course we do not know how a holomorphic automorphism looks like outside a compact set when applying DP; in practice this lack of control has usually been compensated by an iterative use of the Andersén–Lempert theorem to enlarge the compact set to the whole space.

(ii) On compact complex manifolds DP holds trivially because there every vector field is complete and the Lie algebra of holomorphic vector fields is finite dimensional, thus DP does not give further information. On noncompact complex manifolds, DP is a very strong property. For example, a Stein manifold with DP is Oka and elliptic (see [For17, Proposition 5.6.23]).

(iii) Replacing $\mathrm{IVF}(M)$ by holomorphic vector fields which are $\mathbb{R}$-complete turns out to be the same as DP: In [Rit13, Appendix], Ritter starts with $\mathbb{R}$-complete holomorphic vector fields and gives a detailed proof for Andersén–Lempert theorem for Stein manifolds with density property using $\mathbb{R}$-complete holomorphic vector fields. Such a manifold is covered by Fatou–Bieberbach domains [Var00] and thus every bounded plurisubharmonic function is constant. Then by [For96, Corollary 2.2], every $\mathbb{R}$-complete holomorphic vector field is also complete in complex time.

Some geometric properties follow when the holomorphic automorphism group is big enough to allow DP.

Proposition 2.29 *Let* M *be a Stein manifold with the density property. Then* *(a) Finitely many complete holomorphic vector fields span the tangent bundle of M.*

(b) The group $\mathrm{Aut}(M)$ *of holomorphic automorphisms acts transitively on M.*

(c) The group $\mathrm{Aut}(M)$ *of holomorphic automorphisms acts k-transitively on M for all* $k \in \mathbb{N}$.

Proof (a) DP implies that $\mathrm{Lie}(M)$ span the tangent space T_pM at any point $p \in M$. Observe that the Lie bracket $[V, W]$ of complete vector fields V, W can be approximated by a linear combination

$$[V, W] = \lim_{t \to 0} \frac{\Phi_t^*(V) - V}{t}$$

where Φ_t is the flow generated by W, which in turn can be approximated by $\Phi_t^*(V)/t - V/t$ for small $t \neq 0$, uniformly on compact subsets of M. Since $\Phi_t^*(V)/t$ and V/t are complete, the closure of the linear span of $\mathrm{IVF}(M)$ equals the closure of $\mathrm{Lie}(M)$. Suppose that at a point $p_0 \in M$, there are no finitely many complete vector fields spanning $T_{p_0}M$. Then there exists a nonzero linear functional $l \colon T_{p_0}M \to \mathbb{C}$ such that l annihilates any complete vector field. But this would imply l is zero on the closure of the linear span of $\mathrm{IVF}(M)$ restricted to $T_{p_0}M$, a contradiction to M being Stein. If finitely many vector fields $V_1, \dots, V_n$ span the tangent space at one point, they span the tangent space at all points outside a proper analytic subset A, which may consist of countably many irreducible components $A_1, A_2, \dots$.

Now it suffices to find a holomorphic automorphism Φ which takes every $A_i, i = 1, 2, \dots$ into the complement of A. Such a Φ induces complete vector fields $\Phi_*(V_1), \dots, \Phi_*(V_n)$ which span the tangent space at a generic point in each A_i. Then together with $V_1, \dots, V_n$ they span the tangent space at each point outside an analytic subset B of smaller dimension than A. Then an induction on dimension gives the result.

To construct Φ consider an exhausting sequence of compact subsets $K_1 \subset K_2 \subset \dots$ in X such that $\cup_i K_i = X$ and a closed imbedding $\jmath \colon X \hookrightarrow \mathbb{C}^m$. Replacing the balls in (2.2) by K_i we get a complete metric on $\mathrm{Aut}(M)$, viewing each automorphism Φ as continuous map from X to $\mathbb{C}^m$ via $\jmath$. For each $i = 1, 2, \dots$ set

$$Z_i = \{\Psi \in \mathrm{Aut}(M) : \Psi(A_i) \cap (M \setminus A) \neq \emptyset\}$$

which is open in $\mathrm{Aut}(M)$. On the other hand, since $\mathrm{IVF}(M)$ generates the tangent space at each point of M we can choose $V \in \mathrm{IVF}(M)$ transversal to A_i. Then for any $\Psi \in \mathrm{Aut}(M)$, its composition with the flow of V lies in Z_i, which shows that Z_i is everywhere dense. By the Baire category theorem, the intersection of all Z_i is nonempty, which yields the desired Φ.

(b) At a point $p \in M$, there are complete vector fields $V_1, \dots, V_n, n = \dim M$ spanning the tangent space T_pM. The composed map

$$\Phi_p \colon \mathbb{C}^n \to M, \quad (t_1, \dots, t_n) \mapsto \Phi^n_{t_n} \circ \Phi^{n-1}_{t_{n-1}} \circ \dots \circ \Phi^1_{t_1}(p)$$

where $\Phi^j_{t_j}$ is the time-t_j flow of the vector field V_j, has full rank at $t = 0$. By the inverse function theorem Φ_p is a local biholomorphism from a neighborhood of 0 to a neighborhood of p, hence any $\mathrm{Aut}(M)$-orbit is open, which also implies that each orbit is closed. Restricting to each component of M, we see that there is only one orbit.

(c) We need

> [Var00, Theorem 2] Let M be a Stein manifold with DP, K a compact set in M, $p, q \in M$ two points outside the $\mathcal{O}(X)$-convex hull of K, and $x_1, \ldots, x_m \in K$. Then there exists $\Phi \in \mathrm{Aut}(M)$ fixing $x_i, i = 1, \ldots, m$, $\Phi(p) = q$, and close to the identity on K.

Choose the compact set K to be a disjoint union of small neighborhoods of $x_1, \ldots, x_m$ respectively. Then p, q are outside the $\mathcal{O}(X)$-convex hull of K. The Claim says that we can move one point in a given $(m+1)$-tuple to a prescribed position while keeping other m points fixed. Repeat this. □

The generalization taken by Varolin from $\mathbb{C}^n$ to general Stein manifolds enlarges the scope of Andersén–Lempert theory and begs the question: How big is the family of Stein manifolds admitting DP? Varolin himself [Var01] and in collaboration with Tóth [TV00; TV06] came up with the first classes of manifolds enjoying DP, including semisimple Lie groups and homogeneous spaces of semisimple complex Lie groups with trivial center. However, a lack of general method persisted.

The methodological breakthrough happened around 2006, when Kaliman and Kutzschebauch [KK08a] crafted an effective criterion for DP. The point is to find an $\mathcal{O}(M)$-submodule in $\mathrm{VF}(M)$ contained in the closure of $\mathrm{Lie}(\mathrm{IVF}(M))$ and then use transitivity to create more submodules so that their sum coincides with $\mathrm{VF}(M)$.

At first they introduced this method in the algebraic setting with an algebraic analogue of DP.

Definition 2.30 ([Var01; KK08a]) We say that a smooth complex algebraic variety M has the *algebraic density property (ADP)* if the Lie algebra $\mathrm{Lie}_{\mathrm{alg}}(M) = \mathrm{Lie}(\mathrm{IVF}_{\mathrm{alg}}(M))$ generated by the set $\mathrm{IVF}_{\mathrm{alg}}(M)$ of $\mathbb{C}$-complete algebraic vector fields on M is the Lie algebra of all algebraic vector fields on M. In short, ADP means

$$\mathrm{Lie}_{\mathrm{alg}}(M) = \mathrm{VF}_{\mathrm{alg}}(M).$$

Remark 2.31 Since the flow of a complete algebraic vector field is (holomorphic but) not necessarily algebraic, ADP is like a bridge connecting smooth affine varieties to DP on the holomorphic side. One can consider a stronger algebraic property by using only algebraic overshears of locally nilpotent vector fields as Lie generators, then the algebraic manifold enjoys the so-called algebraic overshear density property, see [AK24]. In particular, similar to the fact that the holomorphic density property implies the holomorphic flexibility, the algebraic overshear density property implies that the tangent space at every point is spanned by locally nilpotent vector fields [AK24, Proposition 3.1].

Example 2.32 For $n \geq 2$ the Euclidean space $\mathbb{C}^n$ has ADP: Let $z = (z_1, \ldots, z_n)$ be complex coordinates of $\mathbb{C}^n$ and $V_i = \partial/\partial z_i$. For any $f_i \in \ker V_i$, the vector fields $f_i V_i$ and $z_j f_i V_i$ are complete. Hence for $i \neq j$

$$f_i f_j V_j = [f_i V_i, z_i f_j V_j] - [z_i f_i V_i, f_j V_j] \in \mathrm{Lie}_{\mathrm{alg}}(\mathbb{C}^n).$$

Since the linear span of $\ker V_i \cdot \ker V_j$ equals $\mathbb{C}[z]$, $\mathrm{Lie}_{\mathrm{alg}}(\mathbb{C}^n)$ contains all polynomial vector fields proportional to V_j.

Proposition 2.33 *For a smooth complex affine algebraic variety M, the algebraic density property implies the density property.*

Proof By Theorem 2.6(A), there are global algebraic sections $s_1, \ldots, s_N \in \mathrm{VF}_{\mathrm{alg}}(M)$ of the tangent sheaf $\mathcal{T}_M$ that generate the stalk at every point. For any holomorphic section $s \in \mathrm{VF}(M)$ of $\mathcal{T}_M$ over an $\mathcal{O}(M)$-convex compact set $K \subset X$, by Theorem 2.9, $s = \sum f_i s_i$ with $f_i \in \mathcal{O}(K)$. Approximating the holomorphic functions f_i by global functions in $\mathbb{C}[M]$, we see that $\mathrm{VF}_{\mathrm{alg}}(M)$ is dense in $\mathrm{VF}(M)$. $\square$

Here we explain the criterion for the holomorphic case, which is an adaptation of the original algebraic case in [KK08a], cf. [Leu16; For17]. The crucial first step is a way of finding a submodule contained in $\overline{\mathrm{Lie}}(M)$.

Definition 2.34 A holomorphic vector field V in $\overline{\mathrm{Lie}}(M)$ is *stable* if for every $f \in \mathcal{O}(M)$ we have $fV \in \overline{\mathrm{Lie}}(M)$.

Obviously the submodule of $\mathrm{VF}(M)$ generated by any collection of such stable vector fields is contained in $\overline{\mathrm{Lie}}(M)$. For an automorphism $\Phi \in \mathrm{Aut}(M)$ and a stable vector field $V \in \overline{\mathrm{Lie}}(M)$, the pushforward $\Phi_* V$ is in $\overline{\mathrm{Lie}}(M)$ and also stable, since $\Phi_*(fV) = (f \circ \Phi^{-1})\Phi_* V$ for any $f \in \mathcal{O}(M)$. The point is that a change of coordinates preserves the complete integrability of a vector field and commutes with summation and Lie brackets. Hence Φ_* preserves $\mathrm{IVF}(M)$ and $\mathrm{Lie}(M)$. For any submodule L, $\Phi_*(L)$ is again a submodule .

Definition 2.35 (i) A pair (V, W) of complete holomorphic vector fields is a *semicompatible pair* if $\overline{\mathrm{Span}}(\ker V \cdot \ker W)$ contains a nontrivial ideal $I \subset \mathcal{O}(M)$.

(ii) A semicompatible pair (V, W) is a *compatible pair* if either (1) there exists $h \in \ker W$ such that $V(h) \in \ker V \setminus \{0\}$, or (2) there exists h such that $V(h) \in \ker V \setminus \{0\}$ and $W(h) \in \ker W \setminus \{0\}$.

By Lemma 2.15, the fields hV and hW in Condition (ii) are complete. In case (1) we have the submodule $I \cdot V(h)W$ of $\mathrm{VF}(M)$ contained in $\overline{\mathrm{Lie}}(M)$: Because for $f \in \ker V, g \in \ker W$, we have $fV, fhV, gW, ghW \in \mathrm{IVF}(X)$ and thus

$$\mathrm{Lie}(M) \ni fgV(h)W = [fV, ghW] - [fhV, gW].$$

While in case (2), we have

$$\mathrm{Lie}(M) \ni fg(V(h)W + W(h)V) = [fV, ghW] - [fhV, gW],$$

hence the submodule is $I \cdot (V(h)W + W(h)V) \subset \overline{\mathrm{Lie}}(M)$. Let us denote by Z the vector field in $\mathrm{Lie}(M)$ associated to the compatible pair (V, W), namely

$$Z = \begin{cases} V(h)W & \text{for } h \in (\ker V^2 \setminus \ker V) \cap \ker W \\ V(h)W + W(h)V & \text{for } h \in (\ker V^2 \setminus \ker V) \cap (\ker W^2 \setminus \ker W) \end{cases}$$

Note that for any $f \in I$, the vector field $fZ \in \mathrm{Lie}(M)$ is stable.

After finding a submodule (that corresponds to a stable vector field), we create more by first ”rotating it at one point”.

Definition 2.36 Let M be a complex manifold and $p \in M$. A finite subset S of the tangent space T_pM is called a *generating set* if the image of S under the action of the isotropy subgroup of p (in $\mathrm{Aut}(M)$) spans T_pM as a complex vector space.

Then the transitivity ensures the existence of sufficiently many submodules such that their sum contains all vector fields.

Proposition 2.37 *Let M be a Stein manifold on which $\mathrm{Aut}(M)$ acts transitively. Assume that there are stable holomorphic vector fields $V_1, \dots, V_m \in \overline{\mathrm{Lie}}(M)$ and a point $p \in M$ such that the vectors $V_1(p), \dots, V_m(p)$ form a generating set for T_pM. Then M has the density property.*

Proof We want to see that every holomorphic vector field V on M can be approximated uniformly on any compact set $K \subset M$ by elements of $\overline{\mathrm{Lie}}(M)$. Passing to the $\mathcal{O}(M)$-convex hull of K we may assume that it is $\mathcal{O}(M)$-convex and let $\Omega \subset\subset M$ be a relatively compact Stein Runge neighborhood of K. By a similar argument as for Proposition 2.29(a), we can add to $V_1, \dots, V_m$ finitely many images of these vector fields by elements of $\mathrm{Aut}(M)$ so that $V_i(p)$ span T_pM for every point $p \in \Omega$. Indeed, since Ω is relatively compact, $\Omega \cap A$ is a finite union of irreducible analytic subsets, where A is the proper analytic subset where $V_i(a), a \in A$ do not span T_aM.

Then the enlarged set $V_1(p), \dots, V_N(p)$ span T_pM for all $p \in \Omega$, which implies that $V_i + \mathfrak{m}_p\mathcal{T}$ span the vector space $\mathcal{T}/\mathfrak{m}_p\mathcal{T}$ for all $p \in \Omega$, where $\mathcal{T}$ is the tangent sheaf of M and $\mathfrak{m}_p$ the maximal ideal of p. By Nakayama's lemma, $V_1, \dots, V_N$ generate the tangent sheaf over Ω. Using Theorem 2.9 we can approximate any global section V uniformly on K by $\sum g_i V_i$ for some $g_i \in \mathcal{O}(M)$, which is in $\overline{\mathrm{Lie}}(M)$ since V_i's are stable. □

There are three ingredients in the recipe: Compatible pair(s), generating set (at a point), and $\mathrm{Aut}(M)$-transitivity.

Theorem 2.38 *Let M be a Stein manifold on which $\mathrm{Aut}(M)$ acts transitively. If there are compatible pairs $(V_i, W_i), i = 1, \dots, N$, such that there is a point $p \in M$ where the associated vectors $Z_i(p)$ form a generating set of T_pM, then M has the density property.*

Proof Let I_i be the ideals and Z_i the vector fields associated to the compatible pairs (V_i, W_i). Take a function $f_i \in I_i \setminus \{0\}$ for each i. Since the set of points $p \in M$ where $Z_i(p)$ form a generating set is open and nonempty, there exists a point $q \in M$ where $f_i(p)Z_i(p)$ form a generating set of T_qM. Since $f_iZ_i \in \mathrm{Lie}(M)$ is stable, the $\mathcal{O}(M)$-module generated by f_iZ_i is contained in $\overline{\mathrm{Lie}}(M)$. Apply Proposition 2.37.□

Example 2.39 The special linear group $\mathrm{SL}_2(\mathbb{C})$ has ADP: Since the adjoint action generates an irreducible representation on the Lie algebra $\mathfrak{sl}_2 = \mathrm{T}_e\mathrm{SL}_2(\mathbb{C})$, any nonzero vector in $\mathfrak{sl}_2$ is a generating set. Denote a matrix in $\mathrm{SL}_2(\mathbb{C})$ by

$$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}.$$

Let us consider

$$V = b_1 \frac{\partial}{\partial a_1} + b_2 \frac{\partial}{\partial a_2}, \quad W = a_1 \frac{\partial}{\partial b_1} + a_2 \frac{\partial}{\partial b_2}.$$

Note that the time-t flow of V is adding t-times the first row to the second row, and vice versa for W. Thus V and W are tangent to $\mathrm{SL}_2(\mathbb{C})$. Clearly, $\mathbb{C}[a_1, a_2] \subset \ker V$ and $\mathbb{C}[b_1, b_2] \subset \ker W$ which implies that $\mathrm{Span}(\ker V \cdot \ker W) = \mathbb{C}[\mathrm{SL}_2(\mathbb{C})]$. Taking a_1 which is in $\ker W$ and $V(a_1) = b_1 \in \ker V$, we see that (V, W) is a compatible pair. Apply Theorem 2.38.

Using this criterion, Kaliman and Kutzschebauch enlarged the classes of manifolds with DP substantially [KK08a], cf. Examples 2.40. Also in 2006, they established ADP for Danielewski surfaces [KK08b] where the criterion does not apply due to the lack of compatible pairs in this particular case.

Example 2.40 We list some families that are known to enjoy DP or even ADP:

(i) $\mathbb{C}^n$ for $n \geq 2$ [AL92]

(ii) complex semisimple Lie groups [TV00]

(iii) homogeneous spaces of complex semisimple Lie groups with trivial center [TV06]

(iv) linear algebraic groups $G < \mathrm{GL}_n(\mathbb{C})$ where the connected component is different from $\mathbb{C}$ or $(\mathbb{C}^*)^m, m \in \mathbb{N}$ [KK08a]

(v) algebraic hypersurfaces in $\mathbb{C}^{n+2}$ of the form $uv = p(z)$, where $u, v \in \mathbb{C}, z \in \mathbb{C}^n$ and the zero fiber of the polynomial $p \in \mathbb{C}[\mathbb{C}^n]$ is smooth and reduced [KK08b]

(vi) analytic hypersurfaces in $\mathbb{C}^{n+2}$ of the form $uv = f(z)$, where $u, v \in \mathbb{C}, z \in \mathbb{C}^n$ and the zero fiber of the holomorphic function $f \in \mathcal{O}(\mathbb{C}^n)$ is smooth and reduced [KK08b]

(vii) homogeneous spaces of the form $X = G/R$ where G is a linear algebraic group and R is a closed proper reductive subgroup, and the connected components of X are different from $\mathbb{C}$ or $(\mathbb{C}^*)^m, m \in \mathbb{N}$ [DDK10]

(viii) affine homogeneous spaces $X = G/H$ where $H < G < \mathrm{GL}_n(\mathbb{C})$ and where the connected components of X are different from $\mathbb{C}$ or $(\mathbb{C}^*)^m, m \in \mathbb{N}$ [DDK10; KK17]

(ix) Koras–Russell cubic threefold and related families [Leu16]

(x) Gizatullin surfaces with reduced degenerate fibre [And18]

(xi) Calogero–Moser spaces $\mathcal{C}_n$ [And21]

(xii) Gromov–Vaserstein fibers [And+23]

For an overview of the density property and its wealth of applications in solving holomorphic problems of geometric nature we refer to the overview articles of Kaliman–Kutzschebauch [KK11], Kutzschebauch [Kut14; Kut20], and Forstnerič–Kutzschebauch [FK22]. In particular, we have omitted the volume-preserving case completely, which has an equally impressive development as DP.

2.4 Symplectic density property

Instead, let us look at the symplectic case. In contrast to holomorphic automorphism groups, only very little is known so far about holomorphic symplectic automorphisms of Stein manifolds that are equipped with a holomorphic symplectic form. Loosely speaking, the technical difficulty lies in the rigidity of the symplectic condition.

The first example was the cotangent bundle $T^*\mathbb{C}^n \cong \mathbb{C}^{2n}$ of the complex Euclidean space with the standard symplectic form, where the group of holomorphic symplectic automorphisms was described by Forstnerič [For96] in 1996.

Deng and Wold [DW22] deduced from Forstnerič's result the Hamiltonian density property for closed coadjoint orbits of complex Lie groups. However, the explicit examples they gave are surfaces where the symplectic form equals the holomorphic volume form and hence these examples are already covered by the volume density property.

Definition 2.41 Let M be a complex manifold with a holomorphic symplectic form ω.

1. We call a holomorphic vector field V on M *symplectic* if $\mathcal{L}_V \omega = 0$.
2. We call a holomorphic vector field V on M *Hamiltonian* if there exists a holomorphic function $H\colon M \to \mathbb{C}$, called the *Hamiltonian function of* V, such that $i_V \omega = \mathrm{d}H$.

Remark 2.42 Since ω is closed, it follows from Cartan's homotopy formula that V is symplectic if and only if $\mathrm{d}i_V\omega = 0$. Further, this implies that every Hamiltonian vector field is symplectic. The holomorphic symplectic vector fields and the holomorphic Hamiltonian vector fields on a complex manifold each form a Lie algebra.

The holomorphic functions on X form a Lie algebra under the *Poisson bracket* $\{\cdot,\cdot\}$. It is defined in such a way that the correspondence between Hamiltonian functions H and K and their respective Hamiltonian vector fields V and W respects the Lie algebra structure:

$$i_{[V,W]}\omega = -\mathrm{d}\{H, K\},$$

see the textbook of Arnol'd [Arn89, §40 Corollary 3, p. 215].

Here we introduce the symplectic density property and the Hamiltonian density property for Stein symplectic manifolds. The Hamiltonian density property was also studied by Deng and Wold [DW22, §3.1].

Definition 2.43 Let M be a complex manifold with a holomorphic symplectic form ω.

1. We say that (M, ω) has the *symplectic density property* if the Lie algebra generated by the complete holomorphic symplectic vector fields on M is dense (in the topology of locally uniform convergence) in the Lie algebra of all holomorphic symplectic vector fields on M.

2. We say that (M,ω) has the *Hamiltonian density property* if the Lie algebra generated by the complete holomorphic Hamiltonian vector fields on M is dense (in the topology of locally uniform convergence) in the Lie algebra of all holomorphic Hamiltonian vector fields on M.

Remark 2.44 We consider the space of functions

$$\mathcal{O}(M) = \{f\colon M \to \mathbb{C} \text{ holomorphic}\}$$

as a Lie algebra with the Poisson bracket $\{\cdot,\cdot\}$. Note that the Hamiltonian density property is equivalent to stating that this Lie algebra contains a dense (in the topology of locally uniform convergence) Lie subalgebra that is generated by those functions that correspond to complete Hamiltonian vector fields. To see this, we only need to observe that for holomorphic functions the locally uniform convergence implies also the locally uniform convergence of the derivatives due to Cauchy estimates.

We also emphasize that we only work with the Lie algebra structure, and not with the Poisson algebra structure, i.e. we are not allowed to multiply functions. The conclusions of density properties only hold when working with Lie combinations of the corresponding vector fields, cf. Lemma 2.25.

Lemma 2.45 *Let (M,ω) be a complex symplectic manifold and h a holomorphic function on M which induces a complete Hamiltonian vector field V. Then $f \circ h$ induces complete Hamiltonian vector field on M for any entire function f in one variable.*

Proof We have

$$\mathrm{d}(f(h)) = f'(h)\mathrm{d}h = f'(h)i_V\omega = i_{f'(h)V}\omega.$$

Since $V(f'(h)) = \{h, f'(h)\} = 0$, the vector field $f'(h)V$ is complete by Lemma 2.15. □

We state a symplectic version of the Andersén–Lempert theorem, which was mentioned in [FK22] for $\mathbb{C}^{2n}$.

Theorem 2.46 *Let M be a Stein manifold with a holomorphic symplectic form ω. Assume that (M,ω) has the symplectic density property or the Hamiltonian density property.*

Let $\Omega \subseteq M$ be a Stein open subset with $H^1(\Omega,\mathbb{C}) = 0$ and let $\varphi_t\colon \Omega \to M, t \in [0,1]$, be a jointly $\mathcal{C}^1$-smooth map such that the following holds:

1. *The map $\varphi_0\colon \Omega \to M$ is the natural embedding.*
2. *The map $\varphi_t\colon \Omega \to M$ is a holomorphic symplectic injection for each $t \in [0,1]$.*
3. *The set $\varphi_t(\Omega) \subset M$ is Runge for each $t \in [0,1]$.*

Then for every compact $K \subset \Omega$ and every $\varepsilon > 0$ and every choice of metric on M that induces its topology, there exists a continuous family $\Phi_t\colon M \to M$ of holomorphic symplectic automorphisms such that

$$\sup_{x\in K} d(\varphi_1(x), \Phi_1(x)) < \varepsilon.$$

Moreover, Φ_t can be written as a finite composition of flows of complete vector fields that are generators of the Lie algebra of holomorphic symplectic resp. Hamiltonian vector fields on M.

On a Stein manifold M the ordinary cohomology $H^1(M, \mathbb{C})$ with coefficients in $\mathbb{C}$ is isomorphic to the holomorphic de Rham cohomology

$$\ker(\mathrm{d}|_{\Omega^1(M)}) / \operatorname{Im}(\mathrm{d}|_{\mathcal{O}(M)})$$

where $\Omega^p(M)$ denotes the sheaf of holomorphic p-forms on M, see [GR79, p. 155]. Hence the condition $H^1(\Omega, \mathbb{C}) = 0$ and Ω being Stein ensure that every symplectic vector field on $\varphi_t(\Omega)$ is Hamiltonian.

Proof The proof is based on the one for DP, see e.g. [Rit13, Appendix]. The part where adjustment is needed is basically the same as for the volume preserving case, namely when the Runge property is used, see Kaliman and Kutzschebauch [KK11, Remark 2.2]; We only need to replace the n-form by the closed 2-form ω. Since we require the first holomorphic de Rham cohomology to vanish, we can assume that any symplectic vector field on Ω is Hamiltonian. This is analogous to the case of the volume density property where the n-form induces a correspondence between vector fields and $(n-1)$-forms through $i_V\omega = \eta$. In that case, $n \geq 2$ and $H^{n-1}(\Omega, \mathbb{C}) = 0$ is required in order to write every closed $(n-1)$-form η as the exterior derivative of an $(n-2)$-form ζ, and then Runge approximation is used for ζ. In our case, this is simply the correspondence between the Hamiltonian function and its Hamiltonian vector field. Therefore we can directly use Runge approximation for the Hamiltonian function, which by Cauchy estimate implies the approximation for its vector field. $\square$

Corollary 2.47 *Let M be a Stein manifold with a holomorphic symplectic form ω. Assume that (M, ω) has the symplectic density property. Then*

(a) Finitely many complete holomorphic symplectic vector fields span the tangent bundle of M.

(b) The group $\mathrm{Aut}_\omega(M)$ of holomorphic symplectic automorphisms acts transitively on M.

(c) The group $\mathrm{Aut}_\omega(M)$ of holomorphic symplectic automorphisms acts k-transitively on M for all $k \in \mathbb{N}$.

Proof For (a) and (b) it is the same as the DP case in Proposition 2.29. Note that $\mathrm{Aut}_\omega(M)$ is closed in $\mathrm{Aut}(M)$.

(c) Cf. [KK11, Remark 2.2]. We need the symplectic version of [Var00, Theorem 2]:

> Let M be a symplectic Stein manifold with SDP, K a compact set in M, $p, q \in M$ two points outside the $\mathcal{O}(X)$-hull $\widehat{K}$ of K, and $x_1, \ldots, x_m \in K$. Then there exists $\Phi \in \mathrm{Aut}_\omega(M)$ fixing $x_i, i = 1, \ldots, m$, $\Phi(p) = q$, and close to the identity on K.

Let γ be a piecewise analytic path connecting p and q such that $\gamma \cap \widehat{K} = \emptyset$. Also let $K_1 = K \cup U$, where U is a small ball around q, and set $\gamma_1 = U \cap \gamma$. In the construction of the automorphism Φ for DP in [Var00, §3], the step where adjustment is needed is a global approximation on M of a holomorphic vector field V on K_1 that is identically zero on $\widehat{K}$ and tangent to γ_1. In the symplectic case the global approximation for V must also be symplectic, thus using the Runge property directly does not work. However, as mentioned in the proof of Theorem 2.46, such an approximation of vector field is equivalent to an approximation of the corresponding Hamiltonian function, using $H^1(U, \mathbb{C}) = 0$. Since K_1 is still $\mathcal{O}(M)$-convex, the Runge property applies to the Hamiltonian function of V on K_1. Now proceed as in the DP case. □

2.4.1 The standard example

Let $(x, y) \in \mathbb{C}^n \times \mathbb{C}^n \cong \mathrm{T}^*\mathbb{C}^n$ and let $\omega = \sum_{k=1}^n dx_k \wedge dy_k$ be the canonical symplectic form on $\mathrm{T}^*\mathbb{C}^n$. The alternating bilinear form on $\mathbb{C}^{2n}$ defined as

$$\tilde{\omega}(u, v) = \sum_{j=1}^{n} u_j v_{n+j} - u_{n+j} v_j, \quad u, v \in \mathbb{C}^{2n}$$

is the standard linear symplectic form on $\mathbb{C}^{2n}$ whose corresponding differential form is ω.

Forstnerič proved the following algebraic Hamiltonian density property in 1996, which is the first known example in the symplectic category.

Proposition 2.48 *[For96, Proposition 5.2] The Lie algebra of polynomial Hamiltonian vector fields on $(\mathbb{C}^{2n}, \omega)$ is generated by complete Hamiltonian vector fields of the form*

$$V(z) = f(\tilde{\omega}(z, v)) \sum_{j=1}^{2n} v_j \frac{\partial}{\partial z_j}, \quad z, v \in \mathbb{C}^{2n}, f \in \mathbb{C}[\mathbb{C}^{2n}].$$

Note that these polynomial vector fields generate the flow

$$\Phi_t(z) = z + t f(\tilde{\omega}(z, v)) v, \quad t \in \mathbb{C}, z \in \mathbb{C}^{2n}$$

which are polynomial symplectic shears of $(\mathbb{C}^{2n}, \omega)$.

In combination with Theorem 2.46, this proposition implies the following.

Corollary 2.49 *[For96, Theorem 5.1] Every holomorphic symplectic automorphism of $\mathrm{T}^*\mathbb{C}^n$ can be approximated uniformly on compacts by compositions of symplectic shears.*

Chapter 3

A criterion for density property

This chapter is organized as follows. In Section 3.1, we prove Theorem 1. The basic idea is to find a suitable $\mathcal{O}(X)$-submodule L of $\mathrm{VF}(X)$, the Lie algebra of all holomorphic vector fields on X, such that L is contained in the closure $\overline{\mathrm{Lie}(X)}$ of the Lie algebra $\mathrm{Lie}(X)$ generated by the $\mathbb{C}$-complete holomorphic vector fields on X. In Subsection 3.1.1, we prove an algebraic version of Theorem 1.

In Section 3.2, we turn our attention to the proof of Theorems 2 and 3.

In Section 3.3, we consider applications of Theorems 1 and 2. First, we give a new proof for the fact that the Danielewski surfaces have the algebraic density property. And then, we prove the density property for the manifolds arising in the Gromov–Vaserstein fibration. These manifolds are in the symplectic case given as the common zero set of an arbitrarily large number of polynomial equations. This makes it impossible to prove the density property by direct calculation as in the groundbreaking works of Andersén–Lempert [AL92] and Varolin [Var01].

3.1 Generalization of the Kaliman–Kutzschebauch criterion

Let X be a complex manifold. We let $\mathrm{VF}(X)$ denote the Lie algebra of all holomorphic vector fields on X. Let $\mathrm{Lie}(X)$ denote the Lie subalgebra of $\mathrm{VF}(X)$ generated by the set of $\mathbb{C}$-complete holomorphic vector fields on X.

We now generalize the idea of compatible pairs in [KK08a].

Definition 3.1 Let $S \subset \mathrm{VF}(X)$ be a finite non-empty set. An **S-admissible** graph is a triple (G, π, ε), where

(i) G is a directed graph

(ii) $\pi\colon \mathrm{Vert}(G) \to S$ is a bijective map

(iii) $\varepsilon\colon \mathrm{Edge}(G) \to \mathcal{O}(X)$ such that

$$\varepsilon(v, w) \in \big(\ker \pi(v)^2 \setminus \ker \pi(v)\big) \cap \ker \pi(w)$$

for all $(v, w) \in \mathrm{Edge}(G)$.

Convention:
Given $S \subset \mathrm{VF}(X)$ and an S-admissible graph (G, π, ε), we write θ for both the vector field $\theta \in S$ and the vertex $\pi^{-1}(\theta) \in \mathrm{Vert}(G)$. Moreover, if $\varepsilon(\theta, \varphi) \in (\ker \theta^2 \setminus \ker \theta) \cap \ker \varphi$, we write

$$\theta \xrightarrow{a_\theta} \varphi,$$

where $a_\theta := \varepsilon(\theta, \varphi)$. And finally, we sometimes write f^θ for holomorphic functions with $f^\theta \in \ker \theta$.

Definition 3.2 An ordered n-tuple $(\theta_1, \ldots, \theta_n)$ of complete holomorphic vector fields on X is called **compatible n-tuple** (of holomorphic vector fields on X) if

1. there exists a finite rooted S-admissible tree (G, π, ε) with $\theta_1 = \pi(\text{root})$ and $S = \{\theta_1, \ldots, \theta_n\}$,
2. there exists a non-zero ideal $I \subset \mathcal{O}(X)$ such that

$$I \subset \overline{\operatorname{span}\left(\prod_{i=1}^{n} \ker \theta_i\right)}.$$

The following theorem is an important part of the proof of Theorem 1.

