On the Plumbing Structure of Fibre Surfaces

Inauguraldissertation

der Philosophisch-naturwissenschaftlichen Fakultät
der Universität Bern

vorgelegt von

Filip Misev

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Leiter der Arbeit:

Prof. Dr. Sebastian Baader

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Prof. Dr. G. Colangelo

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Filip Misev

Contents

Introduction

This thesis is situated in the mathematical field of low-dimensional topology which studies manifolds of dimension four or less and their interrelations. Specifically, it is about fibred links and their Seifert surfaces.

A link L in the three-sphere S^3 is *fibred* if its complement $S^3 \setminus L$ can be filled by a continuous family of surfaces that pairwise intersect in their common boundary L. These surfaces, called *fibre surfaces*, are the main objects of study in this thesis.

Fibred links and their fibre surfaces exist in abundance and appear in various settings – links of singularities, Lorenz links and, more generally, closures of positive braids are known to be fibred. Many other interesting classes of links have a non-trivial intersection with the class of fibred links, for example alternating links, rational links, hyperbolic links and pretzel links each have fibred and non-fibred members. At the same moment, fibredness is an extremal property: for example, the Alexander polynomial of a fibred link is always monic and has the maximal possible degree for a link of its genus.

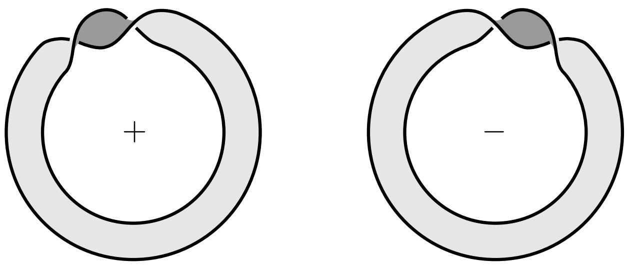

Figure 1: The positive and the negative Hopf band.

The basic example of a fibre surface is the Hopf band, an unknotted annulus with a positive or negative full twist, illustrated in Figure 1. Using a geometric operation called *plumbing*, or *Murasugi sum*, Hopf bands can be glued together to form more complicated fibre surfaces (see for example Figure 2). The reverse operation of deplumbing corresponds to removing a Hopf band, which amounts to a surgery along an arc. By Giroux' work on contact structures and open books in three-manifolds, every fibre surface has a Hopf plumbing structure, that is, it can be obtained from the standard disc by a sequence of Hopf plumbings and deplumbings. In fact, Giroux and Goodman prove in their article [GiGo] that every pair of fibre surfaces in S^3 has a common stabilisation, that is, a third fibre surface which can be reached from both using plumbing only. In particular, every fibre surface can be reached from the standard disc by some sequence of Hopf plumbings, followed by some deplumbings.

Figure 2 shows an example of a fibre surface that can be obtained by Hopf plumbing on the disc. Before we even discuss the precise definition of plumbing in the next chapter, we trust the reader can already clearly see one particular way of constructing this surface using Hopf bands.

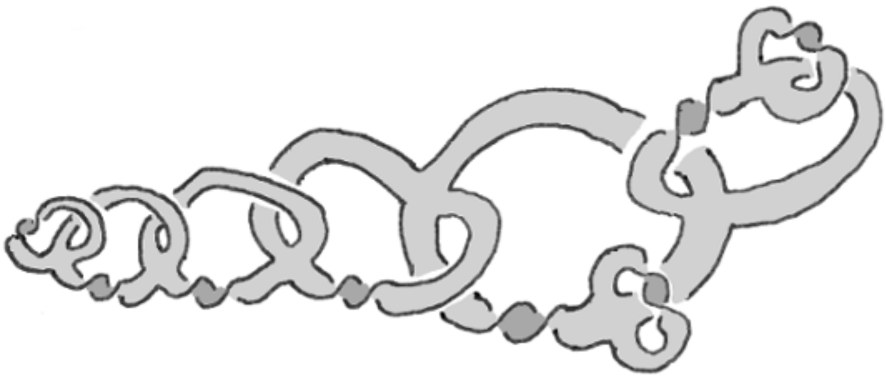

Figure 2: An example of a fibre surface obtained by plumbing Hopf bands.

Our main goal is to study the plumbing structure of fibre surfaces. More precisely, the problem can be formulated as follows. Consider the graph whose vertices are isotopy classes of fibred links (or, equivalently, fibre surfaces) and connect two vertices whenever the associated surfaces are related by a single Hopf plumbing or deplumbing. By Giroux and Goodman's theorem, this graph is connected. We think of the edges as being oriented in the direction of plumbing and decorated by the sign of the Hopf band involved. Furthermore, we can refine our graph by allowing multiple edges between vertices, namely one for each distinct way of plumbing or deplumbing. This can be made precise by representing every vertex by a fixed embedding of the corresponding fibre surface. Under this circumstance, whenever S' is given by plumbing a Hopf band onto S, we obtain an embedding $i : S \hookrightarrow S'$, and i is determined by the plumbing up to isotopy in the target S'. Every isotopy class of embeddings $S \hookrightarrow S'$ coming from a Hopf plumbing gives rise to an edge in our refined graph. Denoting the refined graph by $\mathscr{F}$, we ask for the detailed structure of $\mathscr{F}$.

For example, we can ask the following basic questions:

- What can we say about the degree of a given vertex of $\mathscr{F}$? For which vertices is the number of incoming edges finite?
- Is the number of edges between two given vertices of $\mathscr{F}$ bounded? Is the number of shortest paths between two given vertices bounded?
- Does $\mathscr{F}$ contain loops? Are the lengths of such loops bounded?

Summary of results

In Chapter 1 we collect and briefly discuss the basic notions used in the rest of the thesis (fibred link, fibre surface, monodromy, Murasugi sum and Hopf plumbing, Seifert form, Alexander polynomial) and give some examples of fibre surfaces and fibred links.

Chapter 2 is a short note in which we prove the following statement about the monodromy of a fibred knot.

Theorem 1. *Let K be a fibred knot of genus g with monodromy φ. Then any representation of φ as a product of Dehn twists involves at least $2g$ distinct factors.*

This bound is sharp for fibred knots whose fibre surface is obtained by plumbing Hopf bands. Melvin and Morton gave examples of fibre surfaces of genus two that cannot be obtained as Hopf plumbings [MeMo], showing that the deplumbing operation is indeed necessary for constructing all fibre surfaces. In terms of the graph $\mathscr{F}$ (defined above), there exist vertices that cannot be reached from the standard disc by an oriented path. In Chapter 3, we present another family of examples for this phenomenon, consisting of genus one surfaces with two boundary components (**Theorem 2**).

In Chapter 4, we consider edges pointing to a given vertex of $\mathscr{F}$, that is, different Hopf deplumbings from a given fibre surface. We restrict our attention to two classes of fibred links, namely torus links and arborescent (or *tree-like*) plumbings of positive Hopf bands (the surface in Figure 2 is in fact an example for both). It turns out that the number of such inward edges is infinite in most of the cases, with some interesting exceptions. The precise statements are formulated below in terms of *cutting arcs* that preserve fibredness. These are properly embedded intervals α in a given fibre surface S such that cutting S along α results in another fibre surface. In certain cases, cutting along an arc has the same effect as deplumbing a Hopf band, as indicated in Figure 3. This does not hold for arbitrary cutting arcs that preserve fibredness. However, it is true for fibre surfaces whose monodromy is right-veering (cf. Definition 6), which comprises all positive Hopf plumbings and hence all torus links and positive tree-like Hopf plumbings.

Theorem 3. *Let $n, m \geqslant 4$ or $n = 3, m \geqslant 6$. Then the fibre surface S of the torus link $T(n, m)$ contains infinitely many homologically distinct cutting arcs preserving fibredness.*

Theorem 4. *Let S be the fibre surface obtained by plumbing positive Hopf bands according to a finite tree T. There are, up to isotopy, only finitely many cutting arcs in S preserving fibredness, if and only if T is one of the Coxeter-Dynkin trees A_n, D_n, E_6, E_7 or E_8.*

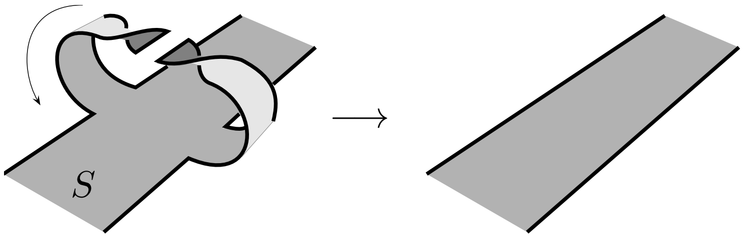

Figure 3: Hopf deplumbing corresponds to cutting along an arc.

The proofs of Theorems 3 and 4 rely on a theorem by Buck et al. [BIRS], characterising cutting arcs that preserve fibredness in terms of the monodromy. It immediately implies the following theorem, which serves our purpose.

Theorem 5 (compare Theorem 1 in [BIRS])**.** *Let L be a fibred link with fibre surface S and right-veering monodromy $\varphi : S \to S$. Then, a cutting arc α preserves fibredness if and only if $\alpha \cap \varphi(\alpha) = \partial\alpha$ after minimising isotopies on α and $\varphi(\alpha)$.*

A simple method for constructing cutting arcs is to repeatedly apply the monodromy to a given cutting arc. If S is a fibre surface with monodromy φ and $\alpha \subset S$ is a cutting arc that preserves fibredness, then $\varphi^n(\alpha)$ is a cutting arc that preserves fibredness, for every $n \in \mathbb{Z}$. This method is not useful for proving Theorem 3 though, since the monodromy of a torus link is periodic. However, we do apply it to prove the following.

Theorem 6. *Let S be a fibre surface whose monodromy $\varphi : S \to S$ is pseudo-Anosov and right-veering. Assume that a Hopf band can be deplumbed from S. Then S contains infinitely many non-isotopic cutting arcs preserving fibredness.*

The contents of Chapter 4 will be published as an article in the Bulletin of the SMF (Société Mathématique de France) and is available on the arXiv preprint server, see [Mi1].

A different approach to the study of plumbing structures is given in Chapter 5, where we look at Hopf bands in arborescent Hopf plumbings, up to homology.

Theorem 7. *Let T be a finite plane tree and $S \subset S^3$ the corresponding positive arborescent Hopf plumbed surface. Then the set of homology classes of Hopf bands $C_1(S)$ is finite if and only if T is spherical.*
In contrast, if T is hyperbolic and ∂S_T is a knot, $C_1(S)$ consists of infinitely many orbits of the monodromy.

Chapter 5 is published as an article preprint on arXiv, see [Mi2].

A short history of fibre surfaces and fibred links

The notion of a fibred knot goes back to the 1960s, when such knots were known as Neuwirth knots. Indeed, Neuwirth [Ne1] had originally studied an algebraic property of knots: for $G = \pi_1(S^3 \setminus K)$ the fundamental group of a knot K, he considered its commutator group $[G, G]$ and proved that $[G, G]$ is a free group of rank $2g(K)$ if it is finitely generated. Here, $g(K)$ denotes the minimal genus of all Seifert surfaces for K. The geometric meaning of this algebraic assumption was subsequently given by Stallings in 1961 [St1], who proved that the complement of a Neuwirth knot has the structure of a surface bundle over the circle – the definition of fibredness we use today. The commutator group plays the role of the fundamental group of the associated fibre surface.

In 1962, Murasugi [Mu] described a geometric glueing operation to construct Seifert surfaces from simple pieces. Using his operation, now known as the Murasugi sum, he proved that the Alexander polynomial detects fibredness among alternating links: if the (reduced) Alexander polynomial of an alternating link has leading coefficient 1, then the link is fibred.

In 1968, Milnor [Mil] proved his fibration theorem for singularities of algebraic hypersurfaces. Namely, every isolated singular point of a complex algebraic plane curve $C = \{f(x, y) = 0\}$ gives rise to a fibred link. Away from C, the argument of the polynomial f defines a locally trivial fibre bundle over S^1 when restricted to a small sphere (real three-dimensional) around the singular point in question. We come back to this construction in the next chapter in more detail.

Stallings' *Constructions of Fibered Knots and Links* [St2] from 1978 consist of three basic geometric operations on fibre surfaces: plumbing (a generalisation of Murasugi's glueing operation to arbitrary fibre surfaces), twisting (a Dehn surgery along an unknotted zero framed curve on a fibre surface) and companionisation. Through these operations, Stallings implicitly gave a myriad of new examples of fibred links and fibre surfaces. Using the plumbing operation, he showed for example that every positive braid link and, more generally, every homogeneous braid link is fibred.

Gabai emphasised the importance and usefulness of Murasugi's and Stallings' operations in his 1983 article *The Murasugi Sum is a Natural Geometric Operation* (see [Ga1]), where he proved that Murasugi sums preserve fibredness (in both directions; that is, a Murasugi sum of two surfaces is a fibre surface if and only if both summands are), incompressibility and minimal genus.

As it turned out in 1982 through Harer's work, Stallings' constructions already encompass all fibre surfaces in S^3. More precisely, Harer proved [Ha, Theorem 1] that every fibred link is related to the unknot via a sequence of Hopf plumbings, Stallings twists, and Hopf deplumbings. He also asked whether the twisting operation could be omitted. A positive answer, relying on Giroux' work on contact structures in 3-manifolds, was given by Giroux-Goodman [GiGo] in 2006: every fibre surface in the three-sphere can be obtained from the standard disc by plumbing and deplumbing Hopf bands. In 1986, Melvin and Morton [MeMo] gave a family of examples showing that deplumbing is indeed necessary for constructing all fibre surfaces.

We have mentioned Murasugi's result that an alternating link whose Alexander polynomial has leading coefficient 1 is fibred. This criterion does not carry over to arbitrary links – take for example the connected sum of any fibred link with any nontrivial knot with Alexander polynomial 1 (the constant polynomial). Knot Floer homology, which can be seen as a refined (or, categorified) version of the Alexander polynomial, does however detect fibredness among all links. An important step was done by Ghiggini, who gave a strategy for proving the conjecture and settled it for genus one knots. Using Ghiggini's strategy, Ni proved the general result in his thesis in 2007 (see [Ni1, Ni2]). The proof also relies on work by Gabai and Ozsváth-Szabó. Another aspect in this context is the uniqueness of minimal genus Seifert surfaces. Fibred links do have a unique Seifert surface of minimal genus, namely their fibre surface (see [Ko]). Juhász proved [Ju] (see also [Ban]) that the top rank of knot Floer homology bounds the number of distinct Seifert surfaces of minimal genus g for a given knot K, as follows: if $n > 0$ is an integer such that $\operatorname{rank} \widehat{HFK}(K, g) < 2^{n+1}$, then K has at most n pairwise disjoint non-isotopic genus g Seifert surfaces.

