Geometric Aspects of Stochastic Processes and Statistical Testing

Inaugural Dissertation
of the Faculty of Science
University of Bern

Presented by
Tommaso Visonà

Supervisors of the doctoral thesis:

Prof. Dr. Ilya Molchanov
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Tommaso Visonà

Abstract

This work uses geometric tools to develop new results in the theories of random sets, valuations, stochastic processes, and statistical testing.

The first part is dedicated to studying the intersection of randomly translated sets. In the theory of random sets, the operations of union and of Minkowski sum have been thoroughly studied, as they agree well with the capacity functional of a random set, which defines its distribution. Then, limit theorems for random sets are mostly derived for their Minkowski sums and unions. Only recently, some limit theorems have been achieved for their intersections. This work expands on some of these results by using the asymptotic properties of integrals over Minkowski differences.

The second contribution came out of necessity in the path of developing the third part of this thesis. The functions defined over the family of convex bodies which satisfy the additivity property, called valuations, are well-studied under assumptions of continuity and invariance under rigid motions. A complete characterisation of planar monotone integer-valued σ-continuous valuations is presented, without assuming invariance under any group of transformations. A construction of the product for valuations of this type is introduced.

In the third part of the thesis, a new family of set-indexed stochastic processes is presented. These processes satisfy the additivity property of valuations, so they are referred to as random valuations. The family of valuations is very rich. To achieve meaningful characterisation results, assumptions in the form of independence and infinite divisibility must be taken into consideration. Under these assumptions and by using tools from the theories of Lévy processes, stochastic geometry, and valuations, we are able to build a rich new theory, which is deeply connected with well-known results of deterministic valuations and integral geometry.

The fourth contribution of this thesis is the development of new methods for testing a cone hypothesis about the mean of a Gaussian distribution, which can be expressed as a constraint testing problem. The proposed tests adapt based on the number of constraints which are violated. It improves the classical (non-adaptive) methods when few constraints are not satisfied, in terms of both simplicity and power. The new tests are shown to have a valid significance level α. Moreover, some possible tools to evaluate the elements of the family of adaptive tests are presented, in terms of risk and power.

Contents

Introduction

This thesis is on the intersection of randomly translated sets, integer-valued valuations, random valuations, and adaptive tests of a cone hypothesis. It essentially builds on one publication, one preprint, and two open projects, either in their journal format, as arXiv preprints, or as the current state of the work.

This introductory chapter gives the motivation and main ideas for each work.

Throughout this introduction, we let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, consisting of a non-empty set Ω, a σ-algebra $\mathcal{A}$, and a probability measure $\mathbb{P}$. We denote by X random variables and random closed sets, and write $\mathbb{E}$ for the expectation with respect to $\mathbb{P}$.

Introduction to Chapter 1

A random closed set X in an Euclidean space is a random element in the family of closed sets in $\mathbb{R}^d$ equipped with the Fell topology. Its distribution is characterised by its capacity functional, which is

$$T_X(K) = \mathbb{P}\left\{X \cap K \neq \varnothing\right\},$$

for each compact set K. Limit theorems for sequences of random closed sets are mostly derived for their Minkowski sums and unions, see Molchanov (2017), as the capacity functional agrees well with these set operations.

The main focus of this chapter is to show limit results for the intersection of randomly translated sets. Only recently, Richey and Sarkar (2022) proved a limit theorem for random sets obtained as the intersection of unit Euclidean balls whose centres form a Poisson point process of growing intensity on the same unit ball.

Following the setting of Marynych and Molchanov (2022). Assume that $\Xi_n = \{\xi_1, \dots, \xi_n\}$ is a set of n i.i.d. points distributed in a set $K \subset \mathbb{R}^d$ according to a probability measure μ and define

$$X_n = \bigcap_{i=1}^{n} (K - \xi_i).$$

In contrast to Marynych and Molchanov (2022), it is not assumed that points are uniformly distributed in K. Moreover, the convexity assumption on K is also dropped. Instead, it is assumed that K is a regular closed set from the extended convex ring, which is the family of locally finite unions of convex bodies, see Kiderlen and Rataj (2006).

The ball hull is the intersection of all unit balls which contain the sample. It is easy to see that the set X_n of all $x \in \mathbb{R}^d$ such that $\Xi_n \subset x + B_1$ satisfies

$$X_n = \{x \in \mathbb{R}^d : \Xi_n \subset x + B_1\} = \bigcap_{i=1}^{n} (\xi_i + B_1),$$

so X_n is the ball hull of a sample $\Xi_n = \{\xi_1, \dots, \xi_n\}$ consisting of points distributed in the unit Euclidean ball. This line of research was initiated in Fodor et al. (2014), in the case that $\xi_1, \dots, \xi_n$ are independent and uniformly distributed. This approach was then generalised in Fodor et al. (2020), by replacing the ball with a convex body in $\mathbb{R}^2$, whose boundary needs to satisfy some requirements. The ball hull model has been extended in Marynych and Molchanov (2022), where the unit ball was replaced by a general convex body in the space of arbitrary dimension, and it was shown that nX_n converges in distribution to the zero cell of a tessellation.

The standard closed convex hull of a set is defined as the intersection of all images, under the action of a group of rigid motions, of a half-space containing the given set. In Kabluchko et al. (2025) a generalisation of this concept is proposed, with a focus on the analysis of the newly defined convex hulls of random samples taken from a fixed convex body.

The main result of this contribution states that, after an appropriate multiplicative scaling, the closure of the complement of the random closed set X_n converges in distribution to the closure of the complement of the zero cell of a tessellation in $\mathbb{R}^d$, whose distribution is determined by the curvature measure of K and the behaviour of μ near the boundary of K. Furthermore, if K is convex, after the same appropriate scaling, then X_n converges in distribution to the zero cell of a tesselation. Some limit results on the expectation of the volume of properly scaled X_n are also presented.

The proof relies on the analysis of the asymptotic properties of a family of measures over Minkowski differences between a set K and a scaled version of another set L, which is

$$K \ominus \varepsilon L := \{x \in \mathbb{R}^d : x + \varepsilon L \subset K\}.$$

In Kiderlen and Rataj (2006), the authors obtain such results for the volume V_d. For each element K of the family of gentle sets, which are closed sets that satisfy some boundary conditions, and each compact set L,

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} V_d(K \setminus K \ominus \varepsilon L) = \int_{\mathbb{S}^{d-1}} h(L, u)^+ S_{d-1}(K, du),$$

where $h(L, u)^+$ is the positive part of the support function of L, and S_{d-1} is the surface area measure of K. This result is expanded by replacing the volume with a general finite measure μ, which is absolutely continuous in a neighbourhood of the boundary of K and has a density function which behaves like a power of order α near the boundary of K. The limit is given by a similar integral, which depends on the support function h^+ and the asymptotic behaviour of the density function of μ near the boundary ∂K. The function of this power then appears in the scaling factor of X_n as n^γ with $\gamma = (1+\alpha)^{-1}$. It is natural that only the behaviour of μ near ∂K matters for the asymptotic of $n^\gamma X_n$ since $K - \xi_i$ for ξ_i within any positive distance of ∂K does not contribute to the intersection.

Introduction to Chapter 2

Let $\mathscr{K}^d$ be the family of convex bodies in $\mathbb{R}^d$, with the convention that the empty set is included in $\mathscr{K}^d$. A valuation $\varphi : \mathscr{K}^d \to \mathbb{R}$ is a real additive map, i.e. for any compact convex sets K and L such that $K \cup L$ is also convex, we have

$$\varphi(K \cup L) + \varphi(K \cap L) = \varphi(K) + \varphi(L),$$

with $\varphi(\varnothing) = 0$.

Many characterisation results for valuations have been achieved under assumptions of invariance under a group of transformations. In most cases, valuations are assumed to be translation invariant or invariant under the group of all rigid motions. Another frequently imposed condition is continuity with respect to the Hausdorff metric on compact convex sets.

The two best-known results are MacMullen's theorem and Hadwiger's theorem. These theorems are presented in the Appendix; the first characterises translation invariant continuous valuations, while the second characterises continuous valuations invariant under rigid motions.

Since the second half of the 1990s, the theory of valuations has undergone rapid development, mainly due to the work of Alesker, with great expansions in depth and applications, see for example Alesker and Fu (2014) or Alesker and Bernig (2012). Up until now, valuations taking values in the group $\mathbb{Z}$ of integers under addition were not thoroughly studied. Clearly, the only continuous valuations with values in $\mathbb{Z}$ are multiples of the Euler characteristic

$$\chi(K) = \mathbb{1}_{K \neq \varnothing}.$$

It is straightforward to see that a sum of Euler characteristics like

$$\varphi(K) = \sum_{i=1}^{N} \chi(K \cap C_i)$$

for convex bodies $C_1, \dots, C_N$ defines an integer-valued monotone valuation. Due to the intersection operation, φ is no longer continuous in the Hausdorff metric, yet it is σ-continuous, which means that $\varphi(K_n) \to \varphi(K)$ whenever $K_n \downarrow K$, as $n \to \infty$. Adding negative terms to this sum preserves additivity and σ-continuity and may still retain the monotonicity property.

This chapter focuses on integer-valued monotone σ-continuous valuations without imposing any invariance assumptions and provides their complete characterisation in dimensions 1 and 2. The main results establish that each integer-valued, monotone, and σ-continuous valuation in dimensions 1 and 2 can be represented as an at most countable sum of Euler characteristics with weights ± 1. The convex bodies C_i necessarily form a locally finite family, and the bodies appearing in the negative terms satisfy a strict admissibility property with respect to the positive ones. In other words, each integer-valued monotone σ-continuous valuation corresponds to a locally finite integer-valued measure on the family of convex bodies.

A key step is to show that each integer-valued σ-continuous valuation is uniquely determined by its values on singletons, which define its support. The support of a valuation φ is the set of points x such that $\varphi(\{x\}) \geq 1$. We show that the intersection of the support with any convex body is polyconvex. For the latter, we apply Eggleston's theorem, see Eggleston (1974), which establishes a connection between polyconvexity and the structure of invisible points. The absence of such a result in dimensions 3 and higher makes it impossible to generalise our technique beyond the planar case.

Introduction to Chapter 3

A random valuation is a stochastic process indexed by the family of convex bodies $\mathscr{K}^d$ in $\mathbb{R}^d$ which satisfies the additivity property of valuations. The distribution of a random valuation is characterised by its finite-dimensional distributions.

A rich source of random valuations is provided by random (signed) measures on the family of closed convex sets $\mathcal{U}$ in $\mathbb{R}^d$ equipped with the Borel σ-algebra generated by the Fell topology. For example, let Z be a locally finite random signed measure on $\mathcal{U}$, it is easy to see that

$$\Phi(K) = Z(\{F \in \mathcal{U} \colon K \cap F \neq \varnothing\}), \qquad K \in \mathscr{K}^d,$$

is a random valuation.

Random valuations have a version that takes paths in the family of valuations, which is very rich. To achieve meaningful characterisation results, we assume infinite divisibility. Every infinitely divisible probability distribution corresponds in a natural way to a Lévy process. Set-indexed Lévy processes were studied in Bass and Pyke (1984) assuming that the values on disjoint sets are independent. The central question in this work was the existence of such a process indexed by a rather general family of sets and such that its paths are sufficiently regular. These processes have been further studied in Herbin and Merzbach (2013), where the authors use a new definition for increment stationarity of set-indexed processes to obtain different characterisations of this class. An exciting result presented in Rosiński (2018) shows the existence of a unique Lévy measure for an infinitely divisible set-indexed process.

On top of assuming infinite divisibility, some form of independence assumption can be taken into consideration. We present a definition for the independence of increments for a random valuation Φ, which is that for each $n \geq 3$ and a nested sequence $L_1 \supset \cdots \supset L_n$ of convex bodies and $L_{n+1} = \varnothing$, the random variables $\Phi(L_i) - \Phi(L_{i+1})$, $i = 1, \ldots, n$, are jointly independent. We show that, if Φ is an infinitely divisible valuation, then Φ has independent increments if and only if its Lévy measure Λ is supported by valuations ψ with two values $\{0, c\}$ and such that ψ is monotone increasing if $c > 0$ and monotone decreasing if $c < 0$.

Introduction to Chapter 4

Let $\mu \in \mathbb{R}^d$ and let A be a real-valued $m \times d$ matrix. A classical testing problem is to test

$$H : A\mu \geq 0 \quad \text{against} \quad H^c : A\mu \ngeq 0,$$

based on a vector $X \sim N(\mu, \Sigma)$, where Σ is a given positive-definite matrix while μ is unknown. This testing model includes testing problems about whether all components of μ are positive or whether $\mu_1 \leq \cdots \leq \mu_d$ against all alternatives. These testing problems are called type B problems in Silvapulle and Sen (2001) and are studied by using the likelihood ratio test. They are referred to as constrained likelihood-ratio tests.

The distribution of the likelihood ratio statistic under H depends on the unknown vector μ, and a least favourable distribution is found at $\mu = 0$, whose distribution is that of a mixture of chi-squared distributions. The likelihood ratio statistic is then evaluated with a quantile of this mixture. The weights of this mixture have an explicit formula only in special cases, and, usually, they are complicated to compute.

To avoid the calculation of mixing proportions, different works, for example Susko (2013), Chen et al. (2018), and Al Mohamad et al. (2020), adopt a form of conditional testing procedure. This line of work was started from ideas of Bartholomew (1961) and Wollan and Dykstra (1986). This type of testing is called adaptive, as it selects adaptively which quantile to compare the likelihood ratio statistic with. The classical approach is then referred to as non-adaptive. The adaptive approach is computationally more efficient and also more powerful in some regions of the parameter space.

In Al Mohamad et al. (2020), the authors show that this adaptive test has a valid level α, and they show that, asymptotically, the power of the proposed test is greater than that of the non-adaptive test when the true vector μ does not violate many constraints of the null hypothesis.

The main goal of Dümbgen (1995) is to find non-randomized tests $x \in \mathbb{R}^d \mapsto \mathbb{1}_{x \notin U}$ of H against all alternatives, where K is the polyhedral convex cone defined by A, i.e. $K = \{x \in \mathbb{R}^d : Ax \geq 0\}$, with small risk

$$R(U) = \sup_{\mu \in K^c} \mathbb{P}_\mu\{U\},$$

under the restriction $\mathbb{P}_\mu\{U\} \geq 1 - \alpha$ for all $\theta \in K^c$ for some fixed level $\alpha \in (0, 1/2)$. This risk function is minimised over the class $\mathcal{A}_\alpha(K)$ of the acceptance regions $U \subseteq \mathbb{R}^d$, which are measurable and satisfy $U + K = U$. The test $\mathbb{1}_{x \notin U}$ for $U \in \mathcal{A}_\alpha(K)$ is monotone, in the sense that

$$\mathbb{1}_{x \notin U} \leq \mathbb{1}_{x+\mu \notin U}, \quad \mu \in K, x \in \mathbb{R}^d.$$

The support of the non-randomised likelihood ratio test belongs to the family $\mathcal{A}_\alpha(K)$. The risk function R is minimised by a test φ in the family $\mathcal{A}_\alpha(K)$, which is referred to as minimax test.

In this chapter, we fix Σ to be the identity matrix and we introduce a general adaptive test. The test adapts based on which face of the polar cone of K the observed X is projected onto. On each such face, a monotone test of level α is defined. Instead of comparing the non-adaptive test with a mixture of quantiles of fixed distributions, only one of the quantiles defined by the monotone tests on one of the faces is used. We show that the adaptive test is also of level α for a certain family of cones. Moreover, we present some possible tools to evaluate the elements of the family of adaptive tests, in terms of the risk function R and power.

Structure of the thesis

The remainder of this thesis consists of four chapters and an appendix, as introduced before: Intersection of randomly translated sets (Chapter 1), Integer-valued valuations (Chapter 2), Random valuations (Chapter 3), and Adaptive tests of a cone hypothesis (Chapter 4). The first two chapters are published research papers or arXiv preprints (available on https://arxiv.org) in their original format, with the exact reference given at the beginning of each chapter. The last two are ongoing projects.

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Chapter 1

Intersection of Randomly Translated Sets

The content of this chapter is a work completed under the supervision of I. Molchanov, and it is published as

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Intersections of Randomly Translated Sets

Tommaso Visonà[1]

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Abstract

Let $\Xi_n = \{\xi_1, \ldots, \xi_n\}$ be a sample of n independent points distributed in a regular closed element K of the extended convex ring in $\mathbb{R}^d$ according to a probability measure μ on k admitting a density function. We consider random sets generated from the intersection of the translations of K by the elements of Ξ_n, namely,

$$X_n = \bigcap_{i=1}^{n} (K - \xi_i).$$

This work aims to show that the scaled closure of the complement of X_n as $n \to \infty$ converges in distribution to the closure of the complement zero cell of a Poisson hyperplane tessellation whose distribution is determined by the curvature measure of K and the behaviour of the density of μ near the boundary of K.

Keywords Minkowski difference · Regular closed random sets · Zero cell of a Poisson tesselation · Intersection of random sets

Mathematics Subject Classification 60D05 · 60G55 · 52A22

1 Introduction

A random closed set in Euclidean space is a random element in the family of closed sets in $\mathbb{R}^d$ equipped with the Fell topology. Limit theorems for random sets are mostly derived for their Minkowski sums and unions, see [1].

Only recently, [2] proved a limit theorem for random sets obtained as the intersection of unit Euclidean balls $x + B_1$ whose centres x form a Poisson point process of growing intensity λ on the same unit ball. It was shown that these random sets scaled by λ converge in distribution as $\lambda \to \infty$ to the zero cell of a Poisson hyperplane

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tessellation. It was also shown that the volumes of these intersection sets converge in distribution.

Many results from [2] follow from the studies of a similar model appearing by taking the ball hull of a sample $\Xi_n = \{\xi_1, \dots, \xi_n\}$ which consists of i.i.d. points uniformly distributed in the unit Euclidean ball. The ball hull

$$Q_n = \bigcap_{x \in \mathbb{R}^d,\, \Xi_n \subset x + B_1} (x + B_1)$$

is the intersection of all (translated) unit balls which contain the sample. It is easy to see that the set X_n of all $x \in \mathbb{R}^d$ such that $\Xi_n \subset x + B_1$ satisfies

$$X_n = \{x : \Xi_n \subset x + B_1\} = \bigcap_{i=1}^{n} (\xi_i + B_1).$$

This line of research was initiated in [3], where the combinatorial structure of the ball polytope Q_n was explored. In particular, it was shown that the expectation of the number of faces of Q_n in dimension 2 converges to a nontrivial limit without taking any normalization. This was generalized in [4], by replacing the ball with a convex body in $\mathbb{R}^2$, whose boundary needs to satisfy some requirements.

The ball hull model has been extended in [5], where the unit ball was replaced by a general convex body in the space of arbitrary dimension and it was shown that the convergence of distribution of nX_n entails the convergence of the combinatorial features of Q_n, namely, its f-vector, in particular, the numbers of vertices and facets. It is shown that the latter convergence holds in distribution together with all moments.

The standard closed convex hull of a set is defined as the intersection of all images, under the action of a group of rigid motions, of a half-space containing the given set. In [6] a generalization of this concept is proposed, with a focus on the analysis of the newly defined convex hulls of random samples taken from a fixed convex body.

Following the setting of [5], we assume that $\Xi_n = \{\xi_1, \dots, \xi_n\}$ is a set of n i.i.d. points distributed in a set $K \subset \mathbb{R}^d$ according to a probability measure μ and define

$$X_n := \bigcap_{i=1}^{n} (K - \xi_i). \tag{1}$$

In contrast to [5], it is not assumed that points are uniformly distributed in K and the convexity assumption on K is also dropped. Instead, it is assumed that K is a regular closed set from the extended convex ring, which is the family of locally finite unions of convex bodies, see [7].

The main result states that, after an appropriate multiplicative scaling, the complement of the closure of the random closed set X_n converges in distribution to the closure of the complement of the zero cell of a tessellation in $\mathbb{R}^d$, whose distribution is determined by the curvature measure of K and the behaviour of μ near the boundary of K. Some limit results on the expectation of the volume of properly scaled X_n are also presented.

The proof is based on two technical results. First, it relies on the analysis of the asymptotic properties of a family of measures over Minkowski differences between a set K and a scaled version of another set L, which is

$$K \ominus \varepsilon L := \{x : x + \varepsilon L \subset K\}.$$

In [7], the authors obtain such results for the volume V_d. For each gentle set K, whose definition can be found in Sect. 2, and each compact set L,

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} V_d(K \setminus K \ominus \varepsilon L) = \int_{\mathbb{S}^{d-1}} h(L, u)^+ S_{d-1}(K, \mathrm{d}u),$$

where $h(L, u)$ is the support function of L and $h(L, u)^+$ is its positive part, and S_{d-1} is the surface area measure of K.

In Sect. 2, we extend this result by replacing the volume with a general finite measure μ, which is absolutely continuous in a neighbourhood of the boundary of K and has density function which behaves like a power of order α near the boundary of K. The limit is given by a similar integral which depends on the support function h and the asymptotic behaviour of the density function of μ near the boundary ∂K. The function of this power then appears in the scaling factor of X_n as n^γ with $\gamma = (1 + \alpha)^{-1}$. It is natural that only the behaviour of μ near ∂K matters for the asymptotic of $n^\gamma X_n$ since $K - \xi_i$ for ξ_i within any positive distance of ∂K does not contribute to the intersection in (1).

The second main technical tool relies on the fact that convergence of the inclusion functionals of regular closed random sets implies the convergence in distribution of the closure of their complements.

2 Asymptotic Properties of Integrals Over Minkowski Differences

By $\mathcal{H}^{d-1}$ we denote the Hausdorff measure of dimension $(d - 1)$, and by $\mathcal{H}^d$ or V_d the Lebesgue measure in $\mathbb{R}^d$. We write dx in integrals with respect to the Lebesgue measure. By $B_r(a)$ we denote the closed ball in $\mathbb{R}^d$ of radius r and centre $a \in \mathbb{R}^d$. Given a set A in $\mathbb{R}^d$, denote by $\mathrm{Int}(A)$ the interior of A, by $\mathrm{cl}(A)$ its closure, and by A^c its complement.

Let $K \subseteq \mathbb{R}^d$ be a closed set. The metric projection $\xi_K : \mathbb{R}^d \setminus \mathrm{exo}(K) \to K$ is defined by letting $\xi_K(a)$ be the unique nearest point to a from K, where the exoskeleton $\mathrm{exo}(K)$ is the set of points which do not admit a unique nearest point in K. The set $\mathrm{exo}(K)$ is measurable and $V_d(\mathrm{exo}(K)) = 0$, see Section 2 in [7]. The reduced normal bundle of K is

$$N(K) := \left\{ \left(\xi_K(z), \frac{z - \xi_K(z)}{\|z - \xi_K(z)\|} \right) : z \notin K \cup \mathrm{exo}(K) \right\}.$$

The set

$$\hat{N}(K) := N(\partial K) \setminus N(K)$$

is called the inner reduced normal bundle. The reach function of K is defined as

$$\delta(K, a, u) := \inf\{t \geq 0 : a + tu \in \text{exo}(K)\}, \quad (a, u) \in N(K),$$

assuming that $\delta(K, a, u) = +\infty$ if the set $\{t \geq 0 : a + tu \in \text{exo}(K)\}$ is empty.

A set $K \subset \mathbb{R}^d$ is called regular closed if it coincides with the closure of its interior. Let $\mathcal{R}$ be the family of regular closed sets. A convex set K is regular closed if and only if its interior is not empty.

A closed set K is said to be gentle if

(G1) $\mathcal{H}^{d-1}(\{a \in B : (a, u) \in N(\partial K), u \in \mathbb{S}^{d-1}\}) < \infty$ for all bounded Borel sets $B \subset \mathbb{R}^d$,
(G2) for $\mathcal{H}^{d-1}$-almost all $a \in \partial K$, there are non-degenerate balls B_i and B_o containing a such that $B_i \subset K$ and $\text{Int}\, B_o \subset K^c$,

see Section 2 in [7]. A gentle set K is not necessarily regular closed, for example, a singleton is gentle. Each convex body in $\mathbb{R}^d$ is gentle.

Let K be a gentle set. For $\mathcal{H}^{d-1}$-almost all $a \in \partial K$, there exists a unique $u \in \mathbb{S}^{d-1}$ such that $(a, u) \in N(K)$ and $(a, -u) \in \hat{N}(K)$. This follows from (G2) since the tangent balls B_i and B_o at a are unique. Let $C_{d-1}(K, \cdot)$ be the image measure of $\mathcal{H}^{d-1}$ on ∂K under the map $a \mapsto (a, u) \in N(K)$, which is defined $\mathcal{H}^{d-1}$-almost everywhere on ∂K and is measurable, see Lemma 6.3 from [8]. The measure $C_{d-1}(K, \cdot)$ is called the curvature measure of K.

We will use the abbreviation for almost all $(a, u) \in N(K)$ instead of for $C_{d-1}(K, \cdot)$-almost all $(a, u) \in N(K)$. The same agreement is used for $\hat{N}(K)$ equipped with the measure $C^*_{d-1}(K, \cdot)$, where $C^*_{d-1}(K, \cdot)$ is the image measure of $C_{d-1}(K, \cdot)$ under the reflection $(a, u) \in N(K) \mapsto (a, -u) \in \hat{N}(K)$, this reflection is well-defined almost everywhere.

If K is gentle, for almost all $(a, u) \in N(K)$, it is possible to express explicitly in terms of the reach function the radii of the inner and outer balls associated with an $a \in \partial K$ as mentioned in (G2). For almost all $(a, u) \in N(K)$, we denote $\delta_\pm := \delta(\partial K, a, \pm u)$, so that $B_o := B_{\delta_+}(a + \delta_+ u)$ and $B_i := B_{\delta_-}(a - \delta_- u)$. Moreover, $\delta_+ > 0$ and $\delta_- > 0$.

For $L \subseteq \mathbb{R}^d$, its support function is defined as

$$h(L, u) := \sup\{\langle x, u\rangle : x \in L\}, \quad u \in \mathbb{R}^d.$$

If $h(L, u) < +\infty$, the supporting hyperplane of L with normal $u \neq 0$ is given by

$$H(L, u) := \{x \in \mathbb{R}^d : \langle u, x\rangle = h(L, u)\}.$$

We denote with $h(L, u)^+$ the positive part of $h(L, u)$.