Theorem 3.3 *Let $\theta_1, \ldots, \theta_n$ be complete holomorphic vector fields on a Stein manifold X. If $(\theta_1, \ldots, \theta_n)$ is a compatible n-tuple, then there exists an $\mathcal{O}(X)$-submodule L of* $\mathrm{VF}(X)$ *with $L \subset \overline{\mathrm{Lie}(X)}$.*

Lemma 3.4 *Let θ be a complete holomorphic vector field on X and $\varphi \in \mathrm{Lie}(X)$ such that $a\varphi \in \mathrm{Lie}(X)$ for some $a \in \mathcal{O}(X)$ with $a \in \ker \theta$. Then we have*

$$[a\theta, \varphi] - [\theta, a\varphi] = -\varphi(a) \cdot \theta \in \mathrm{Lie}(X).$$

Proof For any $f \in \mathcal{O}(X)$ we get

$$\begin{aligned}[a\theta, \varphi](f) &= a\theta(\varphi(f)) - \varphi(a\theta(f)) \\ &= a\theta(\varphi(f)) - \varphi(a) \cdot \theta(f) - a\varphi(\theta(f))\end{aligned}$$

and

$$\begin{aligned}[\theta, a\varphi](f) &= \theta(a\varphi(f)) - a\varphi(\theta(f)) \\ &= \underbrace{\theta(a)}_{=0} \varphi(f) + a\theta(\varphi(f)) - a\varphi(\theta(f)) \\ &= a\theta(\varphi(f)) - a\varphi(\theta(f)).\end{aligned}$$

By assumption, φ and $a\varphi$ are in $\mathrm{Lie}(X)$. Moreover, since θ is complete and $a \in \ker \theta$, the vector fields θ and $a\theta$ are also in $\mathrm{Lie}(X)$. Therefore

$$\begin{aligned}[a\theta, \varphi] - [\theta, a\varphi] &= a\theta \circ \varphi - \varphi(a) \cdot \theta - a\varphi \circ \theta - a\theta \circ \varphi + a\varphi \circ \theta \\ &= -\varphi(a) \cdot \theta \in \mathrm{Lie}(X)\end{aligned}$$

and this proves the claim. □

Lemma 3.5 *Let $S = S_0 \cup \{\theta\}$ be a finite set of vector fields on X and (T, π, ε) an S-admissible rooted tree with $\pi(\mathrm{root}) = \theta$. Assume there is a holomorphic function $a_\theta \in \ker \theta^2 \setminus \ker \theta$. Then $(T, \tilde{\pi}, \varepsilon)$ is S'-admissible, where $S' = S_0 \cup \{a_\theta \theta\}$ and*

$$\tilde{\pi}(v) = \begin{cases} \pi(v) & v \neq \mathrm{root}, \\ a_\theta \theta & v = \mathrm{root}. \end{cases}$$

Proof Clearly, $\tilde{\pi}\colon \mathrm{Vert}(T) \to S'$ is bijective. By definition of $\tilde{\pi}$, it remains to show that

$$\varepsilon(v, \mathrm{root}) \in \left(\ker \tilde{\pi}(v)^2 \setminus \ker \tilde{\pi}(v)\right) \cap \ker(a_\theta \theta).$$

And that is true because $\ker \theta = \ker(a_\theta \theta)$. □

Let S be a finite non-empty set of vector fields on X and (T, π, ε) an S-admissible rooted tree. For $\theta \in S$, let T_θ denote the subtree of T with root $\pi^{-1}(\theta)$.

Corollary 3.6 *Let S be a finite non-empty set of vector fields on X and (T, π, ε) an S-admissible rooted tree. Then $(T_\theta, \pi|_{\mathrm{Vert}(T_\theta)}, \varepsilon|_{\mathrm{Edge}(T_\theta)})$ is $\pi(\mathrm{Vert}(T_\theta))$-admissible.*

A vertex $\theta \in \mathrm{Vert}(T)$ is called a *leaf* if $\mathrm{Vert}(T_\theta) = \{\theta\}$.

Proposition 3.7 *Let S be a finite non-empty set of complete holomorphic vector fields on a Stein manifold X. Assume there is an S-admissible rooted tree (T, π, ε) with root V. Then*

$$\left(\prod_{\varphi \in \mathrm{Vert}(T) \setminus \{V\}} f_\varphi \varphi(a_\varphi) \right) f_V V \in \mathrm{Lie}(X).$$

Proof The proof is by induction on the depth of the tree. Let's start with a leaf $\theta \in \mathrm{Vert}(T)$. Then the subtree T_θ consists of only one vertex and no edges. Hence

$$\left(\prod_{\varphi \in \mathrm{Vert}(T_\theta) \setminus \{\theta\}} f_\varphi \varphi(a_\varphi) \right) f^\theta \theta = f^\theta \theta \in \mathrm{Lie}(X),$$

since $f^\theta \theta$ is complete. This proves the base case.

Now we consider a vertex $\theta \in \mathrm{Vert}(T)$ which is not a leaf. By the induction hypothesis, the proposition is true for all vertices $\psi \in \mathrm{Vert}(T_\theta)$ with $\psi \xrightarrow{a_\psi} \theta$, that is, we have

$$\tilde{\psi} := \left(\prod_{\varphi \in \mathrm{Vert}(T_\psi) \setminus \{\psi\}} f^\varphi \varphi(a_\varphi) \right) f^\psi \psi \in \mathrm{Lie}(X)$$

for all such vertices ψ. By Corollary 3.6, the subtree T_ψ is $S' := \pi(\mathrm{Vert}(T_\psi))$-admissible with root ψ. Moreover, there is a holomorphic function

$$a_\psi \in \left(\ker \psi^2 \setminus \ker \psi\right) \cap \ker \theta$$

by assumption. We can therefore apply Lemma 3.5 to conclude that the subtree T_ψ is S''-admissible, where S'' is obtained by replacing ψ with $a_\psi \psi$. By the induction hypothesis, we get

$$a_\psi \tilde{\psi} \in \mathrm{Lie}(X)$$

for all vertices ψ with $\psi \xrightarrow{a_\psi} \theta$.

In the next step, we let $\{\psi_1, \ldots, \psi_N\}$ denote the set of vertices $\psi \in \text{Vert}(T_\theta)$ such that $\psi \xrightarrow{a_\psi} \theta$. Define

$$W_1 := [a_{\psi_1}\tilde{\psi}_1, f^\theta\theta] - [\tilde{\psi}_1, a_{\psi_1}f^\theta\theta] = \tilde{\psi}_1(a_{\psi_1})f^\theta\theta,$$

for an arbitrary holomorphic function f^θ with $f^\theta \in \ker\theta$. Observe that $\tilde{\psi}_1, a_{\psi_1}\tilde{\psi}_1 \in \text{Lie}(X)$ and $f^\theta\theta, a_{\psi_1}f^\theta\theta$ are even complete, since $f^\theta, a_{\psi_1}f^\theta \in \ker\theta$. Lemma 3.4 implies therefore $W_1 \in \text{Lie}(X)$ for every $f^\theta \in \ker\theta$. In particular, we have $fW_1 \in \text{Lie}(X)$ for all $f \in \ker\theta$. We continue and define

$$W_k := [a_{\psi_k}\tilde{\psi}_k, W_{k-1}] - [\tilde{\psi}_k, a_{\psi_k}W_{k-1}]$$

for $2 \leq k \leq N$. A repeated application of the same reasoning as for W_1 implies that

$$W_k = \pm\tilde{\psi}_1(a_{\psi_1}) \cdots \tilde{\psi}_k(a_{\psi_k})f^\theta\theta \in \text{Lie}(X), \quad 1 \leq k \leq N.$$

It should be mentioned that we can ignore the sign without loss of generality, since -1 can be absorbed by f^θ. In summary, we conclude that

$$\begin{aligned} W_N &= \left(\prod_{k=1}^N \tilde{\psi}_k(a_{\psi_k})\right) f^\theta\theta \\ &= \left(\prod_{k=1}^N f^{\psi_k}\psi_k(a_{\psi_k}) \prod_{\varphi\in\text{Vert}(T_{\psi_k})\setminus\{\psi_k\}} f^\varphi\varphi(a_\varphi)\right) f^\theta\theta \\ &= \left(\prod_{\varphi\in\text{Vert}(T_\theta)\setminus\{\theta\}} f^\varphi\varphi(a_\varphi)\right) f^\theta\theta \in \text{Lie}(X). \end{aligned}$$

Since we chose an arbitrary θ, the proposition follows with $\theta = V$. □

Proof (Proof of Theorem 3.3) Let $(\theta_1, \ldots, \theta_n)$ be a compatible n-tuple of holomorphic vector fields on X and write $S = \{\theta_1, \ldots, \theta_n\}$. By assumption, there exists an S-admissible tree (T, π, ε) with $\pi(\text{root}) = \theta_1$ and a non-zero ideal $I \subset \overline{\text{span}(\prod_{\theta\in\text{Vert}(T)}} \ker\theta)$. By Proposition 3.7, we have

$$\left(\prod_{\varphi\in\text{Vert}(T)\setminus\{\theta_1\}} f^\varphi\varphi(a_\varphi)\right) f^{\theta_1}\theta_1 \in \text{Lie}(X)$$

and therefore

$$\left(\prod_{\varphi\in\text{Vert}(T)\setminus\{\theta_1\}} \varphi(a_\varphi)\right) \text{span}\left(\prod_{\theta\in\text{Vert}(T)} \ker\theta\right) \theta_1 \subset \text{Lie}(X).$$

Thus, since $I \subset \overline{\text{span}(\prod_{\theta\in\text{Vert}(T)} \ker\theta)}$,

$$L := \left(\prod_{\varphi\in\text{Vert}(T)\setminus\{\theta_1\}} \varphi(a_\varphi)\right) I\theta_1 \subset \overline{\text{Lie}(X)}$$

is the desired $\mathcal{O}(X)$-submodule. Recall that the closures are taken w.r.t. to the topology of locally uniform convergence. □

Theorem 3.8 (Theorem 1) *Let X be a homogeneous Stein manifold with finitely many compatible n-tuples $\{(\theta_{k,1}, \ldots, \theta_{k,n})\}_{k=1}^{m}$ of holomorphic vector fields on X such that for some $x_0 \in X$, $\{(\theta_{k,1})_{x_0}\}_{k=1}^{m} \subset T_{x_0}X$ is a generating set. Then X has the density property.*

Proof By Theorem 3.3, there exist non-zero ideals I_k such that $\mathrm{Lie}(X)$ contains the submodules $L_k = I_k\theta_{k,1}$. Hence, there exists a non-zero ideal $J \subset \mathcal{O}(X)$ such that $\mathrm{Lie}(X)$ contains the submodule

$$L = \left\{ \sum_{k=1}^{m} \alpha_k \theta_{k,1} : \alpha_1, \ldots, \alpha_m \in J \right\}.$$

Since $\{(\theta_{k,1})_{x_0}\}_{k=1}^{m}$ remains a generating set under small perturbations of the base point x_0, we can suppose that x_0 does not belong to the zero locus of J. For such x_0 the set $\{V_{x_0} : V \in L\}$ contains a generating set, and by Proposition 2.37 it follows that X has DP. $\square$

3.1.1 Algebraic version of Theorem 1

Let X be a complex affine algebraic manifold. Recall that $\mathrm{VF}_{\mathrm{alg}}(X)$ denote the Lie algebra of all algebraic vector fields on X. Also $\mathrm{Lie}_{\mathrm{alg}}(X)$ denotes the Lie subalgebra of $\mathrm{VF}_{\mathrm{alg}}(X)$ generated by the set of $\mathbb{C}$-complete algebraic vector fields on X. Recall that an algebraic manifold X has ADP if $\mathrm{Lie}_{\mathrm{alg}}(X) = \mathrm{VF}_{\mathrm{alg}}(X)$.

We now recall the following criterion.

Theorem 3.9 *[KK08a, Theorem 1] Let X be a homogeneous (with respect to the group of algebraic automorphisms $\mathrm{Aut}_{\mathrm{alg}}(X)$) affine algebraic manifold. Assume there is a $\mathbb{C}[X]$-submodule L of $\mathrm{VF}_{\mathrm{alg}}(X)$ such that L is contained in $\mathrm{Lie}_{\mathrm{alg}}(X)$. If the fiber $L_p = \{V_p : V \in L\} \subset \mathrm{T}_pX$ over some point $p \in X$ contains a generating set, then X has the algebraic density property.*

Note that the notions of compatible n-tuples and admissible graphs have algebraic analogues.

Definition 3.10 Let $S \subset \mathrm{VF}_{\mathrm{alg}}(X)$ be a finite non-empty set. An S-admissible graph is a triple (G, π, ε), where

(i) G is a directed graph

(ii) $\pi\colon \mathrm{Vert}(G) \to S$ is a bijective map

(iii) $\varepsilon\colon \mathrm{Edge}(G) \to \mathbb{C}[X]$ such that

$$\varepsilon(v, w) \in \left(\ker \pi(v)^2 \setminus \ker \pi(v)\right) \cap \ker \pi(w)$$

for all $(v, w) \in \mathrm{Edge}(G)$.

Definition 3.11 An ordered n-tuple $(\theta_1, \ldots, \theta_n)$ of complete algebraic vector fields on X is a compatible n-tuple (of algebraic vector fields on X) if

(i) there exists a finite rooted S-admissible tree (G, π, ε) with $\theta_1 = \pi(\mathrm{root})$ with $S = \{\theta_1, \ldots, \theta_n\}$,

(ii) there is a non-zero ideal $I \subset \mathbb{C}[X]$ with

$$I \subset \operatorname{span}\left(\prod_{i=1}^{n} \ker \theta_i\right).$$

The following two results are algebraic versions of Proposition 3.7 and Theorem 3.3.

Proposition 3.12 *Let S be a finite non-empty set of complete algebraic vector fields on an algebraic manifold X. Assume there is an S-admissible rooted tree (T, π, ε) with root V. Then*

$$\left(\prod_{\varphi \in \mathrm{Vert}(T)\setminus\{V\}} f_\varphi \varphi(a_\varphi)\right) f_V V \in \mathrm{Lie}_{\mathrm{alg}}(X).$$

Proof Observe that the proofs of Lemma 3.4, Lemma 3.5, Corollary 3.6 and Proposition 3.7 apply in the algebraic setting as well, since we just use that the vector fields form a Lie algebra and also module over the functions. We therefore simply replace the Stein manifold by an algebraic manifold X, holomorphic functions in $\mathcal{O}(X)$ by regular functions in $\mathbb{C}[X]$ and holomorphic vector fields by algebraic vector fields. □

Theorem 3.13 *Let $\theta_1, \ldots, \theta_n$ be complete algebraic vector fields on an algebraic manifold X. If $(\theta_1, \ldots, \theta_n)$ is a compatible n-tuple, then there exists $\mathbb{C}[X]$-submodule L of $\mathrm{VF}_{\mathrm{alg}}(X)$ with $L \subset \mathrm{Lie}_{\mathrm{alg}}(X)$.*

Proof The proof is similar to the one of Theorem 3.3. Let $(\theta_1, \ldots, \theta_n)$ be a compatible n-tuple of algebraic vector fields on X and write $S = \{\theta_1, \ldots, \theta_n\}$. Then there exists an S-admissible tree (T, π, ε) with $\pi(\mathrm{root}) = \theta_1$ and a non-zero ideal $I \subset \operatorname{span}(\prod_{\theta \in \mathrm{Vert}(T)} \ker \theta)$. By Proposition 3.12, we have

$$\left(\prod_{\varphi \in \mathrm{Vert}(T)\setminus\{\theta_1\}} f^\varphi \varphi(a_\varphi)\right) f^{\theta_1} \theta_1 \in \mathrm{Lie}_{\mathrm{alg}}(X)$$

and as in the proof of Theorem 3.8 we conclude that

$$L := \left(\prod_{\theta \in \mathrm{Vert}(T)\setminus\{\theta_1\}} \theta(a_\theta)\right) I\theta_1 \subset \mathrm{Lie}_{\mathrm{alg}}(X)$$

is the desired $\mathbb{C}[X]$-submodule. □

Theorem 3.14 *Let X be a homogeneous affine algebraic manifold with finitely many compatible n-tuples $\{(\theta_{k,1}, \ldots, \theta_{k,n})\}_{k=1}^m$ such that for some $x_0 \in X$, $\{(\theta_{k,1})_{x_0}\}_{k=1}^m \subset \mathrm{T}_{x_0}X$ is a generating set. Then X has the algebraic density property.*

Proof By Theorem 3.13, there exist non-zero ideals I_k such that $\mathrm{Lie}_{\mathrm{alg}}(X)$ contains the submodules $L_k = I_k \theta_{k,1}$. Hence, there exists a non-zero ideal $J \subset \mathbb{C}[X]$ such that $\mathrm{Lie}_{\mathrm{alg}}(X)$ contains the submodule

$$L = \left\{\sum_{k=1}^{m} \alpha_k \theta_{k,1} : \alpha_1, \ldots, \alpha_m \in J\right\}.$$

Since $\{(\theta_{k,1})_{x_0}\}_{k=1}^m$ remains a generating set under small perturbations of the base point x_0, we can suppose that x_0 does not belong to the zero locus of J. For such x_0 the set $\{V_{x_0} : V \in L\}$ contains a generating set. Therefore X has ADP by Theorem 3.9. $\square$

3.2 Holomorphic flexibility and density property

In this section we prove Theorems 2 and 3. The next lemma can be found in [Arz+13, Lemma 4.1].

Lemma 3.15 *Let V be a complete vector field on a complex manifold X, $f \in \mathcal{O}(X)$ a function in the kernel of V and $x \in X$ a point with $f(x) = 0$. Denote the flow of the complete field fV at time t by φ_t (fixing x by our assumption). Then the differential $\mathrm{d}_x\varphi_t$, which is an endomorphism of T_xX, acts on a tangent vector $W \in \mathrm{T}_xX$ as follows*

$$W \mapsto W + t \cdot \mathrm{d}_x f(W) \cdot V_x.$$

We are ready to prove Theorem 3, which extends a result by Varolin, who proved it in the special case when X is a Stein complex Lie group [Var01].

Theorem 3 Suppose X is a holomorphically flexible Stein manifold. Then $X \times \mathbb{C}$ has the density property.

Proof Let us denote the coordinate on $\mathbb{C}$ by t and suppose that V is a complete vector field on X. Then $(V, \partial/\partial t)$ is a compatible pair on $X \times \mathbb{C}$ (where we denote the obvious extension of V to $X \times \mathbb{C}$ again by V). Indeed, $\ker V$ contains all functions of t and $\ker(\partial/\partial t)$ contains all functions on X and the function $a \in \mathcal{O}(X \times \mathbb{C})$ defined by $a := t$ is of degree 1 with respect to $\partial/\partial t$ and in $\ker V$.

If V_i is a spanning set of complete holomorphic vector fields on X, the relevant vectors in the compatible pairs $(V_i, \partial/\partial t)$ span $\mathrm{T}_xX \subset \mathrm{T}(X \times \mathbb{C})$ at any point x. We create a new compatible pair by applying α^* to the pair $(V, \partial/\partial t)$ for a suitable automorphism $\alpha \in \mathrm{Aut}(X \times \mathbb{C})$. We choose a point $x \in X$ where $V(x) \neq 0$ and a holomorphic function $f \in \mathcal{O}(X)$ with $f(x) = 0$ and $\mathrm{d}_x f(V) \neq 0$, which is possible on Stein manifolds by a standard application of Cartan's Theorem B. Now, let α be the time-one flow of the complete field $f \cdot \partial/\partial t$. By Lemma 3.15 we have

$$\alpha^*(V)(x,t) = V(x,t) + \mathrm{d}_x f(V) \frac{\partial}{\partial t}$$

and together with other compatible pairs $(V_i, \partial/\partial t)$ we have a spanning set of $\mathrm{T}_{(x,0)}(X \times \mathbb{C})$.

Note that $X \times \mathbb{C}$ is homogeneous under its group of holomorphic automorphisms, since products of homogeneous manifolds are again homogeneous. Now Proposition 2.37 implies the claim. $\square$

Lemma 3.16 *Let X be a holomorphically flexible Stein manifold and $(\theta_1, \ldots, \theta_n)$ a compatible n-tuple. Assume that there are complete holomorphic fields $V_1, V_2, \ldots, V_N$ which* *span the tangent space* $\mathrm{T}_x X$ *at a point* $x \in X$ *and admit functions* $f_i \in \ker V_i$ *with* $\mathrm{d}_x f_i(\theta_1) \neq 0$ *for* $i = 1, \dots, N$. *Then there is a non-trivial* $\mathcal{O}(X)$*-submodule* L *of* $\mathrm{VF}(X)$ *such that* $L \subset \overline{\mathrm{Lie}(X)}$ *and the fiber* $L_x = \{V_x : V \in L\}$ *contains a generating set.*

Proof Let $J \subset \mathcal{O}(X)$ be the ideal of the compatible n-tuple $(\theta_1, \dots, \theta_n)$. We can assume that the point x is not in the zero locus of J, since spanning the tangent space is an open condition. Theorem 3.3 implies that the compatible n-tuple yields a holomorphic function $f \colon X \to \mathbb{C}$ such that the submodule $L := J \cdot (f\theta_1)$ is contained in $\overline{\mathrm{Lie}(X)}$.

Without loss of generality, we may assume that f_i is zero at x by adding a constant function to it. Let φ_t^i denote the flow of the complete holomorphic vector field $f_i V_i$. Then let $L_i = (\varphi_t^i)^*(L)$ be the submodules obtained by pulling back L by φ_t^i. Observe that the sum $\tilde{L} := L_1 + \ldots + L_N$ is a submodule contained in $\overline{\mathrm{Lie}(X)}$.

By assumption, the point x is fixed by the flow φ_t^i. Hence the differential $\mathrm{d}_x \varphi_t^i$ as an endomorphism of $\mathrm{T}_x X$ acts as

$$W \mapsto W + t \cdot \mathrm{d}_x f(W)(V_i)_x$$

by Lemma 3.15. For $W := (\theta_1)_x$, these image vectors span $\mathrm{T}_x X$ for general t, since the V_i's span $\mathrm{T}_x X$ and $\mathrm{d}_x f_i(W) \neq 0$ by assumption. This proves that the fiber $\tilde{L}_x$ contains a generating set. □

As a corollary we have

Theorem 2 Let X be a Stein manifold and $(\theta_1, \dots, \theta_n)$ a compatible n-tuple. Assume that there are $\mathbb{C}$-complete holomorphic vector fields $V_1, \dots, V_N$ on X which span the tangent bundle $\mathrm{T}X$. If there are functions $f_i \in \ker V_i$ such that $\mathrm{d}_{x_0} f_i(\theta_1) \neq 0$ for some point $x_0 \in X$ and $i = 1, \dots, N$, then X has the density property.

Proof The fact that the complete fields span all tangent spaces implies holomorphic flexibility. Thus Proposition 2.37 implies the result. □

Corollary 3.17 (Theorem 2 - algebraic version) *Let* X *be an affine algebraic manifold and* $(\theta_1, \dots, \theta_n)$ *a compatible* n*-tuple of algebraic vector fields. Assume that there are* $\mathbb{C}$*-complete algebraic vector fields* $V_1, \dots, V_N$ *on* X *with algebraic flows (so-called LNDs) such that the collection* $\{\theta_1, V_1, \dots, V_N\}$ *spans the tangent bundle* $\mathrm{T}X$. *If there exist regular functions* $f_i \in \ker V_i$ *such that* $\mathrm{d}_{x_0} f_i(\theta_1) \neq 0$ *for some point* $x_0 \in X$ *and* $i = 1, \dots, N$, *then* X *has the algebraic density property.*

Proof The idea of the proof is similar to the holomorphic case (cf. Lemma 3.16). We only have to show that the flow of $f_i V_i$ is algebraic. Let $\varphi_i(x, t)$ denote the flow of V_i. Then the flow of $f_i V_i$ is $\varphi_i(x, f_i(x)t)$ and thus algebraic. □

3.3 Applications

3.3.1 Danielewski surfaces

Given a polynomial $p \colon \mathbb{C} \to \mathbb{C}$ with simple zeros, we define the variety

$$D_p := \{(x, y, z) \in \mathbb{C}^3 : xy = p(z)\}$$

called *Danielewski surface*. This is an algebraic manifold, since p has only simple zeros. Furthermore, it is well-known that D_p has the algebraic density property [KK08b, Theorem 1] despite the lack of compatible pairs of complete algebraic vector fields. In this section we give a new, shorter proof of this fact. We show the existence of a compatible 3-tuple and apply the generalized Kaliman–Kutzschebauch criterion (Theorem 2).

The following three are complete algebraic vector fields tangent to the surface D_p:

$$\begin{aligned}\theta_1 &= x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\\ \theta_2 &= p'(z)\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\\ \theta_3 &= p'(z)\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}.\end{aligned}$$

Lemma 3.18 *The vector fields θ_1, θ_2 and θ_3 form a compatible 3-tuple $(\theta_1, \theta_2, \theta_3)$.*

Proof Define the functions $f_x(x,y,z) = x$, $f_y(x,y,z) = y$ and $f_z(x,y,z) = z$. We see that $f_z \in \ker \theta_1$ and

$$f_z \in \ker \theta_i^2 \setminus \ker \theta_i$$

for $i = 2, 3$. Hence there exists a $\{\theta_1, \theta_2, \theta_3\}$-admissible rooted tree (T, π, ε) with root θ_1. More precisely, the map $\varepsilon : \mathrm{Edge}(T) \to \mathbb{C}[D_p]$ can be defined by

$$\varepsilon(\theta_2, \theta_1) := f_z, \quad \varepsilon(\theta_3, \theta_1) := f_z.$$

Moreover, we have $x \in \ker \theta_3, y \in \ker \theta_2$ and $z \in \ker \theta_1$, which shows the existence of a non-zero ideal in

$$\mathrm{span}\,(\ker \theta_1 \cdot \ker \theta_2 \cdot \ker \theta_3)$$

and this finishes the proof. □

Lemma 3.19 *The vector fields θ_1, θ_2 and θ_3 span the tangent bundle $\mathrm{T}D_p$.*

Proof Let $u = (x,y,z) \in D_p$ be a point with $p'(z) \neq 0$. Then the vector fields θ_2 and θ_3 span the tangent space $T_u D_p$. It remains to consider $v = (x,y,z) \in D_p$ with $p'(z) = 0$. Observe that we have $p(z) \neq 0$ for such points, since p has only simple zeros. This implies $xy \neq 0$ and, in particular, $x \neq 0$ and $y \neq 0$. Therefore, the vector fields θ_1 and θ_2 span the tangent space $\mathrm{T}_v D_p$. □

The following is [KK08b, Theorem 1] in the special case where $n = 1$.

Corollary 3.20 *The Danielewski surface D_p has the algebraic density property.*

Proof The conditions of Corollary 3.17 are satisfied by Lemmas 3.18, 3.19 and 3.26. Therefore D_p has ADP. Note that in Corollary 3.17 the algebraic vector field θ_1 need not have algebraic flow. □

3.3.2 Preparation for Gromov–Vaserstein fibers

In this section we take a look at two factorization problems or rather, at a by-product of their proofs. Every holomorphic mapping $f: \mathbb{C}^l \to \mathrm{SL}_n(\mathbb{C})$ can be written as a finite product

$$f = M_1 \cdots M_K,$$

where $M_i: \mathbb{C}^l \to \mathrm{SL}_n(\mathbb{C})$ is a holomorphic mapping of the respective form

$$\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ \star & & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & & \star \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

for i odd and i even, respectively. This was proved by Ivarsson and Kutzschebauch [IK12a].

Similarly, every holomorphic mapping

$$f: \mathbb{C}^l \to \mathrm{Sp}_{2n}(\mathbb{C})$$

can be factorized into a finite product

$$f = N_1 \cdots N_K,$$

where $N_i: \mathbb{C}^l \to \mathrm{Sp}_{2n}(\mathbb{C})$ is a holomorphic mapping of the respective form

$$\begin{pmatrix} I_n & 0 \\ A & I_n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} I_n & A \\ 0 & I_n \end{pmatrix},$$

where I_n denotes the $n \times n$-identity matrix, 0 the $n \times n$-zero matrix and A is a symmetric $n \times n$-matrix, i.e. $A^T = A$. This result can be found in [Sch25].

Both proofs have in common that a suitable polynomial mapping

$$P = (P_1, \ldots, P_k): \mathbb{C}^m \to \mathbb{C}^k, \quad m > k$$

satisfies *nice enough* properties that justify an application of the Oka principle, cf. Section 2.2 for how to apply the Oka principle in the $\mathrm{SL}_n(\mathbb{C})$ case. Here we are only interested in the smooth fibers $P^{-1}(y)$ of these mappings.

Lemma 3.21 *Each smooth fiber $P^{-1}(y)$ is biholomorphic to a product $\mathcal{G} \times \mathbb{C}^L$, where*

$$\mathcal{G} = \{z = (z_1, \ldots, z_m) \in \mathbb{C}^m : p(z) = 0\}$$

is a smooth variety for some polynomial mapping $p = (p_1, \ldots, p_l): \mathbb{C}^m \to \mathbb{C}^l$, $m > l$, and some positive integer L.

We have $l = n$ in the symplectic case and $l = 1$ in the case of the special linear group.

Proof In the case of the special linear group, this statement is implied by the proof of [IK12a, Lemma 3.7]. And in the symplectic case, it follows by Lemma 3.14 and Lemma 3.15 in [Sch25]. □

For $1 \leq i_0 < \cdots < i_l \leq m$, we define the $(l+1)$-tuple of variables $y = (z_{i_0}, \ldots, z_{i_l})$ and the corresponding vector field

$$D_y(p) := \det \begin{pmatrix} \frac{\partial}{\partial z_{i_0}} & \cdots & \frac{\partial}{\partial z_{i_l}} \\ \frac{\partial}{\partial z_{i_0}} p_1(z) & \cdots & \frac{\partial}{\partial z_{i_l}} p_1(z) \\ \vdots & & \vdots \\ \frac{\partial}{\partial z_{i_0}} p_l(z) & \cdots & \frac{\partial}{\partial z_{i_l}} p_l(z) \end{pmatrix}. \tag{3.1}$$

Lemma 3.22 *The vector fields of the form (3.1) are tangent to $\mathcal{G}$.*

Proof Observe that $D_y(p)(p_i) = 0$, since the first row equals the $(i+1)$-th row. □

Let $\mathcal{T} = \{(z_{i_0}, \ldots, z_{i_l}) : 1 \leq i_0 < \cdots < i_l \leq m\}$ be the set of all $(l+1)$-tuples and $\mathcal{V} = \{D_y(p) : y \in \mathcal{T}\}$ the collection of vector fields of the form (3.1). Furthermore, let

$$\Gamma(\mathcal{V}) = \{\alpha^* V : \alpha \in \mathrm{Aut}(X), V \in \mathcal{V}\}$$

denote the set of vector fields generated by $\mathcal{V}$.

We shall sometimes write $\mathcal{G}_{sp}$ and $\mathcal{G}_{sl}$ for the respective varieties. Moreover, since we are only interested in smooth fibers and thus only in smooth varieties $\mathcal{G}$, we refrain from specifying this every time. We will see a classification of the smooth varieties in the next few subsections.

Proposition 3.23 *(i) There are complete holomorphic vector fields $V_1, \ldots, V_N \in \Gamma(\mathcal{V})$ spanning the tangent bundle $\mathrm{T}\mathcal{G}_{sl}$.*

(ii) There are complete holomorphic vector fields $V_1, \ldots, V_N \in \Gamma(\mathcal{V})$ spanning the tangent bundle $\mathrm{T}\mathcal{G}_{sp}$.

Proof (i) See Lemma 5.2 and Lemma 5.3 in [IK12a].
(ii) We refer to [Sch25, Theorem 3.36], which is actually a very difficult and technical proof, since it involves both abstract arguments and many concrete calculations.□

Corollary 3.24 *A smooth fiber $P^{-1}(y) \cong \mathcal{G} \times \mathbb{C}^L$ has the density property.*

Proof The variety $\mathcal{G}$ is holomorphically flexible by the previous lemma. Then, the claim follows by Theorem 3. □

We show that $\mathcal{G}$ also has the density property. The proof is based on an application of Theorem 2.

Theorem 3.25 *(i) There is a compatible m-tuple on $\mathcal{G}_{sl}$ for $n \geq 3$. In particular, $\mathcal{G}_{sl}$ has the density property.*

(ii) There is a compatible m-tuple on $\mathcal{G}_{sp}$ for $n \geq 2$. In particular, $\mathcal{G}_{sp}$ has the density property.

Proof In the following subsections we will show the existence of a compatible m-tuple for each case (Theorem 3.32 for the special linear and Theorem 3.36 for the symplectic case). Then the claims follow immediately from Theorem 2, Proposition 3.23 and Lemma 3.26 below. □

Lemma 3.26 *Let $x \in \mathcal{G}$ and $V \in \Gamma(\mathcal{V})$ with $V_x \neq 0$. Given any tangent vector*

$$W \in \mathrm{T}_x\mathcal{G} \setminus \operatorname{span}(V_x),$$

there is a holomorphic function $f \in \ker V$ with $f(x) = 0$ and $\mathrm{d}_x f(W) \neq 0$.

Proof Note that it suffices to show the claim for vector fields in $\mathcal{V}$, since the conclusion of the lemma is invariant under holomorphic automorphisms. Let $V = D_y(p)$ be a vector field of the form (3.1). Without loss of generality, we may assume that $y = (z_1, \dots, z_{k+1})$ is the corresponding $(k+1)$-tuple, that is, we have

$$V = \sum_{i=1}^{k+1} \alpha_i \frac{\partial}{\partial z_i},$$

where α_i, $i = 1, \dots, k+1$, are regular functions given by

$$\alpha_i = V(z_i) = \det \begin{pmatrix} \frac{\partial}{\partial z_1} p_1(z) & \cdots & \frac{\partial}{\partial z_{i-1}} p_1(z) & \frac{\partial}{\partial z_{i+1}} p_1(z) & \cdots & \frac{\partial}{\partial z_{k+1}} p_1(z) \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial}{\partial z_1} p_k(z) & \cdots & \frac{\partial}{\partial z_{i-1}} p_k(z) & \frac{\partial}{\partial z_{i+1}} p_k(z) & \cdots & \frac{\partial}{\partial z_{k+1}} p_k(z) \end{pmatrix}$$

By assumption, we have $V_x \neq 0$. We may therefore assume, again without loss of generality, that $\alpha_{k+1}(x) \neq 0$. Let $W \in \mathrm{T}_x X \setminus \operatorname{span}(V_x)$.

The vector fields $V_i := D_{y_i}(p)$ corresponding to the $(k+1)$-tuples $y_i = (z_1, \dots, z_k, z_i)$ with $i = k+2, \dots, m$, are given by

$$V_i = \alpha_{k+1} \frac{\partial}{\partial z_i} + \sum_{j=1}^{k} \tilde{\alpha}_{ij} \frac{\partial}{\partial z_j}$$

for some regular functions $\tilde{\alpha}_{ij}, i = k+2, \dots, m, j = 1, \dots, k$. Hence

$$V_x, (V_{k+2})_x, \dots, (V_m)_x$$

form a basis of $\mathrm{T}_x X$. Then

$$W = \lambda V_x + \sum_{i=k+2}^{m} \mu_i (V_i)_x$$

with $\mu_j \neq 0$ for some $j \in \{k+2, \dots, m\}$. Let π_j denote the projection to the j-th component. Then we set $f(z) := \pi_j(z - x) = z_j - x_j$. Observe that $f(x) = 0$ and $f \in \ker V$. It remains to show that $\mathrm{d}_x f(W) \neq 0$. By construction, $\mathrm{d}_x f$ is also the projection to component j, that is, $\mathrm{d}_x f(W) = \mu_j \neq 0$. □

3.3.3 The special linear case

In this subsection we find two complete holomorphic vector fields V_1 and V_2 on $\mathcal{G}$ that together form a compatible 2-tuple (V_1, V_2). For this reason we look more closely at the polynomial mapping $P\colon \mathbb{C}^m \to \mathbb{C}^k$. Recall that for odd K we denote elements in $\mathbb{C}^{n(n-1)/2}$ as

$$Z_K = (z_{K,21}, \dots, z_{K,kl}, \dots, z_{K,n(n-1)}), \quad k > l.$$

and set

$$M_K(Z_K) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ z_{K,21} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ z_{K,n1} & \cdots & z_{K,n(n-1)} & 1 \end{pmatrix}.$$

For even K we proceed similarly and write elements in $\mathbb{C}^{n(n-1)/2}$ as

$$Z_K = (z_{K,12}, \dots, z_{K,kl}, \dots, z_{K,(n-1)n}), \quad k < l,$$

and set

$$M_K(Z_K) = \begin{pmatrix} 1 & z_{K,12} & \cdots & z_{K,1n} \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & z_{K,(n-1)n} \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Take $m = \frac{n(n-1)}{2}$. For a fixed natural number K, the Gromov–Vaserstein fibration

$$P^K = (P_1^K, \dots, P_n^K) \colon (\mathbb{C}^m)^K \to \mathbb{C}^n$$

is given by

$$P^K(Z_1, \dots, Z_K) = e_n^T M_1(Z_1)^{-1} \cdots M_K(Z_K)^{-1}.$$

We now present the definition of the variety $\mathcal{G}$.

