Hyperbolicity and quasiconformal maps on the affine-additive group

Inaugural dissertation
of the Faculty of Science,
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presented by

Elia Bubani

Supervisor of the doctoral thesis:

Prof. Dr. Zoltán Balogh
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The Dean
Prof. Dr. Jean-Louis Reymond

Preface

This thesis consists of results achieved in my PhD studies, which I have conducted under supervision of Prof. Dr. Zoltán Balogh at the Mathematical Institute of the University of Bern. During the time of these studies I have been supported by the Swiss National Science Foundation (Grant numbers 191978 and 10001161).
Some of the results presented here are contained in the preprints [7] and [8], both accepted for publication. The material of such preprints is included in Chapter 4 and Chapters 5, 6, 7, respectively. A preprint that comprises results of Chapter 8 is in preparation.

Elia Bubani
Bern, 18 June, 2025

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I am grateful for all the good fun I had with my bandmates, flatmates (old and current), friends and family. To conclude, I thank JJ for all the pleasant time we share together.

The pictures at pages 56 and 66 are made with GeoGebra® (https://www.geogebra.org/).

Contents

Introduction

In this thesis we present the sub-Riemannian affine-additive group $\mathcal{AA}$. We shall analyze it in the geometric context of sub-Riemannian spaces, explore the associated quasiconformal mapping theory and eventually establish notions of mean and Gaussian curvature for surfaces embedded in it. This introduction is going to contain a description of the context of our research project together with a presentation of our main results.

History and main results.
Due to work of Heinonen and Koskela, [34] the theory of quasiconformal mappings has been developed in the setting of general metric measure spaces satisfying some mild regularity properties. For the related analytic machinery including upper gradients, capacities and Sobolev spaces we refer to the book of Heinonen, [33], or the book of Heinonen, Koskela, Shanmungalingam and Tyson [36]. Let us recall that a homeomorphism $f : X \to Y$ between two metric spaces (X, d_X) and (Y, d_Y) is called *quasiconformal* if there exists $H \geq 1$ such that

$$\limsup_{r\to 0} \frac{\sup_{d_X(p,q)\leq r} d_Y(f(p), f(q))}{\inf_{d_X(p,q)\geq r} d_Y(f(p), f(q))} := H_f(p) \leq H, \tag{1}$$

for all p in X.
An important class of examples where these results apply is the geometric setting of sub-Riemannian spaces, including Heisenberg groups. Motivated by Mostow rigidity [47], the theory of quasiconformal mappings in the Heisenberg group has been developed by Pansu [48] and Korányi and Reimann in [38] and [39]. In our model the *first Heiseberg group* $\mathbb{H}$ is $\mathbb{C}\times\mathbb{R}$ with coordinates $(z = x + iy, t)$ and group operation

$$(z', t') \star (z, t) = (z' + z, t' + t + 2\Im(\overline{z}'z))\,.$$

The contact form of $\mathbb{H}$ is given by

$$\vartheta_{\mathbb{H}} = dt + 2(xdy - ydx).$$

The horizontal sub-bundle $\mathcal{H}_{\mathbb{H}} := \ker \vartheta_{\mathbb{H}}$ of the tangent bundle of $\mathbb{H}$ is spanned by the vector fields

$$X = \partial_x + 2y\partial_t,\ Y = \partial_y - 2x\partial_t\,.$$

The corresponding Carnot-Carathéodory distance $d_{\mathbb{H}}$ is associated to the sub-Riemannian metric $\langle\cdot,\cdot\rangle_{\mathbb{H}}$ making $\{X, Y\}$ an orthonormal frame. For further properties of the Heisenberg group and higher dimensional analogues of this space we refer to the book of Capogna, Danielli, Scott and Tyson [17].
The theory of quasiconformal mappings in Heisenberg groups is rather advanced, examples of non-trivial quasiconformal maps acting between these spaces have been constructed as flows of contact vector fields by Korányi and Reimann [38], [39] and by lifting of planar symplectic maps by Capogna and Tang [18]. Extremal quasiconformal maps that are similar to the planar stretch map, acting between Heisenberg groups were found by Balogh, Fässler and Platis [10]. Using the flow method of Korányi and Reimann, Balogh established in [6] the existence of quasiconformal maps between Heisenberg groups distorting the Hausdorff dimension of Cantor sets in a rather arbitrary fashion.

By a theorem of Darboux (see Theorem 18.19 in Lee's book [41]), every $(2n+1)$-dimensional contact manifold is locally bi-Lipschitz to the n-th Heisenberg group, therefore one would expect that the results of quasiconformal maps could be transposed from the Heisenberg setting to general contact manifolds endowed with a sub-Riemannian metric. However, this turns out not to be the case, as not all contact manifolds are *globally* quasiconformal to the Heisenberg group. Before quoting a remarkable example of this fact we need to define another three dimensional contact manifold: the *roto-translation group* $\mathcal{RT}$ is $\mathbb{C}\times\mathbb{R}$ with coordinates $(z = x+iy, t)$ and group operation

$$(z', t') \star (z, t) = \left(e^{it'}z + z', t' + t\right).$$

The contact form of $\mathcal{RT}$ and the horizontal sub-bundle $\mathcal{H}_{\mathcal{RT}}$ of the tangent bundle of $\mathcal{RT}$ are respectively defined as

$$\vartheta_{\mathcal{RT}} = \sin t\, dx - \cos t\, dy, \quad \mathcal{H}_{\mathcal{RT}} = \ker \vartheta_{\mathcal{RT}}.$$

A basis for $\mathcal{H}_{\mathcal{RT}}$ is given by the vector fields

$$X' = \cos t\,\partial_x + \sin t\,\partial_y,\ Y' = \partial_t\,,$$

and the corresponding Carnot-Carathéodory distance $d_{\mathcal{RT}}$ is associated to the sub-Riemannian metric $\langle\cdot,\cdot\rangle_{\mathcal{RT}}$ which makes $\{X', Y'\}$ orthonormal. In [26], Fässler, Koskela and Le Donne proved that the sub-Riemannian roto-translation group is not globally quasiconformal to the Heisenberg group, in contrast to the fact, that there exists a global contactomorphism between these spaces.

The main object considered in this thesis is the affine-additive group, a three dimensional Lie group endowed with a sub-Riemannian metric. We prove that it is also globally contactomorphic to both, the Heisenberg group $\mathbb{H}$ and (by [26]) to the roto-translation group $\mathcal{RT}$. However, the affine-additive group is not globally quasiconformal to neither the Heisenberg, nor to the roto-translation group. The reason for the non-existence of a global quasiconformal map between these groups is their behaviour at infinity as formulated by Zorich in [56] (see also Holopainen and Rickman [37] and Fässler, Lukyanenko and Tyson [28]). We prove that the affine additive group has a non-vanishing 4-capacity at infinity, thus it is hyperbolic in the terminology of [56], while both the Heisenberg and the roto-translation groups are parabolic, having a vanishing 4-capacity at infinity.

To be more precise we define the affine-additive group $(\mathcal{AA}, \star)$ as the Cartesian product of $\mathbb{R}$ with the hyperbolic right half-plane $\mathbf{H}^1_{\mathbb{C}} := \{(\lambda, t) : \lambda > 0, t \in \mathbb{R}\}$ given by:

$$\mathcal{AA} = \mathbb{R} \times \mathbf{H}^1_{\mathbb{C}},$$

together with the group law

$$(a', \lambda', t') \star (a, \lambda, t) = (a' + a, \lambda'\lambda, \lambda' t + t')$$

and the contact 1-form

$$\vartheta = \frac{dt}{2\lambda} - da.$$

From the topological viewpoint it is one of the eight 3-dimensional Thurston geometries, see Thurston's book [53]. For a detailed presentation of the geometric structure of the the affine-additive group $\mathcal{AA}$ we refer to Chapter 1 of this thesis. At this point, we can say that the Carnot-Carathéodory distance $d_{\mathcal{AA}}$ will be defined as the sub-Riemannian distance on $\mathcal{AA}$ generated by the horizontal vector fields

$$U = \partial_a + 2\lambda\partial_t, \; V = 2\lambda\partial_\lambda,$$

and a sub-Riemannian metric making $\{U, V\}$ an orthonormal frame. The left-invariant Haar measure on the group $\mathcal{AA}$ is given by $d\mu_{\mathcal{AA}} = \frac{da\, d\lambda\, dt}{\lambda^2}$.
As a highlight we have Theorem 1.4.5 which offers a description of the sub-Riemmannian geodesics of $\mathcal{AA}$ obtained by methods deriving from Optimal Control Theory.
In the Chapters 2, 3 and 4 we present the main tools to initially compare the contact

geometries of the spaces $\mathcal{AA}$, $\mathbb{H}$ and $\mathcal{RT}$, then distinguish them under quasiconformality. We summarize Propositions 2.1.2, 3.3, 4.0.1 and Theorems 4.1.1, 4.2.2 as the first main result of this thesis with the following:

Theorem 0.0.1. *The metric measure space $(\mathcal{AA}, d_{\mathcal{AA}}, \mu_{\mathcal{AA}})$ is a locally 4−Ahlfors regular space. It is globally contactomorphic to the first Heisenberg group $\mathbb{H}$. The sub-Riemannian manifold, $(\mathcal{AA}, d_{\mathcal{AA}}, \mu_{\mathcal{AA}})$ is 4 hyperbolic, in particular there is no non-trivial quasiregular map $f : \mathbb{H} \to \mathcal{AA}$.*

We underline in the last statement the inequivalence under quasiconformality between the first Heisenberg group $\mathbb{H}$ and the affine-additive group $\mathcal{AA}$. This means that maps which are quasiconformal in $\mathbb{H}$ are not compatible with the quasiconfomal maps of $\mathcal{AA}$. Therefore it becomes interesting to study quasiconformal maps of the affine-additive group as well as to inquire on further properties of such quasiconformal maps. We are going to introduce the reader to a mapping problem in the complex plane formulated by Grötzsch in 1928.
The Grötzsch problem can be expressed as follows: let $a > 1$, consider a square Q and a rectangle R_a respectively given by

$$Q = (0, 1) \times (0, 1), \quad R_a = (0, a) \times (0, 1).$$

We ask if there is a conformal map which maps the horizontal edges of Q into the corresponding horizontal edges of R_a, and requiring the same condition on the vertical edges. It turns out that there is no such conformal mapping; however, using complex notation, one finds that the linear stretch map given by

$$x + iy \mapsto ax + iy, \tag{2}$$

solves the Grötzsch problem and it is the closest to be conformal. The works of Grötzsch [30], [3], established criteria to measure how to approximate conformality. Now, let again $a > 1$, let $k > 1$ and let us consider the annulus in the complex plane $A(1, a)$ defined as

$$A(1, a) = \{z \in \mathbb{C} : 1 < |z| < a\}.$$

Grötzsch also formulated the analogous problem between two annuli $A(1, a)$ and $A(1, a^k)$ and he proved that the solution is the radial stretch map given by

$$z \mapsto |z|^{k-1} z. \tag{3}$$

Via the formal substitution $z = e^{\xi + i\psi}$ one sees that the radial stretch can be seen as the linear map given by $(\xi, \psi) \mapsto (k\xi, \psi)$. It is also worth mentioning that Astala [4], used the radial stretch map to prove the sharpness on the optimal Sobolev exponent for K-quasiconformal mappings in the complex plane.

An *extremal quasiconformal* map is a minimizer for the mean distortion among some class of quasiconformal mappings, for details see (6). Methods involving the modulus of curve families were used to identify such extremality between annuli in the complex plane by Balogh, Fässler and Platis [9]. Balogh, Fässler and Platis constructed appropriate analogues of linear and radial stretch maps for the first Heisenberg group, [10]. For the latter case the same authors subsequently proved that the radial stretch map is an essentially unique minimizer for the *mean distortion functional*, [11].

In this thesis we search for extremal quasiconformal maps on $\mathcal{AA}$. We implement a suitable version of the modulus method which is evidently useful for our purposes. As a consequence we are able to present new examples of self-mappings of the affine-additive group which share some remarkable features with the linear strecth map (2) and the radial stretch map (3).

Following the work of Balogh, Fässler and Platis [10], we define a mapping having the "minimal stretching property" (MSP) for a given curve family by adapting it to our particular case in the sub-Riemannian framework.

In Chapter 6 we build a modulus method which relies on the minimal stretching property of the map for a given curve family foliating the domain of definition.

In Chapter 7 we show that this designed method detects quasiconformal maps between domains of the affine-additive group. The application is that we obtain extremal stretch maps minimizing the mean distortion functional in the class of all quasiconformal mappings between two such domains.

In order to be more specific, we recall the preliminary metric notion of a quasiconformal mapping given in (1) and we consider its source and target spaces as domains in the affine-additive group. There also exist an analytic as well as a geometric definition for quasiconformal mappings of $\mathcal{AA}$. It turns out that both are equivalent to the above metric definition. This fact was already well-known for quasiconformal maps of $\mathbb{C}$ and also known for general sub-Riemannian manifolds; we illustrate the details for the case of $\mathcal{AA}$ in Chapter 5. Such quasiconformal mappings share Sobolev regularity properties and also satisfy the contact condition, meaning that they preserve the contact form ϑ, i.e.

$$f^*\vartheta = \sigma\vartheta \tag{4}$$

almost everywhere for some non-vanishing smooth function $\sigma : \mathcal{AA} \to \mathbb{R}$. Constructions of diffeomorphisms of $\mathcal{AA}$ satysfying the contact condition are given in Theorem 2.2.4 and

Theorem 2.3.1: such contactomorphisms are presented through the lift of symplectic maps on the hyperbolic half-plane $\mathbf{H}^1_{\mathbb{C}}$ or obtained through the Korányi-Reimann flow method.

The contact condition imposes some rigidity on the quasiconformal mappings of smooth type; on the other hand, the contact condition is a quite straightforward requirement for a quasiconformal map. To be more precise, let $f = (f_1, f_2 + if_3) : \Omega \to \Omega'$ be a quasiconformal mapping between domains in the affine-additive group, and let $f_I = f_2 + if_3$. By defining the complex vector fields

$$Z = \frac{1}{2}(V - iU), \; \overline{Z} = \frac{1}{2}(V + iU), \tag{5}$$

it turns out that the horizontal derivatives Zf_I and $\overline{Z}f_I$ exist both as distributions and almost everywhere. From now on, we will consider quasiconformal mappings to be *orientation preserving*, i.e., $|Zf_I(p)| > |\overline{Z}f_I(p)|$ for almost every $p \in \Omega$.

We then define the *Beltrami coefficient* and the *distortion quotient* as

$$\mu_f(p) = \frac{\overline{Z}f_I}{Zf_I}(p) \text{ and } K(p, f) = \frac{|Zf_I| + |\overline{Z}f_I|}{|Zf_I| - |\overline{Z}f_I|}(p),$$

for points $p \in \mathcal{AA}$ where these expressions exist. In this thesis we shall make an extended use of the square of the distortion quotient $K^2(p, f)$. By letting $K_f = \operatorname{ess\,sup}_p K(p, f)$, we underline that any smooth contact transformation f with $1 \leq K_f < \infty$ is quasiconformal.

Given two domains $\Omega, \Omega' \subseteq \mathcal{AA}$ and a certain given class $\mathcal{F}$ comprising quasiconformal mappings $f : \Omega \to \Omega'$, we may define the deviation of a quasiconformal map from conformality as follows. We say that an $f_0 \in \mathcal{F}$ is *extremal for a mean distortion functional* if

$$\int_\Omega K^2(p, f_0)\rho_0^4(p)\, d\mu_{\mathcal{AA}}(p) = \min_{f \in \mathcal{F}} \int_\Omega K^2(p, f)\rho_0^4(p)\, d\mu_{\mathcal{AA}}(p), \tag{6}$$

for a given density ρ_0. This ρ_0 is extremal for the modulus of a chosen curve family foliating the domain Ω.

The concept of modulus of a curve family is defined in the general metric measure space setting by Heinonen and Koskela in [34]. In our case the modulus $\mathrm{Mod}_4(\Gamma)$ of a curve family Γ is defined as follows. Let $\mathrm{Adm}(\Gamma)$ be the set of *admissible densities*: that is, non-negative Borel functions $\rho : \mathcal{AA} \to [0, \infty]$ such that $\int_\gamma \rho\, d\ell \geq 1$ for all rectifiable curves $\gamma \in \Gamma$. Rectifiability here is understood in terms of the sub-Riemannian distance $d_{\mathcal{AA}}$. Then

$$\mathrm{Mod}_4(\Gamma) = \inf_{\rho \in \mathrm{Adm}(\Gamma)} \int_{\mathcal{AA}} \rho^4(p)\, d\mu_{\mathcal{AA}}(p), \tag{7}$$

see Chapter 5 for details.

It is important to mention that the ρ_0 used in (6) corresponds to the extremal density which attains the infimum for $\mathrm{Mod}_4(\Gamma)$. For example, we mention that the modulus of the curve family connecting the two boundaries of any revolution ring in the first Heisenberg group has been computed by Platis in [50]. The modulus method and its applications to extremal problems for conformal, quasiconformal mappings and the extension of moduli onto Teichmüller spaces is treated in the book of Vasil'ev [54]. Moreover, the notion of modulus of a curve family has been extended to serve as further quasiconformal invariants in the works of Brakalova, Markina and Vasil'ev in [15] and in [16].

An orientation preserving quasiconformal map $f_0 : \Omega \to \Omega'$ between domains in $\mathcal{AA}$ has the *minimal stretching property (MSP)* for a family Γ_0 of horizontal curves in Ω if for all $\gamma \in \Gamma_0$, $\gamma = (\gamma_1, \gamma_I) : [c, d] \to \mathcal{AA}$, one has

$$\mu_{f_0}(\gamma(s))\frac{\dot{\overline{\gamma_I}}(s)}{\dot{\gamma_I}(s)} < 0 \text{ for almost every } s \in (c, d) \text{ with } \mu_{f_0}(\gamma(s)) \neq 0.$$

Note that in the latter definition we require implicitly the expression $\mu_{f_0}(\gamma(s))\frac{\dot{\overline{\gamma_I}}(s)}{\dot{\gamma_I}(s)}$ to be real-valued.

Suppose next that Δ is a domain in $\mathbb{R}^2$. Let $0 \leq c < d$ and let $\gamma : (c, d) \times \Delta \to \Omega$ be a diffeomorphism which foliates a bounded domain Ω in the affine-additive group with the property that

$$\gamma(\cdot, \delta) : [c, d] \to \overline{\Omega}$$

is a horizontal curve with $|\dot{\gamma}(s, \delta)|_H \neq 0$ for all $\delta \in \Delta$ and

$$d\mu_{\mathcal{AA}}(\gamma(s, \delta)) = |\dot{\gamma}(s, \delta)|_H^4 \, ds \, d\nu(\delta)$$

for a measure ν on Δ. We consider the curve family $\Gamma_0 = \{\gamma(\cdot, \delta) : \delta \in \Delta\}$ and it will be shown that

$$\rho_0(p) = \begin{cases} \frac{1}{(d-c)|\dot{\gamma}(\gamma^{-1}(p))|_H}, & p = \gamma(s, \delta) \in \Omega, \\ 0, & p \notin \Omega, \end{cases} \tag{8}$$

is an extremal density for $\mathrm{Mod}_4(\Gamma_0)$.

Let $f_0 : \Omega \to \Omega'$ be an orientation preserving quasiconformal mapping between domains in the affine-additive group. Let γ be a foliation of Ω as described above. Assume as well that f_0 has the MSP for Γ_0; we then say that the distortion quotient $K(\cdot, f_0)$ is *constant along every curve* γ if and only if

$$K(\gamma(s, \delta), f_0) \equiv K_{f_0}(\delta) \quad \text{for all } (s, \delta) \in (c, d) \times \Delta. \tag{9}$$

The following condition for extremality of the mean distortion integral is the main result from Chapter 6.

Theorem 0.0.2. *Assume that f_0 satisfies the minimal stretching property with respect to Γ_0 described as above. Let ρ_0 be the extremal density for Γ_0 and assume $K(\cdot, f_0)$ to be constant along every curve foliating Ω. Let $\Gamma \supseteq \Gamma_0$ be a curve family such that $\rho_0 \in \mathrm{Adm}(\Gamma)$ and let $\mathcal{F}$ be the class of quasiconformal maps $f : \Omega \to \Omega'$ such that*

$$\mathrm{Mod}_4(f_0(\Gamma_0)) \leq \mathrm{Mod}_4(f(\Gamma)).$$

Then

$$\int_\Omega K^2(p, f_0)\rho_0^4(p)\, d\mu_{\mathcal{AA}}(p) \leq \int_\Omega K^2(p, f)\rho_0^4(p)\, d\mu_{\mathcal{AA}}(p)$$

for all $f \in \mathcal{F}$.

Towards a first application of Theorem 0.0.2, we define certain suitable domains in the affine-additive group. Let $k > 0$ and consider two domains Ω and Ω^k which shall be defined in detail in Section 7.1. Further, consider the curve family Γ_0 foliating Ω as well as its extremal density ρ_0 given by (8).

The extremal mapping f_0 will be a version of the *linear stretch map* (2) of the form $f_k : \overline{\Omega} \to \overline{\Omega^k}$:

$$f_k(a, \lambda + it) = (ka, \lambda + ikt).$$

We underline that the distortion $K(\cdot, f_k)$ is constant, implying that

$$\int_\Omega K^2(p, f_k)\rho_0^4(p)\, d\mu_{\mathcal{AA}}(p) = K_{f_k}^2 \int_\Omega \rho_0^4(p)\, d\mu_{\mathcal{AA}}(p). \tag{10}$$

We next formulate the main result of Section 7.1. Denote by $\mathcal{F}_k$ the class of all quasiconformal maps $\overline{\Omega} \to \overline{\Omega^k}$ with prescribed boundary conditions which will be rigorously set up in Section 7.1. The equality (10) allows us to formulate the following

Theorem 0.0.3. *The linear stretch map $f_k : \Omega \to \Omega^k$ is an orientation preserving quasiconformal map. With the above notation for ρ_0, f_k minimizes the mean distortion within the class $\mathcal{F}_k$: for all $f \in \mathcal{F}_k$ we have that*

$$K_{f_k}^2 \int_\Omega \rho_0^4(p)\, d\mu_{\mathcal{AA}}(p) \leq \int_\Omega K^2(p, f)\rho_0^4(p)\, d\mu_{\mathcal{AA}}(p). \tag{11}$$

Towards another application of Theorem 0.0.2, we can also define some suitable domain in the affine-additive group, which looks natural, once we have considered a different type of coordinate system on the affine-additive group. The *cylindrical-logarithmic coordinates* are defined as

$$(a, \lambda + it) = \left(a, e^{\xi + i\psi}\right), \quad (a, \xi, \psi) \in \mathbb{R} \times \mathbb{R} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Let $0 < k < 1$, $r_0 > 1$ and $0 < \psi_0 < \frac{\pi}{2}$. Consider two truncated cylindric shells: D_{r_0,ψ_0} and $D^k_{r_0,\psi_0}$, see for details Section 7.3. Further, consider the curve family Γ_0 foliating D_{r_0,ψ_0} as well as its extremal density ρ_0 given by (8).

The extremal mapping f_0 will be a version of the *radial stretch map* (3) of the form $f_k : \overline{D_{r_0,\psi_0}} \to \overline{D^k_{r_0,\psi_0}}$, expressed in cylindrical-logarithmic coordinates as

$$(a, \xi, \psi) \mapsto \left(a - \frac{\psi}{2} + \frac{1}{2} \tan^{-1}\left(\frac{\tan\psi}{k}\right), k\xi, \tan^{-1}\left(\frac{\tan\psi}{k}\right) \right).$$

We point out that the distortion $K(\cdot, f_k)$ is no longer constant as in Theorem 0.0.3, nevertheless $K(\cdot, f_k)$ is constant along the curves foliating D_{r_0,ψ_0} in the sense of equation (9). Finally, we formulate the main result of Section 7.3. Denote by $\mathcal{F}_k$ the class of all quasiconformal maps $\overline{D_{r_0,\psi_0}} \to \overline{D^k_{r_0,\psi_0}}$ with prescribed boundary conditions (see Section 7.3 for details).

Theorem 0.0.4. *The radial stretch map $f_k : \overline{D_{r_0,\psi_0}} \to \overline{D^k_{r_0,\psi_0}}$ is an orientation preserving quasiconformal map. With the above notation for ρ_0, f_k minimizes the mean distortion within the class $\mathcal{F}_k$: for all $f \in \mathcal{F}_k$ we have that*

$$\int_{D_{r_0,\psi_0}} K^2(p, f_k)\rho_0^4(p)\, d\mu_{\mathcal{AA}}(p) \le \int_{D_{r_0,\psi_0}} K^2(p, f)\rho_0^4(p)\, d\mu_{\mathcal{AA}}(p)\,.$$

In the remaining part of this introduction we focus on the notion of curvature for surfaces in the affine-additive group $\mathcal{AA}$. Thanks to the seminal works of Gauss and Riemann a full understanding of the notion of curvature has received a main role in differential geometry. The purpose of Chapter 8 is to provide notions of horizontal mean curvature and intrinsic Gaussian curvature for Euclidean C^2-smooth surfaces in $\mathcal{AA}$, adopting the so called Riemannian approximation scheme. There are already results in this direction for the roto-translation group $\mathcal{RT}$ by Citti and Sarti [19] and for the Heisenberg group $\mathbb{H}$ by Balogh, Tyson and Vecchi [13] and Diniz and Veloso [22].

The novelty of our approach to this problem is the Riemannian approximation scheme combined with Cartan's formalism, see Clelland [20]. The method of Riemannian approximants counts on a result of paramount importance due to Gromov [29], which, in the context of the affine-additive group, states that the metric space $(\mathcal{AA}, d_{\mathcal{AA}})$ can be obtained as the pointed Gromov–Hausdorff limit of a family of Riemannian manifolds $(\mathcal{AA}, g_\epsilon)$, where g_ϵ is a suitable family of Riemannian metrics. In detail, let us denote by $W = -\partial_a$ the Reeb vector field of $\mathcal{AA}$. In order to make use of the contact structure of $\mathcal{AA}$ we consider the sub-Riemannian metric $\langle \cdot, \cdot \rangle_{\mathcal{AA}}$ which makes U, V an orthonormal basis for $\mathcal{H}_{\mathcal{AA}}$. A possible way to define a

Riemannian inner product is to set $W^\epsilon := \epsilon W$ for every $\epsilon > 0$, and then to extend $\langle \cdot, \cdot \rangle_{\mathcal{AA}}$ to an inner product g_ϵ which makes $\{U, V, W^\epsilon\}$ an orthonormal frame. The family of metric spaces $(\mathcal{AA}, g_\epsilon)$ converges to $(\mathcal{AA}, d_{\mathcal{AA}})$ in the pointed Gromov–Hausdorff sense. By means of Cartan's method on moving frames (see Section 8.1) we derive formulae for the sectional curvature $\overline{K}^\epsilon$ and for the second fundamental form $\mathbf{II}^\epsilon$. By studying the limit case we provide formulae for the horizontal mean curvature H^0 and the horizontal Gaussian curvature K^0. Following this approach with Propositions 8.2.1 and 8.3.1 we will introduce such curvature notions away from characteristic points. We shall consider Euclidean C^2-smooth surfaces $\Sigma = \{p \in \mathcal{AA} : u(p) = 0\}$, whose characteristic set $\mathcal{C}(\Sigma)$ is defined as the set of points $p \in \Sigma$ where $\nabla_H u(p) := (Uu(p), Vu(p)) = (0, 0)$. The explicit expressions of H^0 and K^0 are given in terms of second derivatives of u and read as follows:

$$H^0 = U\left(\frac{Uu}{\|\nabla_H u\|}\right) + V\left(\frac{Vu}{\|\nabla_H u\|}\right) - 2\frac{Vu}{\|\nabla_H u\|},$$

$$K^0 = -2E_1\left(\frac{Wu}{\|\nabla_H u\|}\right) - 4\left(\frac{Wu}{\|\nabla_H u\|}\right)^2,$$

where the differential operator E_1 is defined as $E_1 = \frac{1}{\|\nabla_H u\|}(-Vu\,U + Uu\,V)$.
For both curvatures H^0 and K^0 we will present a list of examples to illustrate the differential geometry of surfaces in $\mathcal{AA}$, calling attention on the constant curvature cases.

Structure of the thesis.

In Chapter 1 we present the affine-additive group $\mathcal{AA}$ in the context of three dimensional contact sub-Riemannian Lie groups. In Chapter 2 we analyze $\mathcal{AA}$ through the lens of contact geometry. In Chapter 3 we present metric measure properties of $\mathcal{AA}$ towards applications in quasiconformal and quasiregular mapping theory. In Chapter 4 we examine the aforementioned applications. In Chapter 5 we elaborate the theory of quasiconformal mappings on $\mathcal{AA}$. In Chapter 6 we formulate a criterion based on the modulus of curve families establishing if a quasiconformal map on $\mathcal{AA}$ is a minimizer for the mean distortion functional. In Chapter 7 we define linear and radial stretch maps on $\mathcal{AA}$ and use the criterion from the previous chapter to prove the extremality of stretch maps for the mean distortion functional. In Chapter 8 we provide notions for the horizontal mean curvature and for the intrinsic Gaussian curvature for surfaces embedded in $\mathcal{AA}$.

Chapter 1

The affine-additive group $\mathcal{AA}$

In this chapter we briefly present the background in which this thesis is located. We will assume a basic level of familiarity with the theory of Lie groups, as presented for instance in [41], as well with Contact and Symplectic Geometry [14].

1.1 Preliminaries on 3-D contact sub-Riemannian Lie groups

The metric spaces considered in this thesis are 3-dimensional Lie groups $\mathbb{G}$ with group multiplication $\star$. We shall assume that $\mathbb{G}$ is equipped with a left-invariant contact form $\vartheta_{\mathbb{G}}$. Using this contact form we define a distribution of planes in the tangent bundle $\mathrm{T}_{\mathbb{G}}$ of $\mathbb{G}$ as $\mathcal{H}_{\mathbb{G}} = \ker \vartheta_{\mathbb{G}}$. Next, a left-invariant sub-Riemannian metric is constructed on $\mathbb{G}$ as follows. If X and Y are left-invariant vector fields such that $\mathcal{H}_{\mathbb{G}} = \mathrm{span}\{X, Y\}$, then a left-invariant sub-Riemannian metric $\langle\cdot,\cdot\rangle_{\mathbb{G}}$ is considered in $\mathcal{H}_{\mathbb{G}}$, making $\{X, Y\}$ an orthonormal basis of $\mathcal{H}_{\mathbb{G}}$.

An absolutely continuous curve $\gamma : [a, b] \to \mathbb{G}$, $\gamma = \gamma(s)$ shall be called horizontal if $\dot\gamma(s) \in \ker(\vartheta_{\mathbb{G}})_{\gamma(s)}$ for almost every $s \in [a, b]$. Then, the horizontal velocity of γ is

$$|\dot\gamma(s)|_{\mathbb{G}} = \sqrt{\langle\dot\gamma(s), X_{\gamma(s)}\rangle^2_{\mathbb{G}} + \langle\dot\gamma(s), Y_{\gamma(s)}\rangle^2_{\mathbb{G}}}.$$

The horizontal length of γ is

$$\ell_{\mathbb{G}}(\gamma) = \int_a^b |\dot\gamma(s)|_{\mathbb{G}}\, ds.$$

The corresponding sub-Riemannian or Carnot-Carathéodory distance $d_{\mathbb{G}}$ associated to the sub-Riemannian metric $\langle\cdot,\cdot\rangle_{\mathbb{G}}$ is defined in $\mathbb{G}$ as follows: let $p, q \in \mathbb{G}$ and consider the family

$\Gamma_{(p,q)}$ of horizontal curves $\gamma : [a,b] \to \mathbb{G}$ such that $\gamma(a) = p$ and $\gamma(b) = q$. Then

$$d_{\mathbb{G}}(p,q) = \inf_{\gamma \in \Gamma_{(p,q)}} \{\ell_{\mathbb{G}}(\gamma)\}. \tag{1.1}$$

We remark that the above definition only depends on the values of $\langle \cdot, \cdot \rangle_{\mathbb{G}}$ on $\mathcal{H}_{\mathbb{G}}$. Moreover, since $\mathcal{H}_{\mathbb{G}}$ is completely non integrable, the distance $d_{\mathbb{G}}$ is finite, geodesic, and induces the manifold topology (see Mitchell [44], Montgomery [45]).
This will make the space $(\mathbb{G}, d_{\mathbb{G}})$ a metric space. We consider the measure $\mu_X = \mu_{\mathbb{G}}$ induced by the contact form $\vartheta_{\mathbb{G}}$ by $\mu_{\mathbb{G}} = \vartheta_{\mathbb{G}} \wedge d\vartheta_{\mathbb{G}}$ (up to a multiplicative constant different from 0) that is also left-invariant. When it will not cause confusion we shall denote by $\mathbb{G}$ the metric measure space $(\mathbb{G}, d_{\mathbb{G}}, \mu_{\mathbb{G}})$.

A well-known example of such a structure is the *first Heisenberg group* $\mathbb{H}$. Its underlying manifold is $\mathbb{C} \times \mathbb{R}$ with coordinates $(z = x + iy, t)$ and the group multiplication $\star$ is given by

$$p' \star p = (z' + z, t' + t + 2\Im(\overline{z}' z))$$

for every $p = (z,t)$ and $p' = (z', t')$ in $\mathbb{C} \times \mathbb{R}$.

