The complement of the open orbit for tame quivers

Inauguraldissertation
der Philosophisch-naturwissenschaftlichen Fakultät
der Universität Bern

vorgelegt von

Andrea Michel

von Bönigen BE

Leiterin der Arbeit:
Prof. Dr. Ch. Riedtmann
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Prof. Dr. G. Colangelo

Contents

1 Preface

In this thesis we present some results about the complement of the open orbit for a special kind of quivers, called tame quivers. It is organized as follows:
The main part of the thesis is Chapter 3, which is self-contained and will be published. The Main Theorem as well as its proof are presented there.
Chapter 2 shall serve the reader as a preparation for Chapter 3 by introducing the subject. In section 2.1 we give a short introduction to (some aspects of) the representation theory of quivers. Of course, this introduction is not intended to be self-contained or complete. However, it contains basic definitions and well-known statements which will be used later on. For more detailed information, the reader is referred to [1] and [16]. In particular, we explain the notion of tame quivers and introduce the definition of reflection functors, which play a key role in the proof of our result. In section 2.2 we deal with the geometric aspects of the topic. In particular, we present the former results of A. Schofield and Ch. Riedtmann, who studied the complement of the open orbit in [12]. In Chapter 4 we apply our result to an example.

Acknowledgments:
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2 Introduction

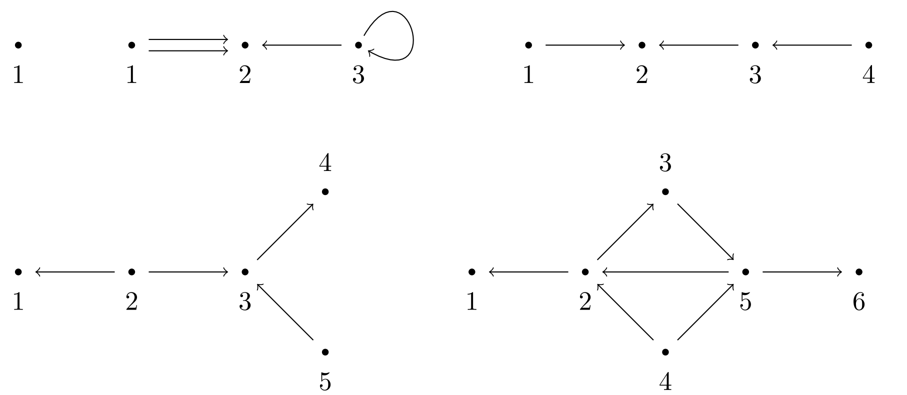
Throughout this chapter, we fix an algebraically closed field k.

2.1 Representations of quivers

2.1.1 Basic definitions

A **quiver** Q is an oriented graph, i.e. consists of a set Q_0 of **vertices** and a set Q_1 of **arrows**. Unless otherwise specified we will always assume that Q is finite, i.e. the sets Q_0 and Q_1 are finite and we identify $Q_0 = \{1, \dots, n\}$.

Example 2.1.



For an arrow $\alpha \in Q_1$ we denote by $t\alpha$ and $h\alpha$ the **tail** and the **head** of α, respectively. A vertex of Q which is the head of some arrows but the tail of none is called a **sink** and a vertex which is the tail of some arrows but the head of none is called a **source**. A **path** of length $l \geq 1$ is a sequence $\pi = \alpha_l \cdots \alpha_1$ of l arrows $\alpha_1, \dots, \alpha_l \in Q_1$ such that $h\alpha_i = t\alpha_{i+1}$ for $i = 1, \dots l-1$. Moreover, for each vertex $a \in Q_0$ there is a path ϵ_a of length zero, the **trivial path**. A non-trivial path is called an **oriented cycle** if $h\alpha_l = t\alpha_1$ and a quiver is called **acyclic** if it does not contain oriented cycles. Furthermore, a quiver is called **connected** if its underlying non-oriented graph is connected.

A **representation** X of a quiver Q (over k) is a family $\{X(a) \colon a \in Q_0\}$ of finite dimensional k-vector spaces together with a family $\{X(\alpha) \colon X(t\alpha) \longrightarrow X(h\alpha) \colon \alpha \in Q_1\}$ of k-linear maps.

Example 2.2.

$$Q = \underset{1}{\bullet} \longleftarrow \underset{2}{\bullet} \longrightarrow \underset{3}{\bullet} \qquad X = k \xleftarrow{\begin{pmatrix}1 & 0\end{pmatrix}} k^2 \xrightarrow{\begin{pmatrix}0 & 1\end{pmatrix}} k$$

A **morphism** $f\colon X \longrightarrow Y$ between two representations X and Y of Q is a familiy

$$\{f(a)\colon X(a) \longrightarrow Y(a)\colon a \in Q_0\}$$

of k-linear maps such that the diagram

$$\begin{array}{ccc} X(t\alpha) & \xrightarrow{X(\alpha)} & X(h\alpha) \\ {\scriptstyle f(t\alpha)}\downarrow & & \downarrow{\scriptstyle f(h\alpha)} \\ Y(t\alpha) & \xrightarrow[Y(\alpha)]{} & Y(h\alpha) \end{array}$$

commutes for all $\alpha \in Q_1$. We denote the finite dimensional k-vector space of morphisms from X to Y by $\mathrm{Hom}(X, Y)$. A homomorphism $f\colon X \longrightarrow Y$ is an **isomorphism** if for every vertex $a \in Q_0$ the map $f(a)\colon X(a) \longrightarrow Y(a)$ is bijective, i.e. is an isomorphism of vector spaces. In this case X and Y are **isomorphic** and we write $X \cong Y$. The category of representations of Q is denoted by $\mathrm{mod}\, Q$.

For two representations X and Y of Q we define the **direct sum** $X \oplus Y$ as the representation given by $(X \oplus Y)(a) = X(a) \oplus Y(a)$ for $a \in Q_0$ and

$$(X \oplus Y)(\alpha) = \begin{pmatrix} X(\alpha) & 0 \\ 0 & Y(\alpha) \end{pmatrix}$$

for $\alpha \in Q_1$. A representation $X \neq 0$ is called **decomposable** if X is the direct sum of two non-zero representations, otherwise it is called **indecomposable**.

Example 2.3. The representation X of Example 2.2 is decomposable as it is the direct sum of the two representations

$$X_1 = k \xleftarrow{\ 1\ } k \xrightarrow{\ 0\ } 0 \qquad \text{and} \qquad X_2 = 0 \xleftarrow{\ 0\ } k \xrightarrow{\ 1\ } k$$

According to the **Theorem of Krull-Schmidt** any representation $X \neq 0$ is isomorphic to a direct sum of indecomposable representations $X_1, \ldots, X_r$ and these representations are unique up to isomorphism and permutation.
A representation X is a **subrepresentation** of Y if $X(a)$ is a linear subspace of $Y(a)$ for all $a \in Q_0$ and $X(\alpha)$ is the restriction of $Y(\alpha)$ to $X(t\alpha)$ for all $\alpha \in Q_1$. Moreover, a subrepresentation X of Y is a **direct summand** of Y if there is a subrepresentation Z of Y such that $Y \cong X \oplus Z$. Note that not every subrepresentation X of Y is a direct summand of Y:

Example 2.4. Let Q be the quiver

$$Q = \underset{1}{\bullet} \longrightarrow \underset{2}{\bullet}$$

and let X and Y be the representations

$$X = 0 \xrightarrow{0} k \qquad \text{and} \qquad Y = k \xrightarrow{1} k$$

Clearly, X is a subrepresentation but not a direct summand of Y.

A representation $X \neq 0$ of a quiver Q is called **simple** if its only subrepresentations are the zero representation and X itself. For a vertex $a \in Q_0$ we denote by $S(a)$ the representation given by

$$S(a)(b) = \begin{cases} k, & \text{if } b = a, \\ 0, & \text{otherwise} \end{cases}$$

and $S(a)(\alpha) = 0$ for all $\alpha \in Q_1$. Clearly, this representation is simple. Moreover, if Q does not contain oriented cycles every simple representation is isomorphic to $S(a)$ for some vertex $a \in Q_0$.

A representation P of Q is called **projective** if for any representations X and Y of Q, any surjective morphism $f\colon X \twoheadrightarrow Y$ and any morphism $g\colon P \to Y$ there exists a morphism $f'\colon P \to X$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle f'}\nwarrow & \uparrow{\scriptstyle g} \\ & & P \end{array}$$

commutes.
Note that a representation P is projective if and only if all indecomposable directs summands of P are projective. In case Q does not contain oriented cycles we may construct all indecomposable projective representations of Q in the following way: Let $a \in Q_0$ be a vertex and define the representation $P(a)$ by

$$P(a)(b) = \bigoplus_{\substack{\pi\colon a \to b \\ \text{is a path} \\ \text{in } Q}} k\pi$$

for all vertices $b \in Q_0$ and $P(a)(\alpha)(\pi) = \alpha\pi$ for any path $\pi\colon a \to b$ and any arrow α with $t\alpha = b$. Then the set $\{P(a)\colon a \in Q_0\}$ forms a complete list of pairwise non-isomorphic indecomposable projective representations of Q.

Example 2.5. Let Q be the quiver

$$Q = \underset{1}{\bullet} \xleftarrow{\alpha_1} \underset{2}{\bullet} \xrightarrow{\alpha_2} \underset{3}{\bullet}$$

The representations $P(1), P(2)$ and $P(3)$ are given by

$$\begin{aligned}
P(1) &= k\epsilon_1 \xleftarrow{\ 0\ } 0 \xrightarrow{\ 0\ } 0 \;\cong\; k \xleftarrow{\ 0\ } 0 \xrightarrow{\ 0\ } 0 \\
P(2) &= k\alpha_1 \xleftarrow{\alpha_1 \circ} k\epsilon_2 \xrightarrow{\alpha_2 \circ} k\alpha_2 \;\cong\; k \xleftarrow{\ 1\ } k \xrightarrow{\ 1\ } k \\
P(3) &= \ 0 \xleftarrow{\ 0\ } 0 \xrightarrow{\ 0\ } k\epsilon_3 \;\cong\; 0 \xleftarrow{\ 0\ } 0 \xrightarrow{\ 0\ } k
\end{aligned}$$

Dually, a representation I of Q is called **injective** if for any representations X and Y of Q, any injective morphism $f\colon X \hookrightarrow Y$ and any morphism $g\colon X \to I$ there exists a morphism $f'\colon Y \to I$ such that the diagram

$$\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
{\scriptstyle g}\downarrow & \swarrow{\scriptstyle f'} & \\
I & &
\end{array}$$

commutes.
Note that a representation I is injective if and only if all indecomposable directs summands of I are injective. In case Q does not contain oriented cycles we may construct all indecomposable injective representations of Q in the following way: Let $a \in Q_0$ be a vertex and define the representation $I(a)$ by

$$I(a)(b) = \bigoplus_{\substack{\pi\colon b \to a \\ \text{is a path} \\ \text{in } Q}} k\pi$$

for all vertices $b \in Q_0$ and

$$I(a)(\alpha)(\pi) = \begin{cases} \pi', & \text{if there is a path } \pi'\colon h\alpha \to a \text{ such that } \pi = \pi'\alpha, \\ 0, & \text{otherwise} \end{cases}$$

for any path $\pi\colon b \to a$ and any arrow α with $t\alpha = b$. Then the set $\{I(a)\colon a \in Q_0\}$ forms a complete list of pairwise non-isomorphic indecomposable injective representations of Q.

Example 2.6. Let Q be the quiver

$$Q = \underset{1}{\bullet} \xleftarrow{\ \alpha_1\ } \underset{2}{\bullet} \xrightarrow{\ \alpha_2\ } \underset{3}{\bullet}$$

The representations $I(1), I(2)$ and $I(3)$ are given by

$$I(1) = k\epsilon_1 \xleftarrow{I(1)(\alpha_1)} k\alpha_1 \xrightarrow{0} 0 \;\cong\; k \xleftarrow{1} k \xrightarrow{0} 0$$

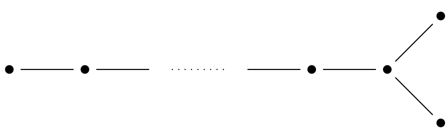
$$I(2) = \;0 \xleftarrow{0} k\epsilon_2 \xrightarrow{0} 0 \;\cong\; 0 \xleftarrow{0} k \xrightarrow{0} 0$$

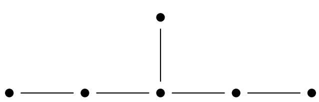
$$I(3) = \;0 \xleftarrow{0} k\alpha_2 \xrightarrow{I(3)(\alpha_2)} k\epsilon_3 \;\cong\; 0 \xleftarrow{0} k \xrightarrow{1} k$$

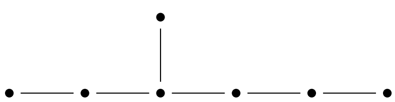
2.1.2 Dynkin and extended Dynkin quivers

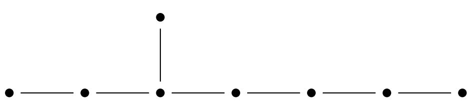
A quiver Q is said to be of **finite representation type** if there are, up to isomorphism, only finitely many indecomposable representations of Q. According to Gabriel [8] a connected quiver Q is of finite representation type if and only if the underlying non-oriented graph $|Q|$ is one of the following Dynkin diagrams:

$\mathbb{A}_n$: ($n \geq 1$ vertices)

$\mathbb{D}_n$:  ($n \geq 4$ vertices)

$\mathbb{E}_6$: 

$\mathbb{E}_7$: 

$\mathbb{E}_8$: 

Moreover, the number of indecomposable representations of Q is given by

$$\begin{cases} \frac{n(n+1)}{2}, & \text{if } |Q| = \mathbb{A}_n, \\ n(n-1), & \text{if } |Q| = \mathbb{D}_n, \\ 36, & \text{if } |Q| = \mathbb{E}_6, \\ 69, & \text{if } |Q| = \mathbb{E}_7, \\ 120, & \text{if } |Q| = \mathbb{E}_8. \end{cases}$$

Example 2.7. The indecomposable representations of

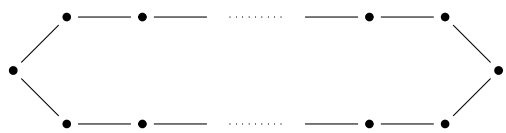
$$Q = \underset{1}{\bullet} \longrightarrow \underset{2}{\bullet}$$

are exactly the representations

$$P(1) = k \xrightarrow{1} k \qquad P(2) = 0 \xrightarrow{0} k \qquad S(1) = k \xrightarrow{0} 0$$

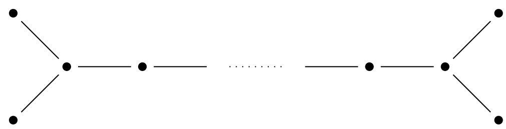
A quiver Q is called a **Dynkin quiver** if $|Q|$ is a disjoint union of Dynkin diagrams. Of course, Q is of finite representation type if and only if all its connected components are. Hence Q is of finite representation type if and only if Q is a Dynkin quiver.
A quiver Q is called an **extended Dynkin quiver** or **Euclidean quiver** if $|Q|$ is a disjoint union of extended Dynkin diagrams:

$\widetilde{\mathbb{A}}_n$:



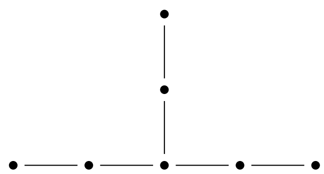
($n+1 \geq 1$ vertices)

$\widetilde{\mathbb{D}}_n$:

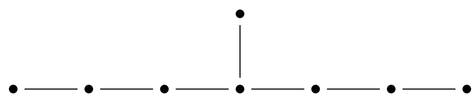


($n+1 \geq 5$ vertices)

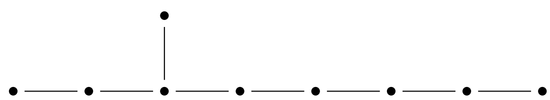
$\widetilde{\mathbb{E}}_6$:



$\widetilde{\mathbb{E}}_7$:



$\widetilde{\mathbb{E}}_8$:



Note that $\widetilde{\mathbb{A}}_0$ has one vertex and one loop and $\widetilde{\mathbb{A}}_1$ has two vertices joined by two edges.

A quiver is called **tame** if in each dimension vector there are at most finitely many one-parameter families of pairwise non-isomorphic indecomposable representations. Otherwise, it is called **wild**. A quiver Q without oriented cycles is tame if and only if $|Q|$ is a disjoint union of Dynkin and extended Dynkin diagrams. For a proof of this statement see e.g. chapter XIX in [17].

The **Euler form** is the bilinear form on $\mathbb{Z}^n$ given by

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i=1}^{n} a_i b_i - \sum_{\alpha \in Q_1} a_{t\alpha} b_{h\alpha},$$

where $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$. Associated to the Euler form is the **Tits form** $q \colon \mathbb{Z}^n \longrightarrow \mathbb{Z}$ given by $q(\mathbf{a}) = \langle \mathbf{a}, \mathbf{a} \rangle$. Suppose Q is a connected quiver. Then q is positive definite if and only if Q is a Dynkin quiver and q is positive semi-definite (but not positive definite) if and only if Q is an extended Dynkin quiver. For a tame quiver Q we call a vector $\mathbf{a} \in \mathbb{N}_0^n$ a **root** of q if $q(\mathbf{a}) \leq 1$. A root is called **real** if $q(\mathbf{a}) = 1$ and **imaginary** otherwise, i.e. if $q(\mathbf{a}) = 0$. If Q is a Dynkin quiver there is a bijection between the isomorphism classes of indecomposable representations of Q and the positive real roots of q. This bijection is given by assigning to an indecomposable representation its dimension vector. [9]

The following example shows that an indecomposable representation is not necessarily determined by its dimension vector if Q is not a Dynkin quiver:

Example 2.8. Let

$$Q = \underset{1}{\bullet} \rightrightarrows \underset{2}{\bullet}$$

and for $\lambda \in k$ let X_λ be the representation

$$X_\lambda = k \underset{\lambda}{\overset{1}{\rightrightarrows}} k$$

Clearly, the representation X_λ is indecomposable but X_λ is not isomorphic to X_μ for $\lambda \neq \mu$.

However, for an extended Dynkin quiver Q the following statement holds true: If X is an indecomposable representation of Q then $\mathbf{dim}\, \boldsymbol{X}$ is a root of q. Conversely, if $\mathbf{a} \in \mathbb{N}_0^n \setminus \{0\}$ is a positive real root of q there is a unique indecomposable representation X of Q such that $\mathbf{dim}\, \boldsymbol{X} = \mathbf{a}$ and if $\mathbf{a} \in \mathbb{N}_0^n \setminus \{0\}$ is a positive imaginary root of q there are infinitely many indecomposable representations X of Q such that $\mathbf{dim}\, \boldsymbol{X} = \mathbf{a}$.

2.1.3 Reflection functors

Let $z \in Q_0$ be a sink of an acyclic quiver Q and denote by $\alpha_j \colon y_j \longrightarrow z$, $j = 1, \dots, s$ the arrows with head z. We denote by zQ the quiver obtained from Q by reversing the arrows $\alpha_1, \dots, \alpha_s$. The **reflection functor** $R_z^+ \colon \operatorname{Rep}(Q) \longrightarrow \operatorname{Rep}(zQ)$ is defined in the following way [9]: For a representation $X \in \operatorname{Rep}(Q)$ the vector space $R_z^+(X)(y)$ is given by

$$R_z^+(X)(y) = \begin{cases} X(y), & \text{if } y \neq z \\ \ker\left(\bigoplus_{j=1}^{s} X(y_j) \xrightarrow{(X(\alpha_1),\ldots,X(\alpha_s))} X(z)\right), & \text{if } y = z. \end{cases}$$

Moreover, $R_z^+(X)(\alpha) = X(\alpha)$ for any arrow $\alpha \neq \alpha_1, \ldots, \alpha_s$ and for $j = 1, \ldots, s$ the map

$$R_z^+(X)(\alpha_j)\colon R_z^+(X)(z) \longrightarrow X(y_j)$$

is the projection from the kernel of $(X(\alpha_1), \ldots, X(\alpha_s))$ to $X(y_j)$. Note that

$$R_z^+(X \oplus Y) = R_z^+(X) \oplus R_z^+(Y)$$

for any representations $X, Y \in \operatorname{Rep}(Q)$. For a morphism $f = (f_y)_{y \in Q_0}\colon X \longrightarrow Y$ the morphism $R_z^+ f\colon R_z^+ X \longrightarrow R_z^+ Y$ is given by $R_z^+(f)_y = f_y$ for $y \neq z$ and $R_z^+(f)_z$ is the restriction of

$$f_{y_1} \oplus \cdots \oplus f_{y_s}\colon \bigoplus_{j=1}^{s} X(y_j) \longrightarrow \bigoplus_{j=1}^{s} Y(y_j) \tag{1}$$

to $R_z^+(X)(z)$. Note that R_z^+ preserves injections.