More recently, intervals with endpoints on the boundary of a Seifert surface turned out to be very useful objects in the context of fibredness, in two ways. First, Baader and Graf [BaGr] (see also Graf's thesis [Gr]) gave a new characterisation of fibre surfaces in 2014. They proved that a Seifert surface S is a fibre surface if and only if every elastic chord lying on the positive side of S with ends attached to the boundary can be taken to the negative side of S through $S^3 \setminus S$. Second, Buck et al. [BIRS] gave a simple characterisation of cutting arcs that preserve fibredness in terms of the monodromy, which we use in Chapter 4 to study deplumbings from given fibre surfaces. In fact, the effect of deplumbing a Hopf band H from a Seifert surface S can equally be achieved by cutting S along a properly embedded arc $\alpha \subset S$ (take a spanning arc of H, compare Figure 3). Passing from Hopf deplumbings to cutting arcs that preserve fibredness, Buck et al. show that a cutting arc either corresponds to a Hopf deplumbing or to a certain generalised Hopf deplumbing. Since generalised deplumbing does not appear for the particular families of fibre surfaces we study here, deplumbing and fibredness-preserving cutting are the same.

While the above short history highlights developments that concern this thesis, fibred links and fibre surfaces have also been looked at from different angles; for example via branched coverings (see Birman [Bir]), in connection with concordance of knots and contact structures (see for example Baker [Bak]), as well as in three-manifolds other than S^3 and in higher dimensions.

Chapter 1

Preliminaries

1 Fibre surfaces and monodromy

Knots, links and Seifert surfaces are understood to be smooth, compact, oriented and embedded in the three-dimensional sphere S^3, and are considered up to smooth ambient isotopy (smooth deformation through a family of diffeomorphisms of the ambient space S^3). For S to be a Seifert surface for a link $L = \partial S$, we always require that the orientations of L and S be compatible and S be connected.

For an introduction to knot theory we refer to Rolfsen's book [Ro] and to the book by Burde and Zieschang [BZH].

Definition 1. A Seifert surface $S \subset S^3$ is called a *fibre surface* if its interior $\mathring{S}$ is the fibre of a locally trivial fibre bundle $p : S^3 \setminus \partial S \to S^1$.

Definition 2. A link $L \subset S^3$ is *fibred* if it is the boundary of a fibre surface.

A fibre surface S always comes with a mapping class $\varphi : S \to S$ called the *monodromy*, fixing ∂S pointwise, which captures how the fibres organise around the boundary link $L = \partial S$. There are several ways of describing the monodromy map; since it plays a crucial role in the study of fibred links and fibre surfaces, we review some of these below. Let $p : S^3 \setminus \partial S \to S^1$ be the locally trivial fibre bundle associated to the fibre surface S.

Monodromy as a glueing map

If we cut open the base circle S^1, we obtain an interval I over which p restricts to a trivial bundle $p^{-1}(I) \cong \mathring{S} \times I$ since I is contractible. The monodromy is the glueing map $\varphi : S \to S$ necessary to recover the original bundle over S^1, that is, $S^3 \cong (S \times [0, 2\pi])/_\sim$, where $(p, 2\pi) \sim (\varphi(p), 0)$ for $p \in S$ and $(p, t) \sim (p, t')$ for $p \in \partial S$. In this way, $S^3 \setminus \partial S$ becomes the mapping torus of $\varphi|_{\mathring{S}}$. The idea of cutting the base circle S^1 open can also be used to construct the infinite cyclic cover of $S^3 \setminus \partial S$. In that picture, the lift of the monodromy generates the group of deck transformations.

Monodromy as the flow of a vector field

We assume that the fibre bundle map p is given by a submersion. Since the fibre surface S is oriented, it carries a field of normal vectors, and so do all the fibres of the fibre bundle p. Organise these normal vectors to a vector field Θ on S^3 that projects to a unit tangent vector field on S^1. By continuity, Θ vanishes on ∂S, where p is not defined. Let φ_t be the flow of Θ and assume that $\mathring{S} = p^{-1}(1)$. By construction, φ_t maps the fibre $p^{-1}(1)$ to the fibre $p^{-1}(e^{it})$, and $\varphi_{2\pi} : S \to S$ is the monodromy. In terms of the previous description, Θ is given by the vertical unit vector field on $S \times [0, 2\pi]$.

Monodromy in terms of proper arcs

An even more pictorial way to understand the monodromy of a fibre surface is given by its action on properly embedded arcs. Under the flow φ_t of the monodromy vector field described above, the fibre surface S turns around its boundary ∂S through S^3. In particular, every figure drawn on S is moved through $S^3 \setminus S$ by φ_t when t runs from 0 to 2π. In this sense, every arc $\alpha \subset S$ with endpoints on ∂S can be dragged from one side of S to the other while keeping its endpoints fixed.

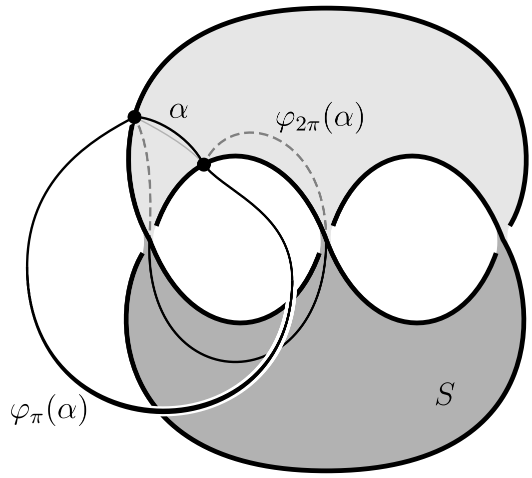

Figure 1.1: A Seifert surface of the positive trefoil knot and a proper arc α.

During this process, the interior of α stays in the different fibres of the fibre bundle and does therefore not intersect S, except at the initial and at the final moment. When it returns to S from below, α takes a new position on S, namely $\varphi_{2\pi}(\alpha)$. Thus, the monodromy can be studied by looking at how it acts on properly embedded arcs. This is explained in detail in Graf's thesis [Gr].

In [BaGr], Baader and Graf prove the following characterisation of fibre surfaces among all Seifert surfaces.

Theorem (see [BaGr], Theorem 2.1)**.** *Let $S \subset S^3$ be a Seifert surface. If every arc $\alpha \subset S$ can be dragged to the other side of S, then S is a fibre surface.*

As a corollary, Baader and Graf obtain a new and simple proof of the fact that Murasugi sums preserve fibredness (see Section 3 for the definition of Murasugi sum). Since it suffices to check the assumption of the above theorem for finitely many arcs that cut the surface into discs, it also provides a practical method to check fibredness of a given Seifert surface in simple cases.

Periodic, pseudo-Anosov and reducible monodromies

As a surface mapping class, the monodromy is subject to the Nielsen-Thurston classification of surface diffeomorphisms into periodic, pseudo-Anosov and reducible mapping classes [Th1, FLP, FM]. The geometry of the fibred link's complement reflects these types as follows [Th2, Th3].

type of monodromy	fibred link complement
periodic	Seifert fibred
pseudo-Anosov	complete hyperbolic metric of finite volume
reducible	existence of an incompressible torus

All types of mapping classes are in fact realised by monodromies of fibred knots, although not all individual mapping classes are. Examples proving the first part of the statement are discussed in detail in the Section 4 below. The next chapter, Chapter 2, discusses an obstruction to the realisability of a given mapping class as monodromy of a fibred knot, in terms of word length in the mapping class group.

2 The Seifert form and the monodromy

For fibre surfaces, the Seifert form has a close relation to the monodromy, as we explain below.

Definition 3. The *Seifert form* of a Seifert surface S is the bilinear form

$$(.\,,.) : H_1(S,\mathbb{Z}) \times H_1(S,\mathbb{Z}) \longrightarrow \mathbb{Z}$$

defined on homology classes of simple closed curves $\alpha, \beta \subset S$ by

$$(\alpha, \beta) = \mathrm{lk}(\alpha, \beta^+),$$

where $\beta^+ \subset S^3 \setminus S$ is the curve obtained by pushing β off S in the normal direction given by the orientation of S, and lk denotes the linking number of two knots in S^3.

If S is a fibre surface and φ_t is the flow of the monodromy vector field, we can take $\beta^+ = \varphi_\pi(\beta)$ in the above definition. This has the following consequence (see [Sa, Lemma 8.3]).

Lemma 1 (see [Sa])**.** *Let S be a fibre surface with monodromy $\varphi : S \to S$ and Seifert form $(.,.)$. Let $\varphi_* : H_1(S,\mathbb{Z}) \to H_1(S,\mathbb{Z})$ be the linear map on homology induced by φ. Then $(v,w) = (\varphi_* w, v)$, for all $v, w \in H_1(S,\mathbb{Z})$.*

Proof. We use the flow φ_t of the monodromy vector field again. By definition of the flow and the monodromy, we have $\varphi = \varphi_{2\pi} = \varphi_\pi \circ \varphi_\pi$. It suffices to check the assertion for $v, w \in H_1(S,\mathbb{Z})$ represented by simple closed curves α, β. Using the symmetry of the linking number and the fact that φ_π arises from an isotopy, it follows

$$\begin{aligned}(v,w) &= \mathrm{lk}(\alpha, \varphi_\pi(\beta)) = \mathrm{lk}(\varphi_\pi(\alpha), \varphi_{2\pi}(\beta)) \\ &= \mathrm{lk}(\varphi_\pi(\alpha), \varphi(\beta)) = \mathrm{lk}(\varphi(\beta), \varphi_\pi(\alpha)) = (\varphi_* w, v).\end{aligned}$$

□

In a basis $e_1, \dots, e_n$ of $H_1(S,\mathbb{Z})$, the Seifert form and the monodromy are given by matrices A and M, respectively. Concretely, if x, y are the coordinate vectors of $v, w \in H_1(S,\mathbb{Z})$, we have $x^\top A y = (v,w)$, and My is the coordinate vector of $\varphi_* w$. The above lemma then reads

$$A = A^\top M, \quad \text{or} \quad M = A^{-\top} A.$$

Indeed, the Seifert matrix A of a fibre surface is always invertible over $\mathbb{Z}$; compare [BZH, Lemma 8.6].

The Alexander polynomial and the monodromy

Let L be a link and let S be a Seifert surface for L. As before, choose a basis of $H_1(S,\mathbb{Z})$ and denote the corresponding Seifert matrix by A. Then the Alexander polynomial $\Delta_L \in \mathbb{Z}[t]$ can be defined as follows:

$$\Delta_L(t) = \det(tA - A^\top) = \det(tA^\top - A).$$

It is symmetric in the following sense: $\Delta_L(1/t) = (-t)^{-n} \Delta_L(t)$, where n denotes the rank of $H_1(S,\mathbb{Z})$. Suppose now that L is a fibred link. Then $\det(A) = \pm 1$ and $M = A^{-\top} A$, as mentioned above. Therefore we obtain the following, where I is the $n \times n$ identity matrix.

$$\Delta_L(t) = \det(tA^\top - A) = \det(A^\top)\det(tI - A^{-\top} A) = \pm \det(tI - M).$$

In other words, the Alexander polynomial of a fibred link is the characteristic polynomial of the monodromy M. In particular, the zeros of Δ_L are the eigenvalues of M. Moreover, Δ_L is monic (that is, its leading coefficient is ± 1) and of degree n. If L is a knot, $n = 2g$, where g is the genus of L (since M acts on the homology of a fibre surface, which has minimal genus). If L is a link with r components, $n = 2g + r - 1$. Note also that the characteristic polynomials of M and M^{-1} coincide.

3 Geometric operations on fibre surfaces

Murasugi sum and Hopf plumbing

Given two disjoint and unlinked Seifert surfaces S_1 and S_2 in the three-sphere and discs $D_1 \subset S_1$, $D_2 \subset S_2$ as shown in Figure 1.2, we can glue them together by matching the discs. The resulting surface is said to be a *Murasugi sum* of S_1, S_2.

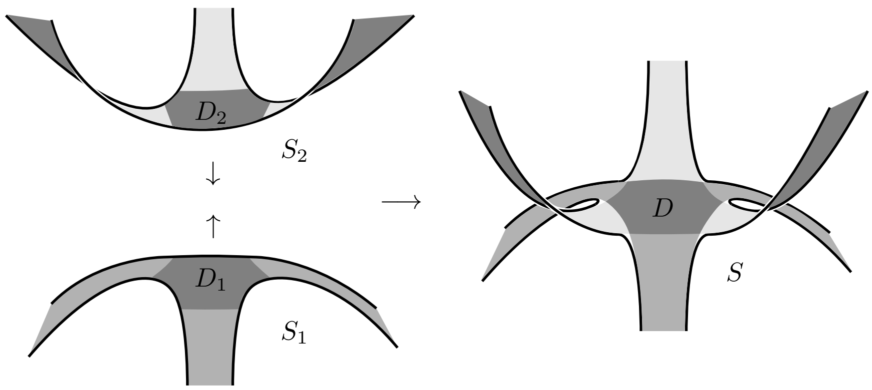

Figure 1.2: The Murasugi sum of two Seifert surfaces.

Here is a more precise definition.

Definition 4. A Seifert surface S is a *Murasugi sum* of two other Seifert surfaces S_1 and S_2 along D if the following conditions hold (after suitable isotopies of S_1 and S_2).

- $S = S_1 \cup S_2$,
- $D = S_1 \cap S_2$ is a disc whose boundary is contained in $\partial S_1 \cup \partial S_2$,
- S_1 and S_2 are separated by a two-sphere.

The last point means that there exist three-dimensional balls $B_1 \supset S_1$ and $B_2 \supset S_2$ whose union is S^3, $B_1 \cap B_2 = \partial B_1 = \partial B_2$ and $\partial B_1 \cap S = \partial B_2 \cap S = D$. This is to prevent the summand surfaces S_1 and S_2 from linking.

The Murasugi sum plays an essential role in the study of fibre surfaces, since it does in fact preserve fibredness in both directions, and the monodromy of a Murasugi sum can easily be computed from the monodromies of the summands:

Theorem ([St2, Ga2])**.** *Let S be a Murasugi sum of two Seifert surfaces S_1, S_2. Then S is a fibre surface if and only if both S_1 and S_2 are fibre surfaces. If φ, φ_1, φ_2 are the monodromies of S, S_1, S_2 respectively, and S_2 is glued on top of S_1 (that is, on the positive side), then $\varphi = \varphi_1 \circ \varphi_2$, where φ_1 and φ_2 are extended to maps on S by the identity.*

One implication was proved by Stallings in 1978 (the Murasugi sum of two fibre surfaces is a fibre surface), whereas the converse is due to Gabai in 1983. The Murasugi sum generalises the boundary connected sum of two Seifert surfaces as shown in Figure 1.3.

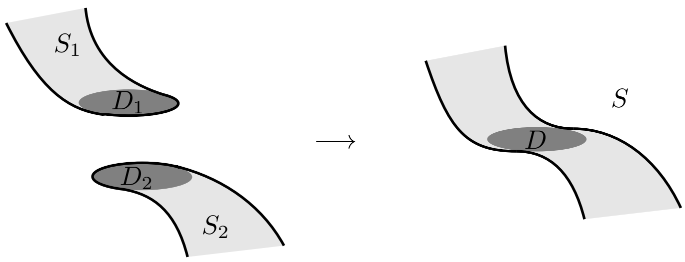

Figure 1.3: The connected sum of two Seifert surfaces is a Murasugi sum.

Gabai generalised the additivity of the genus to Murasugi sums, in the following sense: if S_1 and S_2 are minimal genus Seifert surfaces, then every Murasugi sum S of S_1 and S_2 is of minimal genus for the link ∂S. In addition, the Murasugi sum of two incompressible Seifert surfaces is incompressible [Ga1].