The *contact form* of $\mathbb{H}$ is given by:

$$\vartheta_{\mathbb{H}} = dt + 2\Im(\overline{z} dz) = dt + 2(xdy - ydx).$$

The horizontal sub-bundle $\mathcal{H}_{\mathbb{H}}$ of the tangent bundle is spanned by the vector fields

$$X = \partial_x + 2y\partial_t,\ Y = \partial_y - 2x\partial_t\,.$$

Denote the sub-Riemannian metric in $\mathbb{H}$ by $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ making $\{X, Y\}$ an orthonormal frame. The horizontal length of a curve $\gamma = \gamma(s)$, $s \in [a,b]$, $\gamma(s) = (z(s), t(s))$ is

$$\ell_{\mathbb{H}}(\gamma) = \int_a^b |\dot{z}(s)|\, ds.$$

Denote also the corresponding Carnot-Carathéodory distance by $d_{\mathbb{H}}$. The measure $\mu_{\mathbb{H}}$ is a bi-invariant Haar measure for $\mathbb{H}$ and it coincides with the 3-dimensional Lebesgue measure in $\mathbb{C} \times \mathbb{R}$ denoted with $\mathcal{L}^3$.

The second example is the *roto-translation group* $\mathcal{RT}$ (see Chapter 3 in [17] and [26]). Its underlying manifold is $\mathbb{C} \times \mathbb{R}$ with coordinates $p = (z = x+iy, t)$ and the group multiplication $\star$ is given by

$$p' \star p = \left(e^{it'} z + z', t' + t\right) \in \mathbb{C} \times \mathbb{R}$$

for every $p = (z,t)$ and $p' = (z',t')$ in $\mathbb{C} \times \mathbb{R}$.

The *contact form* of $\mathcal{RT}$ is given by:

$$\vartheta_{\mathcal{RT}} = \sin t\, dx - \cos t\, dy.$$

The horizontal sub-bundle $\mathcal{H}_{\mathcal{RT}}$ of the tangent bundle is spanned by the vector fields

$$X = \cos t\, \partial_x + \sin t\, \partial_y,\ Y = \partial_t\,.$$

Denote the sub-Riemannian metric in $\mathcal{RT}$ by $\langle \cdot, \cdot \rangle_{\mathcal{RT}}$ making $\{X, Y\}$ an orthonormal frame. The horizontal length of a curve $\gamma = \gamma(s)$, $s \in [a,b]$, $\gamma(s) = (z(s), t(s))$ is

$$\ell_{\mathcal{RT}}(\gamma) = \int_a^b |\dot{\gamma}(s)|_{\mathcal{RT}}\, ds,$$

where

$$|\dot{\gamma}(s)|_{\mathcal{RT}} = \sqrt{(\dot{x}(s)\cos t(s) + \dot{y}(s)\sin t(s))^2 + \dot{t}(s)^2}.$$

Denote also the corresponding Carnot-Carathéodory distance by $d_{\mathcal{RT}}$. The measure $\mu_{\mathcal{RT}}$ is again a bi-invariant Haar measure of $\mathcal{RT}$ and it is the 3-dimensional Lebesgue measure in $\mathbb{C} \times \mathbb{R}$.

1.2 The group structure of $\mathcal{AA}$

The main subject of this thesis is the affine-additive group, which we describe below. In particular, after introducing the group, we discuss its sub-Riemannian structure.

Our starting point is the hyperbolic plane, defined as

$$\mathbf{H}^1_{\mathbb{C}} := \{\zeta = \xi + i\eta \in \mathbb{C} : \xi > 0, \eta \in \mathbb{R}\}$$

with the Riemannian metric $g = \frac{|d\zeta|^2}{4\xi^2} = \frac{d\xi^2 + d\eta^2}{4\xi^2}$.
We consider affine transformations on $\mathbf{H}^1_{\mathbb{C}}$, composed by dilations D_λ, $\lambda > 0$, defined by $D_\lambda(\zeta) = \lambda\zeta$, and translations T_t, $t \in \mathbb{R}$, defined by $T_t(\zeta) = \zeta + it$, for $\zeta \in \mathbf{H}^1_{\mathbb{C}}$, resulting in maps of the form

$$M(\lambda, t)(\zeta) = (T_t \circ D_\lambda)(\zeta) = \lambda\zeta + it.$$

It is clear that $\mathbf{H}^1_{\mathbb{C}}$ is in bijection with the set of transformations of the above form: to each point $\xi + i\eta$ we uniquely assign the transformation $M(\xi, \eta)$. Therefore we define a group structure on $\mathbf{H}^1_{\mathbb{C}}$ by considering the composition of any two transformations $M(\lambda', t')$ and $M(\lambda, t)$:

$$(M(\lambda', t') \circ M(\lambda, t))(\zeta) = M(\lambda', t')(\lambda\zeta + it) = \lambda'\lambda\zeta + i(\lambda' t + t') = M(\lambda'\lambda,\ \lambda' t + t')(\zeta).$$

To sum up, (compare to Section 4.4.2 in Petersen's book [49]) the group operation on $\mathbf{H}^1_{\mathbb{C}}$ is given by

$$(\lambda', t') \cdot (\lambda, t) = (\lambda'\lambda,\ \lambda' t + t'). \tag{1.2}$$

This operation is extended over the space $\mathbb{R} \times \mathbf{H}^1_{\mathbb{C}}$ as follows: we take the Cartesian product of the additive group $(\mathbb{R}, +)$ and the group $(\mathbf{H}^1_{\mathbb{C}}, \cdot)$, where $\cdot$ is as in (1.2). Then, if $p' = (a', \lambda', t')$ and $p = (a, \lambda, t)$ are points of $\mathbb{R} \times \mathbf{H}^1_{\mathbb{C}}$ we have

$$p' \star p = (a' + a, \lambda'\lambda, \lambda' t + t'), \tag{1.3}$$

which is again a point in $\mathbb{R} \times \mathbf{H}^1_{\mathbb{C}}$.

Definition 1.2.1. The pair $\mathcal{AA} = (\mathbb{R} \times \mathbf{H}^1_{\mathbb{C}},\ \star)$ shall be called the affine-additive group.

The neutral element of $\mathcal{AA}$ is $e_{\mathcal{AA}} = (0, 1, 0)$ and for $p = (a, \lambda, t) \in \mathcal{AA}$ we have that

$$p^{-1} = \left(-a, \frac{1}{\lambda}, -\frac{t}{\lambda} \right).$$

The centre of $\mathcal{AA}$ is

$$Z(\mathcal{AA}) = \{(a, 1, 0) : a \in \mathbb{R}\} \cong (\mathbb{R}, +).$$

The subgroup $N = \{0\} \times \mathbf{H}^1_{\mathbb{C}} \cong \mathbf{H}^1_{\mathbb{C}}$ of $\mathcal{AA}$ is a normal subgroup of $\mathcal{AA}$, $Z(\mathcal{AA}) \cap N = \{e_{\mathcal{AA}}\}$ and we can therefore write

$$\mathcal{AA} = \mathbb{R} \ltimes \mathbf{H}^1_{\mathbb{C}}.$$

Proposition 1.2.2. *$\mathcal{AA}$ is metabelian.*

Proof. For $p, p' \in \mathcal{AA}$, a straightforward calculation gives

$$p^{-1} \star p'^{-1} \star p \star p' \in \{(0, 1, t) : t \in \mathbb{R}\}.$$

Now, for some $t, t' \in \mathbb{R}$ two elements of the form $(0, 1, t)$ and $(0, 1, t')$ commute:

$$(0, 1, t) \star (0, 1, t') = (0, 1, t + t') = (0, 1, t') \star (0, 1, t).$$

□

We define a 1-form on $\mathcal{AA}$ as follows:

$$\vartheta = \frac{dt}{2\lambda} - da. \tag{1.4}$$

Since $d\vartheta = \frac{1}{2\lambda^2} dt \wedge d\lambda$ we obtain $\vartheta \wedge d\vartheta = \frac{dt \wedge da \wedge d\lambda}{2\lambda^2}$ and thus $(\mathcal{AA}, \vartheta)$ is a contact manifold.

1.3 The sub-Riemannian structure of $\mathcal{AA}$

In what follows we identify the left invariant vector fields and define a left invariant sub-Riemannian metric on the group $\mathcal{AA}$.

Proposition 1.3.1. *The vector fields*

$$U = \partial_a + 2\lambda\partial_t,\ V = 2\lambda\partial_\lambda,\ W = -\partial_a$$

are left-invariant and form a basis for the tangent bundle $T(\mathcal{AA})$ of $\mathcal{AA}$. They satisfy the following Lie bracket relations:

$$[U, W] = [V, W] = 0 \ \text{ and } \ [U, V] = -2(U + W); \tag{1.5}$$

Moreover, a left-invariant measure for $\mathcal{AA}$ is $d\mu_{\mathcal{AA}} = \frac{da\,d\lambda\,dt}{\lambda^2}$.

Proof. By the definition of U, V and W we have the relations:

$$\partial_a = -W, \quad \partial_\lambda = \frac{U}{2\lambda}, \quad \partial_t = \frac{U + W}{2\lambda},$$

and thus $\{U, V, W\}$ is a basis for $T(\mathcal{AA})$. Now we are going to verify that U, V and W are left-invariant. We set $e = e_{\mathcal{AA}} = (0, 1, 0)$ and we define the following three tangent vectors spanning a basis for $T_e(\mathcal{AA})$:

$$U_e = (\partial_a + 2\partial_t)_{|e},\ V_e = (2\partial_\lambda)_{|e},\ W_e = (-\partial_a)_{|e}\,.$$

If we fix a point $p' = (a', \lambda', t') \in \mathcal{AA}$ we can consider the left translation on $\mathcal{AA}$ given by

$$L_{p'}(p) = p' \star p = (a' + a, \lambda'\lambda, \lambda' t + t'),$$

where the Jacobian matrix of the derivative $(L_{p'})_{*,p}$ of $L_{p'}$ evaluated at p is

$$(DL_{p'})_p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda' & 0 \\ 0 & 0 & \lambda' \end{bmatrix}.$$

We construct U, V and W by using $(L_p)_{*,e} : T_e(\mathcal{AA}) \to T_p(\mathcal{AA})$ and verifying that

$$(L_p)_{*,e}\,(\partial_a + 2\partial_t)_{|e} = U_p,\ (L_p)_{*,e}\,(2\partial_\lambda)_{|e} = V_p,\ (L_p)_{*,e}\,(-\partial_a)_{|e} = W_p.$$

This proves the first claim.

The verification of the Lie bracket relations are straightforward for $[U, W] = 0$, $[V, W] = 0$, and for $[U, V]$ we see

$$[U, V] = -4\lambda\partial_t = -2(U + W).$$

The left-invariance of $\mu_{\mathcal{AA}}$ comes from its construction, indeed for $p = (a, \lambda, t) \in \mathcal{AA}$ we have

$$d\mu_{\mathcal{AA}} = \frac{da\, d\lambda\, dt}{|\det(DL_p)_e|} = \frac{da\, d\lambda\, dt}{\lambda^2}.$$

□

As a first consequence of the vector fields relation (1.5) we deduce that the Lie algebra of left invariant vector fields of $\mathcal{AA}$ is not nilpotent. Note, that $\vartheta(U) = \vartheta(V) = 0$ and thus, the horizontal bundle of $\mathcal{AA}$ is $\mathcal{H}_{\mathcal{AA}} = \text{Span}\{U, V\}$. Further, since $d\vartheta(W, \cdot) = 0$, Proposition 1.3.1 implies that W is the *Reeb* vector field.

The sub-Riemannian structure in $\mathcal{AA}$ is defined by a sub-Riemannian metric on $\mathcal{H}_{\mathcal{AA}}$ making $\{U, V\}$ an orthonormal basis. In order to define the sub-Riemannian or Carnot-Carathéodory distance on $\mathcal{AA}$ let $\gamma : [c, d] \to \mathcal{AA}$, $\gamma(s) = (a(s), \lambda(s), t(s))$ be an absolutely continuous curve. Its tangent vector at $\gamma(s)$ is

$$\dot{\gamma}(s) = \frac{\dot{t}(s)}{2\lambda(s)} U_{\gamma(s)} + \frac{\dot{\lambda}(s)}{2\lambda(s)} V_{\gamma(s)} + \left(\frac{\dot{t}(s)}{2\lambda(s)} - \dot{a}(s) \right) W_{\gamma(s)}.$$

The curve γ is a horizontal curve if and only if $\dot{\gamma}(s) \in \ker \vartheta_{\gamma(s)}$ for almost every $s \in [c, d]$. This is equivalent to the ODE

$$\frac{\dot{t}(s)}{2\lambda(s)} - \dot{a}(s) = 0, \text{ a.e. } s \in [c, d]. \tag{1.6}$$

It follows that for a horizontal curve

$$\dot{\gamma}(s) = \frac{\dot{t}(s)}{2\lambda(s)} U_{\gamma(s)} + \frac{\dot{\lambda}(s)}{2\lambda(s)} V_{\gamma(s)} \in (\mathcal{H}_{\mathcal{AA}})_{\gamma(s)}.$$

The horizontal velocity $|\dot{\gamma}|_H$ of γ is now defined by the relation

$$|\dot{\gamma}|_H = \left(\langle \dot{\gamma}, U \rangle_{\mathcal{AA}}^2 + \langle \dot{\gamma}, V \rangle_{\mathcal{AA}}^2 \right)^{1/2} = \frac{\sqrt{\dot{\lambda}^2 + \dot{t}^2}}{2\lambda}. \tag{1.7}$$

Here, $\langle \cdot, \cdot \rangle_{\mathcal{AA}}$ is the sub-Riemannian metric on $\mathcal{H}_{\mathcal{AA}}$. Let $\pi : \mathcal{AA} \to \mathbf{H}^1_{\mathbb{C}}$ denote the canonical projection given by $\pi(a, \lambda, t) = (\lambda, t)$, $(a, \lambda, t) \in \mathcal{AA}$, the horizontal length of γ is then given by

$$\ell(\gamma) = \int_c^d \frac{\sqrt{\dot{\lambda}^2 + \dot{t}^2}}{2\lambda} ds = \ell_h(\gamma_I), \tag{1.8}$$

where $\ell_h(\gamma_I)$ is the hyperbolic length of the projected curve $\gamma_I = \pi \circ \gamma$ in $\mathbf{H}^1_{\mathbb{C}}$.

Conversely, if $\tilde{\gamma}$ is an absolutely continuous curve in $\mathbf{H}^1_{\mathbb{C}}$, $\tilde{\gamma}(s) = (\xi(s), \eta(s))$, $s \in [c,d]$, passing from a point $q_0 = \gamma(s_0)$, then the curve $\gamma : [c,d] \to \mathcal{AA}$ given by $\gamma(s) = (a(s), \lambda(s), t(s))$, where

$$a(s) = \int_{s_0}^{s} \frac{t(u)}{2\lambda(u)}\, du + a_0, \quad \lambda(s) = \xi(s), \quad t(s) = \eta(s),$$

is a horizontal curve passing from a point $p_0 = (a_0, q)$ in the fibre of q.

The corresponding Carnot-Carathéodory distance $d_{\mathcal{AA}}$ associated to the sub-Riemannian metric $\langle \cdot, \cdot \rangle_{\mathcal{AA}}$ is defined for all $p, q \in \mathcal{AA}$ as follows:

$$d_{\mathcal{AA}}(p,q) = \inf_{\gamma \in \Gamma_{p,\,q}} \{\ell(\gamma)\}, \tag{1.9}$$

where $\Gamma_{p,q}$ is the following family of horizontal curves:

$$\Gamma_{p,\,q} = \{\gamma, \gamma : [0,1] \to \mathcal{AA} \text{ horizontal and such that } \gamma(0) = p,\ \gamma(1) = q\}.$$

It is straightforward to prove that the horizontal length (1.7) is invariant under left-translations. The latter fact makes the distance $d_{\mathcal{AA}}$ being invariant under left-translations. As discussed in Section 1.1, we remind that the distance $d_{\mathcal{AA}}$ is finite, geodesic and induces the manifold topology. Our main object of study is the metric measure space $(\mathcal{AA}, d_{\mathcal{AA}}, \mu_{\mathcal{AA}})$.
As a final part of this section we observe that

$$T(\mathcal{AA}) = \mathcal{H}_{\mathcal{AA}} + [\mathcal{H}_{\mathcal{AA}}, \mathcal{H}_{\mathcal{AA}}].$$

Then from Theorem 2 in Mitchell [44] we obtain the following:

Proposition 1.3.2. *The Hausdorff dimension of the sub-Riemannian group $\mathcal{AA}$ is 4.*

1.4 The sub-Riemannian geodesics of $\mathcal{AA}$

In this section we make use of methods of Optimal Control Theory to give an explicit description of the Sub-Riemannian geodesics of $\mathcal{AA}$. We recall that the sub-Riemannian distance on $\mathcal{AA}$ is given by

$$d_{\mathcal{AA}}(p,p') = \inf_{\gamma} \int_0^T \sqrt{\left(\frac{\dot{\lambda}(s)}{2\lambda(s)}\right)^2 + \left(\frac{\dot{t}(s)}{2\lambda(s)}\right)^2}\, ds \tag{1.10}$$

where the infimum is taken over all horizontal curves $\gamma : [0,T] \to \mathcal{AA}$ given by

$$\gamma(s) = (a(s), \lambda(s), t(s)), \text{ with } \gamma(0) = p \text{ and } \gamma(T) = p'.$$

Up to our convenience we shall make use of the curve $\gamma_I : [0,T] \to \mathbf{H}^1_{\mathbb{C}}$, which we recall is defined as $\gamma_I(s) = (\lambda(s), t(s))$, $s \in [0,T]$.
It is important to mention that $\mathcal{H}_{\mathcal{AA}} = \text{span}\{U, V\}$ is a bracket-generating distribution. As a consequence of Theorem 2.1.2 in the book of Montgomery [45], the distance $d_{\mathcal{AA}}$ is geodesic, i.e. the infimum (1.10) is actually a minimum.
We recall that the sub-Riemannian distance $d_{\mathcal{AA}}$ is left-invariant by construction, thus without loss of generality we can assume $\gamma(0) = p = e$ to be the neutral element of $\mathcal{AA}$ and $\gamma(T) = p' \neq e$. Then the length-minimizing property of the geodesic from (1.10) will be equivalent to the following optimal control problem with free time and fixed end-point:

$$\begin{cases} \min_{(u_1,u_2)} \int_0^T \sqrt{u_1^2(s) + u_2^2(s)}\, ds \\ \dot{\lambda} = 2\lambda u_1 \\ \dot{t} = 2\lambda u_2 \\ \dot{a} = u_2 \\ \gamma(0) = e \\ \gamma(T) = p' \end{cases} . \tag{1.11}$$

Notice, that in the above formulation, the curve $\gamma = (a, \lambda, t) : [0,T] \to \mathcal{AA}$ is automatically a horizontal curve, and the control function $u : [0,T] \to \mathbb{R}^2$, defined as $u = (u_1, u_2)$, gives the identity for the horizontal velocity $|\dot{\gamma}|_H = \sqrt{u_1^2 + u_2^2}$. It is straightforward that the two minimization problems (1.10) and (1.11) are equivalent to each other.
In what follows we want to apply Pontryagin's Maximum Principle to describe the geodesics of $\mathcal{AA}$.
However, we will not derive an explicit characterization of the geodesics by a straightforward application of Optimal Control Theory methods. Our current purpose is to change the above setting to an equivalent problem where solving the optimal problem will be easier than solving (1.11). To this end, the following statement will be useful

Proposition 1.4.1. *Without loss of generality we can reparametrize by arc length the horizontal curve $\gamma : [0,T] \to \mathcal{AA}$ such that*

$$\sqrt{u_1^2(s) + u_2^2(s)} = C, \quad s \in [0,T]$$

where $u = (u_1, u_2) = \frac{\dot{\gamma}_I}{2\lambda}$ and C is a positive constant depending only on T and γ.

Proof. Consider $\gamma : [0,T] \to \mathcal{AA}$ to be a horizontal curve joining e with p' such that

$$\ell(\gamma) = \int_0^T \sqrt{u_1^2(s) + u_2^2(s)}\, ds$$

ans satisying $u = \frac{\dot{\gamma}_I}{2\lambda}$. By recalling that $e \neq p'$, we can assume without loss of generality that $v(s) \neq 0$ for all $s \in [0,T]$. Thus we can define an absolutely continuous homeomorphism $\tau : [0,T] \to [0,T]$ by

$$\tau(s) = \frac{T}{\ell(\gamma)} \int_0^s \sqrt{u_1^2(v) + u_2^2(v)}\, dv\,. \tag{1.12}$$

By defining the horizontal curve $\tilde{\gamma} = (\tilde{a}, \tilde{\lambda}, \tilde{t}) : [0,T] \to \mathcal{AA}$ with $\tilde{\gamma}(\sigma) = \gamma \circ \tau^{-1}(\sigma)$, we find the map τ to be the suitable reparametrization. To see this notice that $\tilde{\gamma}(0) = e$, $\tilde{\gamma}(T) = p'$ and $\gamma(s) = \tilde{\gamma} \circ \tau(s)$.
Taking the derivative into the latter relation gives

$$\frac{d\gamma_I}{ds}(s) = \frac{d\tilde{\gamma}_I}{d\sigma}(\tau(s)) \cdot \dot{\tau}(s) = \frac{d\tilde{\gamma}_I}{d\sigma}(\tau(s)) \frac{T}{\ell(\gamma)} \sqrt{u_1^2(s) + u_2^2(s)}, \quad \text{for a.e. } s \in [0,T], \tag{1.13}$$

where we applied first the chain rule and then differentiated the identity (1.12). Now we consider the horizontal velocity given by $|\dot{\tilde{\gamma}}|_H = \frac{\dot{\tilde{\gamma}}_I}{2\tilde{\lambda}}$, then using (1.13) and reordering yields to

$$\sqrt{\tilde{u}_1^2(\sigma) + \tilde{u}_2^2(\sigma)} = \frac{|\frac{d\tilde{\gamma}_I}{d\sigma}(\sigma)|}{2\tilde{\lambda}(\sigma)} = \frac{\ell(\gamma)}{T},$$

for a.e. $\sigma \in [0,T]$. □

Now, let us consider the problem

$$\begin{cases} \min_{(u_1,u_2)} \; \frac{1}{2}\int_0^T u_1^2(s) + u_2^2(s)\, ds \\ \dot{\lambda} = 2\lambda u_1 \\ \dot{t} = 2\lambda u_2 \\ \dot{a} = u_2 \\ \gamma(0) = e \\ \gamma(T) = p' \end{cases} . \tag{1.14}$$

The following proposition states that the two optimal control problems (1.11) and (1.14) are in fact equivalent:

Proposition 1.4.2. *The control problems* (1.11) *and* (1.14) *are equivalent; more precisely, the control u^* is optimal for* (1.11) *if and only if u^* is optimal for* (1.14).

Proof. First, let us suppose that u^* is an optimal control for (1.11), i.e.

$$\int_0^T \sqrt{(u_1^*)^2 + (u_2^*)^2}\, ds \leq \int_0^T \sqrt{u_1^2 + u_2^2}\, ds \tag{1.15}$$

for every admissible control u. By Proposition 1.4.1, we are in the position to assume that $\sqrt{(u_1^*)^2 + (u_2^*)^2}$ is constant. Hence, applying first (1.15) and then the Cauchy–Schwartz inequality with respect to $\sqrt{u_1^2 + u_2^2}$ and $\frac{1}{\sqrt{T}}$ gives

$$\begin{aligned}\int_0^T (u_1^*)^2 + (u_2^*)^2 \, ds &= \left(\int_0^T \sqrt{\frac{(u_1^*)^2 + (u_2^*)^2}{T}} \, ds \right)^2 \le \left(\int_0^T \sqrt{\frac{u_1^2 + u_2^2}{T}} \, ds \right)^2 \\ &\le \left(\int_0^T u_1^2 + u_2^2 \, ds \right) \left(\int_0^T \frac{ds}{T} \right) = \int_0^T u_1^2 + u_2^2 \, ds \, .\end{aligned}$$

This shows that u^* is optimal for the problem (1.14).
Conversely, let us suppose that u^* is optimal for (1.14) and by contradiction let us assume that there exists an admissible control $\tilde{u}$ such that

$$\int_0^T \sqrt{(\tilde{u}_1)^2 + (\tilde{u}_2)^2} \, ds < \int_0^T \sqrt{(u_1^*)^2 + (u_2^*)^2} \, ds \, . \tag{1.16}$$

Again by using Proposition 1.4.1, we may assume $\sqrt{(\tilde{u}_1(s))^2 + (\tilde{u}_2(s))^2}$ to be constant. Applying in order (1.16) and then the Cauchy-Schwartz inequality with respect to $\sqrt{(u_1^*)^2 + (u_2^*)^2}$ and $\frac{1}{\sqrt{T}}$ we obtain

$$\begin{aligned}\int_0^T (\tilde{u}_1)^2 + (\tilde{u}_2)^2 \, ds &= \left(\int_0^T \sqrt{\frac{(\tilde{u}_1)^2 + (\tilde{u}_2)^2}{T}} \, ds \right)^2 < \left(\int_0^T \sqrt{\frac{(u_1^*)^2 + (u_2^*)^2}{T}} \, ds \right)^2 \\ &\le \left(\int_0^T (u_1^*)^2 + (u_2^*)^2 \, ds \right) \left(\int_0^T \frac{1}{T} \, ds \right) = \left(\int_0^T (u_1^*)^2 + (u_2^*)^2 \, ds \right)\end{aligned}$$

this contradicts the optimality of u^* for the problem (1.14), completing the proof. □

In what follows we shall focus our attention to the study of the problem (1.14). Our approach is based on Pontryagin's Maximum Principle. In order to do that, we shall introduce the Hamiltonian:

$$H : \mathcal{AA} \times \mathbb{R}^4 \times \mathbb{R}^2 \to \mathbb{R}$$

$$(X, \Xi, u) \mapsto H(X, \Xi, u) \, ,$$

given by

$$H(X, \Xi, u) = u_2 \xi + 2\lambda u_1 \eta + 2\lambda u_2 \tau - \frac{\omega}{2}(u_1^2 + u_2^2) \, .$$

where $X = (a, \lambda, t)$, $\Xi = (\xi, \eta, \tau, \omega)$ and $u = (u_1, u_2)$.
To the previous we add the notation $\overrightarrow{\xi} = (\xi, \eta, \tau) \in \mathbb{R}^3$ and we denote by an upper index

the dependence of a given vector with respect to the control u, for example X^u, Ξ^u or ω^u. Pontryagin's Maximum Principle applied to our context (see Theorem 12.10 in [1]) gives the following:

Theorem 1.4.3. *Let us consider the control problem (1.14). If v is an optimal control, then there exists a multiplier $\Xi^v \neq 0$ where*

- *$\omega^v = \omega$ is a constant in $\{0, 1\}$,*
- *$(\xi^v, \eta^v, \tau^v) = \overrightarrow{\xi} : [0, T] \to \mathbb{R}^3$ is an absolutely continuous curve,*

such that the following properties hold:

$$v(s) \in \arg\min_{u \in \mathbb{R}^2} H(X^v(s), \Xi^v(s), u), \quad s \in [0, T], \tag{1.17}$$

$$\dot{\overrightarrow{\xi^v}}(s) = -\nabla_X H(X^v(s), \Xi^v(s), v(s)), \quad s \in [0, T], \tag{1.18}$$

$$H(X^v(s), \Xi^v(s), v(s)) = h, \quad s \in [0, T], \tag{1.19}$$

where h is a constant.

Let us start our investigation. For our optimal control we have the following normality property:

Proposition 1.4.4. *Let v be an optimal control for the problem (1.14). Then v is a normal control, i.e. $\omega^v = \omega = 1$.*

Proof. Let us assume by contradiction that $\omega = 0$. Since v is an optimal control, the Maximum Principle (1.17) guarantees that min exists, for every $s \in [0, T]$.
On the other hand we have

$$H(X, \Xi, u) = 2\lambda\eta \cdot u_1 + (\xi + 2\lambda\tau) \cdot u_2,$$

and thus the function

$$u \mapsto H(X^u(s), \Xi^u(s), u)$$

is affine. This implies that the above min exists only if the system

$$\begin{cases} \xi + 2\lambda\tau = 0 \\ \quad 2\lambda\eta = 0 \end{cases} \tag{1.20}$$

holds on $[0, T]$. Since $\lambda > 0$, the second equation of (1.20) gives

$$\eta(s) = 0 \text{ for all } s \in [0, T]. \tag{1.21}$$

With this information equation (1.18) reads as

$$\begin{cases} \dot{\xi} = 0 \\ \dot{\eta} = -2v_2\tau \ . \\ \dot{\tau} = 0 \end{cases} \tag{1.22}$$

From the first and third equations of the latter system we have $\xi(s) = \xi_0$ for all $s \in [0, T]$ and $\tau(s) = \tau_0$ for all $s \in [0, T]$. We examine first the case $\tau_0 = 0$. Thus the first equation of (1.20) gives $\xi(s) = 0$ for all $s \in [0, T]$. This contradicts the assumption $\Xi^v \neq 0$ in Theorem 1.4.3.

In the second case where $\tau_0 \neq 0$ we deduce from the first equation of (1.20) that $\lambda(s) = -\frac{\xi_0}{2\tau_0}$ for all $s \in [0, T]$. Since $\lambda(0) = 1$ the last fact implies that $\lambda(s) = 1$ for all $s \in [0, T]$. Further, from knowing (1.21) and combining it with the second equation of (1.22) we obtain $v_2 = 0$. The horizontality condition of γ written in (1.14) makes us conclude that $\gamma(s) = e$ for all $s \in [0, T]$, which is a contradiction with $\gamma(T) = p' \neq e$. □

Recalling Theorem 1.4.3 we consider the optimal control $v : [0, T] \to \mathbb{R}^2$ for the problem (1.14), the associated curve $\gamma : [0, T] \to \mathcal{AA}$ and the associated multiplier $\overrightarrow{\xi} : [0, T] \to \mathbb{R}^3$. We denote the initial data:

$$\gamma(0) = e, \quad \overrightarrow{\xi}(0) = (\xi_0, \eta_0, \tau_0). \tag{1.23}$$

We are now in the position to state and prove the following

Theorem 1.4.5. *Let (1.23) be the initial data. The sub-Riemannian geodesics $\gamma : [0, T] \to \mathcal{AA}$ are given by the following case distinction:*

(i) if $\eta_0 = \tau_0 = 0$, then $\gamma(s) = (\xi_0 s, 1, 2\xi_0 s)$;

(ii) if $\eta_0 \neq 0$ and $\tau_0 = 0$, then $\gamma(s) = \left(\xi_0 s, e^{4\eta_0 s}, \frac{\xi_0}{2\eta_0}\left(e^{4\eta_0 s} - 1\right)\right)$;

(iii) if $\tau_0 \neq 0$, then $\gamma(s) = (a(s), \lambda(s), t(s))$ is a horizontal curve which satisfies

$$\left(\lambda(s) + \frac{\xi_0}{2\tau_0}\right)^2 + \left(t(s) - \frac{\eta_0}{\tau_0}\right)^2 = r^2,$$

where $r^2 = 1 + \left(\frac{\eta_0}{\tau_0}\right)^2 + \frac{\xi_0}{\tau_0}\left(1 + \frac{\xi_0}{4\tau_0}\right)$.

Proof. We observe that Proposition 1.4.2 grants the study of sub-Riemannian geodesics to be equivalent to describe the solutions of (1.14). Theorem 1.4.3 applied to the control problem

(1.14) and Proposition 1.4.4 give what follows. The Hamiltonian is

$$H(\underbrace{a, \lambda, t}_{X}, \underbrace{\xi, \eta, \tau}_{\vec{\xi}}, \underbrace{u_1, u_2}_{u}) = u_2\xi + 2\lambda u_1\eta + 2\lambda u_2\tau - \frac{1}{2}(u_1^2 + u_2^2),$$

the optimality of the control v for H (1.17) holds and also the costate equation (1.18) holds. The control $v = (v_1, v_2) : [0, T] \to \mathbb{R}^2$ is a minimum for H only if $\nabla_u H = 0$, from the latter expression we obtain that the optimal control v is given by:

$$\begin{cases} v_1 = 2\lambda\eta \\ v_2 = \xi + 2\lambda\tau \end{cases}. \tag{1.24}$$

We rewrite the horizontality condition for γ as

$$\begin{cases} \dot{a} = v_2 \\ \dot{\lambda} = 2\lambda v_1 \\ \dot{t} = 2\lambda v_2 \end{cases}. \tag{1.25}$$

The costate equation reads as:

$$\begin{cases} \dot{\xi} = 0 \\ \dot{\eta} = -2(v_1\eta + v_2\tau) \\ \dot{\tau} = 0 \end{cases}. \tag{1.26}$$

From (1.26) we can write

$$\xi(s) = \xi_0, \quad s \in [0, T],$$

and

$$\tau(s) = \tau_0, \quad s \in [0, T].$$

Next, we replace v_1 and v_2 in the the second equation of (1.26) with the horizontality condition (1.25), this yields to the o.d.e.

$$\dot{\eta} = -\frac{1}{\lambda}\left(\dot{\lambda}\eta + \dot{t}\tau_0\right). \tag{1.27}$$

We solve (1.27) by the *variation of constants* method and we get

$$\eta(s) = \frac{1}{\lambda(s)}(-\tau_0 t(s) + \eta_0), \quad s \in [0, T]. \tag{1.28}$$

Now we insert (1.28) in (1.24) and obtain:

$$\begin{cases} v_1 = 2(-\tau_0 t + \eta_0) \\ v_2 = \xi_0 + 2\tau_0\lambda \end{cases}. \tag{1.29}$$

Consequently we rewrite (1.25) as

$$\begin{cases} \dot{a}(s) = \frac{\dot{t}(s)}{2\lambda(s)} \\ \dot{\lambda}(s) = 4\lambda(s)(-\tau_0 t(s) + \eta_0) \\ \dot{t}(s) = 2\lambda(s)(2\tau_0\lambda(s) + \xi_0) \end{cases} . \tag{1.30}$$

We consider two cases:

- if $\tau_0 = 0$, then (1.30) has an explicit solution, given by $\gamma(s) = \left(\xi_0 s, e^{4\eta_0 s}, \frac{\xi_0}{2\eta_0}\left(e^{4\eta_0 s} - 1\right)\right)$. In the degenerate case, when also $\eta_0 = 0$, we get $\gamma(s) = (\xi_0 s, 1, 2\xi_0 s)$.