If $x \in Q_0$ is a source of Q, there is a dual construction: Let $\alpha_j\colon x \longrightarrow y_j,\ \ j = 1, \ldots, s$ be the arrows with tail x and denote by Qx the quiver obtained from Q by reversing these arrows. The reflection functor $R_x^-\colon \operatorname{Rep}(Q) \longrightarrow \operatorname{Rep}(Qx)$ is given by

$$R_x^-(X)(y) = \begin{cases} X(y), & \text{if } y \neq x \\ \operatorname{coker}\left(X(x) \xrightarrow{(X(\alpha_1),\ldots,X(\alpha_s))^T} \bigoplus_{j=1}^{s} X(y_j)\right), & \text{if } y = x \end{cases}$$

and

$$R_x^-(X)(\alpha) = \begin{cases} X(\alpha), & \text{if } \alpha \neq \alpha_1, \ldots, \alpha_s \\ \pi\big|_{X(y_j)}, & \text{if } \alpha = \alpha_j \end{cases}$$

where $\pi\colon \bigoplus_{j=1}^{s} X(y_j) \longrightarrow R_x^-(X)(x)$ is the projection. Note that $R_x^-(X \oplus Y) = R_x^-(X) \oplus R_x^-(Y)$ for any representations $X, Y \in \operatorname{Rep}(Q)$. For a morphism $f = (f_y)_{y \in Q_0}\colon X \longrightarrow Y$ the morphism $R_x^- f\colon R_x^- X \longrightarrow R_x^- Y$ is given bye $R_x^-(f)_y = f_y$ for $y \neq x$ and $R_x^-(f)_z$ is the map induced by (1). Note that R_x^- preserves surjections.

For a sink $z \in Q_0$ we denote by $\operatorname{mod}(Q)'$ and $\operatorname{mod}(zQ)'$ the full subcategories of $\operatorname{mod} Q$ and $\operatorname{mod} zQ$ whose objects do not contain $S(z)$ as a direct summand, respectively. The reflection functors $R_z^+\colon \operatorname{mod} Q \longrightarrow \operatorname{mod} zQ$ and $R_z^-\colon \operatorname{mod} zQ \longrightarrow \operatorname{mod} Q$ restrict to inverse equivalences between $(\operatorname{mod} Q)'$ and $(\operatorname{mod} zQ)'$. Moreover, any representations $X \in (\operatorname{mod} Q)'$ and $Y \in (\operatorname{mod} zQ)'$ satisfy $\mathbf{dim}\, \boldsymbol{R_z^+ X} = r_z(\mathbf{dim}\, \boldsymbol{X})$ and $\mathbf{dim}\, \boldsymbol{R_z^- Y} = r_z(\mathbf{dim}\, \boldsymbol{Y})$, respectively, where $r_z\colon \mathbb{N}_0^n \longrightarrow \mathbb{N}_0^n$ is given by $r_z(\mathbf{x})_y = \sum_{j=1}^{s} x_{y_j} - x_z$ if $y = z$ and $r_z(\mathbf{x})_y = x_y$ otherwise.

2.1.4 Extension groups

Let Q be an acyclic quiver and X a representation of Q. A **projective resolution** of X is an exact sequence

$$\ldots \longrightarrow P_i \xrightarrow{f_i} P_{i-1} \xrightarrow{f_{i-1}} \ldots \xrightarrow{f_1} P_0 \xrightarrow{f_0} X \longrightarrow 0$$

such that every representation P_i is projective. For a representation Y of Q the induced cochain complex of k-vector spaces yields

$$0 \longrightarrow \mathrm{Hom}(P_0, Y) \xrightarrow{(f_1)_*} \mathrm{Hom}(P_1, Y) \xrightarrow{(f_2)_*} \ldots \xrightarrow{(f_i)_*} \mathrm{Hom}(P_i, Y) \xrightarrow{(f_{i+1})_*} \ldots,$$

where $(f_i)_*\colon \mathrm{Hom}(P_{i-1}, Y) \longrightarrow \mathrm{Hom}(P_i, Y)$ is given by $(f_i)_*(g) = g \circ f_i$. The cohomology groups of this complex are independent of the choice of the projective resolution and $\mathrm{Ext}^i(X, Y)$ is defined as the i-th cohomology group $\mathrm{Ext}^i(X, Y) = \ker(f_{i+1})_* / \operatorname{im}(f_i)_*$ for $i \geq 0$ where we set $(f_0)_* = 0$. Note that $\mathrm{Ext}^0(X, Y)$ is isomorphic to $\mathrm{Hom}(X, Y)$. In our situation there exists a projective resolution of length 2 and hence Ext^i vanishes for $i \geq 2$. We denote the only possibly non-trivial extension group Ext^1 by Ext for short. Hence a short exact sequence

$$0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow X_3 \longrightarrow 0$$

induces long exact sequences

$$\begin{aligned} 0 \longrightarrow &\mathrm{Hom}(X_3, Y) \longrightarrow \mathrm{Hom}(X_2, Y) \longrightarrow \mathrm{Hom}(X_1, Y) \\ \longrightarrow &\mathrm{Ext}(X_3, Y) \longrightarrow \mathrm{Ext}(X_2, Y) \longrightarrow \mathrm{Ext}(X_1, Y) \longrightarrow 0 \end{aligned}$$

and

$$\begin{aligned} 0 \longrightarrow &\mathrm{Hom}(Y, X_1) \longrightarrow \mathrm{Hom}(Y, X_2) \longrightarrow \mathrm{Hom}(Y, X_3) \\ \longrightarrow &\mathrm{Ext}(Y, X_1) \longrightarrow \mathrm{Ext}(Y, X_2) \longrightarrow \mathrm{Ext}(Y, X_3) \longrightarrow 0 \end{aligned}$$

of k-vector spaces. Moreover,

$$\mathrm{Ext}(X_1 \oplus X_2, Y) = \mathrm{Ext}(X_1, Y) \oplus \mathrm{Ext}(X_2, Y)$$

and

$$\mathrm{Ext}(X, Y_1 \oplus Y_2) = \mathrm{Ext}(X, Y_1) \oplus \mathrm{Ext}(X, Y_2).$$

The following interpretation of $\mathrm{Ext}(X, Y)$ is frequently used: A short exact sequence

$$0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$$

is called an **extension** of Y by X. An extension is called **split** if it is equivalent to the **trivial extension**

$$0 \longrightarrow Y \longrightarrow Y \oplus X \longrightarrow X \longrightarrow 0.$$

By definition, two extensions

$$E_1\colon 0 \longrightarrow Y \longrightarrow Z_1 \longrightarrow X \longrightarrow 0$$
$$E_2\colon 0 \longrightarrow Y \longrightarrow Z_2 \longrightarrow X \longrightarrow 0$$

of Y by X are equivalent if and only if there is a morphism $h\colon Z_1 \longrightarrow Z_2$ such that the diagram

$$\begin{array}{cccccccccc} E_1: & 0 & \longrightarrow & Y & \longrightarrow & Z_1 & \longrightarrow & X & \longrightarrow & 0 \\ & & & \| & & \downarrow h & & \| & & \\ E_2: & 0 & \longrightarrow & Y & \longrightarrow & Z_2 & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

commutes. Note that h is in fact an isomorphism according to the Five Lemma. An element in $\mathrm{Ext}(X, Y)$ may be seen as an extension of Y by X, as there is a bijection between the elements of $\mathrm{Ext}(X, Y)$ and the equivalence classes of extensions of Y by X.

2.1.5 The Auslander-Reiten quiver

Let Q be a quiver without oriented cycles. A short exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

in $\operatorname{mod} Q$ is called an **almost split sequence** or **Auslander-Reiten sequence** if the following conditions are satisfied:

i) The sequence does not split.

ii) The representation X is indecomposable and any homomorphism from X to an indecomposable representation which is not an isomorphism factors through Y.

iii) The representation Z is indecomposable and any homomorphism from an indecomposable representation to Z which is not an isomorphism factors through Y.

For every indecomposable non-projective representation Z there is an unique (up to isomorphism) almost split sequence ending with Z and for every indecomposable non-injective representation X there is an unique (up to isomorphism) almost split sequence starting with X. Moreover, if

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

is an almost split sequence the representation X is called the **Auslander-Reiten translation** of Z, denoted by $X = \tau Z$ or, equivalently, $Z = \tau^{-1} X$.

An indecomposable representation X of Q is called **preprojective** or **preinjective** if there exist a vertex $a \in Q_0$ and a natural number $t \geq 0$ such that $X = \tau^{-t}P(a)$ or $X = \tau^t I(a)$, respectively. An indecomposable representation of Q which is neither preprojective nor preinjective is called **regular**. Moreover, an arbitrary representation of Q is called preprojective or preinjective or regular if it is a direct sum of indecomposable preprojective or preinjective or regular representations, respectively. In case Q is a Dynkin quiver every representation is both preprojective and preinjective.

The **Auslander-Reiten quiver** Γ_Q associated with Q is defined in the following way: For each isomorphism class of indecomposable representations of Q there is exactly one vertex in Γ_Q, labelled with a representative of this isomorphism class. For an indecomposable non-injective representation X of Q the number of arrows from the corresponding vertex of Γ_Q to a vertex of Γ_Q corresponding to an indecomposable representation Y is given by the multiplicity of Y in the decomposition of the middle term of the Auslander-Reiten sequence starting with X. For an indecomposable injective representation X the number of arrows from the corresponding vertex to a vertex corresponding to an indecomposable representation Y is given by the multiplicity of X in the decomposition of the middle term of the Auslander-Reiten sequence ending with Y if Y is non-projective and zero otherwise. Note that by the Theorem of Gabriel Γ_Q is a finite quiver if and only if Q is a Dynkin quiver.

Example 2.9. The quiver

$$Q = \underset{1}{\bullet} \xleftarrow{\alpha_1} \underset{2}{\bullet} \xrightarrow{\alpha_2} \underset{3}{\bullet}$$

has exactly 6 indecomposable representations, namely the indecomposable projective representations $P(1), P(2)$ and $P(3)$ (see Example 2.5) and the indecomposable injective representations $I(1), I(2)$ and $I(3)$ (see Example 2.6). The Auslander-Reiten sequences yield

$$\begin{aligned}
&0 \longrightarrow P(1) \longrightarrow P(2) \longrightarrow I(3) \longrightarrow 0\\
&0 \longrightarrow P(2) \longrightarrow I(1) \oplus I(3) \longrightarrow I(2) \longrightarrow 0\\
&0 \longrightarrow P(3) \longrightarrow P(2) \longrightarrow I(1) \longrightarrow 0
\end{aligned}$$

and the Auslander-Reiten quiver is given by

$$\begin{array}{ccccccc}
P(1) & & & & I(3) & & \\
 & \searrow & & \nearrow & & \searrow & \\
 & & P(2) & & & & I(2) \\
 & \nearrow & & \searrow & & \nearrow & \\
P(3) & & & & I(1) & &
\end{array}$$

As in this example the Auslander-Reiten quiver is always drawn such that any vertices corresponding to representations X and τX lie on a (imaginary) horizontal line.

The Auslander-Reiten translation is useful to compute the dimension of $\mathrm{Hom}(X,Y)$: As

$$\mathrm{Hom}(X_1 \oplus X_2, Y_1 \oplus Y_2) = \mathrm{Hom}(X_1,Y_1) \oplus \mathrm{Hom}(X_1,Y_2) \oplus \mathrm{Hom}(X_2,Y_1) \oplus \mathrm{Hom}(X_2,Y_2)$$

we may assume that both X and Y are indecomposable. If X is projective, i.e. $X = P(a)$ for some vertex $a \in Q_0$ then $\dim \mathrm{Hom}(X,Y) = \dim Y(a)$ by **Yoneda's Lemma**. If X is non-projective and Y is projective then $\dim \mathrm{Hom}(X,Y) = 0$ and if X and Y are both non-projective then $\dim \mathrm{Hom}(X,Y) = \dim \mathrm{Hom}(\tau X, \tau Y)$. Hence for a preprojective representation $X = \tau^{-t} P(a)$ the dimension of $\mathrm{Hom}(X,Y)$ is given by $\dim(\tau^t Y)(a)$ if $\tau^t Y$ exists and zero otherwise. Similarly, if Y is preinjective, i.e. $Y = \tau^t I(a)$, then $\dim \mathrm{Hom}(X,Y) = \dim(\tau^{-t} X)(a)$ if $\tau^{-t} X$ exists and $\dim \mathrm{Hom}(X,Y) = 0$ otherwise.
Furthermore, if X, Y and Z are indecomposable preprojective (but not preinjective), regular and preinjective (but not preprojective) representations, respectively, then

$$\mathrm{Hom}(Y,X) = \mathrm{Hom}(Z,X) = \mathrm{Hom}(Z,Y) = 0$$

[1] (Corollary 2.13 in chapter XIII).

In order to compute the dimension of $\mathrm{Ext}(X,Y)$ the **Auslander-Reiten formula** [2] is very helpful: If X is indecomposable and non-projective then $\dim \mathrm{Ext}(X,Y) = \dim \mathrm{Hom}(Y, \tau X)$. Note that $\mathrm{Ext}(X,Y) = 0$ in case X is projective.
Moreover, the Euler form provides a useful connection between Ext and Hom as

$$\langle \mathbf{dim}\, \boldsymbol{X}, \mathbf{dim}\, \boldsymbol{Y} \rangle = \dim \mathrm{Hom}(X,Y) - \dim \mathrm{Ext}(X,Y)$$

for any representations X and Y of Q.

2.2 Geometry of representations

2.2.1 The representation space

For $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n$ we denote by $\mathrm{Rep}\,(Q, \mathbf{d})$ the **representation space** with dimension vector $\mathbf{d}$, that is the set of representations X of Q satisfying $\dim X(j) = d_j$ for $j = 1, \ldots, n$. By choosing a basis for every vector space $X(j)$ a representation X with dimension vector

$$\mathbf{dim}\, \boldsymbol{X} := (\dim X(1), \ldots, \dim X(n)) = \mathbf{d}$$

yields an element of the direct product of $\mathrm{Mat}\,(d_{h\alpha} \times d_{t\alpha}, k)$ over all $\alpha \in Q_1$ and vice versa, hence

$$\mathrm{Rep}\,(Q, \mathbf{d}) = \prod_{\alpha \in Q_1} \mathrm{Mat}\,(d_{h\alpha} \times d_{t\alpha}, k)\,.$$

The group

$$\mathrm{GL}\,(\mathbf{d}) := \prod_{i=1}^{n} \mathrm{GL}\,(d_i, k)$$

acts on $\operatorname{Rep}(Q, \mathbf{d})$ by conjugation, that is

$$(g \cdot X)(\alpha) = g_{h\alpha} \circ X(\alpha) \circ g_{t\alpha}^{-1}$$

for $g = (g_1, \ldots, g_n) \in \mathrm{GL}(\mathbf{d})$ and $X \in \operatorname{Rep}(Q, \mathbf{d})$. By definition, the $\mathrm{GL}(\mathbf{d})$-orbits of $\operatorname{Rep}(Q, \mathbf{d})$ are exactly the isomorphism classes of representations X of Q with dimension vector $\mathbf{dim}\, \boldsymbol{X} = \mathbf{d}$.

The **Zariski topology** for $\operatorname{Rep}(Q, \mathbf{d})$ is defined in the following way: A subset of $\operatorname{Rep}(Q, \mathbf{d})$ is closed if and only if it is the zero set

$$\mathcal{Z}(I) = \{X \in \operatorname{Rep}(Q, \mathbf{d}) : f(X) = 0 \text{ for all } f \in I\}$$

of some ideal $I \subseteq k[\operatorname{Rep}(Q, \mathbf{d})]$, where $k[\operatorname{Rep}(Q, \mathbf{d})]$ denotes the algebra of polynomial functions on $\operatorname{Rep}(Q, \mathbf{d})$. The representation space $\operatorname{Rep}(Q, \mathbf{d})$ is **irreducible**, i.e. it cannot be written as the union of two proper closed sets or, equivalently, every non-empty open set is dense.

Let $X \in \operatorname{Rep}(Q, \mathbf{d})$ be any representation. The orbit $\mathrm{GL}(\mathbf{d}) \cdot X$ of X is **locally closed**, i.e. $\mathrm{GL}(\mathbf{d}) \cdot X$ is open in the closure $\overline{\mathrm{GL}(\mathbf{d}) \cdot X}$. According to the **Artin-Voigt Lemma** [14] the codimension of $\mathrm{GL}(\mathbf{d}) \cdot X$ is given by $\dim \operatorname{Ext}(X, X)$. Furthermore, the orbit of X (and hence its closure as well) is irreducible since $\mathrm{GL}(\mathbf{d})$ is connected.

For X and Y in $\operatorname{Rep}(Q, \mathbf{d})$ the representation X is called a **degeneration** of Y, denoted by $Y \leq_{\deg} X$, if the orbit of X is in the Zariski closure of the orbit of Y, i.e. $X \in \overline{\mathrm{GL}(\mathbf{d}) \cdot Y}$. Clearly, $\leq_{\deg}$ is a partial order on the set of isomorphism classes of representations of Q with dimension vector $\mathbf{d}$.
There is another partial order on this set, namely $\leq_{\hom}$, given by $Y \leq_{\hom} X$ if and only if $\dim \operatorname{Hom}(M, Y) \leq \dim \operatorname{Hom}(M, X)$ for all representations M of Q. In fact, this condition is equivalent to $\dim \operatorname{Hom}(Y, M) \leq \dim \operatorname{Hom}(X, M)$ for all M (see [3] and also [11]).
In case Q is a tame quiver the partial orders $\leq_{\deg}$ and $\leq_{\hom}$ coincide (see [7] and [6]). Thus the geometric problem whether some representation degenerates to some other may be reformulated as an algebraic problem. We will make use of this fact several times.

2.2.2 The open orbit and its complement

A dimension vector $\mathbf{d}$ is called **prehomogeneous** if there is a representation $T \in \operatorname{Rep}(Q, \mathbf{d})$ such that its orbit $\mathrm{GL}(\mathbf{d}) \cdot T$ is open (or, equivalently, dense) in $\operatorname{Rep}(Q, \mathbf{d})$. The Artin-Voigt Lemma implies that T has an open orbit if and only if $\operatorname{Ext}(T, T) = 0$. Note that every dimension vector $\mathbf{d}$ is prehomogeneous if Q is a Dynkin quiver. Indeed, $\operatorname{Rep}(Q, \mathbf{d})$ contains only finitely many orbits in this case since Q is of finite representation type and hence one of them must be dense as $\operatorname{Rep}(Q, \mathbf{d})$ is irreducibe.

Let $\mathbf{d}$ be a sincere prehomogeneous dimension vector and let T have an open orbit in $\operatorname{Rep}(Q, \mathbf{d})$. The complement $\operatorname{Rep}(Q, \mathbf{d}) \setminus \mathrm{GL}(\mathbf{d}) \cdot T$ is a closed subset of $\operatorname{Rep}(Q, \mathbf{d})$ and hence has a decomposition into irreducible components. This decomposition has been studied by Ch. Riedtmann and

A. Schofield in [12]. In order to formulate their result, let

$$T = \bigoplus_{j=1}^{r} T_j^{\lambda_j} \qquad \text{(where } \lambda_j > 0 \text{ and } T_i \not\cong T_j \text{ for } i \neq j\text{)}$$

be the decomposition of T into indecomposable direct summands and set

$$T[\hat{i}] := \bigoplus_{j \neq i} T_j^{\lambda_j}.$$

Following [12] an indecomposable direct summand T_i of T is called **essential** if it is either a submodule or a quotient of some representation in $\operatorname{add} T[\hat{i}]$, the full subcategory of $\operatorname{mod} Q$ whose objects are direct sums of the form

$$\bigoplus_{j \neq i} T_j^{\mu_j} \text{ with } \mu_j \geq 0.$$

Theorem (Ch. Riedtmann and A. Schofield)**.**
Let Q be a quiver without oriented cycles and let $\boldsymbol{d}$ be a sincere prehomogeneous dimension vector. Let the orbit of T be open in $\operatorname{Rep}(Q, \boldsymbol{d})$ *and suppose T is stable. Then* $\operatorname{Rep}(Q, \boldsymbol{d}) \setminus \operatorname{GL}(\boldsymbol{d}) \cdot T$ *has $n - r$ irreducible components of codimension one and there is a bijection between the irreducible components of codimension greater than one and the indecomposable direct summands of T which are essential. Moreover, every component is the closure of an orbit.*

The condition that T must be **stable** is no restriction in case Q is a Dynkin quiver. If Q is not a Dynkin quiver it means the multiplicities λ_j are 'large enough', see section 3.2 for the precise definition.
Note that [12] also contains for each irreducible component of $\operatorname{Rep}(Q, \mathbf{d}) \setminus \operatorname{GL}(\mathbf{d}) \cdot T$ a description of a generic representation, i.e. of a representation such that the closure of its orbit yields the irreducible component.