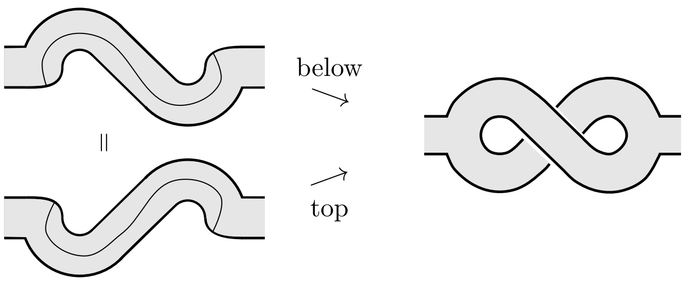

Figure 1.4: Hopf plumbing on top or from below a surface has the same effect.

Another special case of the Murasugi sum is *Hopf plumbing*, where one of the two summands is a positive or negative Hopf band (compare Figure 1.5). In order to define the plumbing of a Hopf band H onto a Seifert surface S, it suffices to specify an embedded interval $\alpha \subset S$ with endpoints on the boundary which is nowhere tangent to ∂S (an *arc* for short). The glueing disc $D \subset S$ is a rectangular neighbourhood of α in S. In fact, the resulting Seifert surface only depends on the sign of the Hopf band and the isotopy class of α in S. Figure 1.4 proves that it does not matter from which side of S the Hopf band is plumbed. However, the side from which the Hopf band is plumbed *does* in general have an effect on the monodromy when writing it as a composition of the summand monodromies.

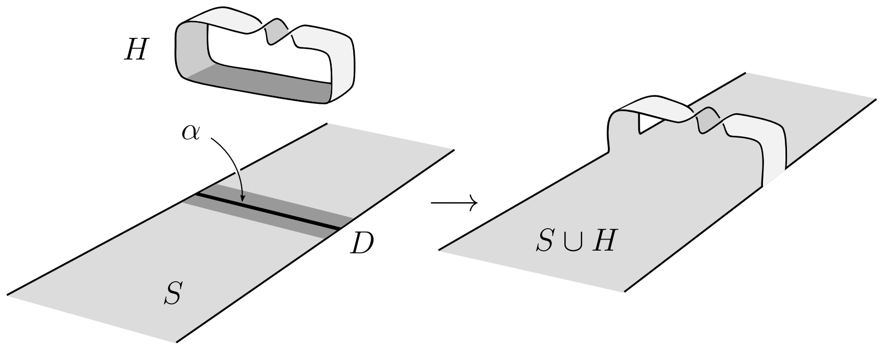

Figure 1.5: Plumbing a Hopf band H to a surface S along α.

Since the monodromy of a positive (negative) Hopf band is a positive (negative) Dehn twist along its core curve (this can be seen by dragging proper arcs, for example), the above theorem has the following corollary.

Corollary 1. *If S is given as an iterated plumbing of Hopf bands on the standard disc, the monodromy of S is a composition of Dehn twists along the core curves of these Hopf bands. The signs of the Dehn twists correspond to those of the Hopf bands.*

4 Examples

4.1 Links of singularities

Given an algebraic curve $C \subset \mathbb{C}^2$ defined by a two-variable polynomial $f \in \mathbb{C}[x, y]$, that is, $C = \{(x, y) \in \mathbb{C}^2 \mid f(x, y) = 0\}$, consider a small sphere $S^3_\varepsilon \subset \mathbb{C}^2 \cong \mathbb{R}^4$ centered at an isolated singular point $q \in C$. The intersection $L = C \cap S^3_\varepsilon$ is a link in the three-sphere S^3_ε. Near q, the curve C is a cone over L with cone point q, so the local topology of the pair $(C, \mathbb{C}^2)$ near q is completely determined by the link L. Milnor [Mil] proved that the argument map

$$\arg(f) = \frac{f}{|f|} : S^3_\varepsilon \setminus L \to S^1$$

is a locally trivial fibre bundle. Thus, every link L arising in this way is a fibred link, and the fibres of $\arg(f)$ are the associated fibre surfaces. In fact, each of these links of complex plane curve singularities is an iterated cable on the unknot, where the so-called Puiseux inequalities characterise the cabling coefficients that can appear (see [BK, EN]).

Torus links

If a, b are positive integers, the (a, b)-cable of the unknot is the *torus link* $T(a, b)$, realised by the polynomial $f = y^a - x^b$. Here, S^3_ε can be taken to be the sphere of radius one and center $(0, 0)$, so the link is explicitly given by

$$L = \{(x, y) \in \mathbb{C}^2 \mid x^b = y^a,\ |x|^2 + |y|^2 = 1\} \subset S^3.$$

In polar coordinates $(x, y) = (re^{i\alpha}, se^{i\beta}) \in L$, the conditions on x and y read

$$r^a = s^b, \quad r^2 + s^2 = 1, \quad a\beta - b\alpha \in 2\pi\mathbb{Z}.$$

Since the pair $(r, s) \in [0, 1]^2$ is uniquely determined by the first two equations, we have $L \subset (r \cdot S^1) \times (s \cdot S^1)$, so L is contained in the surface of a standard torus T^2. By the last condition, L consists of $d = \gcd(a, b)$ parallel curves that each make a/d turns, respectively b/d turns in the meridional and longitudinal direction. In other words, L represents the class (a, b) in $\mathbb{Z}^2 \cong H_1(T^2, \mathbb{Z})$. Note that orientations play a role here: the negative trefoil knot does not appear as the link of a singularity, and neither does the link obtained from $T(2, 2n)$ by switching one of its component's orientations. In fact, every link of a singularity is the closure of a positive braid (see [BK]).

The monodromy of a torus link is periodic

The following description of the monodromy of a torus link $T(a,b)$ is perhaps best suited to see its periodicity. We assume for simplicity that a,b are coprime. Observe that the defining polynomial $f = y^a - x^b$ is *quasihomogeneous* with weights $w(x) = a$, $w(y) = b$ and quasihomogeneous degree ab, that is,

$$f(\zeta^a x, \zeta^b y) = \zeta^{ab} f(x,y), \quad \forall \zeta \in \mathbb{C}.$$

In particular, taking $\zeta = \zeta_t = \exp(2\pi i t/ab)$, we conclude that

$$\varphi_t : \mathbb{C}^2 \to \mathbb{C}^2, \quad (x,y) \mapsto (\zeta_t^a x, \zeta_t^b y)$$

defines a smooth family of diffeomorphisms from the fibre $\{\arg(f) = 1\}$ to the fibre $\{\arg(f) = e^{2\pi i t}\}$. Up to adding a fractional Dehn twist along the fibre surface boundary (in order to have φ_t restrict to the identity on L), φ_t describes the monodromy flow. Since $(\varphi_1)^{ab}$ is the identity, the monodromy of $T(a,b)$ is (freely) isotopic to a periodic diffeomorphism, as is the monodromy of any quasihomogeneous singularity link.

While every fibred knot with periodic monodromy is necessarily a torus knot (see [Gr]), there exist fibred links with periodic monodromy which are not torus links. For example, the polynomial $xy^2 - x^5$ is quasihomogeneous with weights $w(x) = 1$, $w(y) = 2$ and quasihomogeneous degree 5, but the associated link at 0 is not a torus link. However, the monodromy of a singularity does always decompose into periodic pieces in the sense of Nielsen and Thurston's classification, which can be computed using a blow-up diagram of the singularity (compare [AC3]). A fibred link whose monodromy is pseudo-Anosov (such as the figure eight knot) can therefore not be the link of a singularity.

4.2 Positive braids and homogeneous braids

A braid with n strands is *positive* if it can be written as a product of the standard generators $\sigma_1, \ldots, \sigma_{n-1}$ of the braid group on n strands (all of which have to occur at least once), without using their inverses (see Figure 1.6, left, for an example). The i-th standard generator σ_i consists of the i-th strand overcrossing the $(i+1)$-st strand bottom-up, that is, a positive crossing. Closures of positive braids generalise links of singularities. For example, the torus link $T(a,b)$ can be written as the closure of $(\sigma_1 \cdots \sigma_{a-1})^b$. In turn, a positive braid is a special case of a homogeneous braid. A braid is *homogeneous* if each generator σ_i occurs to the same exponent $\varepsilon_i \in \{+1, -1\}$. In other words, a braid is homogeneous if all crossings between any two neighbouring strands are of the same sign. The following theorem is due to Stallings.

Theorem ([St2], Theorem 2)**.** *The closure of any homogeneous braid is fibred, and the associated fibre surface is an iterated plumbing of Hopf bands.*

Proof. Let β be a homogeneous braid and let S be the associated standard Seifert surface of the closure of β. That is, S is constructed by taking a vertical disc for each strand and glueing half-twisted bands, one for each crossing, as in Figure 1.6. Now consider the two leftmost vertical discs D_1, D_2 and between them the top two half-twisted bands b_1, b_2 (if there is only one band, we can reduce the number of strands by a destabilisation). Now connect the two intervals along which b_1 is attached to D_1 and D_2 by an arc $\alpha \subset D_1 \cup b_2 \cup D_2$. Observe that S is obtained from $S \setminus b_1$ by plumbing a Hopf band along α. Here we use that the two half twists of b_1 and b_2 add up to a full twist, which holds by the homogeneity of β. Finally, $S \setminus b_1$ is again the standard Seifert surface of a homogeneous braid, with fewer crossings, and the claim follows by induction on the braid word length. $\square$

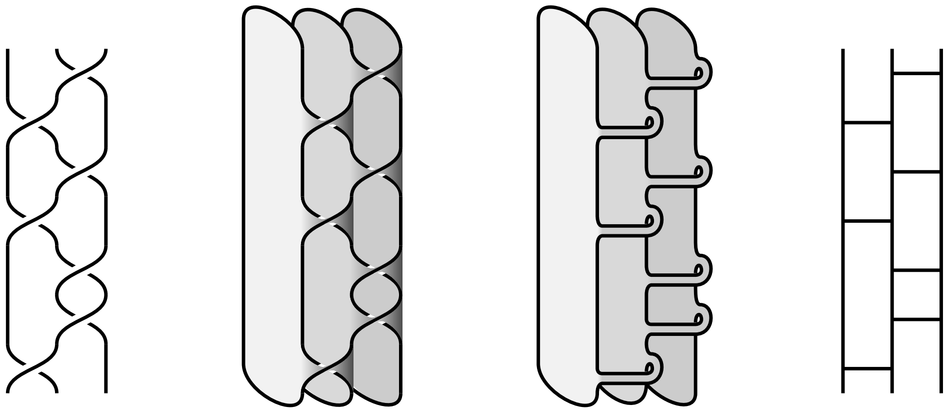

Figure 1.6: Left to right: the positive braid $\sigma_1\sigma_2^2(\sigma_1\sigma_2)^2$, its associated fibre surface, an isotopic copy thereof and the associated brick diagram.

A positive braid can be represented by a brick diagram obtained from a braid diagram (drawn vertically) by replacing every crossing by a short horizontal bar connecting the involved strands, as indicated in Figure 1.6. The brick diagram of a braid can be identified with a spine of the associated standard Seifert surface S, and the bricks correspond to positive Hopf bands in S. The monodromy of S is the product of positive Dehn twists along these Hopf bands, applied in the order from top left to bottom right brick, column by column.

4.3 Other examples

Another interesting family of fibre surfaces are the tree-like Hopf plumbings, given by plumbing Hopf bands together according to a finite plane tree. They are the protagonists of Chapter 5, and they also appear in Chapter 4.

The fibred links given in [MeMo] cannot be obtained by plumbing Hopf bands and are therefore not the closures of homogeneous braids, by the above theorem. Similarly, our examples in Chapter 3 (see Figure 3.1) cannot arise as homogeneous braids for this reason.

Examples of *non-fibred* links can easily be found by realising a non-monic Alexander polynomial. For example, the twist knots are not fibred except for the positive and negative trefoil knot and the figure eight knot. At least for twist knots with an even number of half twists, this can be seen without referring to the Alexander polynomial, as follows. If L is the twist knot with n full twists, a Seifert surface S for L can be constructed by plumbing a Hopf band to an annulus A with n full twists and unknotted core curve. Since L is not the unknot, S has the minimal genus (one). If it is a fibre surface, the annulus A is a fibre surface, since deplumbing preserves fibredness. Using Baader and Graf's characterisation of fibre surfaces in terms of proper arcs (described in Section 1), we only have to check whether the unique non-separating arc $\alpha \subset A$ can be dragged to the other side. If this were possible, α would trace out a disc D under the monodromy flow, and the interiors of D and A would not intersect. In particular, the linking number of the core curve of A with a curve interior to D and parallel to its boundary would have to be zero. This is only possible if $|n| = 1$, that is, if A is a positive or a negative Hopf band.

Chapter 2

The Dehn twist length of the monodromy

In this short chapter, we prove the following theorem, which gives a restriction to the surface mapping classes that are realised by monodromies of fibred knots.

Theorem 1. *Let K be a fibred knot of genus g with monodromy φ. Then any representation of φ as a product of Dehn twists involves at least $2g$ distinct factors.*

Recall from the previous chapter that the monodromy of an iterated Hopf plumbing involving n Hopf bands is a product of n Dehn twists. Hopf plumbing decreases the Euler characteristic of a surface by one. A surface of genus g with one boundary component has Euler characteristic $1 - 2g$. Therefore, if a fibred knot can be obtained from the unknot by Hopf plumbing, its monodromy can be written as a product of $2g$ Dehn twists. So the above bound on the number of Dehn twist factors is sharp.

The proof of Theorem 1 has three main ingredients. Firstly, the Alexander polynomial $\Delta_K(t) \in \mathbb{Z}[t]$ of a fibred knot K with fibre surface Σ equals the characteristic polynomial of the homological action $\varphi_* : H_1(\Sigma, \mathbb{R}) \to H_1(\Sigma, \mathbb{R})$ of the monodromy φ, that is

$$\Delta_K(t) = \chi_{\varphi_*}(t) = \det(t\,\mathrm{id} - \varphi_*).$$

Secondly, if K is a knot, $\Delta_K(1) = 1$. This does not hold for links of more than one component. In fact, $\Delta_L(1) = 0$ if L is a link with at least two components. Thirdly, the homological action of a Dehn twist T about a simple closed curve γ in Σ can be described as follows. Let α be any simple closed curve in Σ and denote by $a = [\alpha]$ and $c = [\gamma]$ the classes of α and γ in $H_1(\Sigma, \mathbb{R})$. Then we have (see [FM])

$$T_*(a) = a + i(a, c)c,$$

where $i(.\,,.)$ denotes the intersection pairing.