For sets K and L in $\mathbb{R}^d$,

$$K \ominus L := \{x \in \mathbb{R}^d : L + x \subseteq K\}$$

is the Minkowski difference of K and L, and the set

$$\check{L} := \{x \in \mathbb{R}^d : -x \in L\}$$

is the reflection of L with respect to the origin.

We recall that for almost all $a \in \partial K$, if $(a, u) \in N(K)$, we have that $\delta(K, a, u) = \delta(\partial K, a, u)$ since $N(K) \subset N(\partial K) = N(K) \cup \hat{N}(K)$. The same applies to $\hat{N}(K)$.

Lemma 2.1 *Let K be a gentle set and let L be a compact set in $\mathbb{R}^d$. Let f be a nonnegative integrable function. Then, for almost all $(a, u) \in \hat{N}(K)$, there exist functions t_+ and t_-, which satisfy*

$$\lim_{\varepsilon \downarrow 0} \frac{t_\pm(\varepsilon)}{\varepsilon} = h(L, -u)^+,$$

and such that

$$\begin{aligned}\int_0^{t_+(\varepsilon)} f(a + tu)\, \mathrm{d}t &\le \int_0^{\delta(\partial K, a, u)} f(a + tu) \mathbf{1}_{K \setminus K \ominus \varepsilon L}(a + tu)\, \mathrm{d}t \\ &\le \int_0^{t_-(\varepsilon)} f(a + tu)\, \mathrm{d}t\end{aligned}$$

for all sufficiently small ε.

Proof Fix $(a, u) \in \hat{N}(K)$ such that the inner ball B_i and outer ball B_o exist as in (G2). Define the half-space

$$H^-_{-u}(h(L, -u)) := \{x \in \mathbb{R}^d : \langle x, -u \rangle \le h(L, -u)\}.$$

There exists an $r > 0$ such that $L \subseteq B_r(0)$, and, consequently, $\varepsilon L \subseteq B_{\varepsilon r}(0)$. The set εL is not only a subset of $B_{\varepsilon r}(0)$, but also of

$$C(\varepsilon) := \varepsilon H^-_{-u}(h(L, -u)) \cap B_{\varepsilon r}(0).$$

Since L is compact, there is a point $l \in \partial L$ such that $\langle l, -u \rangle = h(L, -u)$, which is called the support point, and $\varepsilon l \in \varepsilon(H(L, -u) \cap B_r(0))$.

For all $\varepsilon < r^{-1} \min\{\delta_+, \delta_-, 1\}$ and for $t \in [0, \delta_-)$, we have $(a+tu) \in (K \setminus K \ominus \varepsilon L)$ if and only if $a + tu + \varepsilon L$ is not a subset of K. Hence, for $t \in [0, \delta_-)$,

$$\begin{aligned}\{t : a + tu + \varepsilon L \not\subseteq K\} &\subseteq \{t : a + tu + \varepsilon L \not\subseteq B_i\} \\ &\subseteq \{t : a + tu + C(\varepsilon) \not\subseteq B_i\} \\ &= \{t : a + tu + \varepsilon(H(L, -u) \cap B_r(0)) \not\subseteq B_i\},\end{aligned}$$

and

$$
\begin{aligned}
\{t \,:\, a + tu + \varepsilon L \nsubseteq K\} &= \{t \,:\, (a + tu + \varepsilon L) \cap K^c \neq \emptyset\} \\
&\supseteq \{t \,:\, (a + tu + \varepsilon L) \cap \mathrm{Int}\, B_o \neq \emptyset\} \\
&\supseteq \{t \,:\, (a + tu + \varepsilon l) \in \mathrm{Int}\, B_o\} \\
&\supseteq \{t \,:\, a + tu + \varepsilon(H(L, -u) \cap B_r(0)) \subseteq \mathrm{Int}\, B_o\}.
\end{aligned}
$$

We then define

$$
\begin{aligned}
t_-(\varepsilon) &:= 0 \vee \inf\{t \in (-\delta_+, \delta_-) : a + tu + \varepsilon(H(L, -u) \cap B_r(0)) \subseteq B_i\}, \\
t_+(\varepsilon) &:= 0 \vee \inf\{t \in (-\delta_+, \delta_-) : a + tu + \varepsilon(H(L, -u) \cap B_r(0)) \nsubseteq \mathrm{Int}\, B_o\},
\end{aligned}
$$

where $\vee$ stands for supremum.

The value of $t_-(\varepsilon)$ is the distance between a and a point of the segment $[a, a+\delta_- u]$. Since the set $a+tu+\varepsilon(H(L, -u)\cap B_r(0))$ is invariant under any rotation which keeps u unchanged, $t_-(\varepsilon)$ can be calculated by sectioning this set with any 2-dimensional plane parallel to u which contains a. For each $t \in (0, \delta_-)$, the section of $B_{\delta_-}(a + tu)$ is a circle C. Then $t_-(\varepsilon)$ is the positive part of the sum of $\varepsilon h(L, -u)$ and the distance between a chord of the circle C of length $2(r^2\varepsilon^2 - \varepsilon^2 h(L, -u)^2)^{1/2}$ and the point on the boundary a. Hence,

$$
t_-(\varepsilon) = (\varepsilon h(L, -u) + \delta_- - (\delta_-^2 - (r^2\varepsilon^2 - \varepsilon^2 h(L, -u)^2))^{1/2})^+,
$$

where $t_-(\varepsilon)\varepsilon^{-1} \to h(L, -u)^+$ as $\varepsilon \downarrow 0$.

With the same idea used to calculate $t_-(\varepsilon)$, we can see that

$$
t_+(\varepsilon) = (\varepsilon h(L, -u) - (\delta_+ - (\delta_+^2 - (r^2\varepsilon^2 - \varepsilon^2 h(L, -u)^2))^{1/2}))^+
$$

and $t_+(\varepsilon)\varepsilon^{-1} \to h(L, -u)^+$ as $\varepsilon \downarrow 0$.

Clearly $0 \le t_+(\varepsilon) \le t_-(\varepsilon) < \delta^-$ for $\varepsilon < r^{-1} \min\{\delta_+, \delta_-, 1\}$. The points of discontinuity of $\mathbf{1}_{K\setminus K\ominus\varepsilon L}(a+tu)$ happen for $t \in [t_+(\varepsilon), t_-(\varepsilon)]$. The identity function $\mathbf{1}_{K\setminus K\ominus\varepsilon L}(a + tu)$ is 1 for all $t \in [0, t_+(\varepsilon))$ and is 0 for all $t \ge t_-(\varepsilon)$.

The proof finishes by observing that f is integrable and positive on $[0, \delta^-)$. □

We impose the following conditions on a finite measure μ supported by K.

(M1) The measure μ has compact support, and it is absolutely continuous with density f.

(M2) There exists an $\alpha > -1$ such that, for almost all $(a, u) \in \hat{N}(K)$,

$$
\lim_{t\downarrow 0} \frac{f(a + tu)}{t^\alpha} = \hat{g}(a, u) =: g(a) \in [0, +\infty),
$$

where the function $\hat{g}$ is strictly positive on a subset of ∂K of positive measure and is bounded almost everywhere.

The argument of the limit function $\hat{g}$ is the vector $(a, u) \in \hat{N}(K)$. Since K is gentle, for $\mathcal{H}^{d-1}$-almost all $a \in \partial K$ there is a unique $u \in \mathbb{S}^{d-1}$ such that $(a, u) \in \hat{N}(K)$, then g is well-defined.

As a pointwise limit of measurable functions, g is also measurable. From its definition and since the support of f on K is compact, g has compact support.

Proposition 2.2 *Let K be a gentle set and let L be a compact set in $\mathbb{R}^d$. Let μ be a finite measure on K which has compact support, is absolutely continuous in a neighbourhood of ∂K with density f and satisfies (M2). Then*

$$\lim_{\varepsilon \downarrow 0} \frac{\mu(K \setminus K \ominus \varepsilon^\gamma L)}{\varepsilon} = \int_{N(K)} g(a) \frac{(h(L,u)^+)^{\alpha+1}}{\alpha+1} C_{d-1}(K, \mathrm{d}(a,u)),$$

where $\gamma = (\alpha+1)^{-1}$.

Proof There exists a constant $s > 0$ such that the finite measure μ is absolutely continuous on $\partial K + B_s(0)$ since it is a neighbourhood of ∂K. Moreover, there exists an $r > 0$ such that $L \subseteq B_r(0)$.

For $\varepsilon < (s/r)^{1/\gamma}$, Proposition 4 from [7] yields that

$$\begin{aligned}
&\mu(K \setminus K \ominus \varepsilon^\gamma L) \\
&= \int_{\mathbb{R}^d} f(x) \mathbf{1}_{K \setminus K \ominus \varepsilon^\gamma L}(x)\, \mathrm{d}x \qquad (2)\\
&= \sum_{i=1}^{d} i\kappa_i \int\limits_{N(\partial K)} \int\limits_0^{\delta(\partial K, a, u)} t^{i-1} f(a+tu) \mathbf{1}_{K \setminus K \ominus \varepsilon^\gamma L}(a+tu)\, \mathrm{d}t\, \nu_{d-i}(\partial K, \mathrm{d}(a,u)),
\end{aligned}$$

where the signed measures $\nu_0(\partial K, \cdot), \ldots, \nu_{d-1}(\partial K, \cdot)$ are called support measures of ∂K and the constant κ_i is the volume of the i-dimensional unit ball, for $i = 1, \ldots, d$. The signed measures $\nu_0(\partial K, \cdot), \ldots, \nu_{d-1}(\partial K, \cdot)$ have locally finite total variation from Corollary 2.5 and (2.13) in [8].

The hypotheses of Proposition 4 in [7] require that f is bounded. The statement also holds for unbounded integrable functions. The sequence of functions $f_n := f \mathbf{1}_{\{f \le n\}}$ is monotone and each f_n is bounded. By the monotone convergence, the identity (2) is also satisfied by unbounded integrable functions.

All summands of (2) with $i \ge 2$ are of order $o(\varepsilon)$ as $\varepsilon \downarrow 0$. The support of $\mathbf{1}_{K \setminus K \ominus \varepsilon^\gamma L}$ is contained in $\partial K + B_{2r\varepsilon^\gamma}(0)$, so that

$$\begin{aligned}
&\left| \frac{1}{\varepsilon} \int_{N(\partial K)} \int_0^{\delta(\partial K, a, u)} t^{i-1} f(a+tu) \mathbf{1}_{K \setminus K \ominus \varepsilon^\gamma L}(a+tu)\, \mathrm{d}t\, \nu_{d-i}(\partial K, \mathrm{d}(a,u)) \right| \\
&\le \frac{1}{\varepsilon} \int_0^{2r\varepsilon^\gamma} \int_{N(\partial K)} t^{i-1+\alpha} \frac{f(a+tu)}{t^\alpha}\, \mathrm{d}t\, |\nu_{d-i}|(\partial K, \mathrm{d}(a,u)) \\
&\le \frac{(2r\varepsilon^\gamma)^{i+\alpha}}{(i+\alpha)\varepsilon} \operatorname*{ess\,sup}_{((a,u),t) \in N(\partial K) \times [0, 2r\varepsilon^\gamma)} \frac{f(a+tu)}{t^\alpha} |\nu_{d-i}|(\partial K, \partial K(\varepsilon) \times \mathbb{S}^{d-1}) \to 0,
\end{aligned}$$

as $\varepsilon \downarrow 0$, for $i \neq 1$, $\gamma = (\alpha+1)^{-1}$, $\alpha > -1$, and where

$$\partial K(\varepsilon) := \{x \in \partial K : x + y \in \operatorname{supp} f \text{ for } y \in B_{2r\varepsilon^\gamma}(0)\},$$

which is compact for all sufficiently small ε since $\operatorname{supp} f$ is compact on ∂K. It follows that $|\nu_{d-i}|(\partial K, \partial K(\varepsilon) \times \mathbb{S}^{d-1})$ is finite for each $i = 1, \dots, d-1$, since $\partial K(\varepsilon)$ is bounded and the measures have locally finite total variation. Furthermore, it follows from the assumption (M2) that $f(a+tu)/t^\alpha$ is bounded for almost all $((a,u),t) \in N(\partial K) \times [0, 2r\varepsilon^\gamma)$ for ε small enough.

Proposition 4.1 and Proposition 5.1 from [8] yield that

$$2\nu_{d-1}(\partial K, \cdot) = C_{d-1}(K, \cdot) + C^*_{d-1}(K, \cdot).$$

For $\kappa_1 = 2$,

$$\begin{aligned}
&\lim_{\varepsilon \downarrow 0} \frac{\mu(K \setminus K \ominus \varepsilon^\gamma L)}{\varepsilon} \\
&= \lim_{\varepsilon \downarrow 0} \frac{2}{\varepsilon} \int_{N(\partial K)} \int_0^{\delta(\partial K, a, u)} f(a+tu) \mathbf{1}_{K \setminus K \ominus \varepsilon^\gamma L}(a+tu) \, \mathrm{d}t \, \nu_{d-1}(\partial K, \mathrm{d}(a,u)) \\
&= \lim_{\varepsilon \downarrow 0} \frac{2}{2\varepsilon} \int_{N(K)} \int_0^{\delta_+} f(a+tu) \mathbf{1}_{K \setminus K \ominus \varepsilon^\gamma L}(a+tu) \, \mathrm{d}t \, C_{d-1}(K, \mathrm{d}(a,u)) \\
&\quad + \frac{2}{2\varepsilon} \int_{\hat{N}(K)} \int_0^{\delta_-} f(a+tu) \mathbf{1}_{K \setminus K \ominus \varepsilon^\gamma L}(a+tu) \, \mathrm{d}t \, C^*_{d-1}(K, \mathrm{d}(a,u)) \\
&= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{\hat{N}(K)} \int_0^{\delta_-} f(a+tu) \mathbf{1}_{K \setminus K \ominus \varepsilon^\gamma L}(a+tu) \, \mathrm{d}t \, C^*_{d-1}(K, \mathrm{d}(a,u)),
\end{aligned}$$

since the density function f vanishes outside K. Following Lemma 2.1, for almost all $(a,u) \in \hat{N}(K)$,

$$\begin{aligned}
g(a) \frac{(h(L,-u)^+)^{\alpha+1}}{\alpha+1} &= \lim_{\varepsilon \downarrow 0} \inf_{s \in [0, t_+(\varepsilon^\gamma))} \frac{f(a+su)}{s^\alpha} \frac{1}{\varepsilon} \int_0^{t_+(\varepsilon^\gamma)} t^\alpha \, \mathrm{d}t \\
&\le \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^{t_+(\varepsilon^\gamma)} f(a+tu) \, \mathrm{d}t \\
&\le \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^{\delta_-} f(a+tu) \mathbf{1}_{K \setminus K \ominus \varepsilon^\gamma L}(a+tu) \, \mathrm{d}t \\
&\le \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^{t_-(\varepsilon^\gamma)} f(a+tu) \, \mathrm{d}t, \\
&\le \lim_{\varepsilon \downarrow 0} \sup_{s \in [0, t_-(\varepsilon^\gamma))} \frac{f(a+su)}{s^\alpha} \frac{1}{\varepsilon} \int_0^{t_-(\varepsilon^\gamma)} t^\alpha \, \mathrm{d}t \\
&= g(a) \frac{(h(L,-u)^+)^{\alpha+1}}{\alpha+1}.
\end{aligned}$$

Thus,

$$F_\varepsilon(a,u) := \frac{1}{\varepsilon}\int_0^{\delta_-} f(a+tu)\mathbf{1}_{K\setminus K\ominus\varepsilon^\gamma L}(a+tu)\,\mathrm{d}t \\ \to g(a)\frac{(h(L,-u)^+)^{\alpha+1}}{\alpha+1} \quad \text{as } \varepsilon\downarrow 0$$

for almost all $(a,u)\in\hat{N}(K)$. The limiting function is bounded and has compact support, because g is a bounded function with compact support and L is compact.

The sequence F_ε is bounded for all $\varepsilon>0$ and for almost all $(a,u)\in\hat{N}(K)\cap((\operatorname{supp} f\cap\partial K)\times\mathbb{S}^{d-1})$, which is compact. We can then find an upper bound and apply the dominated convergence theorem.

We conclude by noticing that

$$\int_{N\hat{(}K)} g(a)\frac{(h(L,-u)^+)^{\alpha+1}}{\alpha+1}C^*_{d-1}(K,\mathrm{d}(a,u)) = \\ \int_{N(K)} g(a)\frac{(h(L,u)^+)^{\alpha+1}}{\alpha+1}C_{d-1}(K,\mathrm{d}(a,u)).$$

□

Remark 2.3 For $d\geq 2$, the result does not hold for $\alpha\leq -1$. The reason is due to the summands of (2). There is no $\gamma\in\mathbb{R}$ such that the summand with index $i=1$ converges and others do not converge to 0. For $d=1$ there is only one summand, so this issue does not emerge.

Example 2.4 Let $f(x)=\|\xi_{\partial K}(x)-x\|^\alpha$ with $\alpha\in(-1,\infty)$ and $x\in K$, where K is a gentle compact set, and $f=0$ outside K. Then

$$\lim_{t\downarrow 0}\frac{f(a+tu)}{t^\alpha} = \|u\|^\alpha = 1$$

for almost all $(a,u)\in\hat{N}(K)$. In this case $g(a)=1$ for all $a\in\partial K$ and

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{\mathbb{R}^d} f(x)\mathbf{1}_{K\setminus K\ominus\varepsilon^\gamma L}(x)\,d\mathcal{H}^d(x) = \int_{N(K)}\frac{(h(L,u)^+)^{\alpha+1}}{\alpha+1}C_{d-1}(K,\mathrm{d}(a,u)).$$

3 Limit Theorem for Intersections of Translated Elements of the Extended Convex Ring

A random closed set X in $\mathbb{R}^d$ is a measurable map from a probability space to the space $\mathcal{F}$ of closed sets in $\mathbb{R}^d$ endowed with the Borel σ-algebra generated by the Fell topology, see [1]. The base of the Fell topology consists of finite intersections of the sets $\{F\in\mathcal{F}: F\cap G\neq\emptyset\}$ and $\{F\in\mathcal{F}: F\cap L=\emptyset\}$ for all open G and compact L in $\mathbb{R}^d$.

It is known that the distribution of a random closed set is uniquely determined by its capacity functional defined as

$$T_X(L) := \mathbb{P}\{X \cap L \neq \emptyset\},$$

where L runs through the family of compact sets in $\mathbb{R}^d$. A sequence of random closed sets $(X_n)_{n\geq 1}$ in $\mathbb{R}^d$ converges in distribution to a random closed set X (notation $X_n \xrightarrow{\text{d}} X$) if the corresponding probability measures on $\mathcal{F}$ weakly converge. This is the case if and only if

$$T_{X_n}(L) \to T_X(L) \quad \text{as } n \to \infty$$

for each compact set L such that $T_X(L) = T_X(\text{Int}(L))$, see Theorem 1.7.7 from [1].

We recall that a convex body is a non-empty compact convex set. The extended convex ring $\mathcal{U}$ is a family of closed sets which are locally finite unions of convex bodies of $\mathbb{R}^d$, meaning that each compact set intersects at most a finite number of convex bodies that generate an element of $\mathcal{U}$. We assume that the empty set belongs to $\mathcal{U}$. Clearly, the family $\mathcal{U}$ is closed under intersections.

Let K be a non-empty regular closed element of $\mathcal{U}$, and let μ be a probability measure on K satisfying (M1) and (M2). Consider the set

$$\Xi_n := \{\xi_1, \ldots, \xi_n\}$$

composed of n independent points in K distributed according to μ. Let X_n be a random closed set defined as follows

$$X_n := \bigcap_{i=1}^{n} (K - \xi_i). \tag{3}$$

Let $\mathcal{P}_K := \{(t_i, u_i) : i \geq 1\}$ be a Poisson point process on $(0, \infty) \times \mathbb{S}^{d-1}$ with intensity measure ν, which is the product of an absolutely continuous measure on $(0, \infty)$ with density t^α and a measure $\hat{\nu}$ on $\mathbb{S}^{d-1}$ defined as

$$\hat{\nu}(D) := \int_{N(K)} \mathbf{1}_{\{u \in D\}} g(a) C_{d-1}(K, \text{d}(a, u)), \tag{4}$$

for $D \subset \mathbb{S}^{d-1}$, where $g(a)$ is given by the property (M2) of μ.

The point process $\mathcal{P}_K$ corresponds to the family of hyperplanes $\{x \in \mathbb{R}^d : \langle x, u_i \rangle = t_i\}$, $i \geq 1$, which splits the space into disjoint cells and is said to be a Poisson hyperplane tessellation of $\mathbb{R}^d$, see [9]. The zero cell of this tessellation is the random convex set

$$Z = \bigcap_{i \geq 1} \{x \in \mathbb{R}^d : \langle x, u_i \rangle \leq t_i\}. \tag{5}$$

We now formulate our main result.

Theorem 3.1 *Assume that μ satisfies (M1) and (M2) on K for $\alpha > -1$. If K is a regular closed element of $\mathcal{U}$, then*

$$n^{\gamma}\,\mathrm{cl}(X_n^c) \xrightarrow{d} \mathrm{cl}(Z^c) \quad \text{as } n \to \infty,$$

where $\gamma = (\alpha+1)^{-1}$. Furthermore, if K is a convex set with non-empty interior, then

$$n^{\gamma} X_n \xrightarrow{d} Z \quad \text{as } n \to \infty.$$

Definition 3.2 A random closed set X in $\mathbb{R}^d$ is said to be regular closed if X almost surely belongs to the family $\mathcal{R}$ of regular closed sets.

For further details about regular closed random sets, see Section 1.1.7 of [1].

The following result is a variant of Theorem 7.5 from [6] with an identical proof.

Lemma 3.3 *Let Y and Y_n, $n \in \mathbb{N}$ be regular closed random sets in $\mathbb{R}^d$. If*

$$\mathbb{P}\{L \subseteq Y_n\} \to \mathbb{P}\{L \subseteq Y\} \quad \text{as } n \to \infty$$

for all regular closed compact sets L such that $\mathbb{P}\{L \subseteq Y\} = \mathbb{P}\{L \subseteq \mathrm{Int}\, Y\}$, then $\mathrm{cl}(Y_n^c) \xrightarrow{d} \mathrm{cl}(Y)$ as $n \to \infty$.

Proof The family of regular closed sets is a separating class, see Definition 1.1.48 in [1]. It follows from Corollary 1.7.14 in [1] that for the convergence in distribution it suffices to check that

$$\mathbb{P}\left\{\mathrm{cl}(Y_n^c) \cap L \neq \emptyset\right\} \to \mathbb{P}\left\{\mathrm{cl}(Y^c) \cap L \neq \emptyset\right\} \quad \text{as } n \to \infty,$$

for all regular closed compact L, which are continuity sets for $\mathrm{cl}(Y^c)$. The latter means that

$$\mathbb{P}\left\{\mathrm{cl}(Y^c) \cap L = \emptyset\right\} = \mathbb{P}\left\{\mathrm{cl}(Y^c) \cap \mathrm{Int}(L) = \emptyset\right\}.$$

Fix a regular closed compact set L which is a continuity set. Since

$$\mathbb{P}\left\{\mathrm{cl}(Y^c) \cap L = \emptyset\right\} = \mathbb{P}\{L \subseteq \mathrm{Int}(Y)\}$$

and

$$\mathbb{P}\left\{\mathrm{cl}(Y^c) \cap \mathrm{Int}(L) = \emptyset\right\} = \mathbb{P}\{\mathrm{Int}(L) \subseteq \mathrm{Int}(Y)\},$$

we conclude that

$$\mathbb{P}\{L \subseteq Y\} \le \mathbb{P}\{\mathrm{Int}(L) \subseteq \mathrm{Int}(Y)\} = \mathbb{P}\{L \subseteq \mathrm{Int}(Y)\} \le \mathbb{P}\{L \subseteq Y\},$$

so that $\mathbb{P}\{L \subseteq Y\} = \mathbb{P}\{L \subseteq \mathrm{Int}(Y)\}$.

Let ε_k be a sequence of positive numbers such that $\varepsilon_k \downarrow 0$ as $k \to \infty$, and

$$\mathbb{P}\left\{L + B_{\varepsilon_k} \subseteq Y\right\} = \mathbb{P}\left\{L + B_{\varepsilon_k} \subseteq \mathrm{Int}(Y)\right\}.$$

Sending $n \to \infty$ in the chain of inequalities

$$\mathbb{P}\left\{L + B_{\varepsilon_k} \subseteq Y_n\right\} \le \mathbb{P}\left\{L \subseteq \mathrm{Int}(Y_n)\right\} = \mathbb{P}\left\{\mathrm{cl}(Y_n^c) \cap L = \emptyset\right\} \le \mathbb{P}\left\{L \subseteq Y_n\right\}.$$

Then, following $\mathbb{P}\{L \subseteq Y_n\} \to \mathbb{P}\{L \subseteq Y\}$, we conclude that

$$\begin{aligned}\mathbb{P}\left\{L + B_{\varepsilon_k} \subseteq Y\right\} &\le \liminf_{n\to\infty} \mathbb{P}\left\{\mathrm{cl}(Y_n^c) \cap L = \emptyset\right\} \\ &\le \limsup_{n\to\infty} \mathbb{P}\left\{\mathrm{cl}(Y_n^c) \cap L = \emptyset\right\} \le \mathbb{P}\left\{L \subseteq Y\right\}.\end{aligned}$$

Finally, note that

$$\mathbb{P}\left\{L + B_{\varepsilon_k} \subseteq Y\right\} \uparrow \mathbb{P}\left\{L \subseteq \mathrm{Int}(Y)\right\} = \mathbb{P}\left\{L \subseteq Y\right\} \quad \text{as } k \to \infty.$$

□

Remark 3.4 By Proposition 2 in [7], each regular closed set in the convex ring is gentle. Since properties (G1) and (G2) defining a gentle set are local, each regular closed element of the extended convex ring is also gentle.

In general, the family $\mathcal{R}$ of regular closed sets is not closed under the intersection. The following lemmas show that X_n is a regular closed random set for each $n \in \mathbb{N}$.

Lemma 3.5 *Let A and B be regular closed elements of $\mathcal{U}$. Then*

$$C := \{x \in \mathbb{R}^d : A \cap (B - x) \notin \mathcal{R}\}$$

is measurable, and its Lebesgue measure is zero.