Definition 3.27 For $L \geq 2$, $1 \leq i \leq n$ and $a \in \mathbb{C}^*$, we define

$$\mathcal{G} := \mathcal{G}_{L,i,a} = \{Z \in (\mathbb{C}^m)^L : P_i^L(Z) = a\}.$$

Lemma 3.28 *Let $K \geq 3$ and $a = (a_1, \dots, a_n) \in \mathbb{C}^n \setminus \{0\}$. Then we have*

$$(P^K)^{-1}(a) \cong \mathcal{G}_{K-1,i,a_i} \times \mathbb{C}^l,$$

for some $1 \leq i \leq n$, and some natural number l.

We illustrate the idea of the proof with a simple example. A full proof can be found in the proof of [IK12a, Lemma 3.7]. We consider the case $K = n = 3$ and we assume that $a_3 \neq 0$. We have

$$a = P^3(Z_1, Z_2, Z_3) = P^2(Z_1, Z_2) M_3(Z_3)^{-1}$$

if and only if

$$\begin{aligned} P^2(Z_1, Z_2) = a M_3(Z_3) &= \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ z_{3,21} & 1 & 0 \\ z_{3,31} & z_{3,32} & 1 \end{pmatrix} \\ &= \begin{pmatrix} a_1 + a_2 z_{3,21} + a_3 z_{3,31} & a_2 + a_3 z_{3,32} & a_3 \end{pmatrix}. \end{aligned}$$

We can now express the variables $z_{3,31}$ and $z_{3,32}$ in terms of the others, that is,

$$\begin{aligned} z_{3,32} &= \frac{1}{a_3} \left(P_2^2(Z_1, Z_2) - a_2 \right) \\ z_{3,31} &= \frac{1}{a_3} \left(P_1^2(Z_1, Z_2) - a_1 - a_2 z_{3,21} \right). \end{aligned}$$

Observe that there are no conditions for $z_{3,21}$, therefore, the fiber $(P^3)^{-1}(a)$ is biholomorphic to $\{Z \in (\mathbb{C}^3)^2 : P_3^2(Z) = a_3\} \times \mathbb{C}$.

Proposition 3.29 *Let $L \geq 2$ and $a \in \mathbb{C}^*$. Then*

(1) $\mathcal{G}_{L,i,a}$ is smooth for $1 \leq i < n$,

(2) $\mathcal{G}_{L,n,a}$ is smooth for $a \neq 1$.

Proof Let $L \geq 2$ and $a \in \mathbb{C}^*$. Then we have

$$\mathcal{G}_{L,i,a} \times \mathbb{C}^l \cong (P^{L+1})^{-1}(b) =: \mathcal{F},$$

for some natural number l and some $b = (b_1, \ldots, b_n) \in \mathbb{C}^n$. In [IK12a, Remark 4.1] we have a classification of the singular fibers. We distinguish two cases.

For L odd (and obviously $L+1$ even), we have $b_1 = \cdots = b_{i-1} = 0$ and $b_i = a \neq 0$. Moreover, the fiber $\mathcal{F}$ is regular if and only if $b \neq e_n^T$. And this proves the claim for odd numbers L.

For L even, we have $b_n = \cdots = b_{i+1} = 0$ and $b_i = a \neq 0$. Moreover, the fiber $\mathcal{F}$ is regular if and only if $b_n \neq 1$. And this proves the claim for even numbers L. □

Lemma 3.30 *For $L \leq K$, we have*

$$\frac{\partial}{\partial z_{L,kl}} P_i^K = -P_k^L \cdot e_l^T M_L(Z_L)^{-1} \cdots M_K(Z_K)^{-1} e_i$$

Proof The product rule implies

$$\begin{aligned} 0 &= \frac{\partial}{\partial z_{L,kl}} (M_L(Z_L)^{-1} M_L(Z_L)) \\ &= \frac{\partial}{\partial z_{L,kl}} M_L(Z_L)^{-1} M_L(Z_L) + M_L(Z_L)^{-1} \underbrace{\frac{\partial}{\partial z_{L,kl}} M_L(Z_L)}_{=E_{kl}}, \end{aligned}$$

where E_{kl} denotes the $n \times n$-matrix having a one at entry (k,l) and zeros elsewhere. Hence

$$\frac{\partial}{\partial z_{L,kl}} M_L(Z_L)^{-1} = -M_L(Z_L)^{-1} E_{kl} M_L(Z_L)^{-1}.$$

Another application of the product rule implies

$$\begin{aligned} \frac{\partial}{\partial z_{L,kl}} P_i^K &= e_n^T M_1(Z_1)^{-1} \cdots \frac{\partial}{\partial z_{L,kl}} M_L(Z_L)^{-1} \cdots M_K(Z_K)^{-1} e_i \\ &= -\underbrace{e_n^T M_1(Z_1)^{-1} \cdots M_L(Z_L)^{-1}}_{=P^L} \underbrace{E_{kl} M_L(Z_L)^{-1} \cdots M_K(Z_K)^{-1} e_i}_{=e_l^T M_L(Z_L)^{-1} \cdots M_K(Z_K)^{-1} e_i \cdot e_k} \\ &= -P_k^L \cdot e_l^T M_L(Z_L)^{-1} \cdots M_K(Z_K)^{-1} e_i \end{aligned}$$

and this finishes the proof. □

Lemma 3.31 *For $L < K$ we have $\frac{\partial}{\partial z_{L,kl}} P_i^K \not\equiv 0$ on $\mathcal{G}$. Moreover, we have $\frac{\partial}{\partial z_{K,kl}} P_i^K \not\equiv 0$ on $\mathcal{G}$ if and only if (K is odd and $l \geq i$) or (K is even and $l \leq i$).*

Proof First consider the case $L < K$. For a generic point $Z \in \mathcal{G}$ we have $P_k^L(Z) \neq 0$ and $e_l^T M_L(Z_L)^{-1} \cdots M_K(Z_K)^{-1} e_i \neq 0$. Therefore we get $\frac{\partial}{\partial z_{L,kl}} P_i^K \not\equiv 0$ on $\mathcal{G}$.

For $L = K$ and K even, we have $e_l^T M_K(Z_K) e_i \equiv 0$ for $l > i$ and thus

$$\frac{\partial}{\partial z_{K,kl}} P_i^K \equiv 0, \quad l > i.$$

At a generic point $Z \in \mathcal{G}$, however, we have $P_k^K(Z) \neq 0$ and $e_l^T M_K(Z_K) e_i \not\equiv 0$ for $l \leq i$. Therefore $\frac{\partial}{\partial z_{K,kl}} P_i^K \not\equiv 0$ for $l \leq i$.

Similarly, for $L = K$ and K odd we can conclude that $\frac{\partial}{\partial z_{K,kl}} P_i^K \not\equiv 0$ on $\mathcal{G}$ if and only if $l \geq i$. □

Theorem 3.32 *For $K \geq 2$, the complete holomorphic vector fields*

$$V_1 = \frac{\partial P_i^K}{\partial z_{1,n2}} \frac{\partial}{\partial z_{1,n1}} - \frac{\partial P_i^K}{\partial z_{1,n1}} \frac{\partial}{\partial z_{1,n2}}$$

and

$$V_2 = \frac{\partial P_i^K}{\partial z_{2,2n}} \frac{\partial}{\partial z_{2,1n}} - \frac{\partial P_i^K}{\partial z_{2,1n}} \frac{\partial}{\partial z_{2,2n}}$$

build a compatible 2-tuple (V_2, V_1).

Proof Observe that P_i^K is (at most) linear in each variable $z_{L,kl}$. Therefore the two holomorphic vector fields V_1 and V_2 are $\mathbb{C}$-complete.

We check the two properties of Definition 3.2. Note that each variable $z_{L,kl}$ is in the kernel of V_1 or V_2. Hence we have

$$\overline{\operatorname{span}(\ker V_1 \cdot \ker V_2)} = \mathcal{O}(\mathcal{G})$$

and this implies (1).

For property (2), it remains to find a function $f \in \left(\ker V_1^2 \setminus \ker V_1\right) \cap \ker V_2$. We show that $f = z_{1,n2}$ does the job. Clearly, $f \in \ker V_2$. Moreover,

$$V_1(f) = -\frac{\partial P_i^K}{\partial z_{1,n1}} \not\equiv 0$$

by the above lemma. We prove that

$$\frac{\partial}{\partial z_{1,nj}} \frac{\partial P_i^K}{\partial z_{1,n2}} = 0, \quad 1 \leq j \leq n-1.$$

We can write

$$M_1(Z_1) = \begin{pmatrix} M & 0 \\ z^T & 1 \end{pmatrix} = \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I_{n-1} & 0 \\ z^T & 1 \end{pmatrix},$$

for some lower triangular $(n-1) \times (n-1)$-matrix M with $\frac{\partial}{\partial z_{1,nj}} M = 0$ for $j = 1, \ldots, n-1$, and $z^T = (z_{1,n1}, \ldots, z_{1,n(n-1)})$. Since

$$M_1(Z_1)^{-1} = \begin{pmatrix} I_{n-1} & 0 \\ -z^T & 1 \end{pmatrix} \begin{pmatrix} M^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

we obtain

$$\frac{\partial}{\partial z_{1,nj}} \frac{\partial}{\partial z_{1,n2}} M_1(Z_1)^{-1} = \underbrace{\frac{\partial}{\partial z_{1,nj}} \frac{\partial}{\partial z_{1,n2}} \begin{pmatrix} I_{n-1} & 0 \\ -z^T & 1 \end{pmatrix}}_{=0} \cdot \begin{pmatrix} M^{-1} & 0 \\ 0 & 1 \end{pmatrix} = 0$$

for $j = 1, \ldots, n-1$. By the recursive formula of P^K and the product rule, we conclude that

$$\frac{\partial}{\partial z_{1,nj}} \frac{\partial P_i^K}{\partial z_{1,n2}} = 0.$$

This means, in particular, that $V_1(f) \in \ker V_1$, which was to be proved. □

3.3.4 The symplectic case

Let us start with the definition of a symplectic matrix. Consider the skew-symmetric $2n \times 2n$-matrix

$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

where I_n denotes the $n \times n$-identity matrix and 0 the $n \times n$-zero matrix. Then

$$\mathrm{Sp}_{2n}(\mathbb{C}) = \{M \in \mathbb{C}^{2n \times 2n} : M^T \Omega M = \Omega\}$$

is the symplectic group. As in the previous subsection, we define an elementary mapping

$$M \colon \mathbb{C}^m \to \mathrm{Sp}_{2n}(\mathbb{C}),$$

where $m = n(n+1)/2$. For a natural number K we write elements in $\mathbb{C}^m$ as follows

$$Z_K = (z_{K,11}, \ldots, z_{K,kl}, \ldots, z_{K,nn}), \quad 1 \le k \le l \le n.$$

Observe that

$$\psi(Z_K) = \begin{pmatrix} z_{K,11} & z_{K,12} & \cdots & z_{K,1n} \\ z_{K,12} & z_{K,22} & \cdots & z_{K,2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{K,1n} & z_{K,2n} & \cdots & z_{K,nn} \end{pmatrix}$$

defines an isomorphism $\psi \colon \mathbb{C}^m \to \mathrm{Sym}_n(\mathbb{C})$, where $\mathrm{Sym}_n(\mathbb{C})$ denotes the vector space of symmetric $n \times n$-matrices, that is, matrices $A \in \mathbb{C}^{n \times n}$ with $A^T = A$. By abuse of notation, we let Z_K denote both the vector and the matrix. Then we define for even K

$$M_K(Z_K) = \begin{pmatrix} I_n & Z_K \\ 0 & I_n \end{pmatrix}$$

and for odd K

$$M_K(Z_K) = \begin{pmatrix} I_n & 0 \\ Z_K & I_n \end{pmatrix}.$$

Note that $M_K(Z_K)$ is actually a symplectic matrix.

For a fixed natural number K, the Gromov–Vaserstein fibration

$$P^K = (P_1^K, \ldots, P_{2n}^K) \colon (\mathbb{C}^m)^K \to \mathbb{C}^{2n}$$

is given by

$$P^K(Z_1,\dots,Z_K) = e_{2n}^T M_1(Z_1)\cdots M_K(Z_K).$$

We introduce the notation

$$P_f^K = (P_1^K,\dots,P_n^K), \quad P_s^K = (P_{n+1}^K,\dots,P_{2n}^K).$$

Definition 3.33 Let $a = (a_1,\dots,a_n) \in \mathbb{C}^n \setminus \{0\}$. For $K \geq 2$ even, we define

$$\mathcal{G} := \mathcal{G}_{K,a} := \{Z \in \mathbb{C}^n \times (\mathbb{C}^m)^{K-1} : P_s^K(Z) = a\}.$$

And for $K \geq 3$ odd, we define

$$\mathcal{G} := \mathcal{G}_{K,a} := \{Z \in \mathbb{C}^n \times (\mathbb{C}^m)^{K-1} : P_f^K(Z) = a\}.$$

Lemma 3.34 *Let $K \geq 2$ and $a \in \mathbb{C}^n \setminus \{0\}$. Then*

$$\mathcal{G}_{K,a} \times \mathbb{C}^l \cong (P^{K+1})^{-1}(y)$$

for some natural number l and some $y \in \mathbb{C}^{2n} \setminus \{0\}$.

Proof Without loss of generality, we assume K to be odd (the even case is symmetrical). Set $y = (a,b)$ for an arbitrary $b \in \mathbb{C}^n$. We have $a \neq 0$ by assumption, hence $y \in \mathbb{C}^{2n} \setminus \{0\}$. Now observe that

$$\begin{aligned} \begin{pmatrix} a & b \end{pmatrix} &= P^{K+1}(Z_1,\dots,Z_{K+1}) = P^K(Z_1,\dots,Z_K)M_{K+1}(Z_{K+1}) \\ &= \begin{pmatrix} P_f^K & P_s^K \end{pmatrix} \begin{pmatrix} I_n & Z_{K+1} \\ 0 & I_n \end{pmatrix} \end{aligned}$$

and thus

$$\begin{pmatrix} P_f^K & P_s^K \end{pmatrix} = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} I_n & -Z_{K+1} \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} a & b - aZ_{K+1} \end{pmatrix}.$$

Since we assume $a \neq 0$, we can rearrange the equation $P_s^K = b - aZ_{K+1}$ in such a way that n of the variables $z_{K+1,kl}$ can be expressed. This leads to

$$\underbrace{\{Z \in (\mathbb{C}^m)^K : P_f^K(Z) = a\}}_{\tilde{\mathcal{G}}} \times \mathbb{C}^{m-n} \cong (P^{K+1})^{-1}(y).$$

Observe that there are no conditions placed on the variables $z_{1,kl}$ for $1 \leq k \leq l < n$, by definition of P^{K+1} (since we project the first factor $M_1(Z_1)$ to the last row). Therefore we obtain

$$\tilde{\mathcal{G}} \cong \mathcal{G} \times \mathbb{C}^{m-n}$$

and this finishes the proof. For more details, see [Sch25, Lemma 3.15]. $\square$

Proposition 3.35 *Let $K \geq 2$ and $a \in \mathbb{C}^n \setminus \{0\}$. Then $\mathcal{G}_{K,a}$ is smooth if one of the following properties is satisfied*

(1) K is odd,

(2) K is even and $a \neq e_n^T$.

Proof A classification of the singular fibers of P^K can be found in sections 3.1.1 and 3.2.1 in [Sch25].

Suppose K is odd. Then we have

$$\mathcal{G}_{K,a} \times \mathbb{C}^l \cong (P^{K+1})^{-1}(y), \tag{3.2}$$

with $y = (a, b)$ for some $b \in \mathbb{C}^n$. The only singular fiber of P^{K+1} is the one over $y = (0, e_n^T)$. Since we assume $a \neq 0$, the variety $\mathcal{G}_{K,a}$ is smooth.

For K even, we have Equation (3.2) again, but with $y = (b, a)$ for some $b \in \mathbb{C}^n$. A fiber $(P^{K+1})^{-1}(b, a)$ is singular if and only if $a = e_n^T$. And this proves the claim. □

Theorem 3.36 *Let $K \geq 2$ and $n \geq 2$. Moreover, let $\mathcal{G}$ be smooth. Then there exists a compatible k-tuple on $\mathcal{G}$ for some k. In fact, k turns out to be either two or three.*

Proof Start with the case $n \geq 3$. Consider the $n + 1$ variables

$$x = (z_{1,n1}, \dots, z_{1,nn}, z_{2,11})$$

Then the corresponding vector field (see (3.1))

$$V := \begin{cases} D_x(P_s^K) & K \text{ even,} \\ D_x(P_f^K) & K \text{ odd} \end{cases}$$

is complete on $\mathcal{G}$, since it is affine.

Moreover, the holomorphic vector field

$$\gamma = (z_{1,n3})^2 \frac{\partial}{\partial z_{2,22}} - z_{1,n2} z_{1,n3} \frac{\partial}{\partial z_{2,23}} + (z_{1,n2})^2 \frac{\partial}{\partial z_{2,33}}$$

is complete on $\mathcal{G}$. We claim that (γ, V) is a compatible 2-tuple. Observe that each variable $z_{k,ij}$ is in the kernel of γ or in the kernel of V. Therefore

$$\overline{\operatorname{span}(\ker \gamma \cdot \ker V)} = \mathcal{O}(\mathcal{G}).$$

Furthermore, we have

$$f = z_{2,22} \in \ker V \cap \left(\ker \gamma^2 \setminus \ker \gamma\right)$$

and this proves the claim. Note that this argument works for every $K \geq 2$, hence the theorem is true for $n \geq 3$.

It remains to prove it for $n = 2$. Note that in this case, the vector field γ from the previous step does not exist. We therefore divide this case into two steps.

First, we consider $K \geq 3$. We choose again the vector field V from the previous case and choose

$$\tilde{\gamma} = (P_{n+2}^2)^2 \frac{\partial}{\partial z_{3,11}} - (P_{n+1}^2 P_{n+2}^2) \frac{\partial}{\partial z_{3,12}} + (P_{n+1}^2)^2 \frac{\partial}{\partial z_{3,22}},$$

which is also complete on $\mathcal{G}$. Then we conclude that $(\tilde{\gamma}, V)$ is a compatible 2-tuple with the same reasoning as before.

Finally, we consider $K = 2$. We are not able to find a compatible 2-tuple in this case, instead we have a compatible 3-tuple. Here the variety (in the notation of [IKL23]) is

$$\mathcal{G} = \left\{ (z_2, z_3, w_1, w_2, w_3) \in \mathbb{C}^5 : \begin{pmatrix} w_1 & w_2 \\ w_2 & w_3 \end{pmatrix} \begin{pmatrix} z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right\}$$

for some $(b_1, b_2) \in \mathbb{C}^2 \setminus \{0\}$. The following three vector fields are complete on $\mathcal{G}$:

$$\begin{aligned} V_1 &= -z_2 w_3 \frac{\partial}{\partial z_2} + z_2 w_2 \frac{\partial}{\partial z_3} + (w_1 w_3 - w_2^2) \frac{\partial}{\partial w_1} \\ V_2 &= z_3^2 \frac{\partial}{\partial w_1} - z_2 z_3 \frac{\partial}{\partial w_2} + z_2^2 \frac{\partial}{\partial w_3} \\ V_3 &= z_3^2 \frac{\partial}{\partial z_2} - w_1 z_3 \frac{\partial}{\partial w_2} + (w_1 z_2 - w_2 z_3) \frac{\partial}{\partial w_3} \end{aligned}$$

Then (V_1, V_2, V_3) is a compatible 3-tuple. To prove this, note that $z_2, z_3 \in \ker V_2$, $w_1 \in \ker V_3$ and $w_2, w_3 \in \ker V_1$. Hence

$$\overline{\operatorname{span}(\ker V_1 \cdot \ker V_2 \cdot \ker V_3)} = \mathcal{O}(\mathcal{G}).$$

Moreover, we have

$$w_2 \in \left(\ker V_2^2 \setminus \ker V_2\right) \cap \ker V_1, \quad w_2 \in \left(\ker V_3^2 \setminus \ker V_3\right) \cap \ker V_1$$

which means that there is a $\{V_1, V_2, V_3\}$-admissible rooted tree (T, π, ε) with root V_1. More precisely, the map $\varepsilon \colon \operatorname{Edge}(T) \to \mathcal{O}(\mathcal{G})$ can be defined by

$$\varepsilon(V_2, V_1) := w_2, \quad \varepsilon(V_3, V_1) := w_2.$$

This finishes the proof of the theorem. $\square$

Chapter 4

Symplectic density property for Calogero–Moser spaces

In Section 4.1 we discuss some properties of the Calogero–Moser space and its topology. In particular, the symplectic density property is the same as the Hamiltonian density property for the Calogero–Moser space. In Section 4.2 we take a close look at its coordinate ring and draw a smaller set of algebra generators. Then in Section 4.3 we perform the full computation of Lie-generating all regular functions from two Hamiltonian functions.

4.1 The Calogero–Moser space

Recall the definition given by Wilson [Wil98] c.f. Definition 1.1: Let $\widehat{\mathcal{C}}_n$ be the subvariety of $\mathrm{M}_n(\mathbb{C}) \times \mathrm{M}_n(\mathbb{C})$ given by

$$\operatorname{rank}([X, Y] + \mathrm{id}) = 1$$

where $(X, Y) \in \mathrm{M}_n(\mathbb{C}) \times \mathrm{M}_n(\mathbb{C})$. The group $\mathrm{GL}_n(\mathbb{C})$ acts on $\widehat{\mathcal{C}}_n$ by simultaneous conjugation in both factors:

$$g \cdot (X, Y) = (gXg^{-1}, gYg^{-1})$$

for $g \in \mathrm{GL}_n(\mathbb{C})$. The *Calogero–Moser space* $\mathcal{C}_n$ of n particles is the GIT-quotient $\widehat{\mathcal{C}}_n // \mathrm{GL}_n(\mathbb{C})$.

Lemma 4.1 *[Wil98, Proposition 1.10] Let* $(X, Y) \in \mathrm{M}_n(\mathbb{C}) \times \mathrm{M}_n(\mathbb{C})$ *such that* $[X, Y] + \mathrm{id}$ *is of rank one. If* X *is diagonalizable, then all eigenvalues of* X *are pairwise different,* *and there exists* $g \in \mathrm{GL}_n(\mathbb{C})$ *such that*

$$(gXg^{-1}, gYg^{-1}) = \left(\begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_n \end{pmatrix}, \begin{pmatrix} \beta_1 & (\alpha_1-\alpha_2)^{-1} & \dots & (\alpha_1-\alpha_n)^{-1} \\ (\alpha_2-\alpha_1)^{-1} & \beta_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & (\alpha_{n-1}-\alpha_n)^{-1} \\ (\alpha_n-\alpha_1)^{-1} & \dots & (\alpha_n-\alpha_{n-1})^{-1} & \beta_n \end{pmatrix} \right) \tag{4.1}$$

for $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in \mathbb{C}$ *with* $\alpha_j \neq \alpha_k$ *for* $j \neq k$. *Moreover,* $(g([X,Y] + \mathrm{id})g^{-1})_{jk} = 1$ *for all* $j, k = 1, \dots, n$.

Note that the order of the eigenvalues $\alpha_1, \dots, \alpha_n$ is arbitrary, hence we obtain an $n!$-to-1 covering of the open and dense subset of $\mathcal{C}_n$ where the matrices X are diagonalizable. Alternatively, we can define the injective mapping

$$\Phi\colon \left(\{(\alpha_1, \dots \alpha_n) \in \mathbb{C}^n \,:\, \alpha_j \neq \alpha_k \text{ for } j \neq k\} \times \mathbb{C}^n\right) / S_n \to \mathcal{C}_n$$

that maps $((\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n))$ to (X, Y) according to Equation (4.1). Here, S_n denotes the symmetric group that acts by simultaneous permutations on $(\alpha_1, \dots, \alpha_n)$ and $(\beta_1, \dots, \beta_n)$.

For the definition of vector- or matrix-valued differential forms, see e.g. the textbook of Tu [Tu17, Section 21]. The standard conjugation-invariant symplectic form on $\mathrm{M}_n(\mathbb{C}) \times \mathrm{M}_n(\mathbb{C})$ according to Wilson [Wil98, p. 9] is given as

$$\widetilde{\omega} = \mathrm{tr}(\mathrm{d}X \wedge \mathrm{d}Y) = \sum_{j,k=1}^{n} \mathrm{d}X_{jk} \wedge \mathrm{d}Y_{kj} \tag{4.2}$$

The form $\widetilde{\omega}$ is invariant under conjugation. Following Etingof [Eti07], the action of $\mathrm{GL}_n(\mathbb{C})$ admits a moment map

$$\mu\colon \mathrm{M}_n(\mathbb{C}) \times \mathrm{M}_n(\mathbb{C}) \to \mathfrak{sl}_n(\mathbb{C}),\ (X, Y) \mapsto [X, Y]$$

where we identify the Lie algebra with its dual using the trace form $\langle M, N \rangle = \mathrm{tr}(MN)$. This moment map was first given in [Wil98], following the construction with a unitary group action in Kazhdan, Kostant and Sternberg [KKS78].

Let O_ξ be the coadjoint orbit of the matrix

$$\xi = \mathrm{diag}(-1, -1, \dots, -1, n-1) \in \mathfrak{sl}_n(\mathbb{C}) \cong \mathfrak{sl}_n^*(\mathbb{C})$$

Since the coadjoint action of $\mathrm{PGL}_n(\mathbb{C})$ on $\mathfrak{sl}_n^*(\mathbb{C})$ is given by conjugation, O_ξ consists of traceless matrices T such that $T + \mathrm{id}$ is of rank one. Then $\widehat{\mathcal{C}}_n = \mu^{-1}(O_\xi)$ is the preimage of this orbit, upon which $\mathrm{PGL}_n(\mathbb{C})$ acts freely.

Lemma 4.2 *[Wil98, Corollary 1.5] The group* $\mathrm{PGL}_n(\mathbb{C})$ *acts freely on* $\widehat{\mathcal{C}}_n$.

Proof It suffices to see that for $(X, Y) \in \widehat{\mathcal{C}}_n$, X and Y has no nontrivial common invariant subspace. Since then by Schur's lemma, the only matrices that commute with both X and Y must be scalar, hence trivial in $\mathrm{PGL}_n(\mathbb{C})$. Let $W \neq 0$ be an invariant subspace of $\mathbb{C}^n$ for X, Y. Then the eigenvalues of the commutator $[X, Y]$ on W are part of the $n-1$ copies of -1 and one piece of $n-1$. The sum of eigenvalues of $[X, Y]$ on W must be zero, being the trace of a commutator. It can only be the entire collection of eigenvalues, thus $W = \mathbb{C}^n$. $\square$

The reduction along this orbit

$$\pi\colon \mu^{-1}(O_\xi) \to \mu^{-1}(O_\xi)//\mathrm{PGL}_n(\mathbb{C}) = \mathcal{C}_n$$

gives the symplectic form ω on $\mathcal{C}_n$ which satisfies $\pi^*\omega = i^*\widetilde{\omega}$, where $i\colon \mu^{-1}(O_\xi) \hookrightarrow \mathrm{M}_n(\mathbb{C}) \oplus \mathrm{M}_n(\mathbb{C})$ denotes the inclusion.

The Calogero–Moser space $\mathcal{C}_n$ is diffeomorphic to the Hilbert scheme of n points in the affine plane, $\mathrm{Hilb}_n(\mathbb{C}^2)$. This can be shown using the existence of a hyperkähler structure on $\mathcal{C}_n$, see Wilson [Wil98, Section 8]. For $n > 1$ the Hilbert scheme is not affine or Stein, hence this diffeomorphism cannot be an isomorphism in the algebraic or holomorphic category. The topology of the Hilbert scheme of a plane is well-known; the Borel–Moore homology of $\mathrm{Hilb}_n(\mathbb{C}^2)$ has been calculated by Ellingsrud and Strømme [ES87, Theorem 1.1, (iii)]. In particular, they obtain that the odd homology vanishes. This homology had been introduced by Borel and Moore [BM60] to obtain Poincaré duality for singular cohomology on non-compact manifolds. Since the real dimension of a complex manifold is even, it follows that all odd cohomology vanishes. This implies by the universal coefficient theorem that $H^1(\mathrm{Hilb}_n(\mathbb{C}^2), \mathbb{C}) = 0$ which is a topological invariant, hence $H^1(\mathcal{C}_n, \mathbb{C}) = 0$. Since $\mathcal{C}_n$ is an affine manifold, this implies that both the algebraic and the holomorphic first de Rham cohomology group are trivial, since by the Poincaré lemma they can be computed using resolutions of the sheaf of locally constant complex-valued functions, see e.g. [GR79, p. 80]. We conclude the following:

Lemma 4.3 *We have that $H^1(\mathcal{C}_n, \mathbb{C}) = 0$. Hence all holomorphic symplectic vector fields on $\mathcal{C}_n$ are in fact holomorphic Hamiltonian vector fields.*

The last two sections of this chapter will contain the main part of the proof for the Hamiltonian density property. Instead of dealing with the vector fields directly, we will consider the corresponding Hamilton functions. The following basic remark is crucial for our calculations. It is a consequence of the construction of the Calogero–Moser space by symplectic reduction.

Remark 4.4 Let $f, h\colon \mathcal{C}_n \to \mathbb{C}$ be two (Hamiltonian) regular functions on the Calogero–Moser space $\mathcal{C}_n$ and let $F, H\colon \mathrm{M}_n(\mathbb{C}) \times \mathrm{M}_n(\mathbb{C}) \to \mathbb{C}$ be $\mathrm{GL}_n(\mathbb{C})$-invariant extensions of $f \circ \pi, h \circ \pi\colon \widehat{\mathcal{C}}_n \to \mathbb{C}$, respectively. These extensions exist since $\widehat{\mathcal{C}}_n \subset \mathrm{M}_n(\mathbb{C}) \times \mathrm{M}_n(\mathbb{C})$ is a $\mathrm{GL}_n(\mathbb{C})$-invariant subvariety of a reductive group, and thus

$$\mathbb{C}[\mathrm{M}_n(\mathbb{C}) \times \mathrm{M}_n(\mathbb{C})]^{\mathrm{GL}_n(\mathbb{C})} \to \mathbb{C}[\widehat{\mathcal{C}}_n]^{\mathrm{GL}_n(\mathbb{C})}$$

is surjective. Then

$$\{F, H\}_{\widetilde{\omega}} \circ i = \{f, h\}_\omega \circ \pi$$

and their corresponding vector fields and flows are related in a similar manner. In this way the symplectic reduction relates the Poisson structure on $\mathcal{C}_n$ to the Poisson structure on $\mathrm{M}_n(\mathbb{C}) \times \mathrm{M}_n(\mathbb{C})$, see Marsden and Ratiu [MR86] for an explanation in the context of Poisson reduction. In other words, to obtain Poisson brackets between Hamiltonian functions on $\mathcal{C}_n$, it suffices to compute brackets of the corresponding invariant Hamiltonian functions on $\mathrm{M}_n(\mathbb{C}) \times \mathrm{M}_n(\mathbb{C})$.

Given two Hamiltonian functions F and H on $\mathrm{M}_n(\mathbb{C}) \oplus \mathrm{M}_n(\mathbb{C})$, their Poisson bracket associated with the symplectic form $\widetilde{\omega} = \sum_{j,k} dX_{jk} \wedge dY_{kj}$ is

$$\{F, H\} = \sum_{j,k=1}^{n} \frac{\partial F}{\partial X_{jk}} \frac{\partial H}{\partial Y_{kj}} - \frac{\partial F}{\partial Y_{jk}} \frac{\partial H}{\partial X_{kj}} \tag{4.3}$$

We recall that the Poisson bracket is antisymmetric and satisfies Leibniz's rule, namely

$$\{F, H\} = -\{H, F\}, \quad \{F_1F_2, H\} = F_1\{F_2, H\} + F_2\{F_1, H\}$$

By Leibniz's rule

$$\{F^j, H^k\} = jkF^{j-1}H^{k-1}\{F, H\}, \quad j, k \geq 1.$$

Lemma 4.5 *Hamiltonian functions of the form $H(X)$ or $H(Y)$, depending either only on X or only on Y, and $H = \operatorname{tr} XY$ induce complete vector fields.*

Proof By definition, given a Hamiltonian function $H(X, Y)$, there is a unique vector field V_H satisfying $i_{V_H}\widetilde{\omega} = \mathrm{d}H$. The Hamiltonian vector field V_H is

$$V_H = \sum_{j,k} \frac{\partial H}{\partial Y_{jk}} \frac{\partial}{\partial X_{kj}} - \sum_{j,k} \frac{\partial H}{\partial X_{jk}} \frac{\partial}{\partial Y_{kj}}$$

Now if $H = H(X)$ only depends on X, then the first summand vanishes. We obtain

$$V_H = -\sum_{j,k} \frac{\partial H}{\partial X_{jk}} \frac{\partial}{\partial Y_{kj}}$$

and the coefficients depend only on X, which are constant along any trajectory of a local flow. Hence V_H is complete. For $H = \operatorname{tr} XY$ the associated vector field is

$$V_H = \sum_{j,k} X_{kj} \frac{\partial}{\partial X_{kj}} - \sum_{j,k} Y_{kj} \frac{\partial}{\partial Y_{kj}}$$

which has coefficients linear in each variable and is complete. □

Example 4.6 We give a few examples of Hamiltonian functions, their corresponding vector fields and flows. In particular, the examples given here are complete, i.e. their flows exist for all complex times.