Proof of Theorem 1. Suppose to the contrary that φ could be written as a product of Dehn twists with $n < 2g$ distinct factors $T_1, \ldots, T_n$, where $T_j = T_{c_j}$ denotes a

Dehn twist about a simple closed curve c_j in Σ. Let

$$V = \langle [c_1], \ldots, [c_n] \rangle < H_1(\Sigma, \mathbb{R})$$

be the subspace of $H_1(\Sigma, \mathbb{R})$ spanned by the classes of the c_j. Consider the orthogonal complement of V in $H_1(\Sigma, \mathbb{R})$ with respect to the intersection form i,

$$V^{\perp} = \{x \in H_1(\Sigma, \mathbb{R}) \mid i(x, y) = 0 \quad \forall y \in V\}.$$

Since $\dim V \leqslant n < 2g = \dim H_1(\Sigma, \mathbb{R})$ and i is non-degenerate, we have $\dim V^{\perp} \geqslant 2g - n > 0$, i.e. there exists a non-zero vector $v \in V^{\perp}$. We claim that v is an eigenvector of $\varphi_* : H_1(\Sigma, \mathbb{R}) \to H_1(\Sigma, \mathbb{R})$ for the eigenvalue 1. Indeed, we may write v as a finite linear combination $v = \lambda_1 a_1 + \ldots + \lambda_r a_r$, with $\lambda_k \in \mathbb{R}$ and where $a_1, \ldots, a_r \in H_1(\Sigma, \mathbb{R})$ can be represented by simple closed curves $\alpha_1, \ldots, \alpha_r$ in Σ. For every fixed $k \in \{1, \ldots, r\}$, we have

$$\begin{aligned} T_k(v) &= \sum_{j=1}^{r} \lambda_j T_k(a_j) = \sum_{j=1}^{r} \lambda_j (a_j + i(a_j, c_k) c_k) \\ &= \sum_{j=1}^{r} \lambda_j a_j + i(\sum_{j=1}^{r} \lambda_j a_j, c_k) c_k = v + i(v, c_k) c_k = v + 0 \cdot c_k, \end{aligned}$$

hence $\varphi_*(v) = v$ as claimed. But then, $1 = \Delta_K(1) = \chi_{\varphi_*}(1) = 0$, a contradiction. $\square$

Chapter 3

Fibred links of genus one that cannot be obtained by plumbing Hopf bands

There are infinitely many (hyperbolic) fibred two component links of genus one that cannot be obtained by Hopf plumbing. We show this by finding a simple obstruction to the Alexander polynomials of links that are plumbings of three Hopf bands, as it was done by Melvin and Morton for genus two [MeMo].

Theorem 2. *Let c be an even integer, $|c| \geqslant 6$. Then none of the (two component, genus one, fibred, hyperbolic) links K_c depicted in Figure 3.1 can be obtained by Hopf plumbing. They can all be distinguished by their Alexander polynomials.*

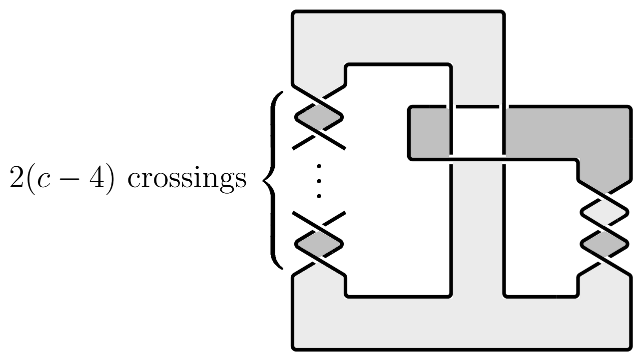

Figure 3.1: The link K_c with $\Delta_{K_c}(t) = t^3 - ct^2 + ct - 1$ with its fibre surface. If $2(c-4) < 0$, the corresponding crossings are negative.

Recall the definition of Hopf plumbing as the Murasugi sum of a Seifert surface S with a Hopf band H from Chapter 1. The resulting surface only depends on the free isotopy class of an arc $\alpha \subset S$ whose neighbourhood is the glueing disc used for the plumbing. Note that Hopf plumbing decreases the Euler characteristic of the surface by one and changes the number of link components by ± 1, depending on whether the endpoints of α lie on the same or on different components of ∂S.

Therefore, a surface obtained from the disc by successively plumbing at most three Hopf bands has either genus zero (and at most four boundary components) or genus one (and at most two boundary components). Since a Hopf band is homeomorphic to an annulus, there are only two types of arcs (up to isotopy). One class consists of arcs joining the two boundary circles and the other arcs have their endpoints on the same boundary component. Thus the only possible Hopf plumbings involving two bands are:

- The positive and negative trefoil fibre surfaces, obtained by plumbing two Hopf bands of the same sign.
- The figure eight knot fibre surface, obtained by plumbing a positive and a negative Hopf band.
- The connected sum of two Hopf bands (which arises by plumbing along an arc that separates off a disc).

In the first two cases, we obtain a torus with one boundary component, whereas the result is a pair of pants in the last case. The next step is to consider plumbing a third Hopf band, which yields already an infinite number of possible links. The following lemma describes the possible Alexander polynomials that can occur.

Lemma 2. *Suppose the surface $S \subset S^3$ is obtained by plumbing three Hopf bands and let $K = \partial S$ be the two component link bounded by S. Then the Alexander polynomial of K takes the form*

$$\Delta_K(t) = t^3 - ct^2 + ct - 1, \quad c = \pm p^2 \pm q^2 \pm pq + \eta,$$

where $\eta \in \{2, 4\}$ and $p, q \in \mathbb{Z}$ are either both zero or coprime.

Proof. We have to study a Hopf plumbing $S = S' \cup H$, where S' itself is built up of two plumbed Hopf bands. Denote the core curves of the three Hopf bands by $e_1, e_2 \subset S'$ and $e_3 \subset H$. Up to orientations to be chosen, (e_1, e_2, e_3) form a basis of $H_1(S, \mathbb{Z})$. In order to compute Δ_K for $K = \partial S$, it suffices to know the Seifert Matrix M with respect to this basis, whose entry M_{ij} is the linking number of e_i with e_j pushed slightly off S in the normal direction. So the diagonal entries will be ± 1, depending on the signs of the respective Hopf bands. If S' was the connected sum of two Hopf bands, e_1 and e_2 are disjoint and do not link, so $M_{12} = M_{21} = 0$ in this case. Otherwise, S' is a torus, and e_1, e_2 intersect transversely in one point. We may choose the orientations and the order of e_1 and e_2 in such a way that $M_{12} = 1$ and $M_{21} = 0$. The remaining entries involve linkings with e_3. If again S' is the connected sum, then S' is a pair of pants, so there are only nine free isotopy classes of arcs $\alpha = e_3 \cap S'$:

- For every boundary component, there is a class of arcs that cobound a disc with the boundary component.
- For every boundary component, there is a class of arcs that do not cobound a disc with the boundary.
- For every pair of distinct boundary components, there is exactly one isotopy class of arcs connecting them.

For the first six cases, one obtains a diagonal Seifert matrix M. (In the first three cases, e_3 can even be chosen to be disjoint from and not linked with e_1, e_2.) In the last three cases, α intersects each of e_1 and e_2 transversely in at most one point. So we have for each $i \in \{1, 2\}$ that $M_{i3}M_{3i} = 0$ and at most one of M_{i3} or M_{3i} is ± 1. However, we may choose the orientations of the e_i, the side of S' on which we plumb H as well as the orientation of S such that $M_{31} = M_{32} = 0$ and $M_{13}, M_{23} \in \{0, 1\}$. Indeed, changing the orientation of S results in transposing the Seifert matrix, so the Alexander polynomial will not be affected (up to factors of $\pm t^{\pm 1}$). Put differently, plumbing H on the other side results in the same link, and it is known that the fibre surface of a fibred link is unique up to isotopy (see [Ko]). On the other hand, if S' is a torus, then $\alpha = e_3 \cap S'$ is determined up to isotopy by two coprime integers p and q (namely the algebraic intersection numbers of α with e_1 and e_2). Again we may choose the orientation of S and the side where the Hopf band H is plumbed, so $M_{31} = M_{32} = 0$ and $M_{13} = p$, $M_{23} = q$ (because the pushed off curve e_3 will link the other curves exactly where they intersected on S). We therefore obtain the Seifert matrix

$$M = \begin{pmatrix} \pm 1 & \varepsilon & p \\ 0 & \pm 1 & q \\ 0 & 0 & \pm 1 \end{pmatrix},$$

where $\varepsilon \in \{0, 1\}$ is the intersection number of e_1 and e_2. We can now compute the Alexander polynomial of $K = \partial S$:

$$\Delta_K(t) = \det(tM - M^\top) = \pm(t-1)^3 - t(t-1)(\pm\varepsilon^2 \pm p^2 \pm q^2 \pm \varepsilon pq).$$

Normalising to have leading coefficient $+1$, we obtain

$$c = 3 \pm \varepsilon \pm p^2 \pm q^2 \pm \varepsilon pq$$

for the two middle coefficients. If S' was a pair of pants (that is, $\varepsilon = 0$), the entries p and q can each take the values 0 or 1, so we find a finite number of possible middle coefficients: $c \in \{1, 2, 3, 4, 5\}$, and they can all be written as claimed. Otherwise (S' is a torus, $\varepsilon = 1$), we have $c = \pm p^2 \pm q^2 \pm pq + \eta$ with $\eta = 3 \pm 1 \in \{2, 4\}$ as claimed. □

Remark 1. One obtains $c = 1, 3, 5$ for the following links in Rolfsen's notation: L4a1 ($c = 1$), L5a1 (Whitehead link, $c = 3$), L6a1 ($c = 5$), whereas $c = 2, 4$ can be achieved by taking the connected sum of the figure eight knot or a (positive or negative) trefoil knot with a (positive or negative) Hopf link. For the connected sum of three Hopf bands, $c = 3$ as well.

Corollary 2. *Assume K is a link with $\Delta_K(t) = t^3 - ct^2 + ct - 1$ such that $c \in 2\mathbb{Z}$ and $|c| \geqslant 6$. Then K cannot be obtained by plumbing three Hopf bands.*

Proof. Suppose to the contrary K were a plumbing of three Hopf bands. Then the middle coefficient c of Δ_K from Lemma 2 reduces to $p^2 + q^2 + pq \mod 2$. If $pq = 0$, then $|c| < 6$. Otherwise p, q are coprime and nonzero, whence p and q cannot be both $0 \mod 2$. But then, $c \equiv p^2 + q^2 + pq \equiv 1 \mod 2$, so c is odd. $\square$

Given any monic polynomial $f(t) \in \mathbb{Z}[t]$ of odd degree with $f(0) = 0$ and $f(t) = -t^{\deg f} f(t^{-1})$, Stoimenow constructs in [Sto] explicitly a fibred hyperbolic two component link K that has Alexander polynomial $f(t)$. Using his algorithm, one obtains the fibred links K_c with Alexander polynomials $t^3 - ct^2 + ct - 1$ depicted in Figure 3.1. These links are fibred, hyperbolic and of genus one by Stoimenow's work. By Corollary 2, they cannot be obtained as plumbings of three Hopf bands if c is even and $|c| \geqslant 6$. We thus obtain the theorem stated at the beginning:

Theorem 2. *Let c be an even integer, $|c| \geqslant 6$. Then none of the (two component, genus one, fibred, hyperbolic) links K_c depicted in Figure 3.1 can be obtained by Hopf plumbing. They can all be distinguished by their Alexander polynomials.*

It would be interesting to study geometric properties of this family of fibred links; for example:

Question. How does the geometric dilatation of the (pseudo-Anosov) monodromy behave with respect to its homological dilatation? Can the monodromy be written as a product of three Dehn twists?

Chapter 4

Cutting arcs for torus links and trees

The content of this chapter will be published in the form of an article in the Bulletin de la Société Mathématique de France and is also available on the arXiv preprint server [Mi1].

Among all torus links, we characterise those arising as links of simple plane curve singularities by the property that their fibre surfaces admit only a finite number of cutting arcs that preserve fibredness. The same property allows a characterisation of Coxeter-Dynkin trees (i.e., A_n, D_n, E_6, E_7 and E_8) among all positive tree-like Hopf plumbings.

1 Introduction

Let $L \subset S^3$ be a fibred link with fibre surface S. Cutting S along a properly embedded interval α (an *arc* for short) results in another Seifert surface S' for another link $\partial S' = L'$. If L' is again a fibred link with fibre S', we say that α *preserves fibredness*. For example, α could be the spanning arc of a plumbed Hopf band, and cutting along α amounts to deplumbing that Hopf band. In [BIRS], Buck et al. give a simple criterion for when an arc preserves fibredness in terms of the monodromy $\varphi\colon S \to S$. As a corollary, they prove that each of the torus links of type $T(2,n)$ admits only a finite number of such arcs up to isotopy. It turns out that among torus links, this is an exception:

Theorem 3. *Let $n, m \geqslant 4$ or $n = 3, m \geqslant 6$. Then the fibre surface S of the torus link $T(n,m)$ contains infinitely many homologically distinct cutting arcs preserving fibredness.*

The remaining torus links $T(2,n)$, $T(3,3)$, $T(3,4)$ and $T(3,5)$ happen to be exactly those torus links that can also be obtained as plumbings of positive Hopf bands according to a finite tree (namely $T(2,n)=A_{n-1}$, $T(3,3)=D_4$, $T(3,4)=E_6$ and $T(3,5) = E_8$), where vertices correspond to positive Hopf bands and edges indicate plumbing.

Theorem 4. *Let S be the fibre surface obtained by plumbing positive Hopf bands according to a finite tree T. There are, up to isotopy, only finitely many cutting arcs in S preserving fibredness, if and only if T is one of the Coxeter-Dynkin trees A_n, D_n, E_6, E_7 or E_8.*

To prove the "only if" part of Theorem 4, we consider orbits of a fixed arc under the monodromy to produce families of arcs that preserve fibredness. The basic idea is that such an orbit is infinite if the monodromy has infinite order. For example, we show that in fact every (prime) positive braid link with pseudo-Anosov monodromy admits infinitely many non-isotopic arcs preserving fibredness. This suggests the following question: is it true that among all (non-split prime) positive braid links, the ADE plane curve singularities are exactly those that admit just a finite number of fibredness preserving arcs up to isotopy?

Plan of the article

We use the shorthand *ADE links* to refer to the links of the positive tree-like Hopf plumbings according to the trees A_n, D_n, E_6, E_7 or E_8. The subsequent section combines a criterion on arcs to preserve fibredness from [BIRS] with the property of monodromies of positive Hopf plumbed surfaces to be right-veering. This allows for the following simple test for an arc to preserve fibredness, in our situation: an arc preserves fibredness if and only if it does not intersect its image under the monodromy (up to free isotopy).

Section 3 contains descriptions of the fibre surfaces and the monodromies of the links we consider (torus links and the ADE links). Alongside, we give a constructive proof of Theorem 3.

In Section 4, we explain the idea of proof for the finiteness result that provides the "if" part of Theorem 4, and list the fibred links obtained by cutting the fibre surfaces of the ADE links along an arc in Table 4.1.

Section 5 accounts for the cases where the monodromy has infinite order. This concerns in particular the positive tree-like Hopf plumbings that correspond to trees different from the ADE trees and settles the "only if" part of Theorem 4.

At the beginning of Section 6, we set up the notation and methods needed for the proof of the finiteness part of Theorem 4, which we split into Proposition 1 (concerning torus links) and Proposition 2 (concerning tree-like Hopf plumbings). The rest of that section is devoted to the proofs of these propositions.