- if $\tau_0 \neq 0$, from the system (1.30) we get the equation

$$2\dot{t}(-\tau_0 t + \eta_0) = \dot{\lambda}(\xi_0 + 2\lambda\tau_0).$$

  Integrating the latter equation on $[0, s]$ gives the relation

$$-\tau_0 t^2 + 2\eta_0 t = \xi_0(\lambda - 1) + \tau_0(\lambda^2 - 1)$$

  which can be rewritten as

$$\left(\lambda + \frac{\xi_0}{2\tau_0}\right)^2 + \left(t - \frac{\eta_0}{\tau_0}\right)^2 = 1 + \left(\frac{\eta_0}{\tau_0}\right)^2 + \frac{\xi_0}{\tau_0}\left(1 + \frac{\xi_0}{4\tau_0}\right).$$

  We want to make sure that $r^2 := 1 + \left(\frac{\eta_0}{\tau_0}\right)^2 + \frac{\xi_0}{\tau_0}\left(1 + \frac{\xi_0}{4\tau_0}\right) \geq 0$ for all $(\xi_0, \eta_0) \in \mathbb{R}^2$ and for all $\tau_0 \neq 0$. To see this let $g : \mathbb{R} \to \mathbb{R}$ be the quadratic function defined as

$$g(s) = s\left(1 + \frac{s}{4}\right), \; s \in \mathbb{R}.$$

  It comes straightforward that

$$g(s) \geq -1, \; s \in \mathbb{R},$$

  granting the inequality

$$\frac{\xi_0}{\tau_0}\left(1 + \frac{\xi_0}{4\tau_0}\right) \geq -1.$$

  The proof is complete.

□

Chapter 2

Contact transformations

A contactomorphism is an important type of diffeomorphism between the spaces presented in the previous chapter. It is also referred as contact transformation and it preserves the horizontal sub-bundles of the respective spaces. In detail we provide the following

Definition 2.0.1. Let $\mathbb{G}_1, \mathbb{G}_2 \in \{\mathcal{AA}, \mathbb{H}, \mathcal{RT}\}$, and let ϑ_1, ϑ_2 denote the respective contact forms of $\mathbb{G}_1$ and $\mathbb{G}_2$. A diffeomorphism $f : \mathbb{G}_1 \to \mathbb{G}_2$ is a *contactomorphism* if and only if $f^*\vartheta_2 = \sigma\vartheta_1$, for some $\sigma : \mathbb{G}_1 \to \mathbb{R}$ nowhere vanishing smooth function.

2.1 The contact equivalence between $\mathcal{AA}$, $\mathbb{H}$ and $\mathcal{RT}$

It is well known, that, by Darboux theorem, each three dimensional contact manifold is locally contactomorphic to the Heisenberg group. The purpose of this section is to give a global version of this fact. In detail we will show that the spaces $\mathcal{AA}$, $\mathbb{H}$ and $\mathcal{RT}$ are globally contactomorphic.

We start by quoting what is already known in the literature (Lemma 5.5 in [26]) with the following:

Proposition 2.1.1. *The manifolds* $(\mathcal{RT}, \vartheta_{\mathcal{RT}})$ *and* $(\mathbb{H}, \vartheta_{\mathbb{H}})$ *are globally contactomorphic.*

We are now in the position to present our result.

Proposition 2.1.2. *The manifolds* $(\mathcal{AA}, \vartheta)$ *and* $(\mathbb{H}, \vartheta_{\mathbb{H}})$ *are globally contactomorphic. Moreover, the metric spaces* $(\mathbb{H}, d_{\mathbb{H}})$ *and* $(\mathcal{AA}, d_{\mathcal{AA}})$ *are locally bi-Lipschitz equivalent.*

Proof. We define the smooth contactomorphism $g : (\mathbb{H}, \vartheta_{\mathbb{H}}) \to (\mathcal{AA}, \vartheta)$ explicitely by the formula

$$g(x, y, t) = \left(xe^{-y},\ e^{y},\ \frac{1}{2}(t - 2xy + 4x)\right) \text{ for } (x, y, t) \in \mathbb{H}. \tag{2.1}$$

Clearly, g is a smooth diffeomorphism between $\mathbb{H}$ and $\mathcal{AA}$. Its inverse map $g^{-1} : \mathcal{AA} \to \mathbb{H}$ is given by

$$g^{-1}(a, \lambda, t) = (a\lambda,\ \ln \lambda,\ 2t + 2a\lambda(\ln \lambda - 2)) \ \text{for} \ \ (a, \lambda, t) \in \mathcal{AA}.$$

To check the contact property of g we compute directly:

$$g^*\vartheta \ = \ \frac{(1/2)dt - xdy - ydx + 2dx}{2e^y} - e^{-y}dx + xe^{-y}dy = \frac{dt + 2xdy - 2ydx}{4e^y} = \frac{1}{4e^y}\vartheta_{\mathbb{H}}.$$

Now, since g is contact it preserves the respective horizontal bundles, i.e. $g_*\mathcal{H}_{\mathbb{H}} = \mathcal{H}_{\mathcal{AA}}$ and this implies that $g^*\langle \cdot, \cdot \rangle_{\mathcal{AA}}$ on $\mathcal{H}_{\mathcal{AA}}$ is a smooth multiple of $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ on $\mathcal{H}_{\mathbb{H}}$. This particularly means that g is locally Lipschitz with respect to the sub-Riemannian distances of $d_{\mathbb{H}}$ and $d_{\mathcal{AA}}$. For the same reason we also obtain that $g^{-1} : \mathcal{AA} \to \mathbb{H}$ is locally Lipschitz. Thus we have that $f : \mathbb{H} \to \mathcal{AA}$ is locally bi-Lipschitz. □

Remark 2.1.3. There are established criteria based on differential topology methods to verify contact equivalence between three dimensional manifolds (see Eliashberg [24], [25]). However, in the last proof, we preferred to provide a direct way by presenting the contactomorphism (2.1).

Combining Proposition 2.1.1 and Proposition 2.1.2 we deduce

Proposition 2.1.4. *The manifolds* $(\mathcal{RT}, \vartheta_{\mathcal{RT}})$, $(\mathbb{H}, \vartheta_{\mathbb{H}})$ *and* $(\mathcal{AA}, \vartheta)$ *are all globally contactomorphic to each other.*

2.2 Lift of symplectic maps from $\mathbf{H}^1_{\mathbb{C}}$ to $\mathcal{AA}$

The current objective of this section is to understand the interplay between the symplectic maps of $\mathbf{H}^1_{\mathbb{C}}$ and the contactomorphisms of $\mathcal{AA}$. Let $f : \mathbf{H}^1_{\mathbb{C}} \to \mathbf{H}^1_{\mathbb{C}}$ be a mapping and let us recall the canonical projection $\pi : \mathcal{AA} \to \mathbf{H}^1_{\mathbb{C}}$ given by $\pi(a, \lambda, t) = (\lambda, t)$ for $(a, \lambda, t) \in \mathcal{AA}$, we say that a mapping $F : \mathcal{AA} \to \mathcal{AA}$ is a *lift of* f when we have the diagram property.

$$f \circ \pi = \pi \circ F, \quad \text{on } \mathcal{AA}.$$

An advantage given by obtaining a result in this direction is that by lifting symplectic self-maps of $\mathbf{H}^1_{\mathbb{C}}$ we can present first examples of contactomorphisms $\mathcal{AA} \to \mathcal{AA}$.

Definition 2.2.1. Let ω denote the symplectic form of $\mathbf{H}^1_{\mathbb{C}}$ given by $\omega = \frac{d\xi \wedge d\eta}{4\xi^2}$.
Let $f = (f_1, f_2) : \mathbf{H}^1_{\mathbb{C}} \to \mathbf{H}^1_{\mathbb{C}}$ be a C^1 diffeomorphism, we say that f is *symplectic* if

$$f^*\omega = \omega.$$

Let J_f denote the Jacobian determinant of a given differentiable mapping f. A straightforward characterization of a symplectic map is given by the following

Proposition 2.2.2. *A C^1 diffeomorphism $f = (f_1, f_2) : \mathbf{H}^1_{\mathbb{C}} \to \mathbf{H}^1_{\mathbb{C}}$ is a symplectic map if and only if*

$$J_f(\xi, \eta) = \left(\frac{f_1(\xi, \eta)}{\xi}\right)^2$$

holds for all $(\xi, \eta) \in \mathbf{H}^1_{\mathbb{C}}$.

By making use of the left-invariant vector fields U,V and W (see Proposition 1.3.1), it is straightforward to derive following equivalent characterization for a contactomorphism $F : \mathcal{AA} \to \mathcal{AA}$

Proposition 2.2.3. *A C^1 diffeomorphism $F = (F_1, F_2, F_3) : \mathcal{AA} \to \mathcal{AA}$ is a contactomorphism if and only if the system of p.d.e.s*

$$\begin{cases} UF_3 = 2F_2\, UF_1, \\ VF_3 = 2F_2\, VF_1, \\ WF_3 = 2F_2(\sigma + WF_1), \end{cases} \tag{2.2}$$

holds for some nowhere vanishing smooth function $\sigma : \mathcal{AA} \to \mathbb{R}$.

We provide a lifting result in the following

Theorem 2.2.4. *Let $f : \mathbf{H}^1_{\mathbb{C}} \to \mathbf{H}^1_{\mathbb{C}}$ be a C^1 diffeomorphism. The following statements hold:*

(i) If $F : (\mathcal{AA}, \vartheta) \to (\mathcal{AA}, \vartheta)$ is a C^1 contactomorphism satysfing $F^*\vartheta = \vartheta$ and also F is a lift of f, then f is a symplectic map;

(ii) If f is a C^2 symplectic map, then there exists $F : (\mathcal{AA}, \vartheta) \to (\mathcal{AA}, \vartheta)$ contactomorphism and also F is a lift of f.

Proof. Proof of (i).
Let $p = (a, \lambda, t) \in \mathcal{AA}$ and $q = \pi(p) = (\lambda, t) \in \mathbf{H}^1_{\mathbb{C}}$. From the condition $\pi \circ F = f \circ \pi$ we have that

$$(F_2(p), F_3(p)) = (f_1(q), f_2(q)).$$

By applying U, V, W on F_3 we obtain the relations:

$$\begin{aligned} U\,F_3(p) &= 2\lambda \partial_\eta f_2(q) \\ V\,F_3(p) &= 2\lambda \partial_\xi f_2(q) \\ W\,F_3(p) &= 0. \end{aligned} \tag{2.3}$$

By combining the assumption $F^*\vartheta = \vartheta$ with Proposition (2.2.3) we write

$$\begin{aligned} UF_3(p) &= 2F_2(p)\, UF_1(p),\\ VF_3(p) &= 2F_2(P)\, VF_1(p),\\ WF_3(p) &= 2F_2(p)(1+WF_1(p)). \end{aligned} \tag{2.4}$$

We apply the relations (2.3) into (2.4) and we obtain

$$\begin{aligned} U\,F_1(p) &= \frac{\lambda\partial_\eta f_2(q)}{f_1(q)},\\ V\,F_1(p) &= \frac{\lambda\partial_\xi f_2(q)}{f_1(q)},\\ W\,F_1(p) &= -1. \end{aligned} \tag{2.5}$$

At this point, we apply V to the first equation of (2.5), respectively U to the second one in (2.5) and we subtract them obtaining:

$$[U,V]F_1(p) = U\left(\frac{\lambda\partial_\xi f_2(q)}{f_1(q)}\right) - V\left(\frac{\lambda\partial_\eta f_2(q)}{f_1(q)}\right). \tag{2.6}$$

Recalling the commutator relation $[U,V] = -2(U+W)$ (see Proposition 1.3.1) and combining it with (2.6) we have

$$U\left(\frac{\lambda\partial_\xi f_2(q)}{f_1(q)}\right) - V\left(\frac{\lambda\partial_\eta f_2(q)}{f_1(q)}\right) = 2\left(1 - \frac{\lambda\partial_\eta f_2(q)}{f_1(q)}\right). \tag{2.7}$$

It is a straightforward calculation to verify that (2.7) implies

$$\partial_\xi f_1(q)\,\partial_\eta f_2(q) - \partial_\eta f_1(q)\,\partial_\xi f_2(q) = \left(\frac{f_1(q)}{\lambda}\right)^2, \quad q = (\lambda,t) \in \mathbf{H}^1_{\mathbb{C}}. \tag{2.8}$$

Thanks to Proposition 2.2.2 we see that (2.8) implies $f : \mathbf{H}^1_{\mathbb{C}} \to \mathbf{H}^1_{\mathbb{C}}$ to be a symplectic map.

Proof of (ii).
As before, consider $p = (a,\lambda,t) \in \mathcal{AA}$ and $q = \pi(p) = (\lambda,t) \in \mathbf{H}^1_{\mathbb{C}}$. By defining

$$F_2(p) = f_1(q) \text{ and } F_3(p) = f_2(q), \tag{2.9}$$

we grant that $\pi \circ F = f \circ \pi$.
It remains to define F_1 in order to have F contact. We define F_1 to be the solution of the

following system of p.d.e.s:

$$\begin{aligned} UF_3 &= 2F_2\, UF_1, \\ VF_3 &= 2F_2\, VF_1, \\ WF_3 &= 2F_2(1 + WF_1). \end{aligned} \tag{2.10}$$

It is straightforward to check that with such definition of F_1 we obtain $F^*\vartheta = \vartheta$. Therefore it is left to show that there exists such F_1 as a solution to the system of p.d.e.s (2.10). By using (2.9) in (2.10) we obtain

$$\begin{aligned} Uf_2(q) &= 2f_1(q)\, UF_1(p), \\ Vf_2(q) &= 2f_1(q)\, VF_1(p), \\ Wf_2(q) &= 2f_1(q)(1 + WF_1(p)), \end{aligned}$$

which we simplify as

$$\begin{aligned} \partial_a F_1(p) + 2\lambda \partial_t F_1(p) &= \frac{\lambda \partial_\eta f_2(q)}{f_1(q)}, \\ \partial_\lambda F_1(p) &= \frac{\partial_\xi f_2(q)}{2f_1(q)}, \\ \partial_a F_1(p) &= 1. \end{aligned} \tag{2.11}$$

We rewrite the system (2.11) as

$$\begin{aligned} \partial_a F_1(p) &= 1, \\ \partial_\lambda F_1(p) &= \frac{\partial_\xi f_2(q)}{2f_1(q)}, \\ \partial_t F_1(p) &= \frac{\partial_\eta f_2(q)}{2f_1(q)} - \frac{1}{2\lambda}, \end{aligned} \tag{2.12}$$

and we then define the functions $P, Q, R : \mathcal{AA} \to \mathbb{R}$ as

$$\begin{aligned} P(p) &= 1, \\ Q(p) &= \frac{\partial_\xi f_2(q)}{2f_1(q)}, \\ R(p) &= \frac{\partial_\eta f_2(q)}{2f_1(q)} - \frac{1}{2\lambda}. \end{aligned} \tag{2.13}$$

Since $\mathcal{AA}$ is simply connected, if $\operatorname{curl}(P, Q, R) = (0, 0, 0)$ we conclude that F_1 solving (2.10) exists.

It is true that $\operatorname{curl}(P, Q, R) = (0, 0, 0)$. Indeed, Proposition 2.2.2 applied to f gives:

$$\begin{aligned}\partial_\lambda R(p) - \partial_t Q(p) =& \frac{f_1(q)\partial^2_{\xi\eta} f_2(q) - \partial_\eta f_2(q)\partial_\xi f_1(q)}{2f_1(q)^2} + \frac{1}{2\lambda^2} \\ &- \frac{f_1(q)\partial^2_{\xi\eta} f_2(q) - \partial_\xi f_2(q)\partial_\eta f_1(q)}{2f_1(q)^2} \\ =& -\frac{J_f(q)}{2f_1(q)^2} + \frac{1}{2\lambda^2} = 0.\end{aligned}$$

We observe that P is constant, both Q and R do not depend on a; thus it is easy to verify that

$$\partial_t P - \partial_a R = \partial_a Q - \partial_\lambda P = 0.$$

At this point we can provide a representation formula for $F = (F_1, F_2, F_3)$.
Since left translations are contact transformations, there is no loss of generality in assuming $F_1(0, 1, 0) = 0$. Thanks to the system of p.d.e.s (2.12) we obtain that

$$F_1(a, \lambda, t) = a + G_1(\lambda) + G_2(\lambda, t),$$

$$\text{where } G_1(\lambda) = \int_1^\lambda Q(a, u, 0)\, du \text{ and } G_2(\lambda, t) = \int_0^t R(a, \lambda, v)\, dv.$$

Further we recall that $(F_2(p), F_3(p)) = (f_1 \circ \pi(p), f_2 \circ \pi(p))$. Therefore a representation formula for F is given by

$$F(p) = \begin{bmatrix} a + G_1(\lambda) + G_2(\lambda, t) \\ f_1(\lambda, t) \\ f_2(\lambda, t) \end{bmatrix} \text{ for all } p \in \mathcal{AA}. \tag{2.14}$$

We proceed by proving that $F : \mathcal{AA} \to \mathcal{AA}$ is a bijection. We pick some point $\hat{p} = (\hat{a}, \hat{\lambda}, \hat{t}) \in \mathcal{AA}$ and we will show that there is a unique $p = (a, \lambda, t) \in \mathcal{AA}$ so that $F(p) = \hat{p}$. Since $f : \mathbf{H}^1_{\mathbb{C}} \to \mathbf{H}^1_{\mathbb{C}}$ is an homeomorphism we know that there exists a unique $(\lambda, t) \in \mathbf{H}^1_{\mathbb{C}}$ so that $f(\lambda, t) = (\hat{\lambda}, \hat{t})$, then a is the unique solution to the equation

$$a = \hat{a} - G_1(\lambda) - G_2(\lambda, t).$$

Now, using the representation formula (2.14) we see that $F : \mathcal{AA} \to \mathcal{AA}$ is a C^2 mapping and also that $J_F(p) = J_f(\pi(p))$ for all $p \in \mathcal{AA}$. This last considerations grant us that F is a C^2 diffeomorphism. The proof is now complete. □

Now we present the following

Example 2.2.5. Let $c_1 > 0$, $c_2 \in \mathbb{R}$ and let us consider the symplectic map $f : \mathbf{H}^1_{\mathbb{C}} \to \mathbf{H}^1_{\mathbb{C}}$ given by

$$(f_1(\xi,\eta), f_2(\xi,\eta)) = (c_1\xi,\, c_2\xi + c_1\eta)\,.$$

Thanks to Theorem 2.2.4, in particular the representation formula (2.14), we obtain that $F : \mathcal{AA} \to \mathcal{AA}$ given by

$$F(a,\lambda,t) = \left(a + \frac{c_2}{2c_1}\log\lambda,\, c_1\lambda,\, c_2\lambda + c_1 t\right)$$

is a contact lift of f.
It is important to mention that when $c_2 = 0$ such map f is a particular Möbius transformation of $\mathbf{H}^1_{\mathbb{C}}$, i.e. an orientation preserving isometry of $\mathbf{H}^1_{\mathbb{C}}$ (for details see Section 4.2 in Platis [51]). In general, it holds that

$$F_*U_p = U_{F(p)}, \quad F_*V_p = V_{F(p)} + \frac{c_2}{c_1}U_{F(p)},$$

for $p = (a,\lambda,t) \in \mathcal{AA}$. Therefore, in the specific case $c_2 = 0$ we obtain that the lift F is an isometry of $(\mathcal{AA}, d_{\mathcal{AA}})$.

2.3 The Korányi-Reimann flow method on $\mathcal{AA}$

The flow method is a powerful tool which gives conditions to some vector field in order to generate a one-parameter subgroup of contact transformations. It was first introduced by Libermann in [42], then put to use by Korányi and Reimann in [38], [39] to construct a family contact transformations in Heisenberg groups. We shall explain this method in detail.
Let $X : \mathcal{AA} \to \mathbb{R}^3$ be a smooth vector field, let $p \in \mathcal{AA}$ and let $I \subseteq \mathbb{R}$ be a interval containing 0. Let us denote by $\Phi_X : I \times \mathcal{AA} \to \mathcal{AA}$ the *flow of* X solving the differential equation:

$$\begin{cases} \frac{d}{ds}\Phi_X(s,p) = X(\Phi_X(s,p)), \quad s \in I \\ \Phi_X(0,p) = p \end{cases}. \tag{2.15}$$

Let us consider the family $(G_s)_{s\in I}$ of C^1 smooth mappings $G_s : \mathcal{AA} \to \mathcal{AA}$. We say that the family $(G_s)_{s\in I}$ is *generated by* X if the mappings are given by

$$G_s(\cdot) = \Phi_X(s,\cdot),\; s \in I. \tag{2.16}$$

Theorem 2.3.1. *Let $(G_s)_{s\in I}$ be the family of maps defined in (2.16). Then $G_s : \mathcal{AA} \to \mathcal{AA}$ are contact transformations for all $s \in I$ if and only if the vector field has the form*

$$X = hW + \frac{1}{2}\left((Uh)V - (Vh)U\right) \tag{2.17}$$

for some $h : \mathcal{AA} \to \mathbb{R}$ smooth function.

Proof. In one direction, let X be a smooth vector field, written in the frame $\{U, V, W\}$ as $X = fU + gV + hW$, where $f, g, h : \mathcal{AA} \to \mathbb{R}$ are smooth functions. Let us assume that the family of maps $(G_s)_{s\in\mathbb{R}}$ generated by X is a contact flow, i.e.

$$\frac{d}{ds}G_s(p) = X(G_s(p)) \text{ and } G_s^*\vartheta = \sigma_s\vartheta, \tag{2.18}$$

for some function $s \mapsto \sigma_s(\cdot)$ where $\sigma_s : \mathcal{AA} \to \mathbb{R}$ is a nowhere vanishing smooth function for all $s \in I$. Differentiating the second relation in (2.18) with respect to s yields to

$$L_X\vartheta = \frac{d}{ds}(\sigma_s) \cdot \vartheta, \tag{2.19}$$

where $\frac{d}{ds}(G_s^*\vartheta) = L_X\vartheta$ corresponds to the Lie derivative of ϑ along the direction of the vector field X (see Chapter 3 in Aubin's book [5]). According to the "Cartan's magic formula" (cf. Proposition 3.6 b) in [5]) we have that

$$L_X(\vartheta) = (d\vartheta)X + d(\vartheta(X)).$$

Since $\vartheta(X) = h$, we obtain $L_X(\vartheta) = (d\vartheta)X + dh$. At this point let us insert the latter expression into (2.19) and apply the obtained one forms on both sides to the vector field W. Since W is the Reeb vector field we have the relations $d\vartheta(X, W) = 0$ and $\vartheta(W) = 1$, and we obtain that $dh(W) = \frac{d}{ds}(\sigma_s)$. This implies that (2.19) will take the form:

$$(d\vartheta)X + dh = dh(W)\vartheta.$$

Using that $d\vartheta = \frac{dt\wedge d\lambda}{2\lambda^2}$ and inserting in the above relation the vector field U and V we obtain that $f = -\frac{1}{2}V(h)$ and $g = \frac{1}{2}U(h)$.
In the other direction, let us assume that a smooth vector field is given by the formula

$$X = hW + \frac{1}{2}((Uh)V - (Vh)U)$$

where $h : \mathcal{AA} \to \mathbb{R}$ is a smooth function. We are going to show that the flow $(G_s)_{s\in I}$ generated by X is a contact flow. To check this let us denote by $\vartheta_s = G_s^*\vartheta$ the pullback of ϑ via G_s. Then we can write ϑ_s in the co-frame $\{\vartheta, da, \frac{d\lambda}{2\lambda}\}$ as

$$\vartheta_s = A_s\vartheta + B_s da + C_s\frac{d\lambda}{2\lambda}. \tag{2.20}$$

We shall prove that the functions B_s and C_s are identically 0. Differentiating both sides of (2.20) with respect to s we obtain

$$\frac{d\vartheta_s}{ds} = \frac{dA_s}{ds}\vartheta + \frac{dB_s}{ds}da + \frac{dC_s}{ds}\frac{d\lambda}{2\lambda} \tag{2.21}$$

and, recalling that $\frac{d}{ds}(G_s^*\vartheta) = L_X(\vartheta)$, the latter relation becomes

$$L_X(\vartheta) = \frac{dA_s}{ds}\vartheta + \frac{dB_s}{ds}da + \frac{dC_s}{ds}\frac{d\lambda}{2\lambda}\,. \tag{2.22}$$

Using the Cartan's magic formula on the l.h.s. of (2.22) we rewrite

$$(d\vartheta)X + dh = \frac{dA_s}{ds}\vartheta + \frac{dB_s}{ds}da + \frac{dC_s}{ds}\frac{d\lambda}{2\lambda}\,. \tag{2.23}$$

We insert the vector fields U and V in the one forms at l.h.s. and r.h.s. of (2.23) and this gives respectively

$$d\vartheta(X,U) + U(h) = \frac{dB_s}{ds}, \quad d\vartheta(X,V) + V(h) = \frac{dC_s}{ds}. \tag{2.24}$$

Because of the particular form of

$$X = hW + \frac{1}{2}\left((Uh)V - (Vh)U\right),$$

we obtain that both the l.h.s.s of (2.24) vanish, giving $\frac{dB_s}{ds} = \frac{dC_s}{ds} = 0$. On the other hand $G_0 = Id$ by definition and thus $G_0^*\vartheta = \vartheta$. The latter identity gives that $B_0 = C_0 = 0$, which concludes that $B_s = C_s = 0$ for all $s \in I$ as required. □

By choosing potentials $h : \mathcal{AA} \to \mathbb{R}$, we can apply Theorem 2.3.1 to generate one-parameter subgroups of contact transformations $\mathcal{AA} \to \mathcal{AA}$. We present the following list of examples.

Example 2.3.2. Let $I = \mathbb{R}$ and assume $h(a,\lambda,t) = c_0$ to be a constant function, then $X = c_0 W$ and we have that $G_s(p) = L_{(-c_0 s,1,0)}(p)$, $s \in \mathbb{R}$.

Example 2.3.3. Let $I = \mathbb{R}$ and assume $h(a,\lambda,t) = c_1 a$, then $X = \frac{c_1}{2}V + c_1 aW$ and we have that $G_s(p) = (e^{-c_1 s}a, e^{c_1 s}\lambda, t)$, $s \in \mathbb{R}$.

Example 2.3.4. Let $I = \mathbb{R}$ and assume $h(a,\lambda,t) = c_2\lambda$, then $X = -c_2\lambda U + c_2\lambda W$ and we have that $G_s(p) = (a - 2c_2 s\lambda, \lambda, t - 2c_2 s\lambda^2)$, $s \in \mathbb{R}$.

Example 2.3.5. Let $I = \mathbb{R}_{\geq 0}$ and assume $h(a,\lambda,t) = c_3 t$ with $c_3 \leq 0$, then $X = c_3\lambda V + c_3 tW$ and we have that

$$G_s(p) = \left(a - c_3 st, \frac{\lambda}{1 - 2c_3 s\lambda}, t\right), \; s \in \mathbb{R}_{\geq 0}.$$

Chapter 3

Quasiconformal mappings on metric measure spaces

3.1 Preliminaries

We start this chapter by recalling some concepts and results on the theory of quasiconformal (QC) maps in the setting of general metric measure spaces. For more details we refer to [34], [33] and [36].

Let us recall that a homeomorphism $f : X \to Y$ between two metric spaces (X, d_X) and (Y, d_Y) is called *quasiconformal* if there exists $K \geq 1$ such that

$$\limsup_{r \to 0} \frac{\sup_{d_X(p,q) \leq r} d_Y(f(p), f(q))}{\inf_{d_X(p,q) \geq r} d_Y(f(p), f(q))} := H_f(p) \leq K, \tag{3.1}$$

for all p in X.

A metric measure space is a triple (X, d_X, μ_X) comprising a non empty set X, a distance function d_X and a regular Borel measure μ_X such that (X, d_X) is a complete, and separable metric space and every metric ball has positive and finite measure. This setting will be our standing assumption throughout this thesis.

Given a point $p \in X$ and a radius $r > 0$, we employ the following notation for balls:

$$B_X(p, r) = \{q \in X : d_X(p, q) < r\} \text{ and } \overline{B}_X(p, r) = \{q \in X : d_X(p, q) \leq r\}.$$

Where it will not cause confusion, we will replace $B_X(p, r)$ by $B(p, r)$.

A metric measure space (X, d_X, μ_X) is called *Ahlfors Q-regular*, $Q > 1$, if there exists a constant $C \geq 1$ such that for all $p \in X$ and $0 < r \leq \operatorname{diam} X$, we have

$$C^{-1} r^Q \leq \mu_X(\overline{B}_X(p, r)) \leq C r^Q. \tag{3.2}$$

Further, we say that (X, d_X, μ_X) is *locally Ahlfors Q-regular*, if for every compact subset $V \subset X$, there is a constant $C \geq 1$ and a radius $r_0 > 0$ such that for each point $p \in V$ and each radius $0 < r \leq r_0$ we have

$$C^{-1} r^Q \leq \mu_X(\overline{B}_X(p, r)) \leq C r^Q. \tag{3.3}$$

We briefly discuss two examples of Ahlfors regular metric measure spaces. It straightforward to see that the Euclidean space $(\mathbb{R}^n, d_E, \mathcal{L}^n)$ is Ahlfors n-regular. It is known from the literature (see Theorem 9.27 in Heinonen's book [33]) that the sub-Riemannian Heisenberg group $\mathbb{H}$ is Ahlfors 4-regular.

An important geometric notion in the theory of quasiconformal mappings is the Q-modulus of a curve family. Let us consider Γ to be a family of curves in the metric measure space (X, d_X, μ_X), we say that a Borel function $\rho : X \to [0, \infty]$ is *admissible* for Γ if for every rectifiable $\gamma \in \Gamma$, we have

$$1 \leq \int_\gamma \rho \, d\ell_X \, .$$

Such a ρ shall be also called a density and the set of all densities shall be denoted by $\mathrm{Adm}(\Gamma)$. If $Q > 1$ then the *Q-modulus* of Γ is

$$\mathrm{Mod}_Q(\Gamma) = \inf_{\rho \in \mathrm{Adm}(\Gamma)} \int_X \rho^Q \, d\mu_X.$$

It follows immediately from this definition that if Γ_0 and Γ are two curve families such that each curve $\gamma \in \Gamma$ has a sub-curve $\gamma_0 \in \Gamma_0$, then

$$\mathrm{Mod}_Q(\Gamma) \leq \mathrm{Mod}_Q(\Gamma_0). \tag{3.4}$$

Let us observe that by Theorem 3.8 in Koskela and Wildrick [40], if (X, d_X, μ_X) and (Y, d_Y, μ_Y) are separable metric measure spaces of locally finite measure that are both locally Ahlfors Q-regular for some given $Q > 1$ and $f : X \to Y$ is a quasiconformal map then there exists $H \geq 1$ such that

$$\mathrm{Mod}_Q(\Gamma) \leq H \, \mathrm{Mod}_Q(f(\Gamma)), \tag{3.5}$$

for every curve family Γ in X, i.e., the Mod_Q is quasi-preserved by quasiconformal maps.

For two disjoint compact sets $E, F \subset X$ we consider the quantity $\mathrm{Mod}_Q(E, F) = \mathrm{Mod}_Q(\Gamma)$ where Γ is the set of all rectifiable curves connecting E and F. If $x_0 \in X$ is a fixed point and $0 < r < R < \operatorname{diam} X$, $E = \partial B(x_0, r)$ and $F = \partial B(x_0, R)$ then the quantity $\mathrm{Mod}_Q(E, F) = \mathrm{Mod}_Q(\mathcal{D}(r, R))$ is the so called modulus of the ring domain

$$\mathcal{D}(r, R) = \{x \in X : r < d(x, x_0) < R\}.$$

The following definition is a reformulation in the setting of metric spaces of the corresponding concept by Zorich [56]. For related results we refer also to Holopainen and Rickman [37], Coulhon, Holopainen and Saloff-Coste [21], Fässler, Lukyanenko and Tyson [28].

Definition 3.1.1. The metric measure spaces (X, d_X, μ_X) is Q-parabolic if and only if for some $x_0 \in X$ and $R_0 > 0$ we have

$$\lim_{R\to\infty} \mathrm{Mod}_Q(\mathcal{D}(R_0, R)) = 0. \tag{3.6}$$

Otherwise we call (X, d_X, μ_X) Q-hyperbolic.

Let us note that this property does not depend on the choice of $x_0 \in X$ and $R_0 > 0$, in particular when x_0, R_0 satisfying (3.6) exist then (3.6) holds true for any other choices of $x_0' \in X$, $R_0' > 0$. This means that parabolicity of a metric measure space is a property about the behaviour of the space at infinity.
We also remark that Q-parabolicity of a metric measure space can be defined equivalently by capacity of condensers (see Section 7 in [56] and Definition 4.5.4 in [28]).

The following sufficient condition seems to be known to experts, however we could not locate a precise reference and we include it for the sake of completeness.

Proposition 3.1.2. *Let $Q > 1$ and (X, d_X, μ_X) be a metric measure space such that there exists $x_0 \in X$, $R_0 > 0$ and $K > 0$ such that for all $R > R_0$ we have*

$$\mu_X(B(x_0, R)) \leq KR^Q. \tag{3.7}$$

Then (X, d_X, μ_X) is Q'-parabolic for any $Q' \geq Q$.