Hamiltonian	vector field	flow map $((X, Y), t) \mapsto$
$\operatorname{tr} X^j$	$-jX^{j-1}\frac{\partial}{\partial Y}$	$(X, Y - tjX^{j-1})$
$\operatorname{tr} Y^j$	$jY^{j-1}\frac{\partial}{\partial X}$	$(X + tjY^{j-1}, Y)$
$(\operatorname{tr} X^j)^2$	$-2j(\operatorname{tr} X^j)X^{j-1}\frac{\partial}{\partial Y}$	$(X, Y - 2tj(\operatorname{tr} X^j)X^{j-1})$
$(\operatorname{tr} Y^j)^2$	$2j(\operatorname{tr} Y^j)Y^{j-1}\frac{\partial}{\partial X}$	$(X + 2tj(\operatorname{tr} Y^j)Y^{j-1}, Y)$
$\operatorname{tr} XY$	$X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}$	$(\exp(t)X, \exp(-t)Y)$

Here, we understand terms like $X^{j-1}\frac{\partial}{\partial Y}$ as standard scalar product between the matrix entries of X^{j-1} and of $\frac{\partial}{\partial Y}$.

We will need the following two examples in Lemma 4.7 for the proof of Theorem 1.6.

Lemma 4.7 1. *The Hamiltonian function* $\operatorname{tr} Y + \operatorname{tr} X^3$ *corresponds to the vector field*

$$\operatorname{id}\frac{\partial}{\partial X} - 3X^2\frac{\partial}{\partial Y},$$

which has the complete algebraic flow, polynomial in t

$$(X, Y) \mapsto (X + t\operatorname{id}, Y - 3X^2 t - 3Xt^2 - \operatorname{id} t^3)$$

2. *The Hamiltonian function* $\operatorname{tr} Y^2 + (\operatorname{tr} X)^2$ *corresponds to the vector field*

$$2Y\frac{\partial}{\partial X} - 2(\operatorname{tr} X)\operatorname{id}\frac{\partial}{\partial Y},$$

which has the complete algebraic flow, holomorphic in t

$$(X, Y) \mapsto \big(X + 2Yt + f(t)\operatorname{id}, Y + \dot{f}(t)\operatorname{id}\big)$$

where

$$f(t) = \frac{1}{n}\big(\cos(2\sqrt{n}t) - 1\big)\operatorname{tr} X + \frac{1}{n^{3/2}}(\sin(2\sqrt{n}t) - 2t\sqrt{n})\operatorname{tr} Y$$

Proof 1. The vector field $\operatorname{id}\frac{\partial}{\partial X} - 3X^2\frac{\partial}{\partial Y}$ is a locally nilpotent derivation and finding its flow is a straightforward computation.

2. We rewrite the vector field $2Y\frac{\partial}{\partial X} - 2(\operatorname{tr} X)\operatorname{id}\frac{\partial}{\partial Y}$ as a system of ODEs:

$$\begin{cases} X'_{jk}(t) & = 2Y_{jk}(t) \\ Y'_{jk}(t) & = -2\operatorname{tr} X(t)\delta_{jk} \end{cases}$$

This implies

$$\begin{aligned} (\operatorname{tr} Y)''(t) &= -4n\operatorname{tr} Y(t) \\ (Y_{jj} - Y_{kk})'(t) &= 0 \\ (Y_{jk})'(t) &= 0 \quad \text{for } j \neq k \end{aligned} \tag{4.4}$$

Solving (4.4) with the initial conditions

$$(\operatorname{tr} Y)(t)|_{t=0} = \operatorname{tr} Y, \quad (\operatorname{tr} Y)''(t)\big|_{t=0} = -2n\operatorname{tr} X$$

yields

$$(\operatorname{tr} Y)(t) = \operatorname{tr} Y\cos(2\sqrt{n}t) - \sqrt{n}\operatorname{tr} X\sin(2\sqrt{n}t)$$

We can now obtain the flow map by integration. □

4.2 The ring of invariant functions

Razmyslov [Raz74] and Procesi [Pro76] proved independently the following about invariant functions of tuples of matrices:

Theorem 4.8 *Consider the action of* $\mathrm{GL}_n(\mathbb{C})$ *on* $\mathrm{M}_n(\mathbb{C}) \times \mathrm{M}_n(\mathbb{C}) \times \cdots \times \mathrm{M}_n(\mathbb{C}) = (\mathrm{M}_n(\mathbb{C}))^m$ *by simultaneous conjugation. Then the ring of invariant polynomials on* $(\mathrm{M}_n(\mathbb{C}))^m$ *is generated by*

$$\operatorname{tr}(F_1 \cdot F_2 \cdots F_k)$$

where each of the $F_1, \ldots, F_k$ *is a matrix from one of the* m *factors and* $k \leq n^2$*. Moreover, all relations between the generators are consequences of the Cayley–Hamilton identity.*

Remark 4.9 Known as Noether's conservation law for Hamiltonian systems with symmetry, see Abraham and Marsden [AM78, Theorem 4.2.2], the flow φ_t induced by any Hamiltonian vector field V_H associated to an invariant Hamiltonian function H preserves the fibers of the moment map μ. In our case, one can verify directly that the commutator $[X, Y]$ is constant along the flow φ_t for a Hamiltonian function $H(X, Y)$, which is a product of traces of monomials in X, Y. In particular, the rank condition $\operatorname{rank}([X, Y] + \mathrm{id}) = 1$ is preserved.

So far we have not made use of the rank condition on $\widehat{\mathcal{C}}_n$. In fact, it turns out that this condition is magical in reducing the generating set of the ring of invariant functions on $\widehat{\mathcal{C}}_n$ to a much simpler subset. Let

$$B = [X, Y], \quad A = [X, Y] + \mathrm{id} = B + \mathrm{id}\,.$$

On $\widehat{\mathcal{C}}_n$, A is of rank one. The following identity for the trace of a product of matrices, where one of the matrices has rank one, will be useful for the reduction.

Lemma 4.10 *Let* $M, N, C \in \mathrm{M}_n(\mathbb{C})$*, and assume that* C *is of rank one. Then*

$$\operatorname{tr} MCNC = \operatorname{tr} MC \cdot \operatorname{tr} NC.$$

Proof Since C is of rank one, write $C = vw^t$ for some $v, w \in \mathbb{C}^n \setminus \{0\}$. Then

$$\begin{aligned} \operatorname{tr} MCNC &= \operatorname{tr} Mvw^t Nvw^t = \operatorname{tr} Mv(w^t Nv)w^t \\ &= (w^t Nv) \cdot \operatorname{tr} Mvw^t = \operatorname{tr} NC \cdot \operatorname{tr} MC. \end{aligned}$$

□

We also need the following identities concerning the commutator B.

Lemma 4.11 *For* $k, l \in \mathbb{N}_0$

$$\operatorname{tr} X^k B = 0, \operatorname{tr} Y^l B = 0.$$

And $\operatorname{tr} XYB = \binom{n}{2}$ *on* $\widehat{\mathcal{C}}_n$.

Proof By definition of B

$$\operatorname{tr} X^k B = \operatorname{tr} X^k (XY - YX) = \operatorname{tr} X^{k+1} Y - \operatorname{tr} X^k Y X = 0$$

Similarly $\operatorname{tr} Y^l B = 0$. For $\operatorname{tr} XYB$, because A is of rank one on $\widehat{\mathcal{C}}_n$, we can find $v, w \in \mathbb{C}^n \setminus \{0\}$ such that $A = vw^t$. Then with $\operatorname{tr} A = \operatorname{tr}([X,Y] + \mathrm{id}) = \operatorname{tr} \mathrm{id} = n$,

$$A^2 = vw^t vw^t = v(w^t v)w^t = (\operatorname{tr} A)A = nA.$$

This implies that

$$\begin{aligned} \operatorname{tr} B^2 &= \operatorname{tr} A^2 - 2 \operatorname{tr} A + \operatorname{tr} \mathrm{id} \\ &= n \operatorname{tr} A - 2 \operatorname{tr} A + n \\ &= n(n-1) \end{aligned}$$

Then from

$$\operatorname{tr} XYB = -\operatorname{tr} YXB = -\operatorname{tr} XYB + \operatorname{tr} B^2$$

it follows that $\operatorname{tr} XYB = \binom{n}{2}$ on $\widehat{\mathcal{C}}_n$. □

Definition 4.12 A *matrix monomial* is a map $M \colon \mathrm{M}_n(\mathbb{C}) \times \mathrm{M}_n(\mathbb{C}) \to \mathrm{M}_n(\mathbb{C})$ of the form

$$(X,Y) \mapsto X^{p_1} Y^{q_1} X^{p_2} Y^{q_2} \cdots X^{p_m} Y^{q_m}$$

with $p_1, \dots, p_m, q_1, \dots, q_m \in \mathbb{N}_0$.

The *bidegree* (i,j) of M is given by

$$(i,j) = (p_1 + p_2 + \cdots + p_m, q_1 + q_2 + \cdots + q_m).$$

The *degree* $\deg M$ is defined as $\deg M = i + j$. If $M = 0$, then $\deg M := -\infty$. For a polynomial function in the traces of matrix monomials, its degree is the total degree of the polynomial in the entries of X and Y.

Lemma 4.13 *Let P be a matrix monomial, then on $\widehat{\mathcal{C}}_n$ the trace $\operatorname{tr} PB$ can be written as a polynomial in traces of matrix monomials whose degree is bounded by $\deg P - 2$.*

Proof Use induction on $k = \deg P$. For $k = 1, 2$, the induction base is given by Lemma 4.11:

$$\begin{aligned} &\operatorname{tr} XB = 0, \operatorname{tr} YB = 0 \\ &\operatorname{tr} XYB = \binom{n}{2}, \operatorname{tr} X^2 B = 0, \operatorname{tr} Y^2 B = 0. \end{aligned}$$

For the induction step, first consider $P_1 = MBN$ with matrix monomials M and N. By the identities $B = A - \mathrm{id}$ and $\operatorname{tr} MANA = \operatorname{tr} MA \operatorname{tr} NA$ in Lemma 4.10,

$$\begin{aligned} \operatorname{tr} P_1 B = \operatorname{tr} MBNB &= \operatorname{tr} MANA - \operatorname{tr} MNA - \operatorname{tr} MAN + \operatorname{tr} MN \\ &= \operatorname{tr} MA \cdot \operatorname{tr} NA - \operatorname{tr} MNB - \operatorname{tr} MBN - \operatorname{tr} MN \\ &= \operatorname{tr} MB \cdot \operatorname{tr} NB + \operatorname{tr} M \cdot \operatorname{tr} NB + \operatorname{tr} MB \cdot \operatorname{tr} N \\ &\quad + \operatorname{tr} M \cdot \operatorname{tr} N - \operatorname{tr} MNB - \operatorname{tr} NMB - \operatorname{tr} MN. \end{aligned}$$

By the induction hypothesis, each summand above involves terms of degree bounded by $\deg P_1 - 2$. Hence $\operatorname{tr} P_1 B$ is a sum of products of traces, where each summand has degree $\leq \deg P_1 - 2$.

Next, we show that $\operatorname{tr} PB - \operatorname{tr} X^i Y^j B$ has degree $\leq \deg P - 2$, where $i + j = \deg P$. If $P = X^i Y^j$, we are done. Otherwise the factor YX appears in P. Let $P = MYXN$, with M, N matrix monomials, $\deg P = k$. Then

$$\operatorname{tr} PB = \operatorname{tr} MXYNB - \operatorname{tr} MBNB. \tag{4.5}$$

Thus the factor YX can be replaced with XY at the price of adding the trace of a sum of matrix monomials of degree $\leq \deg P - 2$ following the computation for P_1 above. Continue swapping YX until we obtain $\operatorname{tr} X^i Y^j B$ for some $i, j \in \mathbb{N}_0$, $i + j = \deg P$.

Now we show that $\operatorname{tr} X^i Y^j B$ is a polynomial in traces of matrix monomials whose degrees are bounded by $i + j - 2$. Consider the identity $\operatorname{tr} X^{i+1} B = 0$, which under the Calogero–Moser flow $(X, Y) \mapsto (X + tY^j, Y)$ becomes

$$\operatorname{tr}(X + tY^j)^{i+1} B = 0$$

since $B = [X, Y]$ is invariant under the flow by Remark 4.9. The coefficient of t in this new identity must vanish, which gives

$$0 = \operatorname{tr} X^i Y^j B + \operatorname{tr} X^{i-1} Y^j X B + \cdots + \operatorname{tr} X Y^j X^{i-1} B + \operatorname{tr} Y^j X^i B \tag{4.6}$$

Continue to swap YX to XY for $\operatorname{tr} X^{i-m} Y^j X^m B, m = 1, \ldots, i$, which yields extra summands of degree up to $i + j - 2$ as in Equation (4.5). Then

$$\operatorname{tr} X^{i-m} Y^j X^m B = \operatorname{tr} X^i Y^j B + \text{ terms of degree up to } i + j - 2$$

This and Equation (4.6) imply that

$$0 = (i + 1) \operatorname{tr} X^i Y^j B + \text{ terms of degree up to } i + j - 2$$

hence $\operatorname{tr} X^i Y^j B$ consists of terms of degree bounded by $i + j - 2$. This in turn shows that $\operatorname{tr} PB$ is a sum of products of traces, with each summand of degree $\leq \deg P - 2$. □

Corollary 4.14 *Let P be a matrix monomial of bidegree (i, j). Then on $\widehat{\mathcal{C}}_n$ the trace $\operatorname{tr} P$ is equal to $\operatorname{tr} X^i Y^j$ up to terms of degree $\leq i + j - 4$.*

Proof For one factor YX in $P = MYXN$, we look at

$$\operatorname{tr} MYXN = \operatorname{tr} MXYN - \operatorname{tr} MBN = \operatorname{tr} MXYN - \operatorname{tr} NMB \tag{4.7}$$

By Lemma 4.13, $\operatorname{tr} NMB$ consists of terms of degree $\leq i + j - 4$. Moving all X to the left by swapping YX to XY, we see that the claim is true. □

In this manner, any trace of a matrix monomial in X, Y on $\widehat{\mathcal{C}}_n$ of bidegree (i, j), can be written as $\operatorname{tr} X^i Y^j$ plus a polynomial in traces of matrix monomials in X, Y, whose degrees are bounded by $i + j - 4$.

Proposition 4.15 *The algebra of invariant polynomials on $\mathrm{M}_n(\mathbb{C}) \oplus \mathrm{M}_n(\mathbb{C})$, restricted to $\widehat{\mathcal{C}}_n$, is generated by $\mathcal{B}_n = \{\operatorname{tr} X^i Y^j : i, j \in \mathbb{N}_0, i + j \leq n^2\}$.*

Proof By the invariant theory of $n \times n$ matrices, see [Pro76, Theorem 1.3 and Theorem 3.4 (a)] and [Raz74, final remark], the ring of invariant functions is generated by traces of matrix monomials in X and Y with degree $\leq n^2$. We show that any such trace of degree D is contained in the subalgebra generated by $\mathcal{B}_n$. For $D \leq 2$, we only have

$$f \equiv \text{constant}, \operatorname{tr} X, \operatorname{tr} Y, \operatorname{tr} X^2, \operatorname{tr} Y^2, \operatorname{tr} XY.$$

Assume that traces of degree $\leq D$ are generated by $\mathcal{B}_n$. If we consider a trace function $\operatorname{tr} MN$ of degree $D+1$ as in Corollary 4.14, we can replace it with $\operatorname{tr} X^i Y^j$ plus terms of degree smaller than or equal to $D-3$, which are generated by $\mathcal{B}_n$ according to the induction hypothesis. □

The result of Proposition 4.15 above was also obtained by Etingof and Ginzburg [EG02, Section 11, p. 322, Remark (ii)] where they make use of the Harish-Chandra homomorphism. Our computations in Lemma 4.13 are in fact very similar to theirs in Lemma 12.4 and Lemma 12.5 in [EG02, Appendix A] where they treated the case of $\operatorname{rank}[X, Y] \leq 1$ while we are dealing with $\operatorname{rank}([X, Y] + \mathrm{id}) = 1$.

4.3 Proof of Hamiltonian density property

We consider the Lie algebra of $\mathrm{GL}_n(\mathbb{C})$-invariant polynomials on $\widehat{\mathcal{C}}_n$, with the Poisson bracket $\{\,,\,\}$ as the bracket operation. Let

$$\mathcal{F} = \{\operatorname{tr} Y^2 + (\operatorname{tr} X)^2, \operatorname{tr} Y + \operatorname{tr} X^3\}$$
$$\mathcal{F}_1 = \{\operatorname{tr} Y^2, (\operatorname{tr} X)^2, \operatorname{tr} Y + \operatorname{tr} X^3\}$$

Theorem 4.16 *The Lie algebra generated by the restrictions to $\widehat{\mathcal{C}}_n$ of the polynomials in $\mathcal{F}$ is the entire Lie algebra of invariant polynomials on $\widehat{\mathcal{C}}_n$.*

Remark 4.17 Etingof and Ginzburg proved that $\{\operatorname{tr} Y^2, \operatorname{tr} X^k \ : \ k \geq 0\}$ generates the ring of function on $\mathcal{C}_n$ as a Poisson algebra [EG02, Section 11, p. 321, Equation (11.33)]. Theorem 4.16 improves this result in two ways: First, we do not use the associative multiplication of the Poisson algebra but only the Lie algebra structure. Second, we only need two generators instead of a countable family.

The theorems of the introduction will easily follow from this theorem. Because of Proposition 4.15, it suffices to prove that products of (restrictions of) functions of form $\operatorname{tr} X^a Y^b$ belong to the Lie algebra generated. In Section 4.3.2 we prove a preliminary result which holds even before restricting to $\widehat{\mathcal{C}}_n$:

Proposition 4.18 *The Lie algebra generated by $\mathcal{F}$ on $\mathrm{M}_n(\mathbb{C}) \times \mathrm{M}_n(\mathbb{C})$ contains products of arbitrary powers of the five functions* $\operatorname{tr} X, \operatorname{tr} Y, \operatorname{tr} X^2, \operatorname{tr} Y^2, \operatorname{tr} XY$.

In Section 4.3.1 we first present a formula for the Poisson bracket of two invariant polynomials of the form $\operatorname{tr} X^a Y^b$ (Lemma 4.20). Then we start with the elements of $\mathcal{F}$ and generate $\operatorname{tr} X^a Y^b$ using Lie-combinations (Corollary 4.24). Afterward we proceed to show Proposition 4.18 in Section 4.3.2 and Theorem 4.16 in Section 4.3.3.

Remark 4.19 Since we only need $\operatorname{tr} X, \operatorname{tr} Y, \operatorname{tr} X^2, \operatorname{tr} Y^2, \operatorname{tr} XY$ to generate the ring of invariant functions for $n = 2$, see [Sib68, Theorem 5], Proposition 4.18 proves already the Hamiltonian density property for $\mathcal{C}_2$. However, all the results in this section are also valid for $n > 2$.

4.3.1 Preparations

We need the following formula for our bracket computation.

Lemma 4.20 *For $a, b, c, d \in \mathbb{N}_0$,*

$$\begin{aligned} &\{\operatorname{tr} X^a Y^b, \operatorname{tr} X^c Y^d\} \\ &= \sum_{\substack{1 \le p \le a \\ 1 \le q \le d}} \operatorname{tr} X^{p-1} Y^{d-q} X^c Y^{q-1} X^{a-p} Y^b - \sum_{\substack{1 \le r \le b \\ 1 \le s \le c}} \operatorname{tr} Y^{r-1} X^{c-s} Y^d X^{s-1} Y^{b-r} X^a. \end{aligned}$$

Proof We make use of the product rule for the Poisson bracket and that $\{X_{ij}, Y_{kl}\} = \delta_{jk}\delta_{li}, \{X_{ij}, X_{kl}\} = 0, \{Y_{ij}, Y_{kl}\} = 0$:

$$\begin{aligned} &\{\operatorname{tr} X^a Y^b, \operatorname{tr} X^c Y^d\} \\ &= \sum \{X_{i_1 i_2} \cdots X_{i_a j_1} Y_{j_1 j_2} \cdots Y_{j_b i_1}, X_{k_1 k_2} \cdots X_{k_c m_1} Y_{m_1 m_2} \cdots Y_{m_d k_1}\} \\ &= \sum \sum_{p,q} \widehat{X_{i_p i_{p+1}}} \widehat{Y_{m_q m_{q+1}}} \{X_{i_p i_{p+1}}, Y_{m_q m_{q+1}}\} + \sum \sum_{r,s} \widehat{Y_{j_r j_{r+1}}} \widehat{X_{k_s k_{s+1}}} \{Y_{j_r j_{r+1}}, X_{k_s k_{s+1}}\} \\ &= \sum \sum_{p,q} \widehat{X_{i_p i_{p+1}}} \widehat{Y_{m_q m_{q+1}}} \delta_{i_{p+1} m_q} \delta_{m_{q+1} i_p} - \sum \sum_{r,s} \widehat{Y_{j_r j_{r+1}}} \widehat{X_{k_s k_{s+1}}} \delta_{j_{r+1} k_s} \delta_{k_{s+1} j_r} \\ &= \sum_{\substack{1 \le p \le a \\ 1 \le q \le d}} \operatorname{tr} X^{p-1} Y^{d-q} X^c Y^{q-1} X^{a-p} Y^b - \sum_{\substack{1 \le r \le b \\ 1 \le s \le c}} \operatorname{tr} Y^{r-1} X^{c-s} Y^d X^{s-1} Y^{b-r} X^a \end{aligned}$$

where $\widehat{X_{i_p i_{p+1}}}$ stands for the product of all factors X without $X_{i_p i_{p+1}}$, i.e.

$$\widehat{X_{i_p i_{p+1}}} = X_{i_1 i_2} \cdots X_{i_{p-1} i_p} X_{i_{p+1} i_{p+2}} \cdots X_{i_a j_1} X_{k_1 k_2} \cdots X_{k_c m_1}$$

and we identify $X_{i_a j_1}$ with $X_{i_a i_{a+1}}$, similarly for other terms. $\square$

For later use, we collect the following special cases of Lemma 4.20:

$$\begin{aligned} \{\operatorname{tr} X^a, \operatorname{tr} X^c\} &= \{\operatorname{tr} Y^b, \operatorname{tr} Y^d\} = 0 \\ \{\operatorname{tr} X^a, \operatorname{tr} Y^d\} &= ad \operatorname{tr} X^{a-1} Y^{d-1} \\ \{\operatorname{tr} X^a, \operatorname{tr} XY^d\} &= ad \operatorname{tr} X^a Y^{d-1} \\ \{\operatorname{tr} X^a, \operatorname{tr} X^c Y\} &= a \operatorname{tr} X^{a+c-1} \end{aligned}$$

The last two lines remain valid up to a minus sign if we exchange the roles of X and Y, since the map $(X, Y) \mapsto (Y, X)$ is antisymplectic.

Lemma 4.21 *We have that*

$$\operatorname{Lie}(\mathcal{F}) = \operatorname{Lie}(\mathcal{F}_1) = \operatorname{Lie}(\{\operatorname{tr} Y, \operatorname{tr} Y^2, \operatorname{tr} X^3, (\operatorname{tr} X)^2\})$$

Proof (1) Clearly $\mathrm{Lie}(\mathcal{F}_1) \subset \mathrm{Lie}(\{\operatorname{tr} Y, \operatorname{tr} Y^2, \operatorname{tr} X^3, (\operatorname{tr} X)^2\})$. Conversely, it suffices to show $\operatorname{tr} Y \in \mathrm{Lie}(\mathcal{F}_1)$. We compute

$$\begin{aligned} \{(\operatorname{tr} X)^2, \operatorname{tr} Y + \operatorname{tr} X^3\} &= 2n \operatorname{tr} X \\ \{\operatorname{tr} X, \operatorname{tr} Y^2\} &= 2 \operatorname{tr} Y \end{aligned}$$

which yields $\mathrm{Lie}(\{\operatorname{tr} Y, \operatorname{tr} Y^2, \operatorname{tr} X^3, (\operatorname{tr} X)^2\}) = \mathrm{Lie}(\mathcal{F}_1)$.

(2) Clearly $\mathrm{Lie}(\mathcal{F}) \subset \mathrm{Lie}(\mathcal{F}_1)$. Conversely, it suffices to show $\operatorname{tr} Y^2 \in \mathrm{Lie}(\mathcal{F})$. We compute

$$\begin{aligned} \{\operatorname{tr} Y^2 + (\operatorname{tr} X)^2, \operatorname{tr} Y + \operatorname{tr} X^3\} &= 2n \operatorname{tr} X - 6 \operatorname{tr} X^2 Y \\ \{n \operatorname{tr} X - 3 \operatorname{tr} X^2 Y, \operatorname{tr} Y + \operatorname{tr} X^3\} &= n^2 - 6 \operatorname{tr} XY + 9 \operatorname{tr} X^4 \\ \{n^2 - 6 \operatorname{tr} XY + 9 \operatorname{tr} X^4, \operatorname{tr} Y + \operatorname{tr} X^3\} &= 54 \operatorname{tr} X^3 - 6 \operatorname{tr} Y \end{aligned}$$

which implies $\operatorname{tr} Y, \operatorname{tr} X^3 \in \mathrm{Lie}(\mathcal{F})$ since we obtained a linear combination different from $\operatorname{tr} Y + \operatorname{tr} X^3 \in \mathrm{Lie}(\mathcal{F})$. We further compute

$$\begin{aligned} \{\operatorname{tr} X^3, \operatorname{tr} Y\} &= 3 \operatorname{tr} X^2 \\ \{\operatorname{tr} X^2, \operatorname{tr} Y^2 + (\operatorname{tr} X)^2\} &= 4 \operatorname{tr} XY \\ \{\operatorname{tr} Y^2 + (\operatorname{tr} X)^2, \operatorname{tr} XY\} &= 2(\operatorname{tr} X)^2 - 2 \operatorname{tr} Y^2 \end{aligned}$$

Therefore $\operatorname{tr} Y^2, (\operatorname{tr} X)^2 \in \mathrm{Lie}(\mathcal{F})$ since $\operatorname{tr} Y^2 + (\operatorname{tr} X)^2 \in \mathrm{Lie}(\mathcal{F})$. This completes the proof that $\mathrm{Lie}(\mathcal{F}_1) = \mathrm{Lie}(\mathcal{F})$. □

Lemma 4.22 $\{\operatorname{tr} X^j, \operatorname{tr} Y^j, (\operatorname{tr} X)^2, (\operatorname{tr} Y)^2 \colon j = 1, 2, 3\} \subset \mathrm{Lie}(\mathcal{F})$.

Proof It is straightforward to verify that

$$\begin{aligned} \{\{\{\operatorname{tr} X^3, \operatorname{tr} Y^2\}, \operatorname{tr} Y^2\}, \operatorname{tr} Y^2\} &= 48 \operatorname{tr} Y^3 \\ \{\{(\operatorname{tr} X)^2, \operatorname{tr} Y^2\}, \operatorname{tr} Y^2\} &= 8(\operatorname{tr} Y)^2 \\ \{\operatorname{tr} X^3, \operatorname{tr} Y\} &= 3 \operatorname{tr} X^2 \\ \{\operatorname{tr} X^2, \operatorname{tr} Y\} &= 2 \operatorname{tr} X \end{aligned}$$

□

Next, we make the following simplification.

Lemma 4.23 *For any integer $k \geq 0$*

$$\operatorname{tr} X^k, \operatorname{tr} Y^k, \operatorname{tr} X^k Y, \operatorname{tr} XY^k \in \mathrm{Lie}(\{\operatorname{tr} X^j, \operatorname{tr} Y^j \colon j = 1, 2, 3\})$$

Proof We first show $\operatorname{tr} X^k \in \mathrm{Lie}(\mathcal{F})$ for any integer $k \geq 0$. For $k \leq 3$ the statement is trivial, and hence we proceed by induction on $k \geq 3$. The induction step follows from

$$\begin{aligned} \{\operatorname{tr} X^k, \operatorname{tr} Y^2\} &= 2k \operatorname{tr} X^{k-1} Y \\ \{\operatorname{tr} X^3, \operatorname{tr} X^{k-1} Y\} &= 3 \operatorname{tr} X^{k+1} \end{aligned}$$

Now by $\operatorname{tr} X^k \in \mathrm{Lie}(\mathcal{F})$ and the first line above, $\operatorname{tr} X^{k-1} Y \in \mathrm{Lie}(\mathcal{F})$. Similarly for $\operatorname{tr} Y^k$ and $\operatorname{tr} XY^k$. □

Corollary 4.24 $\operatorname{tr} X^k Y^l \in \operatorname{Lie}(\mathcal{F})$ *for any integers* $k, l \geq 0$.

Proof By Lemma 4.23 $\operatorname{tr} X^{k+1}, \operatorname{tr} Y^{l+1} \in \operatorname{Lie}(\mathcal{F})$. It follows from the second line of the equations listed after Lemma 4.20 with $a = k+1, b = l+1$. □

Lemma 4.25 $(\operatorname{tr} XY)^2, (\operatorname{tr} X^k)^2, (\operatorname{tr} Y^k)^2 \in \operatorname{Lie}(\mathcal{F})$ *for any integer* $k \geq 0$.

Proof By Lemmas 4.22 and 4.23:

$$\operatorname{tr} X^2, \operatorname{tr} Y^2, \operatorname{tr} X^3, \operatorname{tr} Y^3, (\operatorname{tr} X)^2, (\operatorname{tr} Y)^2, \operatorname{tr} XY^2 \in \operatorname{Lie}(\mathcal{F})$$

Therefore

$$\begin{aligned} \{\operatorname{tr} X^3, (\operatorname{tr} Y)^2\} &= 6 \operatorname{tr} X^2 \operatorname{tr} Y \\ \{(\operatorname{tr} X)^2, \operatorname{tr} Y^3\} &= 6 \operatorname{tr} X \operatorname{tr} Y^2 \\ \{\operatorname{tr} X^2, \operatorname{tr} X \operatorname{tr} Y^2\} &= 4 \operatorname{tr} XY \operatorname{tr} X \\ \{(\operatorname{tr} X)^2, \operatorname{tr} XY \operatorname{tr} X\} &= 2(\operatorname{tr} X)^3 \qquad (4.8) \\ \{(\operatorname{tr} X)^3, \operatorname{tr} Y^3\} &= 9(\operatorname{tr} X)^2 \operatorname{tr} Y^2 \qquad (4.9) \end{aligned}$$

as well as

$$\begin{aligned} \{(\operatorname{tr} X)^3, (\operatorname{tr} Y)^2\} &= 6n(\operatorname{tr} X)^2 \operatorname{tr} Y \\ \{\operatorname{tr} XY^2, (\operatorname{tr} X)^2 \operatorname{tr} Y\} &= (\operatorname{tr} X)^2 \operatorname{tr} Y^2 - 4 \operatorname{tr} XY \operatorname{tr} X \operatorname{tr} Y \end{aligned}$$

Taking into account (4.9) we get $\operatorname{tr} XY \operatorname{tr} X \operatorname{tr} Y \in \operatorname{Lie}(\mathcal{F})$. Moreover, from

$$\{\operatorname{tr} X^2 \operatorname{tr} Y, \operatorname{tr} X \operatorname{tr} Y^2\} = 4 \operatorname{tr} XY \operatorname{tr} X \operatorname{tr} Y - n \operatorname{tr} X^2 \operatorname{tr} Y^2$$

we conclude $\operatorname{tr} X^2 \operatorname{tr} Y^2 \in \operatorname{Lie}(\mathcal{F})$. We proceed with

$$\begin{aligned} \{\operatorname{tr} X^2 \operatorname{tr} Y^2, \operatorname{tr} Y^2\} &= 4 \operatorname{tr} XY \operatorname{tr} Y^2 \\ \{\operatorname{tr} X^2, \operatorname{tr} XY \operatorname{tr} Y^2\} &= 2 \operatorname{tr} X^2 \operatorname{tr} Y^2 + 4(\operatorname{tr} XY)^2 \end{aligned}$$

which shows $(\operatorname{tr} XY)^2 \in \operatorname{Lie}(\mathcal{F})$. Hence we can obtain $(\operatorname{tr} X^k)^2$ for any $k \in \mathbb{N}$ from $\operatorname{tr} X^k$ and $(\operatorname{tr} XY)^2$ as follows

$$\begin{aligned} \{\operatorname{tr} X^k, (\operatorname{tr} XY)^2\} &= 2k \operatorname{tr} X^k \operatorname{tr} XY \\ \{\operatorname{tr} X^k, \operatorname{tr} X^k \operatorname{tr} XY\} &= k(\operatorname{tr} X^k)^2. \end{aligned}$$

By symmetry $(\operatorname{tr} Y^k)^2$ is also in $\operatorname{Lie}(\mathcal{F})$. □

The $\mathbb{C}^*$-action (see Example 4.6) corresponding to the Hamiltonian $e = \operatorname{tr} XY$ induces a $\mathbb{Z}$-grading on $\mathbb{C}[\mathrm{M}_n(\mathbb{C}) \times \mathrm{M}_n(\mathbb{C})]$.

Definition 4.26 The *weight* of a monomial $h \colon \mathrm{M}_n(\mathbb{C}) \times \mathrm{M}_n(\mathbb{C}) \to \mathbb{C}$ is defined to be its grade w.r.t. the grading induced by e.

Remark 4.27 Let φ_t denote the flow map corresponding to e, i.e.

$$\varphi_t(X, Y) = (\exp(t)X, \exp(-t)Y)$$

For a monomial, we obtain

$$\varphi_t^*(X_{j_1,k_1} \cdots X_{j_m,k_m} \cdot Y_{j_1,k_1} \cdots Y_{j_n,k_n}) = \\ \exp((m-n)t)X_{j_1,k_1} \cdots X_{j_m,k_m} \cdot Y_{j_1,k_1} \cdots Y_{j_n,k_n},$$

and we can read off its weight $m - n$. In particular, the weight of a trace of a nonzero matrix monomial M of bidegree (m, n) in (X, Y) equals $m - n$. Note that

$$\{X_{jk}, e\} = +X_{jk} \text{ and } \{Y_{jk}, e\} = -Y_{jk}$$

By the Leibniz rule for the Poisson bracket, this implies

$$\{X_{j_1,k_1} \cdots X_{j_m,k_m} \cdot Y_{j_1,k_1} \cdots Y_{j_n,k_n}, e\} = \\ (m-n)X_{j_1,k_1} \cdots X_{j_m,k_m} \cdot Y_{j_1,k_1} \cdots Y_{j_n,k_n}$$

For a monomial $h\colon \mathrm{M}_n(\mathbb{C}) \times \mathrm{M}_n(\mathbb{C}) \to \mathbb{C}$ this implies

$$\{h, e\} = wh \tag{4.10}$$

where $w \in \mathbb{Z}$ is the weight of h. Moreover, let h_1, h_2 be traces of matrix monomials in X, Y. Then

$$w_{h_1 h_2} = w_{h_1} + w_{h_2} = w_{\{h_1,h_2\}}, \tag{4.11}$$

where the last relation holds when $\{h_1, h_2\} \neq 0$.