2 Right-veering surface diffeomorphisms and cutting arcs that preserve fibredness

In the sequel we would like to make statements on the relative position of two arcs α, β in a surface S with boundary (that is, α, β are embedded intervals with endpoints on the boundary of S that are nowhere tangent to ∂S). The following definition will simplify matters.

Definition 5. Let S be an oriented surface with boundary and let $\alpha, \beta \subset S$ be two arcs. A property $P(\alpha, \beta)$ is said to hold *after minimising isotopies on α and β*, if $P(\tilde{\alpha}, \tilde{\beta})$ holds, where $\tilde{\alpha}$ and $\tilde{\beta}$ are obtained from α, β by two isotopies (fixed at the endpoints) that minimise the geometric number of intersections between the two arcs.

The remainder of this section will recall the fact that every positive braid link (that is, the closure of a braid word consisting only of the positive generators of the braid group, without their inverses) is fibred and has so-called *right-veering* monodromy (see below for a definition). The torus links $T(n, m)$ provide examples, since they can be viewed as the closures of the positive braids $(\sigma_1 \cdots \sigma_{n-1})^m$, where the σ_i denote the (positive) standard generators of the braid group.

Definition 6 (see [HKM], Definition 2.1)**.** Let S be an oriented surface with boundary and $\varphi : S \to S$ a diffeomorphism that restricts to the identity on ∂S. Then φ is called *right-veering* if for every arc $\alpha : [0,1] \to S$, the vectors $((\varphi \circ \alpha)'(0), \alpha'(0))$ form an oriented basis after minimising isotopies on α and $\varphi \circ \alpha$.

This means basically that arcs starting at a boundary point of S get mapped "to the right" under a right-veering diffeomorphism $\varphi : S \to S$.

By Stallings' theorem mentioned in Section 4.2 of Chapter 1, every positive braid can be obtained as an iterated plumbing of positive Hopf bands [St2]. Since a Hopf band is a fibre and plumbing preserves fibredness, every positive braid link is fibred. Moreover the monodromy is a product of positive Dehn twists, since the monodromy of a (positive) Hopf band is a (positive) Dehn twist and the monodromy of a plumbing is the composition of the monodromies of the plumbed surfaces (see [Ga2]). A product of positive Dehn twists is right-veering [HKM]. So we conclude that every positive braid link is fibred with right-veering monodromy. Together with a theorem by Buck et al., this property implies the following simple geometric criterion for when an arc preserves fibredness.

Theorem 5 (compare Theorem 1 in [BIRS])**.** *Let L be a fibred link with fibre surface S and right-veering monodromy $\varphi : S \to S$. Then, a cutting arc α preserves fibredness if and only if $\alpha \cap \varphi(\alpha) = \partial\alpha$ after minimising isotopies on α and $\varphi(\alpha)$.*

Proof. This is a special case of Theorem 1 in [BIRS], saying that the arc α preserves fibredness if and only if α is *clean and alternating* or *once unclean and non-alternating* (see Figure 4.1), without the assumption on φ to be right-veering. But for a right-veering map, every arc is alternating, by definition. Finally, α is clean if and only if $\alpha \cap \varphi(\alpha) = \partial\alpha$ after minimising isotopies on α and $\varphi(\alpha)$. $\square$

Figure 4.1: (Adapted from [BIRS])

Remark 2. An arc α is clean if and only if $\varphi^k(\alpha)$ is clean, for all $k \in \mathbb{Z}$. This is clear since $\alpha \cap \varphi(\alpha) = \partial\alpha$ after minimising isotopies if and only if $\varphi^k(\alpha) \cap \varphi^{k+1}(\alpha) = \partial\alpha$ after minimising isotopies. Similarly, if $\tau : S \to S$ is a homeomorphism such that $\varphi \circ \tau \circ \varphi = \tau$, then α is a clean arc if and only if $\alpha' = \tau(\varphi(\alpha))$ is. Indeed, $\alpha \cap \varphi(\alpha) = \partial\alpha \;\Leftrightarrow\; \tau(\alpha) \cap \tau(\varphi(\alpha)) = \partial\alpha' \;\Leftrightarrow\; \varphi(\tau(\varphi(\alpha))) \cap \tau(\varphi(\alpha)) = \partial\alpha' \;\Leftrightarrow\; \varphi(\alpha') \cap \alpha' = \partial\alpha'$.

3 Monodromy of torus links, E_7 and D_n.

The links that correspond to the trees A_n, E_6 and E_8 are torus links, namely $A_{n-1} = T(2,n)$, $E_6 = T(3,4)$ and $E_8 = T(3,5)$. Together with D_4, which is $T(3,3)$, these form the intersection between torus links and positive tree-like Hopf plumbings. In order to prove the "if" direction of Theorem 4, it therefore suffices to study torus links, E_7 and the D_n family.

The monodromies $\varphi : S \to S$ of the links in question are particular examples of *tête-à-tête twists*, a notion invented by A'Campo and further developed by Graf in his thesis [Gr]. This means that there exists a φ-invariant spine $\Gamma \subset S$, called *tête-à-tête graph.* Cutting S along the tête-à-tête graph results in finitely many annuli, on which φ descends to certain twist maps. More precisely, each of these annuli has one component of ∂S as one boundary circle and a cycle consisting of edges of Γ as the other. φ fixes ∂S pointwise and rotates the edge-cycles by some number ℓ of edges. The number $\ell \in \mathbb{Z}$ is called the *twist length* of the corresponding boundary annulus. After an isotopy (fixing the boundary of S), we may therefore assume that φ is periodic except on some annular neighbourhoods of ∂S. It is thus easy to understand the effect of φ on an arc α, up to isotopy, given the combinatorics of the action of φ on Γ and the amount of twisting on each annulus. Note that tête-à-tête twists define periodic mapping classes in the sense that some power is freely isotopic to the identity. However, this isotopy cannot be taken to be fixed on the boundary of S.

In a way dual to the tête-à-tête graph, we will find in each case a finite set of disjoint arcs that are permuted by φ and which decompose S into finitely many polygons, one for each vertex of Γ. The combinatorics of how these polygons are permuted will be used to prove Theorems 3 and 4.

Monodromy of torus links

The fibre surface S of the torus link $T(n,m)$ can be constructed as thickening of a complete bipartite graph on n and m vertices in the following way, as in Figure 4.2. Arrange n collinear points $a_1,\dots,a_n$ (in this order) in a plane and, similarly, another m points $b_1,\dots,b_m$ along a line parallel to the a_i. Connect a_i and b_j by a straight segment k_{ij}, for every $i \in \{1,\dots,n\}$, $j \in \{1,\dots,m\}$. Avoid intersections between the segments by letting k_{ij} pass slightly under k_{pq} if $i > p$ and $j < q$ (in a slight thickening of the plane containing the points a_i and b_j). Use the blackboard-framing to thicken a_i, b_j, k_{ij} to disks A_i, B_j and bands K_{ij}. Choose the thickness of the bands K_{ij} so that they do not intersect outside the disks A_i, B_j. It can be seen that $S := \bigcup_i A_i \cup \bigcup_j B_j \cup \bigcup_{i,j} K_{ij} \subset \mathbb{R}^3 \subset S^3$ is isotopic to the minimal Seifert surface of $T(n,m)$ in S^3 (compare [Ba]). In addition, the

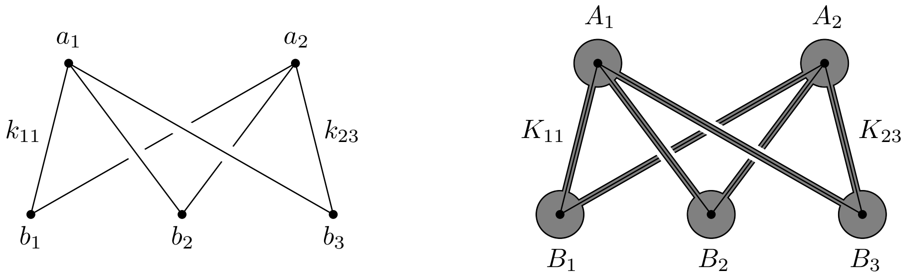

Figure 4.2: The complete bipartite graph on 2 and 3 vertices and blackboard framed thickening.

monodromy $\varphi : S \to S$ is a tête-à-tête twist along the above graph. In each of the $\gcd(n,m)$ complementary annuli, φ fixes ∂S pointwise and rotates the edge-cycles two edges to the right with respect to the orientation of S. Using this description, it is possible to see that φ acts on the graph as follows: $\varphi(a_i) = a_{i-1}$, $\varphi(b_j) = b_{j+1}$,

$\varphi(k_{ij}) = k_{i-1,j+1}$, where the indices i, j are to be taken modulo n, m respectively. A subarc of α that travels near k_{ij} will be mapped to a subarc of $\varphi(\alpha)$ that travels near $k_{i-1,j+1}$. The edges k_{ij} induce a decomposition of ∂A_i into circular arcs lying between points of the form $k_{ij} \cap \partial A_i$ (and the same for B_j). If $n, m \geqslant 3$, it is hence meaningful to speak of points on ∂A_i *between* k_{ij} and $k_{i,j+1}$.

Theorem 3. *Let $n, m \geqslant 4$ or $n = 3, m \geqslant 6$. Then the fibre surface S of the torus link $T(n, m)$ contains infinitely many homologically distinct cutting arcs preserving fibredness.*

Proof. For $n, m \geqslant 4$ consider the following arcs in S, using the notation from above (compare Figure 4.3):

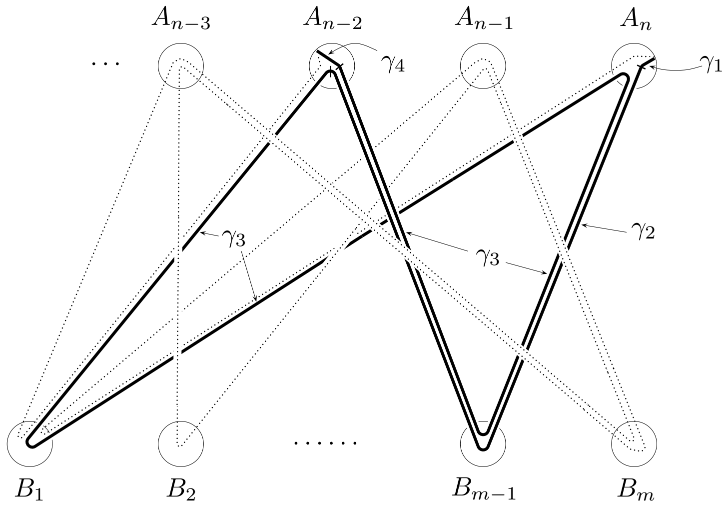

Figure 4.3: The arc $\alpha_1 = \gamma_1 * \gamma_2 * \gamma_3 * \gamma_4$ (solid line) and its image under the monodromy (dotted line). Note that these two arcs do not intersect, except at their endpoints.

- Let γ_1 be a straight segment starting at a point of ∂A_n between k_{n1} and k_{nm} and ending at the vertex a_n.
- Let γ_2 start at a_n, follow the edges $k_{n,m-1}$ and $k_{n-2,m-1}$, thus ending at a_{n-2}.
- γ_3 starts at a_{n-2}, runs along $k_{n-2,1}$, k_{n1}, $k_{n,m-1}$, $k_{n-2,m-1}$ and ends again at a_{n-2}.
- γ_4 is a straight segment from a_{n-2} to a point of ∂A_{n-2} between $k_{n-2,1}$ and $k_{n-2,m}$.

From $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ we can build an infinite family $(\alpha_r)_{r\in\mathbb{N}}$ of arcs in S, taking $\alpha_r = \gamma_1 * \gamma_2 * \underbrace{\gamma_3 * \ldots * \gamma_3}_{r-\text{times}} * \gamma_4$. Here, $*$ denotes concatenation of paths. Replacing the r consecutive copies of γ_3 by r parallel copies, the α_r can be thought of as embedded arcs. It is now easy to check that α_r and its image $\varphi_*\alpha_r$ under the monodromy φ have only their endpoints in common. In addition, φ is right-veering since it is a composition of right-handed Dehn twists (compare Section 4.2 of Chapter 1). Using Theorem 5 it follows that each α_r preserves fibredness. Finally, the α_r are homologically pairwise distinct. This can be seen in the following way: let $[c] \in H_1(S,\mathbb{Z})$ be the cycle represented by a simple closed curve c whose image is $k_{nm} \cup k_{n-1,m} \cup k_{n-1,m-1} \cup k_{n,m-1}$. After an isotopy, c will intersect α_r transversely in $r+1$ points. Now, the linear form on $H_1(S,\partial S,\mathbb{Z})$ that sends α to $i(c,\alpha)$, the number of intersections with c (counted with signs), defines an element c^* of $H^1(S,\partial S,\mathbb{Z})$ such that $c^*(\alpha_r) = r+1$, hence the claim.

If $n = 3, m \geqslant 6$, take the following arcs (compare Figure 4.4):

- γ_1 is a straight segment from a point of ∂A_3 between k_{31} and k_{3m} to a_3.
- γ_2 starts at a_3, follows the edges $k_{3,m-1}$ and $k_{2,m-1}$, thus ending at a_2.
- γ_3 starts at a_2, follows k_{23}, k_{13}, k_{11}, k_{31}, $k_{3,m-1}$ and $k_{2,m-1}$, ending at a_2.
- γ_4 is a straight segment from a_2 to a point of ∂A_2 between k_{22} and k_{23}.

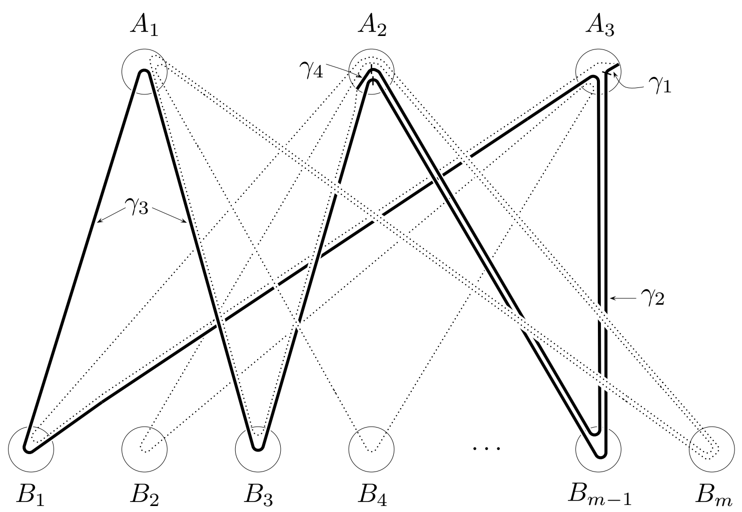

Figure 4.4: The arc α_1 (solid line) and its image under the monodromy (dotted line) for a $T(3,m)$ torus link, $m \geqslant 6$. Again, the two arcs do not intersect.