In [15], A. Schofield presented polynomials whose zeros are the irreducible components of codimension one. In this thesis, we suppose Q is a tame quiver and generalise the idea of A. Schofield to obtain for each irreducible component of codimension greater than one an ideal in the polynomial ring $k[\operatorname{Rep}(Q, \mathbf{d})]$ whose zero set is this component.
In case Q is the equioriented Dynkin quiver of type $\mathbb{A}_n$ K. Baur and L. Hille described in [4] the irreducible components of no matter what codimension of $\operatorname{Rep}(Q, \mathbf{d}) \setminus \operatorname{GL}(\mathbf{d}) \cdot T$ without using the results of Ch. Riedtmann and A. Schofield. We will apply our result to this case and show that it may be seen as a generalisation of the one in [4].

3 The complement of the open orbit for tame quivers

Let Q be a tame quiver and $\mathbf{d}$ a prehomogeneous dimension vector. We consider the complement of the open orbit of the representation space $\mathrm{Rep}\,(Q,\mathbf{d})$ and generalise the idea of A. Schofield [15] to obtain for each irreducible component of codimension greater than one an ideal in the polynomial ring $k\,[\mathrm{Rep}\,(Q,\mathbf{d})]$ whose zero set is this component. Moreover, we compare our result with the one of K. Baur and L. Hille, who found for each irreducible component some defining rank conditions in case Q is the equioriented Dynkin quiver of type $\mathbb{A}_n$ [4].

3.1 Introduction

Let k be an algebraically closed field and let $Q = (Q_0, Q_1)$ be a tame quiver, i.e. every connected component of Q is either a Dynkin quiver of type $\mathbb{A}_m$, $\mathbb{D}_m$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$ or an extended Dynkin quiver of type $\widetilde{\mathbb{A}}_m$, $\widetilde{\mathbb{D}}_m$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$. We identify the vertex set Q_0 with $Q_0 = \{1,\ldots,n\}$. For $\mathbf{d}\in\mathbb{N}^n$ we define

$$\mathrm{Rep}\,(Q,\mathbf{d}) := \prod_{\alpha\in Q_1} \mathrm{Mat}\,(d_{h\alpha}\times d_{t\alpha}, k)\,,$$

where $t\alpha$ and $h\alpha$ denote the tail and the head of $\alpha\in Q_1$, respectively. The group

$$\mathrm{GL}\,(\mathbf{d}) := \prod_{i=1}^{n} \mathrm{GL}\,(d_i, k)$$

acts on $\mathrm{Rep}\,(Q,\mathbf{d})$ by conjugation, that is

$$(g\cdot X)\,(\alpha) = g_{h\alpha}\circ X\,(\alpha)\circ g_{t\alpha}^{-1}$$

for $g = (g_1,\ldots,g_n)\in\mathrm{GL}\,(\mathbf{d})$ and $X\in\mathrm{Rep}\,(Q,\mathbf{d})$. Clearly, the $\mathrm{GL}\,(\mathbf{d})$-orbits of $\mathrm{Rep}\,(Q,\mathbf{d})$ are exactly the isomorphism classes of representations X of Q with dimension vector

$$\mathbf{dim}\,\boldsymbol{X} = (\dim X(1),\ldots,\dim X(n)) = \mathbf{d}.$$

Throughout this chapter we suppose w.l.o.g. that $\mathbf{d}$ is sincere, i.e. $d_j > 0$ for $j = 1,\ldots,n$.
A dimension vector $\mathbf{d}\in\mathbb{N}^n$ is called prehomogeneous if there is a representation $T\in\mathrm{Rep}\,(Q,\mathbf{d})$ such that its orbit $\mathrm{GL}\,(\mathbf{d})\cdot T$ is open in $\mathrm{Rep}\,(Q,\mathbf{d})$ with respect to the Zariski topology. Note that in case Q is a Dynkin quiver every dimension vector $\mathbf{d}$ is prehomogeneous. Indeed, since Q is of finite representation type [8], $\mathrm{Rep}\,(Q,\mathbf{d})$ contains only finitely many orbits and hence one of them is dense (and thus open) as $\mathrm{Rep}\,(Q,\mathbf{d})$ is irreducibe.
Since the support of a representation having an open orbit never contains oriented cycles we assume throughout this chapter that Q has no oriented cycles. Thus the category $\mathrm{mod}\,Q$ of representations of Q is an abelian category of global dimension one and we denote the only possibly non-trivial extension group Ext^1 by Ext for short. Note that by [14], T has an open orbit if and only if

$\mathrm{Ext}\,(T,T)=0$.
For a prehomogeneous dimension vector let

$$T=\bigoplus_{j=1}^{r}T_j^{\lambda_j}\qquad(\text{where } \lambda_j>0 \text{ and } T_i\not\simeq T_j \text{ for } i\neq j)$$

be the decomposition of T into indecomposable direct summands and set

$$T[\hat{i}]:=\bigoplus_{j\neq i}T_j^{\lambda_j}.$$

An indecomposable direct summand T_i of T is called essential if it is either a submodule or a quotient of some representation in $\mathrm{add}\,T[\hat{i}]$, the full subcategory of $\mathrm{mod}\,Q$ whose objects are direct sums of the form

$$\bigoplus_{j\neq i}T_j^{\mu_j}\text{ with }\mu_j\geq 0.$$

The complement $\mathrm{Rep}\,(Q,\mathbf{d})\setminus\mathrm{GL}\,(\mathbf{d})\cdot T$ has been studied by Ch. Riedtmann and A. Schofield in [12]. Provided T is stable (which is no restriction in case Q is a Dynkin quiver; otherwise it means the multiplicities λ_j are 'large enough', see section 3.2 for the precise definition) they proved the following: There are $n-r$ irreducible components of codimension one and there is a bijection between the irreducible components of codimension greater than one and the indecomposable direct summands of T which are essential. Moreover, they showed that every component is the closure of an orbit and gave a description of these orbits. We will recall these results in more detail in section 3.2.
Furthermore, A. Schofield presented polynomials whose zeros are the irreducible components of codimension one in [15]. In fact, he proved these components are given by

$$\mathcal{D}_j=\{X\in\mathrm{Rep}\,(Q,\mathbf{d}):\ \mathrm{Hom}(X,S_j)\neq 0\text{ or }\ \mathrm{Ext}(X,S_j)\neq 0\}\,,\qquad j=r+1,\dots,n,$$

where $S_{r+1},\dots,S_n$ denote the simple objects of the category $T^{\perp}$ (see section 3.2), which he proved is equivalent to the category of representations of a quiver having $n-r$ vertices. This may be reformulated in the following way: Mapping $X\in\mathrm{Rep}\,(Q,\mathbf{d})$ to an injective resolution

$$0\longrightarrow S_j\longrightarrow I_1\longrightarrow I_2\longrightarrow 0$$

of S_j in $\mathrm{mod}\,Q$ gives the long exact sequence

$$0\longrightarrow\mathrm{Hom}(X,S_j)\longrightarrow\mathrm{Hom}(X,I_1)\xrightarrow{g_j}\mathrm{Hom}(X,I_2)\longrightarrow\mathrm{Ext}(X,S_j)\longrightarrow 0.$$

As the Euler form $\langle\mathbf{d},\mathbf{dim}\boldsymbol{S_j}\rangle$ vanishes [15] we have $\dim\mathrm{Hom}(X,S_j)=\dim\mathrm{Ext}(X,S_j)$ (see section 3.2) and hence $\mathrm{Hom}(X,S_j)\neq 0$ or $\mathrm{Ext}(X,S_j)\neq 0$ if and only if the map g_j is not an isomorphism.

Decomposing

$$I_1 = \bigoplus_{k=1}^{r_1} I(a_k) \qquad \text{and} \qquad I_2 = \bigoplus_{l=1}^{r_2} I(b_l),$$

where $I(c)$ denotes the indecomposable injective representation corresponding to the vertex c, the map g_j is isomorphic to the map $X^t(g_j)\colon \bigoplus_{k=1}^{r_1} X(a_k) \longrightarrow \bigoplus_{l=1}^{r_2} X(b_l)$ and hence

$$\mathcal{D}_j = \{X \in \mathrm{Rep}\,(Q, \mathbf{d}) :\ \det X(g_j) = 0\}\,, \qquad j = r+1, \ldots, n.$$

We generalise this idea as follows to obtain for each irreducible component of codimension greater than one an ideal in the polynomial ring $k\,[\mathrm{Rep}\,(Q, \mathbf{d})]$ whose zero set is this component: Let

$$T_B = \bigoplus_{j=r+1}^{n} T_j$$

be the Bongartz completion (see section 3.2) of T. For T_i essential as a submodule we set

$$U_i := T_i / \operatorname{tr}_{T[\hat{\imath}] \oplus T_B} T_i,$$

where $\operatorname{tr}_{T[\hat{\imath}]\oplus T_B} T_i$ is the trace of $T[\hat{\imath}] \oplus T_B$ in T_i, i.e. the sum of all images of maps from $T[\hat{\imath}] \oplus T_B$ to T_i. We will prove in section 3.4 that U_i is indecomposable; in fact, U_i is the unique indecomposable and hence simple representation in $(T[\hat{\imath}] \oplus T_B)^{\perp}$. This notion allows us to formulate our result:

Theorem 3.1. *Let Q be a tame quiver and $\boldsymbol{d} \in \mathbb{N}^n$ a sincere prehomogeneous dimension vector such that T is stable, where $\mathrm{GL}\,(\boldsymbol{d}) \cdot T$ is the open orbit in $\mathrm{Rep}\,(Q, \boldsymbol{d})$. Let T_i be essential as a submodule and denote by T_B the Bongartz completion of T. The irreducible component $\mathcal{C}_i$ corresponding to T_i is given by*

$$\mathcal{C}_i = \{X \in \mathrm{Rep}\,(Q, \boldsymbol{d}) :\ \mathrm{Ext}(X, U_i) \neq 0\}\,,$$

where $U_i = T_i / \operatorname{tr}_{T[\hat{\imath}]\oplus T_B} T_i$.

Again, an ideal in the polynomial ring $k\,[\mathrm{Rep}\,(Q, \mathbf{d})]$ whose zero set is this component is obtained by mapping $X \in \mathrm{Rep}\,(Q, \mathbf{d})$ to an injective resolution of U_i. Note however that unlike before, the dimensions $\dim \mathrm{Hom}(X, U_i)$ and $\dim \mathrm{Ext}(X, U_i)$ never agree. In fact (see section 3.2),

$$\begin{aligned} \dim \mathrm{Hom}(X, U_i) - \dim \mathrm{Ext}(X, U_i) &= \langle \mathbf{dim}\, \boldsymbol{X}, \mathbf{dim}\, \boldsymbol{U_i} \rangle = \langle \mathbf{dim}\, \boldsymbol{T}, \mathbf{dim}\, \boldsymbol{U_i} \rangle \\ &= \dim \mathrm{Hom}(T, U_i) - \dim \mathrm{Ext}(T, U_i) = \lambda_i > 0, \end{aligned}$$

where the last equality will be proved in Lemma 3.7. Therefore, $\mathrm{Ext}(X, U_i) \neq 0$ is equivalent to the condition that all minors of order $\dim \mathrm{Hom}(X, I_2) = \sum_{l=1}^{r_2} d_{b_l}$ of

$$X(g_i)\colon \bigoplus_{l=1}^{r_2} X(b_l) \longrightarrow \bigoplus_{k=1}^{r_1} X(a_k)$$

are zero.

Note that the description of $\mathcal{C}_i$ in case T_i is essential as a quotient is simply obtained by dualizing the Theorem above: Let V_i be the unique indecomposable representation in ${}^{\perp}(T[\hat{i}] \oplus T_{DB})$, where T_{DB} denotes the dual Bongartz completion (see section 3.2) of T. The irreducible component corresponding to T_i is then given by

$$\mathcal{C}_i = \{X \in \mathrm{Rep}\,(Q, \mathbf{d}) :\ \mathrm{Ext}(V_i, X) \neq 0\}$$

and an ideal in the polynomial ring $k\,[\mathrm{Rep}\,(Q, \mathbf{d})]$ whose zero set is this component is obtained by applying $\mathrm{Hom}(\cdot, X)$ to a projective resolution of V_i.

From now on we let $\mathbf{d} \in \mathbb{N}^n$ be a sincere prehomogeneous dimension vector and denote by $\mathrm{GL}\,(\mathbf{d}){\cdot}T$ the open orbit in $\mathrm{Rep}\,(Q, \mathbf{d})$. Moreover, we assume T_i is essential as a submodule and denote by U_i the representation $U_i = T_i/\,\mathrm{tr}_{T[\hat{i}]\oplus T_B}\,T_i$, where T_B denotes the Bongartz completion of T.

In case Q is the equioriented Dynkin quiver of type $\mathbb{A}_n$ K. Baur and L. Hille described in [4] the irreducible components (of no matter what codimension) of the complement of the open orbit without using the results of Ch. Riedtmann and A. Schofield. Instead, they used rank conditions to define closed varieties in the complement, some of them turning out to be exactly the irreducible components. In section 3.3 we apply our result to this case and show that our rank condition derived above is precisely the rank condition found by K. Baur and L. Hille. In this sense, our result may be seen as a generalisation of the one in [4].

The main idea of the proof of the Theorem is the following: It is quite easy to see that the component $\mathcal{C}_i$ is contained in the set

$$\mathcal{E}_i := \{X \in \mathrm{Rep}\,(Q, \mathbf{d}) :\ \mathrm{Ext}(X, U_i) \neq 0\}\,.$$

For the other inclusion, we first prove that we may restrict ourselves to the case where the multiplicities λ_j are 'large enough' for $j = 1, \ldots, r$ and we show the statement is true in case T_i is simple and projective. In a second step, we consider in section 3.7.2 the easier case where Q is just a Dynkin quiver and use, depending on the dimension vector, either reflection functors or an induction argument on n to handle an arbitrary essential T_i. Finally, we prove in section 3.7.3 the statement for an arbitrary tame quiver. The additional difficulty in the latter case is due to the fact that not every representation is preprojective (see section 3.2), which means we cannot always derive the result from the special case where T_i is simple and projective. However, this problem can be solved by considering also reflections at sources and 'cutting out' a source (instead of considering just reflections at sinks and 'cutting out' a sink which suffices in the Dynkin case), meaning the principal ideas remain the same. These arguments as well as the reduction to large multiplicities are based on comparing the complement of the open orbit for (suitable) different dimension vectors. This is easily possible, since the irreducible components are closures of orbits as T is supposed to be stable (see section 3.2).

This chapter is organized as follows: In section 3.2 we introduce some basic notions and recall in particular the results of Ch. Riedtmann and A. Schofield and in section 3.3 we study as announced the special case where Q is the equioriented Dynkin quiver of type $\mathbb{A}_n$. In section 3.4 we prove the inclusion $\mathcal{C}_i \subseteq \mathcal{E}_i$ whereas the rest of the proof is presented in section 3.7. The sections 3.5 and 3.6 contain the necessary preparatory considerations. Note that the sections 3.5.2 and 3.6.2 are not needed for the proof in case Q is a Dynkin quiver.

3.2 Notations and Preliminaries

For $X, Y \in \operatorname{Rep}(Q, \mathbf{d})$ the representation X is called a degeneration of Y, denoted by $Y \leq_{\deg} X$, if the orbit of X is in the Zariski closure of the orbit of Y, i.e. $X \in \overline{\operatorname{GL}(\mathbf{d}) \cdot Y}$. Clearly, $\leq_{\deg}$ is a partial order on the set of isomorphism classes of representations of Q with dimension vector $\mathbf{d}$. There is another partial order on this set, namely $\leq_{\hom}$, given by $Y \leq_{\hom} X$ if and only if $\dim \operatorname{Hom}(M, Y) \leq \dim \operatorname{Hom}(M, X)$ for all representations M of Q. In fact, this condition is equivalent to $\dim \operatorname{Hom}(Y, M) \leq \dim \operatorname{Hom}(X, M)$ for all M (see [3] and also [11]).
Since Q is a tame quiver the partial orders $\leq_{\deg}$ and $\leq_{\hom}$ coincide (see [7] and [6]). Thus the geometric problem whether some representation degenerates to some other may be reformulated as an algebraic problem. We will make use of this fact several times. In particular, we note that the following 'cancellation' holds true: If $X \oplus N$ is a degeneration of $Y \oplus N$ for some representation N, then X is a degeneration of Y.

Let M be any representation of Q. The right and left perpendicular categories $M^{\perp}$ and ${}^{\perp}M$ are defined as the full subcategories of representations X of Q such that $\operatorname{Hom}(M, X) = \operatorname{Ext}(M, X) = 0$ and $\operatorname{Hom}(X, M) = \operatorname{Ext}(X, M) = 0$, respectively. Because these subcategories are closed under direct sums, direct summands, extensions, images, kernels and cokernels they are exact abelian categories. In case $\operatorname{Ext}(M, M) = 0$ the category $M^{\perp}$ is equivalent to the category of representations of some quiver having no oriented cycles and $n - r$ vertices, where r is the number of non-isomorphic indecomposable direct summands of M [15]. Note that this statement is also true for ${}^{\perp}M$. Denoting by τ the Auslander-Reiten translation the Auslander-Reiten formulae [2] yield

$$\dim \operatorname{Ext}(X, Y) = \dim \operatorname{Hom}(Y, \tau X),$$
$$\dim \operatorname{Hom}(X, Y) = \dim \operatorname{Ext}(Y, \tau X)$$

if X does not contain a direct summand which is projective. Clearly, this implies $M^{\perp} = {}^{\perp}\tau M$ for any representation M which does not contain a projective direct summand.

For a vertex $y \in Q_0$ we denote by $P(y)$ the indecomposable projective representation corresponding to y. An indecomposable representation X of Q is called preprojective if $X = \tau^{-t}P(y)$ for some $t \geq 0$ and some vertex $y \in Q_0$. The notion of an indecomposable preinjective representation of Q is defined dually and an indecomposable representation of Q is called regular if it is neither preprojective nor preinjective. Moreover, an arbitrary representation of Q is called preprojective or preinjective or regular if it is a direct sum of indecomposable preprojective or preinjective or regular representations, respectively. Note that in case Q is a Dynkin quiver every representation is both preprojective and preinjective. Let X, Y and Z be an indecomposable preprojective (but not preinjective), regular and preinjective (but not preprojective) representation, respectively. Then $\operatorname{Hom}(Y, X) = \operatorname{Hom}(Z, X) = \operatorname{Hom}(Z, Y) = 0$ [1] (Corollary 2.13 in chapter XIII).

Recall that a representation is called a tilting module if it contains exactly n different indecomposable direct summands and has no self-extensions. If X is a tilting module, then X cannot be a regular representation [17] (Lemma 3.4 in chapter XVII).

Recall that the Euler form is the bilinear form on $\mathbb{Z}^n$ given by $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i - \sum_{\alpha \in Q_1} x_{t\alpha} y_{h\alpha}$, where $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$. For two representations X and Y of Q the Euler form $\langle \mathbf{dim}\, \boldsymbol{X}, \mathbf{dim}\, \boldsymbol{Y} \rangle$ may be computed as $\langle \mathbf{dim}\, \boldsymbol{X}, \mathbf{dim}\, \boldsymbol{Y} \rangle = \dim \mathrm{Hom}(X, Y) - \dim \mathrm{Ext}(X, Y)$. Associated to the Euler form is the Tits form $q \colon \mathbb{Z}^n \longrightarrow \mathbb{Z}$ given by $q(\mathbf{x}) = \langle \mathbf{x}, \mathbf{x} \rangle$. Since Q is a tame quiver this quadratic form is positive semi-definite. More precisely, if Q is a Dynkin quiver, q is positive definite and there is a bijection between the isomorphism classes of indecomposable representations of Q and the positive roots of q, that is the set of $\mathbf{x} \in \mathbb{N}_0^n \setminus \{0\}$ satisfying $q(\mathbf{x}) = 1$. This bijection is given by assigning to an indecomposable representation its dimension vector. [9]

The following statements are proved by D. Happel and C. M. Ringel in [10] (Lemma 4.1. and Corollary 4.2.): For two indecomposable representations X and Y satisfying $\mathrm{Ext}(X, Y) = 0$ any non-zero map from Y to X is either injective or surjective. This implies in particular that for two non-isomorphic indecomposable direct summands T_j and T_l of T the vector spaces $\mathrm{Hom}(T_j, T_l)$ and $\mathrm{Hom}(T_l, T_j)$ cannot both be non-zero.