Proof Assume that $A := \cup_{i=1}^{\infty} L_i$ and $B := \cup_{j=1}^{\infty} K_j$, where L_i and K_j are convex bodies in $\mathbb{R}^d$ for each $i, j \in \mathbb{N}$. Since A and B are regular closed, we can also assume without loss of generality that L_i and K_j have not-empty interior. Define

$$\begin{aligned}C_{ij} &:= \{x \in \mathbb{R}^d : L_i \cap (K_j - x) \ne \mathrm{cl}\left(\mathrm{Int}(L_i \cap (K_j - x))\right)\} \\ &= \{x \in \mathbb{R}^d : L_i \cap (K_j - x) \ne \emptyset, \mathrm{Int}(L_i) \cap \mathrm{Int}(K_j - x) = \emptyset\},\end{aligned}$$

where the second equality follows from the fact that L_i and K_j are convex bodies.

The set of possible translations of K_j, that intersect L_i, is the Minkowski sum $\check{K}_j + L_i$, which is a convex body itself. Since we consider the translations by taking the opposite of a point in $\mathbb{R}^d$, we have that

$$\begin{aligned}\check{C}_{ij} &= (\check{K}_j + L_i) \setminus (\mathrm{Int}(\check{K}_j) + \mathrm{Int}(L_i))\\ &= (\check{K}_j + L_i) \setminus \mathrm{Int}(\check{K}_j + L_i)\\ &= \partial(\check{K}_j + L_i).\end{aligned}$$

In general, $\mathrm{Int}(\check{K}_j) + \mathrm{Int}(L_i) \subseteq \mathrm{Int}(\check{K}_j + L_i)$, but, in this case, we have equality because the two sets are convex bodies. Therefore, $V_d(C_{ij}) = 0$, since C_{ij} is the boundary of a convex body.

We now show that $C \subseteq \cup_{i,j=1}^{\infty} C_{ij}$. If this is the case, then C is measurable since the σ-algebra is complete and any subset of a measurable set of null measure is measurable, and

$$V_d(C) \le V_d(\cup_{i,j=1}^{\infty} C_{ij}) \le \sum_{i,j=1}^{\infty} V_d(C_{ij}) = 0.$$

We have

$$\begin{aligned}C &= \{x \in \mathbb{R}^d : \cup_{i=1}^{\infty} L_i \cap (\cup_{j=1}^{\infty} K_j - x) \neq \mathrm{cl}\left(\mathrm{Int}(\cup_{i=1}^{\infty} L_i \cap \cup_{j=1}^{\infty}(K_j - x))\right)\}\\ &= \{x \in \mathbb{R}^d : \cup_{i,j=1}^{\infty}(L_i \cap (K_j - x)) \supsetneq \mathrm{cl}\left(\mathrm{Int}(\cup_{i=1}^{\infty} L_i \cap \cup_{j=1}^{\infty}(K_j - x))\right)\}.\end{aligned}$$

We recall that, given two countable families $(A_i)_{i\ge 1}$ and $(B_i)_{i\ge 1}$ of subsets of $\mathbb{R}^d$, such that $A_i \supseteq B_i$ for each $i \in \mathbb{N}$ and $\cup_{i=1}^{\infty} A_i \supsetneq \cup_{i=1}^{\infty} B_i$, then there exist at least one $i \in \mathbb{N}$ such that $A_i \supsetneq B_i$. If $x \in C$, then

$$\begin{aligned}\cup_{i,j=1}^{\infty}(L_i \cap (K_j - x)) &\supsetneq \mathrm{cl}\left(\mathrm{Int}(\cup_{i=1}^{\infty} L_i) \cap \mathrm{Int}(\cup_{j=1}^{\infty}(K_j - x))\right)\\ &\supseteq \cup_{i,j=1}^{\infty} \mathrm{cl}\left(\mathrm{Int}(L_i \cap (K_j - x))\right),\end{aligned}$$

so there are $i, j \in \mathbb{N}$ such that $x \in C_{ij}$. Then $C \subseteq \cup_{i,j=1}^{\infty} C_{ij}$. □

Lemma 3.6 *Let μ be an absolutely continuous measure on a regular closed set K, which is an element of $\mathcal{U}$. Then X_n is a regular closed random set for each $n \in \mathbb{N}$.*

Proof. The proof relies on the induction. The step $n = 1$ is clear. Assume that X_{n-1} is a.s. regular closed. Define

$$A_n := \{(x_1, \dots, x_n) \in (\mathbb{R}^d)^n : \cap_{i=1}^{n}(K - x_i) \in \mathcal{R}\},$$

for $n \in \mathbb{N}$. Then $\mathbb{P}\{(\xi_1, \dots, \xi_{n-1}) \in A_{n-1}\} = 1$. We recall the notation $\Xi_n := \{\xi_1, \dots, \xi_n\}$. Since K and X_{n-1} are almost surely regular closed elements of $\mathcal{U}$, Lemma 3.5 yields that

$$\begin{aligned}&\mathbb{P}\{X_n = \mathrm{cl}(\mathrm{Int}(X_n))\} = \mathbb{E}\Big(\mathbb{P}\{X_n = \mathrm{cl}(\mathrm{Int}(X_n)) | \Xi_{n-1}\}\Big)\\ &= \mathbb{E}\Big(\mathbf{1}_{\Xi_{n-1} \in A_{n-1}} \mathbb{P}\{X_n = \mathrm{cl}(\mathrm{Int}(X_n)) | \Xi_{n-1}\}\Big)\end{aligned}$$

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$$= \mathbb{E}\Big(\mathbf{1}_{\Xi_{n-1}\in A_{n-1}}\mathbb{P}\{\{\xi : X_{n-1}\cap(K-\xi) = \mathrm{cl}(\mathrm{Int}(X_{n-1}\cap(K-\xi)))\}|\Xi_{n-1}\}\Big)$$
$$= 1. \qquad \square$$

Proof of Theorem 3.1 Assume that K is a regular closed element of $\mathcal{U}$. Let L be a compact set in $\mathbb{R}^d$. Then

$$\begin{aligned}\mathbb{P}\left\{L \subseteq n^{\gamma} X_n\right\} &= \mathbb{P}\left\{n^{-\gamma} L \subseteq (K-\xi_i) \text{ for all } i = 1,\dots,n\right\}\\ &= \left(1-\mathbb{P}\left\{n^{-\gamma} L \nsubseteq (K-\xi)\right\}\right)^n\\ &= \left(1-\mathbb{P}\left\{\xi + n^{-\gamma} L \nsubseteq K\right\}\right)^n\\ &= \left(1-\mathbb{P}\left\{\xi \notin K \ominus n^{-\gamma} L\right\}\right)^n.\end{aligned}$$

Since K is a regular closed element of $\mathcal{U}$, it is gentle, see Remark 3.4, and L is compact, Proposition 2.2 yields that

$$\begin{aligned}\lim_{n\to\infty} n\mathbb{P}\left\{\xi \notin K \ominus n^{-\gamma} L\right\} &= \lim_{n\to\infty} n\mu(K \setminus K \ominus n^{-\gamma} L)\\ &= \int_{N(K)} g(a)\frac{(h(L,u)^+)^{\alpha+1}}{\alpha+1} C_{d-1}(K, \mathrm{d}(a,u)) < \infty.\end{aligned}$$

Hence,

$$\lim_{n\to\infty} \mathbb{P}\left\{L \subseteq n^{\gamma} X_n\right\} = \exp\left(-\int_{N(K)} g(a)\frac{(h(L,u)^+)^{\alpha+1}}{\alpha+1} C_{d-1}(K, \mathrm{d}(a,u))\right).$$

Let Z be the zero cell of the tessellation generated by the point process $\mathcal{P}_K$. The random convex set Z satisfies

$$\begin{aligned}\mathbb{P}\{L \subseteq Z\} &= \mathbb{P}\left\{h(L,u)^+ \le t \text{ for all } (t,u) \in \mathcal{P}_K\right\}\\ &= \exp(-\nu(\{(t,u) \in (0,\infty)\times\mathbb{S}^{d-1} : h(L,u)^+ > t\}))\\ &= \exp\left(-\int_{N(K)} g(a)\frac{(h(L,u)^+)^{\alpha+1}}{\alpha+1} C_{d-1}(K, \mathrm{d}(a,u))\right).\end{aligned}$$

It follows that,

$$\lim_{n\to\infty} \mathbb{P}\left\{L \subseteq n^{\gamma} X_n\right\} = \mathbb{P}\{L \subseteq Z\}.$$

The zero cell Z of the tessellation is a regular closed random set. Indeed, Z is a convex closed subset of $\mathbb{R}^d$. Moreover, all half-spaces from (5) contain the origin in their interior almost surely, so that the interior of Z is not empty with probability one.

The convergence of $n^{\gamma}\,\mathrm{cl}(X_n^c)$ in distribution to $\mathrm{cl}(Z^c)$ follows by Lemma 3.3, taking into account that $n^{\gamma} X_n$ is a regular closed random set by Lemma 3.6 for each $n \in \mathbb{N}$ and Z is a regular closed random set.

Let K be a non-empty regular closed convex set. By Lemma 7.4 in [6] and the continuity theorem, the sequence of random sets $n^\gamma X_n$ converges in distribution Z as $n \to \infty$. □

It depends on the support of $\hat{\nu}$ in (4), whether the zero cell Z is unbounded or bounded almost surely. The support of $\hat{\nu}$ is contained in a closed hemisphere of $\mathbb{S}^{d-1}$ if and only if Z is unbounded almost surely. Indeed, assume that the support of $\hat{\nu}$ is contained in a closed hemisphere of $\mathbb{S}^{d-1}$, then the dual cone of the cone generated by the support of $\hat{\nu}$ is contained in Z a.s. by construction. Assume that Z is unbounded a.s., it follows from the construction of Z that there is at least a fixed $u \in \mathbb{S}^{d-1}$ such that no hyperplane generated by $\mathcal{P}_K$ intersects the cone generated by u. Then the support of $\hat{\mu}$ must be contained in the dual cone of the cone generated by u, which is a closed hemisphere of $\mathbb{S}^{d-1}$.

Assume that K is a convex body whose interior is non-empty, the sequence of random convex bodies $n^\gamma X_n$ generated by K can converge in distribution to a random set which is unbounded almost surely.

Example 3.7 Let K be the unit ball centred in the origin and let μ be a probability measure whose limit of the density function g has support contained in a closed hemisphere so that the support of $\hat{\nu}$ is also contained in a closed hemisphere. Then the sequence of random sets $n^\gamma X_n$ converges in distribution to a random set which is unbounded almost surely.

It is straightforward to deduce the convergence of all power moments of the volume restricted to a compact set.

Proposition 3.8 *Let M be a compact set and let K be a gentle set. Let $n^\gamma X_n$ and Z be defined as in* (3) and (5) *respectively. Then, for every $m \in \mathbb{N}$,*

$$\mathbb{E}V_d(n^\gamma X_n \cap M)^m \to \mathbb{E}V_d(Z \cap M)^m \quad \text{as } n \to \infty.$$

Proof Following the same steps of proof of Theorem 3.1 and by Proposition 2.2, we notice that, for $x_1, \ldots, x_m \in \mathbb{R}^d$,

$$\begin{aligned}\mathbb{P}\left\{x_1, \ldots, x_m \in n^\gamma X_n\right\} = \mathbb{P}\left\{\{x_1, \ldots, x_m\} \subseteq n^\gamma X_n\right\} \to \\ \mathbb{P}\left\{\{x_1, \ldots, x_m\} \subseteq Z\right\} = \mathbb{P}\left\{x_1, \ldots, x_m \in Z\right\},\end{aligned}$$

as $n \to \infty$, even if K is a gentle set.

Then, by the dominated convergence theorem,

$$\begin{aligned}\mathbb{E}V_d(n^\gamma X_n \cap M)^m &= \int_M \cdots \int_M \mathbb{P}\left\{x_1, \ldots, x_m \in n^\gamma X_n\right\} dx_1 \ldots dx_m \\ &\to \int_M \cdots \int_M \mathbb{P}\left\{x_1, \ldots, x_m \in Z\right\} dx_1 \ldots dx_m \quad \text{as } n \to \infty \\ &= \mathbb{E}V_d(Z \cap M)^m,\end{aligned}$$

since M is bounded. □

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In general, it does not hold that if $n^\gamma X_n$ and Z are bounded almost surely for every $n \in \mathbb{N}$, then the sequence $\mathbb{E}V_d(n^\gamma X_n)$ converges to $\mathbb{E}V_d(Z)$.

If K is convex, then X_n is also convex. In this case, it is possible to consider its intrinsic volumes V_j, for $j = 0, \ldots, d$, which are defined by the Steiner formula, see Theorem 3.10 in [9].

Corollary 3.9 *Let K be a convex body with a non-empty interior. Let μ be a probability measure which satisfies assumptions (M1) and (M2). Let $n^\gamma X_n$ and Z be defined as in (3) and (5) respectively. Furthermore, assume that the support of $\hat{\nu}$, as defined in (4), is not contained in a hemisphere of $\mathbb{S}^{d-1}$. Then, for $j = 0, \ldots, d$,*

$$V_j(n^\gamma X_n) = n^{\gamma j} V_j(X_n) \xrightarrow{d} V_j(Z) \quad \text{as } n \to \infty.$$

Proof The intrinsic volumes V_j are continuous with respect to the convergence in the Fell topology restricted to the family of convex bodies, see Remark 3.22 in [9]. The random closed set X_n is almost surely a convex body with non-empty interior since K is a convex body with non-empty interior. Since the support of $\hat{\mu}$ is not contained in a hemisphere of $\mathbb{S}^{d-1}$, Z is almost surely a convex body with non-empty interior. Then the convergence in distribution of $n^\gamma X_n$ to Z is assured from the second part of the statement of Theorem 3.1. By the continuity theorem, for $j = 0, \ldots, d$,

$$V_j(n^\gamma X_n) = n^{\gamma j} V_j(X_n) \xrightarrow{d} V_j(Z) \quad \text{as } n \to \infty.$$

□

Example 3.10 Let K be the union of the two disjoint balls $B_1(0)$ and $B_1(x)$, with $|x| > 2$, and let μ be the uniform distribution on $B_1(0)$. The set of sample points $\Xi_n = \{\xi_1, \ldots, \xi_n\}$ is a subset of $B_1(0)$ almost surely. The random set nX_n is the disjoint union of the convex random body

$$n\tilde{X}_n := n\big(\cap_{i=1}^n (B_1(0) - \xi_i)\big),$$

and its translate by nx.

The set $n\tilde{X}_n$ converges in distribution to the zero cell Z and the set $n\tilde{X}_n + nx$ converges in distribution to the empty set as $n \to \infty$.

Then

$$V_d(nX_n) = 2V_d\big(n\tilde{X}_n\big) \xrightarrow{d} 2V_d(Z) \quad \text{as } n \to \infty.$$

But nX_n also converges in distribution to Z. Hence, $V_d(nX_n)$ does not converge in distribution to $V_d(Z)$ as $n \to \infty$. In particular, from Proposition 5.4 in [5],

$$\mathbb{E}V_d(nX_n) = 2\mathbb{E}V_d\big(n\tilde{X}_n\big) \to 2\mathbb{E}V_d(Z) \quad \text{as } n \to \infty,$$

so that $\mathbb{E}V_d(nX_n)$ does not converge to $\mathbb{E}V_d(Z)$ as $n \to \infty$.

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Declarations

Conflict of interest The author declares no competing interests.

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Additional Comments on Theorem 3.1

By Theorem 3.1, it holds that $n^\gamma \operatorname{cl}(X_n^c) \xrightarrow{\mathrm{d}} \operatorname{cl}(Z^c)$ as $n \to \infty$, when K is a regular closed element of the extended convex ring. While it appears natural that $n^\gamma X_n \xrightarrow{\mathrm{d}} Z$, since X_n and Z are regular closed for each $n \in \mathbb{N}$, no proof or counterexample could be found.

To try to achieve this result, two main approaches were followed.

The first approach is to study the continuity properties of the complement function in the Fell topology. In particular, checking whether the function

$$C(\cdot) = \operatorname{cl}(\cdot^c) : \mathcal{R} \cap \mathcal{U} \mapsto \mathcal{R}, \quad K \to \operatorname{cl}(K^c),$$

is continuous. Following from Lemma 7.4 in Kabluchko et al. (2025), it holds that the operation $\operatorname{cl}(\cdot^c)$ from the family of regular closed convex sets to the family of closure of their complements is bicontinuous. This does not hold for $C(\cdot)$. Let $(\operatorname{cl}(F_n^c))_{n\in\mathbb{N}}$ be a sequence of closure of complements of elements of $\mathcal{R} \cap \mathcal{U}$ in $\mathbb{R}^2$ defined as follows:

$$\operatorname{cl}(F_n^c) = \operatorname{cl}([-1,1]^2 \setminus (\mathbb{R}^+ \times [-\frac{1}{n}, \frac{1}{n}])),$$

which is a box with a decreasing gap. Following Theorem 4.0.1 in the Appendix, we can see that $\operatorname{cl}(F_n^c) \to [-1,1]^2$ in the Fell topology, as $n \to \infty$. Meanwhile $F_n = \operatorname{cl}(\operatorname{cl}(F_n^c)^c)$ converges to $\operatorname{cl}(\mathbb{R}^2 \setminus [-1,1]^2) \cup \{0\} \times [0,1]$, which is not the closure of the complement of $[-1,1]^2$.

The second is to use the properties of the finite-dimensional distributions of random closed sets, given that $\mathcal{F}$ is a compact set with a countable base. The family $\mathcal{F}$ of closed sets is compact in the Fell topology, so no tightness conditions are required for the convergence in distribution, see Section 1.7 in Molchanov (2017). It follows that each sequence X_n of random closed sets has a subsequence which converges in distribution. For a random closed set X, recall that the functional $I_X(L) = \mathbb{P}\{L \subseteq X\}$ is called the inclusion functional of X. Consider the random function $\mathbb{1}_{x\in X}$. By Kolmogorov's extension theorem, its distribution is determined by its finite-dimensional distributions, which can be expressed in the following terms $\mathbb{P}\{x_1, \dots, x_m \in X\} = I_X(\{x_1, \dots, x_m\})$ for $x_1, \dots, x_m \in \mathbb{R}^d$, for more details see Subsection 1.1.7 in Molchanov (2017). Moreover, if X is a regular closed random set, its finite-dimensional distributions uniquely determine its distribution. The proof of Theorem 3.1 shows that the inclusion functional of $n^\gamma X_n$ over all the compact sets converges to the inclusion functional of Z, which implies the convergence of the finite-dimensional distributions. This is not sufficient to show the convergence in distribution since each finite subset of $\mathbb{R}^d$ is not a continuity set for Z, see Definition 1.7.5 in Molchanov (2017).

Open Problem

Let $K, L \in \mathcal{K}^d$ such that $K \ominus L \neq \varnothing$. Let $\xi_1, \dots, \xi_n$ be i.i.d points on K with respect to an absolutely continuous probability measure μ. Define the random convex body

$$X_n = \cap_{i=1}^n (L + \xi_i).$$

Set

$$p_n = \mathbb{P}\{X_n \neq \varnothing\},$$

with $p_0 = 1$, for $n \in \mathbb{N}_0$. Interestingly, the event $\{X_n \neq \varnothing\}$ is equivalent to the event

$$\{L \ominus \mathrm{conv}\{\xi_1, \ldots, \xi_n\} \neq \varnothing\},$$

i.e. X_n is non-empty if and only if the convex hull of $\xi_1, \ldots, \xi_n$ is contained in a translation of L.

An open problem is to characterise p_n in terms of K and L. A general formula for p_n appears hard to achieve, except in specific cases. For example, define $K = \times_{i=1}^d [0, a_i]$ and $L = \times_{i=1}^d [0, b_i]$ with $b_i < a_i$ for each $i = 1, \ldots, d$, and let μ be the uniform distribution on K. Then

$$p_n = \Pi_{i=1}^d P_{i,n}$$

for $P_{i,n} = \mathbb{P}\{\max_{1 \leq l \leq k \leq n} |\eta_l - \eta_k| \leq b_i\}$, where $\eta_1, \ldots, \eta_n$ are i.i.d random variables uniformly distributed on $[0, a_i]$.

Proposition 1.0.1. *Let τ be defined as follows*

$$\tau = \min\{k \geq 1 : X_k = \varnothing\}.$$

Then τ is a stopping time with respect to the natural filtration of the sequence $(\xi_i)_{i \geq 1}$. τ is finite almost surely and

$$\mathbb{E}(\tau) = \sum_{n=0}^{\infty} p_n.$$

Proof. Since $K \ominus L \neq \varnothing$, it is straightforward to see that τ is a stopping time which is finite almost surely. Moreover

$$\mathbb{E}(\tau) = \sum_{n=0}^{\infty} \mathbb{P}\{\tau > n\} = \sum_{n=0}^{\infty} \mathbb{P}\{X_n \neq \varnothing\} = \sum_{n=0}^{\infty} p_n.$$

□

Chapter 2

Integer-valued Valuations

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Integer-valued valuations

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Abstract

We obtain a complete characterization of planar monotone σ-continuous valuations taking integer values, without assuming invariance under any group of transformations. We further investigate the consequences of dropping monotonicity or σ-continuity and give a full classification of line valuations. We also introduce a construction of the product for valuations of this type.

Keywords: valuation, normal cone, polyconvex set, product of valuations

MSC2020: 52A10

1 Introduction

A valuation φ is an additive map from the family of compact convex subsets of a finite-dimensional vector space to an abelian semigroup. Additivity means that, for any compact convex sets K and L such that $K \cup L$ is also convex, the following identity holds:

$$\varphi(K \cup L) + \varphi(K \cap L) = \varphi(K) + \varphi(L),$$

see [8, Chapter 6] for a detailed exposition. Additionally, we will always include the empty set in the domain of φ and assume that $\varphi(\varnothing) = 0$. Most of the literature on valuations focuses on valuations with values in the set of real or complex numbers or in the family of compact convex sets equipped with Minkowski addition.

A common assumption in the study of valuations is their invariance under a group of transformations. In most cases, valuations are assumed to be translation invariant, meaning that $\varphi(K+x) = \varphi(K)$ for all translations x. Alternatively, valuations are also studied under the assumption of rotation invariance or invariance under the group of all rigid motions. Another frequently imposed condition is continuity with respect to the Hausdorff metric on compact convex sets. This condition is sometimes relaxed to σ-continuity, which requires that $\varphi(K_n) \to \varphi(K)$ whenever $K_n \downarrow K$.

Let $\mathcal{K}^d$ be the family of convex bodies (i.e., compact convex sets) in $\mathbb{R}^d$. While the empty set is typically not considered a convex body, we adopt the convention that it is included in $\mathcal{K}^d$. By Hadwiger's theorem, any real-valued continuous and invariant under rigid motions valuation on $\mathcal{K}^d$ can be expressed as a weighted sum of the intrinsic volumes $V_i(K)$, $i = 0, \dots, d$. Furthermore, MacMullen's theorem states that the vector space of all continuous translation-invariant valuations can be decomposed into a direct sum of subspaces consisting of valuations that are homogeneous of order $k = 0, \dots, d$. A more refined result is given by the theorem of Klain and Schneider. It states that if φ is a continuous simple translation-invariant valuation, then

$$\varphi(K) = cV_d(K) + \int_{\mathbb{S}^{d-1}} f(u)\, \mathrm{d}S_{d-1}(K, u),$$

where $c \in \mathbb{R}$, f is an odd continuous function, and $S_{d-1}(K, \cdot)$ stands for the area measure of K. Here, simplicity means that φ vanishes on all lower-dimensional sets.

In this paper, we consider valuations taking values in the group $\mathbb{Z}$ of integers under addition. Clearly, the only continuous valuations with values in $\mathbb{Z}$ are multiples of the Euler characteristic

$$\chi(K) = \begin{cases} 1, & K \neq \varnothing, \\ 0, & K = \varnothing. \end{cases}$$

It is straightforward to see that a sum of Euler characteristics like

$$\varphi(K) = \sum_{i=1}^{N} \chi(K \cap C_i)$$

for convex bodies $C_1, \dots, C_N$ defines an integer-valued monotone σ-continuous valuation. Due to the intersection operation, φ is no longer continuous in the Hausdorff metric. Adding negative terms to this sum preserves additivity and σ-continuity and may still retain the monotonicity property, as our examples demonstrate.

Our paper focuses on integer-valued monotone σ-continuous valuations without imposing *any invariance assumptions* and provides their complete characterization in dimensions 1 and 2. In the main results, we establish that each integer-valued, monotone, and σ-continuous valuation in dimensions 1 and 2 can be represented as an at most countable sum of Euler characteristics with weights ± 1. The convex bodies C_i necessarily form a locally finite family, and the bodies appearing in the negative terms satisfy a strict admissibility property with respect to the positive ones. In other words, each integer-valued monotone σ-continuous valuation corresponds to a locally finite integer-valued measure on the family of convex bodies.

A key step in proving the representation involves the support F of a valuation φ, which is the set of points x such that $\varphi(\{x\}) \geq 1$. We show that each integer-valued σ-continuous valuation is uniquely determined by its values on singletons and that the intersection of F

with any convex body is polyconvex. For the latter, we apply Eggleston's theorem, which links polyconvexity to the structure of invisible points. The absence of such a result in dimensions 3 and higher makes it impossible to generalize our technique beyond the planar case.

The main result in dimension 2 is proved in Section 3. In Section 4 we characterise all real-valued valuations on the line. In Section 5, we introduce countably generated valuations, which generalize the weighted sums of Euler characteristics discussed above. We then define the multiplication of such valuations by arbitrary σ-continuous ones and examine the properties of this product. Section 6 contains a collection of open problems and conjectures.

2 Preliminaries on integer-valued valuations

A set function $\varphi : \mathcal{K}^d \to \mathbb{R}$ is called *monotone* if $\varphi(K) \le \varphi(L)$ whenever $K \subset L$. In particular, this implies nonnegativity: $\varphi(K) \ge \varphi(\varnothing) = 0$.

Following [8, p. 338], we call a valuation φ *σ-continuous* if

$$\varphi(K) = \lim_{n\to\infty} \varphi(K_n) \tag{1}$$

for any sequence (K_n) of convex bodies such that $K_n \downarrow K$. First of all, we note that, for integer-valued valuations, it suffices to check σ-continuity only at singletons.