As above, we get a family $(\alpha_r)_{r\in\mathbb{N}}$ of homologically distinct arcs preserving fibredness, where $\alpha_r = \gamma_1 * \gamma_2 * \underbrace{\gamma_3 * \ldots * \gamma_3}_{r-\text{times}} * \gamma_4$, using the curve with image $k_{3m} \cup k_{1m} \cup k_{1,m-2} \cup k_{3,m-2}$ to distinguish the α_r. □

Monodromy of E_7 and D_n

In order to obtain a similar model for the fibre surface S of E_7 or D_n, start with two disjoint planar disks D, D' in $\mathbb{R}^3$ and connect them by half twisted bands $b_1, \ldots, b_n$, where $n = 7$ in the case of E_7. The embedded surface $S' = D \cup D' \cup b_1 \cup \ldots \cup b_n$ is then a fibre surface for the $T(2,n)$ torus link. Let $p \in \partial D$ be a point between b_2 and b_3 in the case of D_n, respectively between b_3 and b_4 in the case of E_7. Let I be an arc in D from a point of ∂D between b_1 and b_n to p. Finally, define S to be the surface obtained from S' by plumbing a positive Hopf band along I below the surface S'. Denote the core curve of that plumbed Hopf band by e_1 (so $e_1 \cap S' = I$). Each pair of consecutive bands b_i, b_{i+1}, $1 \leqslant i \leqslant n$, gives rise to a closed curve e_{i+1} that runs from D to D' through b_i and back to D through b_{i+1}. The incidence graph for the system of curves $e_1, \ldots, e_n$ in S is exactly the respective Coxeter-Dynkin tree E_7 or D_n (compare Figure 4.5). The e_i are core

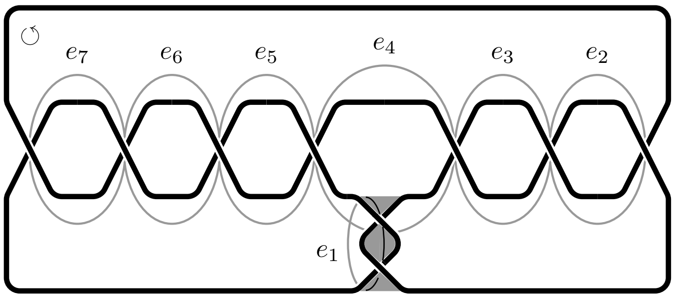

Figure 4.5: E_7 fibre surface with homology basis coming from the plumbing tree. φ is the product of the right handed Dehn twists on the cuves e_i.

curves of positive Hopf bands and S is a tree-like positive Hopf plumbing according to the respective tree. In particular, the monodromy φ of S is the product of the right handed Dehn twists about the curves $e_2, e_3, \ldots, e_n, e_1$, in this order. Just as in the case of torus links, we will find a finite number of disjoint arcs in S that are permuted (up to free isotopy) by φ and such that these arcs cut S into polygons. For E_7, let k_1 be the spanning arc of b_7, and let $k_{i+1} = \varphi^i(k_1)$, $i = 1, \ldots, 8$, up to free isotopy (compare Figure 4.6). Up to free isotopy, $\varphi(k_9) = k_1$. This can

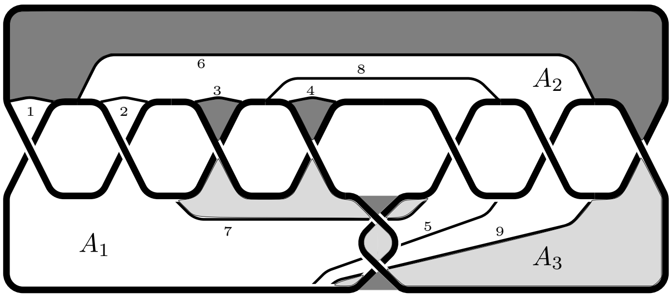

Figure 4.6: Decomposition of the surface into three hexagons A_1, A_2, A_3. Hexagon A_3 is shaded grey. The monodromy permutes the intervals k_i (marked $1, 2, \dots, 9$) cyclically.

be seen by applying the Dehn twists about the e_j to the k_i, as described above. Another more visual way to see this is via *dragging arcs*. Imagine the arcs k_i to be elastic bands whose ends are attached to the surface boundary and whose interiors are pushed slightly off the surface into the positive normal direction. Applying the monodromy φ amounts to dragging the arc through the complement of S to the negative side of the surface, while its endpoints stay fixed on ∂S. Since we are only interested in the position of $\varphi(k_i)$ up to free isotopy, the endpoints of the dragging arc may move freely along ∂S during that process. Let A_1, A_2, A_3 be the three disc components of $S \setminus \bigcup_{i=1}^{9} k_i$. The boundary of A_j alternates between parts of ∂S and the k_i. We choose the order as in Figure 4.7, where the components of $\partial A_j \cap \partial S$ are shrunk to points.

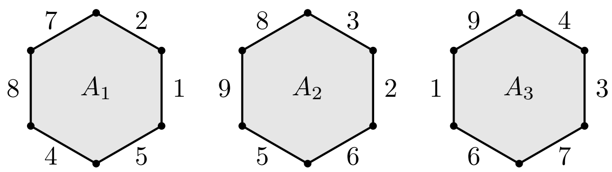

Figure 4.7: Edges with the same label are glued. The monodromy sends A_j to A_{j+1}, indices taken modulo 3, such that edge k_i is sent to edge k_{i+1}, modulo 9.

Examination of the action of φ on the k_i reveals that the A_i are cyclically permuted by φ, in the order $A_1 \mapsto A_2 \mapsto A_3 \mapsto A_1$. In Figure 4.7, the A_i are drawn in such a way that $A_1 \mapsto A_2 \mapsto A_3$ by translation to the right, and A_3 is mapped to A_1 by a translation, followed by a clockwise rotation through 1/3. To obtain the tête-à-tête graph Γ, put a vertex in the middle of each hexagon A_j and connect them by edges through the center of every k_i, connecting the vertices of the adjacent hexagons. The tête-à-tête twist lengths on the two boundary annuli are 1 and 2, respectively.

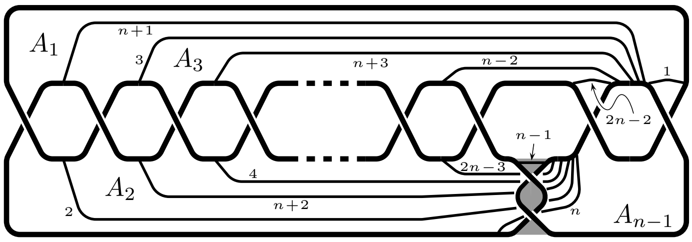

Figure 4.8: Decomposing arcs $k_1, \dots, k_{2n-2}$ on the fibre surface of D_n for odd n.

For the case of D_n, n odd, take k_1 to be the spanning arc of b_1 and let $k_{i+1} = \varphi^i(k_1)$, $i = 1, \dots, 2n-3$. As before, we have $\varphi(k_{2n-2}) = k_1$, and the k_i decompose S into $n-1$ disks $A_1, \dots, A_{n-1}$, as in Figure 4.8. In Figure 4.10 (top), φ maps $A_1 \mapsto A_2 \mapsto \cdots \mapsto A_{n-1}$ by right translations and sends A_{n-1} back to A_1 by a rotation of 180°.

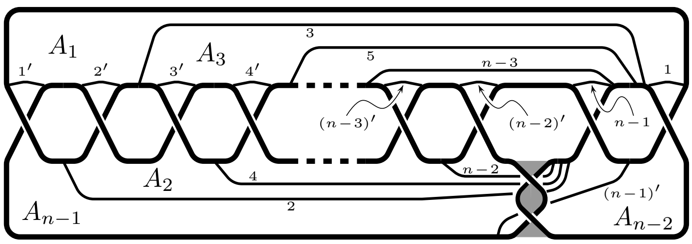

Figure 4.9: Decomposing arcs $k_1, \dots, k_{n-1},\ k'_1, \dots, k'_{n-1}$ on the fibre surface of D_n for even n.

If n is even, we use two orbits of intervals instead of one: define $k_1, \dots, k_{n-1}$ and $k'_1, \dots, k'_{n-1}$ by letting k_1, k'_1 be the spanning arcs of b_1, b_n respectively and $k_{i+1} = \varphi^i(k_1)$, $k'_{i+1} = \varphi^i(k'_1)$. Again, the union of the k_i and the k'_i decomposes S into disks $A_1, \dots, A_{n-1}$ (see Figure 4.9). In Figure 4.10 (bottom), the monodromy maps $A_1 \mapsto A_2 \mapsto \cdots \mapsto A_{n-1} \mapsto A_1$ by translations. The tête-à-tête graphs for

D_n have one vertex at the center of each square and edges pass through the k_i and k_i'. Twist lengths on the boundary annuli are 1, $n-2$ for odd n, and 1, 2, $\frac{n}{2}-1$ for even n.

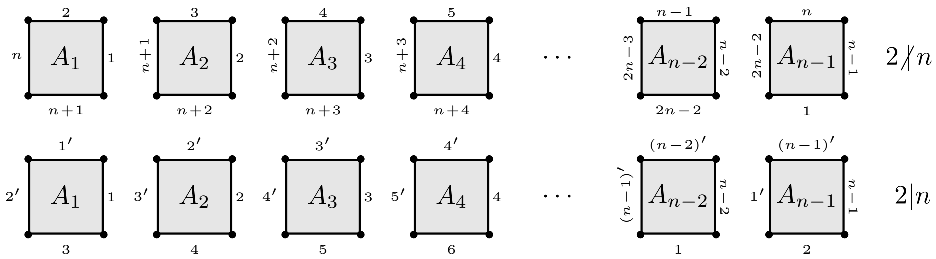

Figure 4.10: Description of the monodromy of D_n, for odd n (top) and for even n (bottom).

4 The finite cases.

In [BIRS, Corollary 2], Buck et al. show that $T(2,n)$ admits only finitely many arcs preserving fibredness (up to isotopy). More precisely, they show that every clean arc is isotopic (free on the boundary) to an arc that is contained in one of the disks A_1, A_2 from the above description of the monodromy of torus links. Apart from this infinite family of torus links, there are only three more torus links with just a finite number of arcs that preserve fibredness:

Proposition 1. *The torus links $T(3,3)$, $T(3,4)$ and $T(3,5)$ admit, up to isotopy (free on the boundary), only a finite number of cutting arcs that preserve fibredness.*

For positive tree-like Hopf plumbed surfaces we similarly obtain:

Proposition 2. *The positive tree-like Hopf plumbings associated to any of the Coxeter-Dynkin trees A_n, D_n, E_6, E_7 or E_8 admit, up to isotopy (free on the boundary), only a finite number of cutting arcs that preserve fibredness.*

The proofs of Propositions 1 and 2 are rather technical and will be given in Section 6. Nevertheless, the idea is very simple: let S be the fibre surface of any torus link $T(n,m)$, given as thickening of a complete bipartite graph on n and m vertices, or of D_n or E_7, as described in Section 3. An arc $\alpha \subset S$ is determined up to isotopy by its endpoints and by the sequence of bands K it passes through. Now start listing all possible such sequences that yield clean arcs, for increasing length of the sequence. In order to prove finiteness of this list, we use three Lemmas, also given in Section 6. The intuitive meaning of Lemma 3 and Lemma 4 can be phrased as follows: if α and $\varphi(\alpha)$ intersect and this intersection seemingly cannot be removed by an isotopy, then α is indeed unclean. Lemma 5 asserts that a clean arc cannot stay in the complement of the graph for a distance of more than ℓ consecutive bands, where ℓ is the tête-à-tête twist length on the corresponding boundary annulus (for example, $\ell = 2$ for all torus links).

This is made precise in Section 6, using a notion of *arcs in normal position* (cf. Definition 7). Along with this case-by-case analysis, one can find all possible fibred links obtained from $A_{n-1} = T(2,n)$, $D_4 = T(3,3)$, D_n, $E_6 = T(3,4)$, E_7 and $E_8 = T(3,5)$ by cutting along an arc. Consult Table 4.1 for a complete list.

From	one obtains by cutting along a clean arc
$T(2,n)$	$T(2,n-1)$, $T(2,m_1)\#T(2,m_2)$ for $m_1+m_2=n$
$T(3,3)$	$T(2,4)$, $(T(2,2)\#T(2,2)\#T(2,2))$[*1]
$T(3,4)$	D_5, $T(2,6)$, $T(2,5)\#T(2,2)$, $T(2,3)\#T(2,3)\#T(2,2)$, $(T(2,3)\#T(2,2)\#T(2,3))$[*2]
$T(3,5)$	E_7, D_7, $T(2,8)$, $(D_5\#T(2,3))$[*3], $T(2,5)\#T(2,4)$, $T(2,7)\#T(2,2)$, $T(3,4)\#T(2,2)$, $T(2,5)\#T(2,3)\#T(2,2)$, $(T(2,5)\#T(2,2)\#T(2,3))$[*4]
D_n	$T(2,n)$, D_{n-1}, $D_{m_1}\#T(2,m_2)$ for $m_1+m_2=n$, $T(2,2)\#T(2,2)\#T(2,n-2)$
E_7	E_6, D_6, $T(2,7)$, $T(2,4)\#T(2,2)\#T(2,3)$, $T(2,6)\#T(2,2)$, $T(2,5)\#T(2,3)$

$K_1\#K_2$ denotes the connected sum of K_1 and K_2, D_n denotes the closure of the braid $\sigma_1^{n-2}\sigma_2\sigma_1^2\sigma_2$, $n \geqslant 3$, and E_n denotes the closure of the braid $\sigma_1^{n-3}\sigma_2\sigma_1^3\sigma_2$, $n = 6, 7, 8$.

[*1] chain of four successive unknots.

[*2] both Hopf link components are summed to one trefoil knot each.

[*3] both possible sums appear (trefoil summed with the unknot component of D_5 as well as trefoil summed with the trefoil component of D_5).

[*4] one component of the Hopf link in the middle is summed to $T(2,5)$ and the other is summed to the trefoil.

Table 4.1: Fibred links obtained from the exceptional torus links by cutting along an arc.