Let us recall the definition of the Bongartz completion T_B of $T = \bigoplus_{j=1}^{r} T_j$ [5]: For $j = 1, \dots, r$ let μ_j be the dimension of $\mathrm{Ext}(T_j, kQ)$, where kQ denotes the quiver algebra of Q, understood as a representation of Q. There is an exact sequence

$$0 \longrightarrow kQ \longrightarrow \tilde{T} \longrightarrow \bigoplus_{j=1}^{r} T_j^{\mu_j} \longrightarrow 0$$

such that for $l = 1, \dots, r$ the map $\mathrm{Hom}(T_l, \bigoplus_{j=1}^{r} T_j^{\mu_j}) \longrightarrow \mathrm{Ext}(T_l, kQ)$ (obtained from mapping T_l to this exact sequence) is surjective. The representation $\tilde{T}$ is independent (up to isomorphism) of the choice of such an exact sequence. Moreover, $T \oplus \tilde{T}$ has exactly n pairwise non-isomorphic indecomposable direct summands and satisfies $\mathrm{Ext}(T \oplus \tilde{T}, T \oplus \tilde{T}) = 0$. The Bongartz completion T_B is defined as the direct sum $T_B = \bigoplus_{j=r+1}^{n} T_j$ consisting of those indecomposable direct summands of $\tilde{T}$ which do not occur as direct summands of T. The notion of the dual Bongartz completion T_{DB} is defined dually, i.e. T_{DB} is the dual of the Bongartz completion of the dual representation of T.

In order to be able to formulate the result of Ch. Riedtmann and A. Schofield we recall the notion of source and sink maps: For pairwise non-isomorphic indecomposable representations $X, Y_1, \dots, Y_l$ a map $f \colon X \longrightarrow Y := \bigoplus_{i=1}^{l} Y_i^{\mu_i}$ (with $\mu_i \geq 0$) is called a source map from X to $\mathrm{add}\,(Y_1 \oplus \ldots \oplus Y_l)$ if first any map from X to some Y_j factors through f and second if $g \circ f$ still has this property for an endomorphism g of Y, then g is an automorphism. The notion of a sink map is defined dually. Both sink and source maps exist and are unique up to isomorphism, i.e. if $f_1 \colon X \longrightarrow \bigoplus Y_i^{\mu_i}$ and

$f_2 \colon X \longrightarrow \bigoplus Y_i^{\nu_i}$ are source maps from X to $\operatorname{add}(Y_1 \oplus \ldots \oplus Y_l)$, then there is an isomorphism $h \colon \bigoplus Y_i^{\mu_i} \longrightarrow \bigoplus Y_i^{\nu_i}$ such that $f_2 = h \circ f_1$.

The following may be found in [12]: For an indecomposable direct summand T_j of T the source map from T_j to $\operatorname{add} T[\hat{j}]$ as well as the sink map from $\operatorname{add} T[\hat{j}]$ to T_j is either injective or surjective. Moreover, the following Lemma holds true.

Lemma 3.2.

i) Let T_j be an indecomposable direct summand of T. If the source map from T_j to $\operatorname{add} T[\hat{j}]$ is injective, then the sink map from $\operatorname{add} T[\hat{j}]$ to T_j is injective as well.

ii) Let T_j be an indecomposable direct summand of T_B. The source map from T_j to $\operatorname{add} T$ is injective if the support of T equals Q_0.

In particular, the sink map from $\operatorname{add} T$ to T_j is injective for $j = r+1, \ldots, n$. Furthermore, the quotients $T_j / \operatorname{tr}_T T_j$ for $j = r+1, \ldots, n$ are exactly the indecomposable projective representations in $T^{\perp}$, see [12].

Sink and source maps may be used to characterize the Bongartz completion and the dual Bongartz completion of T: Given a representation $M = M_{r+1} \oplus \cdots \oplus M_n \in \operatorname{mod} Q$ (where $M_{r+1}, \ldots, M_n$ are indecomposable and pairwise non-isomorphic) such that $T \oplus M$ is a tilting module the sink map from $\operatorname{add}(T \oplus M[\hat{j}])$ to M_j is injective for $j = r+1, \ldots, n$ if and only if M is the Bongartz completion of T [13]. Analogously, M is the dual Bongartz completion of T if and only if the source map from M_j to $\operatorname{add}(T \oplus M[\hat{j}])$ is surjective for $j = r+1, \ldots, n$.

Let T_j be an indecomposable direct summand of T and denote by

$$f_j^+ : T_j \longrightarrow T_j^+ \qquad \text{and} \qquad f_j^- : T_j^- \longrightarrow T_j$$

the source map from T_j to $\operatorname{add} T[\hat{j}]$ and the sink map from $\operatorname{add} T[\hat{j}]$ to T_j, respectively. If Q is a Dynkin quiver the representations T_j^+ and T_j^- are direct summands of T since $\dim \operatorname{Hom}(T_k, T_l)$ is at most one for any pair (T_k, T_l) of non-isomorphic indecomposable direct summands of T. Indeed, assuming $\dim \operatorname{Hom}(T_k, T_l) \neq 0$ implies $\dim \operatorname{Hom}(T_l, T_k) = 0$ and hence $\langle \mathbf{dim}\, \boldsymbol{T_l}, \mathbf{dim}\, \boldsymbol{T_k} \rangle = 0$. The positive definiteness of $\langle \mathbf{dim}\, \boldsymbol{T_l} - \mathbf{dim}\, \boldsymbol{T_k}, \mathbf{dim}\, \boldsymbol{T_l} - \mathbf{dim}\, \boldsymbol{T_k} \rangle$ thus yields $0 < 2 - \langle \mathbf{dim}\, \boldsymbol{T_k}, \mathbf{dim}\, \boldsymbol{T_l} \rangle$ as $\mathbf{dim}\, \boldsymbol{T_k}$ and $\mathbf{dim}\, \boldsymbol{T_l}$ are roots of q, respectively. Combining this with

$$0 = \dim \operatorname{Ext}(T_k, T_l) = \dim \operatorname{Hom}(T_k, T_l) - \langle \mathbf{dim}\, \boldsymbol{T_k}, \mathbf{dim}\, \boldsymbol{T_l} \rangle$$

implies $\dim \operatorname{Hom}(T_k, T_l) = \langle \mathbf{dim}\, \boldsymbol{T_k}, \mathbf{dim}\, \boldsymbol{T_l} \rangle = 1$ as desired.
For the same reason the representation T_j^{++}, defined by the property that there is a source map $g_j^+ : T_j \longrightarrow T_j^{++}$ from an indecomposable direct summand T_j of T_B to $\operatorname{add} T$, is a direct summand of T. However, if Q is not a Dynkin quiver the representations T_j^+, T_j^- and T_j^{++} are not necessarily direct summands of T, respectively, which leads to the following definition [12]: The representation T is called stable if T_j^{++} for $j = r+1, \ldots, n$ as well as T_j^+ for T_j essential as a submodule and T_j^- for T_j essential as a quotient are direct summands of T, respectively.

Supposing T is stable, we recall the result of Ch. Riedtmann and A. Schofield in more detail [12]: As already mentioned, the irreducible components of codimension greater than one of the complement of the open orbit are in bijection with the indecomposable direct summands T_j of T which are essential, i.e. which are either a submodule or a quotient of some representation in $\operatorname{add} T[\hat{j}]$. More precisely, if T_j is essential, then the closure of the set of representations in $\operatorname{Rep}(Q, \mathbf{d})$ which contain $T_j^{\lambda_j+1}$ as a direct summand is an irreducible component of codimension $\lambda_j + 1$, denoted by $\mathcal{C}_j$. Moreover, for T_i essential as a submodule the irreducible component $\mathcal{C}_i$ is the closure of the orbit of

$$W_i := T_i^{\lambda_i+1} \oplus Y_i \oplus R_i,$$

where Y_i denotes the cokernel of the injective source map $f_i^+ : T_i \longrightarrow T_i^+$ from T_i to $\operatorname{add} T[\hat{i}]$ and R_i is given by $T[\hat{i}] = T_i^+ \oplus R_i$. Note that there is a dual description in case T_j is essential as a quotient. Furthermore, the representation Y_i has the following properties:

Lemma 3.3. *The representation Y_i is indecomposable and satisfies* $\operatorname{Ext}(Y_i \oplus T[\hat{i}], Y_i \oplus T[\hat{i}]) = 0$.

From now on, we fix the notation introduced above.

In case T is not stable the complement $\operatorname{Rep}(Q, \mathbf{d}) \setminus \operatorname{GL}(\mathbf{d}) \cdot T$ is still the union of $n - r$ irreducible components of codimension one and closed irreducible subsets $\mathcal{C}_j$ (which are in bijection with the direct summands of T which are essential) of greater codimension. However, there might be inclusions among the sets $\mathcal{C}_j$ and the irreducible components are not necessarily closures of orbits. [12]

Recall that $\mathcal{E}_i$ is the set $\mathcal{E}_i = \{X \in \operatorname{Rep}(Q, \mathbf{d}) : \operatorname{Ext}(X, U_i) \neq 0\}$. In order to prove Theorem 3.1 it is enough to prove

Theorem 3.4. *Let Q be a tame quiver, $\boldsymbol{d} \in \mathbb{N}^n$ a sincere prehomogeneous dimension vector and denote by $\operatorname{GL}(\boldsymbol{d}) \cdot T$ the open orbit in $\operatorname{Rep}(Q, \boldsymbol{d})$. Let T_i be essential as a submodule and assume T_i^+ is a direct summand of T. Then the closure of the orbit of W_i equals $\mathcal{E}_i$.*

Indeed, if T is stable, then T_i^+ is a direct summand of T and the closure of the orbit of W_i is the component $\mathcal{C}_i$.
The only reason we prove this version of the statement instead of the original one is purely technical: In the proof we will use reflection functors, which do not necessarily preserve the stability of T and thus it is more convenient to avoid this notion in the proof. Instead, we suppose throughout this chapter that T_i^+ is a direct summand of T and denote by $\mathcal{W}_i$ the closure of the orbit of W_i. Note that we do not know whether $\mathcal{W}_i$ is an irreducible component or just an irreducible subset.

A vertex $z \in Q_0$ is called sink if there is no arrow in Q_1 with tail z but there are $s \geq 1$ arrows $\alpha_j : y_j \longrightarrow z$, $j = 1, \ldots, s$ with head z. Note that the vertices y_j are not necessarily pairwise distinct. Dually, a vertex $x \in Q_0$ is called source if there is no arrow in Q_1 with head x but there are $s \geq 1$ arrows $\alpha_j : x \longrightarrow y_j$, $j = 1, \ldots, s$ with tail x.

From now on, $z \in Q_0$ denotes a sink and $x \in Q_0$ denotes a source and we set

$$\Delta(z) := \sum_{j=1}^{s} d_{y_j} - d_z \qquad \text{and} \qquad \Delta(x) := \sum_{j=1}^{s} d_{y_j} - d_x,$$

respectively. Moreover, we denote the simple representation supported at a vertex $y \in Q_0$ by $S(y)$. Note that $\dim \mathrm{Ext}(T, S(z)) = \nu + \Delta(z)$, where ν denotes the number of indecomposable direct summands of T isomorphic to $S(z)$. This implies

Remark 3.5. *The representation $S(z)$ is a direct summand of T if and only if $\Delta(z) < 0$.*

Indeed, if $S(z)$ is a direct summand of T, then $0 = \dim \mathrm{Ext}(T, S(z)) = \nu + \Delta(z)$ and hence $\Delta(z) < 0$. Conversely, if $\Delta(z) < 0$ then $\nu > \dim \mathrm{Ext}(T, S(z)) \geq 0$. Dually, $S(x)$ is a direct summand of T if and only if $\Delta(x) < 0$.
As a consequence, $S(z)$ lies in $T^{\perp}$ if and only if $\Delta(z) = 0$. In addition, if $S(z)$ lies in $T^{\perp}$ there exists an indecomposable direct summand T_j of T_B such that $T_j / \mathrm{tr}_T T_j = S(z)$ (see section 3.2) since $S(z)$ is indecomposable and projective. This is only possible if T_j equals $S(z)$ and hence we obtain

Remark 3.6. *The representation $S(z)$ is a direct summand of T_B if and only if $\Delta(z) = 0$.*

3.3 The equioriented Dynkin quiver of type $\mathbb{A}_n$

Let us consider the case where

$$Q = \quad \underset{1}{\bullet} \xleftarrow{\alpha_1} \underset{2}{\bullet} \xleftarrow{\alpha_2} \underset{3}{\bullet} \xleftarrow{\alpha_3} \underset{4}{\bullet} \cdots\cdots \underset{n-1}{\bullet} \xleftarrow{\alpha_{n-1}} \underset{n}{\bullet}$$

is the equioriented Dynkin quiver of type $\mathbb{A}_n$, i.e. $t\alpha_j = j+1$ and $h\alpha_j = j$ for $j = 1, \ldots, n-1$. For $1 \leq a \leq b \leq n$ we denote by $[a, b]$ the indecomposable representation supported at the vertices $\{a, a+1, \ldots, b\}$. Note that every indecomposable representation of Q is of this form. Moreover, we denote by $P(c) = [1, c]$, $I(c) = [c, n]$ and $S(c) = [c, c]$ the indecomposable projective, injective and simple representation corresponding to the vertex c, respectively. It is well-known that

$$\dim \mathrm{Hom}([a, b], [c, d]) = \begin{cases} 1, & \text{if } a \leq c \text{ and } c \leq b \leq d \\ 0, & \text{otherwise} \end{cases}$$

and

$$\dim \mathrm{Ext}([a, b], [c, d]) = \begin{cases} 1, & \text{if } c \leq a-1 \text{ and } a \leq d+1 \leq b \\ 0, & \text{otherwise.} \end{cases}$$

This information is contained in the Auslander-Reiten quiver (see figure 1) as follows: The dimension $\dim \mathrm{Hom}([a, b], [c, d])$ equals 1 if and only if the representation $[c, d]$ lies in the rectangle given by the vertices $[a, b]$, $S(b)$, $I(b)$ and $I(a)$, which is equivalent to the condition that $[a, b]$ lies in the

rectangle given by the vertices $[c,d]$, $S(c)$, $P(c)$ and $P(d)$. Using the Auslander-Reiten formulae we obtain that $\dim \mathrm{Ext}([a,b],[c,d]) = 1$ if and only if $[c,d]$ is not injective and $[a,b]$ lies in the rectangle given by the vertices $[c+1,d+1]$, $S(d+1)$, $I(d+1)$ and $I(c+1)$, which is the case if and only if $[a,b]$ is not projective and $[c,d]$ lies in the rectangle given by the vertices $[a-1,b-1]$, $S(a-1)$, $P(a-1)$ and $P(b-1)$.
Furthermore, note that if for two indecomposable representations $X=[x_1,x_2]$ and $Y=[y_1,y_2]$ of Q with $\mathrm{Ext}(Y,X)=0$ there is a non-zero homomorphism $f\colon X\longrightarrow Y$, then either $x_1=y_1$ and f is injective or $x_2=y_2$ and f is surjective. (Of course, this is just a special case of the Lemma of D. Happel and C. M. Ringel, see section 3.2.) In terms of the Auslander-Reiten quiver (see figure 1) this implies that X and Y both lie on the same downward or upward diagonal, respectively.

Recall that any dimension vector $\mathbf{d}\in\mathbb{N}^n$ is prehomogeneous. Let $T_i=[a,b]$ be essential as a submodule and recall that T_i^+ is a direct summand of T since Q is a Dynkin quiver. Our first goal is to show that $U_i = T_i/\operatorname{tr}_{T[\hat{i}]\oplus T_B} T_i$ is indecomposable and to describe its position on the Auslander-Reiten quiver.
By the observation above T contains an indecomposable direct summand $T_j=[a,\tilde{b}]$ with $\tilde{b}>b$ lying below T_i on the same downward diagonal D_a of the Auslander-Reiten quiver. Moreover, every indecomposable direct summand of $T[\hat{i}]\oplus T_B$ contributing to the trace $\operatorname{tr}_{T[\hat{i}]\oplus T_B} T_i$ lies on D_a above T_i: This is obvious in case T_i and T_j are projective. Otherwise, this follows from the fact that there is no indecomposable direct summand of $T[\hat{i}]\oplus T_B$ lying in the rectangle (marked in red) with vertices $\tau T_j=[a-1,\tilde{b}-1]$, $S(a-1)$, $P(a-1)$ and $P(\tilde{b}-1)$ as $\mathrm{Ext}(T_j, T[\hat{i}]\oplus T_B)=0$, see figure 1. Consequently, if there is no indecomposable direct summand of $T[\hat{i}]\oplus T_B$ lying on D_a above T_i, then $U_i=T_i$. Otherwise, the trace $\operatorname{tr}_{T[\hat{i}]\oplus T_B} T_i$ is given by the indecomposable direct summand of $T[\hat{i}]\oplus T_B$ on D_a above T_i closest to T_i, i.e. $\operatorname{tr}_{T[\hat{i}]\oplus T_B} T_i=[a,c-1]$ for some $c>a$. This implies that $U_i=[c,b]$ is indecomposable and its position on the Auslander-Reiten quiver is therefore obtained by a rectangle whose upper vertex exceeds the Auslander-Reiten quiver by one unit, see figure 1.

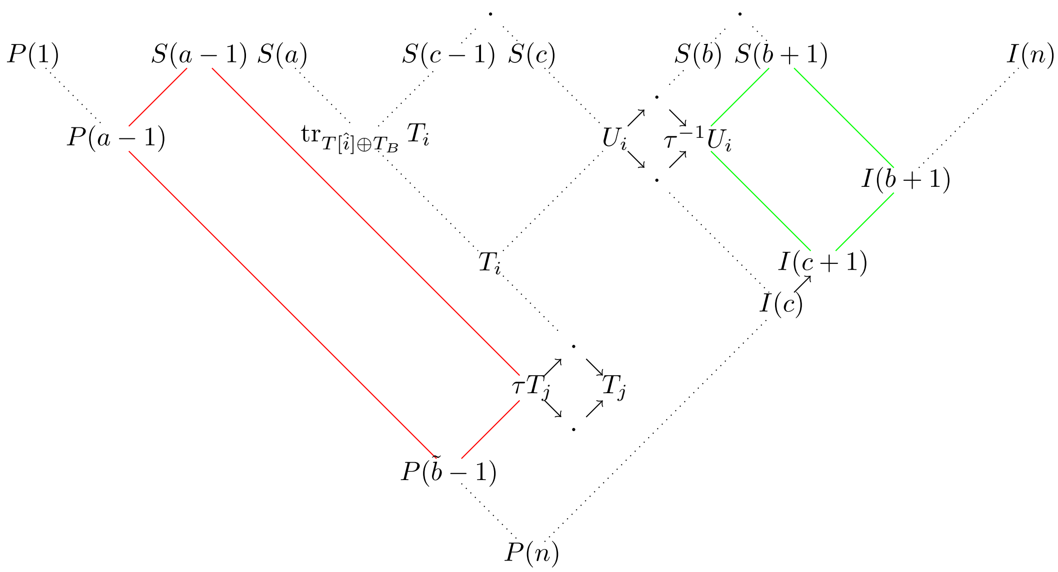


Figure 1: The Auslander-Reiten quiver for the equioriented Dynkin quiver of type $\mathbb{A}_n$

Furthermore, according to Theorem 3.1 a representation $X \in \mathrm{Rep}(Q, \mathbf{d})$ is in $\mathcal{C}_i$ if and only if X contains an indecomposable direct summand V such that $\mathrm{Ext}(V, U_i) \neq 0$. In terms of the Auslander-Reiten quiver this means V is one of the representations in the rectangle (marked in green) with vertices $\tau^{-1}U_i$, $I(c{+}1)$, $I(b{+}1)$ and $S(b{+}1)$. Thus, X contains such a direct summand V if and only if there is a homomorphism from X to $I(b+1)$ which does not factor through $I(c)$. Of course, this is exactly the condition $\mathrm{rk}\, g_i < \dim \mathrm{Hom}(X, I(b+1))$ stated in section 3.1, where $g_i\colon \mathrm{Hom}(X, I(c)) \longrightarrow \mathrm{Hom}(X, I(b+1))$ is obtained by mapping X to

$$0 \longrightarrow U_i \longrightarrow I(c) \longrightarrow I(b+1) \longrightarrow 0,$$

which is an injective resolution of U_i. As explained in section 3.1, this is equivalent to the condition $\mathrm{rk}\, X(g_i) < d_{b+1}$, where

$$X(g_i) = X(\alpha_c) \circ X(\alpha_{c+1}) \circ \cdots \circ X(\alpha_b)\colon k^{d_{b+1}} \longrightarrow k^{d_c}.$$

Indeed, $\mathrm{rk}\, X(g_i) < d_{b+1}$ if and only if X contains a direct summand $[d, e]$ such that $b+1 \leq e$ and $c+1 \leq d \leq b+1$, which is the case if and only if $\mathrm{Ext}(X, U_i = [c, b]) \neq 0$.