Proof. We shall consider the ring domain $\mathcal{D}(R_0, R) = \{x \in X : R_0 < d_X(x, x_0) < R\}$ for $R > R_0$. Our purpose is to show that

$$\lim_{R\to\infty} \mathrm{Mod}_{Q'}(\mathcal{D}(R_0, R)) = 0.$$

To do this, we consider the integer $N \in \mathbb{N}$ defined by the property that $2^N R_0 \geq R > 2^{N-1} R_0$. Note, that if $R \to \infty$, then $N \to \infty$. Consider the density

$$\rho_N(x) = \begin{cases} \frac{3}{N} \cdot \frac{1}{d_X(x_0, x)} & \text{if } x \in \mathcal{D}(R_0, R) \\ 0 & \text{otherwise.} \end{cases}$$

Let us check that the ρ_N is an admissible density for the curve family Γ connecting $\partial B(x_0, R_0)$ and $\partial B(x_0, R)$. To do so we consider the integers $1 < k < N$ and denote by $B_k = B(x_0, 2^k R_0)$ and $D_k = B_k \setminus B_{k-1}$. For $\gamma \in \Gamma$ denote by $\gamma_k = D_k \cap \gamma$. By this notation, we observe that the length of γ_k, $\ell_X(\gamma_k) \geq 2^{k-1}R_0$ and if $x \in \gamma_k$, then $\rho_N(x) \geq \frac{3}{N} \cdot \frac{1}{2^k R_0}$. Using this information we can write

$$\int_\gamma \rho_N \, d\ell_X \geq \sum_{k=2}^{N-1} \int_{\gamma_k} \rho_N \, d\ell_X \geq \sum_{k=2}^{N-1} \frac{3}{N} \cdot \frac{1}{2^k R_0} \ell(\gamma_k) \geq \frac{3(N-2)}{2N} \geq 1,$$

if $N \geq 6$. Note, that by our assumption on the upper of the measure (3.7) we have that $\mu_X(B_k) \leq K 2^{kQ} R_0^Q$. Using this upper estimate on the measure of B_k, the assumption $Q' \geq Q$ and the fact that for $x \in B_k$ we have $\rho(x) \leq \frac{3}{N} \frac{1}{2^{k-1}R_0}$, we can estimate

$$\mathrm{Mod}_{Q'} \mathcal{D}(R_0, R) \leq \int_{\mathcal{D}(R_0,R)} \rho_N^{Q'} d\mu_X \leq \sum_{k=1}^{N} \int_{D_k} \rho_N^{Q'} d\mu_X \leq \sum_{k=1}^{N} \int_{B_k} \left(\frac{3}{N} \frac{1}{2^{k-1}R_0} \right)^{Q'} d\mu_X =$$

$$= \sum_{k=1}^{N} \left(\frac{3}{N} \frac{1}{2^{k-1}R_0} \right)^{Q'} \mu_X(B_k) \leq K \left(\frac{6}{N} \right)^{Q'} R_0^{Q-Q'} \sum_{k=1}^{N} 2^{k(Q-Q')} \to 0 \text{ as } N \to \infty.$$

Since $R \to \infty$ implies that $N \to \infty$ we obtain the statement. □

As already discussed the Heisenberg group $(\mathbb{H}, d_{\mathbb{H}}, \mu_{\mathbb{H}})$ is an Ahlfors 4-regular metric measure space; hence, by applying Proposition 3.1.2, it turns out that $\mathbb{H}$ is 4-parabolic.

As expected, our next statement is a formulation of the fact that a parabolic metric measure space cannot be quasiconformally equivalent to a hyperbolic one. In order to formulate the statement we recall that a metric space is proper, if its closed metric balls are compact.

Theorem 3.1.3. *Let $Q > 1$ and let (X, d_X, μ_X), $(X', d_{X'}, \mu_{X'})$ be two locally Ahlfors Q-regular metric measure spaces. Assume that both spaces are proper and (X, d_X, μ_X) is hyperbolic and $(X', d_{X'}, \mu_{X'})$ is a parabolic space. Then there is no QC map $f : X \to X'$.*

Proof. Assume by contradiction that there is a QC map $f : X \to X'$. Since (X, d_X, μ_X) is assumed to be hyperbolic, there exist a point $x_0 \in X$, $R_0 > 0$, a sequence $R_n \to \infty$, and a number $M > 0$ such that

$$\mathrm{Mod}_Q(\Gamma_n) \geq M > 0, \, n \geq n_0,$$

where Γ_n is the set of curves connecting $\partial B_X(x_0, R_0)$ and $\partial B_X(x_0, R_n)$. By the relation (3.5) there exists $H \geq 1$ such that

$$\mathrm{Mod}_Q(f(\Gamma_n)) \geq \frac{\mathrm{Mod}_Q(\Gamma_n)}{H} \geq \frac{M}{H} > 0.$$

Let us denote by $y_0 = f(x_0) \in X'$. Since X is proper, $\bar{B}_X(x_0, R_0)$ is compact and thus $f(B_X(x_0, R_0))$ is bounded in X'. We conclude that there exists a number $R'_0 > 0$ such that $f(B_X(x_0, R_0)) \subseteq B_{X'}(y_0, R'_0)$. Let us denote by

$$R'_n := \min\{d_{X'}(f(x_0), f(x)) : x \in \partial B_X(x_0, R_n)\}.$$

We claim that $R'_n \to \infty$. For otherwise, we find a sequence $x_n \in X$ with $d_X(x_0, x_n) = R_n$ such that $d'_X(f(x_0), f(x_n)) \leq M'$ for some fixed constant $M' > 0$. Since the space X' is a proper metric space, we obtain that (up to a subsequence) $f(x_n) \to y$ for some $y \in X'$. Let us denote by $x_1 = f^{-1}(y) \in X$ the preimage of y. Since f is a homeomorphism we have that $f(B_X(x_1, r))$ is a neighborhood of $y \in X'$ for any fixed $r > 0$. Since $f(x_n) \to y$ we must have that for n large enough $f(x_n) \in f(B_X(x_1, r))$, which is a contradiction to the injectivity of f.

Let us note that any curve in $f(\Gamma_n)$ has a sub-curve connecting $\partial B_{X'}(y_0, R'_0)$ and $\partial B_{X'}(y_0, R'_n)$. This implies by (3.4) that

$$\mathrm{Mod}_Q(D(R'_0, R'_n)) \geq \mathrm{Mod}_Q(f(\Gamma_n)) \geq \frac{M}{H},$$

which is a contradiction to the parabolicity of $(X', d_{X'}, \mu_{X'})$, concluding the proof. □

3.2 Metric measure properties of $\mathcal{AA}$, $\mathbb{H}$ and $\mathcal{RT}$

The equivalence of contact structures between $\mathcal{AA}$, $\mathbb{H}$ and $\mathcal{RT}$ presented in Chapter 2 has consequences in the theory just discussed in the previous section. In fact Proposition 2.1.2 gives the following:

Proposition 3.2.1. *The metric measure space $(\mathcal{AA}, d_{\mathcal{AA}}, \mu_{\mathcal{AA}})$ is locally Ahlfors 4-regular.*

Proof. We are going to prove a stronger property for $(\mathcal{AA}, d_{\mathcal{AA}}, \mu_{\mathcal{AA}})$: there exist a $C \geq 1$ and a $r_0 > 0$ such that

$$C^{-1} r^4 \leq \mu_{\mathcal{AA}}(\overline{B}_{\mathcal{AA}}(p, r)) \leq C r^4, \tag{3.8}$$

for all $0 < r \leq r_0$ and for all $p \in \mathcal{AA}$.

Due to the left-invariance of both the sub-Riemannian distance $d_{\mathcal{AA}}$ and the measure $\mu_{\mathcal{AA}}$, it suffices to prove (3.8) for balls $B_{\mathcal{AA}}(e, r)$ centered at the neutral element $e = e_{\mathcal{AA}}$. We have that $(\mathbb{H}, \vartheta_{\mathbb{H}})$ and $(\mathcal{AA}, \vartheta)$ are globally contactomorphic thanks to Proposition 2.1.2, so let us consider the map

$$g : (\mathbb{H}, d_{\mathbb{H}}, \mathcal{L}^3) \to (\mathcal{AA}, d_{\mathcal{AA}}, \mu_{\mathcal{AA}}), \quad g^*\vartheta = \sigma \vartheta_{\mathbb{H}},$$

given in (2.1). Since $g(e_{\mathbb{H}}) = e$, where $e_{\mathbb{H}} = (0,0,0)$ is the neutral element of $\mathbb{H}$ and $g : \mathbb{H} \to \mathcal{AA}$ is locally bi-Lipschitz, we have the inclusions

$$g(B_{\mathbb{H}}(e_{\mathbb{H}}, L^{-1}r)) \subseteq B_{\mathcal{AA}}(e, r) \subseteq g(B_{\mathbb{H}}(e_{\mathbb{H}}, Lr))$$

for some fixed number $L \geq 1$ and any $0 \leq r \leq 1$. Since $g^*\mu_{\mathcal{AA}} = \sigma^2\mu_{\mathbb{H}} = \sigma^2\mathcal{L}^3$ (up to multiplicative constants different from 0) and $\mathcal{L}^3(B_{\mathbb{H}}(e_{\mathbb{H}}, r)) = Cr^4$ for some fixed constant $C > 0$, the claim follows. □

Due to Proposition 2.1.4 we obtain that the same statement holds for the roto-translation group $\mathcal{RT}$.

In order to present the last result of this chapter we need a control from above for the measure of metric balls in $\mathcal{RT}$, provided by Corollary 5.9 in [26]:

Proposition 3.2.2. *There exists $R_0 > 0$, and $C_0 > 0$ such that if $B_{\mathcal{RT}}(e_{\mathcal{RT}}, r)$ is the open CC-ball of centre $e_{\mathcal{RT}}$ and radius r then:*

$$\mathcal{L}^3(B_{\mathcal{RT}}(e_{\mathcal{RT}}, r)) \leq C_0 r^3, \ \textit{for all } r \geq R_0. \tag{3.9}$$

The remarkable result of Fässler, Koskela and Le Donne states that in contrast to the fact that both spaces $(\mathbb{H}, d_{\mathbb{H}}, \mu_{\mathbb{H}})$ and $(\mathcal{RT}, d_{\mathcal{RT}}, \mu_{\mathcal{RT}})$ are 4-parabolic and by Proposition 2.1.1 locally bi-Lipschitz equivalent, they are still not QC equivalent (see Corollary 1.2 in [26]).

Now, applying Propositions 3.1.2 and 3.2.2, the following statement follows:

Proposition 3.2.3. *The metric measure spaces $(\mathbb{H}, d_{\mathbb{H}}, \mu_{\mathbb{H}})$ and $(\mathcal{RT}, d_{\mathcal{RT}}, \mu_{\mathcal{RT}})$ are locally 4-Ahlfors regular and 4-parabolic.*

Chapter 4

Hyperbolicity of $\mathcal{AA}$

Our current objective is first to prove hyperbolicity of the affine-additive group $\mathcal{AA}$, then we shall focus on consequences of this fact.

We recall that the formal notion of an hyperbolic metric measure space (X, d_X, μ_X) is given Definition 3.1.1. Let us observe first, that hyperbolicity of (X, d_X, μ_X) holds, if there exists a compact set $E \subset X$ and sequence of compact sets $F_n \subseteq X$ such that

$$\operatorname{dist}(E, F_n) := \inf\{d_X(x, y) : x \in E, y \in F_n\} \to \infty$$

and

$$\liminf_{n\to\infty} \operatorname{Mod}_Q(E, F_n) > 0.$$

To see this, let us pick $x_0 \in E$. We shall consider $R_n = \inf\{d_X(x_0, y) : y \in F_n\}$. Note, that $R_n \to \infty$ and any curve connecting E and F_n must have a sub-curve connecting $\partial B(x_0, R_0)$ and $\partial B(x_0, R_n)$. Thus, by (3.4) we have the inequality

$$\operatorname{Mod}_Q(\mathcal{D}(R_0, R_n)) \geq \operatorname{Mod}_Q(E, F_n).$$

Since $\liminf_{n\to\infty} \operatorname{Mod}_Q(E, F_n) > 0$ we obtain that (X, d_X, μ_X) is hyperbolic.

The main idea of this section is to construct compact sets E and F_n in $\mathcal{AA}$ with the above properties. This is explicitly done as follows:

Let $n \in \mathbb{N}$, $n \geq 2$. We define

$$E = \{(a, 1, t) \in \mathcal{AA} : a \in [-1, 1] \text{ and } t \in [-1, 1]\},$$
$$F_n = \left\{\left(a, \frac{1}{n}, t\right) \in \mathcal{AA} : a \in [-1, 1] \text{ and } t \in [-1, 1]\right\}.$$

Next, for each such n we define the following curve families of piecewise smooth horizontal curves:

$$\Gamma_n = \{\gamma, \gamma : [0, 1] \to \mathcal{AA} \text{ such that } \gamma(0) \in E \text{ and } \gamma(1) \in F_n\}. \tag{4.1}$$

The following estimate holds.

Proposition 4.0.1. *With the above notation, there exists some $M > 0$ such that* $\mathrm{Mod}_4(\Gamma_n) > M$ *for all* $n \in \mathbb{N}$, $n \geq 2$.

Proof. We consider the sub-family $\Gamma_n^0 \subset \Gamma_n$ which comprises curves $\gamma : [0,1] \to \mathcal{AA}$ given by

$$\gamma(s) = \left(a, 1 - \left(1 - \frac{1}{n}\right) s, t\right), \; x \in [-1,1],\, t \in [-1,1].$$

It is straightforward to check that the curves in Γ_n^0 are horizontal with $\gamma(0) \in E$ and $\gamma(1) \in F_n$ for all $n \in \mathbb{N}$, $n \geq 2$. Further, from (1.7) we obtain that

$$|\dot{\gamma}(s)|_H = \frac{1 - \frac{1}{n}}{2\left(1 - \left(1 - \frac{1}{n}\right) s\right)}.$$

If now $\rho \in \mathrm{Adm}(\Gamma_n^0)$, then we have

$$\int_0^1 \rho\left(a, 1 - \left(1 - \frac{1}{n}\right) s, t\right) \frac{1 - \frac{1}{n}}{2\left(1 - \left(1 - \frac{1}{n}\right) s\right)} ds \geq 1,$$

which, under integration by substitution with $\lambda(s) = 1 - \left(1 - \frac{1}{n}\right) s$, gives

$$\int_{\frac{1}{n}}^1 \frac{\rho(a, \lambda, t)}{2\lambda}\, d\lambda \geq 1,\, \forall n \geq 2. \tag{4.2}$$

Next, by integrating (4.2) with respect to $x \in [-1,1]$ and $t \in [-1,1]$, we obtain

$$\int_{-1}^1 \int_{-1}^1 \int_{\frac{1}{n}}^1 \frac{\rho(a, \lambda, t)}{\lambda}\, da\, d\lambda\, dt \geq 8,\, \forall n \geq 2. \tag{4.3}$$

At this point, for $n \geq 2$ we define the sets

$$P_n = \left\{(a, \lambda, t) \in \mathcal{AA} : \; a \in [-1,1], \lambda \in \left[\frac{1}{n}, 1\right],\, t \in [-1,1]\right\}$$

and we apply Hölder's inequality in (4.3) with respect to $\frac{\rho(a,\lambda,t)}{\sqrt{\lambda}} \cdot \frac{\mathcal{X}_{P_n}(a,\lambda,t)}{\sqrt{\lambda}}$ and with conjugated exponents 4 and $\frac{4}{3}$, to obtain

$$\left(\int_{\mathcal{AA}} \rho^4(a, \lambda, t) \frac{da\, d\lambda\, dt}{\lambda^2}\right)^{\frac{1}{4}} \left(\int_{\mathcal{AA}} \mathcal{X}_{P_n}(a, \lambda, t) \frac{1}{\lambda^{\frac{2}{3}}}\, da\, d\lambda\, dt\right)^{\frac{3}{4}} \geq 8, \quad \forall n \geq 2. \tag{4.4}$$

Now we observe that

$$\int_{\mathcal{AA}} \mathcal{X}_{P_n}(a, \lambda, t) \frac{1}{\lambda^{\frac{2}{3}}}\, da\, d\lambda\, dt \leq 4 \int_0^1 \frac{1}{\lambda^{\frac{2}{3}}}\, d\lambda = 12, \quad \forall n \geq 2.$$

The latter inequality combined with (4.4) gives

$$\int_{\mathcal{AA}} \rho^4(a, \lambda, t) \frac{da\, d\lambda\, dt}{\lambda^2} \geq \frac{2^4}{3^3}.$$

Finally, the proof is concluded by taking the infimum over all $\rho \in \mathrm{Adm}(\Gamma_n^0)$. $\square$

4.1 Quasiconformal inequivalence between $\mathcal{AA}$, $\mathbb{H}$ and $\mathcal{RT}$

We are about to notice the importance of hyperbolicity of the affine-additive group $\mathcal{AA}$ as a quasiconformal invariant. In detail we state and prove the following

Theorem 4.1.1. *There are no global QC maps $\mathcal{AA} \to \mathbb{H}$ or $\mathcal{AA} \to \mathcal{RT}$.*

Proof. The proof is an immediate consequence of Theorem 3.1.3. Indeed, Proposition 3.2.3 grants that both the metric measure spaces $\mathbb{H}$ and $\mathcal{RT}$ are locally 4-Ahlfors regular and 4-parabolic. On the other hand Propositions 3.2.1 and 4.0.1 give that the metric measure space $\mathcal{AA}$ is locally 4-Ahlfors regular and 4-hyperbolic. □

4.2 Consequences in quasiregular mapping theory

The hyperbolicity of the affine-additive group has not only consequences in quasiconformal mapping theory but it also sets constraints for a more general class of mappings. Consider a map f between metric spaces and let us recall the quantity $H_f(\cdot)$ from (3.1). Further we define B_f as the branch set (i.e., the set of points where f does not define a local homeomorphism). We make use of the following definition of quasiregular (QR) maps from Fässler, Lukyanenko and Peltonen [27].

Definition 4.2.1. Let M and N be any sub-Riemannian manifolds among $\mathbb{H}$, $\mathcal{RT}$ and $\mathcal{AA}$. We call a mapping $f : M \to N$ K-quasiregular if it is constant, or if:

(1) f is a branched cover onto its image (i.e., continuous, discrete, open and sense-preserving),

(2) $H_f(\cdot)$ is locally bounded on M,

(3) $H_f(p) \leq K$ for almost every $p \in M$,

(4) the branch set B_f and its image have measure zero.

A mapping is said to be *quasiregular* if it is K-quasiregular for some $1 \leq K < \infty$.

From the definition it is follows that every QC map is QR. On the other hand, the class of QR maps can be substantially larger than the class of QC maps.

Let us recall, that by Theorem 4.8.1 from [28] if $f : \mathbb{H} \to N$ is a QR map where N is 4-hyperbolic then f must be constant. Applying this statement to our situation, we obtain the following result:

Theorem 4.2.2. *If $f : \mathbb{H} \to \mathcal{AA}$ is a quasiregular map, then f is constant.*

In contrast to the previous statement, we note that there can be plenty of examples of QR maps $f : \mathcal{AA} \to \mathbb{H}$. One such map is the following:

Example 4.2.3. Let $f : \mathcal{AA} \to \mathbb{H}$ be the map defined by

$$f(a, \lambda, t) = (-\sqrt{\lambda}\cos a, \sqrt{\lambda}\sin a, t), \ (a, \lambda, t) \in \mathcal{AA}.$$

By a direct calculation one can verify the contact property of f, namely $f^*\vartheta_{\mathbb{H}} = 2\lambda\vartheta$. Moreover, denoting $f(a, \lambda, t) = (x, y, t) \in \mathbb{H}$, one can check that $f_*U = yX - xY$ and $f_*V = xX + yY$. Using this, we have that

$$f_*(\alpha U + \beta V) = (\alpha y + \beta x)X + (\beta y - \alpha x)Y,$$

for any $\alpha, \beta \in \mathbb{R}$.

Since $\{U, V\}$, resp. $\{X, Y\}$, is the orthonormal basis in the sub-Riemannian metric of $\mathcal{AA}$, resp. $\mathbb{H}$, we obtain that

$$|f_*(\alpha U + \beta V)|_{\mathbb{H}} = \sqrt{(\alpha^2 + \beta^2)(x^2 + y^2)}$$

and therefore

$$H_f(a, \lambda, t) = \frac{\max\{|f_*(\alpha U + \beta V)|_{\mathbb{H}} : \alpha^2 + \beta^2 = 1\}}{\min\{|f_*(\alpha U + \beta V)|_{\mathbb{H}} : \alpha^2 + \beta^2 = 1\}} = 1,$$

for every point $(a, \lambda, t) \in \mathcal{AA}$. See also Proposition 2.4 in [27] for a different way to compute the value of $H_f(\cdot)$.

Furthermore, note, that a direct computation, gives $\det f_* = \frac{1}{2}$; and thus f is a local diffeomorphism at every point. This means that the branch set B_f of f is empty, and thus f is an immersion of $\mathcal{AA}$ into $\mathbb{H}$. Consequently we conclude that f is 1-quasiregular.

Further examples of QR maps $g : \mathcal{AA} \to \mathbb{H}$ can be obtained as compositions $g = h \circ f$ where $h : \mathbb{H} \to \mathbb{H}$ is a QC map of the Heisenberg group.

With Example 4.2.3 we answer Question 3.20 in Guo, Nicolussi Golo, Williams and Xuan [31].

Chapter 5

Quasiconformal mappings on $\mathcal{AA}$

In this chapter we present the general theory of quasiconformal mappings in the affine-additive group $\mathcal{AA}$. Such theory makes good advantage of the differential geometry features of $\mathcal{AA}$ given by its sub-Riemannian structure.

5.1 Complex differential structure of $\mathcal{AA}$

The use of complex notation on $\mathbf{H}^1_{\mathbb{C}}$ will turn out to be convenient from now on, so we apply slight changes to the setting of $\mathcal{AA}$. The hyperbolic plane is now given by

$$\mathbf{H}^1_{\mathbb{C}} := \{\lambda + it \in \mathbb{C} : \lambda > 0,\, t \in \mathbb{R}\}.$$

Based on this we reintroduce the group operation on $\mathbf{H}^1_{\mathbb{C}}$ by

$$(\lambda' + it') \star_0 (\lambda + it) = \lambda'(\lambda + it) + it'.$$

We rewrite the group operation on $\mathcal{AA}$ as follows: if $p' = (a', \lambda' + it')$ and $p = (a, \lambda + it)$ belong to $\mathbb{R} \times \mathbf{H}^1_{\mathbb{C}}$, we have

$$p' \star p = (a' + a, \lambda'(\lambda + it) + it') \in \mathbb{R} \times \mathbf{H}^1_{\mathbb{C}}. \tag{5.1}$$

We recall that $\mathcal{AA}$ is a three dimensional Lie group and, thanks to Proposition 1.3.1, a basis of the tangent bundle $T(\mathcal{AA})$ comprising left invariant vector fields is given by U, V and W. The complex vector fields (shortly CVF) $Z, \overline{Z}$, are given by

$$Z = \frac{1}{2}(V - iU),\ \overline{Z} = \frac{1}{2}(V + iU).$$

We notice that they satisfy the non-trivial commutator identity $[Z,\overline{Z}] = (\overline{Z}-Z)+iW$. The Lie algebra of left invariant vector fields of the affine-additive group admits a grading

$$\mathrm{span}_{\mathbb{R}}\{\mathrm{Im}Z, \mathrm{Re}Z\} \oplus \mathrm{span}_{\mathbb{R}}\{W\}.$$

The elements of the first layer are referred as *horizontal left invariant vector fields.* The horizontal bundle $\mathcal{H}_{\mathcal{AA}}$ is the subbundle of the tangent bundle $T(\mathcal{AA})$ whose fibers are the *horizontal subspaces*

$$\mathcal{H}_{p,\mathcal{AA}} = \mathrm{span}_{\mathbb{R}}\{\mathrm{Im}Z_p, \mathrm{Re}Z_p\}, \quad p \in \mathcal{AA}.$$

We remind that the contact form for $\mathcal{AA}$ is $\vartheta = \frac{dt}{2\lambda} - da$. We recall also that a contact transformation $f : \Omega \to \Omega'$ on $\mathcal{AA}$ is a diffeomorphism between domains Ω and Ω' in $\mathcal{AA}$ which preserves the contact structure, i.e.

$$f^*\vartheta = \sigma\vartheta, \tag{5.2}$$

for some non-vanishing smooth function $\sigma : \mathcal{AA} \to \mathbb{R}$. Through the identification of $\mathcal{AA}$ with $\mathbb{R}\times\mathbf{H}^1_{\mathbb{C}}$ we write $f = (f_1, f_I)$, $f_I = f_2 + if_3$. Applying Proposition 2.2.3 with respect to the CVFs $Z, \overline{Z}$ gives that a contact map f is determined by the following system of p.d.e.s

$$\begin{aligned} Zf_3 &= 2f_2Zf_1, \\ \overline{Z}f_3 &= 2f_2\overline{Z}f_1, \\ Wf_3 &= 2f_2(\sigma + Wf_1). \end{aligned} \tag{5.3}$$

5.2 Background results on quasiconformal mappings

A *metric definition* of quasiconformality (in the sense of Heinonen and Koskela [34]) is given in terms of the sub-Riemannian distance on the affine-additive group as follows. If Ω, Ω' are two domains in $\mathcal{AA}$, a homeomorphism $f : \Omega \to \Omega'$ is called *quasiconformal* if there exists $1 \le H < \infty$ such that

$$\limsup_{r\to 0} \frac{\sup_{d_{\mathcal{AA}}(p,q)\le r} d_{\mathcal{AA}}(f(p), f(q))}{\inf_{d_{\mathcal{AA}}(p,q)\ge r} d_{\mathcal{AA}}(f(p), f(q))} =: H_f(p) \le H, \quad \text{for all } p \in \mathcal{AA}. \tag{5.4}$$

Any smooth and metric quasiconformal map between domains in $\mathcal{AA}$ is locally a contact transformation; this fact comes as a consequence of Proposition 3.3 in Balogh, Bubani and Platis [7] and Theorem 1 in Korányi and Reimann [38].
Let $\mathcal{L}^3$ be the three dimensional Lebesgue measure on $\mathbb{R}\times\mathbf{H}^1_{\mathbb{C}}$: we point out that $\mu_{\mathcal{AA}} \ll \mathcal{L}^3$

and therefore the terminology "almost everywhere" is well defined on $\mathcal{AA}$ in the sense of $\mathcal{L}^3$. In general, quasiconformal maps on $\mathcal{AA}$ do not need to be smooth, but rather they belong to an apposite class of Sobolev mappings and they satisfy the contact conditions almost everywhere. Explicitly, let $1 \leq p < \infty$ and let Ω be a domain in $\mathcal{AA}$. We say that a function $u : \Omega \to \mathbb{C}$ belongs to the *horizontal Sobolev space*, $u \in HW^{1,p}(\Omega, \mathbb{C})$, if $u \in L^p(\Omega, \mathbb{C})$ and there exist functions $v, w \in L^p(\Omega, \mathbb{C})$ such that

$$\int_\Omega v\varphi \, d\mu_{\mathcal{AA}} = -\int_\Omega uZ\varphi \, d\mu_{\mathcal{AA}}, \quad \text{and} \quad \int_\Omega w\varphi \, d\mu_{\mathcal{AA}} = -\int_\Omega u\overline{Z}\varphi \, d\mu_{\mathcal{AA}}$$

for all $\varphi \in C_0^\infty(\Omega, \mathbb{R})$. For such a function $u \in HW^{1,p}(\Omega, \mathbb{C})$, we denote by Zu and $\overline{Z}u$ the weak horizontal complex derivatives v and w. This definition is compatible with the theory of upper gradients on Carnot-Carathéodory spaces formulated in Hajłasz and Koskela [32]. A map $f = (f_1, f_I) : \Omega \to \mathcal{AA}$ is said to belong to $HW^{1,p}(\Omega, \mathcal{AA})$ if and only if f_1, f_I are in $HW^{1,p}(\Omega, \mathbb{C})$. It is straightforward to define the local horizontal Sobolev spaces $HW^{1,p}_{loc}$.
We have that $(\mathcal{AA}, d_{\mathcal{AA}}, \mu_{\mathcal{AA}})$ is a locally 4-Ahlfors regular space, cf. Proposition 3.5 in [7]. Combining the last fact with Theorem 11.20 in [32], Theorem 1.1 in Balogh, Koskela and Rogovin [12] and Proposition 3.1 in Shanmugalingam [52] we deduce the following result for metric quasiconformal maps on $\mathcal{AA}$.

Proposition 5.2.1. *Let $f : \Omega \to \Omega'$ be a quasiconformal mapping between domains $\Omega, \Omega' \subseteq \mathcal{AA}$. Then the pointwise derivatives $(ReZ)f$ and $(ImZ)f$ exist almost everywhere and coincide with the distributional derivatives almost everywhere.*

We recall that Proposition 1.3.2 grants the Hausdorff dimension of $(\mathcal{AA}, d_{\mathcal{AA}})$ to be equal to 4. Based on the latter fact the corresponding Sobolev class for quasiconformal mappings in the affine-additive group is $HW^{1,4}_{loc}$.
A mapping $f \in HW^{1,4}_{loc}(\Omega, \mathcal{AA})$ is called *weakly contact* if the system of p.d.e.s (5.3) holds almost everywhere in Ω. For such a mapping, we define the *formal tangent map*

$$(f_*)_p : T_p\mathcal{AA} \to T_{f(p)}\mathcal{AA},$$

for almost every $p \in \Omega$. We express it in terms of the bases given by $\mathcal{B}_p^{CVF} = \{Z_p, \overline{Z}_p, W_p\}$ and $\mathcal{B}_{f(p)}^{CVF} = \{Z_{f(p)}, \overline{Z}_{f(p)}, W_{f(p)}\}$ as

$$f_* = \begin{bmatrix} Zf_I/2f_2 & \overline{Z}f_I/2f_2 & * \\ Z\overline{f_I}/2f_2 & \overline{Z}\,\overline{f_I}/2f_2 & * \\ 0 & 0 & \sigma \end{bmatrix},$$

and define the *formal horizontal differential* to be the restriction $D_H f(p) : \mathcal{H}_{p,\mathcal{AA}} \to \mathcal{H}_{f(p),\mathcal{AA}}$ given by

$$D_H f(p) = \begin{bmatrix} Zf_I/2f_2 & \overline{Z}f_I/2f_2 \\ Z\overline{f_I}/2f_2 & \overline{Z}\,\overline{f_I}/2f_2 \end{bmatrix}.$$

Using the commutator relation together with the system (5.3), we find $\sigma = \frac{1}{4f_2^2}\left(|Zf_I|^2 - |\overline{Z}f_I|^2\right)$ almost everywhere. This yields

$$\det(f_*)_p = \frac{1}{(2f_2(p))^4}\left(|Zf_I(p)|^2 - |\overline{Z}f_I(p)|^2\right)^2 \text{ a.e. in } \Omega. \tag{5.5}$$

Further, set $p \in \mathcal{AA}$ and $r > 0$; we define the volume derivative for f with respect to $\mu_{\mathcal{AA}}$ the limit

$$\mathcal{J}_{\mu_{\mathcal{AA}}}(p, f) = \lim_{r\to 0} \frac{\mu_{\mathcal{AA}}(f(B_{\mathcal{AA}}(p, r)))}{\mu_{\mathcal{AA}}(B_{\mathcal{AA}}(p, r))}. \tag{5.6}$$

An interesting property is formulated in the following:

Lemma 5.2.2. *Let $f : \Omega \to \Omega'$ be a weakly contact transformation. Then the identity*

$$\mathcal{J}_{\mu_{\mathcal{AA}}}(p, f) = \det(f_*)_p,$$

holds almost everywhere in Ω.

Proof. We observe first that a limit argument and the change of variables theorem induce that at almost every point $p = (a, \lambda + it) \in \Omega$ we have

$$\mathcal{J}_{\mu_{\mathcal{AA}}}(p, f) = \frac{\lambda^2}{f_2^2(p)}\mathcal{J}(p, f).$$

Here, $\mathcal{J}(p, f)$ corresponds to the determinant of the tangent map $(f_*)_p : T_p\mathcal{AA} \to T_{f(p)}\mathcal{AA}$ considered as a linear map with respect to the canonical bases $\mathcal{B}_p^{Can} = \{\partial_{a|_p}, \partial_{\lambda|_p}, \partial_{t|_p}\}$ and $\mathcal{B}_{f(p)}^{Can} = \{\partial_{a|_{f(p)}}, \partial_{\lambda|_{f(p)}}, \partial_{t|_{f(p)}}\}$. The change of bases formula describing the compositions

$$\mathcal{B}_p^{Can} \mapsto \mathcal{B}_p^{CVF} \mapsto \mathcal{B}_{f(p)}^{CVF} \mapsto \mathcal{B}_{f(p)}^{Can}$$

leads to:

$$\mathcal{J}_{\mu_{\mathcal{AA}}}(p, f) = \frac{1}{(2f_2(p))^4}\left(|Zf_I(p)|^2 - |\overline{Z}f_I(p)|^2\right)^2.$$

□

We consider the curve family

$$\Gamma_1 = \{\gamma, \gamma : [0, 1] \to \mathcal{AA} \text{ horizontal curve with } \gamma(0) = p \text{ and } |\dot{\gamma}|_H = 1\}$$

and we define the quantity $\|D_H f(p)\| = \max\{|(f \circ \gamma)^{\cdot}|_H : \gamma \in \Gamma_1\}$. Using the complex notation we find the explicit formula

$$\|D_H f(p)\| = \frac{|Zf_I(p)| + |\overline{Z} f_I(p)|}{2 f_2(p)} \text{ a.e. in } \Omega.$$

The analytic definition of quasiconformality in $\mathcal{AA}$ is now in order:

Definition 5.2.3 (Analytic definition). A homeomorphism $f : \Omega \to \Omega'$ between domains Ω, Ω' in $\mathcal{AA}$ is K-*quasiconformal* if $f \in HW^{1,4}_{loc}(\Omega, \mathcal{AA})$ is weakly contact, and there exists a constant $1 \leq K < \infty$ such that

$$\|D_H f(p)\|^4 \leq K \mathcal{J}_{\mu_{\mathcal{AA}}}(p, f) \quad \text{for almost every } p \in \Omega. \tag{5.7}$$

A map is *quasiconformal*, if it is K-quasiconformal for some K.