Lemma 4.28 *Let $h \in \mathrm{Lie}(\mathcal{F})$ with weight $w_h \neq 0$, let $k \in \mathbb{N}, p \in \mathbb{N}_0$. Then $h^k e^p \in \mathrm{Lie}(\mathcal{F})$.*

Proof First we show that $he^p \in \mathrm{Lie}(\mathcal{F})$ by induction. For $p = 0$ it is obvious. From Lemma 4.25 we have $e^2 \in \mathrm{Lie}(\mathcal{F})$. The induction step is given by $\{he^p, e^2\} = 2w_h he^{p+1}$. Next we show $h^k e^p \in \mathrm{Lie}(\mathcal{F})$ by induction on k. The case $k = 1$ is clear. The induction step is given by $\{h^k e^p, h\} = -pw_h h^{k+1} e^{p-1}$. □

4.3.2 Proof of Proposition 4.18

Let

$$a = \operatorname{tr} X,\ b = \operatorname{tr} Y,\ c = \frac{1}{2} \operatorname{tr} X^2,\ d = \frac{1}{2} \operatorname{tr} Y^2,\ e = \operatorname{tr} XY$$

and $\mathcal{A} = \{a, b, c, d, e\}$. Notice that $\mathcal{A} \subset \mathrm{Lie}(\mathcal{F})$ since $e = \{c, d\}$.

We consider only products of a, b, c, d, e and show that these products are contained in the Lie algebra generated by $\mathcal{F}$. We make good use of the grading induced by $e = \operatorname{tr} XY$ and its consequence Lemma 4.28. We generate the products of any powers in a, b, c, d, e in (4.12), Lemmas 4.29, 4.30 and 4.31 in the following order

$$a, b \to a, b, c \to a, b, c, e \to a, b, c, d, e.$$

The brackets of elements in $\mathcal{A}$ are:

$$\{a,b\} = n,\ \{a,c\} = 0,\ \{a,d\} = b,\ \{a,e\} = a,\ \{b,c\} = -a,$$
$$\{b,d\} = 0,\ \{b,e\} = -b,\ \{c,d\} = e,\ \{c,e\} = 2c,\ \{d,e\} = -2d.$$

These elements form a Lie subalgebra of $\mathrm{Lie}(\mathcal{F})$.

Since the only element of $\mathcal{A}$ with weight 0 is e, we obtain

$$a^i, b^j, c^k, d^l \in \mathrm{Lie}(\mathcal{F}), \quad i,j,k,l \in \mathbb{N}$$

by applying Lemma 4.28 to $h \in \{a,b,c,d\}$. Hence

$$\{a^{i+1}, b^{j+1}\} = n(i+1)(j+1)\, a^i b^j \in \mathrm{Lie}(\mathcal{F}) \tag{4.12}$$

Now we raise the power of c.

Lemma 4.29 *$a^i b^j c^k \in \mathrm{Lie}(\mathcal{F})$ for any $i,j,k \geq 0$.*

Proof By (4.12) we have $a^i b^{j+1}, c^k \in \mathrm{Lie}(\mathcal{F})$. Considering

$$\{a^i b^{j+1}, c^{k+1}\} = (j+1)(k+1) a^{i+1} b^j c^k$$

we get any monomial in a,b,c when the power of a is positive. To obtain $b^j c^k$ we use

$$\{ab^j c^k, b\} = nb^j c^k - jka^2 b^{j-1} c^{k-1}$$

Since the second term is in $\mathrm{Lie}(\mathcal{F})$ we have $b^j c^k \in \mathrm{Lie}(\mathcal{F})$ as well. □

Next, we raise the power of e.

Lemma 4.30 *$a^i b^j c^k e^p \in \mathrm{Lie}(\mathcal{F})$ for any integers $i,j,k,p \geq 0$.*

Proof By Lemma 4.29 we have in $\mathrm{Lie}(\mathcal{F})$ any monomial in a,b,c, thus

$$\mathrm{Lie}(\mathcal{F}) \ni \{a^i b^{j+1} c^k, ae\} = -n(j+1) a^i b^j c^k e + (i-j-1+2k) a^{i+1} b^{j+1} c^k$$

which yields $a^i b^j c^k e \in \mathrm{Lie}(\mathcal{F})$. To raise the power of e consider

$$\begin{aligned}&\{a^i b^{j+1} c^k e^p, ae\}\\ &= -n(j+1) a^i b^j c^k e^{p+1} - pa^{i+1} b^{j+1} c^k e^p + (i-j-1+2k) a^{i+1} b^{j+1} c^k e^p\end{aligned}$$

which inductively yields any monomial in a,b,c,e. □

Finally we generate all monomials in a,b,c,d,e.

Lemma 4.31 *$a^i b^j c^k d^l e^p \in \mathrm{Lie}(\mathcal{F})$ for any integers $i,j,k,l,p \geq 0$.*

Proof We proceed by induction on l. Lemma 4.30 gives the base case $l = 0$. The induction step follows from

$$\begin{aligned}&\{a^i b^j c^k e^p, d^l\}\\ =&\{a^i, d^l\} b^j c^k e^p + \{c^k, d^l\} a^i b^j e^p + \{e^p, d^l\} a^i b^j c^k\\ =&\, il\, a^{i-1} b^{j+1} c^k d^{l-1} e^p + kl\, a^i b^j c^{k-1} d^{l-1} e^{p+1} + 2pl\, a^i b^j c^k d^l e^{p-1}\end{aligned}$$ □

This finishes the proof of Proposition 4.18.

4.3.3 Proof of Theorem 4.16

To prove Theorem 4.16, we need to prove that for all integers $m \geq 1$ and $p_k, q_k \geq 0$ $(k = 1, 2, \ldots, m)$

$$\prod_{k=1}^{m} \operatorname{tr} X^{p_k} Y^{q_k} \in \operatorname{Lie}(\mathcal{F}) \tag{$*$}$$

on $\widehat{\mathcal{C}}_n$.

We will do this by multiple inductions on m, on the degree D of the product, and on another index that will occur in the course of the proof. The various induction proofs will be formulated as separate lemmas. When $D = 0$, $(*)$ represents the constant function 1, which is in $\operatorname{Lie}(\mathcal{F})$ by $\{\operatorname{tr} X, \operatorname{tr} Y/n\} = 1$. The overall inductive assumption is then that with some $D = 0, 1, 2, \ldots$

$$(*) \text{ holds when } \sum_k (p_k + q_k) \leq D \tag{$**$}$$

We use the notation

$$f \sim g$$

if f and g are of the same degree and are identical up to terms of degree $\leq \deg f - 4$. When $\deg f \leq D + 4$, then by $(**)$ the difference $f - g$ is in $\operatorname{Lie}(\mathcal{F})$. Furthermore, in this notation Corollary 4.14 and Proposition 4.20 imply

$$\{\operatorname{tr} X^a Y^b, \operatorname{tr} X^c Y^d\} \sim (ad - bc) \operatorname{tr} X^{a+c-1} Y^{b+d-1}$$

The proof of Theorem 4.16 starts with the generating set $\mathcal{F}$. Corollary 4.24 provides the first step for $\operatorname{tr} X^p Y^q$. Here, we continue with taking in factors of the form $\operatorname{tr} X^i$.

Lemma 4.32 *Assume that the inductive assumption $(**)$ holds, and let*

$$p, q, m \in \mathbb{N}_0, i_1, \ldots, i_m \in \mathbb{N}$$

with $p + q + \sum i_k \leq D + 4$. Then

$$\operatorname{tr} X^p Y^q \prod_{k=1}^{m} \operatorname{tr} X^{i_k} \in \operatorname{Lie}(\mathcal{F}).$$

Proof By induction on m; the case $m = 0$ is Corollary 4.24. Assume the lemma holds for some $m \geq 0$, and consider a product with $(m+1)$ factors of the form $\operatorname{tr} X^i$

$$f = \operatorname{tr} X^p Y^q \prod_{k=0}^{m} \operatorname{tr} X^{i_k}, \text{ where } p + q + \sum_{k=0}^{m} i_k \leq D + 4$$

Since $i_0 \geq 1$ we have

$$p + q + 1 + \sum_{k=1}^{m} i_k \leq D + 4 + 1 - i_0 \leq D + 4 \tag{4.13}$$

then by the induction assumption on m

$$\operatorname{tr} X^p Y^{q+1} \prod_{k=1}^{m} \operatorname{tr} X^{i_k} \in \operatorname{Lie}(\mathcal{F}).$$

We now use $\operatorname{Lie}(\mathcal{F}) \ni \{(\operatorname{tr} X)^2, \operatorname{tr} X^{i_0} Y\} = 2 \operatorname{tr} X \operatorname{tr} X^{i_0}$ to compute

$$\begin{aligned}
&\operatorname{Lie}(\mathcal{F}) \ni \{\operatorname{tr} X \operatorname{tr} X^{i_0}, \operatorname{tr} X^p Y^{q+1} \prod_{k=1}^{m} \operatorname{tr} X^{i_k}\} \qquad (4.14)\\
&= (q+1) \operatorname{tr} X^p Y^q \operatorname{tr} X^{i_0} \prod_{k=1}^{m} \operatorname{tr} X^{i_k} + i_0(q+1) \operatorname{tr} X^{p+i_0-1} Y^q \operatorname{tr} X \prod_{k=1}^{m} \operatorname{tr} X^{i_k}\\
&\quad + i_0 \cdot (\text{terms of degree} \leq p+q+i_0-5) \cdot \operatorname{tr} X \prod_{k=1}^{m} \operatorname{tr} X^{i_k}
\end{aligned}$$

By the induction assumption

$$\operatorname{tr} X^{p+i_0-1} Y^{q+1} \prod_{k=1}^{m} \operatorname{tr} X^{i_k} \in \operatorname{Lie}(\mathcal{F})$$

since $p + q + \sum_{k=0}^{m} i_k \leq D + 4$. Applying Lemma 4.20 and Corollary 4.14 we then obtain

$$\begin{aligned}
\operatorname{Lie}(\mathcal{F}) \ni &\{(\operatorname{tr} X)^2, \operatorname{tr} X^{p+i_0-1} Y^{q+1} \prod_{k=1}^{m} \operatorname{tr} X^{i_k}\}\\
= &2(q+1) \operatorname{tr} X^{p+i_0-1} Y^q \operatorname{tr} X \prod_{k=1}^{m} \operatorname{tr} X^{i_k} \qquad (4.15)
\end{aligned}$$

On the other hand, since $p + q + \sum_{k=0}^{m} i_k \leq D + 4$, the last term of (4.14) is in $\operatorname{Lie}(\mathcal{F})$ by $(**)$. Together with (4.14) and (4.15) this implies $f \in \operatorname{Lie}(\mathcal{F})$ and the proof is complete. □

Next we aim for more factors of the form $\operatorname{tr} X^p Y^q$.

Lemma 4.33 *Assume that the inductive assumption $(**)$ holds, and let $l \in \mathbb{N}, m \in \mathbb{N}_0$, $i_1, \ldots, i_l \in \mathbb{N}, p_1, \ldots, p_m, q_1, \ldots, q_m \in \mathbb{N}_0$ with $\sum_a i_a + \sum_b (p_b + q_b) \leq D + 4$. Then*

$$\left(\prod_{a=1}^{l} \operatorname{tr} X^{i_a} \right) \prod_{b=1}^{m} \operatorname{tr} X^{p_b} Y^{q_b} \in \operatorname{Lie}(\mathcal{F}).$$

Proof By induction on m; Lemma 4.32 is the induction base $m = 1$. Suppose the lemma holds for some $m \geq 1$, and consider a product with $m + 1$ factors of the form $\operatorname{tr} X^p Y^q$

$$f = \left(\prod_{a=1}^{l} \operatorname{tr} X^{i_a} \right) \prod_{b=0}^{m} \operatorname{tr} X^{p_b} Y^{q_b}$$

where

$$\sum_{a=1}^{l} i_a + \sum_{b=0}^{m} (p_b + q_b) \leq D + 4. \qquad (4.16)$$

If $q_c = 0$ for some $c \in \{1, \ldots, m\}$, then it follows from the induction assumption on m that

$$f = \left(\prod_{a=1}^{l} \operatorname{tr} X^{i_a} \right) \operatorname{tr} X^{p_c} \prod_{\substack{b=0 \\ b \neq c}}^{m} \operatorname{tr} X^{p_b} Y^{q_b} \in \operatorname{Lie}(\mathcal{F})$$

It remains to consider the case $q_b \geq 1$ for all $b = 1, \ldots, m$. By (4.16)

$$p_0 + 1 + \sum_{b=1}^{m} (p_b + q_b) \leq D + 4, \quad q_0 + 1 + \sum_{a=1}^{l} i_a \leq D + 4$$

Hence by the induction assumption on m

$$\operatorname{tr} X^{p_0+1} \prod_{b=1}^{m} \operatorname{tr} X^{p_b} Y^{q_b}, \quad \operatorname{tr} Y^{q_0+1} \prod_{a=1}^{l} \operatorname{tr} X^{i_a} \in \operatorname{Lie}(\mathcal{F})$$

Therefore

$$\begin{aligned} \operatorname{Lie}(\mathcal{F}) &\ni \{\operatorname{tr} X^{p_0+1} \prod_{b=1}^{m} \operatorname{tr} X^{p_b} Y^{q_b}, \operatorname{tr} Y^{q_0+1} \prod_{a=1}^{l} \operatorname{tr} X^{i_a}\} \\ &\sim (p_0+1)(q_0+1) \left(\prod_{a=1}^{l} \operatorname{tr} X^{i_a} \right) \operatorname{tr} X^{p_0} Y^{q_0} \prod_{b=1}^{m} \operatorname{tr} X^{p_b} Y^{q_b} \\ &+ (q_0+1) \operatorname{tr} X^{p_0+1} \left(\prod_{a=1}^{l} \operatorname{tr} X^{i_a} \right) \sum_{b=1}^{m} p_b \operatorname{tr} X^{p_b - 1} Y^{q_b + q_0} \prod_{c \neq b} \operatorname{tr} X^{p_c} Y^{q_c} \\ &- \operatorname{tr} X^{p_0+1} \operatorname{tr} Y^{q_0+1} \sum_{a=1}^{l} \sum_{b=1}^{m} i_a q_b \operatorname{tr} X^{p_b + i_a - 1} Y^{q_b - 1} \left(\prod_{d \neq a} \operatorname{tr} X^{i_d} \right) \prod_{c \neq b} \operatorname{tr} X^{p_c} Y^{q_c} \end{aligned} \tag{4.17}$$

Each summand on the third line of Equation (4.17) has m factors $\operatorname{tr} X^p Y^q$, is in $\operatorname{Lie}(\mathcal{F})$ by the induction assumption and its degree satisfies the condition (4.16).

Next, for $(q_1, \ldots, q_m) = (1, \ldots, 1)$ each summand on the last line of Equation (4.17) has m factors $\operatorname{tr} X^p Y^q$ (including $\operatorname{tr} Y^{q_0+1}$) and is in $\operatorname{Lie}(\mathcal{F})$ by the induction assumption on m. This implies that $f \in \operatorname{Lie}(\mathcal{F})$ with this choice of $q_b = 1, b = 1, \ldots, m$.

Suppose that $f \in \operatorname{Lie}(\mathcal{F})$ for all $(q_1, \ldots, q_m) \prec (s_1, \ldots, s_m)$ in lexicographic order. Then also $f \in \operatorname{Lie}(\mathcal{F})$ with $(q_1, \ldots, q_m) = (s_1, \ldots, s_m)$, since the terms on the last line of Equation (4.17) have $(q_1, \ldots, q_m) \prec (s_1, \ldots, s_m)$ as exponents in the powers of Y. An induction on $(q_1, \ldots, q_m)$ in lexicographic order shows $f \in \operatorname{Lie}(\mathcal{F})$ for any choice of q_b satisfying (4.16), and the proof is complete. □

Lemma 4.34 *Assume that the inductive assumption* (∗∗) *holds, and let* $m \in \mathbb{N}_0, p_1, \ldots, p_m, q_1, \ldots, q_m \in \mathbb{N}_0$ *with* $\sum_k (p_k + q_k) \leq D + 4$. *Then*

$$\prod_{k=1}^{m} \operatorname{tr} X^{p_k} Y^{q_k} \in \operatorname{Lie}(\mathcal{F}).$$

Proof By induction on m; $m = 1$ follows from Corollary 4.24. Suppose the lemma holds for some $m \geq 1$, and consider an $(m+1)$-fold product

$$f = \prod_{k=0}^{m} \operatorname{tr} X^{p_k} Y^{q_k}, \text{ where } \sum_{k=0}^{m} (p_k + q_k) \leq D + 4$$

Lemmas 4.32 and 4.33 imply that $\operatorname{tr} Y^{q_0+1}$, $\operatorname{tr} X^{p_0+1} \prod_{k=1}^{m} \operatorname{tr} X^{p_k} Y^{q_k} \in \operatorname{Lie}(\mathcal{F})$. Therefore

$$\begin{aligned}
\operatorname{Lie}(\mathcal{F}) \ni & \{\operatorname{tr} X^{p_0+1} \prod_{k=1}^{m} \operatorname{tr} X^{p_k} Y^{q_k}, \operatorname{tr} Y^{q_0+1}\} \\
& \sim (p_0+1)(q_0+1) \operatorname{tr} X^{p_0} Y^{q_0} \prod_{k=1}^{m} \operatorname{tr} X^{p_k} Y^{q_k} \\
& \quad + (q_0+1) \operatorname{tr} X^{p_0+1} \sum_{k=1}^{m} p_k \operatorname{tr} X^{p_k-1} Y^{q_0+q_k} \prod_{a \neq k} \operatorname{tr} X^{p_a} Y^{q_a}
\end{aligned}$$

The summands on the third line have degree $\leq D+4$ and are in $\operatorname{Lie}(\mathcal{F})$ by Lemma 4.33. Hence $f \in \operatorname{Lie}(\mathcal{F})$ and the proof is complete. □

Proof (Proof of Theorem 4.16) Lemma 4.34 gives the induction step from degree D up to degree $D+4$ and hence finishes the proof. □

Proof (Proof of Theorem 1.4) Proposition 4.15 and Theorem 4.16 established that all algebraic invariant functions on $\widehat{\mathcal{C}_n}$ are contained in the Lie algebra that is generated by the Hamiltonian functions in

$$\mathcal{F} = \{\operatorname{tr} Y^2 + (\operatorname{tr} X)^2, \operatorname{tr} Y + \operatorname{tr} X^3\}$$

Hence, on $\mathcal{C}_n$ we obtain all holomorphic Hamiltonian functions by taking limits: Since $\mathcal{C}_n$ is an affine variety, we may assume that it is a closed subvariety inside some $\mathbb{C}^N$. Every holomorphic function on $\mathcal{C}_n$ extends to a holomorphic function on $\mathbb{C}^N$, and thus its extension can be approximated uniformly on compacts of $\mathbb{C}^N$ by polynomials which in turn restrict to regular functions on $\mathcal{C}_n$ that approximate the holomorphic function uniformly on compacts of $\mathcal{C}_n$. This proves the Hamiltonian density property. □

Proof (Proof of Theorem 1.5) This follows from the preceding Theorem 1.4 and Lemma 4.3. □

Proof (Proof of Theorem 1.6) We apply the Andersén–Lempert Theorem, i.e. Theorem 2.46 and the fact that $\mathcal{C}_n$ has the Hamiltonian density property with $\mathcal{F}$ as above being the generators. The flows of $\mathcal{F}$ are complete according to Lemma 4.7. Then the corresponding flow maps generate the identity component of the group of holomorphic symplectic automorphisms: In Theorem 2.46, choose Φ_t to be a path that connects any given holomorphic symplectic automorphism to the identity and use the conclusion in the last paragraph of said theorem. □

Alternatively, we can use $\mathcal{F}_1$ with three generators whose flows are algebraic.

Chapter 5

Holomorphic approximation of symplectic diffeomorphisms for $\mathcal{C}_n^{\mathbb{R}}$ in $\mathcal{C}_n$

This chapter is organized as follows: Section 5.1 introduces the setup. In Section 5.2, we prove Theorem 1.11, which claims that the real Calogero–Moser space $\mathcal{C}_n^{\mathbb{R}}$ has the complex one as a symplectic complexification. Section 5.3 applies the symplectic density property of $\mathcal{C}_n$ to express real algebraic symplectic vector fields as Lie combinations of complete symplectic vector fields (Proposition 5.9). This allows for local approximation of real vector fields on $\mathcal{C}_n^{\mathbb{R}}$ by real Lie combinations of complete vector fields whose real-time flows preserve $\mathcal{C}_n^{\mathbb{R}}$ (Proposition 5.12). Section 5.4 extends this local approximation to automorphisms on Saturn-like subsets of $\mathcal{C}_n$. This step is crucial in the push-out method, which constructs a holomorphic automorphism that is close to the identity inside a complex ball (Theorem 5.16). Section 5.5 combines local approximation with the push-out method to obtain global approximation of symplectic diffeomorphisms by holomorphic automorphisms (Theorem 5.20). Section 5.6 examines an alternative totally real submanifold for which the complex Calogero–Moser space serves as a complexification.

5.1 Preparation

5.1.1 Hamiltonian diffeomorphism

The objects to be approximated are symplectic diffeomorphisms which are isotopic to the identity.

Let (M, Ω) be a smooth symplectic manifold without boundary. A *symplectic isotopy* is a jointly smooth map $\varphi\colon [0,1] \times M \to M$ such that φ_t is symplectic for every t in $[0,1]$ and $\varphi_0 = \mathrm{id}$. The isotopy φ_t is generated by a smooth family of vector fields V_t with $\mathrm{d}\varphi_t/\mathrm{d}t = V_t \circ \varphi_t$, $\varphi_0 = \mathrm{id}$. As φ_t is symplectic, the vector fields V_t are symplectic by Cartan's formula

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\varphi_t^*\Omega = \varphi_t^* L_{V_t}\Omega = \varphi_t^*\mathrm{d}(\iota(V_t)\Omega)$$

where L_{V_t} is the Lie derivative with respect to V_t. A *Hamiltonian isotopy* is a symplectic isotopy such that the closed one-form $\iota(V_t)\Omega$ is exact, namely V_t is Hamiltonian, for all t. If the manifold M is simply connected, then every symplectic isotopy is Hamiltonian (the smooth t-dependence of the Hamiltonian functions can be achieved by fixing a reference point p_0 in M and choosing $H_t(p_0) = 0$ for all t in $[0, 1]$). Moreover, we call a symplectomorphism which is the endpoint of a Hamiltonian isotopy a *Hamiltonian diffeomorphism*. Hamiltonian diffeomorphisms form a normal subgroup of the group of symplectic diffeomorphisms, see [MS17, §3.1].

5.1.2 Pointwise seminorm

Next, we explain the notation $\|\cdot\|_{C^k(\cdot)}$, $k \geq 1$, following Manne–Wold–Øvrelid [MWØ11, §2.2] and refer to [GG73, Ch. II] for details. Let $V \subset M$ be a subset and $p \in V$. Consider the equivalence relation on germs of C^k-smooth complex-valued functions at p: $f_p \sim g_p$ if and only if $f - g$ vanishes to k-th order at p. Denote by $\mathcal{J}_p^k$ the set of equivalence classes, which forms a finite dimensional vector space. The union

$$\mathcal{J}^k(M, V) = \bigcup_{p \in V} \mathcal{J}_p^k$$

can be endowed with the structure of a complex vector bundle over V. Each C^k-smooth function f on M induces a continuous section $\mathcal{J}^k(f)$ of $\mathcal{J}^k(M, V)$ by $\mathcal{J}^k(f)(p) = [f_p]$. Choose fiberwise a norm $|\cdot|$ on $\mathcal{J}^k(M, V)$ which varies continuously with respect to p. The pointwise seminorm $\|\cdot\|_{C^k(\cdot)}$ for f is simply the norm of the induced k-jet

$$\|f\|_{C^k(p)} := |\mathcal{J}^k(f)(p)|$$

For a compact $K \subset M$, we set

$$\|f\|_{C^k(K)} := \sup_{p \in K} \|f\|_{C^k(p)}$$

For a smooth mapping $\Phi \colon M \to M$ the pointwise seminorm with respect to a local smooth chart $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \colon U \to \alpha(U) \subset \mathbb{R}^m$ is

$$\|\Phi\|_{C^k(p)} := \sum_{j=1}^{m} \|\alpha_j \circ \Phi\|_{C^k(p)}$$

Similarly we have $\|\Phi\|_{C^k(K)}$ by taking the supremum over a compact K. A different local chart $\beta \colon U' \to \beta(U')$ yields another pointwise seminorm $\|\cdot\|'_{C^k(\cdot)}$ satisfying

$$C(p)^{-1}\|\Phi\|_{C^k(p)} \leq \|\Phi\|'_{C^k(p)} \leq C(p)\|\Phi\|_{C^k(p)}$$

where C is a positive continuous function on $U \cap U'$, therefore preserving the approximation on compact subsets (up to rescaling).

5.1.3 Complexification

We will need the notion of complexification in the symplectic context.

Definition 5.1 [DW22] Let $(M_{\mathbb{R}}, \omega_{\mathbb{R}})$ be a smooth symplectic manifold of dimension $2n$, $n \geq 1$. A *symplectic complexification* (M, ω, τ) of $(M_{\mathbb{R}}, \omega_{\mathbb{R}})$ is a holomorphic symplectic manifold M of complex dimension $2n$, together with a holomorphic symplectic form ω, an antiholomorphic involution $\tau\colon M \to M$ and a smooth map $\jmath\colon M_{\mathbb{R}} \to M$, satisfying

(i) The map $\jmath$ is a proper embedding.

(ii) The image of $M_{\mathbb{R}}$ is the fixed-point set of τ.

(iii) The pullback $\jmath^*\omega$ coincides with $\omega_{\mathbb{R}}$.

(iv) The pullback $\tau^*\omega$ is the complex-conjugated $\overline{\omega}$.

Remark 5.2 (a) Property (i) implies that $\jmath(M_{\mathbb{R}})$ is closed in M, thus we may identify $M_{\mathbb{R}}$ with $\jmath(M_{\mathbb{R}})$ as a totally real submanifold of maximal dimension in M.

(b) We also refer to $M_{\mathbb{R}}$ as the *real form* of M (with respect to τ).

Let $\mathrm{Aut}_{\omega}(M)$ be the group of holomorphic symplectic automorphisms of M and $\mathrm{Diff}_{\omega_{\mathbb{R}}}(M_{\mathbb{R}})$ the group of symplectic diffeomorphisms from $M_{\mathbb{R}}$ onto itself. Also denote by $\tau\mathrm{Aut}_{\omega}(M)$ the group of holomorphic symplectic automorphisms of M which preserve the real form, i.e.

$$\tau\mathrm{Aut}_{\omega}(M) = \{\Phi \in \mathrm{Aut}_{\omega}(M) : \Phi(M_{\mathbb{R}}) = M_{\mathbb{R}}\}$$

By property (iii), $\Phi|_{M_{\mathbb{R}}}$ lies in $\mathrm{Diff}_{\omega_{\mathbb{R}}}(M_{\mathbb{R}})$. Moreover, (iii) implies the real parts of τ-compatible (cf. Definition 2.12) ω-symplectic vector fields V, W are real $\omega_{\mathbb{R}}$-symplectic on $\mathcal{C}_n^{\mathbb{R}}$:

$$\omega(V, W) = \jmath^*\omega(\alpha^{-1}(V), \alpha^{-1}(W)) = \omega_{\mathbb{R}}(\alpha^{-1}(V), \alpha^{-1}(W))$$

where $\alpha\colon \mathrm{T}M \to \mathrm{T}^{1,0}M$ is the $\mathbb{R}$-isomorphism connecting smooth vector fields with holomorphic vector fields.

Let $H \in \mathcal{O}(M)$ be a holomorphic function and V the holomorphic Hamiltonian vector field associated to H, namely $\mathrm{d}H = i_V\omega$. One may compare the following lemma with [AF24, Theorem 1.3].

Lemma 5.3 *Under the settings of Definition 5.1, H is τ-compatible up to a constant if and only if V is τ-compatible.*

Proof Applying τ^* to $\mathrm{d}H = i_V\omega$ yields

$$\mathrm{d}\tau^*H = \tau^*\mathrm{d}H = \tau^*(i_V\omega) = i_{\tau^*V}\overline{\omega}$$

where the last equality follows from (iv). □

As at the beginning of Section 5.1.1, the notion of Hamiltonian isotopy can be defined for holomorphic symplectic manifold as well. If such a holomorphic isotopy Φ_t is in $\tau\mathrm{Aut}_{\omega}(M)$ for every t, then their restriction $\Phi_t|_{M_{\mathbb{R}}} = \jmath^*\Phi_t$ is a Hamiltonian isotopy on the real form $(M_{\mathbb{R}}, \omega_{\mathbb{R}})$ because τ-compatible Hamiltonian vector fields have τ-compatible Hamiltonian functions.

5.2 The real Calogero–Moser space

Now, we choose a real form for the Calogero–Moser space $\mathcal{C}_n$ by taking the conjugation τ on $\mathcal{M} = \mathrm{M}_n(\mathbb{C}) \oplus \mathrm{M}_n(\mathbb{C}) \oplus \mathbb{C}^n \oplus (\mathbb{C}^n)^*$

$$\tau(X, Y, v, w) = (X^*, Y^*, iw^*, iv^*)$$

The fixed-point set of τ is

$$\mathcal{M}^\tau = \{(X, Y, v, w) \in \mathcal{M} : X^* = X, Y^* = Y, v = iw^*\}$$

In particular, the first two components consist of Hermitian matrices.

It is straightforward to check for $g \in \mathrm{GL}_n(\mathbb{C})$,

$$\tau(g \cdot (X, Y, v, w)) = g' \cdot \tau(X, Y, v, w) \tag{5.1}$$

where $g' = (g^*)^{-1}$. Thus the $\mathrm{GL}_n(\mathbb{C})$-orbit through $z \in \mathcal{M}$ is being taken to an orbit through $\tau(z)$. On the other hand, if (X, Y, v, w) lies in $\mu^{-1}(iI_n)$ then $\tau(X, Y, v, w)$ is contained in $\mu^{-1}(iI_n)$

$$[X^*, Y^*] + (iw^*)(iv^*) = X^*Y^* - Y^*X^* - w^*v^* = iI_n$$

Therefore the conjugation on $\mathcal{M}$ induces a conjugation on $\mathcal{C}_n$, which we also denote by τ.

The original setup in [KKS78] is actually compatible with this conjugation. The real Calogero–Moser space was constructed as symplectic reduction over the unitary group in [KKS78, pp. 491–494], started with

$$\mathcal{M}_h = \mathfrak{h}(n) \oplus \mathfrak{h}(n) \oplus \mathbb{C}^n$$

where $\mathfrak{h}(n)$ denotes the $\mathbb{R}$-vector space of Hermitian square matrices of size n. As a real vector space, $\mathcal{M}_h$ is of dimension $2n(n+1)$ and is isomorphic to $\mathcal{M}^\tau \subset \mathcal{M}$ under $(A, B, a) \mapsto (A, B, a, ia^*)$.

The unitary group $\mathrm{U}(n)$ acts on $\mathcal{M}_h$ as

$$u \cdot (A, B, a) = (uAu^*, uBu^*, ua), \; u \in \mathrm{U}(n)$$

which is the restriction of the $\mathrm{GL}_n(\mathbb{C})$-action on $\mathcal{M}^\tau \cong \mathcal{M}_h$.

Let $\mathfrak{u}(n)$ be the Lie algebra of the unitary group $\mathrm{U}(n)$, which consists of skew-Hermitian matrices. Take the real moment map

$$\mu_{\mathbb{R}} \colon \mathcal{M}_h \to \mathfrak{u}(n), \quad (A, B, a) \mapsto [A, B] + iaa^*$$

which is the restriction of μ on $\mathcal{M}^\tau$. Denote by $\omega_{\mathbb{R}}$ the pullback of the holomorphic form ω to $\mathcal{M}^\tau$ under the inclusion $\mathcal{M}^\tau \hookrightarrow \mathcal{M}$, which is a real symplectic form in the following real coordinates on $\mathcal{M}^\tau$

$$X_{jj}, Y_{jj}, \mathrm{Re}(v_j), \mathrm{Im}(v_j), \; j = 1, \ldots, n,$$
$$\mathrm{Re}(X_{jk}), \mathrm{Re}(Y_{kj}), \mathrm{Im}(X_{jk}), \mathrm{Im}(Y_{kj}), \; j < k$$

On the fixed-point set $\mathcal{M}^\tau$

$$\mathrm{d}v_j \wedge \mathrm{d}w_j = i\,\mathrm{d}v_j \wedge \mathrm{d}\overline{v_j} = 2\,\mathrm{d}\,\mathrm{Re}(v_j) \wedge \mathrm{d}\,\mathrm{Im}(v_j)$$

Then for $(A, B, a, ia^*) \in \mathcal{M}^\tau$ the real symplectic form $\omega_{\mathbb{R}}$ is

$$\begin{aligned}\omega_{\mathbb{R}} = &\sum_j \big(\mathrm{d}A_{jj} \wedge \mathrm{d}B_{jj} + 2\mathrm{d}\,\mathrm{Re}(a_j) \wedge \mathrm{d}\,\mathrm{Im}(a_j)\big) \\ &+ 2\sum_{j<k} \big(\mathrm{d}\,\mathrm{Re}(A_{jk}) \wedge \mathrm{d}\,\mathrm{Re}(B_{kj}) - \mathrm{d}\,\mathrm{Im}(A_{jk}) \wedge \mathrm{d}\,\mathrm{Im}(B_{kj})\big)\end{aligned}$$

The $\mathrm{GL}_n(\mathbb{C})$-invariance of ω implies the $\mathrm{U}(n)$-invariance of $\omega_{\mathbb{R}}$, which induces a real symplectic form on the quotient over $\mathrm{U}(n)$. With the symplectic form $\omega_{\mathbb{R}}$ and the moment map $\mu_{\mathbb{R}}$, we have the following.

Definition 5.4 The *real Calogero–Moser space* $(\mathcal{C}_n^{\mathbb{R}}, \omega_{\mathbb{R}})$ is the real symplectic reduction $\mu_{\mathbb{R}}^{-1}(iI_n) / \mathrm{U}(n)$.

In Lemma 4.2 it is shown that A, B have no nontrivial common invariant subspace. If $u \in \mathrm{U}(n)$ fixes $(A, B, a) \in \mu_{\mathbb{R}}^{-1}(iI_n)$, then u is a scalar and by $u \cdot a = a$ must be the identity. We see that $\mathrm{U}(n)$ acts freely on $\mu_{\mathbb{R}}^{-1}(iI_n)$ and the quotient is smooth.