As already mentioned, this example has been studied before by K. Baur and L. Hille in [4]. In the remaining part of this section we recall their result and compare it with ours.
Denoting (for $i' < j'$) by $X_{(i',j')}$ the map

$$X_{(i',j')} = X(\alpha_{i'}) \circ X(\alpha_{i'+1}) \circ \cdots \circ X(\alpha_{j'-1})\colon k^{d_{j'}} \longrightarrow k^{d_{i'}}$$

(and setting $d_0 := 0$ for convenience) K. Baur and L. Hille proved in particular the following statement: For each pair $(i', j') \in Q_0^2$ satisfying

i) $i' < j'$ and $d_{i'} > d_{j'}$,

ii) $d_{l'} > d_{i'}$ for all $i' < l' < j'$,

iii) $d_{l'} \geq d_{i'}$ for all $m' < l' < i'$, where $m' < i'$ is the maximal index with $d_{m'} < d_{j'}$

the set

$$Y_{i',j'} := \left\{ X \in \mathrm{Rep}\,(Q, \mathbf{d}) :\ \mathrm{rk}\, X_{(i',j')} < d_{j'} \right\}$$

is an irreducible component of codimension $d_{i'} - d_{j'} + 1$.
Our goal is to show that such a component is the irreducible component corresponding to a suitable T_i which is essential as a submodule and that the rank conditions $\mathrm{rk}\, X_{(i',j')} < d_{j'}$ and $\mathrm{rk}\, X(g_i) < d_{b+1}$ coincide. Note that K. Baur and L. Hille also proved a dual statement, which corresponds to the case where T_i is essential as a quotient.
In order to obtain such inequalities for the entries of the dimension vector and to understand the roles of i' and j' in terms of our point of view we determine in figure 2 more carefully the positions of the indecomposable direct summands of T on the Auslander-Reiten quiver in case $T_i = [a, b]$ is essential as a submodule. Recall that the source map from T_i to $\mathrm{add}\, T[\hat{\imath}]$ is denoted by $f_i^+\colon T_i \longrightarrow T_i^+$. We consider the general case where $T_i^+ = T_j \oplus T_k$ consists of two indecomposable

direct summands, namely $T_j = [a, \tilde{b}]$ for some $\tilde{b} > b$ and $T_k = [\tilde{c}+1, b]$ for some $\tilde{c} < b$. Note that the case where $T_i^+ = T_j$ is indecomposable may be seen as a special case of this by allowing T_k to be a 'virtual representation' exceeding the Auslander-Reiten quiver by one unit. We denote the upward diagonal through T_i by D_b. By definition of the source map f_i^+ there is no indecomposable direct summand of T lying on D_b between T_i and T_k. We indicate this on the Auslander-Reiten quiver by a dotted red line. Moreover, there may be indecomposable direct summands of T on the downward diagonal D_a above T_i. We denote by $T_l = [a, d-1]$ the one closest to T_i and indicate this by a dotted red line between T_l and T_i. Again, the case where there is no such direct summand corresponds to the case where T_l is a 'virtual representation'. Using $\mathrm{Ext}(T \oplus T_B, T \oplus T_B) = 0$ we additionally obtain up to four red rectangles where there is no indecomposable direct summand of $T \oplus T_B$. In figure 2 we draw the general case where there are four actual rectangles. Note however that all of them may possibly have width zero (i.e. may be just lines) and that only the one corresponding to $\tau^{-1}T_i$ exists in any case.

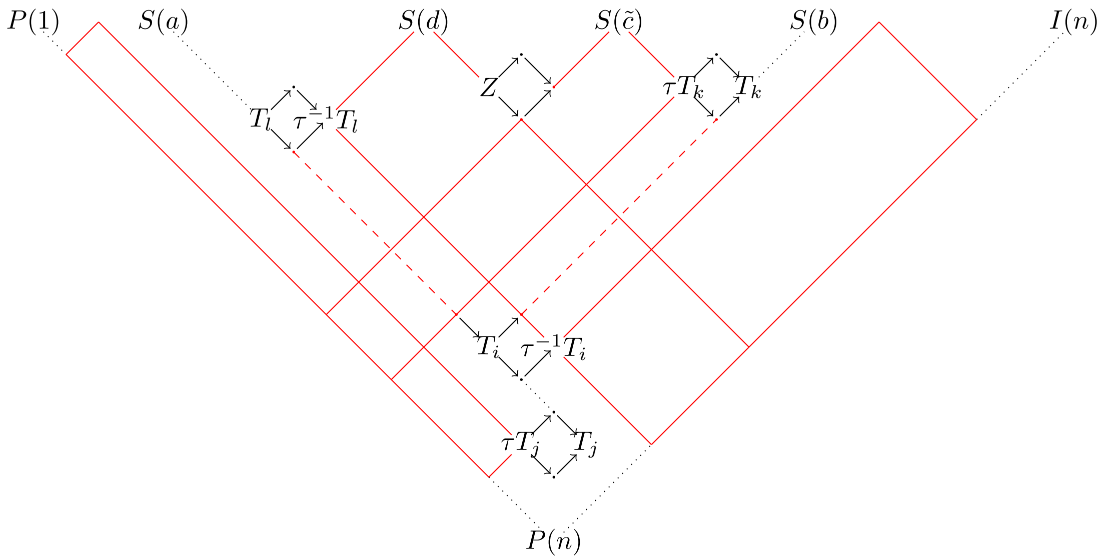


Figure 2: The indecomposable direct summands of T on the Auslander-Reiten quiver

For a vertex $x \in Q_0$ the value of d_x is given by the sum of the multiplicities of those indecomposable direct summands of T which lie in the rectangle with vertices $S(x)$, $I(x)$, $P(n)$ and $P(x)$. Identifying $i' := \tilde{c}$ and $j' := b+1$ we therefore see that the dimension vector $\mathbf{d}$ satisfies $d_{i'} > d_{j'}$ as on the one hand T_i is supported at $\tilde{c}$ but not at $b+1$ and on the other hand any other direct summand of T supported at $b+1$ is also supported at $\tilde{c}$. Note that in particular this implies $d_{i'} - d_{j'} + 1 = \lambda_i + 1$. Similarly, we have $d_{l'} > d_{i'}$ for all $i' < l' < j'$. Moreover, $d_{a-1} < d_{b+1}$ and $d_a, d_{a+1}, \cdots d_{\tilde{c}-1} \geq d_{b+1}$, which means the maximal index $m' < i'$ with $d_{m'} < d_{j'}$ is $m' = a-1$. Hence the condition $d_{l'} \geq d_{i'}$ for all $m' < l' < i'$ is satisfied as well because $d_a, d_{a+1}, \ldots, d_{\tilde{c}-1} \geq d_{\tilde{c}}$. The irreducible component $Y_{i',j'}$ is thus given by $Y_{i',j'} = \{X \in \mathrm{Rep}\,(Q, \mathbf{d}) : \mathrm{rk}\, X_{(\tilde{c}, b+1)} < d_{b+1}\}$. In order to prove that this is the component $\mathcal{C}_i$ it is therefore enough to show that $\tilde{c} = c$, i.e. that U_i lies on D_b just below T_k. This is the case if and only if the trace $\mathrm{tr}_{T[\hat{i}] \oplus T_B} T_i$ equals $[a, \tilde{c}-1]$. Of course we are done if $T_l = [a, \tilde{c}-1]$, so we assume (as in figure 2) that T_l lies above $[a, \tilde{c}-1]$ and we have to show that $[a, \tilde{c}-1]$ is an indecomposable direct summand of T_B. It is easy to see that the indecomposable representation $Z := [d, \tilde{c}-1]$ lies in $T^{\perp}$. Moreover, if $\mathrm{Ext}(Z, [x,y]) \neq 0$ for some indecomposable representation $[x,y]$, then $\mathrm{Hom}(T_l, [x,y]) \neq 0$ in case $a \leq x$ and $\mathrm{Ext}(T_j, [x,y]) \neq 0$ otherwise, hence $[x,y] \notin T^{\perp}$. This proves that Z is projective in $T^{\perp}$ and thus there is an indecomposable direct summand T_m of T_B such that $T_m / \operatorname{tr}_T T_m = Z$ (see section 3.2), which is only possible if T_m equals $[a, \tilde{c}-1]$.

3.4 The inclusion $\mathcal{W}_i \subseteq \mathcal{E}_i$

We now return to the general case where Q is any tame quiver and $\mathbf{d} \in \mathbb{N}^n$ any prehomogeneous dimension vector such that T_i^+ is a direct summand of T for T_i essential as a submodule. Recall that the representation U_i is defined as $U_i = T_i / \operatorname{tr}_{T[\hat{i}] \oplus T_B} T_i$, where T_B denotes the Bongartz completion of T, and $\mathcal{E}_i$ is the set $\mathcal{E}_i = \{X \in \mathrm{Rep}\,(Q, \mathbf{d}) : \mathrm{Ext}(X, U_i) \neq 0\}$. Moreover, $\mathcal{W}_i$ is the closure of the orbit of $W_i = T_i^{\lambda_i+1} \oplus Y_i \oplus R_i$, where Y_i is the cokernel of the injective source map $f_i^+ \colon T_i \longrightarrow T_i^+$ from T_i to $\operatorname{add} T[\hat{i}]$ and R_i is given by $T[\hat{i}] = T_i^+ \oplus R_i$. In this section we prove that $\mathcal{W}_i$ is contained in $\mathcal{E}_i$. Note that $\mathcal{E}_i$ is a closed subset of $\mathrm{Rep}\,(Q, \mathbf{d})$ as explained in section 3.1.

Lemma 3.7. *The module U_i is the unique indecomposable representation in $(T[\hat{i}] \oplus T_B)^{\perp}$ and satisfies $\mathrm{Ext}(T_i, U_i) = 0$ and $\dim \mathrm{Hom}(T_i, U_i) = 1$. In particular, if $x \in Q_0$ is a source of Q, then U_i is different from $S(x)$.*

Proof: The source map from T_i to $\operatorname{add}(T[\hat{i}] \oplus T_B)$ is injective since there is an injection from T_i to $\operatorname{add} T[\hat{i}]$. According to Lemma 3.2 i) the sink map $h_i \colon A \longrightarrow T_i$ from $\operatorname{add}(T[\hat{i}] \oplus T_B)$ to T_i is then injective as well. This means (see section 3.2) that T_i is the Bongartz completion of $T[\hat{i}] \oplus T_B$ and hence $U_i = T_i / \operatorname{tr}_{T[\hat{i}] \oplus T_B} T_i$ is the unique indecomposable (projective) representation in $(T[\hat{i}] \oplus T_B)^{\perp}$ (see section 3.2). Moreover, $\operatorname{tr}_{T[\hat{i}] \oplus T_B} T_i = \operatorname{im} h_i \cong A \in \operatorname{add}(T[\hat{i}] \oplus T_B)$ implies $\mathrm{Ext}(T_i, \operatorname{tr}_{T[\hat{i}] \oplus T_B} T_i) = 0$ as well as $\mathrm{Hom}(T_i, \operatorname{tr}_{T[\hat{i}] \oplus T_B} T_i) = 0$ and we find $\mathrm{Ext}(T_i, U_i) = 0$ and $\dim \mathrm{Hom}(T_i, U_i) = 1$ by mapping T_i to the exact sequence

$$0 \longrightarrow \operatorname{tr}_{T[\hat{i}] \oplus T_B} T_i \longrightarrow T_i \longrightarrow U_i \longrightarrow 0.$$

For the last statement we compute

$$\langle \mathbf{dim}\, \boldsymbol{T}, \mathbf{dim}\, \boldsymbol{U_i} \rangle = \dim \mathrm{Hom}(T, U_i) - \dim \mathrm{Ext}(T, U_i) = \lambda_i$$

and suppose towards a contradiction that U_i equals $S(x)$ for some source x of Q. This implies $\lambda_i = \langle \mathbf{dim}\, \boldsymbol{T}, \mathbf{e_x} \rangle = d_x$ and hence $T_i(x)$ must be zero as T_i is essential as a submodule. Thus $0 = \mathrm{Hom}(T_i, S(x)) = \mathrm{Hom}(T_i, U_i)$, a contradiction. □

As a consequence we obtain the following

Corollary 3.8. *The set $\mathcal{W}_i$ is contained in $\mathcal{E}_i$.*

Proof: Applying $\mathrm{Hom}(\cdot, U_i)$ to the exact sequence

$$0 \longrightarrow T_i \longrightarrow T_i^+ \longrightarrow Y_i \longrightarrow 0$$

yields $\dim \mathrm{Ext}(Y_i, U_i) = 1$ by Lemma 3.7 and therefore $\overline{\mathrm{GL}\,(\mathbf{d}) \cdot W_i} \subseteq \mathcal{E}_i$, since this set is closed. □

3.5 Reflections

Recall that a vertex $z \in Q_0$ is called sink if there is no arrow in Q_1 with tail z but there are $s \geq 1$ arrows $\alpha_j : y_j \longrightarrow z$, $j = 1, \ldots, s$ with head z. For a sink $z \in Q_0$ we denote by zQ the quiver obtained from Q by reversing the arrows $\alpha_1, \ldots, \alpha_s$. The reflection functors $R_z^+ : \operatorname{mod} Q \longrightarrow \operatorname{mod} zQ$ and $R_z^- : \operatorname{mod} zQ \longrightarrow \operatorname{mod} Q$ restrict to inverse equivalences between $(\operatorname{mod} Q)'$ and $(\operatorname{mod} zQ)'$, the full subcategories of $\operatorname{mod} Q$ and $\operatorname{mod} zQ$ whose objects do not contain $S(z)$ as a direct summand. Moreover, any representations $X \in (\operatorname{mod} Q)'$ and $Y \in (\operatorname{mod} zQ)'$ satisfy $\mathbf{dim}\, \boldsymbol{R_z^+ X} = r_z(\mathbf{dim}\, \boldsymbol{X})$ and $\mathbf{dim}\, \boldsymbol{R_z^- Y} = r_z(\mathbf{dim}\, \boldsymbol{Y})$, respectively, where $r_z : \mathbb{N}_0^n \longrightarrow \mathbb{N}_0^n$ is given by $r_z(\mathbf{x})_y = \sum_{j=1}^{s} x_{y_j} - x_z$ if $y = z$ and $r_z(\mathbf{x})_y = x_y$ otherwise. [9]

3.5.1 Reflection at a sink

Let $z \in Q_0$ be a sink of Q and assume $\Delta(z) > 0$. Recall from Remark 3.5 and Remark 3.6 that $S(z)$ cannot be a direct summand of $T \oplus T_B$. In particular

$$\dim \operatorname{Ext}(R_z^+ T, R_z^+ T) = \dim \operatorname{Ext}(T, T) = 0,$$

so the dimension vector $r_z(\mathbf{d})$ is prehomogeneous and the orbit of $R_z^+ T$ is the open orbit in $\operatorname{Rep}(zQ, r_z(\mathbf{d}))$. Moreover, $R_z^+ T_i$ is essential as a submodule because R_z^+ preserves injections and $\dim(R_z^+ T)(z) = \Delta(z) > 0$ ensures $r_z(\mathbf{d})$ is sincere.
Applying $\operatorname{Hom}(\cdot, S(z))$ to the exact sequence

$$0 \longrightarrow T_i \longrightarrow T_i^+ \longrightarrow Y_i \longrightarrow 0$$

yields $Y_i \in (\operatorname{mod} Q)'$ and hence the sequence

$$0 \longrightarrow R_z^+ T_i \longrightarrow R_z^+ T_i^+ \longrightarrow R_z^+ Y_i \longrightarrow 0$$

is exact in $\operatorname{mod} zQ$. Since $R_z^+ f_i^+ : R_z^+ T_i \longrightarrow R_z^+ T_i^+$ is exactly the source map from $R_z^+ T_i$ to $\operatorname{add} R_z^+ T[\hat{\imath}]$ the representation $W_i^+ := W_i(R_z^+ T_i)$ equals $R_z^+ W_i$.

Lemma 3.9. *Let $z \in Q_0$ be a sink of Q such that $\Delta(z) > 0$. The Bongartz completion of $R_z^+ T$ is $R_z^+ T_B$ and $U_i^+ := R_z^+ T_i / \operatorname{tr}_{R_z^+(T[\hat{\imath}] \oplus T_B)} R_z^+ T_i$ equals $R_z^+ U_i$.*

Proof: The representation $R_z^+(T \oplus T_B)$ contains exactly n pairwise non-isomorphic indecomposable direct summands and $\operatorname{Ext}(R_z^+(T \oplus T_B), R_z^+(T \oplus T_B)) = 0$. Moreover, the sink map $f_j : T_j^- \longrightarrow T_j$ from $\operatorname{add}(T \oplus T_B[\hat{\jmath}])$ to T_j is injective for $j = r+1, \ldots, n$ because T_B is the Bongartz completion of T (see section 3.2). Thus for $j = r+1, \ldots, n$ the sink map from $\operatorname{add}(R_z^+(T \oplus T_B[\hat{\jmath}]))$ to $R_z^+ T_j$ is injective as well, since it is precisely the map $R_z^+ f_j : R_z^+ T_j^- \longrightarrow R_z^+ T_j$ and hence $R_z^+ T_B$ is the Bongartz completion of $R_z^+ T$ (see section 3.2).
For the second statement, note that $U_i \neq S(z)$ since $\dim \operatorname{Hom}(T_i, U_i) = 1$ by Lemma 3.7. Hence the representation $R_z^+ U_i$ is the unique indecomposable representation in $(R_z^+(T[\hat{\imath}] \oplus T_B))^\perp$ by Lemma 3.7, but so is U_i^+. □

3.5.2 Reflection at a source

Let $x \in Q_0$ be a source of Q and assume $\Delta(x) > 0$ and $Y_i \neq S(x)$. The representation $S(x)$ cannot be a direct summand of $T \oplus T_B$ and the orbit of $R_x^- T$ is the open orbit in $\mathrm{Rep}\,(Qx, r_x(\mathbf{d}))$. Moreover, the dimension vector $r_x(\mathbf{d})$ is sincere.
The assumption $Y_i \neq S(x)$ implies the sequence

$$0 \longrightarrow R_x^- T_i \longrightarrow R_x^- T_i^+ \longrightarrow R_x^- Y_i \longrightarrow 0$$

is exact in $\mathrm{mod}\, Qx$ and thus $R_x^- T_i$ is essential as a submodule since there is an injection from $R_x^- T_i$ to $\mathrm{add}\, R_x^- T[\hat{i}]$. More precisely, $W_i^- = R_x^- W_i$, where $W_i^- := W_i(R_x^- T_i)$.