It will be later proved that a K-quasiconformal map in the analytic sense has a K-quasiconformal inverse (see Proposition 5.2.4). This fact together with Theorem 3.8 in Koskela and Wildrick [40] induce that a homeomorphism is quasiconformal according the analytic sense if and only if it is quasiconformal according to the metric one.
It can be proven that $\mathcal{J}_{\mu_{\mathcal{AA}}}(\cdot, f) \neq 0$ a.e. for a quasiconformal mapping f. The above considerations show that for a quasiconformal map $f : \Omega \to \Omega'$ between domains in the affine-additive group the following holds:

$$K(p, f)^2 = \frac{\|D_H f(p)\|^4}{\mathcal{J}_{\mu_{\mathcal{AA}}}(p, f)} = \left(\frac{|Zf_I(p)| + |\overline{Z} f_I(p)|}{|Zf_I(p)| - |\overline{Z} f_I(p)|} \right)^2 \text{ a.e. in } \Omega.$$

By setting $K(\cdot, f)^2 = 1$ at the points where $\mathcal{J}_{\mu_{\mathcal{AA}}}(\cdot, f) = 0$, we obtain that $K(\cdot, f)^2$ is a measurable function on Ω which is finite almost everywhere. A quasiconformal map $f : \Omega \to \Omega'$ between domains in the affine-additive group is called *orientation preserving* if

$$\det D_H f(p) > 0 \quad \text{for almost every } p \in \Omega.$$

By recalling the expressions defined in the introduction

$$\mu_f(p) = \frac{\overline{Z} f_I(p)}{Z f_I(p)}, \quad K(p, f) = \frac{|Zf_I(p)| + |\overline{Z} f_I(p)|}{|Zf_I(p)| - |\overline{Z} f_I(p)|}$$

and by defining

$$\|\mu_f\|_\infty = \operatorname{ess\,sup}_p |\mu_f(p)|, \quad K_f = \operatorname{ess\,sup}_p K(p, f),$$

the explicit relation between $\|\mu_f\|_\infty$ and K_f can now be written explicitly as follows:

$$K(p, f) = \frac{1 + |\mu_f(p)|}{1 - |\mu_f(p)|}, \quad K_f = \frac{1 + \|\mu_f\|_\infty}{1 - \|\mu_f\|_\infty}. \tag{5.8}$$

We present a result concerning the inverse of a quasiconformal mapping with the following

Proposition 5.2.4. *Let $f : \Omega \to \Omega'$ be a quasiconformal mapping between domains $\Omega, \Omega' \subseteq \mathcal{AA}$. Then also the inverse $f^{-1} : \Omega' \to \Omega$ is a quasiconformal mapping and the following holds*

$$K(p, f) = K(f(p), f^{-1}) \text{ a.e. in } \Omega. \tag{5.9}$$

Proof. To prove that $f^{-1} : \Omega' \to \Omega$ is quasiconformal in the metric sense one can consider the argument in the proof of Proposition 20 in [39] and adapt it to the context of $\mathcal{AA}$. We will show that the identity (5.9) is true.

Since $f : \Omega \to \Omega'$ is quasiconformal, Proposition 5.2.1 grants that the derivatives Uf and Vf exist almost everywhere in Ω. Let $\mathcal{J}(p, f)$ denote the determinant of the tangent map $(f_*)_p : T_p\mathcal{AA} \to T_{f(p)}\mathcal{AA}$ considered as a linear map with respect to the canonical bases $\mathcal{B}_p^{Can} = \{\partial_{a|_p}, \partial_{\lambda|_p}, \partial_{t|_p}\}$ and $\mathcal{B}_{f(p)}^{Can} = \{\partial_{a|_{f(p)}}, \partial_{\lambda|_{f(p)}}, \partial_{t|_{f(p)}}\}$.

Further, applying Lemma 5.2.2 for f quasiconformal, it holds that $\mathcal{J}(\cdot, f) \neq 0$ almost everywhere in Ω. The following derivatives are therefore well defined for almost every $p = (a, \lambda + it) \in \Omega$:

$$(\partial_a f_2^{-1})_{|f(p)} = \frac{(\partial_a f_3 \partial_t f_2 - \partial_a f_2 \partial_t f_3)_{|p}}{\mathcal{J}(p, f)}, \quad (\partial_\lambda f_2^{-1})_{|f(p)} = \frac{(\partial_a f_1 \partial_t f_3 - \partial_t f_1 \partial_a f_3)_{|p}}{\mathcal{J}(p, f)},$$

$$(\partial_t f_2^{-1})_{|f(p)} = \frac{(\partial_a f_2 \partial_t f_1 - \partial_a f_1 \partial_t f_2)_{|p}}{\mathcal{J}(p, f)}, \quad \text{and}$$

$$(\partial_a f_3^{-1})_{|f(p)} = \frac{(\partial_a f_2 \partial_\lambda f_3 - \partial_\lambda f_2 \partial_a f_3)_{|p}}{\mathcal{J}(p, f)}, \quad (\partial_\lambda f_3^{-1})_{|f(p)} = \frac{(\partial_a f_3 \partial_\lambda f_1 - \partial_a f_1 \partial_\lambda f_3)_{|p}}{\mathcal{J}(p, f)},$$

$$(\partial_t f_3^{-1})_{|f(p)} = \frac{(\partial_a f_1 \partial_\lambda f_2 - \partial_a f_2 \partial_\lambda f_1)_{|p}}{\mathcal{J}(p, f)}.$$

Recalling that $Z = \frac{1}{2}(V - iU)$, we obtain:

$$(Zf_I^{-1})_{|f(p)} = \frac{1}{2\mathcal{J}(p, f)}(\partial_a f_2 \partial_\lambda f_3 - \partial_\lambda f_2 \partial_a f_3 + 2f_2(\partial_a f_1(\partial_\lambda f_2 + \partial_t f_3) - \partial_a f_3 \partial_t f_1 - \partial_\lambda f_1 \partial_a f_2)$$
$$+ i\,(\partial_a f_2 \partial_t f_3 - \partial_t f_2 \partial_a f_3 + 2f_2(\partial_a f_1(\partial_t f_2 - \partial_\lambda f_3) + \partial_a f_3 \partial_\lambda f_1 - \partial_a f_2 \partial_t f_1)))_{|p}.$$

Now, we make use of the three contact equations (5.3) by replacing $(\partial_a f_1)_{|p}$, $(\partial_\lambda f_1)_{|p}$ and $(\partial_t f_1)_{|p}$ in the last identity. This yields to:

$$(Zf_I^{-1})_{|f(p)} = \frac{\sigma(p) f_2(p)}{\lambda \mathcal{J}(p, f)}(\overline{Z}\,\overline{f_I})_{|p},$$

where $\sigma : \mathcal{AA} \to \mathbb{R}$ is the nowhere vanishing smooth function appearing in (5.3). Thanks to the relation $\overline{Zf_I} = \overline{Z}\,\overline{f_I}$ we rewrite

$$(Zf_I^{-1})_{|f(p)} = \frac{\sigma(p) f_2(p)}{\lambda \mathcal{J}(p, f)}(\overline{Zf_I})_{|p}. \tag{5.10}$$

Analogously, we find that

$$(\overline{Z}f_I^{-1})_{|f(p)} = \frac{\sigma(p)f_2(p)}{2\lambda\mathcal{J}(p,f)}\big((Uf_3 - Vf_2)_{|p} - i(Uf_2 + Vf_3)_{|p}\big)$$
$$= -\frac{\sigma(p)f_2(p)}{\lambda\mathcal{J}(p,f)}(\overline{Z}f_I)_{|p}. \tag{5.11}$$

Referring to (5.10) and (5.11) we conclude

$$K(f(p), f^{-1}) = \frac{\left|Zf_I^{-1}(f(p))\right| + \left|\overline{Z}f_I^{-1}(f(p))\right|}{\left|Zf_I^{-1}(f(p))\right| - \left|\overline{Z}f_I^{-1}(f(p))\right|} = K(p,f) \quad \text{a.e. in } \Omega. \tag{5.12}$$

□

We state below a change of variable formula for integration in the case of quasiconformal mappings on the affine-additive group.

Proposition 5.2.5. *Let $f : \Omega \to \Omega'$ be a quasiconformal mapping between domains $\Omega, \Omega' \subseteq \mathcal{AA}$. Then the following transformation formula holds: if $u : \mathcal{AA} \to \mathbb{R}$ is a measurable non-negative function, then the function $p \mapsto (u \circ f)(p)\mathcal{J}_{\mu_{\mathcal{AA}}}(p,f)$ is measurable and we have*

$$\int_\Omega (u \circ f)(p)\mathcal{J}_{\mu_{\mathcal{AA}}}(p,f)\, d\mu_{\mathcal{AA}}(p) = \int_{\Omega'} u(q)\, d\mu_{\mathcal{AA}}(q).$$

For a proof of this result we indicate Theorem 2 in Vodop'yanov's work [55].

In order to present an important feature of quasiconformal mappings on the affine-additive group we recall the construction of Hausdorff dimension for the metric space $(\mathcal{AA}, d_{\mathcal{AA}})$. For $a \in (0,\infty)$, the *Hausdorff a-dimensional outer measure* of a set $A \subseteq \mathcal{AA}$ is given by

$$H_a(A) = \lim_{\epsilon \to 0}\left(\inf_{\mathcal{B}} \sum_{B_i \in \mathcal{B}} (\operatorname{diam} B_i)^a\right),$$

where the infimum is taken over all countable coverings $\mathcal{B} = (B_i)_{i \in I}$ of A by sets B_i with the diameter condition $\operatorname{diam} B_i < \epsilon$ for all $i \in I$. The *Hausdorff dimension* of A is defined by

$$\dim_H(A) = \inf\{a > 0 : H_a(A) = 0\}$$

In what follows we want to highlight the existence of non-smooth quasiconformal maps of $\mathcal{AA}$ that distort the Hausdorff dimension $\dim_H$ of certain Cantor sets in $\mathcal{AA}$ in an arbitrary fashion.

Proposition 5.2.6. *For any s and t such that $0 < s < t < 4$ there exist Cantor sets $C_s \subset \mathcal{AA}$ and $C_t \subset \mathcal{AA}$ such that $\dim_H(C_s) = s$ and $\dim_H(C_t) = t$ and a QC map $F : \mathcal{AA} \to \mathcal{AA}$ such that $F(C_s) = C_t$.*

Proof. The proof is based on the corresponding result in [6] for the case of the Heisenberg group. In fact Theorem 1.1 in [6] states that if $0 < s < t < 4$ there exist Cantor sets $K_s \subset \mathbb{H}$ and $K_t \subset \mathbb{H}$ and a QC map $G : \mathbb{H} \to \mathbb{H}$ such that $\dim_H(K_s) = s$, $\dim_H K_t = t$, $K_s \subset B_{\mathbb{H}}(e, 1)$, $K_t \subset B_{\mathbb{H}}(e, 1)$, $G(K_s) = K_t$ and $G = id_{\mathbb{H}}$ outside of $B_{\mathbb{H}}(e, 1)$. We note that the map $F : \mathcal{AA} \to \mathcal{AA}$ defined by $F = g \circ G \circ g^{-1}$ is a QC map. To see this observe that $G : B_{\mathbb{H}}(e, 1) \to B_{\mathbb{H}}(e, 1)$ is quasiconformal and the map $g : B_{\mathbb{H}}(e, 1) \to g(B_{\mathbb{H}}(e, 1))$ is bi-Lipschitz. This shows that $F : g(B_{\mathbb{H}}(e, 1)) \to g(B_{\mathbb{H}}(e, 1))$ is a QC map. On the other hand since $G = id_{\mathbb{H}\setminus B_{\mathbb{H}}(e,1)}$, this implies that the map $F_{\mathbb{H}\setminus g(B_{\mathbb{H}}(e,1))} = id_{\mathbb{H}\setminus g(B_{\mathbb{H}}(e,1))}$. Thus F is a global QC map and it does satisfy the properties in the statement for the sets $C_s = g(K_s)$ and $C_t = g(K_t)$. □

5.3 Modulus of a curve family in $\mathcal{AA}$

The definition of modulus for a family of curves in general metric measure spaces was given in Chapter 3. In the previous section we presented that the Hausdorff dimension of the affine-additive group is 4. This indicates that a notion of conformally invariant Q-modulus of a curve family will require $Q = 4$.

Before defining the 4-modulus we need to recall definition and properties of curves in the affine-additive group. Any curve γ in $\mathcal{AA}$ shall be always considered continuous. The points on a curve $\gamma : [c, d] \to \mathcal{AA}$ are denoted by

$$\gamma(s) = (\gamma_1(s), \gamma_I(s)) \in \mathbb{R} \times \mathbf{H}^1_{\mathbb{C}}.$$

An absolutely continuous curve $\gamma : [c, d] \to \mathcal{AA}$ (in the Euclidean sense) is called *horizontal* if

$$\dot{\gamma}(s) \in \ker \vartheta_{\gamma(s)} \quad \text{for almost every } s \in [c, d].$$

The length of a horizontal curve corresponds to

$$\ell(\gamma) = \int_c^d |\dot{\gamma}(s)|_H \, ds.$$

We say that γ is *rectifiable* when $\ell(\gamma)$ is finite, moreover we say that γ is *locally rectifiable* when all its closed sub-curves are rectifiable.

If $\gamma : [c,d] \to \mathcal{AA}$ is a rectifiable curve, the line integral over γ of a Borel function $\rho : \mathcal{AA} \to [0,\infty]$ is defined as

$$\int_\gamma \rho \, d\ell = \int_c^d \rho(\gamma(s))|\dot\gamma(s)|_H \, ds,$$

and in case γ is only locally rectifiable, we set

$$\int_\gamma \rho \, d\ell = \sup \left\{ \int_{\gamma'} \rho \, d\ell : \gamma' \text{ is a rectifiable subcurve of } \gamma \right\}.$$

For curves $\gamma : (c,d) \to \mathcal{AA}$ we shall employ the notion of local rectifiability.
The modulus $\mathrm{Mod}_4(\Gamma)$ of a curve family Γ is defined as follows. Let $\mathrm{Adm}(\Gamma)$ be the set of *admissible densities*: that is, non-negative Borel functions $\rho : \mathcal{AA} \to [0,\infty]$ such that $\int_\gamma \rho \, d\ell \geq 1$ for all rectifiable curves $\gamma \in \Gamma$. Then

$$\mathrm{Mod}_4(\Gamma) = \inf_{\rho \in \mathrm{Adm}(\Gamma)} \int_{\mathcal{AA}} \rho^4(p) \, d\mu_{\mathcal{AA}}(p), \tag{5.13}$$

It is worth to say that a family which consists only of curves that are not locally rectifiable has modulus zero, see the book of Heinonen, Koskela, Shanmugalingam and Tyson [36]. All quasiconformal mappings of the affine-additive group are absolutely continuous on almost every curve, see the works of Heinonen, Koskela, Shanmugalingam and Tyson [35] and Shanmugalingam [52]. To sum up, given a quasiconformal map $f : \Omega \to \Omega'$ between domains in the affine-additive group and given a family Γ of closed rectifiable curves in Ω, we have

$$\mathrm{Mod}_4 \left(\gamma \in \Gamma : f \circ \gamma \text{ not absolutely continuous}\right) = 0.$$

The next type of modulus inequality adapts the statement and the proof of Theorem 18 in [10] to the geometric setting of the affine-additive group. We can now prove the following:

Proposition 5.3.1. *Suppose that $f : \Omega \to \Omega'$ is a quasiconformal map between two domains in $\mathcal{AA}$ and Γ is a family of curves in Ω. Then*

$$\mathrm{Mod}_4(f(\Gamma)) \leq \int_\Omega K^2(p,f)\rho^4(p) \, d\mu_{\mathcal{AA}}(p) \text{ for all } \rho \in \mathrm{Adm}(\Gamma). \tag{5.14}$$

Proof. Let Γ_0 be the family of all rectifiable curves in Γ on which f is absolutely continuous (the non-rectifiable curves have modulus zero). Since f is quasiconformal, we have $\mathrm{Mod}_4(\Gamma) = \mathrm{Mod}_4(\Gamma_0)$. Throughout the proof we shall assume f to be differentiable in the sense of [43] on curves in Γ_0 almost everywhere on their domain of definition, for otherwise one can consider the argument in the beginning of the proof on Theorem 18 in [10].

We now take an arbitrary admissible density $\tilde{\rho} \in \mathrm{Adm}(f(\Gamma))$ and we assign to it a pull-back density $\rho_{\tilde{\rho}}$ defined by

$$\rho_{\tilde{\rho}}(p) = \begin{cases} \tilde{\rho}(f(p)) \frac{|Zf_I(p)| + |\overline{Z} f_I(p)|}{2f_2(p)}, & p \in \Omega \\ \\ 0, & p \in \mathcal{AA} \setminus \Omega. \end{cases}$$

We will show that $\rho_{\tilde{\rho}}$ is admissible for Γ_0. To this end, let $\gamma : [a, b] \to \Omega$ be an arbitrary curve in Γ_0. By definition of Γ_0, it is rectifiable and therefore it has a parametrization by arc-length, $\tilde{\gamma} = (\tilde{a}, \tilde{\lambda} + i\tilde{t}) : [0, \ell(\gamma)] \to \Omega$. We know that f is quasiconformal and that $\tilde{\gamma}(s)$ is horizontal; due to Lemma 6.2.2 this reasoning leads to

$$|(f \circ \tilde{\gamma})^{\cdot}(s)|_H = \frac{1}{2f_2(\tilde{\gamma}(s))} \left| Zf_I(\tilde{\gamma}(s)) \frac{\dot{\tilde{\gamma}}_I(s)}{2\tilde{\lambda}(s)} + \overline{Z} f_I(\tilde{\gamma}(s)) \frac{\dot{\overline{\tilde{\gamma}}}_I(s)}{2\tilde{\lambda}(s)} \right| \text{ for a.e. } s \in [0, \ell(\gamma)]$$

which gives

$$|(f \circ \tilde{\gamma})^{\cdot}(s)|_H \le \frac{|Zf_I(\tilde{\gamma}(s))| + |\overline{Z} f_I(\tilde{\gamma}(s))|}{2f_2(\tilde{\gamma}(s))} |\dot{\tilde{\gamma}}(s)|_H \text{ for a.e. } s \in [0, \ell(\gamma)].$$

We notice that $f \circ \tilde{\gamma}$ is absolutely continuous and the latter inequality yields

$$\begin{aligned} \int_\gamma \rho_{\tilde{\rho}} \, d\ell &= \int_0^{\ell(\gamma)} \rho_{\tilde{\rho}}(\tilde{\gamma}(s)) |\dot{\tilde{\gamma}}(s)|_H \, ds \\ &= \int_0^{\ell(\gamma)} \tilde{\rho}(f(\tilde{\gamma}(s))) \frac{|Zf_I(\tilde{\gamma}(s))| + |\overline{Z} f_I(\tilde{\gamma}(s))|}{2f_2(\tilde{\gamma}(s))} |\dot{\tilde{\gamma}}(s)|_H \, ds \\ &\ge \int_0^{\ell(\gamma)} \tilde{\rho}(f(\tilde{\gamma}(s))) |(f \circ \tilde{\gamma})^{\cdot}(s)|_H \, ds = \int_{f \circ \tilde{\gamma}} \tilde{\rho} \, d\ell = \int_{f \circ \gamma} \tilde{\rho} \, d\ell \ge 1. \end{aligned}$$

We deduce that $\rho_{\tilde{\rho}} \in \mathrm{Adm}(\Gamma_0)$. Making use of (6.5), the previous fact allows us to conclude as follows:

$$\begin{aligned} \mathrm{Mod}_4(\Gamma_0) &= \inf_{\rho \in \mathrm{Adm}(\Gamma_0)} \int_\Omega \rho^4(p) \, d\mu_{\mathcal{AA}}(p) \\ &\le \int_\Omega \rho_{\tilde{\rho}}^4(p) \, d\mu_{\mathcal{AA}}(p) = \int_\Omega \tilde{\rho}^4(f(p)) \frac{(|Zf_I(p)| + |\overline{Z} f_I(p)|)^4}{(2f_2(p))^4} \, d\mu_{\mathcal{AA}}(p) \\ &= \int_\Omega \tilde{\rho}^4(f(p)) \left(\frac{|Zf_I(p)| + |\overline{Z} f_I(p)|}{|Zf_I(p)| - |\overline{Z} f_I(p)|} \right)^2 \frac{\left(|Zf_I(p)|^2 - |\overline{Z} f_I(p)|^2\right)^2}{(2f_2(p))^4} \, d\mu_{\mathcal{AA}}(p) \\ &= \int_\Omega \tilde{\rho}^4(f(p)) K^2(p, f) \mathcal{J}_{\mu_{\mathcal{AA}}}(p, f) \, d\mu_{\mathcal{AA}}(p) \\ &= \int_{\Omega'} \tilde{\rho}^4(q) K^2(f^{-1}(q), f) \, d\mu_{\mathcal{AA}}(q), \end{aligned}$$

for all $\tilde{\rho} \in \mathrm{Adm}(f(\Gamma))$. Where the last equality in this chain of computations is obtained through the change of variable from Proposition 5.2.5. We may apply the previous inequality to the quasiconformal map f^{-1} and the curve family $f(\Gamma)$. Thus

$$\mathrm{Mod}_4(f(\Gamma)) \leq \int_{\Omega} K^2(f(p), f^{-1})\rho^4(p)\, d\mu_{\mathcal{AA}}(p), \tag{5.15}$$

for all $\rho \in \mathrm{Adm}(\Gamma)$. Proposition 5.2.4 gives the identity

$$K(p, f) = K(f(p), f^{-1}) \text{ a.e. in } \Omega. \tag{5.16}$$

Combining (5.15) and (5.16) we obtain the desired result. □

Remark 5.3.2. We want to underline two other important consequences following from the proof of Proposition 5.3.1. In the first place, we have that

$$\mathrm{Mod}_4(\Gamma) \leq \int_{\Omega'} K^2(f^{-1}(q), f)\tilde{\rho}^4(q)\, d\mu_{\mathcal{AA}}(q) \text{ for all } \tilde{\rho} \in \mathrm{Adm}(f(\Gamma)),$$

and thus

$$\frac{1}{K_f^2}\mathrm{Mod}_4(\Gamma) \leq \mathrm{Mod}_4(f(\Gamma)) \leq K_f^2\, \mathrm{Mod}_4(\Gamma). \tag{5.17}$$

Moreover, from (5.17) we evince that Mod_4 is conformally invariant, i.e. invariant under 1-quasiconformal mappings.

Chapter 6

The modulus method on $\mathcal{AA}$

With the present chapter we aim to establish a method based on the modulus of curve families in the affine-additive group which detects extremal quasiconformal mappings for the mean distortion functional (see condition (6) in the Introduction).

6.1 Minimal stretching property and extremality

We begin this section with a result concerning an extremal density for the modulus of a curve family foliating a bounded domain in the affine-additive group.

Proposition 6.1.1. *Suppose Δ is a domain in $\mathbb{R}^2$. Let $0 \le c < d$ and let*

$$\gamma : (c, d) \times \Delta \to \Omega$$

be a diffeomorphism which foliates a bounded domain Ω in the affine-additive group with the property that

$$\gamma(\cdot, \delta) : [c, d] \to \overline{\Omega}$$

is an horizontal curve with $|\dot{\gamma}(s, \delta)|_H \neq 0$ for all $\delta \in \Delta$ and

$$d\mu_{\mathcal{AA}}(\gamma(s, \delta)) = |\dot{\gamma}(s, \delta)|_H^4 \, ds \, d\nu(\delta)$$

for a measure ν on Δ. Then

$$\rho_0(p) = \begin{cases} \frac{1}{(d-c)|\dot{\gamma}(\gamma^{-1}(p))|_H}, & p = \gamma(s, \delta) \in \Omega, \\ \\ 0, & p \notin \Omega, \end{cases} \tag{6.1}$$

is an extremal density for the curve family $\Gamma_0 = \{\gamma(\cdot,\delta) : \delta \in \Delta\}$ *with*

$$\mathrm{Mod}_4(\Gamma_0) = \frac{1}{(d-c)^3}\int_\Delta d\nu(\delta).$$

Here, $\dot\gamma(s,\delta) = \frac{\partial}{\partial s}\gamma(s,\delta)$ *for* $(s,\delta) \in (c,d)\times\Delta$.

Proof. We show first that $\rho_0 \in \mathrm{Adm}(\Gamma_0)$: this is because for any $\gamma(\cdot,\delta) \in \Gamma_0$ we have

$$\int_{\gamma(\cdot,\delta)} \rho_0\, d\ell = \int_c^d \rho_0(\gamma(s,\delta))|\dot\gamma(s,\delta)|_H\, ds = 1.$$

Since we assume the measure decomposition $d\mu_{\mathcal{AA}}(\gamma(s,\delta)) = |\dot\gamma(s,\delta)|_H^4\, ds\, d\nu(\delta)$, a direct computation yields

$$\begin{aligned}\int_\Omega \rho_0^4(p)\, d\mu_{\mathcal{AA}}(p) &= \int_\Delta\int_c^d \rho_0^4(\gamma(s,\delta))|\dot\gamma(s,\delta)|_H^4\, ds\, d\nu(\delta)\\ &= \frac{1}{(d-c)^3}\int_\Delta d\nu(\delta)\,.\end{aligned}$$

Consequently,

$$\mathrm{Mod}_4(\Gamma_0) \le \frac{1}{(d-c)^3}\int_\Delta d\nu(\delta).$$

For the reverse inequality, consider an arbitrary density $\rho \in \mathrm{Adm}(\Gamma_0)$. By using the admissibility of ρ and then Hölder's inequality with conjugated exponents 4 and $\frac{4}{3}$, we have

$$\begin{aligned}1 \le \int_{\gamma(\cdot,\delta)} \rho\, d\ell &= \int_c^d \rho(\gamma(s,\delta))|\dot\gamma(s,\delta)|_H\, ds\\ &\le \left(\int_c^d \rho^4(\gamma(s,\delta))|\dot\gamma(s,\delta)|_H^4\, ds\right)^{\frac14}(d-c)^{\frac34}\,,\end{aligned}$$

for every $\delta \in \Delta$. Thus

$$\frac{1}{(d-c)^{\frac34}} \le \left(\int_c^d \rho^4(\gamma(s,\delta))|\dot\gamma(s,\delta)|_H^4\, ds\right)^{\frac14}.$$

We raise the latter inequality to the 4-th power and then we integrate with respect to $d\nu$ over Δ to eventually obtain

$$\frac{1}{(d-c)^3}\int_\Delta d\nu(\delta) \le \int_\Delta\int_c^d \rho^4(\gamma(s,\delta))|\dot\gamma(s,\delta)|_H^4\, ds\, d\nu(\delta) = \int_\Omega \rho^4(p)\, d\mu_{\mathcal{AA}}(p).$$

Our result follows by taking the infimum over all densities $\rho \in \mathrm{Adm}(\Gamma_0)$. □

6.2 Minimization of the mean distortion

Following the work of Balogh, Fässler and Platis [10], we define the minimal stretching property as follows:

Definition 6.2.1. We say that an orientation preserving quasiconformal map $f_0 : \Omega \to \Omega'$ between domains in $\mathcal{AA}$ has the *minimal stretching property (MSP)* for a family Γ_0 of horizontal curves in Ω if for all $\gamma \in \Gamma_0$, $\gamma : [c, d] \to \mathcal{AA}$, one has

$$\mu_{f_0}(\gamma(s))\frac{\dot{\overline{\gamma_I}}(s)}{\dot{\gamma_I}(s)} < 0 \text{ for almost every } s \in [c, d] \text{ with } \mu_{f_0}(\gamma(s)) \neq 0. \tag{6.2}$$

If a map f_0 has the MSP for a curve family Γ_0, this means geometrically that Γ_0 consists of curves which are tangential to the direction of the least stretching of f_0. To make this precise, we state and prove the following

Lemma 6.2.2. *Let $f = (f_1, f_2 + if_3) : \Omega \to \mathcal{AA}$ be a quasiconformal map on a domain $\Omega \subseteq \mathcal{AA}$. Let Γ be a curve family:*

$$\Gamma = \{\gamma(\cdot) = (a(\cdot), \lambda(\cdot) + it(\cdot)), \gamma : [c, d] \to \Omega \textit{ horizontal}\}.$$

Then there exists a sub-family $\Gamma' \subset \Gamma$ of curves with $\mathrm{Mod}_4(\Gamma') = 0$ and such that

$$(f_I \circ \gamma)^{\cdot}(s) = \frac{1}{2\lambda(s)}\left(Zf_I(\gamma(s))\dot{\gamma_I}(s) + \overline{Z}f_I(\gamma(s))\dot{\overline{\gamma_I}}(s)\right) \quad \textit{for a.e.} \quad s \in (c, d), \tag{6.3}$$

for all $\gamma \in \Gamma \setminus \Gamma'$.

Proof. If $\gamma \in \Gamma$ is absolutely continuous then since f is quasiconformal, we have that the image $f \circ \gamma$ is absolutely continuous up to a sub-family of curves having zero 4-modulus, see [12]. Therefore we can choose γ such that $f \circ \gamma$ is differentiable almost everywhere. We choose such an absolutely continuous curve $\gamma : [c, d] \to \mathcal{AA}$ given by $\gamma(s) = (a(s), \lambda(s)+it(s))$ and by using the chain rule we write

$$(f_I \circ \gamma)^{\cdot}(s) = \nabla f_2(\gamma(s)) \cdot \dot{\gamma}(s) + i\,\nabla f_3(\gamma(s)) \cdot \dot{\gamma}(s), \quad \text{for a.e. } s \in (c, d).$$

Using the o.d.e. (1.6) which holds for γ as well as the identities

$$\dot{\lambda}(s) = \frac{\dot{\gamma_I}(s) + \dot{\overline{\gamma_I}}(s)}{2}, \quad \dot{t}(s) = \frac{\dot{\gamma_I}(s) - \dot{\overline{\gamma_I}}(s)}{2i},$$

we obtain

$$(f_I \circ \gamma)^{\cdot}(s) = \frac{1}{2\lambda(s)}\left(Zf_I(\gamma(s))\dot{\gamma_I}(s) + \overline{Z}f_I(\gamma(s))\dot{\overline{\gamma_I}}(s)\right), \quad \text{for a.e. } s \in (c, d).$$

□

Recalling that $|\dot{\gamma}(s)|_H = \frac{|\dot{\gamma}_I(s)|}{2\lambda(s)}$, for an orientation preserving quasiconformal map f we have:

$$\left(\frac{|Zf_I(\gamma(s))| - |\overline{Z}f_I(\gamma(s))|}{2f_2(\gamma(s))}\right)|\dot{\gamma}(s)|_H \leq |(f\circ\gamma)^{\cdot}(s)|_H \leq \left(\frac{|Zf_I(\gamma(s))| + |\overline{Z}f_I(\gamma(s))|}{2f_2(\gamma(s))}\right)|\dot{\gamma}(s)|_H$$

for almost every s. If a map f_0 has the MSP for a family Γ_0, then by (6.2) we have equality

$$|(f_0\circ\gamma)^{\cdot}(s)|_H = \left(\frac{|Z(f_0)_I(\gamma(s))| - |\overline{Z}(f_0)_I(\gamma(s))|}{2(f_0)_2(\gamma(s))}\right)|\dot{\gamma}(s)|_H\,. \tag{6.4}$$

Secondly, we are about to provide a result in which conditions both on the foliation of the domain as well as on the quasiconformal map are being set in order to obtain the equality in the modulus inequality for the mean distortion.

Before getting into the details, we need to briefly recall a concept essential for our next arguments. Denote by $B_{\mathcal{AA}}(p,r)$ the open ball with respect to the distance $d_{\mathcal{AA}}$, centered at $p\in\mathcal{AA}$ and with radius $r>0$. Let $f:\Omega\to\mathcal{AA}$ be a quasiconformal map on a domain $\Omega\subseteq\mathcal{AA}$ and define the volume derivative for f with respect to $\mu_{\mathcal{AA}}$ to be the limit

$$\mathcal{J}_{\mu_{\mathcal{AA}}}(p,f) := \lim_{r\to 0}\frac{\mu_{\mathcal{AA}}(f(B_{\mathcal{AA}}(p,r)))}{\mu_{\mathcal{AA}}(B_{\mathcal{AA}}(p,r))}.$$

The following identity holds:

$$\mathcal{J}_{\mu_{\mathcal{AA}}}(p,f) = \frac{1}{(2f_2(p))^4}\left(|Zf_I(p)|^2 - |\overline{Z}f_I(p)|^2\right)^2, \tag{6.5}$$

almost everywhere in Ω.