Lemma 5.5 *The real Calogero–Moser space $\mathcal{C}_n^{\mathbb{R}}$ is simply connected.*

Proof Since any Hermitian matrix is diagonalizable by conjugation of the unitary group, we can consider A in diagonal form with decreasing diagonal entries $x_1, x_2, \ldots, x_n$. Recall that $(A, B, a) \in \tilde{\mathcal{C}}_n^{\mathbb{R}}$ satisfies $[A, B] + iaa^* = iI_n$. As the commutator has zeros on the diagonal, a is of the form $(e^{i\theta_1}, \ldots, e^{i\theta_n})$. Use

$$u = \mathrm{diag}(e^{-i\theta_1}, \ldots, e^{-i\theta_n})$$

to reduce all components of a to one. Then the off-diagonal entry b_{jk} of B satisfies

$$b_{jk}(x_j - x_k) + i = 0$$

making $x_j \neq x_k$ and $b_{jk} = -i/(x_j - x_k)$, while the real diagonal entries $y_1, \ldots, y_n$ of the Hermitian matrix B remain free. We have

$$(x_1, \ldots, x_n, y_1, \ldots, y_n)$$

with $x_1 > x_2 > \cdots > x_n$ is a system of global coordinates for $\mathcal{C}_n^{\mathbb{R}}$, which is a convex open set in $\mathbb{R}^{2n}$. $\square$

With the notation $\tilde{\mathcal{C}}_n^{\mathbb{R}} = \mu_{\mathbb{R}}^{-1}(iI_n)$, $\tilde{\mathcal{C}}_n = \mu^{-1}(iI_n)$ we have

$$\begin{tikzcd} \tilde{\mathcal{C}}_n \arrow[r, "\tau"] \arrow[d] & \tilde{\mathcal{C}}_n \arrow[d] \\ \mathcal{C}_n \arrow[r, "\tau"] & \mathcal{C}_n \end{tikzcd} \qquad \begin{tikzcd} \tilde{\mathcal{C}}_n^{\mathbb{R}} \arrow[r, "j"] \arrow[d] & \tilde{\mathcal{C}}_n \arrow[d] \\ \mathcal{C}_n^{\mathbb{R}} \arrow[r, "\jmath"] & \mathcal{C}_n \end{tikzcd}$$

The map $\jmath$ takes an orbit $\mathrm{U}(n) \cdot z$ with $z \in \tilde{\mathcal{C}}_n^{\mathbb{R}}$ to the orbit $\mathrm{GL}_n(\mathbb{C}) \cdot z$, and j is the inclusion map. The first diagram commutes due to Equation (5.1), while the second commutes naturally. To simplify the notation we use $\mathrm{U} = \mathrm{U}(n)$ and $\mathrm{G} = \mathrm{GL}_n(\mathbb{C})$.

Remark 5.6 In general, cf. [AF24, Remark 2.1], the image of the real symplectic quotient space is only one connected component of the conjugation-fixed part of the complex symplectic quotient. For the Calogero–Moser space, we have $\jmath(\mathcal{C}_n^{\mathbb{R}}) = \mathcal{C}_n^{\tau}$, which is equivalent to the connectedness of $\mathcal{C}_n^{\tau}$.

Next, we show that the complex Calogero–Moser space $(\mathcal{C}_n, \omega, \tau)$ is a symplectic complexification of the real Calogero–Moser space $(\mathcal{C}_n^{\mathbb{R}}, \omega_{\mathbb{R}})$.

Proof (of Theorem 1.11) We verify the conditions of Definition 5.1. Condition (iii) follows from the definition of $\omega_{\mathbb{R}}$ as the pullback of ω under $\jmath$.

For (iv), the pullback $\tau^*\omega$ is the complex conjugated form

$$\begin{aligned}\tau^*\omega &= \tau^* \operatorname{tr}(\mathrm{d}X \wedge \mathrm{d}Y + \mathrm{d}v \wedge \mathrm{d}w) \\ &= \operatorname{tr}(\mathrm{d}X^* \wedge \mathrm{d}Y^* + i^2 \mathrm{d}w^* \wedge \mathrm{d}v^*) \\ &= \operatorname{tr}(\mathrm{d}\overline{X} \wedge \mathrm{d}\overline{Y} + \mathrm{d}\overline{v} \wedge \mathrm{d}\overline{w}) \\ &= \overline{\omega}\end{aligned}$$

To show (ii) that $\jmath(\mathcal{C}_n^{\mathbb{R}})$ is the fixed-point set of τ, let $g \in \mathrm{G}, z \in \tilde{\mathcal{C}}_n^{\mathbb{R}}$. By Equation (5.1)

$$\tau(g \cdot z) = g' \cdot \tau(z) = g' \cdot z \in \mathrm{G} \cdot z$$

thus the orbit $\mathrm{G} \cdot z$ is stable under τ and $\jmath(\mathcal{C}_n^{\mathbb{R}})$ is a subset of $\mathcal{C}_n^{\tau}$.

Conversely, we show that when a G-orbit in $\tilde{\mathcal{C}}_n$ is stable under τ, then it meets $\tilde{\mathcal{C}}_n^{\mathbb{R}}$ at one U-orbit. Let z_0 be a point in a τ-stable G-orbit. Take the U-invariant function

$$p_z \colon \mathrm{G} \to \mathbb{R}_{\geq 0}, \quad g \mapsto \|g \cdot z\|^2$$

where

$$\begin{aligned}\|(X, Y, v, w)\|^2 &= \|X\|^2 + \|Y\|^2 + \|v\|^2 + \|w\|^2 \\ &= \operatorname{tr} XX^* + \operatorname{tr} YY^* + \operatorname{tr} vv^* + \operatorname{tr} w^*w\end{aligned}$$

Clearly $\|\tau(z)\| = \|z\|$. By geometric invariant theory, see [Nak99, §3.1], p_z is convex on the double coset $\mathrm{U} \setminus \mathrm{G} / \mathrm{G}_z$ and attains minimum exactly when $\mu_1(g \cdot z) = 0$, where

$$\mu_1(X, Y, v, w) = \frac{1}{2}\{[X, X^*] + [Y, Y^*] + vv^* - w^*w\}$$

Since the G-action is free, the isotropy group G_z is trivial. Recall from [Wil98, §8] that the complex space $\mathcal{C}_n$ is homeomorphic to the hyperkähler quotient

$$\mathcal{C}_n = \mu^{-1}(iI_n) / \mathrm{G} \cong (\mu_1^{-1}(0) \cap \mu^{-1}(iI_n)) / \mathrm{U}$$

Thus each orbit $\mathrm{G} \cdot z$ meets the U-stable set $\mu_1^{-1}(0)$ at exactly one U-orbit, where p_z attains minimum.

By $\tilde{\mathcal{C}}_n^{\mathbb{R}} \subset \mu_1^{-1}(0)$, we consider $z_0 = (X, Y, v, w) \in \mu_1^{-1}(0) \cap \tilde{\mathcal{C}}_n$. Since the orbit $\mathrm{G} \cdot z_0$ is τ-stable, there exists $h_0 \in \mathrm{G}$ such that

$$\tau(z_0) = h_0 \cdot z_0$$

By the convexity of p_{z_0} on $\mathrm{U}\setminus\mathrm{G}$, $\mu_1(z_0) = 0$, and $\|\tau(z_0)\| = \|z_0\|$, it follows that $h_0 \in \mathrm{U}$. Then

$$z_0 = \tau(\tau(z_0)) = \tau(h_0 \cdot z_0) = h_0 \cdot \tau(z_0) = h_0^2 \cdot z_0$$

where the third equality is due to Equation (5.1) and $h_0' = (h_0^*)^{-1} = h_0$. By [Wil98, Corollary 1.4] (cf. Lemma 4.2), if $g \in \mathrm{G}$ fixes a point $z \in \tilde{\mathcal{C}}_n$, then g is the identity. This implies that $h_0^2 = I_n$. In combination with $h_0 \in \mathrm{U}$, we see that h_0 is also Hermitian. Hence h_0 is of the form uDu^* for some $u \in \mathrm{U}$, and $D = \mathrm{diag}(d_1, \ldots, d_n)$ with entries either 1 or -1. Replacing z_0 by $u^* \cdot z_0$, we may assume that $h_0 = D$. Then from $\tau(z_0) = D \cdot z_0$ we have

$$X^* = DXD, \quad Y^* = DYD$$

which implies that X and Y are normal. Hence there exists $u_1 \in \mathrm{U}$ which diagonalizes X.

Move to the point $z_1 = u_1 \cdot z_0 = (X_1, Y_1, v_1, w_1)$ on the same U-orbit. Because $z_1 \in \tilde{\mathcal{C}}_n$

$$iI_n = [X_1, Y_1] + v_1 w_1 = [X_1, Y_1] + iv_1 v_1^* u_1 D u_1^* \tag{5.2}$$

where the second equality follows from $h_1 = u_1 D u_1^*$ and

$$\tau(z_1) = h_1 \cdot z_1 \implies iv_1^* = w_1 u_1 D u_1^*$$

Thanks to X_1 being diagonal, the commutator $[X_1, Y_1]$ has zeros on the diagonal. Comparing the j-th diagonal entry of Equation (5.2) yields

$$i = i\, d_j |(u_1^* v_1)_j|^2$$

which implies that $d_j = 1$. Therefore, $h_0 = I_n$ and $\tau(z_0) = z_0$. Since τ commutes with the U-action, the entire orbit $\mathrm{U} \cdot z_0$ is contained in $\tilde{\mathcal{C}}_n^{\mathbb{R}}$. This shows that $\mathcal{C}_n^\tau \subset \jmath(\mathcal{C}_n^{\mathbb{R}})$.

For (i): The injectivity of $\jmath \colon \mathcal{C}_n^{\mathbb{R}} \to \mathcal{C}_n$ is equivalent to that each G-orbit through a point $z \in \tilde{\mathcal{C}}_n^\tau$ meets $\tilde{\mathcal{C}}_n^{\mathbb{R}}$ at exactly one U-orbit. By Equation (5.1), this condition implies that the G-orbit is τ-stable. Thus the injectivity follows from the above discussion, but we also give a direct proof: We show that for $z_1, z_2 \in \tilde{\mathcal{C}}_n^{\mathbb{R}}$ if their U-orbits have empty intersection, then their G-orbits have empty intersection. Suppose otherwise, that $g \cdot z_1 = z_2$ for some $g \in \mathrm{G}$. Equation (5.1) and τ-invariance yield $g' \cdot z_1 = z_2$. By [Wil98, Corollary 1.4], $g^{-1} g' = I_n$. It follows that $g \in \mathrm{U}$ and $\jmath$ is injective. Note that in the more general setting of [AF24, Theorem 3.2], the injectivity of $\jmath$ is also a consequence of the free G-action on $\tilde{\mathcal{C}}_n^{\mathbb{R}}$.

To check that $\jmath$ is an immersion, we refer to [AF24, Theorem 3.2].

The properness of $\jmath$ follows from $\mathcal{C}_n^{\mathbb{R}} = \mathcal{C}_n \cap \mathbb{R}^q$, see Remark 5.8. This shows (i) of Definition 5.1. □

5.3 τ-symplectic density property

It is well-known that instead of the moment map $\mu\colon \mathcal{M} \to \mathrm{M}_n(\mathbb{C})$ and the preimage $\mu^{-1}(iI_n)$ of a coadjoint-invariant point, it is equivalent to consider the moment map

$$\hat{\mu}\colon \mathrm{M}_n(\mathbb{C}) \oplus \mathrm{M}_n(\mathbb{C}) \to \mathfrak{sl}_n^* \cong \mathfrak{sl}_n, \quad (X,Y) \mapsto [X,Y]$$

and take the complex Calogero–Moser space as the symplectic reduction

$$\hat{\mu}^{-1}(\mathrm{G}\cdot\xi)/\,\mathrm{G}$$

along a coadjoint orbit $\mathrm{G}\cdot\xi$, where any off-diagonal entry of ξ is $-i$ and diagonal entries 0. For the real counterpart, the real Calogero–Moser space $\mathcal{C}_n^{\mathbb{R}}$ is the quotient $\hat{\mu}_{\mathbb{R}}^{-1}(\mathrm{U}\cdot\xi)/\,\mathrm{U}$ along the coadjoint orbit $\mathrm{U}\cdot\xi$ where the real moment map is the restriction of $\hat{\mu}$

$$\hat{\mu}_{\mathbb{R}}\colon \mathfrak{h}(n) \oplus \mathfrak{h}(n) \to \mathfrak{u}(n), \quad (A,B) \mapsto [A,B]$$

Let

$$\widehat{\mathcal{C}}_n = \hat{\mu}^{-1}(\mathrm{G}\cdot\xi) \subset \mathrm{M}_n(\mathbb{C}) \oplus \mathrm{M}_n(\mathbb{C}), \quad \widehat{\mathcal{C}}_n^{\mathbb{R}} = \hat{\mu}_{\mathbb{R}}^{-1}(\mathrm{U}\cdot\xi) \subset \mathfrak{h}(n) \oplus \mathfrak{h}(n)$$

Then $\widehat{\mathcal{C}}_n$ is the variety of $\mathrm{M}_n(\mathbb{C}) \oplus \mathrm{M}_n(\mathbb{C}) \cong \mathbb{C}^{2n^2}$ consisting of pair of matrices (X,Y) such that $\operatorname{rank}([X,Y] - iI_n) = 1$. We also take the conjugation τ on $\widehat{\mathcal{C}}_n$ as $\tau(X,Y) = (X^*, Y^*)$. This induces a conjugation on the $\mathbb{C}$-algebra $\mathbb{C}[\widehat{\mathcal{C}}_n]$ of regular functions such that on scalars it is the complex conjugation. As mentioned in Section 5.1, of special interest are the *τ-compatible functions*, namely holomorphic functions $F\colon \widehat{\mathcal{C}}_n \to \mathbb{C}$ satisfying $F \circ \tau = \overline{F}$. A τ-compatible function is in particular real-valued on the real Calogero–Moser space.

Lemma 5.7 *The $\mathbb{R}$-algebra of real-valued* U*-invariant algebraic functions on $\widehat{\mathcal{C}}_n^{\mathbb{R}}$ can be generated by* $\{\operatorname{tr} A^j B^k : j+k \leq n^2\}$.

Proof Let f be a real-valued U-invariant algebraic function on $\widehat{\mathcal{C}}_n^{\mathbb{R}}$. Then f is a real polynomial in the entries of X and Y. Extend f to a real complex algebraic function F on $\widehat{\mathcal{C}}_n$

$$(X,Y) \mapsto \frac{1}{2}(f(X,Y) + \overline{f(\tau(X,Y))})$$

The unitary group U is a totally real submanifold of maximal dimension in G and elements of $\mathbb{C}[\widehat{\mathcal{C}}_n]$ are holomorphic, hence F is G-invariant. By Etingof and Ginzburg [EG02, §11, p. 322, Remark (ii)], the $\mathbb{C}$-algebra $\mathbb{C}[\widehat{\mathcal{C}}_n]^{\mathrm{G}}$ of G-invariants on $\widehat{\mathcal{C}}_n$ can be generated by $\{\operatorname{tr} X^j Y^k : j+k \leq n^2\}$. Write F in terms of these generators over $\mathbb{C}$. Separating the scalars into real and imaginary parts and collecting terms, we get $F = F_1 + iF_2$, where $F_l, l = 1,2$ are in the $\mathbb{R}$-algebra generated by these trace functions. From Equation (5.3), the generators $\operatorname{tr} X^j Y^k$ are real, thus F_l is real-valued on $\mathcal{C}_n^{\mathbb{R}}$. Thus $f = F|_{\mathcal{C}_n^{\mathbb{R}}} = F_1|_{\mathcal{C}_n^{\mathbb{R}}}$. ☐

Remark 5.8 Since the $\mathbb{C}$-algebra $\mathbb{C}[\widehat{\mathcal{C}}_n]^{\mathrm{G}}$ of G-invariants on $\widehat{\mathcal{C}}_n$ can be generated by $\{\operatorname{tr} X^j Y^k : j+k \leq n^2\}$, we choose a minimal generating set $\mathcal{G}$. Denote by q the cardinality of $\mathcal{G}$, then the Calogero–Moser space $\mathcal{C}_n$ is a smooth affine variety in

$$\mathbb{C}^q \cong \operatorname{Spec} \mathbb{C}[\mathcal{G}]$$

On the generating set $\mathcal{G}$, the conjugation $\tau(X,Y) = (X^*, Y^*)$ on $\widehat{\mathcal{C}}_n$ acts as complex conjugation due to the cyclicity of the trace

$$\tau^*(\operatorname{tr} X^j Y^k) = \operatorname{tr}(X^*)^j (Y^*)^k = \operatorname{tr} \overline{Y}^k \overline{X}^j = \overline{\operatorname{tr} X^j Y^k} \tag{5.3}$$

Hence τ descends to the complex conjugate on the coordinates of $\mathbb{C}^q$. The real Calogero–Moser space $\mathcal{C}_n^{\mathbb{R}}$, being the fixed-point set of τ in $\mathcal{C}_n$, is indeed the intersection of $\mathcal{C}_n$ with $\mathbb{R}^q$. This also shows that $\mathcal{C}_n^{\mathbb{R}}$ is totally real in $\mathcal{C}_n$.

Proposition 5.9 *On the real Calogero–Moser space $(\mathcal{C}_n^{\mathbb{R}}, \omega_{\mathbb{R}})$ every real algebraic Hamiltonian vector field can be written as **real** Lie combination of complete algebraic Hamiltonian vector fields, each of them associated to a* G*-invariant function from*

$$\mathcal{F} = \{\operatorname{tr} Y, \operatorname{tr} Y^2, \operatorname{tr} X^3, (\operatorname{tr} X)^2\}$$

Proof It suffices to consider the Hamiltonian functions. Since $\mathcal{C}_n^{\mathbb{R}}$ is a symplectic reduction, an algebraic Hamiltonian function f corresponds to a U-invariant $\hat{f}$ on $\widehat{\mathcal{C}}_n^{\mathbb{R}}$ and by Lemma 5.7 the algebraic U-invariants as a $\mathbb{R}$-algebra are generated by $\{\operatorname{tr} A^j B^k : j+k \leq n^2\}$. The computation in Chapter 4 shows that any algebraic G-invariant is contained in the complex Lie algebra generated by the four functions in $\mathcal{F}$. We point out that in Chapter 4 all formulae come with real coefficients, where the condition is that $[X,Y] + I_n$ has rank one. However we have a different condition, namely $[X,Y] - iI_n$ has rank one, which introduces an imaginary factor.

For a real-valued algebraic U-invariant $\hat{f}$ on $\widehat{\mathcal{C}}_n^{\mathbb{R}}$, we can first extend it to a G-invariant polynomial F on $\widehat{\mathcal{C}}_n$ as in the proof of Lemma 5.7. Then write it as a complex Lie combination of invariant polynomials in $\mathcal{F}$ and separate it into real and imaginary parts $F = F_1 + iF_2$. Fortunately, since F and the generating functions $\operatorname{tr} X^j Y^k$ are real-valued on $\mathcal{C}_n^{\mathbb{R}}$, F_2 is zero on $\widehat{\mathcal{C}}_n^{\mathbb{R}}$. Hence $\hat{f}$ is in the real Lie algebra generated by $\mathcal{F}$. □

Let M be a complex manifold with a real form (N, τ). Recall from Lemma 2.13 that a holomorphic vector field V on M is τ-compatible if $\tau_* V = \overline{V}$, which implies $\alpha^{-1}(V_p) = 2\operatorname{Re}(V_p) \in \mathrm{T}_p N$ for all $p \in N$, where

$$\alpha\colon \mathrm{T}M \hookrightarrow \mathrm{T}M \otimes_{\mathbb{R}} \mathbb{C} \to \mathrm{T}^{1,0}M,\; V \mapsto \frac{1}{2}(V - iJV)$$

is the $\mathbb{R}$-isomorphism between $\mathrm{T}M$ and $\mathrm{T}^{1,0}M$ (Definition 2.12). In particular the $\mathbb{R}$-flow of V preserves the submanifold. In fact, the following version of density property is appropriate for Carleman approximation.

Definition 5.10 Let $(M_{\mathbb{R}}, \omega_{\mathbb{R}})$ be smooth symplectic manifold and (M, ω, τ) a symplectic complexification of $(M_{\mathbb{R}}, \omega_{\mathbb{R}})$. We say that (M, ω, τ) admits the *τ-symplectic density property*, if the complex Lie algebra generated by $\mathbb{C}$-complete τ-compatible holomorphic symplectic vector fields is dense in the Lie algebra of holomorphic symplectic vector fields with respect to the compact-open topology.

For the purpose of approximation in the next section, it suffices for $\mathcal{C}_n$ to have this density property with respect to τ.

Lemma 5.11 *The Calogero–Moser space $\mathcal{C}_n$ admits the τ-symplectic density property.*

Proof Since the Lie generators in $\mathcal{F}$ are all τ-compatible with respect to τ by (5.3), the claim follows from Lemma 5.3 and the symplectic density property of $\mathcal{C}_n$. $\square$

The following is a parametric approximation on compacts for real symplectic vector fields on the real Calogero–Moser space. Apart from details, proofs in Chapter 5 starting from the following proposition are inspired by those in [DW22].

Proposition 5.12 *Let Z be a compact subset of $\mathbb{R}^N$ for some natural number N, V_z a continuous family of smooth Hamiltonian vector fields on the real Calogero–Moser space $\mathcal{C}_n^{\mathbb{R}}$. For any $\varepsilon > 0, k \in \mathbb{N}$, and compact $K \subset \mathcal{C}_n^{\mathbb{R}}$, there exists a continuous family of complete holomorphic Hamiltonian vector fields $W_{z,1}, \ldots, W_{z,m}$ on the complex Calogero–Moser space $\mathcal{C}_n$, and a real Lie combination $L(W_{z,1}, \ldots, W_{z,m})$, all having real-time flows that preserve $\mathcal{C}_n^{\mathbb{R}}$, such that*

$$\|V_z - \alpha^{-1}(L(W_{z,1}, \ldots, W_{z,m}))\|_{C^k(K)} < \varepsilon$$

where $\alpha \colon \mathrm{T}M \to \mathrm{T}^{1,0}M$ is the $\mathbb{R}$-isomorphism between $\mathrm{T}M$ and $\mathrm{T}^{1,0}M$.

Proof Fix $z \in Z$. In order to approximate Hamiltonian vector fields in the C^k-norm, by $dH = \omega_{\mathbb{R}}(V_H, \cdot)$ it suffices to approximate the Hamiltonian function f_z of V_z in the C^{k+1}-norm.

On the compact subset $K \subset \mathcal{C}_n^{\mathbb{R}}$ we approximate the smooth function f_z in the C^{k+1}-norm by a τ-compatible polynomial P_z, using the Weierstrass approximation theorem for compact subsets in $\mathbb{R}^q$ (see e.g. [Nar85, §1.6.2]). Since $\mathcal{C}_n$ is a submanifold in $\mathbb{C}^q$, approximation of functions on $\mathbb{C}^q$ in the C^{k+1}-norm on $\mathbb{C}^q$ implies the corresponding approximation on $\mathcal{C}_n$. By Proposition 5.9 the lift of P_z corresponds to a real Lie combination of finitely many U-invariants from $\mathcal{F}$ on $\widehat{\mathcal{C}}_n^{\mathbb{R}}$, hence we can approximate f_z in the C^{k+1}-norm by a real Lie combination $L(g_{z,1}, \ldots, g_{z,m})$ of functions $g_{z,1}, \ldots, g_{z,m}$ which correspond to complete vector fields.

Next consider the extensions of the U-invariant functions to $\widehat{\mathcal{C}}_n$ by requiring G-invariance. Let $W_{z,1}, \ldots, W_{z,m}$ be the corresponding complete Hamiltonian vector fields on $\mathcal{C}_n$. For each j, $W_{z,j}$ is the Hamiltonian vector field of a Hamiltonian function in $\mathcal{F}$. Since $\omega_{\mathbb{R}} = \jmath^*\omega$ where $\jmath \colon \mathcal{C}_n^{\mathbb{R}} \hookrightarrow \mathcal{C}_n$, the real-valued U-invariant functions induce, after being extended by G-invariance to $\widehat{\mathcal{C}}_n$, τ-compatible Hamiltonian vector fields. The τ-compatibility is stable under real scalar multiplication, summation and taking Lie brackets, thus the real Lie combination $L(W_{z,1}, \ldots, W_{z,m})$ remains τ-compatible. $\square$

5.4 Local approximation

To transfer the approximation in Proposition 5.12 for vector fields to one for symplectic diffeomorphisms, we need some preparation.

From vector fields to flow maps

A parametric version of Theorem 2.24 with extra uniform condition can be found in [DW22, Theorem 3.3], while the same statement with approximation in the

C^k-norm was proved in the unpublished Diplomarbeit of B. Schär [Sch07] and cited in [DW22, Theorem 3.4].

To apply the (symplectic) density property for approximation of holomorphic automorphisms, a further ingredient is that the nearness of vector fields in the C^k-norm implies the nearness of their flows in the C^k-norm. This is the content of [DW22, Lemma 3.2].

Mergelyan approximation on admissible sets

Recall that a Stein compact is a compact subset which admits a basis of open Stein neighborhoods. A compact $\mathcal{O}(X)$-convex subset Z in a Stein space M admits a basis of open Stein neighborhoods of the form

$$\{p \in M : |f_1(p)| < 1, \dots, |f_N(p)| < 1\}$$

for some $f_1, \dots, f_N \in \mathcal{O}(M)$, hence it is Stein compact.

Definition 5.13 A compact set S in a complex manifold M is called *admissible*, if it is of the form $S = K \cup Z$, where K is a totally real submanifold (possibly with boundary), S and Z are Stein compacts.

The next theorem is a parametric version of [FFW20, Theorem 20]. An inspection of the proof in [FFW20, pp. 29-31] shows that the same proof enhanced with parameter works with the following adjustment: For the case when the function has support in $S \setminus K$, apply [Car58, Lemma 3] to the linear continuous operator in the additive Cousin problem to find a solution which depends continuously on the parameter; for the general case, use the parametric Oka-Weil theorem. Furthermore, in Proposition 2 of [FFW20, §6.1], the approximation is obtained by taking convolution with the Gaussian kernel. Thus the continuous dependence on the parameter also holds for the approximation.

Actually, this parametric version was mentioned already in [For20, §5] with the hint that it can be obtained from the nonparametric case by applying a continuous partition of unity (similar to the proof of the parametric Oka-Weil theorem), which was carried out in detail in [GA23, Lemma 4.10].

Theorem 5.14 *[FFW20, Theorem 20] Let $S = K \cup Z$ be an admissible set in a complex manifold M, with K a totally real submanifold (possibly with boundary) of class C^k. Then for any $f \colon [0,1] \times S \to \mathbb{C}$ with $f_t \in C^k(S) \cap \mathcal{O}(Z)$ for $t \in [0,1]$ and f continuous in t, there exists a t-family of sequences $f_{j,t} \in \mathcal{O}(S)$ such that*

$$\lim_{j \to \infty} \sup_{t \in [0,1]} \|f_{j,t} - f_t\|_{C^k(S)} = 0.$$

Sublevel sets in $\mathcal{C}_n$

For p_0 in $\mathcal{C}_n^{\mathbb{R}}$, the mapping

$$\mathbb{C}^q \to \mathbb{C}^q, \quad p \mapsto \|\tau(p) - p_0\|^2$$

has a strictly positive definite Levi form, because by Remark 5.8 the conjugation τ interchanges a coordinate of $\mathbb{C}^q$ with its complex conjugation and is thus antiholomorphic. Next, an easy computation shows that the composition of a smooth strictly plurisubharmonic exhaustion function with an antiholomorphic diffeomorphism is strictly plurisubharmonic.

Definition 5.15 We take the strictly plurisubharmonic exhaustion function

$$\rho\colon \mathcal{C}_n \to \mathbb{R}_{\geq 0}, \quad p \mapsto \|p - p_0\|^2 + \|\tau(p) - p_0\|^2$$

where the norm $\|\cdot\|$ is the restriction of the standard norm of $\mathbb{C}^q$. Consider the closed sublevel set

$$Z_R = \{p \in \mathcal{C}_n : \rho(p) \leq R\} \tag{5.4}$$

Denote by

$$Z_R^{\mathbb{R}} = Z_R \cap \mathcal{C}_n^{\mathbb{R}}$$

the intersection of Z_R with the real part.

With these tools at hand, we show an approximation of Andersén-Lempert type for symplectic diffeomorphisms of $\mathcal{C}_n^{\mathbb{R}}$ onto itself locally on Saturn-like sets.

Theorem 5.16 *Let $\varphi\colon [0,1] \times \mathcal{C}_n^{\mathbb{R}} \to \mathcal{C}_n^{\mathbb{R}}$ be a Hamiltonian isotopy and $R \geq 0$ such that $\varphi_t|_{Z_R^{\mathbb{R}}} = \mathrm{id}_{Z_R^{\mathbb{R}}}$ for all t in $[0,1]$. Then for any $k \in \mathbb{N}$, any $b \in (0,R)$ and any $a > b$, there exists a sequence of holomorphic Hamiltonian isotopies $\Phi_j\colon [0,1] \times \mathcal{C}_n \to \mathcal{C}_n$ such that $\Phi_{j,t} \in \tau\mathrm{Aut}_\omega(\mathcal{C}_n)$ for all t in $[0,1]$ and*

$$\|\Phi_{j,t} - \varphi_t\|_{C^k(Z_a^{\mathbb{R}})} \xrightarrow{j\to\infty} 0 \tag{5.5}$$

$$\|\Phi_{j,t} - \mathrm{id}\|_{C^k(Z_b)} \xrightarrow{j\to\infty} 0 \tag{5.6}$$

uniformly for t in $[0,1]$.

Proof Let $K \subset \mathcal{C}_n^{\mathbb{R}}$ be a compact subset of class C^{k+1} such that

$$\bigcup_{t\in[0,1]} \varphi_t(Z_a^{\mathbb{R}}) \subset\subset \mathrm{Int}(K)$$

This condition suffices for our purpose of approximating φ_t. In Step 1 we switch to the smooth family of vector fields V_t generating φ_t and approximate V_t on K. By Equation (5.7), to obtain approximation of flows from approximation of vector field, it suffices to impose the inclusion that the image of $Z_a^{\mathbb{R}}$ under φ_t for all t is contained in K. On this compact K we invoke Proposition 5.12 to approximate smooth vector fields by Lie combinations of complete τ-compatible vector fields.

Step 1: By assumption φ_t is a Hamiltonian isotopy, thus there exist a family of Hamiltonian functions P_t and the corresponding family of Hamiltonian vector fields V_t such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t = V_t \circ \varphi_t, \quad t \in [0,1] \tag{5.7}$$

By the second assumption on φ_t we can extend V_t to be identically zero in an open neighborhood of Z_b.

For a fixed $m \in \mathbb{N}$, fixed $t \in [0,1]$, and $j \in \{0,1,\dots,m-1\}$, denote by $\varphi_s^{(j)}$ the time-s flow map of the time independent vector field $V_{jt/m}$. We can choose m large enough so that $\varphi_{t/m}^{(j)}$ is well-defined. Now we take

$$\varphi_t^m := \varphi_{t/m}^{(m-1)} \circ \varphi_{t/m}^{(m-2)} \circ \cdots \circ \varphi_{t/m}^{(1)} \circ \varphi_{t/m}^{(0)}$$

By [DW22, Corollary 3.5] the composition φ_t^m of flow maps converges to φ_t uniformly in the C^k-norm in p and uniformly in t as m tends to infinity. Therefore for a large enough m, it will be enough to show that each t-parameter family of flow maps $\varphi_s^{(j)}$ can be approximated in the C^k-norm in p by a t-parameter family of holomorphic automorphisms in $\tau\mathrm{Aut}_\omega(\mathcal{C}_n)$ uniformly in (t,s). Then by [DW22, Lemma 3.2], that approximation for vector field implies approximation for its flow, we may simply approximate the t-parameter family of Hamiltonian vector fields $V_{jt/m}$ uniformly in the C^k-norm in p and uniformly in t, by complete τ-compatible vector fields.

Step 2: Here we will use Proposition 5.12 to approximate the real vector field $V_{jt/m}$ on K with the extra requirement from (5.6) that the approximation is uniformly close to zero on $Z_b^{\mathbb{R}}$. Let $P_{j,t} = P_{jt/m}$ denote the Hamiltonian function for $V_{jt/m}$ on $\mathcal{C}_n^{\mathbb{R}}$ and extend $P_{j,t}$ to be zero on Z_b. This extension is allowed since $\varphi_t|_{Z_R^{\mathbb{R}}} = \mathrm{id}_{Z_R^{\mathbb{R}}}$ for all t in $[0,1]$ and $b < R$.

To find a τ-compatible function on $\mathcal{C}_n$ approximating $P_{j,t}$, we recall (from Remark 5.8) that

$$\mathcal{C}_n^{\mathbb{R}} = \mathcal{C}_n \cap \mathbb{R}^q \subset \mathbb{C}^q$$

Since $\mathcal{C}_n$ is an affine variety in $\mathbb{C}^q$, the $\mathcal{O}(\mathcal{C}_n)$-convex subset $Z_b \subset \mathcal{C}_n$ is polynomially convex in $\mathbb{C}^q$ by Cartan's Theorem B. By a result of Chirka and Smirnov [SC91, Theorem 2], the union of a compact set in $\mathbb{R}^q$ and a polynomially convex compact set in $\mathbb{C}^q$, which is symmetric with respect to $\mathbb{R}^q$, is polynomially convex. This yields the polynomial convexity of $K \cup Z_b$, and therefore its Stein compactness.

Since K is a totally real submanifold of class C^{k+1} and $K \cup Z_b$ is an admissible set in $\mathbb{C}^q$ (cf. Definition 5.13), we can use the above parametric version of [FFW20, Theorem 20] to approximate $P_{j,t}|_{K \cup Z_b}$ in the C^{k+1}-norm by a function $P'_{j,t}$ which is holomorphic in a neighborhood of $K \cup Z_b \subset \mathbb{C}^q$. Then by the Oka-Weil theorem with parameters (c.f. Theorem 2.10), there is a t-parameter family of holomorphic polynomials $Q_{j,t}$ which approximates $P'_{j,t}$ on $K \cup Z_b$. Replacing $Q_{j,t}$ by

$$\frac{1}{2}\left(Q_{j,t} + \overline{\tau^* Q_{j,t}}\right)$$

makes it real-valued on $\mathcal{C}_n^{\mathbb{R}}$. By Theorem B, restricting $Q_{j,t}$ to the affine variety $\mathcal{C}_n$ yields the desired τ-compatible function. Therefore, the associated Hamiltonian vector field $V_{j,t}$ for $Q_{j,t}$ approximates the real Hamiltonian vector field $V_{jt/m}$ for $P_{j,t}$ on K and zero on Z_b. As in the proof of Proposition 5.12, there are complete τ-compatible Hamiltonian vector fields $W_{t,1}, \dots, W_{t,M}$ on $\mathcal{C}_n$, such that a Lie combination $L(W_{t,1}, \dots, W_{t,M})$ of them approximates $V_{j,t}$.