Lemma 3.10. *Let $x \in Q_0$ be a source of Q such that $\Delta(x) > 0$. The Bongartz completion of $R_x^- T$ is $R_x^- T_B$ and $U_i^- := R_x^- T_i / \operatorname{tr}_{R_x^-(T[\hat{i}] \oplus T_B)} R_x^- T_i$ equals $R_x^- U_i$.*

Proof: Let T_B^- be the Bongartz completion of $R_x^- T$. The vertex x is a sink of the quiver Qx and satisfies $\Delta_{Qx}(x) = \sum_{j=1}^{s} r_x(\mathbf{d})(y_j) - r_x(\mathbf{d})(x) = d_x > 0$. According to Lemma 3.9 the Bongartz completion of $R_x^+ R_x^- T = T$ is thus $T_B = R_x^+ T_B^-$, which implies $T_B^- = R_x^- R_x^+ T_B^- = R_x^- T_B$. Moreover, $R_x^- U_i$ is the unique indecomposable representation in $(R_x^- T[\hat{i}] \oplus T_B^-)^\perp$ according to Lemma 3.7, but so is U_i^-. □

3.6 Perpendicular Categories

3.6.1 Cutting out a sink

Let $z \in Q_0$ be a sink of Q and denote by $Q[\hat{z}]$ the quiver obtained from Q by deleting the vertex z and the arrows $\alpha_1, \ldots, \alpha_s$. For $M \in \mathrm{mod}\, Q$ we define the representation $M[\hat{z}] \in \mathrm{mod}\, Q[\hat{z}]$ as the restriction of M to $Q[\hat{z}]$ and for a morphism $f\colon M_1 \longrightarrow M_2$ we define $f[\hat{z}]$ as the restriction of f to $M_1[\hat{z}]$. Conversely, we define a functor $F\colon \mathrm{mod}\, Q[\hat{z}] \longrightarrow \mathrm{mod}\, Q$ in the following way: For a representation $N \in \mathrm{mod}\, Q[\hat{z}]$ we let $F(N)(y) = N(y)$ for $y \neq z$ and $F(N)(\alpha) = N(\alpha)$ for $\alpha \neq \alpha_1, \ldots, \alpha_s$. Moreover we set $F(N)(z) = \bigoplus_{j=1}^{s} N(y_j)$ and let $F(N)(\alpha_l)\colon N(y_l) \longrightarrow \bigoplus_{j=1}^{s} N(y_j)$ be the inclusion for $l = 1, \ldots, s$. For a morphism $f = (f_j)_{j \in Q_0 \setminus \{z\}}\colon N_1 \longrightarrow N_2$ we define $F(f)_y = f_y$ for $y \neq z$ and $F(f)_z = f_{y_1} \oplus \cdots \oplus f_{y_s}\colon \bigoplus_{j=1}^{s} N_1(y_j) \longrightarrow \bigoplus_{j=1}^{s} N_2(y_j)$.
It is easy to verify that $\mathrm{mod}\, Q[\hat{z}]$ is equivalent (via these functors) to the full subcategory of $\mathrm{mod}\, Q$ whose objects have the property that

$$(M(\alpha_1), \ldots, M(\alpha_s))\colon \bigoplus_{j=1}^{s} M(y_j) \longrightarrow M(z)$$

is an isomorphism. Note that a representation $M \in \mathrm{mod}\, Q$ lies in this subcategory if and only if $\dim M(z) = \sum_{j=1}^{s} \dim M(y_j)$ and M does not contain $S(z)$ as a direct summand, which is the case if and only if M lies in ${}^\perp S(z)$. Furthermore, the two functors are adjoint by construction and hence

$$\dim \mathrm{Hom}_Q(F(N), M) = \dim \mathrm{Hom}_{Q[\hat{z}]}(N, M[\hat{z}]) \tag{2}$$

for $M \in \operatorname{mod} Q$ and $N \in \operatorname{mod} Q[\hat{z}]$.

We suppose $\Delta(z) \leq 0$ and set $\mathbf{d}' := \mathbf{d} + \Delta(z) \cdot \mathbf{e_z}$. Note that every representation $M \in \operatorname{Rep}(Q, \mathbf{d})$ is of the form $M = M' \oplus S(z)^{-\Delta(z)}$ for a representation $M' \in \operatorname{Rep}\big(Q, \mathbf{d}'\big)$, as any indecomposable representation $M \neq S(z)$ satisfies $\dim M(z) \leq \sum_{j=1}^{s} \dim M(y_j)$. In particular, the open orbit in $\operatorname{Rep}(Q, \mathbf{d})$ is given by $T = T' \oplus S(z)^{-\Delta(z)}$ and thus $\mathbf{d}'$ is prehomogeneous as well. By definition of $\mathbf{d}'$ and according to Remark 3.5 the representation $S(z)$ is not a direct summand of T' and hence $T' \in {}^{\perp}S(z)$.

Lemma 3.11. *Let $z \in Q_0$ be a sink of Q such that $\Delta(z) \leq 0$. The Bongartz completion of $T = T' \oplus S(z)^{-\Delta(z)}$ is given by*

$$T_B = \begin{cases} T'_B, & \text{if } \Delta(z) < 0, \\ T'_B \oplus S(z), & \text{if } \Delta(z) = 0, \end{cases}$$

where T'_B denotes the Bongartz completion of T' in ${}^{\perp}S(z)$.

Proof: In case $\Delta(z) = 0$ the representation $S(z)$ is a direct summand of T_B according to Remark 3.6. Thus $S(z)$ is a direct summand of $T \oplus T_B$ in both cases and hence $\operatorname{Ext}(T \oplus T_B, S(z)) = 0$. This implies $(T \oplus T_B)[\widehat{S(z)}] \in {}^{\perp}S(z)$ and it remains to show that

$$\begin{cases} T_B, & \text{if } \Delta(z) < 0, \\ T_B[\widehat{S(z)}], & \text{if } \Delta(z) = 0 \end{cases}$$

is the Bongartz completion of T' in ${}^{\perp}S(z)$. Let $\Delta(z) < 0$ and let T_j be an indecomposable direct summand of T_B. The sink map from $\operatorname{add}(T \oplus T_B[\hat{\jmath}])$ to T_j is injective (see section 3.2) and the sink map from $\operatorname{add}(T' \oplus T_B[\hat{\jmath}])$ to T_j factors through this map and therefore cannot be surjective, hence (see section 3.2) it is injective. Similarly, if $\Delta(z) = 0$ and T_j is an indecomposable direct summand of $T_B[\widehat{S(z)}]$ the sink map from $\operatorname{add}(T' \oplus (T_B[\hat{\jmath}])[\widehat{S(z)}])$ to T_j factors through the sink map from $\operatorname{add}(T' \oplus T_B[\hat{\jmath}])$, which is injective. □

Let us suppose $T_i \neq S(z)$. The source map from T_i to $\operatorname{add} T'[\hat{\imath}]$ is isomorphic to the source map from T_i to $\operatorname{add} T[\hat{\imath}]$ and thus T_i is essential as a submodule for $T'[\hat{\imath}]$ and

$$0 \longrightarrow T_i \longrightarrow T_i^+ \longrightarrow Y_i \longrightarrow 0$$

is exact in ${}^{\perp}S(z)$. Moreover, the representation $W'_i := T_i^{\lambda_i+1} \oplus Y_i \oplus R'_i$, where R'_i is given by $T'[\hat{\imath}] = T_i^+ \oplus R'_i$, lies in ${}^{\perp}S(z)$ and satisfies $W_i = W'_i \oplus S(z)^{-\Delta(z)}$. Note that U_i does not lie in ${}^{\perp}S(z)$ in general.
The equivalence of ${}^{\perp}S(z) \subseteq \operatorname{mod} Q$ and $\operatorname{mod} Q[\hat{z}]$ described above thus implies the dimension vector $\mathbf{d}[\hat{\mathbf{z}}] := (d_1, \ldots, d_{z-1}, d_{z+1}, \ldots, d_n) \in \mathbb{N}^{n-1}$ is prehomogeneous since the orbit of $T[\hat{z}]$ is open in $\operatorname{Rep}(Q[\hat{z}], \mathbf{d}[\hat{\mathbf{z}}])$. Moreover, $T_i[\hat{z}]$ is essential as a submodule, the source map from $T_i[\hat{z}]$ to $\operatorname{add} T[\hat{\imath}][\hat{z}]$ is the map $f_i^+[\hat{z}] \colon T_i[\hat{z}] \longrightarrow T_i^+[\hat{z}]$ and

$$0 \longrightarrow T_i[\hat{z}] \longrightarrow T_i^+[\hat{z}] \longrightarrow Y_i[\hat{z}] \longrightarrow 0$$

is exact.

Lemma 3.12. *Let $z \in Q_0$ be a sink such that $\Delta(z) \leq 0$ and assume $T_i \neq S(z)$. The Bongartz completion of $T[\hat{z}]$ is $T_B[\hat{z}]$, the representation U_i satisfies $\dim U_i(z) = 0$ and $T_i[\hat{z}] / \operatorname{tr}_{(T[\hat{i}] \oplus T_B)[\hat{z}]} T_i[\hat{z}]$ equals $U_i[\hat{z}]$.*

Proof: The first statement follows directly from Lemma 3.11. Moreover, $U_i(z) = 0$ as $S(z)$ is a direct summand of $T[\hat{i}] \oplus T_B$. In particular, $U_i[\hat{z}]$ is indecomposable and in view of Lemma 3.7 it remains to prove $U_i[\hat{z}] \in ((T[\hat{i}] \oplus T_B)[\hat{z}])^{\perp}$. Since $U_i(z) = 0$ we have

$$\langle \mathbf{dim}\, \boldsymbol{M}, \mathbf{dim}\, \boldsymbol{U_i} \rangle_Q = \langle \mathbf{dim}\, \boldsymbol{M}[\hat{\boldsymbol{z}}], \mathbf{dim}\, \boldsymbol{U_i}[\hat{\boldsymbol{z}}] \rangle_{Q[\hat{z}]}$$

and $\dim \operatorname{Hom}(M, U_i) = \dim \operatorname{Hom}(M[\hat{z}], U_i[\hat{z}])$ for any representation M of Q. Consequently, $\dim \operatorname{Hom}((T[\hat{i}] \oplus T_B)[\hat{z}], U_i[\hat{z}]) = \dim \operatorname{Hom}(T[\hat{i}] \oplus T_B, U_i) = 0$ and

$$\begin{aligned} \dim \operatorname{Ext}((T[\hat{i}] \oplus T_B)[\hat{z}], U_i[\hat{z}]) &= -\langle \mathbf{dim}\, (\boldsymbol{T}[\hat{\boldsymbol{i}}] \oplus \boldsymbol{T_B})[\hat{\boldsymbol{z}}], \mathbf{dim}\, \boldsymbol{U_i}[\hat{\boldsymbol{z}}] \rangle_{Q[\hat{z}]} \\ &= -\langle \mathbf{dim}\, \boldsymbol{T}[\hat{\boldsymbol{i}}] \oplus \boldsymbol{T_B}, \mathbf{dim}\, \boldsymbol{U_i} \rangle_Q = \dim \operatorname{Ext}(T[\hat{i}] \oplus T_B, U_i) = 0 \end{aligned}$$

according to Lemma 3.7. □

Proposition 3.13. *Let $z \in Q_0$ be a sink such that $\Delta(z) \leq 0$ and assume $T_i \neq S(z)$. If Theorem 3.4 holds true for the pair $(\operatorname{Rep}(Q[\hat{z}], \mathbf{d}[\hat{\mathbf{z}}]), T_i[\hat{z}])$, then $\mathcal{W}_i = \mathcal{E}_i$.*

Proof: Let $X = X' \oplus S(z)^{-\Delta(z)} \in \operatorname{Rep}(Q, \mathbf{d})$ satisfy $\operatorname{Ext}(X, U_i) \neq 0$. It is enough to prove that X' is a degeneration of W_i' since then $W_i = W_i' \oplus S(z)^{-\Delta(z)} \leq_{\deg} X' \oplus S(z)^{-\Delta(z)} = X$. In order to do so, we first prove that we may assume w.l.o.g. $X' \in {}^{\perp}S(z)$. Decomposing $X' = X'' \oplus S(z)^{\mu}$ such that $S(z)$ is not a direct summand of X'' we obtain the short exact sequence

$$0 \longrightarrow S(z)^{\mu} \longrightarrow F(X''[\hat{z}]) \longrightarrow X'' \longrightarrow 0.$$

Since $F(X''[\hat{z}]) = F(X'[\hat{z}])$ this implies X' is a degeneration of $F(X'[\hat{z}])$. Moreover, applying $\operatorname{Hom}(\cdot, U_i)$ and using $U_i(z) = 0$ yields $\dim \operatorname{Ext}(F(X'[\hat{z}]), U_i) = \dim \operatorname{Ext}(X'', U_i) \neq 0$ and hence we may suppose w.l.o.g. $X' \in {}^{\perp}S(z)$.
Furthermore, $U_i(z) = 0$ implies $\dim \operatorname{Ext}(X'[\hat{z}], U_i[\hat{z}]) = \dim \operatorname{Ext}(X', U_i) \neq 0$. Hence $X'[\hat{z}]$ is a degeneration of $W_i'[\hat{z}]$ according to Lemma 3.12 and the assumption. Since $Q[\hat{z}]$ is still a tame quiver this is equivalent to $W_i'[\hat{z}] \leq_{\hom} X'[\hat{z}]$ (see section 3.2). Using $W_i', X' \in {}^{\perp}S(z)$ and (2) thus implies

$$\begin{aligned} \dim \operatorname{Hom}_Q(W_i', M) &= \dim \operatorname{Hom}_Q(F(W_i'[\hat{z}]), M) = \dim \operatorname{Hom}_{Q[\hat{z}]}(W_i'[\hat{z}], M[\hat{z}]) \\ &\leq \dim \operatorname{Hom}_{Q[\hat{z}]}(X'[\hat{z}], M[\hat{z}]) = \dim \operatorname{Hom}_Q(F(X'[\hat{z}]), M) \\ &= \dim \operatorname{Hom}_Q(X', M) \end{aligned}$$

for all indecomposable representations $M \in \operatorname{mod} Q$. Therefore $W_i' \leq_{\hom} X'$ and consequently $W_i' \leq_{\deg} X'$ (see section 3.2) as desired. □

3.6.2 Cutting out a source

Let $x \in Q_0$ be a source of Q and denote by $Q[\hat{x}]$ the quiver obtained from Q by deleting the vertex x and the arrows $\alpha_1, \dots, \alpha_s$. The following construction is analogous to the one in section 3.6.1: For $M \in \operatorname{mod} Q$ we define the representation $M[\hat{x}] \in \operatorname{mod} Q[\hat{x}]$ as the restriction of M to $Q[\hat{x}]$ and for a morphism $f\colon M_1 \longrightarrow M_2$ we define $f[\hat{x}]$ as the restriction of f to $M_1[\hat{x}]$. Conversely, we define a functor $G\colon \operatorname{mod} Q[\hat{x}] \longrightarrow \operatorname{mod} Q$ in the following way: For a representation $N \in \operatorname{mod} Q[\hat{x}]$ we let $G(N)(y) = N(y)$ for $y \neq x$ and $G(N)(\alpha) = N(\alpha)$ for $\alpha \neq \alpha_1, \dots, \alpha_s$. Moreover we set $G(N)(x) = \bigoplus_{j=1}^{s} N(y_j)$ and let $G(N)(\alpha_l)\colon \bigoplus_{j=1}^{s} N(y_j) \longrightarrow N(y_l)$ be the projection for $l = 1, \dots, s$. For a morphism $f = (f_j)_{j \in Q_0 \setminus \{x\}}\colon N_1 \longrightarrow N_2$ we define $G(f)_y = f_y$ for $y \neq x$ and $G(f)_x = f_{y_1} \oplus \dots \oplus f_{y_s}\colon \bigoplus_{j=1}^{s} N_1(y_j) \longrightarrow \bigoplus_{j=1}^{s} N_2(y_j)$.
The category $\operatorname{mod} Q[\hat{x}]$ is equivalent (via these functors) to the category $S(x)^{\perp} \subseteq \operatorname{mod} Q$ and

$$\dim \operatorname{Hom}_Q(M, G(N)) = \dim \operatorname{Hom}_{Q[\hat{x}]}(M[\hat{x}], N)$$

for $M \in \operatorname{mod} Q$ and $N \in \operatorname{mod} Q[\hat{x}]$.

We suppose $\Delta(x) \leq 0$ and $Y_i \neq S(x)$. Note that T_i is different from $S(x)$ as well since it is essential as a submodule.
Setting $\mathbf{d}' := \mathbf{d} + \Delta(x) \cdot \mathbf{e_x}$ every representation $M \in \operatorname{Rep}(Q, \mathbf{d})$ is of the form $M = M' \oplus S(x)^{-\Delta(x)}$ for a representation $M' \in \operatorname{Rep}\big(Q, \mathbf{d}'\big)$. In particular, $\mathbf{d}'$ is prehomogeneous, $T = T' \oplus S(x)^{-\Delta(x)}$ and $T' \in S(x)^{\perp}$. Moreover, mapping $S(x)$ to the exact sequence

$$0 \longrightarrow T_i \longrightarrow T_i^{+} \longrightarrow Y_i \longrightarrow 0$$

implies $\operatorname{Ext}(S(x), Y_i) = 0$ and hence $Y_i \in S(x)^{\perp}$ as $Y_i \neq S(x)$ by assumption. Therefore, this sequence is exact in $S(x)^{\perp}$ and $f_i^{+}\colon T_i \longrightarrow T_i^{+}$ is the source map from T_i to $\operatorname{add} T'$ in $S(x)^{\perp}$. Furthermore, the representation $W_i' := T_i^{\lambda_i + 1} \oplus Y_i \oplus R_i'$, where R_i' is given by $T'[\hat{i}] = T_i^{+} \oplus R_i'$, satisfies $W_i = W_i' \oplus S(x)^{-\Delta(x)}$.

Lemma 3.14. *Let $x \in Q_0$ be a source such that $\Delta(x) \leq 0$. The Bongartz completion T_B of $T = T' \oplus S(x)^{-\Delta(x)}$ is given by*

$$T_B = \begin{cases} T_B' \oplus T_n, & \text{if } \Delta(x) = 0, \\ T_B', & \text{if } \Delta(x) < 0, \end{cases}$$

where T_B' is the Bongartz completion of T' in $S(x)^{\perp}$ and T_n is the Bongartz completion of $T' \oplus T_B'$ in $\operatorname{mod} Q$. Moreover, $U_i' := T_i / \operatorname{tr}_{T'[\hat{i}] \oplus T_B'} T_i$ equals U_i.

Proof: The second statement follows immediately from the first one. Indeed, mapping $S(x)$ to the exact sequence

$$0 \longrightarrow \operatorname{tr}_{T[\hat{i}] \oplus T_B} T_i \longrightarrow T_i \longrightarrow U_i \longrightarrow 0$$

implies $\operatorname{Ext}(S(x), U_i) = 0$ and hence $U_i \in S(x)^{\perp}$ as $U_i \neq S(x)$ by Lemma 3.7. Moreover, U_i lies

in $(T'[\hat{i}] \oplus T'_B)^\perp$ by the first statement and Lemma 3.7 and is thus identical to U'_i.
In order to prove the first statement, we first consider the case $\Delta(x) = 0$. By definition of T_n the sink map from add $(T' \oplus T'_B)$ to T_n is injective (see section 3.2) and it remains to prove the sink map from add$(T' \oplus T_n \oplus T'_B[\hat{j}])$ to T'_j is injective as well for any indecomposable direct summand T'_j of T'_B. As the sink map from add$(T' \oplus T'_B[\hat{j}])$ to T'_j is injective by definition of T'_B, the representation T'_j is the Bongartz completion of $T' \oplus T'_B[\hat{j}]$ in $S(x)^\perp$. Thus the source map from T'_j to add$(T' \oplus T'_B[\hat{j}])$ is injective according to Lemma 3.2 ii). Since this map factors through the source map from T'_j to add$(T' \oplus T_n \oplus T'_B[\hat{j}])$, the latter must be injective as well and Lemma 3.2 i) finishes the proof in this case.
In case $\Delta(x) < 0$ the representation $T \oplus T'_B$ contains n pairwise non-isomorphic indecomposable direct summands and satisfies $\mathrm{Ext}(T \oplus T'_B, T \oplus T'_B) = 0$. Moreover, for an indecomposable direct summand T'_j of T'_B the source map from T'_j to add$(T' \oplus T'_B[\hat{j}])$ is injective as we have seen above. Hence the sink map from add$(T' \oplus T'_B[\hat{j}])$ to T'_j is injective by Lemma 3.2 i) and since this map is isomorphic to the sink map from add$(T \oplus T'_B[\hat{j}])$ to T'_j the assertion follows. □

Proposition 3.15. *Let $x \in Q_0$ be a source such that $\Delta(x) \leq 0$ and assume $Y_i \neq S(x)$. If Theorem 3.4 holds true for the pair* $(\mathrm{Rep}\,(Q[\hat{x}], \mathbf{d}[\hat{\mathbf{x}}])\,, T_i[\hat{x}])$*, then $\mathcal{W}_i = \mathcal{E}_i$.*