Proposition 2.1. *Let φ be an integer-valued valuation on $\mathcal{K}^d$ such that (1) holds for all $K = \{x\}$, $x \in \mathbb{R}^d$. Then φ is σ-continuous.*

Proof. Assume that the claim is false, and (1) fails to hold for some $K_n \downarrow K$. Then $\varphi(K_n) \neq \varphi(K)$ for infinitely many n. For all such n and for any hyperplane H_1 meeting K and dividing $\mathbb{R}^d$ into closed half-spaces H_1^- and H_1^+, we have

$$\begin{aligned}\varphi(K) &= \varphi(K \cap H_1^-) + \varphi(K \cap H_1^+) - \varphi(K \cap H_1),\\ \varphi(K_n) &= \varphi(K_n \cap H_1^-) + \varphi(K_n \cap H_1^+) - \varphi(K_n \cap H_1).\end{aligned}$$

Hence, there is $H_1^\bullet \in \{H_1^-, H_1^+, H_1\}$ such that $\varphi(K \cap H_1^\bullet) \neq \varphi(K_n \cap H_1^\bullet)$ for infinitely many n. Proceeding with this division process and choosing H_m and $H_m^\bullet$ at the m-th step in such a way that $K \cap H_1^\bullet \cap \ldots \cap H_m^\bullet$ shrink to a singleton $\{x\}$ as $m \to \infty$, we obtain

$$\varphi(K \cap H_1^\bullet \cap \ldots \cap H_m^\bullet) \neq \varphi(K_n \cap H_1^\bullet \cap \ldots \cap H_m^\bullet) \tag{2}$$

for each fixed m and infinitely many n. However, due to the σ-continuity of φ at $\{x\}$, both sides of (2) converge to $\varphi(\{x\})$ as $m, n \to \infty$ simultaneously. Since both sides are integer-valued, this contradicts (2). □

Denote $H_{u,t}^- = \{x \in \mathbb{R}^d \colon \langle u, x\rangle \le t\}$. The following criterion is useful for verifying the monotonicity of a (not necessarily additive) σ-continuous set function.

Proposition 2.2. *A non-negative σ-continuous set function φ on $\mathcal{K}^d$ is monotone if and only if $\varphi(H^-_{u,t} \cap M)$ is non-decreasing in t for each fixed $u \in \mathbb{S}^{d-1}$ and $M \in \mathcal{K}^d$.*

Proof. The necessity is clear. To prove sufficiency, let $K \subset L$, choose an $x_1 \in \partial K$, and draw through x_1 a supporting hyperplane H_1 to K. Denote by L_1 the part of L cut off by H_1 and containing K. Using the assumption with u orthogonal to H_1 and $M = L$, we get $\varphi(L) \geq \varphi(L_1)$. Proceeding with this process and choosing $x_n \in \partial K$ and H_n at each step in such a way that $L_n \downarrow K$, we obtain, by applying the assumption to $M = L_{n-1}$, that $\varphi(L_{n-1}) \geq \varphi(L_n)$. Thus, $\varphi(L) \geq \varphi(L_n)$, and, by σ-continuity, $\varphi(L) \geq \varphi(K)$. □

Note that, as follows from the proof, this proposition remains valid even if σ-continuity is replaced by a significantly weaker condition $\varphi(K) \leq \sup_{n\geq 1} \varphi(K_n)$ for any $K_n \downarrow K$.

We now give a somewhat unexpected property of integer-valued σ-continuous valuations, which plays a fundamental role in what follows.

Proposition 2.3. *Let φ and φ' be integer-valued σ-continuous valuations on $\mathcal{K}^d$ that coincide on singletons: $\varphi(\{x\}) = \varphi'(\{x\})$ for any $x \in \mathbb{R}^d$. Then $\varphi = \varphi'$.*

Proof. We employ reasoning similar to that used in the proof of Proposition 2.1. Suppose the claim is false and $\varphi(K) \neq \varphi'(K)$ for some $K \in \mathcal{K}^d$. Drawing a hyperplane H that meets K and denoting the closed half-spaces it cuts off by H^- and H^+, we have

$$\begin{aligned}\varphi(K \cap H^-) + \varphi(K \cap H^+) - \varphi(K \cap H) = \varphi(K)&\\ \neq \varphi'(K) = \varphi'(K \cap H^-) + \varphi'(K \cap H^+) - \varphi'(K \cap H).&\end{aligned}$$

Thus, $\varphi(K_1) \neq \varphi'(K_1)$ for some $K_1 \in \{K \cap H^-, K \cap H^+, K \cap H\}$. Proceeding with this process so that $K_n \downarrow \{x\}$ for some $x \in \mathbb{R}^d$, we have $\varphi(K_n) \neq \varphi'(K_n)$ for all n, while, by σ-continuity,

$$\lim_{n\to\infty} \varphi(K_n) = \varphi(\{x\}) = \varphi'(\{x\}) = \lim_{n\to\infty} \varphi'(K_n).$$

This is impossible due to the integer-valued property of φ and φ'. □

Proposition 2.3 implies that no simple (i.e., vanishing on lower-dimensional sets) integer-valued σ-continuous valuations exist. Any such valuation must vanish on singletons and is therefore identically zero.

3 The structure of planar integer-valued valuations

In this section, we describe the structure of planar integer-valued monotone σ-continuous valuations. Recall that the *normal cone* to a closed convex set C at a point $x \in C$ is defined by

$$\mathcal{N}_C(x) = \big\{u \in \mathbb{R}^d \colon \langle u, y - x\rangle \leq 0 \text{ for all } y \in C\big\} \tag{3}$$

and adopt the convention $\mathcal{N}_C(x) = \varnothing$ for $x \notin C$. In particular,

$$\mathbb{1}_{\mathcal{N}_C(x)}(0) = \mathbb{1}_C(x), \tag{4}$$

$$\mathbb{1}_{\mathcal{N}_C(x)}(u) = \mathbb{1}_C(x) \cdot \mathbb{1}\{C \cap \mathring{H}_u^+(x) = \varnothing\}, \quad u \neq 0, \tag{5}$$

where

$$\mathring{H}_u^+(x) = \{y \in \mathbb{R}^d \colon \langle u, y - x\rangle > 0\}.$$

This means that $\mathcal{N}_C(x)$ is empty for $x \notin C$, contains only 0 for $x \in \operatorname{int} C$, and is a non-degenerate closed convex cone for $x \in \partial C$. Also denote $\overline{\mathbb{N}}_0 = \mathbb{N} \cup \{0, \infty\}$.

Definition 3.1.

(i) A family of $N \in \overline{\mathbb{N}}_0$ closed convex sets C_n is said to be *locally finite* if only finitely many of them hit any fixed $K \in \mathcal{K}^d$.

(ii) A locally finite family (C_n^-) of cardinality N^- is said to be *admissible* with respect to a locally finite family (C_n^+) of cardinality N^+ if

$$\sum_{n=1}^{N^-} \mathbb{1}_{\mathcal{N}_{C_n^-}(x)}(u) \leq \sum_{n=1}^{N^+} \mathbb{1}_{\mathcal{N}_{C_n^+}(x)}(u) \tag{6}$$

for all $x, u \in \mathbb{R}^d$.

In particular, (6) implies that $\bigcup_n C_n^- \subset \bigcup_n C_n^+$ by letting $u = 0$ and using (4). Furthermore, (6) yields that $\bigcup_n \partial C_n^- \subset \bigcup_n \partial C_n^+$. Otherwise, for any x violating this inclusion, the right-hand side of (6) vanishes for all $u \neq 0$, while the left-hand side does not for some u.

The simplest example of an integer-valued monotone σ-continuous valuation is provided by the Euler characteristic

$$\chi(K) = \mathbb{1}\{K \neq \varnothing\}, \quad K \in \mathcal{K}^d.$$

The following theorem provides a complete description of such valuations for $d = 2$.

Theorem 3.2. *A function $\varphi : \mathcal{K}^2 \to \mathbb{Z}$ is an integer-valued monotone σ-continuous valuation if and only if there exist $N^+, N^- \in \overline{\mathbb{N}}_0$ and two locally finite families of N^+ and N^- nonempty closed convex sets C_n^+ and C_n^- with the latter being admissible with respect to the former, such that, for any $K \in \mathcal{K}^2$,*

$$\varphi(K) = \sum_{n=1}^{N^+} \chi\big(K \cap C_n^+\big) - \sum_{n=1}^{N^-} \chi\big(K \cap C_n^-\big). \tag{7}$$

The families (C_n^+) and (C_n^-) are not uniquely determined: $(C_n^+), (C_n^-)$ and $(\widetilde{C}_n^+), (\widetilde{C}_n^-)$ define the same valuation if and only if

$$\sum_{n=1}^{N^+} \mathbb{1}_{C_n^+}(x) - \sum_{n=1}^{N^-} \mathbb{1}_{C_n^-}(x) = \sum_{n=1}^{\widetilde{N}^+} \mathbb{1}_{\widetilde{C}_n^+}(x) - \sum_{n=1}^{\widetilde{N}^-} \mathbb{1}_{\widetilde{C}_n^-}(x) \tag{8}$$

for all $x \in \mathbb{R}^2$.

The non-uniqueness of the representation (7) is confirmed by the following example.

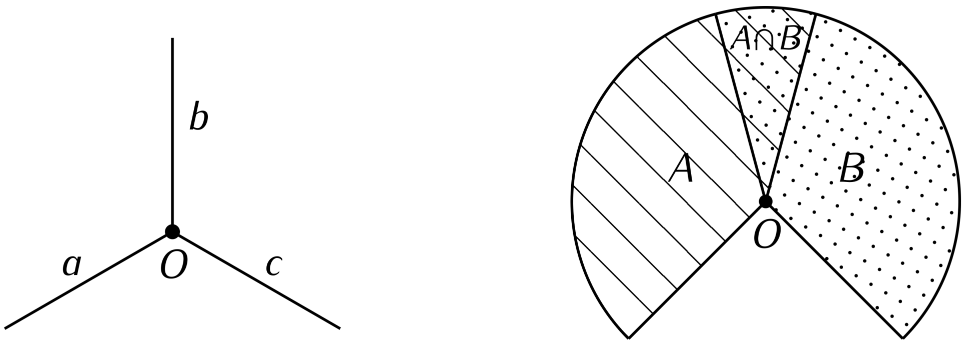
Example 3.3. Let C_1 and C_2 be two convex bodies such that $C_1 \cup C_2$ is convex. Then the valuation $\varphi(K) = \chi(K \cap (C_1 \cup C_2))$ can be alternatively represented as

$$\varphi(K) = \chi(K \cap C_1) + \chi(K \cap C_2) - \chi(K \cap (C_1 \cap C_2)).$$

Since both sides agree on singletons, this follows from Proposition 2.3.

The following examples demonstrate that not all monotone valuations can be constructed using only C_n^+.

Example 3.4. Let a, b and c be segments positioned as shown in Figure 1(a) with O denoting their intersection point.



(a) (b)

Figure 1

Consider the valuation

$$\varphi(K) = \chi(K \cap a) + \chi(K \cap b) + \chi(K \cap c) - \chi(K \cap \{O\}), \quad K \in \mathcal{K}^2.$$

This valuation is integer-valued σ-continuous and monotone. To prove monotonicity, note that $\varphi(K) = 0$ means that K and $a \cup b \cup c$ are disjoint, and $\varphi(K) = 1$ means that K intersects exactly one of these segments. Thus, for $\varphi(K) \le 1$ and $K' \subset K$, we have $\varphi(K') \le \varphi(K)$. If $\varphi(K) = 2$, the latter inequality holds because $\varphi(K') \le 2$ for all $K' \in \mathcal{K}^2$.

Another way to prove the monotonicity of φ is to verify the admissibility of $(\{O\})$ with respect to (a, b, c). From (3), it easily follows that $\mathcal{N}_{\{O\}}(O) = \mathbb{R}^2$, and for any $s \in \{a, b, c\}$, $\mathcal{N}_s(O)$ is a closed half-plane that does not contain $\operatorname{int} s$, with its boundary passing through O and orthogonal to s. Since the union of these three half-planes is the entire plane, (6) holds for $x = O$. At all other points, (6) holds trivially, as its left-hand side vanishes for any u. Hence, φ is monotone by Theorem 3.2.

Example 3.5. Now consider the valuation

$$\varphi = \chi(\cdot \cap A) + \chi(\cdot \cap B) + \chi(\cdot \cap \{O\}) - \chi(\cdot \cap A \cap B),$$

see Figure 1(b). It is also integer-valued σ-continuous and monotone; both proofs of monotonicity are similar to those in Example 3.4. Note that without the term $\chi(\cdot \cap O)$, the valuation would be non-monotone. Altering the positions of the lines bordering $A \cap B$ (while maintaining their nonempty intersection) does not change the valuation but leads to its different representation.

We will precede the proof of Theorem 3.2 with two auxiliary lemmas. For the first one, we call a point set P an *invisibility set* if $\varphi(\{x\}) \geq 1$ for each $x \in P$, and, for any $x, y \in P$, there exists a point $z \in (x, y)$ such that $\varphi(\{z\}) = 0$. We denote the convex hull of a set P by $\operatorname{conv} P$ and write $\operatorname{card} P$ for the cardinality of P. Note that the bound on the cardinality of P in the following result is apparently far from optimal one, but it suffices for our purposes.

Lemma 3.6. *Let φ be an integer-valued monotone valuation on $\mathcal{K}^2$ and $n \in \mathbb{N}$. If there exists an invisibility set P with $\operatorname{card} P \geq 4^n$, then $\varphi(\operatorname{conv} P) > \frac{n}{2}$.*

Proof. We proceed by induction on n. For $n = 1$, the claim is clear. Assume it holds for $n - 1$. Arguing by contradiction, suppose that $\varphi(\operatorname{conv} P) \leq \frac{n}{2}$.

As before, for a line H, we denote by H^- and H^+ the two closed half-planes into which H divides $\mathbb{R}^2$. Draw H in such a way that $\operatorname{card}(P \cap H^-) \geq 2^{2n-1}$ and $\operatorname{card}(P \cap H^+) \geq 2^{2n-1}$. With a slight adjustment, H can always be made to pass through some $x, y \in P$. Mark $z \in (x, y)$ with $\varphi(\{z\}) = 0$, and denote by a, b the intersection points of H and $\partial \operatorname{conv} P$. Connect z by line segments to some $u, v \in \partial \operatorname{conv} P$ in such a way as to divide $P \cap H^-$ and $P \cap H^+$ into four closed convex polygons $Q^{--}, Q^{-+}, Q^{+-}, Q^{++}$ with $\operatorname{card} Q^{ij} \geq 4^{n-1}$, $i, j \in \{-, +\}$, see Figure 2 for $n = 2$.

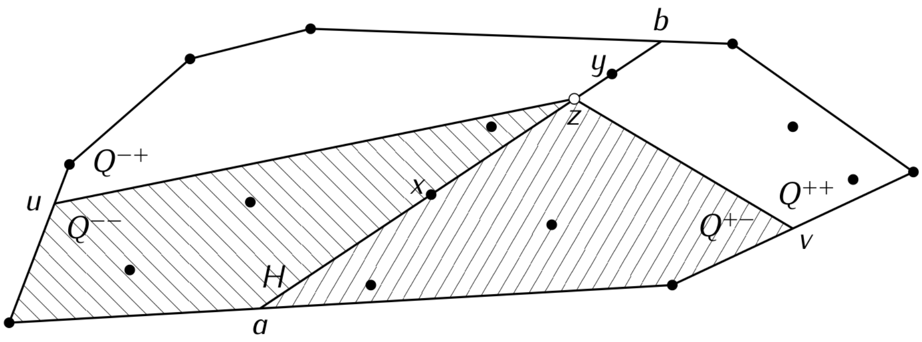


Figure 2

One of the polygons $Q^{--} \cup Q^{+-}$ or $Q^{-+} \cup Q^{++}$ is convex, depending on whether the angle $\angle uzv$ is $\leq \pi$ or $\geq \pi$. Assume the former. Since

$$\varphi([a, z]) = \varphi([a, b]) + \varphi(\{z\}) - \varphi([z, b]) \leq \varphi(\operatorname{conv} P) - \varphi(\{y\}) \leq \frac{n}{2} - 1,$$

applying the induction hypothesis to Q^{--} and Q^{+-} yields the contradiction:

$$\frac{n}{2} \geq \varphi(\operatorname{conv} P) \geq \varphi(Q^{--} \cup Q^{+-}) \\ = \varphi(Q^{--}) + \varphi(Q^{+-}) - \varphi([a,z]) > 2\,\frac{n-1}{2} - \left(\frac{n}{2} - 1\right) = \frac{n}{2}. \quad \square$$

Recall that a set is said to be *polyconvex* if it is a finite union of (not necessarily disjoint) convex sets. In particular, the empty set is also considered as polyconvex.

Lemma 3.7. *Let φ be an integer-valued monotone σ-continuous valuation defined on closed convex subsets of some $W \in \mathcal{K}^2$. Then its support*

$$F = \{x \in W : \varphi(\{x\}) \geq 1\}$$

is polyconvex, and all its convex components are closed.

Before proceeding to the proof, we recall a fact from convex geometry. For $m \geq 2$, a set $S \subset \mathbb{R}^2$ is called *m-convex* if, for any m distinct points in S, at least one of the line segments connecting them lies in S. In particular, 2-convex sets are just convex. According to Eggleston's theorem [3], a closed m-convex set is polyconvex. Note that an extensive literature has been devoted to deriving upper bounds on the number of convex components, see [2], [6], [7], etc.

Proof of Lemma 3.7. We first note that F is closed. Indeed, if $F \ni x_k \to x$, then, for some closed convex neighbourhood V_x of x and some $k \geq 1$, we have by σ-continuity and monotonicity that

$$\varphi(\{x\}) = \varphi(V_x) \geq \varphi(\{x_k\}) \geq 1.$$

This implies $x \in F$.

Take any set P of $m = 4^{2\varphi(W)}$ points from F. At least one of the line segments connecting them lies entirely in F: otherwise, they would form an invisibility set, and by Lemma 3.6, we would arrive at the contradiction $\varphi(W) \geq \varphi(\operatorname{conv} P) > \varphi(W)$. Hence, by Eggleston's theorem, $F = \bigcup_{i=1}^{l} K_i$ for some $l \geq 0$ and convex K_i. Taking the closures of both sides of this equality and recalling that F is closed, we arrive at the desired representation. $\square$

Proof of Theorem 3.2.

Sufficiency. The set function φ given by (7) is an integer-valued σ-continuous valuation, since it is a sum of such valuations and, due to local finiteness, this sum has only finitely many nonzero terms for each K. The only thing that remains to be proved is its monotonicity.

Taking $u = 0$ in (6) and using (4), we have

$$\varphi(\{x\}) = \sum_{n=1}^{N^+} \mathbb{1}_{C_n^+}(x) - \sum_{n=1}^{N^-} \mathbb{1}_{C_n^-}(x) \geq 0, \quad x \in \mathbb{R}^2. \tag{9}$$

We will now show that $K \subset L$ implies $\varphi(K) \leq \varphi(L)$. In particular, combined with (9), this ensures $\varphi(K) \geq 0$ for any K.

Denote by $\mathcal{F}_0$ the family of all sets C_n^+ and C_n^- which appear in (7). Fix $K_0 = K \subset L$, and define $\mathcal{F}_1$ to be the family of all sets from $\mathcal{F}_0$ that hit L while missing K_0. Due to local finiteness, $\mathcal{F}_1$ is finite, and it is possible to find a $\delta > 0$ such that the family of sets from $\mathcal{F}_0$ which hit $L + B_\delta(0)$ while missing K_0 is exactly $\mathcal{F}_1$. Here $+$ stands for the Minkowski addition. Now replace all sets $C_n^\pm$ from the family $\mathcal{F}_1$ by their intersections with L. This does not affect the values of φ on L and its subsets.

We claim that there exist

1) a point $x_1 \in L$ on the boundary of some set from $\mathcal{F}_1$,

2) a supporting line H_1 at x_1 to this set that separates its interior from K_0,

3) a segment S_{ε_1} on H_1 of small length $2\varepsilon_1$ with $\varepsilon_1 < \delta / \operatorname{card} \mathcal{F}_1$ centered at x_1, such that S_{ε_1} hits the same sets from $\mathcal{F}_0$ as $\{x_1\}$ and $\operatorname{conv}(K_0 \cup S_{\varepsilon_1}) \setminus S_{\varepsilon_1}$ does not intersect any set from $\mathcal{F}_1$,

see Figure 3, where, for simplicity, the sets C_n^+ and C_n^-, shown in gray, are depicted as disjoint. The above construction can be carried out by choosing x_1 to be the minimizer r of the function $x \mapsto \inf_{z \in K_0} \|z - x\|$ for all points x from any of the sets in the family $\mathcal{F}_1$. In the case of multiple minimizers, any of them can be chosen. Note as well that this minimizer may belong to several sets, say $C_{i_1}, \dots, C_{i_p}$, from $\mathcal{F}_1$. The r-parallel set K_0^r is smooth at x, so there is a unique supporting line which then becomes H_1. Since any other set from $\mathcal{F}_1$ is farther away from K_0 than r, none of them intersects $\operatorname{conv}(K_0 \cup S_{\varepsilon_1})$ for a sufficiently small segment S_{ε_1} on H_1 centered at x_1. Furthermore, since ∂K_0^r is smooth at x_1, no set from $C_{i_1}, \dots, C_{i_p}$ intersects $\operatorname{conv}(K_0 \cup S_{\varepsilon_1}) \setminus S_{\varepsilon_1}$.

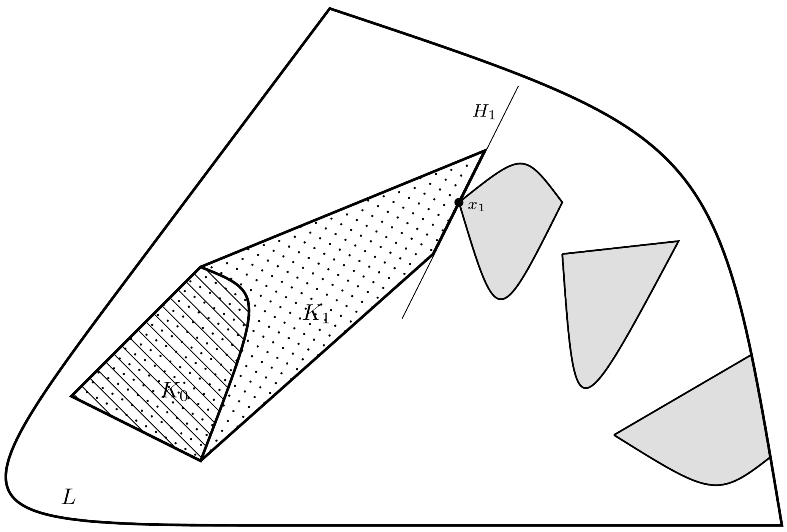


Figure 3

Denote $K_1 = \operatorname{conv}(K_0 \cup S_{\varepsilon_1})$ and observe that $K_0 \subset K_1 \subset L + B_{\varepsilon_1}(0)$. Let $\mathcal{F}_2$ be the family of sets from $\mathcal{F}_0$ (actually, from $\mathcal{F}_1$) that hit L while missing K_1. Now repeat the above process first with $K_1, \mathcal{F}_2, x_2, H_2, \varepsilon_2$ instead of $K_0, \mathcal{F}_1, x_1, H_1, \varepsilon_1$, and, similarly, at the subsequent steps. This process terminates at step $m \leq \operatorname{card} \mathcal{F}_1$, when $\mathcal{F}_{m+1} = \varnothing$. Then $K_m \subset L + B_\varepsilon(0)$ with $\varepsilon = \sum \varepsilon_i < \delta$. Then the sets K_m, L, and $L + B_\varepsilon(0)$ hit the same sets from the collection $\mathcal{F}_0$. Hence, $\varphi(K_m) = \varphi(L)$, and, to prove monotonicity, it remains to show that $\varphi(K_{i-1}) \leq \varphi(K_i)$ for each $i = 1, \dots, m$.

Since $\operatorname{conv}(K_{i-1} \cup S_{\varepsilon_i}) \setminus S_{\varepsilon_i}$ does not hit any set from $\mathcal{F}_i$, the increment of φ between K_{i-1} and K_i is determined exclusively by those C_n^+ and C_n^- that hit S_{ε_i} but miss the open half-plane $\mathring{H}^+_{u_i}(x_i)$ bounded by H_i and containing K_{i-1}. By the choice of ε_i, such a set hits S_{ε_i} if and only if it contains x_i. Hence,

$$\varphi(K_i) - \varphi(K_{i-1}) = \sum_{n=1}^{N^+} \mathbb{1}_{C_n^+}(x_i) \cdot \mathbb{1}\{C_n^+ \cap \mathring{H}^+_{u_i}(x_i) = \varnothing\} - \sum_{n=1}^{N^-} \mathbb{1}_{C_n^-}(x_i) \cdot \mathbb{1}\{C_n^- \cap \mathring{H}^+_{u_i}(x_i) = \varnothing\},$$

which is non-negative by (5) and (6).

Necessity. We first prove that (7) holds with some locally finite families (C_n^+) and (C_n^-) and afterwards address the admissibility of (C_n^-) with respect to (C_n^+). To begin with, assume that φ is supported by a subset of a fixed set $W \in \mathcal{K}^2$. By monotonicity,

$$M_\varphi = \sup_{x \in W} \varphi(\{x\}) \leq \varphi(W) < \infty.$$

We will proceed by induction on M_φ.

If $M_\varphi = 0$, then, comparing φ with the zero valuation using Proposition 2.3, we get $\varphi = 0$, so that the claim holds with $N^+ = N^- = 0$.