5 Arcs for links with infinite order monodromy

Theorem 6. *Let S be a fibre surface whose monodromy $\varphi : S \to S$ is pseudo-Anosov and right-veering. Assume that a Hopf band can be deplumbed from S. Then S contains infinitely many non-isotopic cutting arcs preserving fibredness.*

Proof. By Theorem 5, an arc in S preserves fibredness if and only if it is clean, since the monodromy φ is right-veering by assumption. Let S' be a surface obtained from S by deplumbing a Hopf band and denote by c the core curve of that Hopf band. S' is again a fibre surface because deplumbing preserves fibredness, and φ is the monodromy of S' followed by a Dehn twist along c (compare [Ga2]). Let α be an arc dual to c that does not enter S'. Then, applying φ, only the Dehn twist along c affects α (in particular, the deplumbed Hopf band is necessarily positive since φ is right-veering). It follows that α is clean, and therefore $\varphi^n(\alpha)$ is also clean by Remark 2. Since φ is pseudo-Anosov and α is essential, the length of $\varphi^n(\alpha)$ (with respect to an auxiliary Riemannian metric) grows exponentially as n tends to infinity (compare [FM, Section 14.5]). In particular, the arcs $\varphi^n(\alpha)$ are pairwise non-isotopic and clean. □

Corollary 3. *Let S be a surface obtained by iterated plumbing of positive Hopf bands and suppose the monodromy $\varphi : S \to S$ is pseudo-Anosov. Then S contains infinitely many non-isotopic cutting arcs preserving fibredness.*

Proof. The monodromy φ is a composition of right Dehn twists along the core curves of the Hopf bands used for the construction of S as a Hopf plumbing. This implies that φ is right-veering [HKM, Lemma 2.5]. Now apply Theorem 6 to the last plumbed Hopf band. □

Proposition 3. *Let S be a surface obtained by plumbing positive Hopf bands according to a tree other than A_n, D_n, E_6, E_7 and E_8. Then S contains infinitely many non-isotopic cutting arcs preserving fibredness.*

Proof. The proposition is basically a consequence of A'Campo's work on slalom knots [AC1]. These are tree-like Hopf plumbings whose plumbing tree is obtained from a rooted tree by subdividing every edge except the root edge. A'Campo proved that a slalom knot has pseudo-Anosov monodromy if and only if the corresponding plumbing tree is different from A_{2n}, E_6 and E_8 [AC1, Theorem 1]. In fact, his argument carries over to general trees except the ones of type ADE and the affine Coxeter-Dynkin trees $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$ (compare [AC1, AC2, Th1]). Up to these exceptions, the statement can therefore be seen as a special case of Corollary 3. In the affine cases, the monodromy φ is not pseudo-Anosov, but still has infinite order. In fact, the corresponding links are $T(2,m)$-satellites of $T(2,2)\#T(2,2)$

and $T(2,4)$ respectively, so φ is reducible with periodic reducible pieces. To prove the assertion, we therefore have to find a clean arc α which is not contained in one periodic piece in these cases. To this end, let S be any surface obtained by positive Hopf plumbing according to an affine Coxeter-Dynkin tree. Denote the induced action of the monodromy on $H_1(S, \mathbb{Z})$ by φ_* and let $e_1, \ldots, e_n \in H_1(S, \mathbb{Z})$ be the basis vectors represented by the core curves of the Hopf bands used in the plumbing construction. As mentioned above, φ_* has infinite order. The reason for this is the existence of Jordan blocks of size 2×2 to the eigenvalue -1. We can hence find a vector $v \in H_1(S, \mathbb{Z})$ whose orbit under the monodromy is infinite. Choose $j \in \{1, \ldots, n\}$ such that the j-th coordinate of $(\varphi_*)^k(v)$ is unbounded. Let $\alpha \subset S$ be a spanning arc of the Hopf band with core curve e_j. Then α is clean, since cutting along α yields a connected sum of positive tree-like Hopf plumbings, which is fibred. Moreover we have $|i(v, \varphi^{-k}(\alpha))| = |i(\varphi^k(v), \alpha)| \to \infty$ for $k \to \infty$ by construction, and the arcs $\varphi^{-k}(\alpha)$ are all clean by Remark 2. In particular, there are infinitely many non-isotopic cutting arcs preserving fibredness. □

Proof of Theorem 4. Combine Propositions 2 and 3. □

6 Proof of Propositions 1 and 2

Before we begin with the proofs, some notation and remarks are necessary. Let S be the fibre surface of either $T(n,m)$, D_n or E_7, and let $\varphi : S \to S$ be the monodromy. Precisely as in Section 3, we decompose S into finitely many disjoint polygonal disks A_i (and B_j in the case of torus links) that are glued using bands (K_{ij} for the torus links and neighbourhoods of the k_i, k_i' for D_n and E_7). We use the letter D to denote any of the disks and the letter K to denote any of the bands. Let U be the union of all the disks and let $N \subset S$ be the neighbourhood of the tête-à-tête graph on which φ is assumed to be periodic.

Definition 7. An arc $\alpha \subset S$ is in *normal position* if the following conditions hold:

(a) The endpoints of α lie in ∂U.

(b) For every band K, $\alpha \cap K \setminus U$ consists of finitely many straight segments parallel to the edges of the tête-à-tête graph.

(c) The number of such segments in K is minimal among all arcs isotopic to α.

(d) α intersects the graph transversely in finitely many points of U.

(e) $\alpha \setminus N \subset U$, that is, before α enters N and after it leaves N, it stays in the disks that contain its endpoints.

(f) $\alpha \cap U$ consists of finitely many straight arcs.

Remarks 3 (on normal position)**.**

- Any arc can be brought into normal position by a free isotopy.
- If α is in normal position, then $\varphi(\alpha)$ can be brought into normal position keeping N fixed. Indeed, it suffices to straighten the two subarcs $\varphi(\alpha) \setminus N$ (or, undoing the twisting that occurs in the respective annuli), sliding the endpoints of $\varphi(\alpha)$ along ∂S, see Figure 4.11.

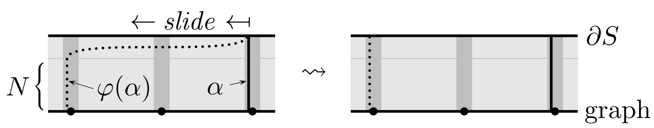

Figure 4.11: How to bring $\varphi(\alpha)$ in normal position, keeping N fixed.

- If α and $\varphi(\alpha)$ are in normal position as above, we may isotope $\varphi(\alpha)$ with endpoints fixed and keeping it in normal position, such that α and $\varphi(\alpha)$ intersect transversely in finitely many points of U. In particular, the sets $\alpha \setminus U$ and $\varphi(\alpha) \setminus U$ are now disjoint (cf. Figure 4.12).

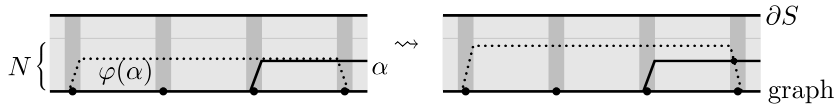

Figure 4.12: How to make $\alpha, \varphi(\alpha)$ intersect transversely, keeping normal position.

- Let α be in normal position and suppose it passes through at least one band. Let K be the first (respectively last) band traversed by α after (before) it starts (ends) at a boundary point p of one of the disks, say D. Then p cannot lie between K and one of the two bands adjacent to K on ∂D. Otherwise an isotopy sliding the starting point (endpoint) of α along ∂K would decrease the number of segments in K, contradicting part (c) of Definition 7.

Remarks 4 (compare the *bigon criterion*, Proposition 1.7 in [FM])**.** Suppose α intersects $\varphi(\alpha)$ in its interior. If α is clean, there must be a bigon $\Delta \subset S$ whose sides consist of a subarc of α and a subarc of $\varphi(\alpha)$. If $\alpha, \varphi(\alpha)$ are in normal position, such Δ takes a particularly simple form:

- Δ cannot be contained in U (i.e., in one of the disks A_i or B_j). This would contradict part (f) of the above Definition 7.

- None of the two sides of Δ is contained in U, since the other side of Δ would have to leave U through one of the bands K and return through the same K. The disc Δ would then yield an isotopy reducing the number of segments of $\alpha \cap K$ or $\varphi(\alpha) \cap K$, contradicting part (c) of Definition 7.
- For every band K, $\Delta \cap K \setminus U$ consists of rectangles with two opposite sides parallel to the edge passing through K.
- $\Delta \cap U$ consists of topological disks δ connected to at least one rectangle.
- Construct a spine T for Δ as follows: put a vertex for each δ and connect two vertices by an edge if the corresponding disks δ connect to the same rectangle. T is a tree, for Δ is contractible. Two of its vertices correspond to the vertices of the bigon Δ. Among the other vertices of T, there is none of degree one because the adjacent edge would correspond to a rectangle in some K whose sides parallel to its core edge both belong to the same arc (α or $\varphi(\alpha)$). In other words, either α or $\varphi(\alpha)$ would pass through K and immediately return through K in the opposite direction. This would contradict part (c) of Definition 7. Therefore, T is a line consisting of some number of consecutive edges, and the two extremal vertices correspond to the vertices of Δ.

For the next two lemmas, note that ∂U is a disjoint union of circles, each partitioned into finitely many circular arcs that alternate between parts of ∂S and the regions where bands attach to a disc. We call the latter *band attaching regions.*

Lemma 3. *Let $\alpha, \varphi(\alpha)$ be in normal position and suppose they intersect in a point $p \in D$, where D is one of the disks A_i (or B_j in the torus link case). Let α', α'' be the components of $\alpha \cap D, \varphi(\alpha) \cap D$ containing p. If no two of the four points $\partial\alpha' \cup \partial\alpha'' \subset \partial D$ lie in the same band attaching region, then α cannot be clean.*

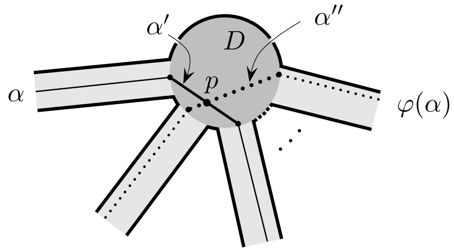

Figure 4.13: α cannot be clean by Lemma 3.

Remark 5. Note that we did not exclude the possibility that one of the endpoints of α or $\varphi(\alpha)$ lie in $\partial\alpha' \cup \partial\alpha''$. In other words, endpoints of α' and $\varphi(\alpha')$ may lie on $\partial S \cap \partial U$ as well as on band attaching regions.

Proof of Lemma 3. If α were clean, there would be a bigon. After possibly removing a certain number of such bigons, we are left with a bigon Δ with vertex p. By Remark 4, Δ has to leave D through one of the adjacent bands. Since one of the sides of Δ is a subarc of α and the other side is a subarc of $\varphi(\alpha)$, we find two points among $\partial\alpha' \cup \partial\alpha''$ that lie in the attaching region of this band, contradicting the assumption on α', α''. □

The second lemma is a generalisation of Lemma 3 to subarcs α', α'' that can pass through bands and visit several disks rather than staying in one disc.

Lemma 4. *Let $\alpha, \varphi(\alpha)$ be in normal position and let α', α'' be subarcs of $\alpha, \varphi(\alpha)$ respectively (not necessarily contained in U). Suppose that the four endpoints of α' and α'' are contained in ∂U and that no two of them lie in the same band attaching region. We further assume that α' and α'' intersect in exactly one point and that α', α'' traverse the same sequence of bands (see Figure 4.14). Then α cannot be clean.*

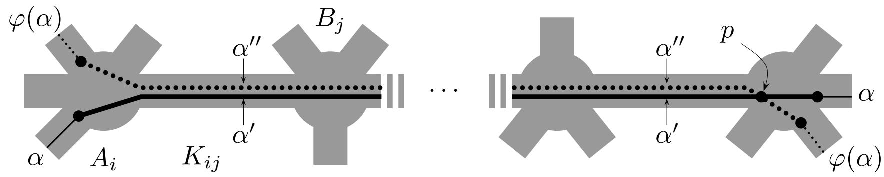

Figure 4.14: α cannot be clean by Lemma 4.

Proof. Assume $\alpha' \cap \alpha'' = \{p\}$, then $p \in U$. As in the proof of Lemma 3, study a bigon Δ that starts at p. Δ consists of a sequence of rectangles as described in Remarks 4. Starting at p, it therefore has to follow α' and α'' through the sequence of bands they traverse. Since p is the only intersection between α' and α'', Δ has to pass through at least one more band. But this is impossible by the assumption on the endpoints of α' and α''. □

In order to formulate the third lemma we recall the description of the monodromy $\varphi : S \to S$ as a tête-à-tête twist from Section 3: cutting the surface S along the tête-à-tête graph Γ results in d annuli $O_1, \ldots, O_d$, where d is the number of components of $\partial S = L$. Each annulus O_i has a link component as one boundary and a cycle consisting of edges of the graph as the other boundary. We call the latter the *graph boundary* of O_i and denote by r_i its length (that is, the number of edges). The monodromy φ keeps the link boundary of O_i fixed and rotates the neighbourhood N of the graph boundary by some number of edges ℓ_i called the *twist length*. In all cases we consider, we have $1 \leqslant \ell_i \leqslant r_i - 1$. A sequence of bands $K^{(1)}, K^{(2)}, \dots$ is *consecutive*, if the set $(\bigcup_r K^{(r)} \cup \bigcup_i A_i \cup \bigcup_j B_j) \setminus \Gamma$ has a connected component that intersects all bands $K^{(r)}$ of the sequence in this order, i.e. it is possible to stay on the same side of the graph when walking along the bands.

Lemma 5. *A clean arc in normal position cannot traverse more than ℓ consecutive bands along an annulus of twist length ℓ.*

Proof. Let α be a clean arc in normal position that traverses n consecutive bands with respect to an annulus O of twist length ℓ. Suppose that $n \geqslant \ell + 1$. We may assume that n is the maximal number of consecutively traversed bands with respect to O. In these bands as well as the adjacent disks, isotope α such that it stays on one side of the graph, keeping it in normal position.

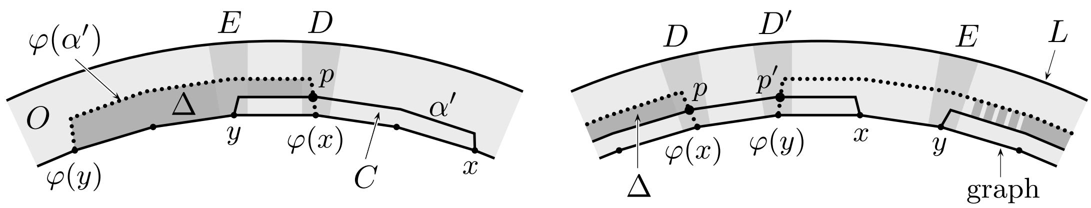

Figure 4.15: A normal arc passing through more than ℓ consecutive bands has to intersect its image under the monodromy (here $\ell = 2$). This can happen in two possible ways. Part of an a priori possible bigon Δ in grey.

In O, we will see a subarc $\alpha' \subset \alpha$ that has exactly its endpoints x, y in common with the graph and travels near the graph boundary for a distance of n consecutive edges (Note that α' cannot have any endpoint on ∂S. This would contradict part (c) of Definition 7. We also have that n cannot exceed the length of the graph boundary of O, since α does not intersect itself). Let C be the disc bounded by α' and the graph. Since $n \geqslant \ell + 1$, the disks C and $\varphi(C)$ have to overlap, and either one or both of the endpoints of $\varphi(\alpha')$ lie in C. Accordingly, the four endpoints of α' and $\varphi(\alpha')$ either appear in the cyclic order $x, \varphi(x), y, \varphi(y)$ or $x, \varphi(y), \varphi(x), y$ for a suitable choice of orientation of the graph boundary of O. Now bring $\varphi(\alpha)$ in normal position transverse to α as described in the Remarks 3. Let D, D', E be the disks (A_i or B_j) containing the points $\varphi(x), \varphi(y), y$, respectively. Note that D, D' coincide if n equals the graph boundary length. Nonetheless, the part of E shown in Figure 4.15 is disjoint from D, D' since $n > \ell \geqslant 1$. In the first case, where $x, \varphi(x), y, \varphi(y)$ appear in this order, we may assume that $\alpha', \varphi(\alpha')$ intersect in exactly one point $p \in D$ (compare Figure 4.15, left). In the second case, where the order is $x, \varphi(y), \varphi(x), y$, we can arrange that $\alpha', \varphi(\alpha')$ intersect in exactly two points $p \in D$, $p' \in D'$, as shown in Figure 4.15, right. However, α is clean, so there must be a bigon in S whose sides consist of a subarc of α and a subarc of $\varphi(\alpha)$, in both cases. After possibly removing a certain number of such bigons, we will be left with a bigon Δ starting at p. Our goal is now to prove that Δ must be attached to the point p in the way shown in Figure 4.15. We will then see that this forces α to pass through one more consecutive band, contradicting the assumption on n being maximal. Firstly, we know from the Remarks 4 that Δ has to leave D and consists of a sequence of rectangles. Let R be the first rectangle in this sequence, i.e. R is contained in a band adjacent to D. Let K^-, K^+ be the two bands adjacent to D that contain segments of α', K^+ being the one that also contains a segment of $\varphi(\alpha')$ (see Figure 4.16). Let β be the component of

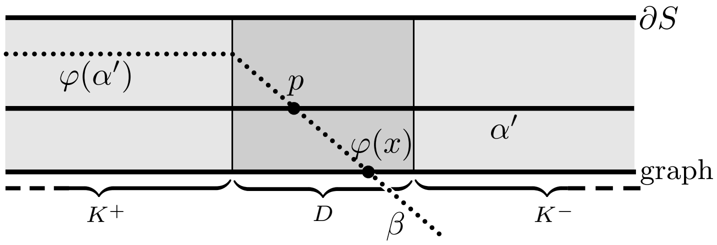

Figure 4.16: Part of the annulus O, where the arcs α' and $\varphi(\alpha')$ intersect in a point $p \in D$.