Proof: Let $X = X' \oplus S(x)^{-\Delta(x)}$ satisfy $\mathrm{Ext}(X, U_i) \neq 0$. As in the proof of Proposition 3.13 it is enough to prove that X' is a degeneration of W'_i and we first show we may assume w.l.o.g. $X' \in S(x)^\perp$. Let $X' = X'' \oplus S(x)^\mu$ be a decomposition such that $S(x)$ is not a direct summand of X''. We have seen in the beginning of the proof of Lemma 3.14 that $U_i \in S(x)^\perp$ and hence $\mathrm{Ext}(X'', U_i) \neq 0$. By definition of G there is an inclusion $X'' \hookrightarrow G(X''[\hat{x}]) = G(X'[\hat{x}])$, i.e. there is a short exact sequence

$$0 \longrightarrow X'' \longrightarrow G(X'[\hat{x}]) \longrightarrow S(x)^\mu \longrightarrow 0$$

and thus X' is a degeneration of $G(X'[\hat{x}])$. Since applying $\mathrm{Hom}(\cdot, U_i)$ yields $\mathrm{Ext}(G(X'[\hat{x}]), U_i) \neq 0$ we may thus suppose w.l.o.g. $X' \in S(x)^\perp$.
Furthermore, $\dim \mathrm{Ext}(X'[\hat{x}], U'_i[\hat{x}]) = \dim \mathrm{Ext}(X', U'_i) = \dim \mathrm{Ext}(X', U_i) \neq 0$ by Lemma 3.14 and hence $X'[\hat{x}]$ is a degeneration of $W'_i[\hat{z}]$ by assumption. The rest of the proof is analogous to the proof of Proposition 3.13. □

3.7 Proof of the Theorem

In this section, we prove Theorem 3.4. Recall that $\mathcal{W}_i$ denotes the closure of the orbit of W_i and that the inclusion $\mathcal{W}_i \subseteq \mathcal{E}_i$ has already been proved in section 3.4. For the other inclusion, we start with the following observation: Increasing the multiplicities $\lambda_1, \ldots, \lambda_r$ does not change the representations T_B, U_i, T_i^+ and Y_i, provided T_i^+ is a direct summand of T. This trivial remark turns out to be very useful, as it allows us to restrict ourselves to 'large' multiplicities, which is helpful as we will see.
In the following Lemma we prove it is enough to consider the case where the multiplicities are 'large enough'. More precisely, we call a multiplicity vector $\boldsymbol{\mu} := (\mu_1, \ldots, \mu_r) \in \mathbb{N}^r$ admissible (for

T_i) if T_i^+ is a direct summand of

$$T(\boldsymbol{\mu}) := \bigoplus_{j=1}^{r} T_j^{\mu_j}.$$

For an admissible multiplicity vector $\boldsymbol{\mu} \in \mathbb{N}^r$ we set

$$\begin{aligned}
\mathbf{d}(\boldsymbol{\mu}) &:= \mathbf{dim}\, \boldsymbol{T(\mu)} = \sum_{j=1}^{r} \mu_j \cdot \mathbf{dim}\, \boldsymbol{T_j}, \\
\mathcal{E}_i\,(\boldsymbol{\mu}) &:= \{X \in \mathrm{Rep}(Q, \mathbf{d}(\boldsymbol{\mu}))\colon\ \mathrm{Ext}(X, U_i) \neq 0\}\,, \\
\mathcal{W}_i(\boldsymbol{\mu}) &:= \overline{\mathrm{GL}\,(\mathbf{d}(\boldsymbol{\mu}))) \cdot W_i(\boldsymbol{\mu})}, \\
W_i(\boldsymbol{\mu}) &:= T_i^{\mu_i+1} \oplus Y_i \oplus R_i(\boldsymbol{\mu}),
\end{aligned}$$

where $R_i(\boldsymbol{\mu})$ is given by $T(\boldsymbol{\mu})[\hat{i}] = T_i^+ \oplus R_i(\boldsymbol{\mu})$.
Moreover, for $\boldsymbol{\mu}, \boldsymbol{\rho} \in \mathbb{N}^r$ we write $\boldsymbol{\mu} \geq \boldsymbol{\rho}$ if and only if $\mu_j \geq \rho_j$ for $j = 1, \ldots, r$.

Lemma 3.16. *Assume there is an admissible multiplicity vector $\boldsymbol{\rho} \in \mathbb{N}^r$ such that $\mathcal{W}_i(\boldsymbol{\mu}) = \mathcal{E}_i(\boldsymbol{\mu})$ holds true for all $\boldsymbol{\mu} \geq \boldsymbol{\rho}$. Then $\mathcal{W}_i(\boldsymbol{\mu}) = \mathcal{E}_i(\boldsymbol{\mu})$ holds true for an arbitrary admissible multiplicity vector $\boldsymbol{\mu} \in \mathbb{N}^r$.*

Proof: Let $\boldsymbol{\mu} \in \mathbb{N}^r$ be an arbitrary admissible multiplicity vector and let $X \in \mathcal{E}_i(\boldsymbol{\mu})$ be any representation. The idea of the proof is to add a representation in $\mathrm{add}\, T$ to X such that the premise of the Lemma may be used. Technically, we set $\nu_j := \max\{\mu_j, \rho_j\}$ for $j = 1, \ldots, r$ and $\boldsymbol{\nu} := (\nu_1, \ldots, \nu_r)$. Clearly, this implies $\boldsymbol{\nu} \geq \boldsymbol{\rho}$. The representation

$$X(\boldsymbol{\nu}) := X \oplus \bigoplus_{j=1}^{r} T_j^{\max\{0, \rho_j - \mu_j\}} \in \mathrm{Rep}(Q, \mathbf{d}(\boldsymbol{\nu}))$$

lies in $\mathcal{E}_i(\boldsymbol{\nu})$ and hence is by assumption a degeneration of

$$W_i(\boldsymbol{\nu}) = W_i(\boldsymbol{\mu}) \oplus \bigoplus_{j=1}^{r} T_j^{\max\{0, \rho_j - \mu_j\}}.$$

Therefore, X is a degeneration of $W_i(\boldsymbol{\mu})$ as we may 'cancel' the representation $\bigoplus_{j=1}^{r} T_j^{\max\{0, \rho_j - \mu_j\}}$ (see section 3.2). □

There are two situations which play an important role in the proof of Theorem 3.4. The first situation is the case where T_i is simple and projective and the second is the case where Y_i is simple and injective. The proof of Theorem 3.4 in these situations is not very difficult and is presented in section 3.7.1. Furthermore, the proof of Theorem 3.4 is simpler in case Q is a Dynkin quiver and not an arbitrary tame quiver. We treat this case in section 3.7.2 and handle the general case in section 3.7.3.

3.7.1 Basic special cases

First, we prove Theorem 3.4 in case T_i is simple and projective:

Lemma 3.17. *If T_i is simple and projective, then $U_i = T_i$ and $\mathcal{W}_i = \mathcal{E}_i$.*

Proof: If T_i is simple and projective, the trace $\mathrm{tr}_{T[\hat{i}]\oplus T_B} T_i$ is zero and hence $U_i = T_i$. Moreover, for $X \in \mathrm{Rep}\,(Q, \mathbf{d})$ we compute

$$\langle \mathbf{dim}\, \boldsymbol{X}, \mathbf{dim}\, \boldsymbol{U_i} \rangle = \langle \mathbf{dim}\, \boldsymbol{T}, \mathbf{dim}\, \boldsymbol{U_i} \rangle = \dim \mathrm{Hom}(T, U_i) - \dim \mathrm{Ext}(T, U_i) = \lambda_i$$

by Lemma 3.7 and hence $\dim \mathrm{Hom}(X, U_i) = \dim \mathrm{Ext}(X, U_i) + \lambda_i$. Consequently, if X satisfies $\mathrm{Ext}(X, U_i) \neq 0$, then $\dim \mathrm{Hom}(X, T_i) = \dim \mathrm{Hom}(X, U_i) \geq \lambda_i + 1$. By assumption this is only possible if X contains $T_i^{\lambda_i+1}$ as a direct summand and hence X is in the closure of the orbit of W_i as $\mathrm{Ext}(Y_i \oplus R_i, Y_i \oplus R_i) = 0$ by Lemma 3.3. □

Next, we prove Theorem 3.4 in case Y_i is simple and injective. Note that this is actually redundant if Q is a Dynkin quiver.

Lemma 3.18. *If $Y_i = S(x)$ is simple and injective, then $U_i = \tau S(x)$ and $\mathcal{W}_i = \mathcal{E}_i$.*

Proof: The second statement follows directly from the first one: If $U_i = \tau S(x)$ then any representation $X \in \mathrm{Rep}\,(Q, \mathbf{d})$ satisfying $\mathrm{Ext}(X, U_i) \neq 0$ must contain $S(x)$ as a direct summand and hence lies in the closure of the orbit of $S(x) \oplus T_i^{\lambda_i+1} \oplus R_i$, which is $\mathcal{W}_i$.
For the first statement it is enough to prove $\tau S(x) \in (T[\hat{i}] \oplus T_B)^{\perp}$ according to Lemma 3.7. This is equivalent to $S(x) \in {}^{\perp}(T[\hat{i}] \oplus T_B)$ by the Auslander-Reiten formulae. By definition of Y_i we have $\mathrm{Ext}(S(x), T_i) \neq 0$ and hence $S(x)$ is not a direct summand of $T[\hat{i}] \oplus T_B$, thus $\mathrm{Hom}(S(x), T[\hat{i}] \oplus T_B) = 0$. Moreover, $\mathrm{Ext}(S(x), T[\hat{i}]) = 0$ according to Lemma 3.3. In order to prove $\mathrm{Ext}(S(x), T_B) = 0$ let T_j be an indecomposable direct summand of T_B. Applying $\mathrm{Hom}(\cdot, T_j)$ to the exact sequence

$$0 \longrightarrow T_i \longrightarrow T_i^{+} \longrightarrow S(x) \longrightarrow 0$$

yields the exact sequence

$$0 \longrightarrow \mathrm{Hom}(T_i^{+}, T_j) \longrightarrow \mathrm{Hom}(T_i, T_j) \longrightarrow \mathrm{Ext}(S(x), T_j) \longrightarrow 0.$$

We suppose $\mathrm{Hom}(T_i, T_j) \neq 0$ as otherwise clearly $\mathrm{Ext}(S(x), T_j) = 0$. The source map $T_j \longrightarrow T_j^{++}$ from T_j to $\mathrm{add}\, T$ is injective according to Lemma 3.2 ii) and we denote its cokernel by Z_j. Note that Z_j is indecomposable according to Lemma 3.3 (applied to the source map from T_j to $\mathrm{add}\, T$). There is no non-zero map from T_j to T_i as $\mathrm{Hom}(T_i, T_j) \neq 0$ (see section 3.2), so $T_j^{++} \in \mathrm{add}\, T[\hat{i}]$ and hence $\mathrm{Ext}(S(x), T_j^{++}) = 0$. Mapping $S(x)$ to the exact sequence

$$0 \longrightarrow T_j \longrightarrow T_j^{++} \longrightarrow Z_j \longrightarrow 0$$

thus implies $\dim \mathrm{Ext}(S(x), T_j) = \dim \mathrm{Hom}(S(x), Z_j)$ and it remains to show $Z_j \neq S(x)$. This follows directly from the fact $\mathrm{Ext}(S(x), T_i) \neq 0$ as $\mathrm{Ext}(T \oplus Z_j, T \oplus Z_j) = 0$ according to Lemma 3.3. □

3.7.2 The Dynkin case

In this section we assume Q is a Dynkin quiver, i.e. every connected component Γ of Q is of type $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$ or $\mathbb{E}_8$. Recall from section 3.1 that any dimension vector $\mathbf{d} \in \mathbb{N}^n$ is prehomogeneous in this case. Moreover, any multiplicity vector $\boldsymbol{\lambda} \in \mathbb{N}^r$ is admissible as explained in section 3.2. Therefore, our goal is to prove the following Theorem:

Theorem 3.19. *Let Q be a Dynkin quiver, $\boldsymbol{d} \in \mathbb{N}^n$ any sincere dimension vector and $\mathrm{GL}\,(\boldsymbol{d}) \cdot T$ the open orbit in $\mathrm{Rep}\,(Q, \boldsymbol{d})$. If T_i is essential as a submodule, then $\mathcal{W}_i = \mathcal{E}_i$.*

The main idea of the proof is to use reflections at sinks to reach the situation where T_i is simple and projective and to make use of Lemma 3.17 afterwards. However, by using reflection functors the following problem occurs: A representation $X \in \mathrm{Rep}\,(Q, \mathbf{d})$ may contain $S(z)$ as a direct summand and thus $R_z^+ X$ does not necessarily lie in $\mathrm{Rep}\,(zQ, r_z(\mathbf{d}))$. Hence there is no use in just applying reflection functors to the set $\mathcal{E}_i$. The following Proposition allows us to solve this problem and may therefore be seen as the key point of the proof of the Theorem. Note that it is also the place we take advantage of the fact that the multiplicities may be chosen 'large enough'. In fact, we set

$$\rho(Q) := \max\{\rho(\Gamma) \colon \Gamma \text{ is a connected component of } Q\}$$

and

$$\rho(\Gamma) := \begin{cases} 1, & \text{if } \Gamma \text{ is of type } A_n, \\ 2, & \text{if } \Gamma \text{ is of type } D_n, \\ 3, & \text{if } \Gamma \text{ is of type } E_6, \\ 4, & \text{if } \Gamma \text{ is of type } E_7, \\ 6, & \text{if } \Gamma \text{ is of type } E_8. \end{cases}$$

Note that actually $\rho(Q) = \max\{\dim \mathrm{Ext}(A, B) \colon A, B \text{ indecomposable representations of } Q\}$.

For a sink $z \in Q_0$ and an indecomposable representation $M \neq S(z)$ we denote by $\Delta_M(z)$ the value

$$\Delta_M(z) := \sum_{j=1}^{s} \dim M(y_j) - \dim M(z) = \dim \mathrm{Ext}(M, S(z)) \geq 0.$$

Proposition 3.20. *Let $z \in Q_0$ be a sink of Q such that $\Delta(z) > 0$. Assume $\lambda_j \geq \rho(Q)$ for $j = 1, \ldots, r$ and let $X \in \mathrm{Rep}\,(Q, \boldsymbol{d})$ satisfy $\mathrm{Ext}(X, U_i) \neq 0$. Then there exists a representation $Y \in \mathrm{Rep}\,(Q, \boldsymbol{d})$ such that X is a degeneration of Y, the representation $S(z)$ is not a direct summand of Y and $\mathrm{Ext}(Y, U_i) \neq 0$.*

Proof: By assumption the representation X has an indecomposable direct summand $V \neq S(z)$ such that $\mathrm{Ext}(V, U_i) \neq 0$. We set $Y := V \oplus Y_1$, where the orbit of Y_1 is open in $\mathrm{Rep}\,(Q, \mathbf{d} - \mathbf{dim}\,\boldsymbol{V})$. Clearly, this implies $Y \leq_{\deg} X$ and $\mathrm{Ext}(Y, U_i) \neq 0$ and it remains to show that $S(z)$ is not a direct summand of Y. According to Remark 3.5 it is enough to prove that

$$\sum_{j=1}^{s} \dim Y_1(y_j) - \dim Y_1(z) = \Delta(z) - \Delta_V(z)$$

is non-negative. The value of $\Delta(z)$ may be computed as $\Delta(z) = \sum_{j=1}^{r} \lambda_j \cdot \Delta_{T_j}(z)$ and since $\Delta(z)$ is positive, there is an indecomposable direct summand T_k of T such that $\Delta_{T_k}(z) > 0$. This implies $\Delta(z) \geq \lambda_k \cdot \Delta_{T_k}(z) \geq \rho(Q) \geq \Delta_V(z)$ by definition of $\rho(Q)$ and the assumption on the multiplicities. □

Recall from section 3.5.1 the definition of the representations W_i^+ and U_i^+ and denote by $\mathcal{W}_i^+$ and $\mathcal{E}_i^+$ the sets $\mathcal{W}_i^+ := \overline{\mathrm{GL}\,(r_z(\mathbf{d})) \cdot W_i^+}$ and $\mathcal{E}_i^+ := \{X^+ \in \mathrm{Rep}\,(zQ, r_z(\mathbf{d})) : \mathrm{Ext}(X^+, U_i^+) \neq 0\}$, respectively. The following Corollary shows the importance of the preceding Proposition:

Corollary 3.21. *Let $z \in Q_0$ be a sink such that $\Delta(z) > 0$ and assume $\lambda_j \geq \rho(Q)$ for $j = 1, \dots, r$. If $\mathcal{W}_i^+ = \mathcal{E}_i^+$, then $\mathcal{W}_i = \mathcal{E}_i$.*

Proof: Let $X \in \mathrm{Rep}\,(Q, \mathbf{d})$ satisfy $\mathrm{Ext}(X, U_i) \neq 0$. According to Proposition 3.20 we may suppose w.l.o.g $S(z)$ is not a direct summand of X. As $\dim \mathrm{Ext}(R_z^+ X, R_z^+ U_i) \neq 0$ this implies $R_z^+ X \in \mathcal{W}_i^+$ by Lemma 3.9 and the assumption. Since the partial orders $\leq_{\deg}$ and $\leq_{\hom}$ coincide (see section 3.2) it is enough to prove $W_i \leq_{\hom} X$. Obviously $\dim \mathrm{Hom}(S(z), W_i) = d_z = \dim \mathrm{Hom}(S(z), X)$. Moreover, $R_z^+ W_i \leq_{\deg} R_z^+ X$ by Lemma 3.9 and thus $R_z^+ W_i \leq_{\hom} R_z^+ X$, which implies

$$\dim \mathrm{Hom}(M, W_i) = \dim \mathrm{Hom}(R_z^+ M, R_z^+ W_i) \leq \dim \mathrm{Hom}(R_z^+ M, R_z^+ X) = \dim \mathrm{Hom}(M, X)$$

for any indecomposable representation $M \neq S(z)$. □

Proof of Theorem 3.19:
The proof is done by induction on n, the number of vertices of Q. In case $n = 2$ the only representation which may possibly occur as an indecomposable direct summand of T which is essential as a submodule is simple and projective. We have seen in Lemma 3.17 that the claim holds true in this case and therefore we may suppose $n \geq 3$. Moreover, we may assume w.l.o.g. $\lambda_j \geq \rho(Q)$ for $j = 1, \dots, r$ according to Lemma 3.16. Note that these inequalities are preserved under reflections at sinks satisfying $\Delta(z) > 0$. Since Q is a Dynkin quiver it is possible to choose a finite sequence of $t \geq 0$ vertices (where z_j is a sink in $z_{j-1} \cdots z_1 Q$ for $j = 1, \dots, t$) such that $R_{z_t}^+ \circ \ldots \circ R_{z_1}^+ T_i$ is simple and projective in $z_t \ldots z_1 Q$. Applying Corollary 3.21 as long as $\Delta_{z_{j-1} \cdots z_1 Q}(z_j) > 0$ we may thus assume w.l.o.g. that either T_i is simple and projective or $\Delta(z_1) \leq 0$. Therefore, the assertion follows using either Lemma 3.17 or Proposition 3.13 and the induction hypothesis. □

3.7.3 The general case

We return to the general case where Q is any tame quiver and $\mathbf{d} \in \mathbb{N}^n$ any prehomogeneous dimension vector such that T_i^+ is a direct summand of T.
Let $z \in Q_0$ be a sink of Q such that $\Delta(z) > 0$. As in the Dynkin case we aim to prove that for a representation $X \in \mathrm{Rep}\,(Q, \mathbf{d})$ satisfying $\mathrm{Ext}(X, U_i) \neq 0$ there exists a representation $Y \in \mathrm{Rep}\,(Q, \mathbf{d})$

such that X is a degeneration of Y, the representation $S(z)$ is not a direct summand of Y and $\mathrm{Ext}(Y, U_i) \neq 0$. The following Lemma shows this is true provided $\Delta(z)$ is 'large enough'.
Recall that we have $\Delta_M(z) = \sum_{j=1}^{s} \dim M(y_j) - \dim M(z) = \dim \mathrm{Ext}(M, S(z)) \geq 0$ for an indecomposable representation $M \neq S(z)$.