Now let $M_\varphi = k$, $k \geq 1$, and suppose the claim has been established for any valuation φ' on W such that $M_{\varphi'} \leq k - 1$. By Lemma 3.7, the support of φ is $F = \bigcup_{i=1}^{l} K_i$ for some $l \geq 1$ and closed convex sets $K_1, \dots, K_l \subset W$. Denote by $\varphi|_L = \varphi(\cdot \cap L)$ the restriction of φ to $L \in \mathcal{K}^2$, and consider the valuation

$$\varphi^* = \chi|_F + \sum_{r=1}^{l} (-1)^{r-1} \sum_{1 \leq i_1 < \dots < i_r \leq l} (\varphi - \chi)|_{K_{i_1} \cap \dots \cap K_{i_r}}. \tag{10}$$

The valuations φ and φ^* coincide on singletons: if $x \in F$ belongs to exactly m sets from $K_1, \dots, K_l$ and $\varphi(\{x\}) = p$, then

$$\varphi^*(\{x\}) = 1 + (p-1) \sum_{r=1}^{m} (-1)^{r-1} \binom{m}{r} = p.$$

Hence, by Proposition 2.3, $\varphi = \varphi^*$. The valuation $\varphi' = (\varphi - \chi)|_{K_{i_1} \cap \dots \cap K_{i_r}}$ is integer-valued monotone σ-continuous, and $M_{\varphi'} \leq k - 1$. Thus, by the induction hypothesis, it is of the required form. Substituting the expression (7) for φ' into (10) yields the required representation for $\varphi^* = \varphi$.

Now consider an integer-valued monotone σ-continuous valuation φ on the entire $\mathcal{K}^2$. For $i, j \in \mathbb{Z}$, denote

$$Q_{i,j} = [i, i+1] \times [j, j+1], \qquad E_{i,j} = [i, i+1] \times \{j\},$$
$$E'_{i,j} = \{i\} \times [j, j+1], \qquad V_{i,j} = \{i\} \times \{j\}.$$

Note that $\mathbb{R}^2 = \bigcup_{i,j\in\mathbb{Z}} Q_{i,j}$, and double, triple, and quadruple intersections of distinct components take the form of $E_{i,j}$, $E'_{i,j}$, or $V_{i,j}$, while the intersections of higher orders are empty. The restriction of φ to any of these sets is an integer-valued monotone σ-continuous valuation as well. Hence, by the reasoning above, these restrictions are of the required form. Applying an analogue of (10) to the countable collection of sets $Q_{i,j}$ with intersections beyond the fourth order being empty, we obtain the required form of φ.

We now prove that (C_n^-) is admissible with respect to (C_n^+). Since $\varphi(\{x\}) \geq 0$ for any x, we have

$$\sum_{n=1}^{N^-} \mathbb{1}_{C_n^-}(x) \leq \sum_{n=1}^{N^+} \mathbb{1}_{C_n^+}(x),$$

which, by (4), implies (6) for $u = 0$. Fix some x and $u \neq 0$, and let

$$\begin{aligned} p^+ &= \sum_{n=1}^{N^+} \mathbb{1}_{C_n^+}(x), & p^- &= \sum_{n=1}^{N^-} \mathbb{1}_{C_n^-}(x), \\ q^+ &= \sum_{n=1}^{N^+} \mathbb{1}_{C_n^+}(x) \cdot \mathbb{1}\{C_n^+ \cap \mathring{H}_u^+(x) = \varnothing\}, & q^- &= \sum_{n=1}^{N^-} \mathbb{1}_{C_n^-}(x) \cdot \mathbb{1}\{C_n^- \cap \mathring{H}_u^+(x) = \varnothing\}. \end{aligned} \tag{11}$$

If (6) is violated for x and u, then by (5), we have $q^+ - q^- < 0$. It follows from (11) that

$$\begin{aligned} &\sum_{n=1}^{N^+} \mathbb{1}_{C_n^+}(x) \cdot \mathbb{1}\{C_n^+ \cap \mathring{H}_u^+(x) \neq \varnothing\} - \sum_{n=1}^{N^-} \mathbb{1}_{C_n^-}(x) \cdot \mathbb{1}\{C_n^- \cap \mathring{H}_u^+(x) \neq \varnothing\} \\ &\quad = (p^+ - q^+) - (p^- - q^-) = (p^+ - p^-) - (q^+ - q^-) > (p^+ - p^-) = \varphi(\{x\}). \end{aligned} \tag{12}$$

Due to local finiteness, there are a disk $B_\varepsilon(x)$ that hits the same C_n^+ and C_n^- as $\{x\}$ and a closed convex set K, approximating $B_\varepsilon(x) \cap \mathring{H}_u^+(x)$ from the inside, that hits the same C_n^+ and C_n^- as $B_\varepsilon(x) \cap \mathring{H}_u^+(x)$. Hence, the left-hand side of (12) is $\varphi(K)$, while the right-hand side is $\varphi(B_\varepsilon(x))$, which contradicts monotonicity.

To prove the final claim of the theorem, it suffices to note that (8) means the equality of the corresponding valuations on singletons. By Proposition 2.3, this implies their overall equality. □

Remark 3.8. Note that, in fact, we constructed the representation (7) with components C_n^+ and C_n^- that are not only closed and convex but also bounded, meaning they belong to $\mathcal{K}^2$. However, using unbounded components is often convenient. For example, for the Euler characteristic χ, we can simply take $N^+ = 1$, $N^- = 0$ and $C_1^+ = \mathbb{R}^2$.

The proof that the admissibility of (C_n^-) with respect to (C_n^+) is both necessary and sufficient for the monotonicity of φ extends to any dimension along the same lines. In other words, if a valuation φ on $\mathcal{K}^d$ has the form (7) with some locally finite families (C_n^+) and (C_n^-), then it is monotone if and only if (C_n^-) is admissible with respect to (C_n^+) in the sense of Definition 3.1 (ii). However, the necessity of the representation (7) beyond the two-dimensional setting remains an open question: the most critical part of the proof relies on an application of Eggleston's theorem, and little is known about its validity in higher dimensions.

The σ-continuity condition imposed in Theorem 3.2 is crucial. If it is omitted, the class of integer-valued monotone valuations expands. This is illustrated by the following examples, which work in spaces of any dimension.

Example 3.9. If $N^- = 0$, then the right-hand side of (7), written in the form of

$$\varphi = \sum_{n=1}^{N^+} \mathbb{1}\{\cdot \cap C_n^+ \neq \varnothing\},$$

defines an integer-valued monotone valuation even if the sets C_n^+ are not necessarily closed.

Example 3.10. For $u \in \mathbb{S}^{d-1}$, denote by H_u the supporting hyperplane of K with outer normal u and set $K_u = K \setminus (K \cap H_u)$. Thus, K_u is K with one exposed face removed. For $N^+ \in \overline{\mathbb{N}}_0$, a set $\{u_n\} \subset \mathbb{S}^{d-1}$ and a locally finite set $\{x_n\} \subset \mathbb{R}^d$, both of cardinality N^+, define

$$\varphi(K) = \sum_{n=1}^{N^+} \mathbb{1}\{x_n \in K_{u_n}\}, \quad K \in \mathcal{K}^d. \tag{13}$$

The monotonicity of (13) is clear. To prove additivity, we first note that, for $K, L \in \mathcal{K}^d$ with convex union,

$$(K \cup L)_u = K_u \cup L_u \quad \text{and} \quad (K \cap L)_u = K_u \cap L_u.$$

The only two non-trivial inclusions here are the direct one in the first equality and the reverse one in the second. Let $H_{u,x}^+$ stand for the open half-space with inner normal u whose boundary contains x. If $x \in (K \cup L)_u$, then x belongs to, say, K, and there exists $y \in H_{u,x}^+ \cap (K \cup L)$. If $y \in K$, we have $x \in K_u$. If, however, $y \in L$, then, due to convexity of $K \cup L$, there exists $z \in [x, y] \cap K \cap L$. If $z = x$, we have $x, y \in L$, and thus $x \in L_u$. If $z \neq x$, then $x, z \in K$, and so $x \in K_u$. This proves the direct inclusion in the first equality.

Now let $x \in K_u \cap L_u$. Then $x \in K \cap L$ and there exist $y_1 \in H_{u,x}^+ \cap K$, $y_2 \in H_{u,x}^+ \cap L$. Again, due to convexity of $K \cup L$, there is $z \in [y_1, y_2] \cap K \cap L$. Hence, $z \in H_{u,x}^+ \cap (K \cap L)$, and so $x \in (K \cap L)_u$. This proves the reverse inclusion in the second inequality.

The additivity of each summand in (13) follows from the identity

$$\begin{aligned}\mathbb{1}\{x_n \in K_{u_n}\} + \mathbb{1}\{x_n \in L_{u_n}\} &= \mathbb{1}\{x_n \in K_{u_n} \cup L_{u_n}\} + \mathbb{1}\{x_n \in K_{u_n} \cap L_{u_n}\} \\ &= \mathbb{1}\{x_n \in (K \cup L)_{u_n}\} + \mathbb{1}\{x_n \in (K \cap L)_{u_n}\}.\end{aligned}$$

The general case follows by linearity and, if necessary, by passing to the limit.

It is interesting to note that, in the one-dimensional case, all discontinuous integer-valued monotone valuations are fully characterized by a combination of these two examples. This follows from Theorem 4.1(iv) in the next section. On the other hand, for σ-continuous valuations with the monotonicity condition dropped, the representation (7) may also fail even in the one-dimensional setting, as demonstrated by Examples 4.2 and 4.3 in the next section. This confirms that the Jordan decomposition does not hold for integer-valued σ-continuous valuations.

4 Valuations on the line

In this section, we will explore the structure of valuations on $\mathcal{K}^1$, with a focus on integer-valued valuations that possess some additional properties such as monotonicity or σ-continuity. In particular, it will be shown that, in the one-dimensional analogue of Theorem 3.2, it is always possible to set $N^- = 0$, thus restricting the right-hand side of (7) to positive terms only. The one-dimensional case is, of course, much simpler than the planar one, which allows us to provide in the following theorem a complete characterization of all one-dimensional valuations with certain properties.

We will use the double angle brackets $\langle\!\langle p, q\rangle\!\rangle$, $-\infty \le p \le q \le \infty$, to denote any of the four types of intervals: closed, semi-open, or open. If $p = -\infty$ or $q = \infty$, the interval on the corresponding side can only be open. If $p = q$, then $\langle\!\langle p, q\rangle\!\rangle = [p, p] = \{p\}$.

Theorem 4.1. *Let φ be an arbitrary valuation on $\mathcal{K}^1 = \{[a, b]: a \le b\}$. Then there exist two unique functions $f, g: \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ such that $\varphi([a, b]) = g(b) - f(a)$ for any $a \le b$. Conversely, any such pair of functions defines a valuation. Moreover,*

(i) φ is integer-valued if and only if f and g are integer-valued;

(ii) φ is monotone if and only if f and g are non-decreasing and $f \le g$;

(iii) φ is σ-continuous if and only if f is left-continuous and g is right-continuous;

(iv) φ is integer-valued and monotone if and only if there exist $N_1, N_2, N_3 \in \overline{\mathbb{N}}_0$, a locally finite family of N_1 intervals $\langle\!\langle p_n, q_n\rangle\!\rangle$, and two locally finite sets of N_2 (resp., N_3) points r_n (resp., s_n), such that, for each $[a, b] \in \mathcal{K}^1$,

$$\varphi([a, b]) = \sum_{n=1}^{N_1} \mathbb{1}\big\{[a, b] \cap \langle\!\langle p_n, q_n\rangle\!\rangle \ne \varnothing\big\} + \sum_{n=1}^{N_2} \mathbb{1}\{r_n \in (a, b]\} + \sum_{n=1}^{N_3} \mathbb{1}\{s_n \in [a, b)\}; \tag{14}$$

(v) φ is integer-valued monotone and σ-continuous if and only if there exist $N \in \overline{\mathbb{N}}_0$ and a locally finite family of N closed intervals $[p_n, q_n]$, such that, for each $[a, b] \in \mathcal{K}^1$,

$$\varphi([a, b]) = \sum_{n=1}^{N} \mathbb{1}\big\{[a, b] \cap [p_n, q_n] \ne \varnothing\big\}.$$

Note that, unlike the terms in the last two sums of (14), $\mathbb{1}\{t \in (a,b)\}$ is not a valuation: additivity is violated, e.g., for $K = [t-1, t]$ and $L = [t, t+1]$. Moreover, (14) can be seen as a combination of Examples 3.9 and 3.10 in the one-dimensional setting.

Proof of Theorem 4.1. The difference $g(b) - f(a)$ clearly satisfies additivity and so defines a valuation. Conversely, for the valuation φ, define

$$\begin{aligned} f(x) &= \begin{cases} \varphi([0,x]) - \varphi(\{x\}), & x \geq 0, \\ \varphi(\{0\}) - \varphi([x,0]), & x < 0, \end{cases} \\ g(x) &= \begin{cases} \varphi([0,x]), & x \geq 0, \\ \varphi(\{x\}) + \varphi(\{0\}) - \varphi([x,0]), & x < 0. \end{cases} \end{aligned} \tag{15}$$

Then $f(0) = 0$ and, for $0 \leq a \leq b$, we have by additivity

$$\varphi([a,b]) = \varphi([0,b]) - \varphi([0,a]) + \varphi(\{a\}) = g(b) - f(a).$$

The other two cases, $a \leq b < 0$ and $a < 0 \leq b$, are treated similarly.

In (i), the "only" part follows from (15), while the "if" part from $\varphi([a,b]) = g(b) - f(a)$. The same equality easily yields both parts in (ii) and (iii).

The "if" part in (iv) follows from Examples 3.9 and 3.10. We now prove the "only if" part in (iv). Let $\varphi(\{c\}) = m = \min_{x \in \mathbb{R}} \varphi(\{x\})$ and $\varphi' = \varphi(\cdot - c) - m$. Then φ' is an integer-valued monotone valuation with $\varphi'(\{0\}) = 0$. Hence, for its functions f and g, we have $f(0) = g(0) = 0$. It follows from the previous claims that f and g are non-decreasing step functions with integer jumps, and $f \leq g$. To each point $x > 0$ where g has a left discontinuity, i.e., $g(x) - g(x-) \geq 1$, we associate a pattern of $g(x) - g(x-)$ consecutive identical entries "$[x$". In a similar manner, handle the right discontinuities of g, denoting their positions as "$(x$", $x \geq 0$. Then proceed similarly with the left and right discontinuities of f, using the notation "$x)$" and "$x]$", respectively. Finally, combine these patterns in increasing order of x into a single, at most countable sequence. In the case of patterns with the same x, they should be arranged in the following order: $[x \ldots (x \ldots x) \ldots x] \ldots$. The resulting sequence encodes both f and g on $[0, \infty)$.

For example, for the functions

$$\begin{aligned} f &= 2\mathbb{1}_{(0,2)} + 3\mathbb{1}_{\{2\}} + 5\mathbb{1}_{(2,4]} + 7\mathbb{1}_{(4,6]} + 10\mathbb{1}_{(6,\infty)}, \\ g &= 3\mathbb{1}_{(0,1]} + 4\mathbb{1}_{(1,2]} + 5\mathbb{1}_{(2,4)} + 6\mathbb{1}_{\{4\}} + 7\mathbb{1}_{(4,6)} + 8\mathbb{1}_{\{6\}} + 12\mathbb{1}_{(6,\infty)}, \end{aligned}$$

using this algorithm, we obtain the following sequence:

(0 (0 (0 0] 0] (1 (2 2) 2] 2] [4 (4 4] 4] [6 (6 (6 (6 (6 6] 6] 6].

Now, for the first opening bracket, find the nearest closing one on the right, note the resulting interval, remove the used pair from the sequence and repeat the procedure. If there are not enough closing brackets, use ∞) as many times as needed. In the above example, we arrive at the following set of intervals:

$$(0,0],\ (0,0],\ (0,2),\ (1,2],\ (2,2],\ [4,4],\ (4,4],\ [6,6],\ (6,6],\ (6,6],\ (6,\infty),\ (6,\infty).$$

The resulting intervals can be real, such as the four types of $\langle\!\langle p,q\rangle\!\rangle$, or virtual, such as $[r,r)$ and $(s,s]$. A virtual interval (t,t) is impossible by construction due to the condition $f\le g$. Along the same lines, a similar list of real and virtual intervals can be constructed on $(-\infty,0]$.

Consider the valuation φ'' constructed according to (14), by incorporating the real intervals $\langle\!\langle p_n,q_n\rangle\!\rangle$ into the terms of the first sum, and the points r_n, s_n defining the virtual intervals into the terms of the second and third sums. Calculating by (15) the functions f_n and g_n corresponding to all six types of terms in (14), it is easy to see that the step functions f and g for φ'' have the same positions and structure of discontinuities as those for φ'. Hence, these functions coincide, and so $\varphi''=\varphi'$. Thus, φ' takes the form of (14). Shifting φ' to the right by c and adding $m=m\mathbb{1}\{[a,b]\cap(-\infty,\infty)\neq\varnothing\}$, we arrive at the required representation for φ.

The "if" part in (v) is clear. The "only if" part follows from (iv) and the fact that all other terms in (14) are easily seen not to be σ-continuous. $\square$

We can now give the examples announced at the end of Section 3, which demonstrate that, even in the one-dimensional case, the representation (7) may fail if the monotonicity condition on the valuation is dropped.

Example 4.2. Let

$$f=0\quad\text{and}\quad g=\sum_{n=1}^{\infty}\mathbb{1}_{\left[\frac{2n-1}{2n},\frac{2n}{2n+1}\right)}.$$

By Theorem 4.1, $\varphi([a,b])=g(b)-f(a)=g(b)$ defines an integer-valued σ-continuous valuation on $\mathcal{K}^1$. Since the pair $(f_{p,q},g_{p,q})=(0,\mathbb{1}_{[p,q)})$ corresponds to the valuation

$$\varphi_{p,q}([a,b])=\mathbb{1}\{b\in[p,q)\}=\mathbb{1}\{[a,b]\cap[p,q]\neq\varnothing\}-\mathbb{1}\{[a,b]\cap\{q\}\neq\varnothing\},$$

we arrive at the representation

$$\begin{aligned}\varphi([a,b])&=\sum_{n=1}^{\infty}\mathbb{1}\{[a,b]\cap C_n^+\neq\varnothing\}-\sum_{n=1}^{\infty}\mathbb{1}\{[a,b]\cap C_n^-\neq\varnothing\}\\&=\sum_{n=1}^{\infty}\chi\{[a,b]\cap C_n^+\}-\sum_{n=1}^{\infty}\chi\{[a,b]\cap C_n^-\},\end{aligned}\tag{16}$$

where $C_n^+=\left[\frac{2n-1}{2n},\frac{2n}{2n+1}\right]$, $C_n^-=\left\{\frac{2n}{2n+1}\right\}$, and $\infty-\infty=0$ by convention. The families (C_n^+) and (C_n^-) are not locally finite.

Example 4.3. Let $f = 0$ and $g(x) = \lfloor \frac{1}{1-x} \rfloor \cdot \mathbb{1}_{(-\infty,1)}$, $x \in \mathbb{R}$. Since $g = \sum_{n=1}^{\infty} \mathbb{1}_{[\frac{n-1}{n},1)}$, the above reasoning leads to (16) with $C_n^+ = [\frac{n-1}{n}, 1]$, $C_n^- = \{1\}$ for all n, and the same convention. This time, the families (C_n^+) and (C_n^-) are neither locally finite, nor is the sum in (8) even well defined.

In both of the above examples, there are no other locally finite families $(\widetilde{C}_n^+)$ and $(\widetilde{C}_n^-)$. Indeed, denoting $y_k = \frac{k}{k+1}$, we have by (16) that $\varphi(\{y_k\}) \neq \varphi(\{y_{k+1}\})$ for $k \geq 1$. Hence, on each interval $[y_k, y_{k+1}]$, there must be a point from some $\partial\widetilde{C}_n^+$ or $\partial\widetilde{C}_n^-$. This contradicts the local finiteness.

5 Multiplication of countably generated valuations

Theorems 3.2 and 4.1(v) lead us to the following general definition.

Definition 5.1. A valuation φ on $\mathcal{K}^d$ is called *countably generated* if there exist $N \in \overline{\mathbb{N}}_0$, a locally finite family of N nonempty closed convex sets C_n, and a set of N real numbers α_n such that

$$\varphi(K) = \sum_{n=1}^{N} \alpha_n \chi\big(K \cap C_n\big), \quad K \in \mathcal{K}^d. \tag{17}$$

While in Definition 5.1 the sets C_n were assumed to be only closed and convex, an equivalent representation with compact C_n follows from the inclusion-exclusion argument used in the proof of Theorem 3.2.

The above theorems show that any integer-valued monotone σ-continuous valuation on $\mathcal{K}^1$ or $\mathcal{K}^2$ is countably generated with all $\alpha_n = 1$ if $d = 1$ and $\alpha_n = \pm 1$ if $d = 2$.

Any countably generated valuation is clearly σ-continuous. Let $\mathbb{V}^d$ stand for the vector space of all σ-continuous valuations on $\mathcal{K}^d$ equipped with the natural operations of addition and multiplication by real numbers, and denote by $\mathbb{G}^d$ its subspace of countably generated valuations. Note that elements of $\mathbb{G}^d$ are completely determined by their values on singletons: if $\varphi, \varphi' \in \mathbb{G}^d$ are defined by $N, (\alpha_n), (C_n)$ and $N', (\alpha_n'), (C_n')$, respectively, then $\varphi = \varphi'$ if and only if

$$\sum_{n=1}^{N} \alpha_n \mathbb{1}_{C_n} = \sum_{n=1}^{N'} \alpha_n' \mathbb{1}_{C_n'}. \tag{18}$$

This can be proved along the same lines as Proposition 2.3.

For a countably generated valuation, *multiplication* by a σ-continuous valuation can be defined as follows. For $\psi \in \mathbb{V}^d$ and $\varphi \in \mathbb{G}^d$ given by (17), define

$$(\varphi \cdot \psi)(K) = \sum_{n=1}^{N} \alpha_n \psi(K \cap C_n), \quad K \in \mathcal{K}^d. \tag{19}$$

The terms on the right-hand side are well defined, since $K \cap C_n \in \mathcal{K}^d$ for all n. If $N = \infty$, only a finite number of them are non-zero due to the local finiteness of (C_n). Finally, the value of the sum on the right-hand side of (19) does not depend on the specific choice of $N, (\alpha_n), (C_n)$ in the representation of φ by (17). Indeed, this sum is the Groemer integral of $\sum_{n=1}^N \alpha_n \mathbb{1}_{K\cap C_n}$ with respect to ψ, see [5]. This integral is well defined for $\psi \in \mathbb{V}^d$ by Theorem 3 in the same paper. It remains to note that, for another set $N', (\alpha'_n), (C'_n)$ corresponding to φ, we have

$$\sum_{n=1}^{N'} \alpha'_n \mathbb{1}_{K\cap C'_n} = \mathbb{1}_K \cdot \sum_{n=1}^{N'} \alpha'_n \mathbb{1}_{C'_n} = \mathbb{1}_K \cdot \sum_{n=1}^{N} \alpha_n \mathbb{1}_{C_n} = \sum_{n=1}^{N} \alpha_n \mathbb{1}_{K\cap C_n}$$

by (18).

In the following proposition, we list the basic properties of this product.

Proposition 5.2. *For fixed $K \in \mathcal{K}^d$,*

(i) $(\varphi, \psi) \mapsto (\varphi \cdot \psi)(K)$ is a bilinear map from $\mathbb{G}^d \times \mathbb{V}^d$ to $\mathbb{R}$;

(ii) $(\varphi \cdot \psi)(K) = (\psi \cdot \varphi)(K)$ on $\mathbb{G}^d \times \mathbb{G}^d$;

(iii) $(\chi \cdot \psi)(K) = \psi(K)$, where χ is the Euler characteristic.

For fixed $\varphi \in \mathbb{G}^d$ and $\psi \in \mathbb{V}^d$,

(iv) $(\varphi \cdot \psi)(\cdot)$ is a σ-continuous valuation, that is, this operation acts from $\mathbb{G}^d \times \mathbb{V}^d$ into $\mathbb{V}^d$, moreover, $(\varphi \cdot \psi)(\{x\}) = \varphi(\{x\})\psi(\{x\})$ for each $x \in \mathbb{R}^d$;

(v) if $\psi \in \mathbb{G}^d$, then $\varphi \cdot \psi \in \mathbb{G}^d$ as well, more precisely, if φ is defined by $N, (\alpha_n), (C_n)$, and ψ by $N', (\alpha'_n), (C'_n)$, then $\varphi \cdot \psi$ is defined by $NN', (\alpha_n \alpha'_m), (C_n \cap C'_m)$.

Proof. (i) follows directly from (19). For

$$\varphi = \mathbb{1}\{\cdot \cap C \neq \varnothing\} \quad \text{and} \quad \psi = \mathbb{1}\{\cdot \cap C' \neq \varnothing\}, \tag{20}$$

we have

$$(\varphi \cdot \psi)(K) = \psi(K \cap C) = \mathbb{1}\{K \cap C \cap C' \neq \varnothing\} = \varphi(K \cap C') = (\psi \cdot \varphi)(K). \tag{21}$$

The general case of (ii) follows by linearity. Statement (iii) directly results from $\chi = \mathbb{1}\{\cdot \cap \mathbb{R}^d \neq \varnothing\}$.

For (iv), if $\varphi = \mathbb{1}\{\cdot \cap C \neq \varnothing\}$, then $(\varphi \cdot \psi)(K) = \psi(K \cap C)$, which is a σ-continuous valuation, then use linearity. The equality in (iv) follows from

$$(\varphi \cdot \psi)(\{x\}) = \sum_{n=1}^{N} \alpha_n \psi(\{x\} \cap C_n) = \sum_{n=1}^{N} \alpha_n \chi(\{x\} \cap C_n)\psi(\{x\}) = \varphi(\{x\})\psi(\{x\}).$$

For (v), under (20), the result follows from (21). In the general case, again use linearity. □

The valuation $\varphi \cdot \psi$ can be naturally called the product of φ and ψ for the following reason. The multiplication of smooth valuations introduced by S. Alesker [1] can be, in the translation-invariant case, succinctly described as follows. Let φ_0 stand for the volume, and define $\varphi_A = \varphi_0(\cdot + A)$, $A \in \mathcal{K}^d$, with $+$ being the Minkowski addition. The Alesker product is defined by setting

$$(\varphi_A \cdot \varphi_B)(K) = \varphi_0\big(\Delta(K) + A \times B\big), \quad K \in \mathcal{K}^d, \tag{22}$$

where $\Delta\colon \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ stands for the diagonal embedding $x \mapsto (x, x)$. This product then extends by linearity and continuity to all pairs of smooth translation-invariant valuations.