For the proof of (6.5), see Lemma 5.2.2 in Chapter 5. Everything is arranged for the following

Proposition 6.2.3. *Let $f_0:\Omega\to\Omega'$ be an orientation preserving quasiconformal map between domains in the affine-additive group. As described above, let γ be the foliation of Ω and let Γ_0 be the curve family. Assume further that f_0 has the MSP for Γ_0 and that*

$$K(\gamma(s,\delta),f_0)\equiv K_{f_0}(\delta) \tag{6.6}$$

for all $(s,\delta)\in(c,d)\times\Delta$. Then

$$\mathrm{Mod}_4(f_0(\Gamma_0)) = \frac{1}{(d-c)^3}\int_{\Delta}K_{f_0}^2(\delta)\,d\nu(\delta)\,. \tag{6.7}$$

Proof. Let $\rho'\in\mathrm{Adm}(f_0(\Gamma_0))$ be an arbitrary density. Since f_0 is quasiconformal, we know that the image under f_0 of an horizontal curve $\gamma(\cdot,\delta)\in\Gamma_0$ is still a horizontal curve up to a sub-family of curves contained in Γ_0 with vanishing 4-modulus. The minimal stretching property of f_0 grants (6.4) and therefore we find

$$
\begin{aligned}
1 &\leq \int_c^d \rho'(f_0 \circ \gamma(s,\delta)) |(f_0 \circ \gamma)^{\cdot}(s,\delta)|_H \, ds \\
&= \int_c^d \rho'(f_0 \circ \gamma(s,\delta)) \frac{\left|Z(f_0)_I(\gamma(s,\delta))\right| - \left|\overline{Z}(f_0)_I(\gamma(s,\delta))\right|}{2(f_0)_2(\gamma(s,\delta))} |\dot\gamma(s,\delta)|_H \, ds.
\end{aligned}
$$

We apply Hölder's inequality with conjugated exponents 4 and $\frac{4}{3}$ to the last relation; this gives

$$
\frac{1}{(d-c)^3} \leq \int_c^d \rho'^4(f_0 \circ \gamma(s,\delta)) \Big(\frac{\left|Z(f_0)_I(\gamma(s,\delta))\right| - \left|\overline{Z}(f_0)_I(\gamma(s,\delta))\right|}{2(f_0)_2(\gamma(s,\delta))} \Big)^4 |\dot\gamma(s,\delta)|_H^4 \, ds.
$$

Now, multiplying both sides by $K^2_{f_0}(\delta)$ and integrating over Δ with respect to $d\nu$ gives

$$
\begin{aligned}
&\frac{1}{(d-c)^3} \int_\Delta K^2_{f_0}(\delta) \, d\nu(\delta) \leq \qquad (6.8) \\
&\int_\Delta \int_c^d \rho'^4(f_0 \circ \gamma(s,\delta)) K^2_{f_0}(\delta) \left(\frac{\left|Z(f_0)_I(\gamma(s,\delta))\right| - \left|\overline{Z}(f_0)_I(\gamma(s,\delta))\right|}{2(f_0)_2(\gamma(s,\delta))} \right)^4 |\dot\gamma(s,\delta)|_H^4 \, ds \, d\nu(\delta).
\end{aligned}
$$

By plugging in the assumption $K_{f_0}(\delta) \equiv K(\gamma(s,\delta), f_0)$ for all $(s,\delta) \in (c,d) \times \Delta$ into (6.5), we obtain the identity

$$
K^2(\gamma(s,\delta), f_0) \left(\frac{\left|Z(f_0)_I(\gamma(s,\delta))\right| - \left|\overline{Z}(f_0)_I(\gamma(s,\delta))\right|}{2(f_0)_2(\gamma(s,\delta))} \right)^4 = \mathcal{J}_{\mu_{\mathcal{AA}}}(\gamma(s,\delta), f_0).
$$

By recomposing the left-invariant Haar measure on $\mathcal{AA}$ through the foliation γ as

$$
|\dot\gamma(s,\delta)|_H^4 \, ds \, d\nu(\delta) = d\mu_{\mathcal{AA}}(p), \;\; p = \gamma(s,\delta) \in \Omega,
$$

we get that (6.8) results into

$$
\frac{1}{(d-c)^3} \int_\Delta K^2_{f_0}(\delta) \, d\nu(\delta) \leq \int_\Omega \rho'^4(f_0(p)) \mathcal{J}_{\mu_{\mathcal{AA}}}(p, f_0) \, d\mu_{\mathcal{AA}}(p). \qquad (6.9)
$$

We apply the change of variable $f_0(p) = q \in \Omega'$; then (6.9) becomes

$$
\frac{1}{(d-c)^3} \int_\Delta K^2_{f_0}(\delta) \, d\nu(\delta) \leq \int_{\Omega'} \rho'^4(q) \, d\mu_{\mathcal{AA}}(q).
$$

Since ρ' was chosen arbitrarily among the admissible densities of $f_0(\Gamma_0)$, the latter inequality shows that

$$
\frac{1}{(d-c)^3} \int_\Delta K^2_{f_0}(\delta) \, d\nu(\delta) \leq \mathrm{Mod}_4(f_0(\Gamma_0)).
$$

For the other inequality consider the push-forward density given by

$$\rho_0'(q) = \begin{cases} \frac{2(f_0)_2(\gamma(s,\delta))}{(d-c)|\dot\gamma(s,\delta)|_H\left(|Z(f_0)_I(\gamma(s,\delta))|-|\overline{Z}(f_0)_I(\gamma(s,\delta))|\right)}, & q = f_0(\gamma(s,\delta)) \in \Omega', \\ \\ 0, & q \notin \Omega'. \end{cases}$$

Thanks to the minimal stretching property of f_0 this density is admissible, i.e. $\rho_0' \in \mathrm{Adm}(f_0(\Gamma_0))$:

$$\int_{f_0\circ\gamma} \rho_0'\, d\ell = \int_c^d \rho_0'(f_0 \circ \gamma(s,\delta))|(f_0\circ\gamma)^{\cdot}(s,\delta)|_H\, ds = \int_c^d \frac{1}{d-c}\, ds = 1.$$

Therefore, via the change of variable $f_0(\gamma(s,\delta)) = q \in \Omega'$ for some $(s,\delta) \in (c,d)\times\Delta$, we obtain:

$$\begin{aligned}
\mathrm{Mod}_4(f_0(\Gamma_0)) &\le \int_{\Omega'} \rho_0'^4(q)\, d\mu_{\mathcal{A}\mathcal{A}}(q) \\
&= \int_\Omega \rho_0'^4(f_0(p))\mathcal{J}_{\mu_{\mathcal{A}\mathcal{A}}}(p,f_0)\, d\mu_{\mathcal{A}\mathcal{A}}(p) \\
&= \int_\Omega \rho_0'^4(f_0(p))\frac{1}{(2(f_0)_2(p))^4}\left(|Z(f_0)_I|^2 - |\overline{Z}(f_0)_I|^2\right)^2(p)\, d\mu_{\mathcal{A}\mathcal{A}}(p) \\
&= \int_\Delta\int_c^d \frac{1}{(d-c)^4|\dot\gamma(s,\delta)|_H^4} K^2(\gamma(s,\delta),f_0)|\dot\gamma(s,\delta)|_H^4\, ds\, d\nu(\delta) \\
&= \frac{1}{(d-c)^4}\int_\Delta\int_c^d K^2(\gamma(s,\delta),f_0)\, ds\, d\nu(\delta) \\
&= \frac{1}{(d-c)^3}\int_\Delta K_{f_0}^2(\delta)\, d\nu(\delta).
\end{aligned}$$

In this way the proof is concluded. □

Before stating the main result of this chapter concerning conditions for extremality of the mean distortion integral we need to define a condition on the quasiconformal distortion. Let $f_0 : \Omega \to \Omega'$ be an orientation preserving quasiconformal mapping between domains in the affine-additive group. Let γ be a foliation of Ω as described in Proposition 6.1.1. Assume as well that f_0 has the MSP for Γ_0; we then say that the distortion quotient $K(\cdot, f_0)$ is *constant along every curve* γ if and only if

$$K(\gamma(s,\delta), f_0) \equiv K_{f_0}(\delta) \quad \text{for all } (s,\delta)\in(c,d)\times\Delta. \tag{6.10}$$

We are in the position to prove the following

Theorem 6.2.4. *Assume that f_0 satisfies the minimal stretching property with respect to Γ_0 described as above. Let ρ_0 be the extremal density for Γ_0 and assume $K(\cdot, f_0)$ to be constant* *along every curve foliating* Ω. *Let* $\Gamma \supseteq \Gamma_0$ *be a curve family such that* $\rho_0 \in \text{Adm}(\Gamma)$ *and let* $\mathcal{F}$ *be the class of quasiconformal maps* $f : \Omega \to \Omega'$ *such that*

$$\text{Mod}_4(f_0(\Gamma_0)) \leq \text{Mod}_4(f(\Gamma)). \tag{6.11}$$

Then

$$\int_\Omega K^2(p, f_0)\rho_0^4(p)\, d\mu_{\mathcal{AA}}(p) \leq \int_\Omega K^2(p, f)\rho_0^4(p)\, d\mu_{\mathcal{AA}}(p)$$

for all $f \in \mathcal{F}$.

Proof. Recalling the definition of $\rho_0 \in \text{Adm}(\Gamma_0)$ given in (6.1) and combining it with Proposition 6.2.3, applied on Γ_0, ρ_0 and f_0, we get that:

$$\int_\Omega K^2(p, f_0)\rho_0^4(p)\, d\mu_{\mathcal{AA}}(p) = \frac{1}{(b-a)^3}\int_\Lambda K_{f_0}^2(\delta)\, d\nu(\delta) = \text{Mod}_4(f_0(\Gamma_0)) \tag{6.12}$$

Thanks to the assumption (6.11), and applying (5.14) from Proposition 5.3.1 with respect to the density $\rho_0 \in \text{Adm}(\Gamma_0) \subseteq \text{Adm}(\Gamma)$, we obtain:

$$\text{Mod}_4(f_0(\Gamma_0)) \leq \text{Mod}_4(f(\Gamma_0)) \leq \int_\Omega K^2(p, f)\rho_0^4(p)\, d\mu_{\mathcal{AA}}(p), \text{ for all } f \in \mathcal{F}. \tag{6.13}$$

By coupling the inequalities (6.12) and (6.13) we obtain the desired result. □

Remark 6.2.5. From the statement of Theorem 6.2.4 we can develop a method which verifies if a candidate quasiconformal map $f_0 : \Omega \to \Omega'$ between domains $\Omega, \Omega' \subset \mathcal{AA}$, is a minimizer for the mean distortion functional. We describe the steps of the method:

1. let $\mathcal{F}$ be a class of quasiconformal mappings $f : \Omega \to \Omega'$, $f \in \mathcal{F}$;

2. let γ be a foliation for Ω which is composed of horizontal curves decomposing the volume measure of $\mathcal{AA}$, i.e. γ verifies the assumptions of Proposition 6.1.1;

3. introduce the curve family Γ_0 and then calculate the extremal density ρ_0 for $\text{Mod}_4(\Gamma_0)$ given by (6.1);

4. verify that the distortion quotient $K(\cdot, f)$ is constant along such horizontal curves foliating Ω, i.e. condition (6.6);

5. check the MSP for f_0 with respect to Γ_0;

6. determine a curve family $\Gamma \supset \Gamma_0$ such that $\text{Mod}_4(f_0(\Gamma_0)) \leq \text{Mod}_4(f(\Gamma))$ for all $f \in \mathcal{F}$ and verify $\rho_0 \in \text{Adm}(\Gamma)$.

Chapter 7

Stretch maps on $\mathcal{AA}$

In this chapter we define linear and radial stretch maps on the affine-additive group, and prove that they are minimizers of the mean quasiconformal distortion functional (6). For the proofs we rely on the modulus method described in Chapter 6, in particular we shall follow the strategy described in Remark 6.2.5.

7.1 The linear stretch map

For $k > 0$, the map $f_k : \mathcal{AA} \to \mathcal{AA}$ given by

$$f_k(a, \lambda + it) = (ka, \lambda + ikt), \tag{7.1}$$

shall be called *linear stretch map*. Its name is justified by the fact that it is a linear map with respect to the cartesian coordinates.
We will present two geometric settings where the linear stretch map turns to be a minimizer for the mean distortion functional, respectively for $k \in (0, 1)$ and for $k > 1$. This distinction is motivated by the Beltrami coefficient $\mu_{f_k} = \frac{1-k}{1+k}$: the two geometric settings have distinct suitable domains, foliations and associated curve families such that the MSP for f_k holds for both cases.

7.1.1 The case $k \in (0, 1)$.

Let $k \in (0, 1)$ and define two domains as follows:

$$\Omega = \left\{ \left(a + \frac{t}{2\lambda}, \lambda + it \right) \in \mathcal{AA} : a \in (0,1), \lambda \in \left(\frac{1}{2}, 1 \right), t \in (0,1) \right\},$$

$$\Omega^k = \left\{ \left(k \left(a + \frac{t}{2\lambda} \right), \lambda + ikt \right) \in \mathcal{AA} : a \in (0,1), \lambda \in \left(\frac{1}{2}, 1 \right), t \in (0,1) \right\}.$$

For a fixed $t \in \mathbb{R}$, we denote

$$\partial\Omega_t = \left\{ \left(a + \frac{t}{2\lambda}, \lambda + it \right) \in \mathcal{AA} : a \in (0,1), \lambda \in \left(\frac{1}{2}, 1 \right) \right\}$$

and consider the class $\mathcal{F}_k$ of all quasiconformal mappings $f : \Omega \to \Omega^k$ which extend homeomorphically to the boundary and we impose conditions

$$f(\partial\Omega_0) = \partial\Omega_0^k \text{ and } f(\partial\Omega_1) = \partial\Omega_1^k.$$

The domain Ω is displayed in the following figure:

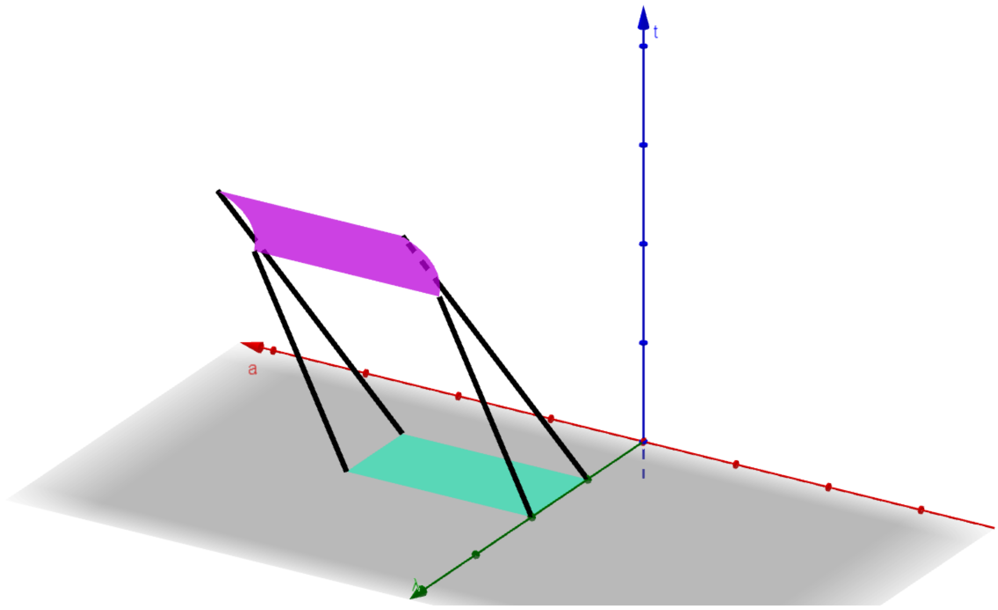

Figure 7.1: $\partial\Omega_0$ is in cyan and $\partial\Omega_1$ is in purple.

With the present setting we prove the following

Theorem 7.1.1. *The linear stretch map $f_k : \Omega \to \Omega^k$ is an orientation preserving quasiconformal map. With the above notation for ρ_0, f_k minimizes the mean distortion within the class $\mathcal{F}_k$: for all $f \in \mathcal{F}_k$ we have that*

$$K_{f_k}^2 \leq \frac{\int_\Omega K^2(\cdot, f)\rho_0^4 \, d\mu_{\mathcal{AA}}}{\int_\Omega \rho_0^4 \, d\mu_{\mathcal{AA}}}. \tag{7.2}$$

Proof. The steps of the proof steps are the ones explained in Remark 6.2.5.
1. The class $\mathcal{F}_k$ is presented above the proof.
2. Let the pair $(a, \lambda) \in (0,1) \times \left(\frac{1}{2}, 1\right)$ and let $\gamma : (0,1) \times (0,1) \times \left(\frac{1}{2}, 1\right) \to \Omega$ be the foliation of Ω given by

$$\gamma(s, a, \lambda) = \left(a + \frac{s}{2\lambda}, \lambda + is\right), \quad (s, a, \lambda) \in (0,1) \times (0,1) \times \left(\frac{1}{2}, 1\right).$$

In this way, the volume element on $\mathcal{AA}$ can be written as

$$d\mu_{\mathcal{AA}}(\gamma(s, a, \lambda)) = \frac{1}{\lambda^2} \, da \, d\lambda \, ds = |\dot{\gamma}(s, a, \lambda)|_H^4 \, ds \, d\nu(a, \lambda),$$

where

$$d\nu(a, \lambda) = 2^4 \lambda^2 \, da \, d\lambda.$$

3. In order to apply Proposition 6.1.1, we consider the family of horizontal curves

$$\Gamma_0 = \left\{\gamma(\cdot, a, \lambda) : a \in (0,1), \lambda \in \left(0, \frac{1}{2}\right)\right\}.$$

An extremal density for Γ_0 is given by formula (6.1): namely, $\rho_0(a, \lambda + it) = 2\lambda \cdot \mathcal{X}_\Omega(a, \lambda + it)$. Hence

$$\mathrm{Mod}_4(\Gamma_0) = \int_{\frac{1}{2}}^1 \int_0^1 2^4 \lambda^2 \, da \, d\lambda = \frac{14}{3}.$$

4. An explicit calculation gives constant distortion quotient K_{f_k}. Indeed,

$$K(\gamma(s, a, \lambda), f_k) \equiv \frac{1}{k}, \quad (s, a, \lambda) \in (0,1) \times (0,1) \times \left(\frac{1}{2}, 1\right),$$

and so $K_{f_k} = \operatorname{ess\,sup}_p K(p, f_k) = \frac{1}{k}$.
5. We observe that f_k has the MSP with respect to the curve family Γ_0. Indeed, for $k \in (0,1)$ we have

$$\mu_{f_0}(\gamma_{a,\lambda}(s)) \frac{\overline{\dot{(\gamma_{a,\lambda})_I}(s)}}{\dot{(\gamma_{a,\lambda})_I}(s)} = \frac{k-1}{1+k} < 0 \quad \text{for all } s \in (0,1).$$

6. Aiming to apply Theorem 6.2.4, we need to find a bigger curve family $\Gamma \supseteq \Gamma_0$ for which ρ_0 is still admissible and such that $\mathrm{Mod}_4(f_k(\Gamma_0)) \le \mathrm{Mod}_4(f(\Gamma))$ for all $f \in \mathcal{F}_k$. A guess for Γ is the family of *all* horizontal curves contained in Ω which are joining the two components $\partial\Omega_0$ and $\partial\Omega_1$. The boundary conditions for maps in the class $\mathcal{F}_k$ provide that the image $f_k(\Gamma)$ is going to be a family of the same type in Ω^k. Using the absolute continuity of quasiconformal mappings on almost every curve up to a negligible family of curves with zero 4-modulus and using the boundary conditions, we may show that

$$\mathrm{Mod}_4(f_k(\Gamma_0)) \le \mathrm{Mod}_4(f(\Gamma)) \quad \text{for all } f \in \mathcal{F}_k$$

We have to check that ρ_0 is admissible for the extended family Γ. Indeed, for a curve $\gamma : [c,d] \to \mathcal{AA}$ with $\gamma \in \Gamma$, we have

$$\int_\gamma \rho_0 \, d\ell = \int_c^d \sqrt{\dot\lambda(s)^2 + \dot t(s)^2}\, ds \ge \int_c^d \dot t(s)\, ds = 1$$

Here we have used for the evaluation of the integral the fact that $s \mapsto t(s)$ is an absolutely continuous function and the conditions $\gamma(c) \in \partial\Omega_0$, $\gamma(d) \in \partial\Omega_1$.
We conclude that $\rho_0 \in \mathrm{Adm}(\Gamma)$ and, from Theorem 6.2.4, it follows that

$$K_{f_k}^2 \int_\Omega \rho_0(p)^4 \, d\mu_{\mathcal{AA}}(p) \le \int_\Omega K(p,f)^2 \rho_0(p)^4 \, d\mu_{\mathcal{AA}}(p) \quad \text{for all } f \in \mathcal{F}_k\,. \tag{7.3}$$

The proof is complete. □

We wish to highlight an important consequence which follows from the last proof: by taking the essential supremum for $K^2(\cdot, f)$ on the r.h.s. of (7.3), we notice that f_k minimizes also the maximal distortion K_f (see (5.8)). We therefore state:

Corollary 7.1.2. *The linear stretch map $f_k : \Omega \to \Omega^k$ is an orientation preserving quasiconformal map such that*

$$K_{f_k} \le K_f,$$

for all $f \in \mathcal{F}_k$.

7.1.2 The case $k > 1$.

Let $k > 1$ and by using a similar notation as in the previous section we consider two domains as follows:

$$\Omega = \left\{(a, \lambda + it) \in \mathcal{AA} : a \in (0,1), \lambda \in \left(\frac{1}{2}, 1\right), t \in (0,1)\right\},$$

$$\Omega^k = \left\{(ka, \lambda + ikt) \in \mathcal{AA} : a \in (0,1), \lambda \in \left(\frac{1}{2}, 1\right), t \in (0,1)\right\}.$$

For a fixed $\lambda > 0$, let

$$\partial\Omega_\lambda = \{(a, \lambda + it) \in \mathcal{AA} : a \in (0,1), t \in (0,1)\}$$

and consider the class $\mathcal{F}_k$ of all quasiconformal mappings $f : \Omega \to \Omega^k$ which extend homeomorphically to the boundary subject to the conditions

$$f(\partial\Omega_{\frac{1}{2}}) = \partial\Omega^k_{\frac{1}{2}} \text{ and } f(\partial\Omega_1) = \partial\Omega^k_1.$$

With the current specific setting we prove a result analogous to Theorem 7.1.1 in the following

Theorem 7.1.3. *The linear stretch map $f_k : \Omega \to \Omega^k$ is an orientation preserving quasiconformal map. With the above notation for ρ_0, f_k minimizes the mean distortion within the class $\mathcal{F}_k$: for all $f \in \mathcal{F}_k$ we have that*

$$K^2_{f_k} \le \frac{\int_\Omega K^2(\cdot, f)\rho_0^4 \, d\mu_{\mathcal{AA}}}{\int_\Omega \rho_0^4 \, d\mu_{\mathcal{AA}}}. \tag{7.4}$$

Proof. Again, the steps of the proof follow the strategy of Remark 6.2.5.

1. The class $\mathcal{F}_k$ is as above.

2. Let the pair $(a, t) \in (0,1) \times (0,1)$ and let $\gamma : \left(\frac{3}{2^{\frac{1}{3}}}, 3\right) \times (0,1) \times (0,1) \to \Omega$ be the foliation of Ω given by

$$\gamma(s, a, t) = \left(a, \frac{s^3}{3^3} + it\right), \quad (s, a, t) \in \left(\frac{3}{2^{\frac{1}{3}}}, 3\right) \times (0,1) \times (0,1).$$

In this way, the volume element on $\mathcal{AA}$ can be written as

$$d\mu_{\mathcal{AA}}(\gamma(s, a, t)) = \frac{3^4}{s^4} \, da \, ds \, dt = |\dot\gamma(s, a, t)|^4_H \, ds \, d\nu(a, t),$$

where

$$d\nu(a, t) = 2^4 \, da \, dt.$$

3. In order to apply Proposition 6.1.1, we consider the family of horizontal curves

$$\Gamma_0 = \{\gamma(\cdot, a, t) : a \in (0,1), t \in (0,1)\}.$$

The extremal density ρ_0 for Γ_0 following from formula (6.1), is given by

$$\rho_0(a, \lambda + it) = c_0 \lambda^{\frac{1}{3}} \cdot \mathcal{X}_\Omega(a, \lambda + it),$$

where $c_0 = \frac{2^{\frac{4}{3}}}{3\left(2^{\frac{1}{3}} - 1\right)}$. This results into

$$\mathrm{Mod}_4(\Gamma_0) = \frac{1}{\left(3 - \frac{3}{2^{\frac{1}{3}}}\right)^3} \int_0^1 \int_0^1 2^4 \, da \, d\lambda = \frac{2^5}{3^3\left(2^{\frac{1}{3}} - 1\right)}.$$

4. An explicit calculation gives constant distortion quotient K_{f_k}, indeed

$$K(\gamma(s,a,t),f_k) \equiv k, \quad (s,a,t) \in \left(\frac{3}{2^{\frac{1}{3}}},3\right) \times (0,1) \times (0,1),$$

and so $K_{f_k} = \operatorname{ess\,sup}_p K(p,f_k) = k$.

5. We observe that f_k has the MSP with respect to the curve family Γ_0. Indeed, for $k>1$ we have

$$\mu_{f_0}(\gamma_{a,t}(s))\frac{\overline{(\dot{\gamma}_{a,t})_I(s)}}{(\dot{\gamma}_{a,t})_I(s)} = \frac{1-k}{1+k} < 0, \quad \text{for all } s \in \left(\frac{3}{2^{\frac{1}{3}}},3\right).$$

6. Now, in order to apply Theorem 6.2.4, we need to find a bigger curve family $\Gamma \supseteq \Gamma_0$ for which ρ_0 is still admissible and such that $\mathrm{Mod}_4(f_k(\Gamma_0)) \leq \mathrm{Mod}_4(f(\Gamma))$ for all $f \in \mathcal{F}_k$. As in the previous proof, a guess for Γ is the family of *all* horizontal curves contained in Ω which are joining the two components $\partial\Omega_{\frac{1}{2}}$ and $\partial\Omega_1$. Using similar arguments as in the previous proof we have

$$\mathrm{Mod}_4(f_k(\Gamma_0)) \leq \mathrm{Mod}_4(f(\Gamma)) \quad \text{for all } f \in \mathcal{F}_k.$$

We have to check that ρ_0 is admissible for the extended family Γ. Indeed, for a curve $\gamma : [c,d] \to \mathcal{AA}$ with $\gamma \in \Gamma$, we have

$$\begin{aligned}\int_\gamma \rho_0 \, d\ell =& c_0 \int_c^d \lambda(s)^{\frac{1}{3}} \frac{\sqrt{\dot{\lambda}(s)^2 + \dot{t}(s)^2}}{2\lambda(s)} \, ds \\ \geq& \frac{c_0}{2} \int_c^d \frac{\dot{\lambda}(s)}{\lambda(s)^{\frac{2}{3}}} \, ds = \frac{3c_0}{2}\left(\lambda(d)^{\frac{1}{3}} - \lambda(c)^{\frac{1}{3}}\right) = 1.\end{aligned}$$

Here we have used for the evaluation of the integral the fact that $s \mapsto \lambda(s)$ is an absolutely continuous function and the conditions $\gamma(c) \in \partial\Omega_{\frac{1}{2}}$, $\gamma(d) \in \partial\Omega_1$.

We conclude that $\rho_0 \in \mathrm{Adm}(\Gamma)$ and, from Theorem 6.2.4, it follows that

$$K_{f_k}^2 \int_\Omega \rho_0(p)^4 \, d\mu_{\mathcal{AA}}(p) \leq \int_\Omega K(p,f)^2 \rho_0(p)^4 \, d\mu_{\mathcal{AA}}(p) \quad \text{for all } f \in \mathcal{F}_k\,.$$

The proof is complete. □

In the same manner as in Corollary 7.1.2, we obtain

Corollary 7.1.4. *The linear stretch map $f_k : \Omega \to \Omega^k$ is an orientation preserving quasiconformal map such that*

$$K_{f_k} \leq K_f,$$

for all $f \in \mathcal{F}_k$.

This case may be viewed as a solution to the Grötzsch problem on the setting of the affine-additive group (see also [2] and [30] for the classical Grötzsch problem on the complex plane, as well as Section 5.2 in [10] for the analogous Grötzsch problem on the Heisenberg group).

7.2 Cylindrical-logarithmic coordinates

In order to construct radial stretch maps it is convenient to set up an appropriate type of coordinate system on the affine-additive group. To this direction, a first step is to consider cylindrical coordinates: recall that $\mathcal{AA}$ identifies to $\mathbb{R} \times \mathbf{H}^1_{\mathbb{C}}$, hence the coordinate map for the cylindrical coordinates is given by $\mathcal{C} : \mathbb{R} \times \mathbb{R}_{>0} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathcal{AA}$ where

$$\mathcal{C}(a, r, \psi) = (a, re^{i\psi}), \quad (a, r, \psi) \in \mathbb{R} \times \mathbb{R}_{>0} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

By applying the transformation $\xi \mapsto e^{\xi} = r > 0$, with $\xi \in \mathbb{R}$, the cylidrical-logarithmic coordinates are defined as $\Phi : \mathbb{R} \times \mathbb{R} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathcal{AA}$ where

$$\Phi(a, \xi, \psi) = \left(a, e^{\xi + i\psi}\right). \tag{7.5}$$

Moreover, the inverse map for this new type of coordinates is explicitly given by

$$\Phi^{-1}(a, \lambda + it) = \left(a, \frac{\log(\lambda^2 + t^2)}{2}, \tan^{-1}\left(\frac{t}{\lambda}\right)\right) \in \mathbb{R} \times \mathbb{R} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad (a, \lambda + it) \in \mathcal{AA}.$$

On the domain

$$\mathbf{A} := \mathbb{R} \times \mathbb{R} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

the map $\Phi : \mathbf{A} \to \mathcal{AA}$ is a smooth diffeomorphism with corresponding Jacobian determinant

$$(\det \Phi_*)_{(a,\xi,\psi)} = e^{2\xi} \neq 0. \tag{7.6}$$

It follows that for each curve $\gamma : [c, d] \to \mathcal{AA}$ and each point $(a, \xi, \psi) = \Phi^{-1}(\gamma(c))$, there exists a unique curve $\tilde{\gamma} = \Phi^{-1} \circ \gamma : [c, d] \to \mathbf{A}$ such that $\tilde{\gamma}(c) = (a, \xi, \psi)$. If γ is absolutely continuous in the Euclidean sense, or if it is C^k for a $k \in \mathbb{N}_0$, then $\tilde{\gamma}$ will be as regular as γ. Further, the same reasoning applies also for continuous mappings from simply connected domains in $\mathcal{AA}$. In detail, every mapping $\tilde{f} : \mathbf{A} \to \mathbf{A}$ yields a well-defined map $f : \mathcal{AA} \to \mathcal{AA}$ by setting $f = \Phi \circ \tilde{f} \circ \Phi^{-1}$.

In what follows, we are going to use only cylindrical-logarithmic coordinates: we will define a quasiconformal map f between domains in the affine-additive group by giving a formula for $\tilde{f}$. On the other hand, we will still work with $\tilde{f}$ in the case where this is convenient.

It turns out that the stretch map has a much neater form in cylindrical-logarithmic coordinates.
In what follows we shall give expression for:

- the contact condition;
- the horizontal vector fields;
- the volume and curve integrals;
- the Beltrami coefficient;
- the MSP condition,

in these particular coordinates. We adopt the notation

$$\tilde{f}(a,\xi,\psi) = (A(a,\xi,\psi), \Xi(a,\xi,\psi), \Psi(a,\xi,\psi))\,.$$

Also, if η is an index running through a, ξ and ψ, we will write $A_\eta = \frac{\partial A}{\partial \eta}$ for a given differentiable function A.

7.2.1 Horizontality, contact condition and minimal stretching property.

In order to apply the modulus method, we will need to understand how the horizontality condition transfers on curves in terms of cylindrical-logarithmic coordinates. The following formulas of horizontality and of line integration for curves in $\mathbf{A}$ are useful.

Proposition 7.2.1. *A curve $\gamma : [c,d] \to \mathcal{AA}$ is horizontal if and only if there exists an absolutely continuous curve*

$$\tilde{\gamma} : [c,d] \to \mathbf{A}, \quad \tilde{\gamma}(s) = (a(s), \xi(s), \psi(s)),$$

with $\Phi \circ \tilde{\gamma} = \gamma$ and

$$\frac{\dot{\psi}(s)}{2} + \frac{\tan\psi(s)}{2}\dot{\xi}(s) - \dot{a}(s) = 0 \quad \textit{for almost every } s \in [c,d]. \tag{7.7}$$

Moreover, for any Borel function $\rho : \mathcal{AA} \to [0, +\infty]$, we have

$$\int_\gamma \rho \, d\ell = \int_c^d \rho(\Phi(\tilde{\gamma}(s))) \frac{\sqrt{\dot{\xi}(s)^2 + \dot{\psi}(s)^2}}{2\cos\psi(s)}\, ds\,. \tag{7.8}$$

Proof. If $\tilde{\gamma} : [c, d] \to \mathbf{A}$ is an absolutely continuous curve satisfying (7.7), then we consider the absolutely continuous curve $\gamma := \Phi \circ \tilde{\gamma}$. Conversely, if $\gamma : [c, d] \to \mathcal{AA}$ is horizontal, we take $\tilde{\gamma} : [c, d] \to \mathbf{A}$ to be $\tilde{\gamma} = \Phi^{-1} \circ \gamma$.
Now, consider two almost everywhere differentiable curves $\gamma : [c, d] \to \mathcal{AA}$ and $\tilde{\gamma} : [c, d] \to \mathbf{A}$ such that $\Phi \circ \tilde{\gamma} = \gamma$ for all $s \in [c, d]$. Let s be a point of differentiability in $[c, d]$. There exists a neighborhood of s where we also have $\Phi \circ \tilde{\gamma} = \gamma$. By applying the latter identity to (1.6) it follows that the condition for a horizontal curve reads as (7.7). Then we obtain:

$$|\dot{\gamma}(s)|_H = \frac{\sqrt{\dot{\xi}(s)^2 + \dot{\psi}(s)^2}}{2 \cos \psi(s)}.$$

For such a horizontal curve $\gamma : [c, d] \to \mathcal{AA}$, the formula for the curve integral follows immediately since $\int_\gamma \rho \, d\ell = \int_c^d \rho(\gamma(s)) |\dot{\gamma}(s)|_H \, ds$. □

Now, we are going to describe the contact form and the contact conditions with respect to the cylindrical-logarithmic coordinates. The cartesian coordinates on $\mathcal{AA}$ can be defined through the diffeomorphism Φ using coordinates (a, ξ, ψ). The expression of the contact form ϑ on $\mathbf{A}$ is

$$\vartheta = \frac{d\psi}{2} + \frac{\tan \psi}{2} d\xi - da \,. \tag{7.9}$$

Proposition 7.2.2. *Let Q be an open set in $\mathbf{A}$ and assume that there exist C^1 maps $\tilde{f} : Q \to \mathbf{A}$ and $f : \Phi(Q) \to \mathcal{AA}$ such that $f = \Phi \circ \tilde{f} \circ \Phi^{-1}$ on Q. Then the following conditions are equivalent:*

(1) the map f is a contact transformation;

(2) there exists a nowhere vanishing function $\tilde{\lambda} : Q \to \mathbb{R}$ such that the map $\tilde{f} = (A, \Xi, \Psi)$ is a C^1 diffeomorphism satisfying the system of p.d.e.s

$$\begin{aligned} &\Psi_\psi + \tan \Psi \, \Xi_\psi - 2A_\psi = \tilde{\lambda} \\ &\Psi_\xi + \tan \Psi \, \Xi_\xi - 2A_\xi = \tilde{\lambda} \tan \psi \\ &2A_a - \Psi_a - \tan \Psi \, \Xi_a = 2\tilde{\lambda} \,. \end{aligned} \tag{7.10}$$

Proof. Since f and $\tilde{f}$ are related by $f = \Phi \circ \tilde{f} \circ \Phi^{-1}$, it is straightfoward to see that f is a C^1 diffeomorphism if and only if $\tilde{f}$ is so.
Now, focusing on the contact conditions, we recall that the map Φ is a diffeomorphism and that the contact form ϑ is given by (7.9). The condition that there exists $\lambda(p) \neq 0$ such that $(f^*\vartheta)_p = \lambda(p)\vartheta_p$ is equivalent to (7.10) with $\tilde{\lambda} = \lambda \circ \Phi$. □

Remark 7.2.3. We wish to underline here that a quasiconformal mapping f is differentiable almost everywhere and contact almost everywhere. Thus the corresponding $\tilde{f}$ satisfies the system of p.d.e.s (7.10) almost everywhere.