Lemma 3.22. *Let $z \in Q_0$ be a sink of Q and let V be an indecomposable direct summand of $X \in \mathrm{Rep}\,(Q, \mathbf{d})$ such that $\mathrm{Ext}(V, U_i) \neq 0$ and $\Delta(z) \geq \Delta_V(z)$. Then there exists a representation $Y \in \mathrm{Rep}\,(Q, \boldsymbol{d})$ such that X is a degeneration of Y, the representation $S(z)$ is not a direct summand of Y and $\mathrm{Ext}(Y, U_i) \neq 0$.*

Proof: The proof will be done by induction on ν, where $X = V \oplus S(z)^{\nu} \oplus X_1$ such that $S(z)$ is not a direct summand of X_1. In case $\nu = 0$ there is nothing to show. If ν is positive a short computation yields

$$\dim \mathrm{Ext}(X_1, S(z)) = -\langle \mathbf{dim}\, \boldsymbol{X_1}, \mathbf{e_z} \rangle = \Delta(z) + \nu - \Delta_V(z) \geq \nu > 0.$$

Hence there is a short exact sequence

$$0 \longrightarrow S(z) \longrightarrow \tilde{X}_1 \longrightarrow X_1 \longrightarrow 0$$

which does not split and thus X is a degeneration of $V \oplus S(z)^{\nu-1} \oplus \tilde{X}_1$. Since $S(z)$ is not a direct summand of $\tilde{X}_1$ we are done by the induction hypothesis. □

In the Dynkin case we achieved to satisfy the condition $\Delta(z) \geq \Delta_V(z)$ by choosing the multiplicities λ_j 'large enough', namely $\lambda_j \geq \rho(Q)$ (see section 3.7.2). In the present case where Q is any tame quiver it is still possible to satisfy the condition by choosing the multiplicities 'large enough'. However, the 'required multiplicities' now depend on X (more precisely, on the chosen indecomposable direct summand V satisfying $\mathrm{Ext}(V, U_i) \neq 0$) and not just on the types of the connected components of the quiver.
Let us fix an indecomposable direct summand T_k of T such that $\Delta_{T_k}(z) > 0$. Note that this is possible since $0 < \Delta(z) = \sum_{j=1}^{r} \lambda_j \cdot \Delta_{T_j}(z)$. Clearly, for a chosen representation V we may satisfy the condition $\Delta(z) \geq \Delta_V(z)$ by increasing λ_k.
The following Corollary is an adapted version of Corollary 3.21. Note that we need a stronger premise since the increase of the multiplicities depends on X.

Corollary 3.23. *Let $z \in Q_0$ be a sink of Q such that $\Delta(z) > 0$. If $\mathcal{W}_i^+(r_z(\boldsymbol{\mu})) = \mathcal{E}_i^+(r_z(\boldsymbol{\mu}))$ for all $\boldsymbol{\mu} \geq \boldsymbol{\lambda}$, then $\mathcal{W}_i = \mathcal{E}_i$.*

Proof: Let $X \in \mathrm{Rep}\,(Q, \mathbf{d})$ satisfy $\mathrm{Ext}(X, U_i) \neq 0$ and choose an indecomposable direct summand V of X such that $\mathrm{Ext}(V, U_i) \neq 0$. Moreover, choose $\eta \in \mathbb{N}_0$ such that

$$(\eta + \lambda_k) \cdot \Delta_{T_k}(z) \geq \Delta_V(z)$$

and consider the representation space $\mathrm{Rep}(Q, \mathbf{d}(\boldsymbol{\mu}))$, where $\mu_k = \lambda_k + \eta$ and $\mu_j = \lambda_j$ for $j \neq k$.

Since

$$\Delta_{\mathbf{d}(\boldsymbol{\mu})}(z) = \Delta_{\mathbf{d}}(z) + \eta \cdot \Delta_{T_k}(z) > 0$$

there is a representation $Y(\boldsymbol{\mu}) \in \mathrm{Rep}(Q, \mathbf{d}(\boldsymbol{\mu}))$ such that $X(\boldsymbol{\mu}) := X \oplus T_k^\eta$ is a degeneration of $Y(\boldsymbol{\mu})$, $\mathrm{Ext}(Y(\boldsymbol{\mu}), U_i) \neq 0$ and $S(z)$ is not a direct summand of $Y(\boldsymbol{\mu})$ by Lemma 3.22. Our goal is to show that $Y(\boldsymbol{\mu})$ is a degeneration of $W_i(\boldsymbol{\mu}) = W_i \oplus T_k^\eta$ since then also $W_i(\boldsymbol{\mu}) \leq_{\deg} X(\boldsymbol{\mu})$ and hence $W_i \leq_{\deg} X$ by 'cancellation' of T_k^η (see section 3.2). In order to prove this it is enough to show $W_i(\boldsymbol{\mu}) \leq_{\hom} Y(\boldsymbol{\mu})$ as the partial orders $\leq_{\deg}$ and $\leq_{\hom}$ coincide (see section 3.2). Obviously $\dim \mathrm{Hom}(S(z), W_i(\boldsymbol{\mu})) = \dim \mathrm{Hom}(S(z), Y(\boldsymbol{\mu}))$. Moreover, applying R_z^+ and using the assumption of the Lemma implies $R_z^+ W_i(\boldsymbol{\mu}) \leq_{\deg} R_z^+ Y(\boldsymbol{\mu})$ and hence $R_z^+ W_i(\boldsymbol{\mu}) \leq_{\hom} R_z^+ Y(\boldsymbol{\mu})$. Therefore,

$$\begin{aligned}\dim \mathrm{Hom}(M, W_i(\boldsymbol{\mu})) &= \dim \mathrm{Hom}(R_z^+ M, R_z^+ W_i(\boldsymbol{\mu})) \\ &\leq \dim \mathrm{Hom}(R_z^+ M, R_z^+ Y(\boldsymbol{\mu})) = \dim \mathrm{Hom}(M, Y(\boldsymbol{\mu}))\end{aligned}$$

for any indecomposable representation $M \neq S(z)$. □

Since Q is not necessarily a Dynkin quiver, T_i is not necessarily preprojective, i.e. there is not necessarily a sequence of $t \geq 0$ vertices (where z_j is a sink in $z_{j-1} \cdots z_1 Q$ for $j = 1, \ldots, t$) such that $R_{z_t}^+ \circ \ldots \circ R_{z_1}^+ T_i$ is simple and projective in $z_t \ldots z_1 Q$. This means we need to extend the proof strategy. The most obvious thing is to consider also reflections at sources and to ensure this leads to a situation where we are able to prove Theorem 3.4 directly. Indeed, we have seen in Lemma 3.18 that the Theorem holds true in case Y_i is simple and injective. In this sense, Lemma 3.18 is the analogue of Lemma 3.17 and the following Proposition is the analogue of Proposition 3.20 and Lemma 3.22, respectively.

Proposition 3.24. *Let $x \in Q_0$ be a source such that $\Delta(x) > 0$ and assume $Y_i \neq S(x)$ and $\lambda_j \geq \dim T_j(x) + 1$ for $j = 1, \ldots, r$. Moreover, let $X \in \mathrm{Rep}\,(Q, \boldsymbol{d})$ satisfy $\mathrm{Ext}(X, U_i) \neq 0$. Then there exists a representation $Y \in \mathrm{Rep}\,(Q, \boldsymbol{d})$ such that X is a degeneration of Y, the representation $S(x)$ is not a direct summand of Y and $\mathrm{Ext}(Y, U_i) \neq 0$.*

Proof: By assumption the representation X has an indecomposable direct summand V such that $\mathrm{Ext}(V, U_i) \neq 0$. The proof will be done by induction on $\dim \mathrm{Ext}(S(x), V)$.

Base case: $\mathrm{Ext}(S(x), V) = 0$

We first consider the case $V \neq S(x)$. Decomposing $X = V \oplus S(x)^\nu \oplus X_1$ such that $S(x)$ is not a direct summand of X_1 we may suppose $\nu > 0$ as otherwise we are done. A short computation using $\mathrm{Ext}(S(x), V) = 0$ yields

$$\dim \mathrm{Ext}(S(x), X_1) = -\langle \mathbf{e_x}, \mathbf{dim}\, \boldsymbol{X_1} \rangle = \Delta(x) + \nu > 0$$

and hence there is an exact sequence

$$0 \longrightarrow X_1 \longrightarrow \tilde{X}_1 \longrightarrow S(x) \longrightarrow 0$$

which does not split. Therefore, X is in the closure of the orbit of $V \oplus S(x)^{\nu-1} \oplus \tilde{X}_1$ and since $S(x)$ is not a direct summand of $\tilde{X}_1$ we are done by induction on ν in this case.

We continue with the case $V = S(x)$. The sink map $T^- \longrightarrow S(x)$ from $\operatorname{add} T$ to $S(x)$ is surjective and we denote its kernel by T''. Note that T'' does not contain $S(x)$ as a direct summand. By the assumption $\lambda_j > \dim T_j(x) = \dim \operatorname{Hom}(T_j, S(x))$ for $j = 1, \dots, r$ we may decompose T as $T = T^- \oplus \bigoplus_{j=1}^{r} T_j \oplus T_R$. We claim $T'' \oplus \bigoplus_{j=1}^{r} T_j \oplus T_R$ has an open orbit in $\operatorname{Rep}(Q, \mathbf{d} - \mathbf{e_x})$ and prove this by checking $\operatorname{Ext}(T \oplus T'', T \oplus T'') = 0$. Applying $\operatorname{Hom}(\cdot, T)$ and $\operatorname{Hom}(T, \cdot)$ to the exact sequence

$$0 \longrightarrow T'' \longrightarrow T^- \longrightarrow S(x) \longrightarrow 0$$

yields $\operatorname{Ext}(T'', T) = 0$ and that the sequence

$$0 \longrightarrow \operatorname{Hom}(T, T'') \longrightarrow \operatorname{Hom}(T, T^-) \longrightarrow \operatorname{Hom}(T, S(x)) \longrightarrow \operatorname{Ext}(T, T'') \longrightarrow 0$$

is exact, respectively. The induced map $\operatorname{Hom}(T, T^-) \longrightarrow \operatorname{Hom}(T, S(x))$ is surjective by definition of the sink map and hence $\operatorname{Ext}(T, T'') = 0$. By applying $\operatorname{Hom}(\cdot, T'')$ to the exact sequence above this implies $\operatorname{Ext}(T'', T'') = 0$ and therefore $\operatorname{Ext}(T \oplus T'', T \oplus T'') = 0$.
Consequently, it is enough to prove there is an indecomposable direct summand $T_l \neq T_i$ such that $\operatorname{Ext}(S(x), T_l) \neq 0$. Indeed, if

$$0 \longrightarrow T_l \longrightarrow \tilde{T}_l \longrightarrow S(x) \longrightarrow 0$$

is an exact sequence which does not split, applying $\operatorname{Hom}(\cdot, U_i)$ to this sequence and using $U_i \in T[\hat{i}]^\perp$ yields $\dim \operatorname{Ext}(\tilde{T}_l, U_i) = \dim \operatorname{Ext}(S(x), U_i) \neq 0$ and hence the representation

$$Y := \tilde{T}_l \oplus \bigoplus_{j \neq l} T_j \oplus T_R \oplus T''$$

has the required properties.
In order to prove the statement above note $\operatorname{Ext}(S(x), T_i) \neq 0$ as U_i is a quotient of T_i and $\operatorname{Ext}(S(x), U_i) \neq 0$. Mapping $S(x)$ to the exact sequence

$$0 \longrightarrow T_i \longrightarrow T_i^+ \longrightarrow Y_i \longrightarrow 0$$

this implies $\operatorname{Ext}(S(x), T_i^+) \neq 0$ because $\operatorname{Hom}(S(x), Y_i) = 0$ as $Y_i \neq S(x)$ by assumption. Since T_i^+ is a direct summand of T there is thus an indecomposable direct summand $T_l \neq T_i$ of T such that $\operatorname{Ext}(S(x), T_l) \neq 0$.

Inductive step: $\operatorname{Ext}(S(x), V) \neq 0$
We suppose $S(x)$ is a direct summand of X as otherwise we are done. Let

$$0 \longrightarrow V \longrightarrow \tilde{V} \longrightarrow S(x) \longrightarrow 0$$

be any exact sequence which does not split. Since $\operatorname{Ext}(V, U_i) \neq 0$ we may write $\tilde{V} = V_1 \oplus R_1$ where V_1 is an indecomposable representation such that $\operatorname{Ext}(V_1, U_i) \neq 0$. Mapping $S(x)$ to the exact sequence above yields the exact sequence

$$0 \longrightarrow \mathrm{Hom}(S(x), S(x)) \longrightarrow \mathrm{Ext}(S(x), V) \longrightarrow \mathrm{Ext}(S(x), \tilde{V}) \longrightarrow 0$$

and thus

$$\dim \mathrm{Ext}(S(x), V_1) \leq \dim \mathrm{Ext}(S(x), \tilde{V}) = \dim \mathrm{Ext}(S(x), V) - 1 < \dim \mathrm{Ext}(S(x), V).$$

Decomposing $X = V \oplus S(x) \oplus X_1$ implies X is in the closure of the orbit of $V_1 \oplus R_1 \oplus X_1$ and we are done by the induction hypothesis. □

Recall from section 3.5.2 the definition of the representations W_i^- and U_i^- and denote by $\mathcal{W}_i^-$ and $\mathcal{E}_i^-$ the sets $\mathcal{W}_i^- := \overline{\mathrm{GL}\,(r_x(\mathbf{d})) \cdot W_i^-}$ and $\mathcal{E}_i^- := \left\{X^- \in \mathrm{Rep}\,(Qx, r_x(\mathbf{d})) : \mathrm{Ext}(X^-, U_i^-) \neq 0\right\}$, respectively. The preceding Proposition allows us to prove the analogue of Corollary 3.21.

Corollary 3.25. *Let $x \in Q_0$ be a source such that $\Delta(x) > 0$ and assume $Y_i \neq S(x)$ and*

$$\lambda_j \geq \dim T_j(x) + 1 \text{ for } j = 1, \ldots, r.$$

If $\mathcal{W}_i^- = \mathcal{E}_i^-$, then $\mathcal{W}_i = \mathcal{E}_i$.

Proof: Replacing Lemma 3.9 by Lemma 3.10, the proof is analogous to the proof of Corollary 3.21. □

Proof of Theorem 3.1:
We may assume w.l.o.g. that Q is connected. In particular, for any vertex $y \in Q_0$ the Theorem is proved for the quiver $Q[\hat{y}]$ since this is a Dynkin quiver. As in the proof of Theorem 3.19 the idea of the proof is to apply reflection functors until we reach a situation where we are able either to prove the statement directly or to use the fact that the statement is true for the quiver obtained by deleting a vertex.
We distinguish the two cases whether or not T contains a preinjective direct summand and consider first the case where there is an indecomposable direct summand T_m which is preinjective. In this case it is possible to choose a finite sequence of $t \geq 0$ vertices (where x_l is a source in $Qx_1 \cdots x_{l-1}$ for $l = 1, \ldots, t$) such that $R_{x_t}^- \circ \ldots \circ R_{x_1}^- T_m$ is simple and injective in $Qx_1 \ldots x_t$. According to Lemma 3.16 we may assume w.l.o.g. $\lambda_j \geq \rho_j$ for $j = 1, \ldots, r$, where

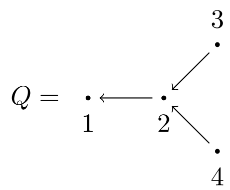
$$\rho_j := \max_{1 \leq l \leq t} \left\{\dim R_{x_{l-1}}^- \circ \ldots \circ R_{x_1}^- T_j(x_l)\right\} + 1, \qquad j = 1, \ldots, r.$$

In view of Corollary 3.25 we may thus assume w.l.o.g. that either T_m is simple and injective or $\Delta(x_1) \leq 0$ or $Y_i = S(x_1)$. Note that the first case implies $T_m = S(x)$ for some source x and hence $\Delta(x) < 0$ (see section 3.2). Therefore, the assertion follows using either Proposition 3.15 and Theorem 3.19 or Lemma 3.18.
We continue with the remaining case where T has no preinjective direct summand. As the source map from any indecomposable direct summand of T_B to $\mathrm{add}\, T$ is injective according to Lemma 3.2 ii) this implies there is no preinjective direct summand of T_B either since there is no non-zero homomorphism from a preinjective to a non-preinjective representation (see section 3.2).

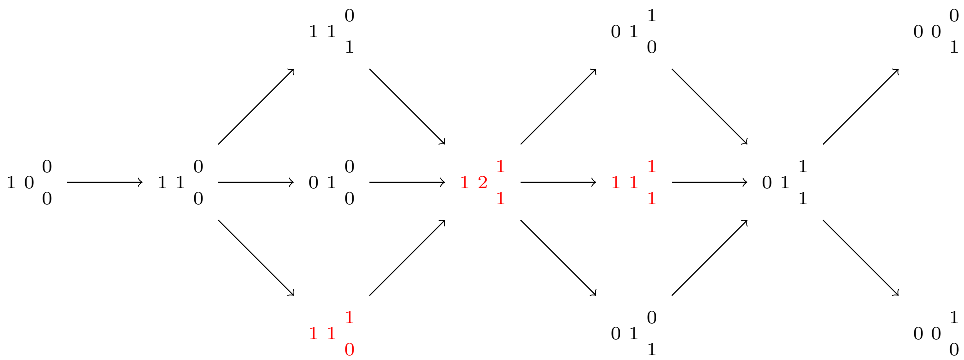
Hence the representation $T \oplus T_B$ must contain an indecomposable direct summand T_m which is preprojective as there is no regular tilting module (see section 3.2). Therefore, it is possible to choose a finite sequence of $t \geq 0$ vertices (where z_l is a sink in $z_{l-1} \cdots z_1 Q$ for $l = 1, \ldots, t$) such that the representation $R^+_{z_t} \circ \ldots \circ R^+_{z_1} T_m$ is simple and projective in $z_t \ldots z_1 Q$. In view of Corollary 3.23 we may thus assume w.l.o.g. that either T_m is simple and projective or $\Delta(z_1) \leq 0$. In the latter case the assertion follows either from Lemma 3.17 (if $T_i = S(z_1)$) or from Proposition 3.13 and Theorem 3.19 (if $T_i \neq S(z_1)$). In case T_m is simple and projective, T_m equals $S(z)$ for some sink z of Q. We suppose $T_m \neq T_i$, as otherwise Lemma 3.17 finishes the proof. If T_m is a direct summand of T, then $\Delta(z) < 0$ according to Remark 3.5 and if T_m is a direct summand of T_B, then $\Delta(z) = 0$ according to Remark 3.6. Therefore, Proposition 3.13 and Theorem 3.19 end the proof. □

4 Example

Let k be an algebraically closed field, let Q be the quiver



and let $\mathbf{d} = (4, 5, 4, 2)$. Recall that $\mathbf{d}$ is prehomogeneous since Q is a Dynkin quiver. Identifying each indecomposable representation of Q with its dimension vector the Auslander-Reiten quiver of Q is given by



where we marked in red the representations

$$T_1 := 1\,1\substack{1\\0}, \quad T_2 := 1\,2\substack{1\\1} \quad \text{and} \quad T_3 := 1\,1\substack{1\\1}.$$

The orbit of $T := T_1^2 \oplus T_2 \oplus T_3$ is the open orbit of $\operatorname{Rep}(Q, \mathbf{d})$ since $\operatorname{Ext}(T, T) = 0$. As Q is a Dynkin quiver T is stable. Note that T_1 is essential as a submodule and T_3 is essential as a quotient, respectively. In order to compute the Bongartz completion T_B and the dual Bongartz completion T_{DB} we notice that there are only two representations M_1 and M_2 such that $T \oplus M_i$ is a tilting module, namely

$$M_1 := 1\,1\substack{0\\1} \quad \text{and} \quad M_2 := 0\,1\substack{1\\0}.$$

Since there is no non-zero map from T to M_1 the sink map from $\operatorname{add} T$ to M_1 is injective and hence M_1 is the Bongartz completion of T. Analogously, M_2 is the dual Bongartz completion of T. Recall that U_1 and V_3 are the unique indecomposable representations in $(T \oplus T_B)^{\perp}$ and ${}^{\perp}(T \oplus T_{DB})$, respectively. Consequently, the representations U_1 and V_3 yield $U_1 = T_1$ and $V_3 = T_3$. Moreover, there is only one indecomposable representation in $T^{\perp}$, namely $S_4 := M_1$. Therefore, the complement $\mathrm{Rep}\,(Q,\mathbf{d}) \setminus \mathrm{GL}\,(\mathbf{d}) \cdot T$ is the union of the three irreducible components

$$\begin{aligned}
\mathcal{C}_1 &= \{X \in \mathrm{Rep}\,(Q,\mathbf{d}) :\ \mathrm{Ext}(X, U_1) \neq 0\}\,, \\
\mathcal{C}_3 &= \{X \in \mathrm{Rep}\,(Q,\mathbf{d}) :\ \mathrm{Ext}(V_3, X) \neq 0\}\,, \\
\mathcal{D}_4 &= \{X \in \mathrm{Rep}\,(Q,\mathbf{d}) :\ \mathrm{Ext}(X, S_4) \neq 0\} \\
&= \{X \in \mathrm{Rep}\,(Q,\mathbf{d}) :\ \mathrm{Hom}(X, S_4) \neq 0\}\,,
\end{aligned}$$

which are of codimension 3, 2 and 1, respectively.

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Erklärung

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Titel der Arbeit:	The complement of the open orbit for tame quivers
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