Except for the multiples of the Euler characteristic, countably generated valuations are neither smooth nor translation-invariant. Therefore, to use this approach, the basic valuation φ_0 needs to be redefined. Let $\varphi_0 = \mathbb{1}\{0 \in \cdot\}$. Then $\varphi_A = \varphi_0(\cdot + A) = \mathbb{1}\{\cdot \cap (-A) \neq \varnothing\}$ for any (not necessarily bounded) nonempty closed convex set A. It follows from (22) that

$$(\varphi_A \cdot \varphi_B)(K) = \mathbb{1}\big\{\Delta(K) \cap \big((-A) \times (-B)\big) \neq \varnothing\big\} = \mathbb{1}\{K \cap (-A) \cap (-B) \neq \varnothing\}, \quad K \in \mathcal{K}^d,$$

which is consistent with the description of the product given in Proposition 5.2(v). This is in line with the intersectional approach to the Alesker product given in [4] within the framework of smooth manifolds.

6 Open problems

In this section, we outline some open problems and conjectures. First, a major issue is to consider the case of general dimensions. This cannot be done by mimicking the proof of Theorem 3.2 due to the absence of a result relating m-convexity and polyconvexity in dimensions 3 and more.

Problem 6.1. Characterize integer-valued monotone σ-continuous valuations in dimensions 3 and higher.

Counterexamples show that it is not possible to obtain meaningful results for valuations which are not σ-continuous. However, relaxing the monotonicity condition may be interesting also in dimension 2.

Problem 6.2. Obtain characterization results under weaker variants of the monotonicity condition, e.g., assuming nonnegativity or local boundedness of variation in the sense of

$$\sup_{L \subset K,\, L \in \mathcal{K}^d} \varphi(L) \le C_K, \quad K \in \mathcal{K}^d,$$

where C_K is a constant depending on K.

Problem 6.3. Which property of an integer-valued monotone σ-continuous valuation φ ensures that its representation (7) contains no negative terms? This question can be posed in general dimension, assuming that the representation (7) holds.

The representation (7) can be interpreted as follows. Consider an integer-valued signed measure on the space of convex closed sets in $\mathbb{R}^2$ of the form

$$\mu = \sum_{n=1}^{N^+} \delta_{C_n^+} - \sum_{n=1}^{N^-} \delta_{C_n^-},$$

where δ_C stands for the unit mass at C. By Remark 3.8, we may assume that this measure is defined only on $\mathcal{K}^2$. Then (7) can be written in the following integral form

$$\varphi(K) = \int_{\mathcal{K}^2} \chi(K \cap C)\, \mu(\mathrm{d}C), \quad K \in \mathcal{K}^2.$$

More generally, by (17), any countably generated valuation on $\mathcal{K}^d$ can be written in the same form with $\mu = \sum_{n=1}^{N} \alpha_n \delta_{C_n}$ for real numbers α_n.

We call a measure μ on $\mathcal{K}^d$ (with its Borel σ-algebra generated by the Hausdorff metric) *locally finite* if $\mu(\mathcal{C}_K) < \infty$ for all $K \in \mathcal{K}^d$, where $\mathcal{C}_K = \{C \in \mathcal{K}^d : K \cap C \neq \varnothing\}$. An arbitrary locally finite signed measure μ on $\mathcal{K}^d$ yields a valuation by letting

$$\varphi(K) = \int_{\mathcal{K}^d} \chi(K \cap C)\, \mu(\mathrm{d}C) = \int_{\mathcal{K}^d} \mathbb{1}\{K \cap C \neq \varnothing\}\, \mu(\mathrm{d}C) = \mu(\mathcal{C}_K). \tag{23}$$

Since $\mathcal{C}_{K_n} \downarrow \mathcal{C}_K$ as $K_n \downarrow K$, this valuation is σ-continuous due to the σ-additivity of the measure μ.

Problem 6.4. Identify σ-continuous valuations on $\mathcal{K}^d$ such that (23) holds for a locally finite signed measure μ on $\mathcal{K}^d$? Note that, as follows from Example 3.3, such a measure need not be unique.

The set of valuations admitting an integral representation of the form (23) is far from being limited to countably generated valuations. For instance, the d-dimensional volume can be expressed in this form with a measure μ concentrated on singletons

$$\mu(\{x\} : x \in B) = \lambda_d(B), \quad B \in \mathcal{B}(\mathbb{R}^d),$$

where λ_d stands for the d-dimensional Lebesgue measure. Similar representations hold for intrinsic volumes.

We conjecture that (23) holds for a very broad family of valuations. Examples 4.2 and 4.3 demonstrate that this does not hold for all σ-continuous valuations, since the families (C_n^+) and (C_n^-) in these examples do not satisfy the local finiteness condition and so the measure μ is not locally finite. This may be explained by the lack of monotonicity in these valuations.

Problem 6.5. Is the family of countably generated valuations dense (in some sense) in the space of all σ-continuous valuations?

The following problems address changing the range of values and/or the definition domain of valuations.

Problem 6.6. Characterize valuations taking values in other semigroups, such as $(\mathbb{Z}/n\mathbb{Z}, +)$, $(\mathbb{Z}/n\mathbb{Z}, \times)$, $(\mathbb{Q}, +)$, etc.

Problem 6.7. Characterize integer-valued valuations on convex functions. It is very likely that this can be done using our methods.

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Chapter 3

Random Valuations

The content of this chapter is an ongoing project in collaboration with I. Molchanov and A. Ilienko.

Random Valuations

Abstract

A random valuation is a stochastic process on the family of convex bodies whose realisations are a.s. additive. The family of valuations is very rich, and to achieve meaningful characterisation results, assumptions in the form of independence and infinite divisibility must be taken into consideration. Under these assumptions and by using tools from the theories of Lévy processes, stochastic geometry, and valuations, we are able to build a rich new theory, which is deeply connected with well-known results of deterministic valuations and integral geometry.

1 Introduction

Most of the literature on valuations focuses on valuations with values in the set of real or complex numbers or in the family of compact convex sets equipped with Minkowski addition.

Let $\mathcal{K}^d$ be the family of convex bodies (i.e., compact convex sets) in $\mathbb{R}^d$. While the empty set is typically not considered a convex body, we adopt the convention that it is included in $\mathcal{K}^d$. A valuation φ is an additive map from $\mathcal{K}^d$ to an abelian semigroup. Additivity means that,

$$\varphi(K \cup L) + \varphi(K \cap L) = \varphi(K) + \varphi(L) \tag{1}$$

for any compact convex sets K and L such that $K \cup L$ is also convex, see [9, Chapter 6]. Additionally, we will always assume that $\varphi(\varnothing) = 0$.

A random valuation is a stochastic process indexed by the family of convex bodies in $\mathbb{R}^d$ which satisfies the additivity property of valuations. To achieve meaningful characterisation results, we assume that the distribution is infinitely divisible. Each infinitely divisible probability distribution corresponds in a natural way to a Lévy process. Set-indexed Lévy processes were studied in [1] assuming that the values on disjoint sets are independent. The central question in this work was the existence of such a process indexed by a rather general family of sets and such that its paths are sufficiently regular. These processes have been further studied in [4], where the authors use a new definition for increment stationarity of set-indexed processes to obtain different characterisations of this class. An exciting result presented in [7] shows the existence of a unique Lévy measure for an infinitely divisible set-indexed process.

2 Basic properties of random valuations

Let $\mathbb{R}^{\mathcal{K}^d}$ denote the space of all functions $\varphi : \mathcal{K}^d \to \mathbb{R}$, and let $\mathcal{B}^{\mathcal{K}^d}$ be its cylindrical σ-algebra. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space.

Definition 2.1. A *random valuation* Φ is a stochastic process on $\mathcal{K}^d$ whose realisations are a.s. additive: with probability 1,

$$\Phi(K \cup L) + \Phi(K \cap L) = \Phi(K) + \Phi(L)$$

for all $K, L, K \cup L \in \mathcal{K}^d$, and $\Phi(\varnothing) = 0$. The distributions of $(\Phi(K_1), \dots, \Phi(K_n))$ for $n \geq 1$, $K_1, \dots, K_n \in \mathcal{K}^d$ are called the finite-dimensional distributions of Φ.

A random valuation Φ is said to be *separable* if there exist a countable family $\mathcal{D} \subset \mathcal{K}^d$ and a set $\Omega_0 \subset \Omega$ of full probability such that, for each $\omega \in \Omega_0$ and every $K \in \mathcal{K}^d$, there is a sequence $(K_n \in \mathcal{D}, n \geq 1)$ with $K_n \to K$ in the Hausdorff metric ρ_H and $\Phi(K_n) \to \Phi(K)$. The separability property implies separability in probability, obtained by replacing the a.s. convergence $\Phi(K_n) \to \Phi(K)$ with convergence in probability. Since $\mathcal{K}^d$ is separable in the Hausdorff metric, a random valuation is separable a.s. (resp., in probability) whenever it is continuous a.s. (respectively, in probability).

These separability conditions are formulated in terms of distributions and therefore do not imply any continuity property of the realisations of Φ. Recall that there are two basic notions of continuity for a deterministic valuation φ:

(C1) continuity in the Hausdorff metric;

(C2) σ-continuity, meaning that $\varphi(K_n) \to \varphi(K)$ whenever $K_n \downarrow K$.

It is clear that (C1) implies (C2). Moreover, if φ is monotone, then σ-continuity is equivalent to the upper semicontinuity of φ. For a random valuation Φ, these continuity conditions are understood as applying to almost all realisations of Φ.

Lemma 2.2. *If Φ is a separable random valuation, then $C_\Phi = \{\Phi$ is continuous$\}$ is an $\mathcal{F}$-measurable event.*

Proof. Let $\tilde{C}_\Phi$ denote the event that Φ is locally uniformly continuous on $\mathcal{D}$:

$$\tilde{C}_\Phi = \bigcap_{r\geq 1} \bigcap_{m\geq 1} \bigcup_{n\geq 1} \bigcap_{\substack{K,L\in\mathcal{D},\\ K,L\subset B_r(0),\\ d_H(K,L)\leq n^{-1}}} \big\{|\Phi(L) - \Phi(K)| \leq m^{-1}\big\}.$$

Since all intersections and unions are countable, we have $\tilde{C}_\Phi \in \mathcal{F}$.

To prove the lemma, it suffices to show that $\Omega_0 \cap \tilde{C}_\Phi = \Omega_0 \cap C_\Phi$. The reverse inclusion follows from the local compactness of $\mathcal{K}^d$ w.r.t. the Hausdorff metric: any continuous function on $\mathcal{K}^d$ is locally uniformly continuous there, and hence also on $\mathcal{D}$. To prove the direct inclusion, let $K_n \to K$. By the definition of separability, choose $K^i, K^i_n \in \mathcal{D}$ with $K^i \to K$ and $K^i_n \to K_n$ such that $\Phi(K^i) \to \Phi(K)$ and $\Phi(K^i_n) \to \Phi(K_n)$ for all $\omega \in \Omega_0$. Hence,

$$|\Phi(K_n) - \Phi(K)| \le |\Phi(K_n) - \Phi(K^i_n)| + |\Phi(K^i_n) - \Phi(K^i)| + |\Phi(K^i) - \Phi(K)|,$$

and, therefore,

$$\limsup_{n\to\infty} |\Phi(K_n) - \Phi(K)| \le \limsup_{n\to\infty} \limsup_{i\to\infty} |\Phi(K^i_n) - \Phi(K^i)|,$$

which vanishes by the local uniform continuity of Φ on $\mathcal{D}$. □

If the separability assumption on Φ is replaced by its (joint) measurability with respect to $\mathcal{F} \otimes \mathcal{B}(\mathcal{K}^d)$, then the events that many other path properties hold turn out to be $\mathcal{F}$-measurable as well. As usual, we assume that the underlying probability space is complete.

Lemma 2.3. *If Φ is measurable with respect to $\mathcal{F} \otimes \mathcal{B}(\mathcal{K}^d)$, then the following events are $\mathcal{F}$-measurable:*

$$\{\Phi \textit{ is continuous}\}, \quad \{\Phi \textit{ is } \sigma\textit{-continuous}\}, \quad \{\Phi \textit{ is monotone}\},$$
$$\{\Phi \textit{ is invariant under a given group of Borel automorphisms of } \mathcal{K}^d\}.$$

Proof. Since all these events are shown to be measurable in the same way, we consider only $C^\sigma_\Phi = \{\Phi \text{ is } \sigma\text{-continuous}\}$. Define

$$S = \Big\{(\omega, K, (K_i)_{i\ge1}) \in \Omega \times \mathcal{K}^d \times (\mathcal{K}^d)^{\mathbb{N}} \colon K_i \downarrow K, \limsup_{i\to\infty} |\Phi(K_i) - \Phi(K)| > 0\Big\}.$$

The $\mathcal{F} \otimes \mathcal{B}(\mathcal{K}^d)$ -measurability of Φ readily implies that S belongs to $\mathcal{F} \otimes \mathcal{B}(\mathcal{K}^d) \times (\mathcal{B}(\mathcal{K}^d))^{\mathbb{N}}$. Then, $C^\sigma_\Phi = \big(\operatorname{proj}_\Omega S\big)^{\complement}$.

By the measurable projection theorem (see, e.g., Theorem 2.12 in [3]), $\operatorname{proj}_\Omega S$ is universally measurable with respect to $\mathcal{F}$ and, in particular, belongs to the completion $\overline{\mathcal{F}}$. Since the probability space is complete, $C^\sigma_\Phi \in \mathcal{F}$, as claimed. □

Lemma 2.4. *Each σ-continuous random valuation Φ is separable.*

Proof. The family of polytopes with vertices in $\mathbb{Q}^d$ is countable and dense in $\mathcal{K}^d$; in particular, each $K \in \mathcal{K}^d$ can be approximated from above by elements of this family. □

The family of all valuations is very rich, and some invariance conditions are necessary to arrive at meaningful characterisation results.

If almost all realisations of Φ are translation invariant and continuous, then, following McMullen's theorem (see, e.g., Theorem 6.3.5 in [9]), Φ can be decomposed into the sum of homogeneous valuations of orders $k = 0, \dots, d$, and each summand is a random valuation itself.

Proposition 2.5. *Assume that* Φ *is a random valuation whose almost all realisations are continuous and translation invariant. Then*

$$\Phi(K) = \sum_{i=0}^{d} \Phi_i(K), \tag{2}$$

where each Φ_i *is a random valuation whose almost all realisations are continuous, translation invariant, and homogeneous of degree* i*.*

Proof. By McMullen's theorem, (2) holds for almost all realisations of Φ. Each Φ_i is additive and homogeneous of degree i on $\mathcal{K}^d$. For each $K \in \mathcal{K}^d$, we evaluate Φ at rK for $r > 0$, which shows that

$$\Phi(rK) = \sum_{i=0}^{d} \Phi_i(rK) = \sum_{i=0}^{d} r^i \Phi_i(K).$$

Thus, $\big(\Phi_0(K), \dots, \Phi_d(K)\big)$ is the solution of the above system of linear equations for $d+1$ different values of r. This solution is a linear transform of the values $\Phi(r_0 K), \dots, \Phi(r_d K)$, and so each $\Phi_i(K)$ is a random variable for all $K \in \mathcal{K}^d$.

For each $i = 0, \dots, d$ and for all finite subsets of $\mathcal{K}^d$ as $\{K_1, \dots, K_n\}$, the distributions of the random vectors $(\Phi_i(K_1), \dots, \Phi_i(K_n))$ determine the finite-dimensional distributions of a stochastic process on $\mathcal{K}^d$, which is additive and thus is a random valuation. □

A direct consequence of the previous result and Hadwiger's theorem (see, e.g., Theorem 6.4.14 in [9]) is the characterisation of random valuations whose almost all realisations are continuous and invariant under rigid motions.

Corollary 2.6. *Assume that* Φ *is a random valuation whose almost all realisations are continuous and invariant under rigid motions. Then*

$$\Phi(K) = \sum_{i=0}^{d} \xi_i V_i(K), \tag{3}$$

where $(\xi_0, \dots, \xi_d)$ *is a random vector, and* $V_0, \dots, V_d$ *stand for the intrinsic volumes.*

A rich source of random valuations is provided by random (signed) measures on the family $\mathcal{C}$ of closed convex sets in $\mathbb{R}^d$ equipped with the Borel σ-algebra generated by the Fell topology. A measure μ on $\mathcal{C}$ is said to be locally finite if μ is finite on the families $\{F \in \mathcal{C} \colon F \cap K \neq \varnothing\}$ for all compact sets K.

Proposition 2.7. *Let* Z *be a locally finite random signed measure on* $\mathcal{C}$*. Then*

$$\Phi(K) = Z(\{F \in \mathcal{C} \colon K \cap F \neq \varnothing\}), \qquad K \in \mathcal{K}^d,$$

is a σ*-continuous random valuation. Furthermore,* Φ *is a.s. non-negative if* Z *is a.s. non-negative. In this case,* Φ *is necessarily monotone.*

Proof. Since Z is a random measure, $\Phi(K)$ is $\mathcal{F}$-measurable for any $K \in \mathcal{K}^d$. Therefore, Φ is a stochastic process on $\mathcal{K}^d$.

Assume that $K, L \in \mathcal{K}^d$ are such that $K \cup L \in \mathcal{K}^d$. Define

$$\mathcal{A}_K = \{F \in \mathcal{C} \colon K \cap F \neq \varnothing\}.$$

Since Z is a random measure and its support is contained in the family of closed convex sets, we get that

$$\Phi(K) + \Phi(L) = Z(\mathcal{A}_K) + Z(\mathcal{A}_L) = Z(\mathcal{A}_{K\cup L}) + Z(\mathcal{A}_{K\cap L}) = \Phi(K \cup L) + \Phi(K \cap L) \quad \text{a.s.}$$

Since $\mathcal{A}_\varnothing = \varnothing$, we have $\Phi(\varnothing) = Z(\varnothing) = 0$ a.s.

Assume that $(K_n)_{n\geq 1}$ is a sequence in $\mathcal{K}^d$ such that $K_n \downarrow K$. Then $\mathcal{A}_{K_n} \downarrow \mathcal{A}_K$ as $n \to \infty$. Since Z is a random measure,

$$\lim_{n\to\infty} \Phi(K_n) = \lim_{n\to\infty} Z(\mathcal{A}_{K_n}) = Z(\mathcal{A}_K) = \Phi(K) \quad \text{a.s.}$$

If Z is non-negative, then Φ is clearly also non-negative and monotone, since $\mathcal{A}_L \subset \mathcal{A}_K$ for $L \subset K$. □

3 Infinitely divisible valuations

A random valuation Φ is said to be *infinitely divisible*, if, for each $n \geq 2$, Φ is equal in distribution to the sum $\Phi_{1n} + \ldots + \Phi_{nn}$, where $\Phi_{1n}, \ldots, \Phi_{nn}$ are n i.i.d. random valuations. In this case we say that Φ is an ID valuation.

An ID valuation is an infinitely divisible element in the group of all additive functions $\mathcal{K}^d \to \mathbb{R}$ with addition. Note that the family of all additive functions is a closed subset of the family of all functions $\mathbb{R}^{\mathcal{K}^d}$ equipped with the pointwise convergence.

Let Φ be an ID valuation, and let $\mathcal{I} = \{K_1 \ldots, K_m\}$ be a finite collection of sets from $\mathcal{K}^d$. Then the random vector $\Phi_\mathcal{I} = (\Phi(K_1), \ldots, \Phi(K_m))$ is infinitely divisible. By the Lévy-Khinchin representation, there exists a unique triplet $(\Sigma_\mathcal{I}, \Lambda_\mathcal{I}, b_\mathcal{I})$ such that, for each $u \in \mathbb{R}^m$,

$$\mathbf{E} e^{\imath\langle u,\Phi_I\rangle} = \exp\Big\{ -\frac{1}{2}\langle u, \Sigma_\mathcal{I} u\rangle + \imath\langle u, b_\mathcal{I}\rangle + \int_{\mathbb{R}^m\setminus\{0\}} (e^{\imath\langle u,x\rangle} - 1 - \imath\langle u, x\rangle \mathbb{1}_{\|x\|\leq 1} \Lambda_\mathcal{I}(dx) \Big\},$$

where $\Sigma_\mathcal{I}$ is a non-negative definite $m\times m$ matrix, $b_\mathcal{I} \in \mathbb{R}^m$, and $\Lambda_\mathcal{I}$ is a Lévy measure on $\mathbb{R}^m$. The uniqueness of the triplet $(\Sigma_\mathcal{I}, \Lambda_\mathcal{I}, b_\mathcal{I})$ implies that for $\mathcal{I} \subset \mathcal{I}'$, the matrix $\Sigma_{\mathcal{I}'}$ restricted to $\mathcal{I} \times \mathcal{I}$ equals $\Sigma_\mathcal{I}$, the vector $b_{\mathcal{I}'}$ restricted to $\mathcal{I}$ equals $b_\mathcal{I}$, and $\Lambda_\mathcal{I}$ is the projection of $\Lambda_{\mathcal{I}'}$.

The matrix $\Sigma_\mathcal{I}$ determines the Gaussian component of Φ. In particular, each ID valuation is the sum of two independent random valuations, one with only the Gaussian component and the other determined by $b_\mathcal{I}$ and $\Lambda_\mathcal{I}$. Assume that Φ does not have the Gaussian component.

With the tools developed in [7], it is possible to patch together the measures $\Lambda_\mathcal{I}$ to come up with a single measure on $\mathcal{B}^{\mathcal{K}^d}$ which admits them as projections. A measure Λ on $(\mathbb{R}^{\mathcal{K}^d}, \mathcal{B}^{\mathcal{K}^d})$ is said to be a *Lévy measure* if the following two conditions hold:

(L1) for all $K \in \mathcal{K}^d$,

$$\int \min(1, |\psi(K)|^2)\Lambda(d\psi) < \infty, \quad K \in \mathcal{K}^d; \tag{4}$$

(L2) for all $A \in \mathcal{B}^{\mathcal{K}^d}$, $\Lambda(A) = \Lambda_*(A \setminus O_{\mathcal{K}^d})$, where Λ_* is the inner measure and $O_{\mathcal{K}^d}$ is the function $\psi\colon \mathcal{K}^d \to \mathbb{R}$ which is identically zero.

Condition (L2) is satisfied if there exists a countable set $\mathcal{J} \subset \mathcal{K}^d$ such that

$$\Lambda(\{\psi \in \mathbb{R}^{\mathcal{K}^d} \colon \psi(K) = 0 \text{ for all } K \in \mathcal{J}\}) = 0.$$

Theorem 2.8 and Corollary 2.18 from [7] imply the following result.

Theorem 3.1. *Let Φ be a separable in probability ID valuation without Gaussian component. Then there exists a unique Lévy measure Λ on $(\mathbb{R}^{\mathcal{K}^d}, \mathcal{B}^{\mathcal{K}^d})$ (which is necessarily σ-finite) and a deterministic valuation φ such that, for any $m \geq 1$, finite family $\{K_1, \dots, K_m\} \subset \mathcal{K}^d$, and $u \in \mathbb{R}^m$,*

$$\mathbf{E}\exp\Big\{\imath \sum_{j=1}^m u_j \Phi(K_j)\Big\} = \exp\Big\{\imath \sum_{j=1}^m u_j \varphi(K_j) + \int_{\mathbb{R}^{\mathcal{K}^d}} \Big(e^{\imath \sum_{j=1}^m u_j \psi(K_j)} - 1 - \imath \sum_{j=1}^m u_j \psi(K_j) \mathbb{1}_{\|(\psi(K_1),\dots,\psi(K_m))\| \leq 1}\Big)\Lambda(d\psi)\Big\}.$$

Remark 3.2. If Φ is an ID valuation which takes values in a subset of $\mathbb{R}^{\mathcal{K}^d}$ closed with respect to pointwise convergence, then its Lévy measure is supported by the same subset. This applies to monotone random valuations or to non-negative ones. Indeed, denote by μ_n the distribution of Φ_{1n} such that the sum of its n independent copies equals to Φ in distribution. By [2, Lemma 4.3.12], Λ is the vague limit of $n\mu_n$ in the family of all non-negative functions on $\mathcal{K}^d$ in the topology of pointwise convergence with the zero function excluded. If the realisations of Φ almost surely belong to a family $\mathcal{V}$ which is closed in the pointwise convergence, then [6, Lemma 4.1] implies that Λ vanishes on the complement to $\mathcal{V}$.

The Lévy measure Λ can be seen as the projective limit of $\Lambda_{\mathcal{I}}$, for each finite subset $\mathcal{I}$ of $\mathcal{K}^d$.

If, instead of (4), we assume that

$$\int \min(1, |\psi(K)|)\Lambda(d\psi) < \infty, \quad K \in \mathcal{K}^d,$$

then Λ is the Lévy measure of an ID valuation. If the weaker condition (4) is imposed, then it is harder to ensure regularity properties of the ID valuation with such Lévy measure.

A random valuation is even in distribution if $\Phi(K)$ has the same distribution as $\Phi(\check{K})$, where $\check{K} = \{-x : x \in K\}$; it is odd in distribution if $\Phi(K)$ has the same distribution as

$-\Phi(\check{K})$. If the equalities hold almost surely, then Φ is said to be a.s. even (odd). The uniqueness of the Lévy measure implies that an ID valuation is even in distribution if Λ is invariant under transformations which maps a valuation ψ to the valuation $\psi'(K) = \psi(\check{K})$, and it is a.s. even if Λ is supported by even valuations. Similar statements hold for odd valuations.

Let N be a completely random measure on the family of valuations with mean Λ. Note that $N\big(\{\psi\colon |\psi(K)| > \varepsilon\}\big)$ has Poisson distribution with parameter $\Lambda\big(\{\psi\colon |\psi(K)| > \varepsilon\}\big) < \infty$, and so is almost surely finite. Proposition 2.10 from [7] implies the following.

Proposition 3.3. *A separable in probability ID valuation Φ without Gaussian component has the same distribution as*

$$\tilde{\Phi}(K) = \int \psi(K)\big[N(d\psi) - \mathbb{1}_{|\psi(K)|\leq 1}\Lambda(d\psi)\big] + \varphi(K), \quad K \in \mathcal{K}^d, \tag{5}$$

where $\varphi(K)$ is a deterministic valuation and N is the Poisson random measure on the family of all valuations with intensity Λ.