Step 3: Finally, let $\psi_s^{t,a}$ be the time-s flow map of $W_{t,a}$ for $a = 1, \ldots, M$. Let $\Psi_s^{t,j}$ be an algorithm consistent with (cf. Lemma 2.25) the Lie combination $L(W_{t,1}, \ldots, W_{t,M})$. To see how the approximation operates on the level of automorphisms and to keep the notation manageable, we consider the simple case when $L(W_{t,1}, \ldots, W_{t,M})$ is the sum. Then

$$\Psi_s^{t,j} = \psi_s^{t,M} \circ \cdots \circ \psi_s^{t,1}$$

is an algorithm consistent with the sum $W_t = \sum_a W_{t,a}$, whose time-(t/m) flow map can be approximated by

$$(\Psi_{t/(ml)}^{t,j})^l, \quad l \to \infty$$

in the C^k-norm according to [DW22, Theorem 3.4]. Since W_t approximates $V_{j,t}$, which in turn approximates $V_{jt/m}$ on K and zero on Z_b. By [DW22, Lemma 3.2] we have that on $Z_a^{\mathbb{R}}$ the flow of W_t approximates the flow $\varphi_{t/m}^{(j)}$ of $V_{jt/m}$. Therefore

$$(\Psi_{t/(ml)}^{t,j})^l \xrightarrow{l \to \infty} \varphi_{t/m}^{(j)}$$

which is the missing link in Step 1. □

Remark 5.17 There is constraint only on b since it needs to be smaller than R for which φ_t is already the identity on the real part $Z_R^{\mathbb{R}}$. However we may choose a, which determines where the automorphisms approximate the given family of diffeomorphisms, as large as the situation requires.

5.5 Global approximation

5.5.1 Interlude

We digress, and take a close look at the classical Carleman approximation for functions, see e.g. [FFW20, Theorem 8].

Theorem 5.18 (Carleman 1927 [Car27]) *Given continuous functions $f\colon \mathbb{R} \to \mathbb{C}$ and $\varepsilon\colon \mathbb{R} \to \mathbb{R}_+$, there exists an entire function $F \in \mathcal{O}(\mathbb{C})$ such that*

$$|F(x) - f(x)| < \varepsilon(x)$$

for all $x \in \mathbb{R}$.

Proof Set $\overline{\Delta} = \{z \in \mathbb{C} : |z| \leq 1\}$ and take for $j \in \mathbb{N}_0$ the union of a disc and an interval

$$K_j = j\overline{\Delta} \cup [-j-2, j+2], \quad \varepsilon_j = \min\{\varepsilon(x) : |x| \leq j+2\}$$

We have $\varepsilon_j \geq \varepsilon_{j+1} > 0$ for all j. Next we construct a sequence of continuous functions $f_j\colon (j+1/3)\Delta \cup \mathbb{R} \to \mathbb{C}$ which satisfies the following conditions for all $j \in \mathbb{N}$

(a_j) the function f_j is holomorphic on $(j + \frac{1}{3})\Delta$

(b_j) for $x \in \mathbb{R}$ with $|x| \geq j + \frac{2}{3}$: $f_j(x) = f(x)$

(c_j) on K_{j-1}: $|f_j - f_{j-1}| < \frac{\varepsilon_{j-1}}{2^{j+1}}$

To start the induction base $j = 0$, take a smooth cutoff function $\chi\colon \mathbb{R} \to [0,1]$ such that $\chi(x) = 1$ for $|x| \leq 1/3$ and $\chi(x) = 0$ for $|x| \geq 2/3$. Apply Mergelyan's theorem to the function $f|_{K_0}$ on the compact $K_0 = [-2,2]$ to obtain a holomorphic polynomial h approximating $f|_{K_0}$ on K_0. Now set

$$f_0(x) = \begin{cases} h(x), & x \in \frac{1}{3}\Delta \\ \chi(x)h(x) + (1-\chi(x))f(x), & x \in \mathbb{R} \end{cases}$$

Roughly speaking, it amounts to glue the function h on the disc and f on the real line outside $[-2/3, 2/3]$ together continuously. By construction we have (a_0) and (b_0).

For the induction step $(j-1) \to j$, apply Mergelyan's theorem for $f_{j-1}|_{K_{j-1}}$ and K_{j-1} to get a holomorphic polynomial h_j such that

$$|h_j - f_{j-1}| < \frac{\varepsilon_{j-1}}{2^{j+1}}$$

on K_{j-1}. Then choose a smooth cutoff function $\chi_j\colon \mathbb{R} \to [0,1]$ such that $\chi_j(x) = 1$ for $|x| \leq j + 1/3$ and $\chi_j(x) = 0$ for $|x| \geq j + 2/3$. Again we glue by setting

$$f_j(x) = \begin{cases} h_j(x), & x \in (j + \frac{1}{3})\Delta \\ \chi_j(x)h_j(x) + (1-\chi_j(x))f_{j-1}(x), & x \in \mathbb{R} \end{cases}$$

By construction (a_j) and (b_j) are fulfilled. It thus remains to check (c_j) that f_j approximates f_{j-1} on $K_{j-1} = (j-1)\overline{\Delta} \cup [-j-1, j+1]$:

- On $(j + 1/3)\Delta$ by definition of h_j.
- For $x \in \mathbb{R}$ with $|x| \geq j + 2/3$ we have $\chi_j(x) = 0$, which implies that $f_j(x) = f_{j-1}(x)$.
- When $x \in \mathbb{R}$ with $j + 1/3 \leq |x| < j + 2/3$, we have
$$|f_j(x) - f_{j-1}(x)| = |\chi_j(x)h_j(x) - \chi_j(x)f_{j-1}(x)| \leq |h_j(x) - f_{j-1}(x)|$$

Finally, by virtue of (b_j) we have $f_0 = f_1 = \cdots = f_{k-1}$ on $\{|x| \geq k\}$ for any $k \in \mathbb{N}$. Combining this with (c_j) we can see that $\{f_j\}$ converges to an entire function $F \in \mathcal{O}(\mathbb{C})$ such that for any $k \in \mathbb{N}_0$ the following estimate holds on $\{k \leq |x| \leq k+1\}$

$$|F(x) - f(x)| \leq \sum_{j=0}^{\infty} |f_{j+1}(x) - f_j(x)| < \sum_{j=k-1}^{\infty} \frac{\varepsilon_j}{2^{j+1}} \leq \varepsilon_{k-1} \leq \varepsilon(x).$$

□

The local approximation is delivered by the following.

Theorem 5.19 (Mergelyan's theorem) *If K is a compact set in $\mathbb{C}$ with connected complement, then every function, which is continuous on K and holomorphic in the interior* $\mathrm{Int}(K)$*, can be approximated uniformly on K by holomorphic polynomials.*

From a bird's eye view, Theorem 5.16 is the analogue of Mergelyan's theorem, while ideas of gluing and pushing out present in the next section are analogous to the classical Carleman approximation.

5.5.2 From local to global approximation

The next statement is the counterpart of [DW22, Theorem 1.2] and likewise is a consequence of Theorem 5.16. Namely we will construct the global approximation by applying local approximation successively.

Theorem 5.20 *Let $\varphi \in \mathrm{Diff}_{\omega_{\mathbb{R}}}(\mathcal{C}_n^{\mathbb{R}})$ be a symplectic diffeomorphism which is smoothly isotopic to the identity. Then for any positive continuous function ε on $\mathcal{C}_n^{\mathbb{R}}$, there exists a holomorphic symplectic automorphism $\Phi \in \tau\mathrm{Aut}_\omega(\mathcal{C}_n)$ such that*

$$\|\Phi - \varphi\|_{C^k(p)} < \varepsilon(p)$$

for all p in $\mathcal{C}_n^{\mathbb{R}}$.

Remark 5.21 (i) The key idea parallels the proof of the classical Carleman approximation on $\mathbb{R} \subset \mathbb{C}$ by entire functions, see Theorem 5.18. Theorem 5.16 ensures the local approximation and requires the holomorphic convexity of Saturn-like sets, and a combination with the push-out method in [For99] delivers the wanted holomorphic automorphism in the limit.

(ii) To use Theorem 5.16, we need an isotopy ψ_t of Hamiltonian diffeomorphisms *equal* to the identity on a compact subset $K \subset \mathcal{C}_n^{\mathbb{R}}$. In the proof to follow, we usually end up with an isotopy φ_t only *approximating* the identity on K. Thus an extra interpolation is necessary. The goal is to construct a family ψ_t which is equal to φ_t on the complement of an open neighborhood containing K. To achieve this, we multiply the time dependent Hamiltonian function H_t of φ_t with a cutoff function γ, which is 0 inside K and 1 outside a neighborhood U_1 of K. The new Hamiltonian function γH_t induces a slightly different vector field W_t with flow map ψ_t.

Moreover, we assume that φ_t is close to the identity on U_2, equivalently V_t is close to zero on an open set U_2 containing U_1. This additional buffer zone $U_2 \setminus U_1$ is used to secure that the new flow ψ_t is equal to φ_t on the complement of U_2: A point p close to U_2 might flow along V_t into U_2, but since V_t is by assumption small on U_2, the trajectory will stay outside U_1, where the function γ is 1 and the flow is simply φ_t. Inside K, ψ_t is the identity by the definition of γ.

The Hamiltonian vector field W_t for γH_t is associated to the 1-form

$$\mathrm{d}(\gamma H_t) = \gamma \mathrm{d}H_t + H_t \mathrm{d}\gamma$$

The flow map ψ_t exists for time $t \in [0,1]$ since H_t is close to 0 on $U_1 \setminus K$.

Proof (of Theorem 5.20) [1] By Lemma 5.5 every symplectic isotopy in $\mathrm{Diff}_{\omega_{\mathbb{R}}}(\mathcal{C}_n^{\mathbb{R}})$ is a Hamiltonian isotopy. Thus there exists a smooth Hamiltonian isotopy α_t in $\mathrm{Aut}_{\omega_{\mathbb{R}}}(\mathcal{C}_n^{\mathbb{R}}), t \in [0,1]$ with $\alpha_0 = \mathrm{id}$ and $\alpha_1 = \varphi$.

Assume that we have a holomorphic symplectic automorphism Φ_j with $\Phi_j(\mathcal{C}_n^{\mathbb{R}}) = \mathcal{C}_n^{\mathbb{R}}$, a Hamiltonian isotopy $\psi_{j,t}$ of $\mathcal{C}_n^{\mathbb{R}}$, real numbers R_j, S_j with $R_j \geq j-1, S_j \geq R_j + 1$ such that

[1]For automorphisms, uppercase greek letters denote holomorphic symplectic automorphisms, while lowercase symplectic diffeomorphisms.

(1_j) The image of Z_{R_j} under Φ_j is contained in Z_{S_j}.

(2_j) For $j \geq 2$
$$\|\Phi_j - \Phi_{j-1}\|_{C^k(Z_{R_{j-1}})} < \varepsilon_j$$

(3_j) At time zero $\psi_{j,0} \colon \mathcal{C}_n^{\mathbb{R}} \to \mathcal{C}_n^{\mathbb{R}}$ is the identity.

(4_j) On $Z^{\mathbb{R}}_{S_j}$ we have $\psi_{j,t}$ is the identity for t in $[0,1]$.

(5_j) For all p in $\mathcal{C}_n^{\mathbb{R}}$
$$\|\psi_{j,1} \circ \Phi_j - \varphi\|_{C^k(p)} < \varepsilon(p)$$

The induction hypothesis is that given any positive ε_j, $(1_j) - (5_j)$ can be realized.

Induction base: For $j = 1$ take $R_1 = 0$. In this case we have
$$Z_0 = Z_0^{\mathbb{R}} = \{p_0\} \subset \mathcal{C}_n^{\mathbb{R}}$$
from (5.4) and by the choice of the strictly plurisubharmonic exhaustion function. Choose S_1 so that
$$\alpha_1\left(Z_1^{\mathbb{R}}\right) \subset\subset Z^{\mathbb{R}}_{S_1} \tag{5.8}$$
Choose $r > S_1$. Then by (5.8)
$$Z_0^{\mathbb{R}} \subset Z_1^{\mathbb{R}} \subset\subset (\alpha_1)^{-1}\left(Z^{\mathbb{R}}_{S_1}\right) \subset (\alpha_1)^{-1}\left(Z^{\mathbb{R}}_{r}\right) \tag{5.9}$$
Using Theorem 5.16 we get a holomorphic Hamiltonian isotopy $A_t \in \tau\mathrm{Aut}_\omega(\mathcal{C}_n)$ approximating α_t on the compact subset
$$\bigcup_{t\in[0,1]} (\alpha_t)^{-1}\left(Z^{\mathbb{R}}_{r+3}\right)$$
of $\mathcal{C}_n^{\mathbb{R}}$. Since A_t approximates α_t on this compact, we have
$$(A_t)^{-1}\left(Z^{\mathbb{R}}_{r+2}\right) \subset\subset (\alpha_t)^{-1}\left(Z^{\mathbb{R}}_{r+3}\right) \tag{5.10}$$
In particular, A_1 approximates α_1 on $A_1^{-1}\left(Z^{\mathbb{R}}_{r+2}\right)$, hence $\alpha_t \circ A_t^{-1}\big|_{\mathcal{C}_n^{\mathbb{R}}}$ is close to the identity on $Z^{\mathbb{R}}_{r+2}$. Choose $\Phi_1 = A_1$.

To construct a Hamiltonian isotopy $\psi_{1,t}$ we interpolate between the identity on $Z_r^{\mathbb{R}}$ and $\alpha_t \circ A_t^{-1}\big|_{\mathcal{C}_n^{\mathbb{R}}}$ outside $Z^{\mathbb{R}}_{r+2}$. More precisely, let Q_t be the Hamiltonian function associated to the Hamiltonian isotopy $\alpha_t \circ A_t^{-1}\big|_{\mathcal{C}_n^{\mathbb{R}}}$. Fix $\psi_{1,t}$ to be the identity on $Z_r^{\mathbb{R}}$ by multiplying Q_t with a cutoff function γ, which is zero on $Z_r^{\mathbb{R}}$ and one outside $Z^{\mathbb{R}}_{r+1}$. Namely, we take $\psi_{1,t}$ to be the isotopy of symplectic diffeomorphisms which comes from the Hamiltonian function γQ_t.

By the choice of r we have that $Z^{\mathbb{R}}_{S_1}$ is contained in $Z_r^{\mathbb{R}}$, which implies that $\psi_{1,t}$ is the identity on $Z^{\mathbb{R}}_{S_1}$ for all t in $[0,1]$. This shows (4_1).

To see that (5_1) is satisfied, let $p \in \mathcal{C}_n^{\mathbb{R}}$ and consider separately

(i) $p \in A_1^{-1}\left(Z_r^{\mathbb{R}}\right)$: $\psi_{1,1}$ is the identity at $A_1(p)$ and A_1 approximates α_1 by (5.10).

(ii) $p \notin A_1^{-1}\left(Z_{r+2}^{\mathbb{R}}\right)$: $\psi_{1,1}$ is equal to $\alpha_1 \circ A_1^{-1}$ at $A_1(p)$ by the choice of γ.

(iii) $p \in A_1^{-1}\left(Z_{r+2}^{\mathbb{R}} \setminus Z_r^{\mathbb{R}}\right)$: A_1 approximates α_1 by (5.10) and $\psi_{1,1} \circ \alpha_1$ is the interpolation between $\mathrm{id} \circ \alpha_1$ and $\alpha_1 \circ A_1^{-1} \circ \alpha_1$. Here $\alpha_1 \sim A_1$ implies $A_1^{-1} \circ \alpha_1 \sim \mathrm{id}$.

Last, let us check that (1_1) holds. By the choice of A_t, we may assume that there exists a small positive δ which is less than one, such that

$$\Phi_1\left(Z_0^{\mathbb{R}}\right) \subset \alpha_1\left(Z_\delta^{\mathbb{R}}\right) \subset \alpha_1\left(Z_1^{\mathbb{R}}\right) \subset\subset Z_{S_1}^{\mathbb{R}}$$

The first inclusion follows from the fact that $\Phi_1 = A_1$ approximates α_1 by (5.9) and the last inclusion is due to the choice of S_1 in (5.8). This concludes the induction base.

Induction step: Take $R_{j+1} = S_j + 1$ and choose $a > \max\{R_j + 1, R_{j+1}\}$ such that

$$\Phi_j(Z_{R_{j+1}}) \subset\subset Z_a \tag{5.11}$$

By Theorem 5.16 there exists a holomorphic Hamiltonian isotopy $\Psi_{j,t}$ in $\tau\mathrm{Aut}_\omega(\mathcal{C}_n)$, which approximates the identity near Z_{S_j} and $\psi_{j,t}$ near $Z_{a+2}^{\mathbb{R}}$. Thus

$$\sigma_{j,t} = \left(\Psi_{j,t}\big|_{\mathcal{C}_n^{\mathbb{R}}}\right)^{-1} \circ \psi_{j,t}$$

approximates the identity on $Z_{a+2}^{\mathbb{R}}$. Being the composition of two Hamiltonian isotopies, $\sigma_{j,t}$ is also a Hamiltonian isotopy.

Moreover take a cutoff function χ on $\mathcal{C}_n^{\mathbb{R}}$ such that it is zero on $Z_a^{\mathbb{R}}$ and equal to one outside $Z_{a+1}^{\mathbb{R}}$. Let P_t be the Hamiltonian function associated to the Hamiltonian isotopy $\sigma_{j,t}$ and let $\tilde{\sigma}_{j,t}$ be the flow map of the vector field whose Hamiltonian function is χP_t. Then $\tilde{\sigma}_{j,t}$ is the identity on $Z_a^{\mathbb{R}}$, close to the identity on $Z_{a+2}^{\mathbb{R}}$, and equal to $\sigma_{j,t}$ outside $Z_{a+2}^{\mathbb{R}}$.

Then we have on $\mathcal{C}_n^{\mathbb{R}}$

$$\Psi_{j,t} \circ \tilde{\sigma}_{j,t} \sim \psi_{j,t} \tag{5.12}$$

$$(\Psi_{j,t} \circ \tilde{\sigma}_{j,t})^{-1} \sim (\psi_{j,t})^{-1} \tag{5.13}$$

by the choices of $\Psi_{j,t}, \sigma_{j,t}, \tilde{\sigma}_{j,t}$. Next, choose $S_{j+1} > a$ so that

$$\Psi_{j,1}(Z_a) \subset\subset Z_{S_{j+1}} \tag{5.14}$$

Set $b = S_{j+1} + 1$ and pick c and d so that

$$Z_{b+2}^{\mathbb{R}} \subset\subset \varphi\left(Z_c^{\mathbb{R}}\right) \tag{5.15}$$

$$\Phi_j\left(Z_c^{\mathbb{R}}\right) \subset\subset Z_d^{\mathbb{R}} \tag{5.16}$$

$$Z_c^{\mathbb{R}} \subset\subset \psi_{j,1}\left(Z_d^{\mathbb{R}}\right) \tag{5.17}$$

Next, apply Theorem 5.16 to obtain an isotopy $\Sigma_{j,t} \in \tau\mathrm{Aut}_\omega(\mathcal{C}_n)$, which approximate $\tilde{\sigma}_{j,t}$ on $Z_d^{\mathbb{R}}$ and the identity on Z_a. Set $\Phi_{j+1} = \Psi_{j,1} \circ \Sigma_{j,1} \circ \Phi_j$. The above choices of c and d allow us to approximate on $Z_c^{\mathbb{R}}$

$$\varphi \sim \psi_{j,1} \circ \Phi_j \sim \Psi_{j,1} \circ \tilde{\sigma}_{j,1} \circ \Phi_j \sim \Psi_{j,1} \circ \Sigma_{j,1} \circ \Phi_j = \Phi_{j+1} \tag{5.18}$$

where the first approximation comes from (5_j), the second by (5.12), and the third due to (5.16). Combining this with (5.15) we have

$$Z_{b+2}^{\mathbb{R}} \subset\subset \Phi_{j+1}\left(Z_c^{\mathbb{R}}\right) \tag{5.19}$$

Furthermore take the Hamiltonian isotopy

$$\hat{\sigma}_{j,t} = \psi_{j,t} \circ (\Psi_{j,t} \circ \Sigma_{j,t})^{-1}\Big|_{\mathcal{C}_n^{\mathbb{R}}}$$

and let $\hat{P}_t$ be the corresponding Hamiltonian function for $\hat{\sigma}_{j,t}$. Moreover let λ be a cutoff function on $\mathcal{C}_n^{\mathbb{R}}$ such that it is zero on $Z_b^{\mathbb{R}}$ and equal to one outside $Z_{b+1}^{\mathbb{R}}$. Finally let $\psi_{j+1,t}$ denote the flow map of the vector field whose Hamiltonian function is $\lambda\hat{P}_t$.

We go through the five conditions for the induction step:

(1_{j+1}) By the definition of Φ_{j+1} we have that

$$\Phi_{j+1}(Z_{R_{j+1}}) = \Psi_{j,1} \circ \Sigma_{j,1} \circ \Phi_j(Z_{R_{j+1}})$$

The claim follows from the fact that $\Sigma_{j,t}$ approximates the identity on Z_a and from the choices of a, S_{j+1} in (5.11), (5.14) respectively.

(2_{j+1}) Here we want to estimate

$$\|\Phi_{j+1} - \Phi_j\|_{C^k(Z_{R_j})} = \|\Psi_{j,1} \circ \Sigma_{j,1} \circ \Phi_j - \Phi_j\|_{C^k(Z_{R_j})}$$

By (1_j) we have

$$\Phi_j(Z_{R_j}) \subset Z_{S_j}$$

The estimate follows from the fact that $\Psi_{j,1}$ and $\Sigma_{j,1}$ approximate the identity on Z_{S_j}.

(3_{j+1}) A flow map at time zero is the identity.

(4_{j+1}) Because $\psi_{j+1,t}$ is obtained by interpolating between the identity on $Z_b^{\mathbb{R}}$ and $\hat{\sigma}_{j,t}$ outside $Z_{b+2}^{\mathbb{R}}$, it follows that $\psi_{j+1,t}(p) = (p)$ for p near $Z_{S_{j+1}}^{\mathbb{R}}$ because $b = S_{j+1} + 1$.

(5_{j+1}) To see that

$$\|\psi_{j+1,1} \circ \Phi_{j+1} - \varphi\|_{C^k(p)} < \varepsilon(p)$$

for all p in $\mathcal{C}_n^{\mathbb{R}}$ we consider separately:

(i) On the complement of $(\Phi_{j+1})^{-1}\left(Z_c^{\mathbb{R}}\right)$, we have $\lambda = 1$ at $\Phi_{j+1}(p)$ thanks to (5.19). Then $\psi_{j+1,1}$ is

$$\hat{\sigma}_{j,1} = \psi_{j,1} \circ (\Psi_{j,1} \circ \Sigma_{j,1})^{-1}$$

which together with $\Phi_{j+1} = \Psi_{j,1} \circ \Sigma_{j,1} \circ \Phi_j$ reduces this case to (5_j).

(ii) On $\Phi_j^{-1}\left(Z_d^{\mathbb{R}}\right)$, we have by the choice of $\Sigma_{j,1}$ that

$$\Phi_{j+1} = \Psi_{j,1} \circ \Sigma_{j,1} \circ \Phi_j \sim \Psi_{j,1} \circ \tilde{\sigma}_{j,1} \circ \Phi_j \sim \psi_{j,1} \circ \Phi_j \tag{5.20}$$

which approximates φ by the induction hypothesis (5_j). Moreover, this implies that $\Psi_{j,1} \circ \Sigma_{j,1} \sim \psi_{j,1}$ on $Z_d^{\mathbb{R}}$. Then on

$$\Phi_{j+1} \circ \Phi_j^{-1}\left(Z_d^{\mathbb{R}}\right) = \Psi_{j,1} \circ \Sigma_{j,1}\left(Z_d^{\mathbb{R}}\right)$$

we have

$$\begin{aligned} \hat{\sigma}_{j,1} &= \psi_{j,1} \circ \left(\Psi_{j,1} \circ \Sigma_{j,1}\right)^{-1} \\ &\sim \psi_{j,1} \circ \left(\psi_{j,1}\right)^{-1} = \mathrm{id} \end{aligned}$$

Since

$$Z_{b+2}^{\mathbb{R}} \subset \Phi_{j+1}(Z_c^{\mathbb{R}}) \subset \Phi_{j+1} \circ \Phi_j^{-1}(Z_d^{\mathbb{R}})$$

this justifies our interpolation $\psi_{j+1,t}$ between the identity on $Z_b^{\mathbb{R}}$ and $\hat{\sigma}_{j,t}$ outside $Z_{b+2}^{\mathbb{R}}$. Hence $\psi_{j+1,1}$ is either close to the identity (reducing to the above (5.20)) or equal to

$$\hat{\sigma}_{j,1} = \psi_{j,1} \circ (\Psi_{j,1} \circ \Sigma_{j,1})^{-1}\Big|_{\mathcal{C}_n^{\mathbb{R}}}$$

which composed with Φ_{j+1} again reduces it to (5_j).

(iii) It suffices to consider the above two cases, because by the choice of d in (5.17) and the approximation $\psi_{j,1} \sim \Psi_{j,1} \circ \tilde{\sigma}_{j,1}$ in (5.12)

$$Z_c^{\mathbb{R}} \subset\subset \Psi_{j,1} \circ \Sigma_{j,1}\left(Z_d^{\mathbb{R}}\right) = \Phi_{j+1} \circ \Phi_j^{-1}\left(Z_d^{\mathbb{R}}\right)$$

it follows

$$(\Phi_{j+1})^{-1}\left(Z_c^{\mathbb{R}}\right) \subset\subset (\Phi_j)^{-1}\left(Z_d^{\mathbb{R}}\right)$$

This completes the induction step.

Concluding, to see that $\lim_{j\to\infty} \Phi_j$ exists, we underline the connection between the above induction and the push-out method, see e.g. [For99, Proposition 5.1]. The holomorphic symplectic automorphism $\Phi_{j+1} \circ \Phi_j^{-1} = \Psi_{j,1} \circ \Sigma_{j,1}$ approximates the identity on Z_{R_j}. Since $R_j \geq j-1$, the sublevel set Z_{R_j} exhausts $\mathcal{C}_n$ in the limit. Additionally we have $R_{j+1} = S_j + 1 > R_j + 2$. Choose $\{\varepsilon_j\}_{j\in\mathbb{N}} \subset \mathbb{R}_{>0}$ to have finite sum such that

$$0 < \varepsilon_{j+1} < \mathrm{dist}(Z_{R_j}, \mathcal{C}_n \setminus Z_{R_{j+1}})$$

where $\mathrm{dist}(\cdot,\cdot)$ is the distance function on $\mathbb{C}^q$ restricted to $\mathcal{C}_n$. Therefore the limit

$$\lim_{j\to\infty} \Phi_{j+1} \circ \Phi_j^{-1}$$

exists uniformly on compacts on

$$\bigcup_{j=1}^{\infty} \left(\Phi_{j+1} \circ \Phi_1^{-1}\right)^{-1} (Z_{R_j}) = \mathcal{C}_n$$

and

$$\Phi = (\lim_{j\to\infty} \Phi_{j+1} \circ \Phi_j^{-1}) \circ \Phi_1 = \lim_{j\to\infty} \Phi_j$$

is a holomorphic symplectic automorphism of $\mathcal{C}_n$ with $\Phi(\mathcal{C}_n^{\mathbb{R}}) = \mathcal{C}_n^{\mathbb{R}}$. Moreover, (1_j) and (4_j) guarantee that $\psi_{j,1} \circ \Phi_j$ is Φ_j on $Z_{R_j}^{\mathbb{R}}$. Then (5_j) says that Φ_j approximates φ on $Z_{R_j}^{\mathbb{R}}$ and thus in the limit Φ approximates φ. □

5.6 A different real form

Let us consider another antiholomorphic involution σ on $\mathcal{M}$

$$\sigma(X, Y, v, w) = (\bar{X}, \bar{Y}, \bar{v}, \bar{w})$$

and the corresponding subgroup $\mathrm{G}_r = \mathrm{GL}_n(\mathbb{R})$ acting on

$$\mathcal{M}_{\mathbb{R}} = \mathrm{M}_n(\mathbb{R}) \oplus \mathrm{M}_n(\mathbb{R}) \oplus \mathbb{R}^n \oplus (\mathbb{R}^n)^*$$

by the restriction of the G-action. This gives the moment map

$$\mu_r \colon \mathcal{M}_{\mathbb{R}} \to \mathrm{M}_n(\mathbb{R}), \quad (X, Y, v, w) \mapsto [X, Y] + vw$$

Since the complex quotient $\mathcal{C}_n$ is Hausdorff and all G-orbits in $\widehat{\mathcal{C}}'_n$ are closed, the G-action is proper. Thus G_r also acts properly and freely on $\widehat{\mathcal{C}}^r_n = \mu_r^{-1}(-I_n)$. Notice that we consider here the complex Calogero–Moser space $\mathcal{C}'_n$ with a different rank condition $[X, Y] + vw = -I_n$, which is isomorphic to $\mathcal{C}_n$ in the main text.

Moreover, the real symplectic form on $\mathcal{M}_{\mathbb{R}}$ is given by $\omega_r = \omega|_{\mathcal{M}_{\mathbb{R}}}$. Then the real symplectic reduction $\widehat{\mathcal{C}}^r_n / \mathrm{G}_r$ is a real symplectic manifold $\mathcal{C}^r_n$ equipped with an induced symplectic form also denoted by ω_r.

The above conjugation maps a G-orbit to another G-orbit

$$\sigma(g \cdot z) = \bar{g} \cdot \sigma(z)$$

for $g \in \mathrm{G}$ and $z \in \mathcal{M}$. Hence it induces a conjugation on $\mathcal{C}_n$. It is clear that the conjugation commutes with the G_r-action. From the above we have $\widehat{\mathcal{C}}^r_n = (\widehat{\mathcal{C}}'_n)^\sigma$, the fixed-point set of the conjugation on $\widehat{\mathcal{C}}'_n = \mu^{-1}(-I_n)$. The map $\jmath' \colon \mathcal{C}^r_n \to \mathcal{C}'_n$ takes a G_r-orbit to the G-orbit containing it.

$$\begin{array}{ccc} \widehat{\mathcal{C}}'_n & \xrightarrow{\sigma} & \widehat{\mathcal{C}}'_n \\ \downarrow & & \downarrow \\ \mathcal{C}'_n & \xrightarrow{\sigma} & \mathcal{C}'_n \end{array} \qquad \begin{array}{ccc} \widehat{\mathcal{C}}^r_n & \xrightarrow{\jmath'} & \widehat{\mathcal{C}}'_n \\ \downarrow & & \downarrow \\ \mathcal{C}^r_n & \xrightarrow{\jmath'} & \mathcal{C}'_n \end{array}$$

Lemma 5.22 *The map $\jmath'$ is surjective.*

Proof Take a σ-stable G-orbit $\mathrm{G}\cdot z$. We show that it meets $\widehat{\mathcal{C}}_n^r$ at one point. As in the proof of Theorem 1.11 using GIT, we see from $\|\sigma(z)\| = \|z\|$ that for $z_0 \in \mu_1^{-1}(0) \cap \widehat{\mathcal{C}}_n'$ there exists $h_0 \in \mathrm{U}$ such that

$$\sigma(z_0) = h_0 \cdot z_0$$

Then

$$z_0 = \sigma(\sigma(z_0)) = \sigma(h_0 \cdot z_0) = \bar{h}_0 \cdot \sigma(z_0) = \bar{h}_0 h_0 \cdot z_0$$

which implies that $\bar{h}_0 h_0 = I_n$. This in turn gives $h_0^{-1} = h_0^* = \bar{h}_0$. Then h_0 is also symmetric, thus there exists another symmetric unitary matrix s such that $h_0 = s^2$. Replace z_0 by $s \cdot z_0$

$$\sigma(s \cdot z_0) = \bar{s}s^2 \cdot z_0 = s \cdot z_0$$

Namely $s \cdot z_0 \in \widehat{\mathcal{C}}_n^r$. □

The rest of the proof for the following is similar to the one for Theorem 1.11.

Theorem 5.23 *The complex Calogero–Moser space $(\mathcal{C}_n', \omega, \sigma)$ is a symplectic complexification of the smooth symplectic manifold $(\mathcal{C}_n^r, \omega_r)$.*

The computation in [AH25] shows that any algebraic G-invariant on $\widehat{\mathcal{C}}_n'$ is contained in the complex Lie algebra generated by the four functions $\operatorname{tr} Y, \operatorname{tr} Y^2, \operatorname{tr} X^3, (\operatorname{tr} X)^2$, and all formulae appearing in the calculation leading to it come with real coefficients. This implies the τ-symplectic density property of $\mathcal{C}_n'$ with respect to $\mathcal{C}_n^r$. On the other hand, the antiholomorphic involution σ induces the complex conjugation on the algebra generators $\operatorname{tr} X^j Y^k$. Hence the real form $\mathcal{C}_n^r$ is the intersection of the complex Calogero–Moser space $\mathcal{C}_n'$ with $\mathbb{R}^q$ (observe that $\widehat{\mathcal{C}}_n' = \mu^{-1}(-I_n)$ and $\tilde{\mathcal{C}}_n = \mu^{-1}(iI_n)$ are different sets in $\mathcal{M}$), where q is the cardinality of a minimal generating set $\mathcal{G}$ for the $\mathbb{C}$-algebra $\mathbb{C}[\widehat{\mathcal{C}}_n']^{\mathrm{G}}$ with generators of the form $\operatorname{tr} X^j Y^k$. Therefore the same construction of holomorphic Hamiltonian automorphisms to approximate Hamiltonian diffeomorphisms on $\mathcal{C}_n^r$ stays intact. However we do not know whether $\mathcal{C}_n^r$ has trivial first de Rham cohomology, hence we approximate only Hamiltonian diffeomorphisms.

Theorem 5.24 *Let φ be a Hamiltonian diffeomorphism of the symplectic manifold $(\mathcal{C}_n^r, \omega_r)$. Then for a positive continuous function ε on $\mathcal{C}_n^r$, there exists a holomorphic symplectic automorphism $\Phi \in \tau\mathrm{Aut}_\omega(\mathcal{C}_n')$ such that*

$$\|\Phi - \varphi\|_{C^k(p)} < \varepsilon(p)$$

for all p in $\mathcal{C}_n^r$.

Chapter 6

Bounds for holomorphic matrix factorization

The main theme of this chapter is Linear Algebra questions when the matrices are not "constant" matrices with entries in the field of complex numbers but rather holomorphic functions. These interesting problems of holomorphically depending Linear Algebra can be formulated as the study of algebraic properties of the general linear groups $\mathrm{GL}_n(\mathcal{O}(X))$, the special linear groups $\mathrm{SL}_n(\mathcal{O}(X))$ and the symplectic groups $\mathrm{Sp}_{2n}(\mathcal{O}(X))$ over the ring of holomorphic functions $\mathcal{O}(X)$ on a reduced Stein space X. In this setting of matrix groups over rings our results can be viewed as results in K-theory, just for the special ring $R = \mathcal{O}(X)$.

Our first result is concerned with the classical K-theoretic question about the K_1 groups of the ring $R = \mathcal{O}(X)$ and bounded generation of the corresponding elementary subgroups with concrete bounds. Since we assume that the Stein space X has finite dimension n (defined as the complex dimension of the manifold $X \setminus \mathrm{Sing}(X)$, called the smooth part of X), these rings are interesting rings since they have finite Bass stable rank $\mathrm{bsr}(\mathcal{O}(X)) = \lfloor \frac{1}{2} \dim X \rfloor + 1$ as established by Alexander Brudnyi in [Bru19].