Below, we give expressions for the vector fields Z and $\overline{Z}$ in terms of cylindrical-logarithmic coordinates. Straightforward calculations yield

$$Z = e^{-i\psi}\cos\psi(\partial_\xi - i\partial_\psi) - \frac{i}{2}\partial_a\,, \tag{7.11}$$

$$\overline{Z} = e^{i\psi}\cos\psi(\partial_\xi + i\partial_\psi) + \frac{i}{2}\partial_a\,. \tag{7.12}$$

Let f and $\tilde{f} = (A, \Xi, \Psi)$ be C^1 maps as in Proposition 7.2.2. The Beltrami coefficient of f is given by

$$\mu_f(\Phi(a,\xi,\psi)) = \left(\frac{\overline{Z}(\Xi + i\Psi)}{Z(\Xi + i\Psi)}\right)_{|(a,\xi,\psi)}\,. \tag{7.13}$$

Now assume in addition that f is an orientation preserving quasiconformal map. Let $\tilde{\Gamma}$ be a family of C^1 curves

$$\tilde{\gamma} : [a,b] \to \mathbf{A}, \quad \tilde{\gamma}(s) = (a(s), \xi(s), \psi(s))$$

such that

$$\frac{\dot{\psi}(s)}{2} + \frac{\tan\psi(s)}{2}\dot{\xi}(s) - \dot{a}(s) = 0 \quad \text{for all } s \in (a,b),$$

and

$$\frac{\dot{\xi}(s) - i\dot{\psi}(s)}{\dot{\xi}(s) + i\dot{\psi}(s)}\left(\frac{\overline{Z}(\Xi + i\Psi)}{Z(\Xi + i\Psi)}\right)_{|\tilde{\gamma}(s)} < 0, \tag{7.14}$$

for $s \in (a,b)$ with $\mu_f(\Phi(\tilde{\gamma}(s)) \neq 0$. Then f has the MSP for the family $\Gamma = \{\Phi \circ \tilde{\gamma} : \tilde{\gamma} \in \tilde{\Gamma}\}$.

7.3 The radial stretch map

In order to construct an analogue of the radial stretch map $z \mapsto |z|^{k-1}z$ in the setting of $\mathcal{AA}$, we detect a suitable domain where this radial stretch map will be defined. This domain happens to be a truncated cylindrical shell, parallel to the a-axis of $\mathcal{AA}$.
In detail, for $0 < \psi_0 < \frac{\pi}{2}$ and $r_0 > 1$ we define

$$D_{r_0,\psi_0} = \left\{\left(a + \frac{\tan\psi}{2}\xi,\, e^{\xi + i\psi}\right) \in \mathcal{AA} : a \in (0,1),\, \psi \in (0,\psi_0),\, \xi \in (0, \log r_0)\right\}. \tag{7.15}$$

Furthermore, we define the following subsets of $\partial D_{r_0,\psi_0}$:

$$E = \left\{ \left(a, e^{i\psi}\right) \in \mathcal{AA} : a \in (0,1),\ \psi \in (0,\psi_0) \right\}, \tag{7.16}$$

$$F = \left\{ \left(a + \frac{\tan\psi}{2}\log r_0,\ r_0 e^{i\psi}\right) \in \mathcal{AA} : a \in (0,1),\ \psi \in (0,\psi_0) \right\}. \tag{7.17}$$

By varying $a \in (0,1)$ and $\psi \in (0,\psi_0)$, we see that E and F are connected by horizontal curves $\gamma_{a,\psi} : [0, \log r_0] \to D_{r_0,\psi_0}$ given by

$$\gamma_{a,\psi}(s) = \left(a + \frac{\tan\psi}{2}s, e^{s+i\psi}\right).$$

Before we proceed, we shall give a volume formula with respect to the logarithmic-cylindrical coordinates and then apply it to the particular case of D_{r_0,ψ_0}. Let $\Omega \subseteq \mathcal{AA}$ be a measurable set and let $Q \subseteq \mathbf{A}$ be an open set such that its image $\Phi(Q) = \Omega$. Then a function $h : \Omega \to \mathbb{R}$ is integrable if and only if $(h \circ \Phi)|\det \Phi_*|$ is integrable on Q and in this case we have

$$\int_\Omega h(p)\, d\mu_{\mathcal{AA}}(p) = \int_Q \frac{h(\Phi(a,\xi,\psi))}{\cos^2\psi}\, d\mathcal{L}^3(a,\xi,\psi).$$

For every integrable function $h : D_{r_0,\psi_0} \to \mathbb{R}$ we have

$$\int_{D_{r_0,\psi_0}} h(p)\, d\mu_{\mathcal{AA}}(p) = \int_0^{\psi_0}\int_0^{\log r_0}\int_{\frac{\tan\psi}{2}\xi}^{1+\frac{\tan\psi}{2}\xi} \frac{h(\Phi(a,\xi,\psi))}{\cos^2\psi}\, da\, d\xi\, d\psi.$$

D_{r_0,ψ_0} for the case $r_0 = e$, $\psi_0 = \frac{\pi}{4}$ is in the following figure.

7.3.1 Proof of extremality.

In this section we construct the radial stretch map on the affine-additive group. We shall prove the extremality of the radial stretch map with respect to the mean distortion integral and then discuss the properties of the radial stretch map in the remarks.
Let $0 < k < 1$; we start by considering logarithmic-polar coordinates $(\xi,\psi) \in \mathbb{R} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ on $\mathbf{H}^1_{\mathbb{C}}$ with symplectic form $\omega = \frac{d\xi \wedge d\psi}{4\cos^2\psi}$ and, with respect to the same coordinates, we set the 1-form τ on $\mathbf{H}^1_{\mathbb{C}}$ given by $\tau = \frac{d\psi}{2} + \frac{\tan\psi}{2}d\xi$. We introduce the symplectic and planar radial stretch map $g_k : \mathbb{R} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, defined as

$$g_k(\xi,\psi) = \left(k\xi, \tan^{-1}\left(\frac{\tan\psi}{k}\right)\right).$$

Now, let $p = \Phi(a,\xi,\psi) \in \mathcal{AA}$, take γ to be an horizontal path joining the neutral element $e_{\mathcal{AA}} = (0,1,0) \in \mathcal{AA}$ with p and we construct, in cylindrical-logarithmic coordinates,

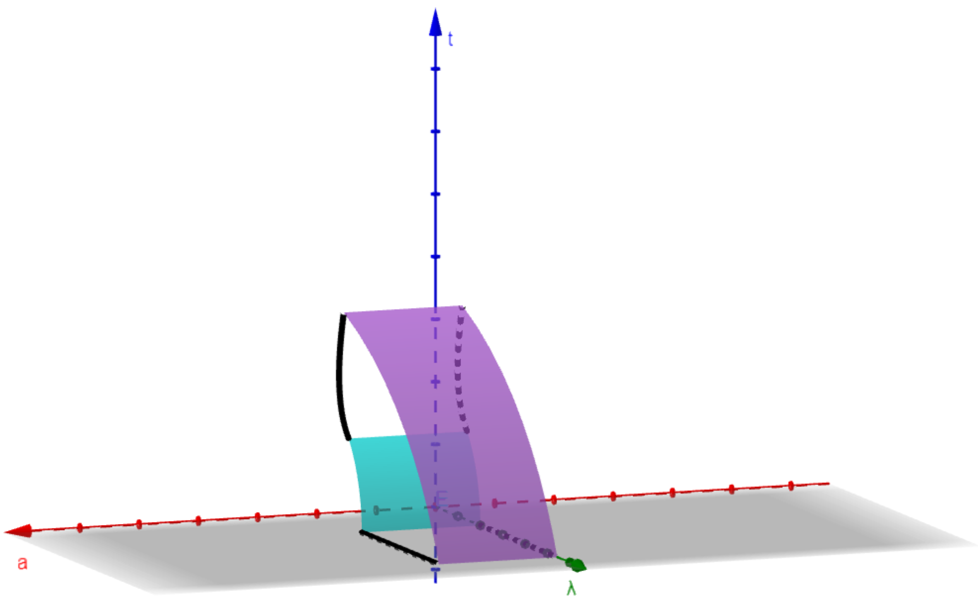

Figure 7.2: Domain $D_{e,\frac{\pi}{4}}$ with E in cyan and F in purple.

$\tilde{f}_k : \mathbf{A} \to \mathbf{A}$, i.e. the lift of g_k as

$$\begin{aligned} \tilde{f}_k(a,\xi,\psi) &= \left(\int_{\pi(\gamma)} g_k^*\tau,\, k\xi,\, \tan^{-1}\left(\frac{\tan\psi}{k}\right)\right) \\ &= \left(a - \frac{\psi}{2} + \frac{1}{2}\tan^{-1}\left(\frac{\tan\psi}{k}\right),\, k\xi,\, \tan^{-1}\left(\frac{\tan\psi}{k}\right)\right). \end{aligned} \tag{7.18}$$

Let $r_0 > 1$, $0 < \psi_0 < \frac{\pi}{2}$ and let also the domain D_{r_0,ψ_0}. Consider another truncated cylindrical shell $D^k_{r_0,\psi_0}$. In cylindrical-logarithmic coordinates those domains are given by

$$\begin{aligned} D_{r_0,\psi_0} &= \left\{\Phi\left(a + \frac{\tan\psi}{2}s, s, \psi\right) \in \mathcal{AA} : a \in (0,1),\, \psi \in (0,\psi_0),\, s \in (0, \log r_0)\right\}, \\ D^k_{r_0,\psi_0} &= \left\{\Phi\left(a + \frac{\tan\psi}{2}s - \frac{\psi}{2} + \frac{1}{2}\tan^{-1}\left(\frac{\tan\psi}{k}\right),\, ks,\, \tan^{-1}\left(\frac{\tan\psi}{k}\right)\right) \in \mathcal{AA} : \right. \\ &\qquad \left. a \in (0,1),\, \psi \in (0,\psi_0),\, s \in (0,\log r_0)\right\}. \end{aligned}$$

Now, we setup a precise boundary condition for a mapping problem. The subsets E and F of $\partial D_{r_0,\psi_0}$ (see (7.16)) are given in cylindrical-logarithmic coordinates by

$$\begin{aligned} E =& \{\Phi\,(a,0,\psi) \in \mathcal{AA} : a \in (0,1), \psi \in (0,\psi_0)\}, \\ F =& \left\{\Phi\left(a + \frac{\tan\psi}{2}, 1, \psi\right) \in \mathcal{AA} : a \in (0,1), \psi \in (0,\psi_0)\right\}, \end{aligned}$$

respectively. Also, we consider the following subsets of $\partial D^k_{r_0,\psi_0}$:

$$\begin{aligned} E^k =& \left\{\Phi\left(a - \frac{\psi}{2} + \frac{1}{2}\tan^{-1}\left(\frac{\tan\psi}{k}\right), 0, \tan^{-1}\left(\frac{\tan\psi}{k}\right)\right) \in \mathcal{AA} :\ a \in (0,1), \psi \in (0,\psi_0)\right\}, \\ F^k =& \left\{\Phi\left(a + \frac{\tan\psi}{2} - \frac{\psi}{2} + \frac{1}{2}\tan^{-1}\left(\frac{\tan\psi}{k}\right), k\log r_0, \tan^{-1}\left(\frac{\tan\psi}{k}\right)\right) \in \mathcal{AA} :\right. \\ & \left. a \in (0,1), \psi \in (0,\psi_0)\right\}. \end{aligned}$$

Denote by $\mathcal{F}_k$ the class of all quasiconformal maps $\overline{D_{r_0,\psi_0}} \to \overline{D^k_{r_0,\psi_0}}$ with the property that they map homeomorphically the component E to E^k and the component F to F^k, respectively. We can now prove the following

Theorem 7.3.1. *The radial stretch map $f_k : \overline{D_{r_0,\psi_0}} \to \overline{D^k_{r_0,\psi_0}}$ is an orientation preserving quasiconformal map. With the above notation for ρ_0, f_k minimizes the mean distortion within the class $\mathcal{F}_k$: for all $f \in \mathcal{F}_k$ we have that*

$$\int_{D_{r_0,\psi_0}} K^2(p,f_k)\rho_0^4(p)\, d\mu_{\mathcal{AA}}(p) \le \int_{D_{r_0,\psi_0}} K^2(p,f)\rho_0^4(p)\, d\mu_{\mathcal{AA}}(p)\,.$$

Proof. We prove first that $f_k : D_{r_0,\psi_0} \to D^k_{r_0,\psi_0}$ is a quasiconformal map. The assumptions of Proposition 7.2.2 are satisfied by the smooth map $\tilde{f}_k : \mathbf{A} \to \mathbf{A}$; thus the stretch map $f_{k\,|D_{r_0,\psi_0}}$ is a smooth contact transformation onto its image. Formula (7.13) yields that

$$\mu_{f_k}(\Phi(a,\xi,\psi)) = e^{2i\psi}\frac{k^2-1}{k^2+2\tan^2\psi+1}, \quad (a,\xi,\psi) \in \mathbf{A},$$

proving $\|\mu_{f_k}\|_\infty < 1$. We have therefore proved that f_k is a smooth orientation preserving quasiconformal map on D_{r_0,ψ_0} with

$$\|\mu_{f_k}\|_\infty = \frac{1-k^2}{1+k^2} < 1 \text{ and } K_{f_k} = \frac{1}{k^2} < \infty.$$

We next prove that

$$\int_{D_{r_0,\psi_0}} K^2(p,f_k)\rho_0(p)^4\, d\mu_{\mathcal{AA}}(p) \le \int_{D_{r_0,\psi_0}} K^2(p,f)\rho_0(p)^4\, d\mu_{\mathcal{AA}}(p),$$

with $\rho_0(a,\lambda,t) = (\log r_0)^{-1}\frac{2\lambda}{|\lambda+it|}$ for all $f \in \mathcal{F}_k$. In order to do so, we once more follow the steps of the proof as in Remark 6.2.5.

1. The class $\mathcal{F}_k$ is as above.
2. Let $\Delta = (0,1)\times(0,\psi_0)$; we define

$$\tilde{\gamma} : (0,\log r_0)\times\Delta \to \mathbf{A}, \quad \tilde{\gamma}(s,a,\psi) = (a(s),\, \xi(s),\, \psi(s)) = \left(a + \frac{\tan\psi}{2}s,\, s,\, \psi\right), \tag{7.19}$$

and

$$\gamma : (0,\log r_0)\times\Delta \to D_{r_0,\psi_0}, \quad \gamma(s,a,\psi) = \Phi(\tilde{\gamma}(s,a,\psi)).$$

The smooth diffeomorphism γ has nowhere vanishing Jacobian determinant $\det\gamma_*(s,a,\psi) = e^{2s}$. Further, for each fixed $(a,\psi)\in\Delta$ the curve

$$\gamma(\cdot,a,\psi) : (0,\log r_0) \to D_{r_0,\psi_0}, \quad s \mapsto \Phi\left(a + \frac{\tan\psi}{2}s, s, \psi\right),$$

is horizontal: indeed, we observe that

$$\frac{\dot{\psi}(s)}{2} + \frac{\tan\psi(s)}{2}\dot{\xi}(s) - \dot{a}(s) = 0, \quad s\in(0,\log r_0),$$

and we use Proposition 7.2.1. Additionally,

$$|\dot{\gamma}(s,a,\psi)|_H = \frac{1}{2\cos\psi} \neq 0 \quad \text{for all } (s,a,\psi)\in(0,\log r_0)\times\Delta.$$

In this way, by introducing $\delta = (a,\psi)\in\Delta$, the volume element on $\mathcal{AA}$ may be written as

$$d\mu_{\mathcal{AA}}(\gamma(s,\delta)) = \frac{1}{\cos^2\psi}\,ds\,da\,d\psi = |\dot{\gamma}(s,a,\psi)|_H^4\,ds\,d\nu(a,\psi),$$

where

$$d\nu(a,\psi) = 2^4\cos^2\psi\,da\,d\psi.$$

3. Our model curve family is

$$\Gamma_0 = \{\gamma(\cdot,a,\psi) : (a,\psi)\in(0,1)\times(0,\psi_0)\}.$$

According to Proposition 6.1.1, an extremal density for Γ_0 is ρ_0 defined by

$$\rho_0(p) = \begin{cases} \log(r_0)^{-1}2\cos\psi, & \text{if } p = \gamma(s,a,\psi)\in D_{r_0,\psi_0}, \\ 0, & \text{if } p\notin D_{r_0,\psi_0}, \end{cases}$$

and also

$$\mathrm{Mod}_4(\Gamma_0) = \left(\frac{2}{\log r_0}\right)^3(\psi_0 + \sin\psi_0\cos\psi_0).$$

4. Since we have

$$K(\gamma(s,a,\psi),f_k) = \frac{1}{k^2\cos^2\psi+\sin^2\psi},\ (s,a,\psi)\in(0,\log r_0)\times\Delta,$$

we notice that the distortion $K(\gamma(s,a,\psi),f_k)$ does not depend on $s\in(0,\log r_0)$, but only on $\psi\in(0,\psi_0)$. This means that the distortion $K(\cdot,f_k)$ is constant along every curve γ in the sense of (6.10).

5. We use the criterion given in (7.14) to verify the MSP for f_k with respect to the curve family Γ_0. We check straightforwardly that

$$\frac{\dot\xi(s)-i\dot\psi(s)}{\dot\xi(s)+i\dot\psi(s)}\left(\frac{\overline{Z}(\Xi+i\Psi)}{Z(\Xi+i\Psi)}\right)_{|\tilde\gamma(s)} = \frac{k^2-1}{k^2+2\tan^2\psi+1}<0,$$

for all $s\in(0,\log r_0)$. This holds true for all k such that $0<k<1$. In this way, due to Proposition 6.2.3, we obtain

$$\begin{aligned}\mathrm{Mod}_4(f_k(\Gamma_0)) =& \frac{2^4}{(\log r_0)^3}\int_0^{\psi_0}\int_0^1 K_{f_k}^2(a,\psi)\cos^2\psi\,da\,d\psi\\ =&\frac{2^4}{(\log r_0)^3}\int_0^{\psi_0}\frac{\cos^2\psi}{(k^2\cos^2\psi+\sin^2\psi)^2}\,d\psi\\ =&\int_{D_{r_0,\psi_0}} K^2(p,f_k)\rho_0^4(p)\,d\mu_{\mathcal{AA}}(p).\end{aligned} \tag{7.20}$$

6. We now define a bigger curve family $\Gamma\supseteq\Gamma_0$ for which ρ_0 is still admissible and such that $\mathrm{Mod}_4(f_k(\Gamma_0))\le\mathrm{Mod}_4(f(\Gamma))$ for all $f\in\mathcal{F}_k$. A typical guess for Γ is the family of *all* absolutely continuous and almost everywhere horizontal curves contained in D_{r_0,ψ_0} which are joining the two components E and F. The boundary conditions for maps in the class $\mathcal{F}_k$ assure us that the image $f_k(\Gamma)$ is going to be a family of the same type in $D^k_{r_0.\psi_0}$. Using the absolute continuity of quasiconformal mappings on almost every curve up to a negligible family of curves with zero 4-modulus and using the boundary conditions, we can show that

$$\mathrm{Mod}_4(f_k(\Gamma_0))\le\mathrm{Mod}_4(f(\Gamma))\quad\text{for all } f\in\mathcal{F}_k. \tag{7.21}$$

Eventually, we have to check that ρ_0 is admissible for the extended family Γ. Observe that from Proposition 7.2.1 it follows that for a curve $\gamma:[c,d]\to\mathcal{AA}$ with $\gamma\in\Gamma$, we have

$$\begin{aligned}\int_\gamma\rho_0\,d\ell =&\frac{1}{\log r_0}\int_c^d\sqrt{\dot\xi(s)^2+\dot\psi(s)^2}\,ds\ge\frac{1}{\log r_0}\int_c^d\dot\xi(s)\,ds\\ =&\frac{1}{\log r_0}(\log r_0-0)=1\,.\end{aligned}$$

Here, to evaluate the integral we have used the fact that $s \mapsto \xi(s)$ is an absolutely continuous function and the conditions $\gamma(c) \in E$, $\gamma(d) \in F$.
We conclude that $\rho_0 \in \mathrm{Adm}(\Gamma)$ and from Theorem 6.2.4 it follows that

$$\int_{D_{r_0,\psi_0}} K^2(p, f_k)\rho_0^4(p)\, d\mu_{\mathcal{AA}}(p) \leq \int_{D_{r_0,\psi_0}} K^2(p, f)\rho_0^4(p)\, d\mu_{\mathcal{AA}}(p) \quad \text{for all } f \in \mathcal{F}_k\,.$$

The proof is complete. □

Remark 7.3.2. It is straightforward to show that the map f_k, $k > 1$ is quasiconformal with $K_{f_k} = k^2$. Indeed, it is enough to recover the same arguments from the first part of the above proof. However, proving extremality in the case $k > 1$ requires a different argument.

Remark 7.3.3. In cartesian coordinates, the map $f_k = \Phi \circ \tilde{f}_k \circ \Phi^{-1} : \mathcal{AA} \to \mathcal{AA}$, is given by

$$f_k(a, \lambda + it) = \left(a - \frac{1}{2}\tan^{-1}\left(\frac{t}{\lambda}\right) + \frac{1}{2}\tan^{-1}\left(\frac{t}{\lambda k}\right), \left(\frac{(\lambda^2+t^2)^k}{\lambda^2 k^2 + t^2}\right)^{\frac{1}{2}} \cdot (\lambda k + it) \right).$$

Remark 7.3.4. By writing the coordinate map $\varphi : \mathbb{R} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbf{H}^1_{\mathbb{C}}$ as $\varphi(\xi, \psi) = e^{\xi + i\psi}$, $(\xi, \psi) \in \mathbb{R} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and setting $\check{f}_k = \varphi \circ g_k \circ \varphi^{-1} : \mathbf{H}^1_{\mathbb{C}} \to \mathbf{H}^1_{\mathbb{C}}$, we notice that the map $f_k : \mathcal{AA} \to \mathcal{AA}$ has the lifting property $\pi \circ f_k = \check{f}_k \circ \pi$.

Remark 7.3.5. Making use of the formal substitution $k = -1$, we obtain that the map $f_{-1} : \mathcal{AA} \to \mathcal{AA}$ given by

$$(a, \lambda + it) \mapsto \left(a - \tan^{-1}\left(\frac{t}{\lambda}\right), \frac{-\lambda + it}{|\lambda + it|^2} \right),$$

is a contactomorphism with $f_{-1}^*\vartheta = \vartheta$ and also a conformal map (1-quasiconformal).

7.4 Open question

In this final section we want to discuss the minimality of f_k for the maximal distortion K_{f_k}, see (5.8). Let $r_0 > 1$, $\psi_0 \in \left(0, \frac{\pi}{2}\right)$ and we make use of the same notation as in the proof of Theorem 7.3.1, reminding that the curve family $\Gamma \supseteq \Gamma_0$ consists of all horizontal curves contained in D_{r_0,ψ_0} which connect the two boundary components E and F of $\partial D_{r_0,\psi_0}$. By coupling the modulus inequality given in (7.21) with the right inequality in (5.17), we obtain the chain of inequalities

$$\frac{\mathrm{Mod}_4(f_k(\Gamma))}{\mathrm{Mod}_4(\Gamma)} \leq \frac{\mathrm{Mod}_4(f(\Gamma))}{\mathrm{Mod}_4(\Gamma)} \leq K_f^2, \quad f \in \mathcal{F}_k. \tag{7.22}$$

Now, notice that if we had

$$K_{f_k}^2 = \frac{\mathrm{Mod}_4(f_k(\Gamma))}{\mathrm{Mod}_4(\Gamma)}, \tag{7.23}$$

we would conclude

$$K_{f_k} \le K_f, \quad f \in \mathcal{F}_k.$$

On the other hand, this is not the case because (7.23) does not hold for all $\psi_0 \in \left(0, \frac{\pi}{2}\right)$. To this end, we recall that the density ρ_0 is still admissible for the larger family $\Gamma \supseteq \Gamma_0$. This gives the modulus identity $\mathrm{Mod}_4(\Gamma) = \mathrm{Mod}_4(\Gamma_0)$ and thus

$$\mathrm{Mod}_4(\Gamma) = \left(\frac{2}{\log r_0}\right)^3 (\psi_0 + \sin\psi_0 \cos\psi_0).$$

To ρ_0 we can assign a pushforward density $f_{k\#}\rho_0$ given by

$$f_{k\#}\rho_0(q) = \begin{cases} \frac{2(f_k)_2(f_k^{-1}(q))}{|Z(f_k)_I(f_k^{-1}(q))| - |\overline{Z}(f_k)_I(f_k^{-1}(q))|}\rho_0(f_k^{-1}(q)), & \text{if} \quad q \in D^k_{r_0,\psi_0}, \\ 0, & \text{if} \quad q \notin D^k_{r_0,\psi_0}. \end{cases}$$

Based on the proof of Theorem 7.3.1, it is straightforward to see that $f_{k\#}\rho_0 \in \mathrm{Adm}(f_k(\Gamma))$, giving an analogous modulus identity $\mathrm{Mod}_4(f_k(\Gamma)) = \mathrm{Mod}_4(f_k(\Gamma_0))$. We refer to (7.20) and this gives

$$\mathrm{Mod}_4(f_k(\Gamma)) = \left(\frac{2}{\log r_0}\right)^3 k^{-3} \left(\frac{k \sin 2\psi_0}{1 + k^2 + (k^2 - 1)\cos 2\psi_0} + \tan^{-1}\left(\frac{\tan\psi_0}{k}\right)\right).$$

For $\psi_0 \in (\frac{\pi}{4}, \frac{\pi}{2})$ we observe that the function

$$k \mapsto \frac{k^{\frac{3}{2}} \sin 2\psi_0}{1 + k^2 + (k^2 - 1)\cos 2\psi_0} + k^{\frac{1}{2}} \tan^{-1}\left(\frac{\tan\psi_0}{k}\right), \quad k \in (0, 1),$$

is monotone increasing (by a direct calculation the derivative is positive for $k \in (0, 1)$) and thus bounded from above by $\psi_0 + \sin\psi_0 \cos\psi_0$. Therefore

$$\frac{\mathrm{Mod}_4(f_k(\Gamma))}{\mathrm{Mod}_4(\Gamma)} \le k^{-\frac{7}{2}} < k^{-4} = K_{f_k}^2.$$

In the case $\psi_0 \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ we see that equality does not necessarily hold for the stretch map f_k and the curve family Γ. Despite the latter inequality holds strictly it could still be true that f_k is minimal for the maximal distortion and it is an open question.

Chapter 8

Riemannian approximation scheme

In this chapter we make use of a Riemannian approximation scheme to provide notions of horizontal mean curvature and of intrinsic Gaussian curvature for an Euclidean C^2-smooth surface in the affine-additive group $\mathcal{AA}$ away from characteristic points. Our approach will combine such Riemannian approximation scheme with Cartan's formalism, see Clelland's book [20]. Consider a surface Σ embedded in $\mathcal{AA}$, by means of Cartan's structural equations we derive formulae for the sectional curvature $\overline{K}^{\epsilon}_{\Sigma}$ and for the second fundamental form $\mathbf{II}^{\epsilon}_{\mathbf{\Sigma}}$. By studying the limit case we provide formulae for the horizontal mean curvature H^0 and the intrinsic Gaussian curvature K^0.

8.1 The method of moving frames

Before looking at the Riemannian approximation scheme we need to consider the differential geometry setup of $\mathcal{AA}$ as a Riemmannian manifold with a complex structure on the horizontal sub-bundle of the tangent bundle $\mathcal{T}(\mathcal{AA})$.
Let ϑ be the contact form of $\mathcal{AA}$ given by (1.4), we recall that the horizontal bundle $\mathcal{H}_{\mathcal{AA}}$ is given by

$$\mathcal{H}_{\mathcal{AA}} = \text{span}\{U, V\},$$

where U, V as in Proposition 1.3.1. We denote with $\langle \cdot, \cdot \rangle_{\mathcal{AA}}$ the sub-Riemannian metric defined on $\mathcal{H}_{\mathcal{AA}}$ and we associate the corresponding Carnot-Carathéodory distance $d_{\mathcal{AA}}$ (see Chapter 1 for more details). We define a complex structure $J : \mathcal{H}_{\mathcal{AA}} \to \mathcal{H}_{\mathcal{AA}}$ by letting

$$J(U) = V, \quad J(V) = -U.$$

Again according to Proposition 1.3.1 we denote with W the Reeb vector field.
Later on we will need the following (Proposition 12.17 in Lee's book [41]):

Proposition 8.1.1. *For any differential* 1*-form* ω *and vector fields* X_1*,* X_2 *it holds*

$$d\omega(X_1, X_2) = X_1(\omega(X_2)) - X_2(\omega(X_1)) - \omega([X_1, X_2]).$$

We are now ready to implement the Riemannian approximation scheme. First, let us define $W^\epsilon = \epsilon W$, for $\epsilon > 0$. We consider a family of Riemannian metrics $(g_\epsilon)_{\epsilon>0}$ on $\mathcal{AA}$ such that $\{U, V, W^\epsilon\}$ becomes an orthonormal basis. The choice of this specific family of Riemannian metrics on $\mathcal{AA}$ is indicated by the following theorem

Theorem 8.1.2. *The family of metric spaces* $(\mathcal{AA}, g_\epsilon)$ *converges to* $(\mathcal{AA}, d_{\mathcal{AA}})$ *in the pointed Gromov–Hausdorff sense as* $\epsilon \to 0^+$*.*

The result holds in the wider class of Carnot-Carathédory spaces, we refer to Chapter 2 in Gromov's book [29] or Chapter 1 in Monti's PhD thesis [46].
An *affine connection* is a basic concept in Riemannian geometry which serves as a useful computational tool.

Definition 8.1.3. Let $\mathcal{X}(M)$ be the set of C^∞-smooth vector fields on a manifold M. Let $D(M)$ be the ring of real-valued C^∞-smooth functions on M. An affine connection ∇ on M is a mapping

$$\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$$

which is denoted with $(X, Y) \mapsto \nabla_X Y$ and which satisfies the following properties:

i) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$;

ii) $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$;

iii) $\nabla_X(fY) = f\nabla_X Y + X(f)Y$;

in which $X, Y, Z \in \mathcal{X}(M)$ and $f, g \in D(M)$.