The integral in (5) is understood as its principal value, that is, as

$$\lim_{\varepsilon\downarrow 0} \int_{|\psi(K)|\geq\varepsilon} \psi(K)\big[N(d\psi) - \mathbb{1}_{|\psi(K)|\leq 1}\Lambda(d\psi)\big].$$

The representation given at (5) involves the compensating term arising from integration with respect to Λ. The compensating term vanishes if Φ is symmetric, that is, Φ has the same distribution as $-\Phi$.

Remark 3.4. If Φ is non-negative, then it does not have a Gaussian component. Its Lévy measure is supported by non-negative valuations and satisfies

$$\int \min(1, \psi(K))\Lambda(d\psi) < \infty, \quad K \in \mathcal{K}^d. \tag{6}$$

In this case, it is possible to work with the Laplace transform of the finite-dimensional distributions, which becomes

$$\mathbf{E}\exp\Big\{-\sum_{j=1}^m u_j\Phi(K_j)\Big\} = \exp\Big\{-\sum_{j=1}^m u_j\varphi(K_j) - \int_{\mathbb{R}^{\mathcal{K}^d}} \Big(1 - e^{-\sum_{j=1}^m u_j\psi(K_j)}\Big)\Lambda(d\psi)\Big\}$$

for $u \in \mathbb{R}_+^m$, where φ is a deterministic valuation.

If Φ is non-negative, then the compensating term in (5) also vanishes and then

$$\Phi(K) = \int \psi(K)N(d\psi) + \varphi(K), \tag{7}$$

where the integral exists in the conventional sense. If Φ is also integrable, then

$$\mathbf{E}\Phi(K) = \int \psi(K)\Lambda(d\psi) + \varphi(K).$$

The constant term φ vanishes if the essential infimum of $\Phi(K)$ is zero for all K.

Since the family of σ-continuous valuations is not closed in the topology of pointwise convergence (unlike the families of monotone or nonnegative valuations), it is not possible to use the argument from Remark 3.2 to pass the σ-continuity property from Φ to the support of its Lévy measure.

Lemma 3.5. *Assume that Φ is a separable in probability ID valuation, which is monotone. If Φ is σ-continuous and integrable, then its Lévy measure is supported by σ-continuous valuations.*

Proof. Since the family of all monotone valuations is closed in the topology of pointwise convergence, the Lévy measure is supported by a subset of all monotone valuations. Thus,

$$\mathbf{E}e^{-t\Phi(K)} = \exp\Big\{ -\varphi(K) - \int (1 - e^{-t\psi(K)})\Lambda(d\psi)\Big\}.$$

Assume that $K_n \downarrow K$. Then $\Phi(K_n) \to \Phi(K)$ almost surely. The dominated convergence theorem yields that the Laplace transforms converge, and so

$$\int (e^{-t\psi(K)} - e^{-t\psi(K_n)})\Lambda(d\psi) \to 0 \quad \text{as } n \to \infty.$$

Since ψ is monotone, this yields that

$$\psi(K_n) \downarrow \psi(K) \quad \text{as } n \to \infty$$

for Λ-almost all valuations ψ. □

4 Independence of increments

A random valuation Φ is said to have *independent increments* if, for each $n \geq 3$ and a nested sequence $L_1 \supset \cdots \supset L_n$ of convex bodies and $L_{n+1} = \varnothing$, the random variables $\Phi(L_i) - \Phi(L_{i+1})$, $i = 1, \dots, n$, are jointly independent. For instance, if $(K_t)_{t\geq 0}$ is a nested increasing family of convex bodies, then $\Phi(K_t)$, $t \geq 0$, is a stochastic process with independent increments.

Proposition 4.1. *An ID valuation has independent increments if and only if $\Phi(K) - \Phi(L)$ and $\Phi(L) - \Phi(M)$ are independent for all convex bodies $K \supset L \supset M$.*

Proof. Necessity is trivial and we prove only sufficiency. Let $L_1 \supset \cdots \supset L_n \supset L_{n+1} = \varnothing$ be a nested collection of sets from $\mathcal{K}^d$. Define $\xi_i = \Phi(L_i) - \Phi(L_{i+1})$, $i = 1, \dots, n$. The imposed condition yields that the random variables ξ_i and ξ_{i+1} are independent for each $i = 1, \dots, n$. The Lévy measure of the pair (ξ_i, ξ_{i+1}) is supported by the union of the axes in $\mathbb{R}^2$, see, e.g., Exercise 12.10 from [8]. This Lévy measure is the projection of the Lévy measure of the whole vector $(\xi_1, \dots, \xi_n)$, and so this latter Lévy measure is also supported by the axes in $\mathbb{R}^n$, meaning that the components are jointly independent. □

Lemma 4.2. *Let Φ be an ID valuation with the Lévy measure Λ, and let $a_1, \dots, a_n$ and $b_1, \dots, b_m$ be real numbers and $K_1, \dots, K_n$ and $L_1, \dots, L_m$ be convex bodies. Then the finite sums $\sum a_i \Phi(K_i)$ and $\sum b_j \Phi(L_j)$ are independent if and only if Λ is supported by valuations ψ such that $\sum a_i \psi(K_i) = 0$ or $\sum b_j \psi(L_j) = 0$.*

Proof. The statement follows from the already mentioned Exercise 12.10 from [8]. □

The next result characterises ID valuations with independent increments. Recall that we always assume that random valuations are separable in probability.

Theorem 4.3. *Let Φ be an ID valuation. Then Φ has independent increments if and only if its Lévy measure Λ is supported by valuations ψ with two values $\{0, c\}$ and such that ψ is monotone increasing if $c > 0$ and monotone decreasing if $c < 0$.*

Proof. By Proposition 4.1 and Lemma 4.2, Φ has independent increments if and only if Λ is supported by ψ such that

$$\psi(K) = \psi(L) \quad \text{or} \quad \psi(L) = \psi(M) \tag{8}$$

for all $K, L, M \in \mathcal{K}^d$ with $M \subset L \subset K$. Let $K \in \mathcal{K}^d$ with $c = \psi(K) \neq 0$. Taking $M = \varnothing$ we obtain that $\psi(L) = 0$ or $\psi(L) = \psi(K) = c$ for all $L \subset K$. If $K \subset W$ for $W \in \mathcal{K}^d$, then $c = \psi(K) = \psi(W)$, since $\Phi(K)$ does not vanish. For each $L \in \mathcal{K}^d$, take $W \in \mathcal{K}^d$ such that $(K \cup L) \subset$W. Then $\psi(L)$ is either zero or $\psi(L) = \psi(W) = c$. If $c > 0$, then $\psi(K) \geq \psi(L)$ whenever $L \subset K$. If $c < 0$, then ψ is decreasing.

In the other direction, if ψ is monotone and takes two values 0 and c, then (8) holds. For example, if $c > 0$ and ψ is increasing, then $\psi(L)$ is either 0 or is equal to $\psi(K)$ for $L \subset K$. □

Note that ψ from the support of Λ specified in Theorem 4.3 may be two-valued with different c's, also combining positive and negative values. If Φ is monotone, then $\Phi(K) \geq \Phi(\varnothing) = 0$ for all $K \in \mathcal{K}^d$ and so Φ is nonnegative. For deterministic valuations, the nonnegativity property does not imply monotonicity, e.g., the valuation

$$\Phi(K) = \mathbb{1}_{K \cap L \neq \varnothing} - \mathbb{1}_{K \cap M \neq \varnothing}$$

is nonnegative, but not monotone.

Corollary 4.4. *If Φ is a nonnegative ID valuation with independent increments, then Φ is necessarily monotone.*

The representation from Theorem 4.3 can be further specified if the random valuation Φ is nonnegative and σ-continuous. Recall that $\mathcal{C}$ denotes the family of all convex closed sets in $\mathbb{R}^d$.

Theorem 4.5. *Let Φ be a nonnegative σ-continuous ID valuation with independent increments. Then its Lévy measure Λ is the pushforward of a Borel measure ν on $\mathcal{C} \times \mathbb{R}_+$ by the map*

$$(F, t) \mapsto \psi(K) = t\mathbb{1}_{K \cap F \neq \varnothing}, \quad K \in \mathcal{K}^d, \tag{9}$$

and

$$\Phi(K) = \sum_{(F_i, t_i) \in \eta} t_i \mathbb{1}_{K \cap F_i \neq \varnothing},$$

where η is the Poisson process on $\mathcal{C} \times \mathbb{R}_+$ with intensity ν. The measure ν necessarily satisfies

$$\int \min(1, r)\mathbb{1}_{F \cap K \neq \varnothing} \nu(d(F, r)) < \infty. \quad K \in \mathcal{K}^d. \tag{10}$$

Proof. By Theorem 4.3, Λ is supported by valuations ψ with values in $\{0, c\}$ for some $c > 0$ (which depends on ψ). By Lemma 3.5, all such ψ are σ-continuous.

It follows from [5] that a nonnegative σ-continuous valuation ψ with two values is uniquely determined by its support, which is the set F of all points x such that $\psi(\{x\}) > 0$ and then

$$\psi(K) = c\mathbb{1}_{F \cap K \neq \varnothing}$$

for some $c > 0$. This set F is necessarily nonempty, since otherwise ψ would have empty support and so would be zero. By σ-continuity of ψ we immediately obtain that F is closed.

Let $L = \operatorname{conv}(\{x, y\})$ for any $x, y \in F$, and choose $z \in L$. Since $x \in L$, we have $\psi(L) = \psi(K)$ or $\psi(L) = \psi(\{x\})$, so in all cases $\psi(L) = \psi(K)$. By the same argument, $\psi(L_1) = \psi(L_2) = \psi(K)$ for the segments with end-points x, z and y, z, respectively. By additivity,

$$\psi(\{z\}) = \psi(L_1) + \psi(L_2) - \psi(L) = \psi(K).$$

Thus, F is a convex set.

Property (10) follows from (6). □

The Poisson process η on $\mathcal{C} \times \mathbb{R}_+$ from Theorem 4.5 defines a random measure Z on $\mathcal{C}$ by letting

$$Z(\mathcal{A}) = \sum_{F_i \in \mathcal{A}} t_i$$

for all measurable $\mathcal{A} \subset \mathcal{C}$. In this way the representation in Theorem 4.5 can be formulated as follows.

Corollary 4.6. *Assume that Φ is a nonnegative σ-continuous ID valuation with independent increments. Then*

$$\Phi(K) = Z(\{F \in \mathcal{C} : F \cap K \neq \varnothing\})$$

for a Poisson random measure Z on $\mathcal{C}$.

Example 4.7. Let Λ be the pushforward of the product of the Lebesgue measure and the Dirac measure δ_1 under the map $(x, r) \mapsto (B_1(x), r)$. Furthermore, let $\varphi(K) = 0$. Then

$$\mathbf{E}e^{-t\Phi(K)} = \exp\Big\{ - \big(1 - e^{-t}\big) V_d(K^1)\Big\},$$

where $K^1 = K + B_1(0)$ is the 1-envelope of K. By a direct check it is easy to see that

$$\Phi(K) = \eta(K^1),$$

where η is the unit intensity Poisson process on $\mathbb{R}^d$.

We now characterise ID valuation satisfying a weaker independence condition. We say that a random valuation Φ has *convex-independent increments* if the random variables $\Phi(K) - \Phi(K \cap L)$ and $\Phi(L) - \Phi(K \cap L)$ are independent for all $K, L \in \mathcal{K}^d$ such that $K \cup L$ is convex.

Lemma 4.8. *If Φ is a random valuation with independent increments, then Φ has convex-independent increments.*

Proof. By assumption, $\Phi(K \cup L) - \Phi(L)$ and $\Phi(L) - \Phi(K \cap L)$ are independent. It remains to notice that $\Phi(K \cup L) - \Phi(L)$ is a.s. equal to $\Phi(K) - \Phi(K \cap L)$ by the additivity property of Φ. □

A function $g : \mathbb{R}^d \to \mathbb{R}_+$ is said to be *quasi-concave* if $\{x : g(x) \geq t\}$ is convex for all $t \in \mathbb{R}_+$. For any $K \in \mathcal{K}^d$ denote

$$g^\vee(K) = \sup\{g(x) : x \in K\}.$$

Theorem 4.9. *Let Φ be a monotone σ-continuous ID valuation. Then Φ has convex-independent increments if and only if its Lévy measure Λ is supported by valuations*

$$\psi(K) = g^\vee(K)$$

for a family of quasi-concave functions $g : \mathbb{R}^d \to \mathbb{R}_+$.

Proof. By Lemma 4.2, Λ is supported by ψ such that

$$\psi(K) = \psi(K \cap L) \quad \text{or} \quad \psi(L) = \psi(K \cap L) \tag{11}$$

for all $K, L \in \mathcal{K}^d$ with $K \cup L \in \mathcal{K}^d$. Furthermore, all ψ from the support of Λ are monotone and σ-continuous. Taking into account the monotonicity properties of ψ, we have that

$$\psi(K \cup L) = \psi(K) + \psi(L) - \psi(K \cap L) = \max(\psi(K), \psi(L)).$$

Define $g(x) = \psi(\{x\})$, $x \in \mathbb{R}^d$. Consider any hyperplane H which intersects the relative interior of K and so splits K into two convex sets K_1 and K_2 whose union is K. Assume that $g(K_1) = g(K)$. Splitting K_1, we obtain a sequence K_n, $n \geq 1$, which shrinks to a point

x and such that $\psi(K_n) = \psi(K)$ for all n. By σ-continuity, $g(x) = \psi(\{x\}) = \psi(K)$, while the construction shows that $g(x) = g^\vee(K)$.

Consider the segment $[x, y]$ and a point z in its relative interior. Then

$$g([x, y]) + g(z) = g^\vee([x, z]) + g^\vee([z, y]).$$

Without loss of generality assume that $g([x, y]) = g^\vee([z, y])$. Then $g(z) = g^\vee([x, z]) \geq g(x)$. Thus, $g(z) \geq \min(g(x), g(y))$, and so g is quasi-concave.

In the other direction, if g is quasi-concave, then (11) holds, so that Φ has convex-independent increments. □

The following result shows that its possible to slightly weaken the independence of increments property if Φ is assumed to be σ-continuous and monotone.

Proposition 4.10. *Assume that Φ is a σ-continuous monotone ID valuation. Then Φ has independent increments if and only if $\Phi(K) - \Phi(L)$ and $\Phi(L)$ are independent for all $K, L \in \mathcal{K}^d$ such that $L \subset K$.*

Proof. By Lemma 4.2, Λ is supported by ψ such that

$$\psi(K) = \psi(L) \quad \text{or} \quad \psi(L) = 0$$

for all $K, L \in \mathcal{K}^d$ with $L \subset K$. Furthermore, all ψ from the support of Λ are monotone and σ-continuous.

Since ψ is monotone, for $K, L \in \mathcal{K}^d$ such that $K \cup L$ is convex, we have $\psi(K \cup L) = \psi(K \cap L)$ or $\psi(K \cap L) = 0$. Furthermore, $\psi(K \cup L) = \psi(K)$ or $\psi(K) = 0$. If $\psi(K \cap L) = 0$ and $\psi(K) = 0$, then $\psi(K) = \psi(K \cap L)$. If $\psi(K \cap L) = 0$ and $\psi(K \cup L) = \psi(K)$, then $\psi(L) = 0$ by additivity and $\psi(L) = \psi(K \cap L)$. If $\psi(K \cup L) = \psi(K \cap L)$, then $\psi(K) = \psi(K \cap L)$ by monotonicity.

Thus, (11) holds, so that $\psi(K) = g^\vee(K)$ for $g(x) = \psi(\{x\})$, $x \in \mathbb{R}^d$. Therefore,

$$g^\vee(K) = g^\vee(L) \quad \text{or} \quad g^\vee(L) = 0.$$

In particular for $L = \{x\}$ for $x \in K$ we have that $g^\vee(K) = g(x)$ or $g(x) = 0$. Thus, g is a constant or zero on K. Letting K grow yields the conclusion. □

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Chapter 4

Adaptive Tests of a Cone Hypothesis

The material presented in this chapter is an ongoing project under the supervision of Lutz Dümbgen.

Adaptive Tests of a Cone Hypothesis

Abstract

We propose a new type of test for a cone hypothesis which adapts based on the number of constraints that are violated. It improves the classical non-adaptive model when few constraints are not satisfied, in terms of both simplicity and power. The new tests are shown to have a valid significance level α in the case of the negative cone. Moreover, some possible tools to evaluate the elements of the family of adaptive tests are presented, in terms of risk and power.

1 Introduction

Let $X \sim N(\mu, I_d)$, where $\mu \in \mathbb{R}^d$, and let A be any real-valued $m \times d$ matrix. A well-known testing problem is to test

$$H : A\mu \geq 0 \quad \text{against} \quad H^c : A\mu \ngeq 0.$$

This problem can be translated into another testing setting, whether μ belongs to a fixed polyhedral convex cone or not.

Let K be a closed convex cone in $\mathbb{R}^d$. In general, a convex set is polyhedral if it is a finite intersection of half-spaces. Then K is polyhedral if there exists a matrix A such that

$$K = \{x \in \mathbb{R}^d : Ax \geq 0\}.$$

Let K° be the polar cone corresponding to K. That is

$$K^\circ = \{x \in \mathbb{R}^d : x \cdot y \leq 0,\ y \in K\} = \text{cone}(-a_1, \ldots, -a_m),$$

where $a_1, \ldots, a_m$ are the columns of A^T. The reflection of K with respect to the origin is $-K$.

The testing problem H has been thoroughly studied. In [2], the author shows that a particular test, referred to as minimax test, minimises a specific risk function over a certain class of non-randomised tests, which also contains the likelihood ratio test. In [1], the authors present a new type of tests in the likelihood ratio case.

2 Associated normal variables restricted to a convex cone

A bounded measurable function $f : \mathbb{R}^d \to \mathbb{R}$ is called monotone with respect to a convex cone K if

$$f(x + \mu) \geq f(x)$$

for $x \in \mathbb{R}^d$ and $\mu \in K$. A linear transform T is inclusive for a convex cone K if $T(K) \supseteq K$. The following theorem extends the main result presented in [3].

Theorem 2.1. *Let K be a convex cone whose interior is non-empty. Let X be a multivariate normal with mean vector 0 whose covariance matrix Σ is inclusive for K. Then f and g are monotone with respect to K and $-K^\circ$ respectively if and only if*

$$Cov(f(X), g(X)) \geq 0.$$

Proof. The proof follows that of the main theorem in [3]. First, assume that the functions f and g are continuously differentiable with bounded partial derivatives. As shown in [3], the following functional

$$F(\lambda) = \int_{\mathbb{R}^d} \phi(x) f(x) g_\lambda(x) dx,$$

where ϕ is the density of X and

$$g_\lambda(x) = \int_{\mathbb{R}^d} \phi_\lambda(\lambda x - y) g(y) dy$$

for $\phi_\lambda(x) = (1 - \lambda^2)^{-d/2} \phi((1 - \lambda^2)^{-1/2} x)$, has derivative

$$F'(\lambda) = \frac{1}{\lambda} \int_{\mathbb{R}^d} \phi(x) \Big(\nabla f(x)^T \Sigma \nabla g_\lambda(x) \Big) dx.$$

It's observed that $F(\lambda)$ is continuous in λ and that $F(0) = \mathbb{E}f(X)\mathbb{E}g(X)$ and $F(1) = \mathbb{E}(f(X)g(X))$. If $F'(\lambda)$ is positive, then the main statement holds.

Notice that for $\mu \in K \cap \mathbb{S}^{d-1}$ and $x \in \mathbb{R}^d$, we have that

$$0 \leq \lim_{t \downarrow 0} \frac{f(x+t\mu) - f(x)}{t} = \nabla f(x) \cdot \mu,$$

which implies that $\nabla f(x) \in -K^\circ$ for each $x \in \mathbb{R}^d$. Similarly, we can see that $\nabla g(x) \in K$. Furthermore, since $\frac{\partial g_\lambda}{\partial x_i} = \frac{\partial \phi_\lambda * g}{\partial x_i} = \phi_\lambda * \frac{\partial g}{\partial x_i}$ for each $i = 1, \ldots, d$, we also have that $\nabla g_\lambda(x) \in K$.

Since Σ is inclusive for K, it follows that $\Sigma \nabla g_\lambda(x) \in K$ for each $x \in \mathbb{R}^d$ and that $F'(\lambda)$ is positive. The other steps of the proof remain unchanged. □

Remark 2.2. It is easy to see that the previous result is an expansion of the theorem from [3]. Let K be the positive cone, then $K^\circ = -K$ and f and g are monotone, i.e. they are non-decreasing functions of each of the separate variables $x_1, \ldots, x_d$. Furthermore, the matrix Σ is inclusive for K if and only if $\sigma_{ij} \geq 0$ for each $i, j = 1, \ldots, d$.

We conjecture that a similar result can be achieved in the case that X is a standard normal variable restricted to the convex cone.

Conjecture 2.3. *Let K° be the polar cone of a closed convex cone K in $\mathbb{R}^d$. Let $f : \mathbb{R}^d \to \mathbb{R}^+$ be a bounded measurable function such that*

$$f(x+\alpha) \leq f(x) \text{ for } x \in \mathbb{R}^d, \alpha \in K, \tag{1}$$

Let X be a standard random Gaussian variable. Then

$$Cov(f(X), \exp(X \cdot \mu) | X \in K^\circ) \leq 0.$$

In [2], the class $\mathcal{A}(K)$ of acceptable regions is introduced, which are measurable sets $U \subseteq \mathbb{R}^d$ that satisfy $U + K = U$. In particular, the non-randomized test $\mathbf{1}_{x \notin U}$ for $U \in \mathcal{A}_\alpha(K)$ is monotone, in the sense that

$$\mathbf{1}_{x \in U} \leq \mathbf{1}_{x+\mu \in U}, \quad \mu \in K, x \in \mathbb{R}^d.,$$

which is equivalent to

$$\mathbf{1}_{x \notin U} \geq \mathbf{1}_{x+\mu \notin U}, \quad \mu \in K, x \in \mathbb{R}^d.$$

Assuming that the Conjecture 2.3 holds, it is straightforward to achieve the next statement.

Corollary 2.4. *Let K° be the polar cone of a closed convex cone K in $\mathbb{R}^d$. For each $\mu \in K$,*

$$\frac{\int_{K^\circ} h(x) e^{-\frac{1}{2}||x-\mu||^2} dx}{\int_{K^\circ} e^{-\frac{1}{2}||x-\mu||^2} dx} \leq \frac{\int_{K^\circ} h(x) e^{-\frac{1}{2}||x||^2} dx}{\int_{K^\circ} e^{-\frac{1}{2}||x||^2} dx}, \tag{2}$$

for each measurable function $h : \mathbb{R}^d \to \{0, 1\}$ such that

$$h(x + \nu) \leq h(x) \quad x \in \mathbb{R}^d, \nu \in K.$$

Proof. The inequality (2) is equivalent to

$$\int_{K^\circ} \int_{K^\circ} h(x) e^{x \cdot \mu} dM_d(x) dM_d(y) \leq \int_{K^\circ} \int_{K^\circ} h(x) e^{y \cdot \mu} dM_d(x) dM_d(y),$$

where M_d is the Gaussian measure of mean 0 and covariance matrix I_d. This is equivalent to

$$Cov(h(X), \exp(X \cdot \mu) | X \in K^\circ) \leq 0.$$

We conclude by noticing that h satisfies the condition (1) of Conjecture 2.3. □

The previous statement can be proven without the use of the Conjecture 2.3 in the case that K is the negative cone, i.e.

$$K = \{x \in \mathbb{R}^d : x_1 \leq 0, \ldots, x_d \leq 0\},$$

where its polar cone is the positive cone.

Lemma 2.5. *Let K° be the polar cone of the negative cone K. Let $f : \mathbb{R}^d \to \mathbb{R}^+$ be a bounded measurable function such that*

$$f(x + \alpha) \leq f(x) \text{ for } x \in \mathbb{R}^d, \alpha \in K.$$

Then

$$\frac{\int_{K^\circ} h(x) e^{-\frac{1}{2}||x-\mu||^2} dx}{\int_{K^\circ} e^{-\frac{1}{2}||x-\mu||^2} dx} \leq \frac{\int_{K^\circ} h(x) e^{-\frac{1}{2}||x||^2} dx}{\int_{K^\circ} e^{-\frac{1}{2}||x||^2} dx},$$

for each $\mu \in K$.

Proof. The proof works on induction over the dimension d. The case $d = 1$ follows from Lemma 1 in [1].

Assume that the statement holds up to dimension $d-1$. The function f is a nondecreasing function of the separate variables $x_1, \ldots, x_d$. For $\mu \in K$ and $\bar{x} = (x_2, \ldots, x_d)$, following from the induction hypothesis, we have

$$\begin{aligned}
&\frac{\int_{K^\circ} h(x) e^{-\frac{1}{2}||x-\mu||^2} dx}{\int_{K^\circ} e^{-\frac{1}{2}||x-\mu||^2} dx} \\
&= \frac{\int_0^\infty \cdots \int_0^\infty \frac{\int_0^\infty h(x_1,\bar{x}) \exp{(x_1\mu_1)} \exp{(-\frac{1}{2}x_1^2)} dx_1}{\int_0^\infty \exp{(x_1\mu_1)} \exp{(-\frac{1}{2}x_1^2)} dx_1} \exp{(\bar{x}\cdot\bar{\mu})} \exp(-\frac{1}{2}||\bar{x}||^2) d\bar{x}}{\int_0^\infty \cdots \int_0^\infty \exp{(\bar{x}\cdot\bar{\mu})} \exp(-\frac{1}{2}||\bar{x}||^2) d\bar{x}} \\
&\le \frac{\int_0^\infty \cdots \int_0^\infty \frac{\int_0^\infty h(x_1,\bar{x}) \exp{(-\frac{1}{2}x_1^2)} dx_1}{\int_0^\infty \exp{(-\frac{1}{2}x_1^2)} dx_1} \exp{(\bar{x}\cdot\bar{\mu})} \exp(-\frac{1}{2}||\bar{x}||^2) d\bar{x}}{\int_0^\infty \cdots \int_0^\infty \exp{(\bar{x}\cdot\bar{\mu})} \exp(-\frac{1}{2}||\bar{x}||^2) d\bar{x}} \\
&\vdots \\
&\le \frac{\int_{K^\circ} h(x) e^{-\frac{1}{2}||x||^2} dx}{\int_{K^\circ} e^{-\frac{1}{2}||x||^2} dx}.
\end{aligned}$$

□

Remark 2.6. As Lemma 1 in [1], if we assume that h is radial, i.e. $h(rv) = h(r)$ for $v \in \mathbb{S}^{d-1}$ and $r \ge 0$, then we can drop the assumption that $x \cdot y \ge 0$ for each $x, y \in K$. We can see that $X = RV$, where $V \sim Unif(\mathbb{S}^{d-1} \cap K^\circ)$, $R^2 \sim \chi_d^2$, and R and V are independent. Then

$$Cov(h(X), \exp(X \cdot \mu) | X \in K^\circ) = Cov(h(RV), \exp(RV \cdot \mu)).$$

By the law of total covariance,

$$\begin{aligned}
Cov(h(RV), \exp(RV \cdot \mu)) &= \mathbb{E}(Cov(h(RV), \exp(RV \cdot \mu) | V)) \\
&\quad + Cov(\mathbb{E}(h(RV)|V), \mathbb{E}(\exp(RV \cdot \mu)|V)).
\end{aligned}$$

The first component is always negative in the case that h is non-decreasing in R, since $\exp(RV \cdot \mu)$ is non-increasing in R. If we assume that h depends only on R, i.e. $h(RV) = h(R)$, which is equivalent to the likelihood ratio case, it is easy to see that the second component vanishes. It implies that the conditional covariance is negative with the two previous assumptions, so that (2) holds.