Theorem 6.1 (Holomorphic Factorization) Let n, d be natural numbers. Then

(a) there exists a minimal upper bound $t = t(n,d) \in \mathbb{N}$ such that every null-homotopic holomorphic mapping $f\colon X \to \mathrm{Sp}_{2n}(\mathbb{C})$ from a d-dimensional reduced Stein space X factorizes into a product of at most t unitriangular factors. Moreover, we have

$$t(n,d) \leq t(1,d).$$

(b) $t(n,1) = 4$ and $t(n,2) \leq t(1,2) \leq 5$ for all $n \geq 1$.

The straightforward generalizations of our results to non-reduced Stein spaces can be achieved as in the aforementioned paper by Brudnyi. Here we consider only reduced Stein spaces. Our results rely on a combination of K-theoretic methods (Theorem 6.8) and deep results from complex analysis, the solution to the so-called Gromov-Vaserstein problem (see Theorem 6.4), about unitriangular (either upper triangular with 1 along the diagonal or lower triangular with 1 along the diagonal) factorization of holomorphic matrices. Since Theorem 6.8 is formulated in [VSS11] only for the so-called UL factorization, i.e., for one with even number of factors, we give a complete proof for any (also odd) number of factors in section 6.1.3.

Next we consider our first application: The product of exponentials for the above mentioned groups $\mathrm{GL}_n(\mathcal{O}(X))$, $\mathrm{SL}_n(\mathcal{O}(X))$, $\mathrm{Sp}_{2n}(\mathcal{O}(X))$. We give a simple lower bound for all of these cases, prove the existence of an upper bound for the symplectic case, and give new upper bounds for the symplectic case for Stein spaces of dimension 1 and dimension 2.

Theorem 6.2 (Product of Exponentials) Let n, d be natural numbers. Then

(a) there exists a minimal upper bound $e = e_{\mathrm{Sp}}(2n, d) \in \mathbb{N}$ such that every null-homotopic holomorphic mapping $f\colon X \to \mathrm{Sp}_{2n}(\mathbb{C})$ from a reduced Stein space X of dimension d is a product of at most e exponentials, that is, there exist $A_1, ..., A_e\colon X \to \mathfrak{sp}_{2n}(\mathbb{C})$ such that

$$f(x) = \exp(A_1(x)) \cdots \exp(A_e(x)).$$

(b) $2 \leq e_{\mathrm{Sp}}(2n, 1) \leq e_{\mathrm{Sp}}(2n, 2) \leq 3$.

We give another result in this direction, namely (Proposition 6.14) that the number of exponentials for the general linear group over the ring $\mathcal{O}(X)$ of holomorphic functions on a Stein space X is at least 2. For 2×2 matrices this has been proved before by Kutzschebauch and Studer in [KS19].

Our second application of the existence of uniform factorization is the fact that the path-connected component of the group $\mathrm{Sp}_{2n}(\mathcal{O}(X))$, which is equal to the elementary symplectic group $\mathrm{Ep}_{2n}(\mathcal{O}(X))$, admits Kazhdan's property (T) for $n \geq 2$. The corresponding result for the path-connected components of the groups $\mathrm{SL}_n(\mathcal{O}(X))$, $n \geq 3$ is due to Ivarsson and Kutzschebauch [IK14].

Theorem 6.3 (Kazhdan's Property (T)) Let $n \geq 2$ and let X be a Stein manifold with finitely many connected components. Then $\mathrm{Ep}_{2n}(\mathcal{O}(X)) = (\mathrm{Sp}_{2n}(\mathcal{O}(X)))_0$ has Kazhdan's property (T).

Since the solution to the Gromov-Vaserstein problem involves the so-called Oka principle, called the most beautiful principle in analysis by René Thom, it is natural to compare the K-theoretic questions for the ring $\mathcal{O}(X)$ with the corresponding questions for the ring $\mathcal{C}(X)$ of continuous complex-valued functions on the Stein space X. More precisely, the Oka principle vaguely stated says that under certain conditions, the existence of a continuous solution implies the existence of a holomorphic solution. Let $t(n, d, \mathcal{C}, \mathcal{O})$ (see [IK12b]) be the minimal number such that all null-homotopic holomorphic mappings, from a Stein space of dimension d into $\mathrm{SL}_n(\mathbb{C})$, factorize as a product of $t(n, d, \mathcal{C}, \mathcal{O})$ continuous unitriangular matrices (starting with a lower triangular one) and let $t(n, d, \mathcal{O})$ be the minimal number that all null-homotopic holomorphic mappings, from Stein spaces of dimension d into $\mathrm{SL}_n(\mathbb{C})$, factorize as a product of $t(n, d, \mathcal{O})$ holomorphic unitriangular matrices (starting with a lower triangular one). In [IK12b] it is proved that $t(2, 1, \mathcal{C}, \mathcal{O}) = t(2, 1, \mathcal{O}) = 4$ and that if a fixed holomorphic matrix $A \in \mathrm{SL}_2(\mathcal{O}(X))$

factorizes as a product of N continuous unitriangular matrices, then it factorizes as a product of $N+2$ holomorphic unitriangular matrices. Moreover, it is proved that the famous Cohn example [Coh66] $A_0 \in \mathrm{SL}_2(\mathcal{O}(\mathbb{C}^2))$ factorizes as a product of 4 continuous unitriangular matrices, but not less than 5 holomorphic unitriangular matrices. Thus the question remained whether $t(2,2,\mathcal{C},\mathcal{O})$ is equal to 4 or to 5. In the last section of this paper, we give an answer by proving $t(2,2,\mathcal{C},\mathcal{O}) = t(2,2,\mathcal{O}) = 5$.

For matrices of bigger size it follows from [IK12a], $\mathrm{bsr}(\mathcal{O}(X)) = 1$ when $\dim X = 1$ and [VSS11] that $t(n,1,\mathcal{C},\mathcal{O}) = t(n,1,\mathcal{O}) = 4$ for all $n \geq 3$. Note that $4 \leq t(n,2,\mathcal{C},\mathcal{O}) \leq t(n,2,\mathcal{O}) \leq 5$ (see [Bru19, Remark 1.2]). Also if we denote the corresponding numbers for the symplectic group Sp_{2n} with the subscript symp, one has by Theorem 6.8

$$4 \leq t_{\mathrm{symp}}(2n,2,\mathcal{C},\mathcal{O}) \leq t_{\mathrm{symp}}(2n,2,\mathcal{O}) \leq 5$$

for all $n \geq 2$.

6.1 Preparation

6.1.1 Elementary Generators

Consider $n \times n$ matrices with entries in a commutative ring R with 1. Let $E_{ij}, i,j = 1,\ldots,n$ be the matrix with 0 everywhere except at the (i,j) entry with 1. Let I be the $n \times n$ identity matrix. Then $I + rE_{ij}$ is upper diagonal with 1 along the diagonal for $i < j, r \in R$, and lower diagonal with 1 along the diagonal for $i > j, r \in R$. A product of matrices $I + rE_{ij}$ for $i < j, r \in R$ is upper triangular with 1 along the diagonal, similarly for $i > j$ lower triangular. These upper and lower triangular matrices with 1 along the diagonal are called *unitriangular*. Let $\mathrm{E}_n(R)$ be the group generated by $I + rE_{ij}, i \neq j, r \in R$, and let $\mathrm{SL}_n(R)$ be the set of matrices with determinant 1. Over the complex numbers, $\mathrm{E}_n(\mathbb{C}) = \mathrm{SL}_n(\mathbb{C})$.

Next, the symplectic group $\mathrm{Sp}_{2n}(\mathbb{C})$ can be represented as isometries of $\mathbb{C}^{2n}$ with respect to a nondegenerate, skew-symmetric bilinear form. A convenient choice for the Gramian matrix of this bilinear form is

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{6.1}$$

where 0 denotes the $n \times n$ zero matrix. We index rows and columns by $1,2,\ldots,n$, $-1,-2,\ldots,-n$. In the block notation

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{C}),$$

the symplectic condition $MJM^T = J$ gives rise to three simple types of J-symplectic matrices:

- (i): $\begin{pmatrix} I & B \\ 0 & I \end{pmatrix}$, upper triangular with symmetric $B = B^T$.

- (ii): $\begin{pmatrix} I & 0 \\ C & I \end{pmatrix}$, lower triangular with symmetric $C = C^T$.
- (iii): $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$, block diagonal with invertible $A \in \mathrm{GL}_n(\mathbb{C})$ and $D = (A^{-1})^T$.

If block A in type (iii) is upper triangular, then $D = (A^{-1})^T$ is lower triangular. In fact, A and D are simultaneously upper or lower triangular in another basis. This new basis can be obtained from the old one by reversing the order of the last n basis elements, giving a Gramian matrix

$$\tilde{J} = \begin{pmatrix} 0 & L \\ -L & 0 \end{pmatrix}, \tag{6.2}$$

where L is the $n \times n$ matrix with 1 along the skew-diagonal. Notice that symplectic matrices of type (i) and (ii) remain upper or lower triangular with respect to $\tilde{J}$, respectively.

Over the ring R, the elementary J-symplectic generators for the elementary symplectic group $\mathrm{Ep}_{2n}(R)$ are similar to those for the special linear group. For example, type (i) corresponds to $I_{2n} + r(E_{i,-j} + E_{j,-i}), i \neq j$ and $I_{2n} + rE_{i,-i}$, while type (iii) corresponds to $I_{2n} + r(E_{ij} - E_{-j,-i}), i \neq j$ [Car72, p. 186]. The subgroup U of upper unitriangular J-symplectic matrices are generated by

$$\begin{pmatrix} I & r(E_{ij} + E_{ji}) \\ 0 & I \end{pmatrix}, \quad \begin{pmatrix} I & rE_{ii} \\ 0 & I \end{pmatrix}, \quad \begin{pmatrix} I + rE_{ij} & 0 \\ 0 & I - rE_{ji} \end{pmatrix}, \qquad i < j,\ r \in R.$$

Notice that the last form has upper triangular counterpart as $\tilde{J}$-symplectic matrices. Similarly one finds the generators for the subgroup U^- of lower unitriangular J-symplectic matrices. From now on, we shall abbreviate J-symplectic as symplectic. Over the complex numbers, $\mathrm{Ep}_{2n}(\mathbb{C}) = \mathrm{Sp}_{2n}(\mathbb{C})$.

6.1.2 Factorization of holomorphic mappings into $\mathrm{Sp}_{2n}(\mathbb{C})$

Let X be a finite-dimensional Stein space and let $\mathcal{O}(X)$ be the ring of holomorphic functions on X. Then $\mathrm{Sp}_{2n}(\mathcal{O}(X))$ denotes the symplectic group with entries in $\mathcal{O}(X)$. Interpreting this group as holomorphic mappings from X to $\mathrm{Sp}_{2n}(\mathbb{C})$, we denote by $(\mathrm{Sp}_{2n}(\mathcal{O}(X)))_0$ the path-connected component containing the identity. From [Sch25, Theorem 1.1] we cite the following

Theorem 6.4 *Let n, d be natural numbers and let X be a reduced Stein space of dimension d. Then $\mathrm{Ep}_{2n}(\mathcal{O}(X)) = (\mathrm{Sp}_{2n}(\mathcal{O}(X)))_0$. Moreover, there is a natural number $K(n,d)$, depending only on n and d, such that each null-homotopic matrix M, i.e. $M \in (\mathrm{Sp}_{2n}(\mathcal{O}(X)))_0$, can be written as a product of no more than $K(n,d)$ symplectic matrices of type (i) and (ii).*

Remark 6.5 Observe that elementary matrices in $\mathrm{Ep}_{2n}(\mathcal{O}(X))$ of type (i) and (ii) are null-homotopic, since we can multiply the off-diagonal entries by t, that is,

$$\begin{pmatrix} I_n & tB \\ 0 & I_n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} I_n & 0 \\ tC & I_n \end{pmatrix}.$$

Therefore null-homotopy of $M \in \mathrm{Sp}_{2n}(\mathcal{O}(X))$ is a necessary condition. In the case where X is contractible, every matrix is null-homotopic. The same is true if X is a Stein space of dimension $d \in \{1,2\}$: On the one hand, $\pi_1(\mathrm{Sp}_{2n}(\mathbb{C})) = \pi_2(\mathrm{Sp}_{2n}(\mathbb{C})) = 0$ (see [Hal15]) and on the other, X is homotopy equivalent to a CW complex of real dimension d (see [For17, Corollary 3.11.2]).

Remark 6.6 When $\dim X = 1$ (or $\dim X = 2$), we will see in Theorem 6.11 that 4 (or 5) factors are sufficient for the unitriangular factorization, that is, factoring into the unipotent subgroups $U^{\pm}$. However, the set of elementary factors in Theorem 6.4 are taken from type (i) and (ii), and only the existence of an upper bound depending on d and n is guaranteed. For the same factorization over a field, 5 factors are optimal [JLX22]. Thus we find it natural to ask the following question.

Problem 6.7 Let X be a Stein space of dimension d. Is there a bound for the optimal number $K(n,d)$ of factorizing a null-homotopic holomorphic mapping from X to $\mathrm{Sp}_{2n}(\mathbb{C})$ into factors of type (i) and of type (ii), such that $K(n,d)$ is independent of n?

6.1.3 Reduction to smaller matrices

To obtain factorization estimates for holomorphic matrices of arbitrary size, we will make use of the Tavgen reduction. This appears in the setting of elementary Chevalley groups, see [VSS11] and references therein for notation and a background on Chevalley groups. Let Φ be a reduced irreducible root system of rank $l \geq 2$ and let R be a commutative ring with 1. We choose an order on Φ and a system of fundamental roots $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$. Each root $\alpha \in \Phi$ is an integral sum of the fundamental roots

$$\alpha = \sum_{i=1}^{l} k_i(\alpha)\alpha_i,$$

where the integer coefficients $k_i(\alpha)$ are either all ≥ 0 or all ≤ 0. For $r = 1$ and $r = l$, we define the following subsets of Φ

$$\begin{aligned} \Delta_r &= \{\alpha \in \Phi : k_r(\alpha) = 0\}, \\ \Sigma_r &= \{\alpha \in \Phi : k_r(\alpha) > 0\}, \quad \Sigma_r^- = \{\alpha \in \Phi : k_r(\alpha) < 0\}. \end{aligned}$$

Δ_r is itself a root system of rank $l-1$. On the level of Dynkin diagram, we obtain Δ_r from Φ by taking away the first ($r = 1$) or the last ($r = l$) fundamental root. The elementary Chevalley group $E(\Phi, R)$ of type Φ over R is generated by root subgroups $X_\alpha, \alpha \in \Phi$

$$E(\Phi, R) = \{x_\alpha(r) \mid \alpha \in \Phi, r \in R\}.$$

The positive unipotent subgroup $U(\Phi, R)$ is generated by the root subgroups of positive roots

$$U(\Phi, R) = \{x_\alpha(r) \mid \alpha \in \Phi^+, r \in R\}.$$

Similarly, $U^-(\Phi, R) = \{x_\alpha(r) \mid \alpha \in \Phi^-, r \in R\}$. The following theorem was originally proved by Oleg Tavgen and adapted in [VSS11], where the number of factors is even. For our estimates, we need the same result allowing odd number of factors. We remark that the shape of the starting factor, upper or lower, is also immaterial.

Theorem 6.8 (Tavgen-VSS) *Let Φ be a reduced irreducible root system of rank $l \geq 2$ and let R be a commutative ring with 1. Suppose that for subsystems $\Delta = \Delta_1, \Delta_l$ of rank $l-1$ the elementary Chevalley group $E(\Delta, R)$ admits a unitriangular factorization with L factors*

$$E(\Delta, R) = U^-(\Delta, R)U(\Delta, R) \cdots U^{\pm}(\Delta, R).$$

Then the elementary Chevalley group $E(\Phi, R)$ admits a unitriangular factorization with the same number of factors

$$E(\Phi, R) = U^-(\Phi, R)U(\Phi, R) \cdots U^{\pm}(\Phi, R).$$

Proof We take

$$Y = U^-(\Phi, R)U(\Phi, R) \cdots U^{\pm}(\Phi, R).$$

Y is a nonempty subset of $E(\Phi, R)$, in particular it contains 1. Since the group $E(\Phi, R)$ is generated by the following root elements $X = \{x_\alpha(r) \mid \alpha \in \pm\Pi, r \in R\} \subset E(\Phi, R)$. Notice that the generating set X is symmetric, i.e. $X^{-1} = X$. We claim that $x_\alpha(r)Y \subset Y$ for $\alpha \in \pm\Pi$: Since $l \geq 2$, α lies in at least one of the subsystems Δ_1, Δ_l. Suppose that α belongs to $\Delta = \Delta_r$, then we consider the Levi decomposition

$$U(\Phi, R) = U(\Delta, R) \ltimes E(\Sigma, R), \qquad U^-(\Phi, R) = U^-(\Delta, R) \ltimes E(\Sigma^-, R),$$

where $\Sigma = \Sigma_r$ and $E(\Sigma, R) = \langle x_\alpha(r) \mid \alpha \in \Sigma, r \in R\rangle$. Since $U^{\pm}(\Delta, R)$ normalizes $E^{\pm}(\Sigma, R)$ [Car72, Theorem 8.5.2], we can rewrite Y as

$$\begin{aligned}
Y &= U^-(\Phi, R)U(\Phi, R) \cdots U^{\pm}(\Phi, R) \\
&= U^-(\Delta, R)E(\Sigma^-, R)\, U(\Delta, R)E(\Sigma, R)\, \cdots\, U^{\pm}(\Delta, R)E(\Sigma^{\pm}, R) \\
&= \left(U^-(\Delta, R)U(\Delta, R) \cdots U^{\pm}(\Delta, R)\right) E(\Sigma^-, R)E(\Sigma, R) \cdots E(\Sigma^{\pm}, R) \\
&= E(\Delta, R)\, E(\Sigma^-, R)E(\Sigma, R) \cdots E(\Sigma^{\pm}, R),
\end{aligned}$$

where the last step follows from the assumption. For $\alpha \in \Delta$, $x_\alpha(r)$ is an element in $E(\Delta, R)$, hence $x_\alpha(r)Y \subset Y$.

Lemma 6.9 shows that $Y = E(\Phi, R)$. $\square$

The proof of the following lemma is an easy exercise.

Lemma 6.9 *Let G be a group and $Y \subset G$ be a nonempty subset. Given a symmetric, generating subset X of G, if $XY \subset Y$, then Y is the group G.*

6.2 Number of unitriangular factors

In this section, we prove Theorem 6.1. The first part of (a) is Theorem 6.4 and the second part of (a) follows from Theorem 6.8.

Before we prove part (b), we restate a result from [IK12b].

Theorem 6.10 *Let X be a two-dimensional Stein space and let $f\colon X \to \mathrm{SL}_2(\mathbb{C})$ be a holomorphic mapping. Then there exist holomorphic mappings $g_1, \ldots, g_5\colon X \to \mathbb{C}$ such that*

$$f(x) = \begin{pmatrix} 1 & 0 \\ g_1(x) & 1 \end{pmatrix} \begin{pmatrix} 1 & g_2(x) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ g_3(x) & 1 \end{pmatrix} \begin{pmatrix} 1 & g_4(x) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ g_5(x) & 1 \end{pmatrix}.$$

Let us formulate part (b) of Theorem 6.1 in more detail.

Theorem 6.11 *Let X be a reduced Stein space of dimension 1 or 2, and let f be a holomorphic mapping from X to $\mathrm{Sp}_{2n}(\mathbb{C})$. Then there exist holomorphic mappings*

$$g_1, g_2, \ldots, g_t\colon X \to \mathbb{C}^{n(2n-1)}$$

such that

$$f(x) = M_1(g_1(x))M_2(g_2(x)) \cdots M_t(g_t(x)),$$

where $t = 4$ for $\dim X = 1$ and $t \le 5$ for $\dim X = 2$.

Here M_j, $j \in \mathbb{N}$, is respectively defined as

$$M_j(g_j(x)) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ g_j(x) & & 1 \end{pmatrix} \quad \text{and} \quad M_j(g_j(x)) = \begin{pmatrix} 1 & & g_j(x) \\ & \ddots & \\ 0 & & 1 \end{pmatrix},$$

for j odd and j even, respectively.

Proof According to the above theorem from [IK12b], Theorem 6.1 (a) and Remark 6.5, we have

$$t(n, 2) \le t(1, 2) \le 5.$$

The ring $\mathcal{O}(X)$ has Bass stable rank 1 if X is a one-dimensional Stein space (see e.g. [Bru19, Theorem 1.1]). Then $t(n, 1) = 4$ by [VSS11, Theorem 1]. $\square$

6.3 Number of exponential factors

Let $\mathcal{A}$ be a m-convex Fréchet algebra with 1 (see [Mic52]). The exponential of a $n \times n$ matrix A is given by the exponential series. Let $\mathrm{Exp}_n(\mathcal{A})$ denote the subgroup of $\mathrm{GL}_n(\mathcal{A})$ generated by exponentials and $e(n, \mathcal{A})$ be the minimal number such that any matrix in $\mathrm{Exp}_n(\mathcal{A})$ factorizes as a product of $e(n, \mathcal{A})$ exponentials. Let $t(n, \mathcal{A})$ be the minimal number such that any element in the elementary Chevalley group $E(\Phi, \mathcal{A}) \subset \mathrm{GL}_n(\mathcal{A})$ factorizes as a product of $t(n, \mathcal{A})$ unitriangular matrices. When no such number exists, set $t(n, \mathcal{A}) = \infty$.

Proposition 6.12

$$e(n,\mathcal{A}) \le \lfloor \frac{1}{2} t(n,\mathcal{A}) \rfloor + 1.$$

Proof Observe that for a nilpotent matrix N,

$$\log(I+N) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} N^k$$

is a finite sum. Thus every unipotent matrix A can be written as the exponential of $\log A$. Also under conjugation an exponential remains an exponential, since

$$BAB^{-1} = \exp(B \cdot \log A \cdot B^{-1})$$

for any invertible $n \times n$-matrix B. To estimate the number of exponential factors, we use a trick presented in [Bru22, Lemma 2.1]. Let K be an odd natural number and $U_1 U_2 \cdots U_K$ is a product of K unitriangular factors, where the factors with odd (respectively even) indices are upper (respectively lower) unitriangular. Let $\Psi_{2k+1} := U_1 U_3 \cdots U_{2k+1}$ denote the product of the first $2k+1 \le K$ upper unitriangular factors and note that Ψ_{2k+1} is upper unitriangular, thus exponential as well. We prove the following formula

$$\prod_{i=1}^{K} U_i = \left(\prod_{i=1}^{\frac{K-1}{2}} \Psi_{2i-1} U_{2i} \Psi_{2i-1}^{-1} \right) \Psi_K$$

by induction on K. For $K = 1$, this is trivially true. Let $K > 1$ be an odd natural number. By the induction hypothesis, we may assume that the formula is true for $K-2$. Therefore

$$\begin{aligned}
\prod_{i=1}^{K} U_i &= \left(\prod_{i=1}^{K-2} U_i \right) U_{K-1} U_K \\
&= \left(\prod_{i=1}^{\frac{K-3}{2}} \Psi_{2i-1} U_{2i} \Psi_{2i-1}^{-1} \right) \underbrace{\Psi_{K-2} U_{K-1} U_K}_{=(\Psi_{K-2} U_{K-1} \Psi_{K-2}^{-1}) \Psi_{K-2} U_K} \\
&= \left(\prod_{i=1}^{\frac{K-1}{2}} \Psi_{2i-1} U_{2i} \Psi_{2i-1}^{-1} \right) \Psi_K,
\end{aligned}$$

and the last equality is true, since $\Psi_K = \Psi_{K-2} U_K$.

By the introducing paragraph of the proof, the formula implies the estimate for odd K. If K is even, then we simply have an additional lower unitriangular factor and thus an additional exponential and this yields the estimate for even K. □

Proof (Proof of Theorem 6.2) Part (a) follows from part (a) of Theorem 6.1 and the fact, that every unitriangular matrix can be written as an exponential.

For part (b), we set $\mathcal{A} = \mathrm{Sp}_{2n}(\mathcal{O}(X))$ and apply part (b) of Theorem 6.1 and Proposition 6.12 to obtain the upper bounds.

The lower bound follows from the fact, that the exponential map $\exp\colon \mathfrak{sp}_{2n}(\mathbb{C}) \to \mathrm{Sp}_{2n}(\mathbb{C})$ is not surjective.

Finally, note that $e_{Sp}(n,d) \leq e_{Sp}(n,d+1)$ for every $n,d \in \mathbb{N}$, since a mapping $f\colon X \to \mathrm{Sp}_{2n}(\mathbb{C})$ can be considered as a mapping $X \times \mathbb{C} \to \mathrm{Sp}_{2n}(\mathbb{C}), (x,z) \mapsto f(x)$. Hence $e_{Sp}(n,1) \leq e_{Sp}(n,2)$. □

Note that Theorem 6.2, part (a) has an analogue for the special linear group $\mathrm{SL}_n(\mathbb{C})$ (cf. [DK19]), that is, there exists a natural number $e = e_{SL}(n,d)$ such that every null-homotopic holomorphic mapping $X \to \mathrm{SL}_n(\mathbb{C})$ from a d-dimensional Stein space is a product of at most e exponentials. Let $e_{SL}(n, \mathcal{O}(X))$ denote the minimal number such that every null-homotopic holomorphic map $X \to \mathrm{SL}_n(\mathbb{C})$ from a fixed d-dimensional Stein space X is a product of at most $e_{SL}(n,\mathcal{O}(X))$ exponentials. Clearly, we have $e_{SL}(n,\mathcal{O}(X)) \leq e_{SL}(n,d)$.

Corollary 6.13 *Let n be a natural number and X a finite-dimensional reduced Stein space. Then*

$$e(n,\mathcal{O}(X)) \leq e_{SL}(n,\mathcal{O}(X)).$$

Proof For $f \in \mathrm{GL}_n(\mathcal{O}(X))$ null-homotopic, composition with the determinant $\det \circ f\colon X \to \mathbb{C}^*$ is also null-homotopic. Thus there exists a holomorphic function $g\colon X \to \mathbb{C}$ such that $\exp \circ g = \det \circ f$. Since $\exp(-\frac{g}{n} I_n) f$ is in $\mathrm{SL}_n(\mathcal{O}(X))$ and for every matrix A the following equation is satisfied

$$\exp\left(\frac{g}{n} I_n\right)\exp(A) = \exp\left(\frac{g}{n} A\right)$$

the claim follows. □

Proposition 6.14 *Let X be a Stein space with $\dim X > 0$ and let $n \geq 2$. Then $e(n,\mathcal{O}(X)) \geq 2$.*

Proof The proof is essentially the same as in [Bru22]. Let $X' \subset X$ be an irreducible component with $\dim X' > 0$. Then there exist two distinct points $x_1, x_2 \in X'$, we choose a holomorphic function $h \in \mathcal{O}(X)$ and $x_1, x_2 \in X'$ such that $h(x_1) = 0, h(x_2) = 2\pi i$. Set $g = \exp h$. Let

$$T = \begin{pmatrix} g & 1 \\ 0 & 1 \end{pmatrix}.$$

The same argument in [Bru22] shows that there does not exist $S \in M_2(\mathcal{O}(X))$ with $S^2 = T$ and in particular, T does not have a logarithm. So $e(2,\mathcal{O}(X)) \geq 2$. For $n > 2$, fix $M \in \mathbb{C} \setminus \{0,1\}$ and set

$$T_n = \begin{pmatrix} MI_{n-2} & 0 \\ 0 & T \end{pmatrix}.$$

Suppose that T_n had a logarithm, then there would exist

$$S_n = \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix} \in M_n(\mathcal{O}(X))$$

with the same block partition as T_n and such that $S_n^2 = T_n$. Then we have $S_n T_n = T_n S_n$, which implies that

$$L_2(T - MI_2) = 0 \quad \text{and} \quad (T - MI_2)L_3 = 0.$$

On $X' \setminus (\exp h)^{-1}(M)$, $T - MI_2$ is invertible, so $L_2 = L_3 = 0$. By the identity theorem, L_2 and L_3 vanish on X'. But this would imply that $L_4^2 = T$, a contradiction. Hence $e(n, \mathcal{O}(X)) \geq 2$ for all $n \geq 2$. $\square$

6.4 Kazhdan's property (T)

Let G be a topological group, $K \subset G$, $\varepsilon > 0$, H a Hilbert space, and (π, H) a continuous unitary G-representation. A vector $v \in H$ is called *(K, ε)-invariant* if $\|\pi(g)v - v\| < \varepsilon\|v\|$ for all $g \in K$. We say that G has *Kazhdan's property (T)*, if there exist compact $K \subset G$ and $\varepsilon > 0$ such that every continuous unitary G-representation with a (K, ε)-invariant vector contains a nonzero G-invariant vector. (K, ε) is called a *Kazhdan pair* for G. The symplectic group $\mathrm{Sp}_{2n}(R)$ over commutative ring R is called *boundedly elementary generated*, if there exists an integer ν such that every element is a product of at most ν elementary symplectic matrices. Theorem 6.3 is a symplectic version of [IK14, Theorem 2.6].

Proof (Proof of Theorem 6.3) According to Theorem 6.4 and the fact that each factor of type (i) or (ii) is a product of at most $n(n+1)/2$ elementary symplectic generators, $\mathrm{Ep}_{2n}(\mathcal{O}(X))$ is boundedly elementary generated. A finite set of holomorphic functions which generates a dense subring of $\mathcal{O}(X)$ can be constructed as follows. First, embed X into $\mathbb{C}^N$ using Remmert's embedding theorem. Next, the set $S = \{z_1, z_2, \ldots, z_N, \sqrt{2}, i\}$ generates a dense subring of $\mathbb{C}[z_1, z_2, \ldots, z_N]$. Then, $\mathbb{C}[z_1, z_2, \ldots, z_N]|_X$ is dense in $\mathcal{O}(X)$ by the Oka-Weil theorem. Hence, S generates a dense subring of $\mathcal{O}(X)$. Now [Neu03, Theorem 1.1] gives a Kazhdan pair for $\mathrm{Ep}_{2n}(\mathcal{O}(X))$. $\square$

From its definition, Kazhdan's property (T) is preserved under closure. As in [IK14, Theorem 3.1], we can relate Kazhdan's property (T) of $\mathrm{Sp}_{2n}(\mathcal{O}(X))$ to that of its quotient over the closure of the elementary group, with identical proof.

Theorem 6.15 *Let $n \geq 2$ and let X be a Stein space with finite embedding dimension. Then $\mathrm{Sp}_{2n}(\mathcal{O}(X))$ has Kazhdan's property (T) if and only if*

$$\mathrm{Sp}_{2n}(\mathcal{O}(X))/\overline{\mathrm{Ep}_{2n}(\mathcal{O}(X))}$$

has Kazhdan's property (T).

6.5 Continuous versus holomorphic factorization

In the following we present an example showing that $t(2, 2, \mathcal{C}, \mathcal{O}) = 5$. Let X be a two-dimensional Stein space, and let $f\colon X \to \mathrm{SL}_2(\mathbb{C})$ be a holomorphic mapping.

Then f can be written as a product of 5 unitriangular holomorphic matrices by Theorem 6.11. In [IK12b] one finds an example which factorizes into 5 unitriangular holomorphic factors and 4 unitriangular continuous factors. Here we aim to give another example, which factorizes into 5 unitriangular holomorphic factors *and* 5 continuous factors. To this end, we first study what it means to have a factorization with 4 factors. Denote f by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, b, c, d \in \mathcal{O}(X)$ and $ad - bc = 1$. If there exist mappings $g_1, g_2, g_3, g_4 \colon X \to \mathbb{C}$ so that f can be decomposed as

$$f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ g_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & g_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ g_3 & 1 \end{pmatrix} \begin{pmatrix} 1 & g_4 \\ 0 & 1 \end{pmatrix}. \tag{6.3}$$

Bring the first and the fourth factor to the left side, and carry out the multiplications

$$\begin{pmatrix} a & b - ag_4 \\ c - ag_1 & -g_4(c - ag_1) + d - bg_1 \end{pmatrix} = \begin{pmatrix} 1 + g_2 g_3 & g_2 \\ g_3 & 1 \end{pmatrix}.$$

In case $a \neq 0$, the first three equations read

$$\begin{aligned} a &= 1 + g_2 g_3, \\ g_4 &= \frac{1}{a}(b - g_2), \\ g_1 &= \frac{1}{a}(c - g_3), \end{aligned}$$

and the fourth equation follows from the other three. If moreover $a \neq 1$, any choice of $g_3 \colon \{x \in X \mid a(x) \notin \{0, 1\}\} \to \mathbb{C}^*$ gives a factorization in this part of X. The fiber of the fibration $f^*\Phi_4$ (see [IK12b]) over $\{x \in X \mid a(x) \notin \{0, 1\}\}$ is $\mathbb{C}^*$, where

$$\Phi_4 \colon \mathbb{C}^4 \to \mathrm{SL}_2(\mathbb{C}),\ (z_1, z_2, z_3, z_4) \mapsto \begin{pmatrix} 1 & 0 \\ z_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & z_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z_3 & 1 \end{pmatrix} \begin{pmatrix} 1 & z_4 \\ 0 & 1 \end{pmatrix}.$$

When $a = 0$, then

$$1 + g_2 g_3 = 0, \quad g_2 = b, \quad g_3 = c, \quad 1 = -cg_4 + d - bg_1.$$

Notice that g_2 and g_3 are prescribed as b and c, respectively, and the fiber of $f^*\Phi_4$ here is $\mathbb{C}$. For $a = 1$, the fiber is the cross of axis.

Consider the following holomorphic mapping $f \colon \mathbb{C}^2 \to \mathrm{SL}_2(\mathbb{C})$

$$f(z, w) = \begin{pmatrix} (zw - 1)(zw - 2) & (zw - 1)z + (zw - 2)z^2 \\ h_1(z, w) & h_2(z, w) \end{pmatrix},$$

where the functions in the second row are chosen such that $f(z, w)$ has determinant 1. The existence of such polynomial functions follows from Hilbert's Nullstellensatz, or if one is looking for holomorphic functions from a standard application of Theorem B. For this observe that the functions in the first row have no common zeros.

Suppose that there were continuous $g_1, g_2, g_3, g_4 \colon \mathbb{C}^2 \to \mathbb{C}$ such that f factorizes as in Equation (6.3). Then on $zw = 1$, $g_2(z, w) = -z^2$ and on $zw = 2$, $g_2(z, w) = z$.

Denote by ξ_1, ξ_2 the roots of $(D-1)(D-2) = 1$, and choose a continuous curve $\gamma\colon [0,1] \to \mathbb{C} \setminus \{\xi_1, \xi_2\}$ such that $\gamma(0) = 1$ and $\gamma(1) = 2$. Then g_2 induces a family of continuous self-maps of $\mathbb{C}^*$

$$F\colon [0,1] \times \mathbb{C}^* \to \mathbb{C}^*, (t, \theta) \mapsto g_2(\theta, \frac{1}{\theta}\gamma(t)).$$

connecting between $F(0,\theta) = -\theta^2$ and $F(1,\theta) = \theta$. But since these two self-maps of $\mathbb{C}^*$ have different degrees, we find a contradiction.

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