It is well known that every Riemannian manifold is equipped with a privileged affine connection: the Levi-Civita connection ∇ (see Theorem 3.6, Chapter 2 in do Carmo's book [23]). This is the unique affine connection which is compatible with the given Riemannian metric and symmetric. Let $\epsilon > 0$, we explain in detail such aforementioned properties for the case of $(\mathcal{AA}, g_\epsilon)$:

$$Xg_\epsilon(Y, Z) = g_\epsilon(\nabla_X Y, Z) + g_\epsilon(Y, \nabla_X Z)$$

and

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

for all $X, Y, Z \in \mathcal{X}(\mathcal{AA})$.
Continuing with notation, we consider vector fields X, Y and write them with respect to the orthonormal basis $\{U, V, W^\epsilon\}$ as

$$X = x_1U + x_2V + x_3^\epsilon W^\epsilon, \quad Y = y_1U + y_2V + y_3^\epsilon W^\epsilon,$$

thus obtaining

$$g_\epsilon(X, Y) = x_1y_1 + x_2y_2 + x_3^\epsilon y_3^\epsilon.$$

Let us fix once for all the assumptions we will make on the surface Σ through this whole chapter. We will say that a surface $\Sigma \subset (\mathcal{AA}, g_\epsilon)$ is *regular* if

$$\Sigma \text{ is a Euclidean } C^2\text{-smooth compact and oriented surface.} \tag{8.1}$$

In particular we will assume that there exists a C^2-smooth function $u : \mathcal{AA} \to \mathbb{R}$ such that

$$\Sigma = \{(a, \lambda, t) \in \mathcal{AA} : u(a, \lambda, t) = 0\}$$

and also assume that the standard gradient satisfies the condition $\nabla_{\mathbb{R}^3} u \neq 0$. We say that a point $p \in \Sigma$ is called *characteristic* if

$$\nabla_H u(a, \lambda, t) := (Uu, Vu)_{|(a,\lambda,t)} = (0, 0). \tag{8.2}$$

Our study will be local and away from characteristic points of Σ. We may also define the *characteristic set* of Σ as

$$\mathcal{C}(\Sigma) = \{(a, \lambda, t) \in \Sigma : \nabla_H u(a, \lambda, t) = (0, 0)\}. \tag{8.3}$$

We follow the notation adopted both in the book of Capogna, Danielli, Pauls and Tyson [17] and in the work of Balogh, Tyson and Vecchi [13], and we define first

$$p = Uu, \quad q = Vu, \quad r = W^\epsilon u.$$

We then define

$$\begin{aligned} l = \|\nabla_H u\| := \sqrt{(Uu)^2 + (Vu)^2}, \quad \overline{p} = \frac{p}{l}, \quad \overline{q} = \frac{q}{l}, \quad \overline{r} = \frac{r}{l} \\ l_\epsilon = \sqrt{(Uu)^2 + (Vu)^2 + (W^\epsilon u)^2}, \quad \overline{r_\epsilon} = \frac{r}{l_\epsilon}, \\ \overline{p_\epsilon} = \frac{p}{l_\epsilon} \quad \text{and} \quad \overline{q_\epsilon} = \frac{q}{l_\epsilon}. \end{aligned} \tag{8.4}$$

In particular $\overline{p}^2 + \overline{q}^2 = 1$. By looking at (8.2) it is straightforward that all the functions in (8.4) are well defined on $\Sigma \setminus \mathcal{C}(\Sigma)$.

Definition 8.1.4. Let $\Sigma \subset (\mathcal{AA}, g_\epsilon)$ be a regular surface and let $u : \mathcal{AA} \to \mathbb{R}$ be as above. The Riemannian unit normal n_Σ to Σ is

$$n_\Sigma = \frac{\nabla_\epsilon u}{\|\nabla_\epsilon u\|} = \overline{p_\epsilon}\, U + \overline{q_\epsilon}\, V + \overline{r_\epsilon}\, W^\epsilon,$$

where $\nabla_\epsilon u = (Uu)U + (Vu)V + (W^\epsilon u)W^\epsilon$ is the Riemannian gradient of u.

Definition 8.1.5. Let $\Sigma \subset (\mathcal{AA}, g_\epsilon)$ be a regular surface and let $u : \mathcal{AA} \to \mathbb{R}$ be as above. We introduce the *moving frame* comprising the ordered vector fields $\{E_1, E_2, n_\Sigma\}$, where E_1, E_2 are given by

$$E_1 = -\overline{q}\, U + \overline{p}\, V,$$

and

$$E_2 = \frac{l}{l_\epsilon}(\overline{r}\, \overline{p}\, U + \overline{r}\, \overline{q}\, V - W^\epsilon).$$

For every point $(a, \lambda, t) \in \Sigma$ we observe that $\{(E_1)_{|(a,\lambda,t)}, (E_2)_{|(a,\lambda,t)}\}$ is an orthonormal basis for the tangent plane $T_{(a,\lambda,t)}\Sigma$. We also notice that for every point $(a, \lambda, t) \in \Sigma$ the normal vector $(n_\Sigma)_{|(a,\lambda,t)}$ is orthogonal to $T_{(a,\lambda,t)}\Sigma$. Later on we will make use of the two following

Lemma 8.1.6. *For $\overline{p}$, $\overline{q}$, l_ϵ and $\overline{r_\epsilon}$ as above, we have*

$$l_\epsilon \to \|\nabla_H u\|, \quad \text{as } \epsilon \to 0^+, \tag{8.5}$$

$$\overline{r_\epsilon} \to 0, \quad \text{as } \epsilon \to 0^+, \tag{8.6}$$

$$\frac{\overline{r_\epsilon}}{l_\epsilon} \to 0, \quad \text{as } \epsilon \to 0^+, \tag{8.7}$$

$$\frac{\overline{r_\epsilon}}{\epsilon\, l_\epsilon} \to \frac{Wu}{\|\nabla_H u\|^2}, \quad \text{as } \epsilon \to 0^+, \tag{8.8}$$

$$\left(\frac{\overline{r_\epsilon}}{\epsilon}\right)^2 \to \frac{(Wu)^2}{\|\nabla_H u\|^2}, \quad \text{as } \epsilon \to 0^+, \tag{8.9}$$

$$\frac{\overline{r_\epsilon}}{\epsilon^2} \sim \frac{Wu}{\epsilon\|\nabla_H u\|}, \quad \text{as } \epsilon \to 0^+. \tag{8.10}$$

Proof. All the limits and the asymptotic follow directly from the definitions in (8.4). □

Lemma 8.1.7. *Let us define*

$$H^H = U(\overline{p}) + V(\overline{q}), \quad Q^H = U(\overline{q}) - V(\overline{p}),$$

then the following identities hold:

$$\begin{aligned} n_\Sigma(\overline{p}) &= (l/l_\epsilon)(-\overline{q}\,Q^H + \overline{r}\,W^\epsilon(\overline{p})), \\ n_\Sigma(\overline{q}) &= (l/l_\epsilon)(\overline{p}\,Q^H + \overline{r}\,W^\epsilon(\overline{q})), \\ E_1(\overline{p}) &= -\overline{q}\,H^H, \\ E_1(\overline{q}) &= \overline{p}\,H^H, \\ E_2(\overline{p}) &= (l/l_\epsilon)(-\overline{r}\,\overline{q}\,Q^H - W^\epsilon(\overline{p})), \\ E_2(\overline{q}) &= (l/l_\epsilon)(\overline{r}\,\overline{p}\,Q^H - W^\epsilon(\overline{q})). \end{aligned}$$

Proof. Applying U or V to the identity $\overline{p}^2 + \overline{q}^2 = 1$ we get

$$\overline{p}\,U(\overline{p}) = -\overline{q}\,U(\overline{q}) \quad \text{and} \quad \overline{p}\,V(\overline{p}) = -\overline{q}\,V(\overline{q}).$$

Now, by making use of the latter two relations, one recovers the identities from the statement through straightforward verifications. □

Let us consider three differential 1-forms given by

$$\omega_1 = \frac{dt}{2\lambda},\ \omega_2 = \frac{d\lambda}{2\lambda},\ \vartheta_\epsilon = \frac{\vartheta}{\epsilon},$$

we define the *moving coframe* as the ordered differential 1-forms $\{\alpha_1, \alpha_2, \alpha_\Sigma\}$ given by

$$\begin{aligned} \alpha_1 &= -\overline{q}\,\omega_1 + \overline{p}\,\omega_2, \\ \alpha_2 &= (l/l_\epsilon)(\overline{r}\,\overline{p}\,\omega_1 + \overline{r}\,\overline{q}\,\omega_2 - \theta_\epsilon), \\ \alpha_\Sigma &= (l/l_\epsilon)\left(\overline{p}\,\omega_1 + \overline{q}\,\omega_2 + \overline{r}\,\theta_\epsilon\right), \end{aligned} \tag{8.11}$$

We observe that the moving frame $\{E_1, E_2, n_\Sigma\}$ is related to the moving coframe $\{\alpha_1, \alpha_2, \alpha_\Sigma\}$ via the duality relations:

$$\begin{aligned} &\alpha_1(E_1) = 1,\ \alpha_1(E_2) = 0,\ \alpha_1(n_\Sigma) = 0, \\ &\alpha_2(E_1) = 0,\ \alpha_2(E_2) = 1,\ \alpha_2(n_\Sigma) = 0, \\ &\alpha_\Sigma(E_1) = 0,\ \alpha_\Sigma(E_2) = 0,\ \alpha_\Sigma(n_\Sigma) = 1. \end{aligned}$$

A direct calculation leads to the following

Proposition 8.1.8. *Let ω_1, ω_2, ϑ_ϵ, α_Σ, α_1 and α_2 be as above, then it holds:*

$$\begin{aligned} \omega_1 &= (l/l_\epsilon)\overline{p}\alpha_\Sigma - \overline{q}\alpha_1 + (l/l_\epsilon)\overline{r}\,\overline{p}\,\alpha_2, \\ \omega_2 &= (l/l_\epsilon)\overline{q}\alpha_\Sigma + \overline{p}\alpha_1 + (l/l_\epsilon)\overline{r}\,\overline{q}\,\alpha_2, \\ \theta_\epsilon &= (l/l_\epsilon)(\overline{r}\alpha_\Sigma - \alpha_2). \end{aligned}$$

Moreover,

$$
\begin{aligned}
\omega_1 \wedge \omega_2 &= (l/l_\epsilon)(\alpha_\Sigma \wedge \alpha_1 - \overline{r}\alpha_1 \wedge \alpha_2), \\
\omega_1 \wedge \theta_\epsilon &= (l/l_\epsilon)\overline{q}\,\overline{r}\,\alpha_\Sigma \wedge \alpha_1 - \overline{p}\,\alpha_\Sigma \wedge \alpha_2 + (l/l_\epsilon)\,\overline{q}\,\alpha_1 \wedge \alpha_2, \\
\omega_2 \wedge \theta_\epsilon &= -(l/l_\epsilon)\overline{p}\,\overline{r}\,\alpha_\Sigma \wedge \alpha_1 - \overline{q}\,\alpha_\Sigma \wedge \alpha_2 - (l/l_\epsilon)\,\overline{p}\,\alpha_1 \wedge \alpha_2
\end{aligned}
$$

and

$$
\begin{aligned}
d\omega_1 &= (2l/l_\epsilon)\,(\alpha_\Sigma \wedge \alpha_1 - \overline{r}\,\alpha_1 \wedge \alpha_2), \\
d\omega_2 &= 0, \\
d\theta_\epsilon &= (2l/(\epsilon l_\epsilon))(\alpha_\Sigma \wedge \alpha_1 - \overline{r}\alpha_1 \wedge \alpha_2).
\end{aligned}
$$

Proposition 8.1.9. *Let α_1, α_2 and α_Σ be as above, then the differential 2-forms $d\alpha_1$, $d\alpha_2$ and $d\alpha_\Sigma$ are given by:*

$$
\begin{aligned}
d\alpha_1 &= A_1\,\alpha_\Sigma \wedge \alpha_1 + A_2\,\alpha_\Sigma \wedge \alpha_2 + A_3\,\alpha_1 \wedge \alpha_2, \\
d\alpha_2 &= B_1\,\alpha_\Sigma \wedge \alpha_1 + B_2\,\alpha_\Sigma \wedge \alpha_2 + B_3\,\alpha_1 \wedge \alpha_2, \\
d\alpha_\Sigma &= C_1\,\alpha_\Sigma \wedge \alpha_1 + C_2\,\alpha_\Sigma \wedge \alpha_2,
\end{aligned}
$$

where

$$
\begin{aligned}
A_1 &= (l/l_\epsilon)\left(H^H - 2\,\overline{q}\right), && (8.12) \\
A_2 &= \overline{q}\,W^\epsilon(\overline{p}) - \overline{p}\,W^\epsilon(\overline{q}), && (8.13) \\
A_3 &= (l/l_\epsilon)\left(-\overline{r}(H^H - 2\,\overline{q})\right), && (8.14) \\
B_1 &= \overline{r}E_1(\log(l_\epsilon/l)) - E_1(\overline{r}) - 2/\epsilon, && (8.15) \\
B_2 &= n_\Sigma(\log(l_\epsilon/l)) - \overline{r}E_2(\log(l_\epsilon/l)) - E_2(\overline{r}), && (8.16) \\
B_3 &= E_1(\log(l_\epsilon/l)) + \overline{r}(2/\epsilon), && (8.17) \\
C_1 &= E_1(\log l_\epsilon), && (8.18) \\
C_2 &= E_2(\log l_\epsilon). && (8.19)
\end{aligned}
$$

Proof. Since $\alpha_\Sigma = du/l_\epsilon$, we have

$$
\begin{aligned}
d\alpha_\Sigma = -(1/l_\epsilon^2)dl_\epsilon \wedge du &= \alpha_\Sigma \wedge d(\log l_\epsilon) \\
&= E_1(\log l_\epsilon)\,\alpha_\Sigma \wedge \alpha_1 + E_2(\log l_\epsilon)\,\alpha_\Sigma \wedge \alpha_2.
\end{aligned}
$$

Next,

$$
\begin{aligned}
d\alpha_1 &= -d\overline{q}\wedge\omega_1 + d\overline{p}\wedge\omega_2 - \overline{q}\,d\omega_1 + \overline{p}\,d\omega_2 \\
&= -\left(n_\Sigma(\overline{q})\alpha_\Sigma + E_1(\overline{q})\alpha_1 + E_2(\overline{q})\alpha_2\right)\wedge\left((l/l_\epsilon)\overline{p}\alpha_\Sigma - \overline{q}\alpha_1 + (l/l_\epsilon)\overline{r}\,\overline{p}\alpha_2\right) \\
&\quad + \left(n_\Sigma(\overline{p})\alpha_\Sigma + E_1(\overline{p})\alpha_1 + E_2(\overline{p})\alpha_2\right)\wedge\left((l/l_\epsilon)\overline{q}\alpha_\Sigma + \overline{p}\alpha_1 + (l/l_\epsilon)\overline{r}\,\overline{q}\alpha_2\right) \\
&\quad - \overline{q}(2l/l_\epsilon)\left(\alpha_\Sigma\wedge\alpha_1 - \overline{r}\,\alpha_1\wedge\alpha_2\right).
\end{aligned}
$$

By using the identities from Lemma 8.1.7 we proceed as follows: the component which multiplies to $\alpha_\Sigma\wedge\alpha_1$ is

$$
\begin{aligned}
A_1 &= \overline{q}n_\Sigma(\overline{q}) + \overline{p}n_\Sigma(\overline{p}) + (l/l_\epsilon)(\overline{p}E_1(\overline{q}) - \overline{q}E_1(\overline{p})) + (l/l_\epsilon)(-2\,\overline{q}) \\
&= (l/l_\epsilon)\left(H^H - 2\,\overline{q}\right).
\end{aligned}
$$

Similarly, the component which multiplies to $\alpha_\Sigma\wedge\alpha_2$ is

$$
\begin{aligned}
A_2 &= (l/l_\epsilon)\left(-\overline{r}\,\overline{p}n_\Sigma(\overline{q}) + \overline{r}\,\overline{q}n_\Sigma(\overline{p}) + \overline{p}E_2(\overline{q}) - \overline{q}E_2(\overline{p})\right) \\
&= \overline{q}\,W^\epsilon(\overline{p}) - \overline{p}\,W^\epsilon(\overline{q}).
\end{aligned}
$$

Finally, the component of $\alpha_1\wedge\alpha_2$ is

$$
\begin{aligned}
A_3 &= (l/l_\epsilon)\left(\overline{r}(\overline{q}E_1(\overline{p}) - \overline{p}E_1(\overline{q}) + 2\,\overline{q})\right) - \overline{q}E_2(\overline{q}) - \overline{p}E_2(\overline{p}) \\
&= (l/l_\epsilon)\left(-\overline{r}(H^H - 2\,\overline{q})\right).
\end{aligned}
$$

The formula for $d\alpha_1$ then follows. Finally, to calculate $d\alpha_2$, we observe first that

$$
\alpha_2 = \overline{r}\alpha_\Sigma - \frac{l_\epsilon}{l}\theta_\epsilon,
$$

hence

$$
\begin{aligned}
d\alpha_2 &= d\overline{r}\wedge\alpha_\Sigma + \overline{r}\,d\alpha_\Sigma - d(l_\epsilon/l)\wedge\theta_\epsilon - (l_\epsilon/l)d\theta_\epsilon \\
&= \left(n_\Sigma(\overline{r})\,\alpha_\Sigma + E_1(\overline{r})\,\alpha_1 + E_2(\overline{r})\,\alpha_2\right)\wedge\alpha_\Sigma \\
&\quad + \overline{r}E_1(\log l_\epsilon)\,\alpha_\Sigma\wedge\alpha_1 + \overline{r}E_2(\log l_\epsilon)\,\alpha_\Sigma\wedge\alpha_2 \\
&\quad - \left(n_\Sigma(l_\epsilon/l)\,\alpha_\Sigma + E_1(l_\epsilon/l)\,\alpha_1 + E_2(l_\epsilon/l)\,a_2\right)\wedge\left((l/l_\epsilon)\overline{r}\alpha_\Sigma - (l/l_\epsilon)\,\alpha_2\right) \\
&\quad - (2/\epsilon)(\alpha_\Sigma\wedge\alpha_1 - \overline{r}\alpha_1\wedge\alpha_2) \\
&= -E_1(\overline{r})\,\alpha_\Sigma\wedge\alpha_1 - E_2(\overline{r})\,\alpha_\Sigma\wedge\alpha_2 \\
&\quad + \overline{r}E_1(\log l_\epsilon)\,\alpha_\Sigma\wedge\alpha_1 + \overline{r}E_2(\log l_\epsilon)\,\alpha_\Sigma\wedge\alpha_2 \\
&\quad + \overline{r}E_1(\log(l_\epsilon/l))\,\alpha_\Sigma\wedge\alpha_1 + \left(n_\Sigma(\log(l_\epsilon/l)) + \overline{r}E_2(\log(l_\epsilon/l))\right)\,\alpha_\Sigma\wedge\alpha_2 + E_1(\log(l_\epsilon/l))\,\alpha_1\wedge\alpha_2 \\
&\quad - (2/\epsilon)(\alpha_\Sigma\wedge\alpha_1 - \overline{r}\alpha_1\wedge\alpha_2)
\end{aligned}
$$

and thus our formula for $d\alpha_2$. □

Corollary 8.1.10. *The Lie brackets* $[n_\Sigma, E_i]$, $i = 1, 2$, *and* $[E_1, E_2]$, *are given by*

$$\begin{aligned}
[n_\Sigma, E_1] &= -C_1 n_\Sigma - A_1 E_1 - B_1 E_2, \\
[n_\Sigma, E_2] &= -C_2 n_\Sigma - A_2 E_1 - B_2 E_2, \\
[E_1, E_2] &= -A_3 E_1 - B_3 E_2.
\end{aligned}$$

Proof. We will only calculate $[E_1, E_2]$, as the other two identities are analogous. We have

$$[E_1, E_2] = \alpha_\Sigma([E_1, E_2])\, n_\Sigma + \alpha_1([E_1, E_2])\, E_1 + \alpha_2([E_1, E_2])\, E_2.$$

On the other hand Proposition 8.1.1 yields to

$$\begin{aligned}
d\alpha_\Sigma(E_1, E_2) &= -\alpha_\Sigma([E_1, E_2]), \\
d\alpha_1(E_1, E_2) &= -\alpha_1([E_1, E_2]), \\
d\alpha_2(E_1, E_2) &= -\alpha_2([E_1, E_2]),
\end{aligned}$$

and our formula follows. □

In order to proceed with the next results we need the following

Definition 8.1.11. Let us denote by ∇ the Levi-Civita connection of $(\mathcal{AA}, g_\epsilon)$ defined and let us consider the moving frame $\{E_1, E_2, n_\Sigma\}$. Let $j \in \{1, 2\}$, $k \in \{1, 2, 3\}$ and let $X_k \in \{E_1, E_2, n_\Sigma\}$, we define the *connection* 1-*forms* η_j^k to be the skew-symmetric differential 1-forms such that

$$\eta_j^k(E_i) = -g_\epsilon(\nabla_{E_i} X_k, E_j), \quad i \in 1, 2.$$

Now we formulate a statement known as Cartan's first structural equation, see Chapter 12.6 in Clelland's book [20].

Proposition 8.1.12. *Let* α_1, α_2 *and* α_Σ *be defined as above, let* $j \in \{1, 2\}$, $k \in \{1, 2, 3\}$ *and let* η_j^k *be the connection* 1-*forms. Then it holds*

$$\begin{aligned}
d\alpha_1 &= \eta_1^2 \wedge \alpha_2 + \eta_1^3 \wedge \alpha_\Sigma, \\
d\alpha_2 &= -\eta_1^2 \wedge \alpha_1 + \eta_2^3 \wedge \alpha_\Sigma, \\
d\alpha_\Sigma &= -\eta_1^3 \wedge \alpha_1 - \eta_2^3 \wedge \alpha_2.
\end{aligned}$$

By writing the connection 1-forms η_j^k with respect to the moving coframe $\{\alpha_1, \alpha_2, \alpha_\Sigma\}$ we have

$$\begin{aligned}
\eta_1^2 &= (\eta_1^2)_1\, \alpha_1 + (\eta_1^2)_2\, \alpha_2 + (\eta_1^2)_3\, \alpha_\Sigma \\
\eta_1^3 &= (\eta_1^3)_1\, \alpha_1 + (\eta_1^3)_2\, \alpha_2 + (\eta_1^3)_3\, \alpha_\Sigma \\
\eta_2^3 &= (\eta_2^3)_1\, \alpha_1 + (\eta_2^3)_2\, \alpha_2 + (\eta_2^3)_3\, \alpha_\Sigma,
\end{aligned}$$

then combining Propositions 8.1.9 and 8.1.12 we get

$$\begin{aligned}
&(\eta_1^2)_1 = A_3, \quad (\eta_1^2)_3 - (\eta_1^3)_2 = A_2, \quad (\eta_1^3)_1 = -A_1, \\
&(\eta_1^2)_3 + (\eta_2^3)_1 = -B_1, \quad (\eta_2^3)_2 = -B_2, \quad (\eta_1^2)_2 = B_3, \\
&(\eta_1^3)_3 = -C_1, \quad (\eta_2^3)_3 = -C_2, \quad (\eta_1^3)_2 - (\eta_2^3)_1 = 0.
\end{aligned}$$

The latter relations result in

$$\begin{aligned}
\eta_1^2 &= A_3\, \alpha_1 + B_3\, \alpha_2 + \frac{A_2 - B_1}{2}\, \alpha_\Sigma, \\
\eta_1^3 &= -A_1\, \alpha_1 + \frac{-A_2 - B_1}{2}\, \alpha_2 - C_1\, \alpha_\Sigma, \\
\eta_2^3 &= \frac{-A_2 - B_1}{2}\, \alpha_1 - B_2\, \alpha_2 - C_2\, \alpha_\Sigma.
\end{aligned} \tag{8.20}$$

We are in the position to formulate Cartan's second structural equation (see again [20]) with the following

Proposition 8.1.13. *Let η_1^2, η_1^3 and η_2^3 be the connection 1-forms. Then the sectional curvatures are given by*

$$\begin{aligned}
\overline{K}^\epsilon(E_1, E_2) &= \eta_2^3 \wedge \eta_1^3(E_1, E_2) - d\eta_1^2(E_1, E_2), \\
\overline{K}^\epsilon(E_1, n_\Sigma) &= -d\eta_1^3(E_1, n_\Sigma) - \eta_2^3 \wedge \eta_1^2(E_1, n_\Sigma), \\
\overline{K}^\epsilon(E_2, n_\Sigma) &= \eta_1^3 \wedge \eta_1^2(E_2, n_\Sigma) - d\eta_2^3(E_2, n_\Sigma).
\end{aligned}$$

We apply the previous Proposition to the specific connection forms given by (8.20) and write the sectional curvatures explicitly with the following

Proposition 8.1.14. *Let $\Sigma \subset (\mathcal{AA}, g_\epsilon)$ be a regular surface and let $u : \mathcal{AA} \to \mathbb{R}$ as above. Let $\{E_1, E_2, n_\Sigma\}$ be the associated moving frame and let A_1, A_2, A_3, B_1, B_2, B_3, C_1 and C_2 be given by (8.12). Then the sectional curvatures are given by:*

$$\begin{aligned}
\overline{K}^\epsilon(E_1, E_2) &= -E_1(B_3) + E_2(A_3) - A_3^2 - B_3^2 + \frac{(A_2 + B_1)^2}{4} - A_1 B_2, \\
\overline{K}^\epsilon(E_1, n_\Sigma) &= E_1(C_1) - n_\Sigma(A_1) - A_1^2 - C_1^2 + \frac{(A_2 + B_1)^2}{4} + C_2 A_3, \\
\overline{K}^\epsilon(E_2, n_\Sigma) &= E_2(C_2) - n_\Sigma(B_2) - B_2^2 - C_2^2 + \frac{(A_2 + B_1)^2}{4} + C_1 B_3.
\end{aligned}$$

At this point we define the second fundamental form $\mathbf{II}^\epsilon$ of the embedding of Σ into $(\mathcal{AA}, g_\epsilon)$:

$$\mathbf{II}^\epsilon = \begin{pmatrix} -g_\epsilon(\nabla^\epsilon_{E_1} n_\Sigma, E_1) & -g_\epsilon(\nabla^\epsilon_{E_1} n_\Sigma, E_2) \\ -g_\epsilon(\nabla^\epsilon_{E_2} n_\Sigma, E_1) & -g_\epsilon(\nabla^\epsilon_{E_2} n_\Sigma, E_2) \end{pmatrix}.$$

By making use of the connection 1-forms given in (8.20), the explicit computation of the second fundamental form in our case gives

$$\mathbf{II}^\epsilon = \begin{pmatrix} \eta_1^3(E_1) & \eta_2^3(E_1) \\ \eta_2^3(E_1) & \eta_2^3(E_2) \end{pmatrix} = \begin{pmatrix} -A_1 & -\frac{1}{2}(A_2 + B_1) \\ -\frac{1}{2}(A_2 + B_1) & -B_2 \end{pmatrix}.$$

The *Riemannian mean curvature* H^ϵ of Σ is

$$H^\epsilon := -\mathrm{Trace}(\mathbf{II}^\epsilon) = A_1 + B_2. \tag{8.21}$$

By means of the Gauss equation (see Theorem 2.5, Chapter 6 in do Carmo's book [23]) we have that the *Riemannian Gaussian curvature* K^ϵ is

$$K^\epsilon := \overline{K}^\epsilon(E_1, E_2) + \det(\mathbf{II}^\epsilon) = E_1(B_3) + E_2(A_3) - A_3^2 - B_3^2. \tag{8.22}$$

8.2 The horizontal mean curvature

We begin this section with the following

Proposition 8.2.1. *Away from characteristic points, the horizontal mean curvature H^0 of a surface $\Sigma \subset \mathcal{AA}$ is given by*

$$H^0 = \lim_{\epsilon \to 0^+} H^\epsilon = U\left(\frac{Uu}{\|\nabla_H u\|}\right) + V\left(\frac{Vu}{\|\nabla_H u\|}\right) - 2\frac{Vu}{\|\nabla_H u\|}. \tag{8.23}$$

Proof. By definition

$$H^\epsilon = A_1 + B_2,$$

where we comply with the notation (8.4) and from Lemma 8.1.7. The result now follows by using the ϵ-estimates from Lemma 8.1.6, indeed

$$A_1 + B_2 \to H^H - 2\,\overline{q}$$

as $\epsilon \to 0^+$. □

We provide the following list of examples

Example 8.2.2 (Planes in $\mathcal{AA}$)**.** We look at the horizontal mean curvature for a plane P in $\mathcal{AA}$ given by

$$P = \{(a, \lambda, t) \in \mathcal{AA} : u(a, \lambda, t) = a - (c_1\lambda + c_2 t) = 0\}$$

with $c_1, c_2 \in \mathbb{R}$. By using (8.23) we obtain that

$$H^0 = 8c_1\lambda^2 \frac{2\lambda(c_1^2 + c_2^2) - c_2}{(4c_1^2\lambda^2 + (1 - 2c_2\lambda)^2)^{3/2}}.$$

This evidences that not all planes have costantly zero mean horizontal curvature, but $H^0(p) = 0$ for all $p \in \Sigma$ if and only if $c_1 = 0$.

Example 8.2.3 (Surfaces independent in λ)**.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^2-smooth function and we consider the surface Σ given by

$$\Sigma_1 = \{(a, \lambda, t) \in \mathcal{AA} : u(a, \lambda, t) = f(a, t) = 0\}$$

and we assume there are no characteristic points. The latter assumption, combined with the fact that $Vu = 0$ identically implies that $Uu \neq 0$ for every point of Σ_1. Without loss of generality we can assume $Uu > 0$, thus having $\|\nabla_H u\| = Uu$ and obtaining that

$$U\left(\frac{Uu}{\|\nabla_H u\|}\right) = 0.$$

Recalling that $Vu = 0$, we conclude

$$H^0 = 0.$$

Example 8.2.4 (Graphs independent in t)**.** Let $f : (0, \infty) \to \mathbb{R}$ be a C^2-smooth function and we consider the surface Σ_2 given by

$$\Sigma_2 = \{(a, \lambda, t) \in \mathcal{AA} : u(a, \lambda, t) = a - f(\lambda) = 0\}.$$

Then by the formula (8.23) we have

$$H^0 = \frac{4\lambda^2(4\lambda(f'(\lambda))^3 - f''(\lambda))}{(4\lambda^2(f'(\lambda))^2 + 1)^{\frac{3}{2}}}.$$

Solving the equation $H^0 = 0$ is equivalent to the o.d.e.

$$4\lambda(f'(\lambda))^3 - f''(\lambda) = 0.$$

Hence the surfaces satisfying $H^0 = 0$ are described by

$$f_{c_1,c_2}(\lambda) = \pm\frac{1}{2}\arctan\left(\frac{\lambda}{\sqrt{c_1 - \lambda^2}}\right) + c_2, \quad c_1, c_2 \in \mathbb{R}.$$

The existence of surfaces of this type is prescribed in the slice $(\lambda, t) \in (0, c_1) \times \mathbb{R}$ thus giving

$$\Sigma_2 = \{(f_{c_1,c_2}(\lambda), \lambda, t) \in \mathcal{AA} : \lambda \in (0, c_1),\ t \in \mathbb{R}\}.$$

8.3 The intrinsic Gaussian curvature

We start by providing the following

Proposition 8.3.1. *Away from characteristic points, the intrinsic Gaussian curvature K^0 of a surface $\Sigma \subset \mathcal{AA}$ is given by*

$$K^0 = \lim_{\epsilon \to 0^+} K^\epsilon = -2E_1\left(\frac{Wu}{\|\nabla_H u\|}\right) - 4\left(\frac{Wu}{\|\nabla_H u\|}\right)^2. \tag{8.24}$$

Proof. Directly from the definition we have

$$K^\epsilon = -E_1(B_3) + E_2(A_3) - A_3^2 - B_3^2\,,$$

where we adopt the notations from (8.4) and from Lemma 8.1.7. The result now follows again by making use of the ϵ-estimates from Lemma 8.1.6, indeed

$$-E_1(B_3) + E_2(A_3) - A_3^2 - B_3^2 \to -E_1\left(2\frac{Wu}{l}\right) - \left(2\frac{Wu}{l}\right)^2$$

as $\epsilon \to 0^+$. □

Example 8.3.2 (Surfaces independent in a)**.** Let $f : \mathbf{H}^1_{\mathbb{C}} \to \mathbb{R}$ be a C^2-smooth function. Then the surfaces Σ given by

$$\Sigma = \{(a, \lambda, t) \in \mathcal{AA} : u(a, \lambda, t) = f(\lambda, t) = 0\}$$

have zero intrinsic Gaussian curvature. This is so because the function u is independent of a. In particular for all cylinders C_R defined as

$$C_R = \{(a, \lambda, t) \in \mathcal{AA} : (\lambda - 1)^2 + t^2 - R^2 = 0\},$$

where $0 < R < 1$, we find zero intrinsic Gaussian curvature. On the other hand the cylinders C'_R given by

$$C'_R = \{(a, \lambda, t) \in \mathcal{AA} : u(a, \lambda, t) = (\lambda - 1)^2 + a^2 - R^2 = 0\},$$

with $0 < R < 1$, have intrinsic Gaussian curvature equal to

$$K^0 = -\frac{4(a^4 + 4a^2(\lambda - 1)\lambda^2(3\lambda - 2) + 4(\lambda - 1)^3\lambda^3)}{(a^2 + 4(\lambda - 1)^2\lambda^2)^2}.$$

Example 8.3.3. The hyperbolic half-plane $\mathbf{H}^1_{\mathbb{C}}$ embedded in $\mathcal{AA}$ given by

$$\mathbf{H}^1_{\mathbb{C}} = \{(a, \lambda, t) \in \mathcal{AA} : u(a, \lambda, t) = a = 0\}$$

has constant intrinsic Gaussian curvature

$$K^0 = -4.$$

Example 8.3.4. For planes P given by

$$P = \{(a, \lambda, t) \in \mathcal{AA} : u(a, \lambda, t) = Aa + B\lambda + Ct + D = 0\}, \quad \text{with } A, B, C, D \in \mathbb{R},$$

we have

$$K^0 = -4A \frac{(A^3 + 6A^2C\lambda + 4A(2B^2 + 3C^2)\lambda^2 + 8C(B^2 + C^2)\lambda^3)}{(A^2 + 4AC\lambda + 4(B^2 + C^2)\lambda^2)^2}.$$

We see that planes with $A = 0$ have zero intrinsic Gaussian curvature, planes with $C = 0$ have negative intrinsic Gaussian curvature equal to

$$K^0 = -\frac{4(A^4 + 8A^2B^2\lambda^2)}{(A^2 + 4B^2\lambda^2)^2}.$$

From the previous formula we see that all planes of type

$$\{(a, \lambda, t) \in \mathcal{AA} : u(a, \lambda, t) = Aa + D = 0\},$$

have constant negative intrinsic Gaussian curvature corresponding to

$$K^0 = -4.$$

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