3 Adaptive Tests

We assume that the Conjecture 2.3 holds.

We recall that a monotone (non-adaptive) test for a convex cone K is a measurable function $\varphi : \mathbb{R}^d \to \{0,1\}$ such that $\varphi(x) \geq \varphi(x+\mu)$ for any $x \in \mathbb{R}^d$ and $\mu \in K$. In particular, the complement of the support of φ belongs to $\mathcal{A}(K)$.

For each $j = 1, \ldots, k$, define the monotone test φ_j for K_j on S_j, which means

$$\varphi_j(x) \geq \varphi_j(x+\nu)$$

for any $x \in S_j$ and $\nu \in K_j$.

Definition 3.1. The function φ is called an adaptive test for K generated by the monotone tests φ_j if

$$\varphi(y) = \sum_{j=1}^{k} \mathbf{1}_{\{\Pi(y|K^\circ) \in \mathrm{ri}(F_j)\}}(y) \varphi_j(P_j y), \quad y \in \mathbb{R}^d. \tag{3}$$

We work under the assumption that conjecture ... holds.

The distribution of $\varphi_j(P_j Y)$ under H depends on $P_j \mu$. For each $j = 1, \ldots, k$, we want to work with any test such that a least favourable distribution is obtained at $\mu = 0$, given that $\Pi(Y|K^\circ) \in \mathrm{ri}(F_j)$. Lemma 1 in [1] establishes that it is always the case for the likelihood ratio test.

We now need some more results to establish that each adaptive test φ is of level α if each φ_j is of level α in S_j, for $j = 1, \ldots, k$.

Lemma 3.2. *Let K be a polyhedral cone. For $\mu \in K$,*

$$\mathbb{P}_\mu(\varphi_j(P_j Y) = 1 | \Pi(Y|K^\circ) \in \mathrm{ri}(F_j)) \leq \mathbb{P}_0(\varphi_j(P_j Y) = 1 | \Pi(Y|K^\circ) \in \mathrm{ri}(F_j)), \tag{4}$$

where $Y \sim N(0, I_d)$.

Proof. Fix $j \in \{1, \ldots, k\}$ and fix $\mu \in K_j$. Let $Z \sim N(0, I_{d_j})$ in S_j. Since the covariance matrix of Y is the identity, then $P_j Y = Z$ in distribution. We want to study the following conditional probability

$$\mathbb{P}(\varphi_j(Z+\mu) = 1 | Z+\mu \in F_j) = \frac{\int_{F_j} \varphi_j(x) e^{-\frac{1}{2}||x-\mu||^2} dx}{\int_{F_j} e^{-\frac{1}{2}||x-\mu||^2} dx}.$$

By Corollary 2.4, since φ_j is a monotone test, we have that

$$\mathbb{P}(\varphi_j(Z+\mu)=1|Z+\mu\in F)\leq \mathbb{P}(\varphi_j(Z)=1|Z\in F).$$

As shown by Lemma 3.14.2 in [4], the event $\{\Pi(Y|K^\circ)\in \mathrm{ri}(F_j)\}$ is equivalent to the event $\{P_jY\in \mathrm{ri}(F_j),(I-P_j)Y\in F_j^\perp\cap K\}$.

To conclude, since P_jY is independent of the event $\{(I-P_j)Y\in F_j^\perp\cap K\}$, we have

$$\begin{aligned}
&\mathbb{P}_\mu(\varphi_j(P_jY)=1|\Pi(Y|K^\circ)\in \mathrm{ri}(F_j))\\
&\quad=\mathbb{P}_\mu(\varphi_j(P_jY)=1|P_jY\in \mathrm{ri}(F_j),(I-P_j)Y\in F_j^\perp\cap K)\\
&\quad=\mathbb{P}_\mu(\varphi_j(P_jY)=1|P_jY\in \mathrm{ri}(F_j))\\
&\quad\leq \mathbb{P}_\mu(\varphi_j(P_jY-P_j\mu)=1|P_jY-P_j\mu\in \mathrm{ri}(F_j))\\
&\quad=\mathbb{P}_0(\varphi_j(P_jY)=1|\Pi(Y|K^\circ)\in \mathrm{ri}(F_j))
\end{aligned}$$

since $P_j\mu\in K_j$, $P_j(Y-\mu)\sim N(0,I_{d_j})$, and $P_j(Y-\mu)=P_jY-P_j\mu$. □

The following result shows that each adaptive test φ, such that each φ_j is a test of level α in S_j, is also a test of level α.

Theorem 3.3. *Let K be a polyhedral cone. If φ_j is a valid test of level α for $H_{|S_j}$ against $H^c_{|S_j}$ for each $j=1,\dots,k$, then*

$$\mathbb{P}_\mu(\varphi(Y)=1)\leq(1-\mathbb{P}_\mu(Y\in K))\alpha\leq(1-\mathbb{P}_0(Y\in K))\alpha<\alpha. \tag{5}$$

Proof.

$$\begin{aligned}
\mathbb{P}_\mu(\varphi(Y)=1)&=\sum_{j=1}^k\mathbb{P}_\mu(\{\varphi_j(P_jY)=1\}\cap\{\Pi(Y|K^\circ)\in \mathrm{ri}(F_j)\})\\
&=\sum_{j=1}^k\mathbb{P}_\mu(\varphi_j(P_jY)=1|\Pi(Y|K^\circ)\in \mathrm{ri}(F_j))\mathbb{P}_\mu(\Pi(Y|K^\circ)\in \mathrm{ri}(F_j))\\
&\leq\alpha\sum_{j=1}^k\mathbb{P}_\mu(\Pi(Y|K^\circ)\in \mathrm{ri}(F_j))\\
&=(1-\mathbb{P}_\mu(\Pi(Y|K^\circ)\in \mathrm{ri}(F_0)))\alpha.\\
&=(1-\mathbb{P}_\mu(Y\in K))\alpha\leq(1-\mathbb{P}_0(Y\in K))\alpha<\alpha.
\end{aligned}$$

□

The adaptive test, like the non-adaptive one, does not exhaust the level α unless $\mu = 0$. When it is possible to calculate the adjustment term $\mathbb{P}_\mu(Y \in K)$, we can adjust the adaptive test and gain more power.

4 Risk

The main goal of the work from [2] is to find non-randomized tests $x \in \mathbb{R}^d \mapsto \mathbf{1}_{x \notin U}$ of

$$H : \mu \in K \quad \text{against} \quad H^c : \mu \notin K,$$

with small risk

$$R(U) = \sup_{\theta \in K^c} \mathbb{P}_\theta(U),$$

under the restriction $\mathbb{P}_\theta U \geq 1 - \alpha$ for all $\theta \in K^c$ for some fixed level $\alpha \in (0, 1/2)$. This risk function is minimised over the class $\mathcal{A}_\alpha(K)$ of the acceptance regions $U \subseteq \mathbb{R}^d$, which are measurable and satisfy $U + K = U$. In particular, the test $\mathbf{1}_{x \notin U}$ for $U \in \mathcal{A}_\alpha(K)$ is monotone.

The likelihood ratio test belongs to the family $\mathcal{A}_\alpha(K)$. The main result of [2], Theorem 2.1, shows that, for K with non-empty interior, the risk function R is minimised by a certain test, which is referred to as minimax test. The main step of the proof is to see that the risk function R can be expressed in the following way

$$R(U) = \sup_{\theta \in \Delta_0} \lim_{r \to \infty} \mathbb{P}_{r\theta}(U), \tag{6}$$

where Δ_0 is a dense subset of ∂K. Furthermore, it is shown that each point of Δ_0 belongs to the relative interior of one of the $d-1$-dimensional faces of K.

We need to establish some results to see how the risk function R behaves over the family of adaptive tests.

Let φ be an adaptive test as in (3) defined on $\mathbb{R}^d$. Let φ^j be the test $\varphi|_{S_j}$.

Lemma 4.1. *The test φ^j is an adaptive test on S_j.*

Proof. The set $\mathcal{F}_j = \{F_0, \ldots, F_{k_j}\}$ of faces of the cone F_j is a subset of the family of faces of K°. Moreover, each subspace spanned by an element of $\mathcal{F}_j$ is contained in S_j. Then

$$\varphi^j(y) = \sum_{i=1}^{k_j} \mathbf{1}_{\{\Pi(y|F_j) \in \mathrm{ri}(F_i)\}}(y) \varphi_i(P_i y), \quad y \in S_j,$$

since $\Pi(y|F_j) = \Pi(y|K^\circ)$, for $y \in S_j$. □

Proposition 4.2. *Let ν be an element of* $\mathrm{ri}(F_j)$, *for $j = 1, \dots, k-1$, and let μ be in K such that $\mu \perp \nu$. Assume that* $\dim(F_j) = d_j$ *for $d_j \in \{1, \dots, d-1\}$. Then*

$$\lim_{t\to\infty} \mathbb{E}_{t\mu+\nu}\varphi(X) = \mathbb{E}_\nu \varphi^j(Y), \tag{7}$$

where $Y \sim N_{d_j}(0, I_{d_j})$.

Proof. For $j = 1, \dots, k-1$, φ can be bounded from above and below. Assume without loss of generality that $F_0, \dots, F_{k_j}$ are the faces of F_j. Set $A_j = \cup_{i=1}^{k_j} C_i$, and similarly $B_j = \cup_{i=k_j+1}^{k} C_j$. Notice that A_j and B_j are a partition of $\mathbb{R}^d$. For $x \in \mathbb{R}^d$, define

$$f(x) = \varphi^j(P_j(x))\mathbf{1}_{A_j}(x) \le \varphi(x)$$

and

$$g(x) = \varphi^j(P_j(x))\mathbf{1}_{A_j}(x) + \mathbf{1}_{B_j}(x) \ge \varphi(x).$$

Set $c_d = (2\pi)^{-d/2}$. Let O_j be the linear subspace orthogonal to S_j, notice that $\nu \in S_j$ and $\mu \in O_j$. Assume without loss of generality that $\overline{x} = (x_1, \dots, x_{d_j}, 0, \dots, 0) \in S_j$ and $\underline{x} = (0, \dots, 0, x_{d_j+1}, \dots, x_d) \in O_j$. Then

$$\begin{aligned}
&\lim_{t\to\infty} \mathbb{E}_{t\mu+\nu} f(X) = \lim_{t\to\infty} c_d \int_{\mathbb{R}^d} f(x) e^{-\frac{1}{2}||x-(t\mu+\nu)||^2} dx \\
&= \lim_{t\to\infty} c_d \int_{\mathbb{R}^d} \varphi^j(P_j(x))\mathbf{1}_{A_j}(x) e^{-\frac{1}{2}||\overline{x}-\nu||^2} e^{-\frac{1}{2}||\underline{x}-t\mu||^2} dx \\
&= \lim_{t\to\infty} c_{d_j} c_{d-d_j} \int_{\mathbb{R}^{d_j}} \int_{\mathbb{R}^{d-d_j}} \varphi^j(\overline{x})\mathbf{1}_{A_j}(x) e^{-\frac{1}{2}||\overline{x}-\nu||^2} e^{-\frac{1}{2}||\underline{x}-t\mu||^2} dx \\
&= \mathbb{E}_\nu \varphi^j(Y).
\end{aligned}$$

Similarly,

$$\lim_{t\to\infty} \mathbb{E}_{t\mu+\nu} g(X) = \mathbb{E}_\nu \varphi^j(Y),$$

since

$$\lim_{t\to\infty} \mathbb{E}_{t\mu+\nu} \mathbf{1}_{B_j}(X) = 0.$$

This yields

$$\lim_{t\to\infty} \mathbb{E}_{t\mu+\nu}\varphi(X) = \mathbb{E}_\nu \varphi^j(Y).$$

□

Let $\mathcal{A}_\alpha^{ad}(K)$ be the class of the adaptive acceptance regions of level α, so each element is the closure of the complement of the support of an adaptive test. The same risk function R is minimised by each element of $\mathcal{A}_\alpha^{a}(K)$. The identity (6) also holds over $\mathcal{A}_\alpha^{ad}(K)$, where θ still belongs to the relative interior of a full-dimensional facet of K. Following Proposition 4.2, by choosing $\nu = 0$ and $\mu = \theta$, we get that, for each $U \in \mathcal{A}_\alpha^{a}(K)$,

$$R(U) = \mathbb{E}_0 \varphi^1(Y),$$

where $Y \sim N_1(0, 1)$. By the definition of the adaptive tests, φ^1 is a test of level α for $K = [0, \infty)$, which is uniquely defined. It follows that each adaptive acceptance region has the same risk under R.

A possible new risk function would need to be adaptive. Let $U \in \mathcal{A}_\alpha^{ad}(K)$ be the closure of the complement of the support of φ. Define

$$R_j(U) = \sup_{\theta \in C_j} \mathbb{P}_\theta(U). \tag{8}$$

Definition 4.3. The adaptive risk function for φ is defined as

$$R^{ad}(U) = \max_{j=1,\ldots,k} R_j(U),$$

where U is the closure of the complement of the support of φ.

By Proposition 4.2 and the identity (6), it follows that

$$R_j(U) = \sup_{\theta \in \mathrm{ri}(F_j)} \mathbb{E}_\theta(1 - \varphi^j(Y)),$$

where $Y \sim N(0, I_{d_j})$.

5 Adaptive Critical Value and Power of the Adaptive Test

Let K be a polyhedral cone in $\mathbb{R}^d$. Let φ_L be the adaptive test whose φ_j are the likelihood ratio tests in S_j. The test φ_L is called adaptive likelihood ratio test, and it is extensively studied in [1].

The likelihood ratio statistic for testing $H : \mu \in K$ against all alternatives for a polyhedral cone K in $\mathbb{R}^d$ is

$$LR(Y) = \min_{\theta \in K} (Y - \theta)^T (Y - \theta) = ||\Pi(Y|K^\circ)||^2,$$

for $Y \sim N(\mu, I_d)$. The distribution of $LR(Y)$ under H depends on μ, and a least favourable null distribution is obtained at $\mu = 0$. So

$$\mathbb{P}_0(LR(Y) \leq c) = \sum_{i=0}^{d} w_{d-i}(d, K)\mathbb{P}(\chi_i^2 \leq c).$$

From Proposition 3.6.1 in [4], $w_{d-i}(d, K)$ is the measure of the set of points which project onto a facet of K° of dimension $d - i$. Usually, we test H against H^c by comparing $LR(Y)$ to the $(1-\alpha)$-quantile of the least favourable distribution. In [1], the authors compare the likelihood ratio test against the adaptive critical value

$$q(K, Y, \alpha) = \sum_{j=1}^{k} \mathbf{1}_{\Pi(Y|K^\circ)\in \mathrm{ri}(F_j)} q_{r_j},$$

where q_{r_j} is the $(1-\alpha)$ quantile of a χ^2 random variable of with $r_j = \mathrm{rank}(P_j)$ degrees of freedom.

The power of the nonadaptive test is

$$\mathbb{P}_\mu(LR(Y) > c(\alpha)),$$

where $c(\alpha)$ is the $(1 - \alpha)$-quantile of the mixture of χ^2 distributions. For the adaptive test, the power is

$$\mathbb{P}_\mu(LR(Y) > q(K, Y, \alpha)).$$

Let $r_L(\alpha)$ be the largest integer such that $c(\alpha) \geq q_{r_L(\alpha)}$. Assume the faces $F_1, \ldots, F_k$ are ordered such that the ranks r_j of the projection matrices P_j onto the linear spaces S_j are ordered, i.e. $r_1 \leq \cdots \leq r_k$. From Proposition 3 in [1], it follows that for each $j = 1, \ldots, k$ such that $j \leq r_L(\alpha)$, if the event $\{\Pi(Y|K^\circ) \in \mathrm{ri}(F_j)\}$ occurs, then the nonrejection of the adaptive test implies the nonrejection of the non adaptive test. For $j > r_L(\alpha)$, the converse occurs.

The same type of study can be done for other adaptive test φ, by defining an adaptive critical value $q_\varphi(K, Y, \alpha)$ and $r_\varphi(\alpha)$.

In most cases, it is difficult to do explicit calculations because of the shape of the cone K and the type of test φ. We now study one simple case where it is straightforward to calculate an explicit $r_\varphi(\alpha)$ for a fixed α and K.

Let K be the negative cone in $\mathbb{R}^d$, i.e.

$$K = \{x \in \mathbb{R}^d, x_1 \leq 0, \ldots, x_d \leq 0\}.$$

Let φ_M be the minimax adaptive test, which means that each φ_j is the minimax test in each S_j. Assume $X \sim N(\mu, I_d)$. In the non-adaptive case, we reject if $\max(X) \geq c$, for $c > 0$. Then

$$\alpha = \mathbb{P}_0(\max(X) \geq c) = 1 - \mathbb{P}(X < c) = 1 - \Phi(c)^d,$$

so

$$c = \Phi^{-1}\big((1-\alpha)^{1/d}\big).$$

In the adaptive case, we reject if $\max(X) \geq c_{d(X)}$, where

$$d(X) = |\{i = 0, \dots, d : X_i > 0\}|,$$

i.e. $d(X)$ is the number of constraints which are not satisfied. Then, for $k \in \{1, \dots, d\}$ and $Z \sim N_k(0, I_k)$,

$$\begin{aligned}\alpha &= \mathbb{P}(\max(X) \geq c_k | d(X) = k)\\ &= \mathbb{P}(\max(Z) \geq c_k | Z \geq 0)\\ &= 1 - (2\Phi(c_k) - 1)^k,\end{aligned}$$

so

$$c_k = \Phi^{-1}\Big(\frac{1 + (1-\alpha)^{1/k}}{2}\Big).$$

It follows that, for $\gamma = (1-\alpha)$, $r_{\varphi_M}(\alpha)$ is the largest $k = 0, \dots, d$ such that

$$\gamma^{1/d} \geq \frac{1 + \gamma^{1/k}}{2},$$

which is equivalent to

$$1 - 2(\gamma^{1/d}) + (\gamma^{1/d})^{d/k} \leq 0.$$

For $\alpha \downarrow 0$, we have

$$\begin{aligned}1 - 2(\gamma^{1/d}) + (\gamma^{1/d})^{d/k} &= 1 - 2\exp(\log(1-\alpha)/d) + \exp(\log(1-\alpha)/k)\\ &= 2\alpha/d - \alpha/k + \mathcal{O}(\alpha^2),\end{aligned}$$

so for α very small, then $r_{\varphi_M}(\alpha) \simeq d/2$.

In Section 6 of [1], the authors show by computational calculations that, for the adaptive likelihood ratio test φ_L and $\alpha = 0.05$,

$$r_L(\alpha) \simeq \lfloor d/2 \rfloor + 1,$$

for d large enough.

Recall the matrix A defining the testing problem and the cone K. Each row a of A defines a constraint for $x \in \mathbb{R}^d$ such that either $a \cdot x \geq 0$ or $a \cdot x < 0$. If x doesn't satisfy i constrains, for $i = 1, \ldots, d-1$, then $\Pi(x|K^\circ)$ projects on a face F_j of K° of dimension less or equal to $d - i$. For a fixed cone K, a fixed $\alpha \in (0, 1)$, and an adaptive test φ, when it is possible to define and calculate $r_\varphi(\alpha)$, we call $r_\varphi(\alpha)$ the power of φ. We would say that φ is the most powerful test for K if it maximises $r_\varphi(\alpha)$ over the family of all the adaptive tests. Equivalently, the most powerful test φ for K would be the one for which the highest number of constraints can be broken, and to be still more powerful than its non-adaptive counterpart.

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Appendix

The Fell Topology and Random Closed Sets

All the definitions and results presented in this section are taken from Chapter 1 in Molchanov (2017) and Chapter 12 in Schneider and Weil (2008).

Let E be a locally compact space with a countable base. Let $\mathcal{F}, \mathcal{G}$ and $\mathcal{C}$ be the family of closed, open, and compact subsets of E respectively. For $A, A_1, \ldots, A_k \subseteq E$ and $k \in \mathbb{N}_0$, one defines the missing and hitting families

$$\mathcal{F}^A := \{F \in \mathcal{F} : F \cap A = \varnothing\} \quad \mathcal{F}_A := \{F \in \mathcal{F} : F \cap A \neq \varnothing\} \tag{1}$$

and

$$\mathcal{F}^A_{A_1,\ldots,A_k} := \mathcal{F}^A \cap \mathcal{F}_{A_1} \cap \cdots \cap \mathcal{F}_{A_k}.$$

The base of the Fell topology is generated by the set system

$$\{F^C \in \mathcal{F} : C \in \mathcal{C}\} \cup \{F_G \in \mathcal{F} : G \in \mathcal{G}\},$$

which is $\cap$-stable. A well-known fact is that $\mathcal{F}$ is a compact space with a countable base. The following theorem presents a useful characterisation of the convergence in the Fell topology. By 'almost all $j \in \mathbb{N}$' we mean all $j \in \mathbb{N}$ with at most finitely many exceptions.

Theorem 4.0.1. *Let $(F_j)_{j\in\mathbb{N}}$ be a sequence in F, and let F be in $\mathcal{F}$. Then the following statements are equivalent:*

(a) $F_j \to F$ in the Fell topology, as $j \to \infty$.

(b) The following conditions hold:

(b_1) If $G \in \mathcal{G}$ and $G \cap F \neq \varnothing$, then $G \cap F_j \neq \varnothing$ for almost all $j \in \mathbb{N}$.

(b_2) If $C \in \mathcal{C}$ and $C \cap F = \varnothing$, then $C \cap F_j = \varnothing$ for almost all $j \in \mathbb{N}$.

By using this result, one can see that the operation of union is continuous in the Fell topology, meanwhile the intersection is not.

A random closed set X in E is a measurable map from a probability space to the space $\mathcal{F}$ of closed sets in E endowed with the Borel σ-algebra generated by the Fell topology.

It is known that the distribution of a random closed set is uniquely determined by its capacity functional defined as

$$T_X(L) = \mathbb{P}\{X \cap L \neq \varnothing\},$$

where L runs through the family of compact sets in E. A sequence of random closed sets $(X_n)_{n\geq 1}$ in E converges in distribution to a random closed set X, i.e. $X_n \xrightarrow{d} X$, if the corresponding probability measures on $\mathcal{F}$ weakly converge. This is the case if and only if

$$T_{X_n}(L) \to T_X(L) \quad \text{as } n \to \infty$$

for each compact set L such that $T_X(L) = T_X(\mathrm{Int}(L))$.

Valuations

All the definitions and results presented in this section are taken from Chapter 6 in Schneider (2014).

Let $\mathscr{K}^d$ be the family of convex bodies (i.e., compact convex sets) in $\mathbb{R}^d$, with the convention that the empty set is included in $\mathscr{K}^d$. A valuation $\varphi : \mathscr{K}^d \to \mathbb{R}$ is a real additive map. Additivity means that, for any compact convex sets K and L such that $K \cup L$ is also convex, the following identity holds:

$$\varphi(K \cup L) + \varphi(K \cap L) = \varphi(K) + \varphi(L),$$

with the condition that $\varphi(\varnothing) = 0$. The family $\mathscr{K}^d$ is equipped with the Hausdorff metric, which is

$$\rho_H(K, L) = \inf\{\epsilon > 0 : K + B_\varepsilon \supseteq L, L + B_\varepsilon \supseteq K\}, \quad K, L \in \mathscr{K}^d.$$

A valuation φ is translation invariant if $\varphi(K + x) = \varphi(K)$ for each $x \in \mathbb{R}^d$, rotation invariant if $\varphi(gK) = \varphi(K)$ for each rotation g, and invariant under rigid motions if it is translation and rotation invariant. A valuation is said to be continuous if it is with respect to the Hausdorff metric. Moreover, it is homogeneous of degree $\alpha \geq 0$ if $\varphi(\lambda K) = \lambda^\alpha \varphi(K)$ for each $\lambda \geq 0$. For translation invariant continuous valuations, only a few degrees of homogeneity are possible.

Theorem 4.0.2 (McMullen)**.** *Let φ be a translation invariant, continuous valuation on $\mathscr{K}^d$. Then, there are continuous, translation invariant valuations $\varphi_0, \dots, \varphi_d$ on $\mathscr{K}^d$ such that φ_i is homogeneous of degree i, $i = 0, \dots, d$, and*

$$\varphi(\lambda K) = \sum_{i=0}^{d} \lambda^i \varphi_i(K),$$

for each $\lambda \geq 0$ and $K \in \mathscr{K}^d$.

By adding the assumption of invariance under rotations, we get the following statement.

Theorem 4.0.3 (Hadwiger)**.** *Let φ be a continuous valuation on $\mathscr{K}^d$ which is invariant under rigid motions. Then, there are constants $c_0, \dots, c_d$ such that*

$$\varphi(K) = \sum_{i=0}^{d} c_i V_i(K),$$

where V_i is the i-th intrinsic volume.

Declaration of consent

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