$\varphi(\alpha) \setminus \{p\}$ that contains $\varphi(x)$. We claim that β cannot leave D through K^- nor K^+. Indeed, if β would leave D through K^-, $\varphi(\alpha)$ would traverse $n+1$ consecutive bands, contradicting the assumption on n being maximal. On the other hand, if β would leave D through K^+, we could reduce the number of segments in $\varphi(\alpha) \cap K^+$, contradicting the normal position of $\varphi(\alpha)$, i.e. part (c) of Definition 7. In contrast, α leaves D, starting from p in both directions, through K^- and K^+. Consider now the subarcs of α and $\varphi(\alpha)$ that constitute two opposite sides of the rectangle R. Since R is contained in a band adjacent to D, these two subarcs arrive at ∂D through the same band, and they connect directly to $p \in D$. Therefore, we must have $R \subset K^+$, since K^+ is the only band containing two subarcs of α and $\varphi(\alpha)$ that directly connect to $p \in D$. Furthermore, R has to be the region enclosed by $\alpha' \cap K^+$ and $\varphi(\alpha') \cap K^+$. Following $\varphi(\alpha')$ in the direction from $\varphi(x)$ to p, we see that it leaves D through K^+ as one of the sides of R and continues staying on the same side of the graph for exactly $n-1$ more edges. By assumption, α' and $\varphi(\alpha')$ intersect at most in p and p'. In addition, no intersection between α' and $\varphi(\alpha)$ (or between α and $\varphi(\alpha')$) can occur in any of the $n-\ell$ disks visited by $\varphi(\alpha')$ on its way after leaving D and until it arrives in E. Indeed, the first such intersection would necessarily be the ending vertex of Δ, implying that it were an intersection between α' and $\varphi(\alpha')$. However, none of the points p, p' lies in any of these disks. Therefore, the bigon Δ has to pass through at least $n-\ell+1$ rectangles through consecutive bands starting at p. Similarly, the sides of these rectangles that are subarcs of α have to traverse at least $n-\ell+1$ consecutive bands starting at p. We obtain a contradiction to the maximality of n, because α' ends after $n-\ell$ bands starting from p, since φ rotates the graph boundary by ℓ edges. This finishes the proof. □

Proof of Proposition 1. We will concentrate on the most complicated case of the torus knot $T(3,5)$. It contains all difficulties appearing in the proofs for $T(3,3)$ and $T(3,4)$ which go along the same lines with fewer cases to consider. For each link appearing in Table 4.1 of Section 4, we will indicate one (but not every) possible choice of a cutting arc that yields the link in question. Let hence S be the fibre surface of $T(3,5)$ and let $\alpha \subset S$ be any arc that preserves fibredness, i.e. a clean arc. Bring α into normal position using an isotopy (not fixing the boundary), cf. Remarks 3. Since φ permutes the vertices $\{a_i\}$ cyclically as well as the vertices $\{b_j\}$, it suffices to show that there are only finitely many clean arcs starting at a point of ∂A_1 or at a point of ∂B_1, up to isotopy. We may further assume that α starts either at a point of ∂A_1 between k_{11} and k_{15} or at a point of ∂B_1 between k_{21} and k_{31}.

<u>*Case A.*</u> α starts at ∂A_1, between k_{11} and k_{15}. Then, α cannot continue through either of the bands K_{11} nor K_{15} by the last item of Remarks 3. So, either α stays in A_1 (and there are only four such arcs up to isotopy), or it continues through K_{12}, K_{13} or K_{14}. If α stays in A_1, the links obtained by cutting are E_7 (e.g. if α ends between k_{11} and k_{12}) and D_7 (e.g. if α ends between k_{12} and k_{13}).

<u>*Case A.1.*</u> α continues through K_{12}. Arriving in B_2, there are three possibilities: either α ends at a point of ∂B_2 between k_{22} and k_{32} (and cutting along α yields $T(3,4)\#T(2,2)$), or it continues through K_{22} or K_{32} (ending at other points of ∂B_2 is impossible by the last item of Remarks 3).

<u>*Case A.1.1.*</u> α continues through K_{22}. Arriving in A_2, α can end at a point of ∂A_2 (cutting yields $T(2,7)\#T(2,2)$ if α ends between k_{24} and k_{25}, and $T(2,3)$ summed with the unknot component of D_5 if α ends between k_{23} and k_{24}), or it can continue through K_{23} or K_{24}. It cannot continue through K_{21}, since K_{12}, K_{22}, K_{21} is a sequence of three consecutive bands, so α would not be clean by Lemma 5. Finally, α cannot continue through K_{25}. If it did, α and $\varphi(\alpha)$ would intersect in a point of A_1, and Lemma 3 would imply that α cannot be clean (see Figure 4.17, top left). Note that we do not know whether the mentioned intersection is the only one since we do not know how α ends.

Case A.1.1.1. α continues through K_{23}. From B_3, it cannot continue through K_{13}, for K_{22}, K_{23}, K_{13} are consecutive (Lemma 5). If it continues through K_{33} it cannot continue through any band adjacent to A_3. Indeed, K_{23}, K_{33}, K_{32} are consecutive, so α cannot continue through K_{32}. If it would continue through K_{34} or K_{35} or K_{31}, we could apply Lemma 4 to the band K_{33} to show that α is not clean (see Figure 4.17).

Case A.1.1.2. α continues through K_{24}. If it ends in B_4 between k_{14} and k_{34}, we obtain $T(2,5)\#T(2,4)$ after cutting. Otherwise, it can continue from B_4 through K_{14} or through K_{34}.

Case A.1.1.2.1. If it continues through K_{14}, it cannot go further. Firstly, K_{24}, K_{14}, K_{15} are consecutive, so K_{15} is no option (Lemma 5). Neither can it proceed through K_{11} (this would produce a self-intersection of α) nor K_{12} (for otherwise we could apply Lemma 3 to an intersection between α and $\varphi(\alpha)$ in A_1). If it continues through K_{13}, it cannot go on through K_{23} since K_{14}, K_{13}, K_{23} are consecutive (Lemma 5). Suppose it continues through K_{33}. From A_3, it cannot proceed through any of K_{31}, K_{35}, K_{34}, for otherwise we could apply Lemma 4 to the bands K_{13} and K_{33}, with an intersection between α and $\varphi(\alpha)$ occuring in A_3 (see Figure 4.17 left). However, α cannot continue through K_{32} either, because we could again apply Lemma 4, this time for the band K_{24} and an intersection in A_2 (see Figure 4.17 right).

Case A.1.1.2.2. α continues from B_4 through K_{34}. If it ends in A_3 between k_{35} and k_{31}, cutting yields $T(2,5)\#T(2,2)\#T(2,3)$. Otherwise, it cannot continue from A_3 through K_{33} since K_{24}, K_{34}, K_{33} are consecutive. Neither can it proceed through K_{32} (apply Lemma 3 to A_3). So α can only continue through K_{35} or K_{31}.

Case A.1.1.2.2.1. If it continues through K_{35}, the only option to go further is through K_{15}, since K_{34}, K_{35}, K_{25} are consecutive. From A_1 (compare Figure 4.17), it cannot continue through K_{11} nor K_{12} (apply Lemma 4 to K_{15} with an intersection occuring in A_1). Neither can it continue through K_{14}, since K_{35}, K_{15}, K_{14} are consecutive. So it has to go through K_{13}. Arriving in B_3, it cannot continue through K_{23} (apply Lemma 4 to K_{34} with an intersection occuring in B_4). Therefore α has to continue through K_{33}. From A_3, it cannot proceed further. Firstly, K_{32} is not an option (otherwise apply Lemma 4 to K_{34} and K_{24} with an intersection in A_2). Neither can it go through K_{34} or K_{35} (apply Lemma 4 to K_{15}, K_{13}, K_{33} with an intersection occuring in A_3). Finally, it cannot pass through K_{31} either (apply Lemma 4 to the bands K_{15}, K_{13}, K_{33} with an intersection occuring in A_3).

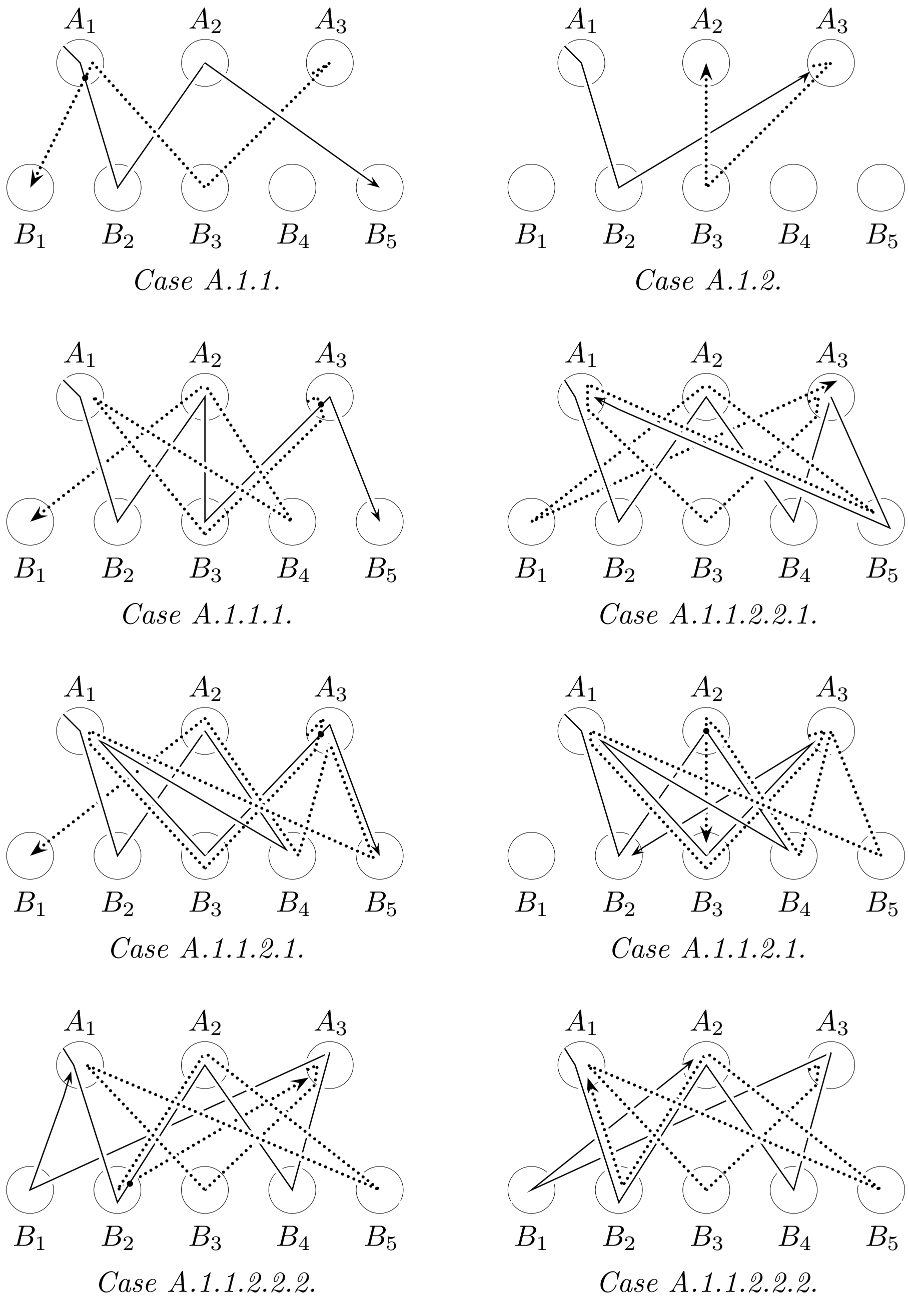

Figure 4.17: Schematic illustration for a selection of the cases in the proof of Proposition 1. The arc α is drawn as solid line, whereas $\varphi(\alpha)$ is shown as a dotted line.

Case A.1.1.2.2.2. If it continues through K_{31} and arrives in B_1, it cannot proceed through K_{11} (apply Lemma 4 to K_{22} with an intersection occuring in B_2, see Figure 4.17 left). So it has to go through K_{21}. From A_2, it cannot proceed through K_{22}, for K_{31}, K_{21}, K_{22} are consecutive. Neither can it go through either of K_{23} nor K_{24} (apply Lemma 3 to an intersection occuring in A_2, see Figure 4.17 right). Finally, K_{25} can be ruled out by Lemma 4, applied to the bands K_{22} and K_{12}, with an intersection occuring in A_1.

Case A.1.2. α continues through K_{32} (see Figure 4.17). Arriving in A_3, it cannot continue through any band. Firstly, K_{12}, K_{32}, K_{33} are consecutive, so α cannot continue through K_{33}. If it would continue through any of the other bands adjacent to A_3, α would intersect $\varphi(\alpha)$ in a point of A_3 such that we could apply Lemma 3 to obtain a contradiction to α being clean.

Case A.2. α proceeds through K_{13}. If it ends in B_3 between k_{23} and k_{33}, we obtain $T(2,8)$ after cutting. From B_3, it can continue through K_{23} or through K_{33}.

Case A.2.1. α continues through K_{23}. It cannot go on via K_{22}, for K_{13}, K_{23}, K_{22} are consecutive. Neither can it continue through K_{21} or K_{25} by Lemma 3 applied to an intersection in A_1. If it next passes through K_{24}, it cannot go on through K_{14}, because K_{23}, K_{24}, K_{14} are consecutive. Proceeding through K_{34}, it can end in A_3 between k_{31} and k_{32} (this yields $T(2,5)\#T(2,3)\#T(2,2)$). However, the only possibility for α to go further is via K_{32}, for K_{24}, K_{34}, K_{33} are consecutive (so K_{33} is no option), and α cannot continue through K_{35} nor K_{31} by applying Lemma 4 to the band K_{34} with an intersection of $\alpha, \varphi(\alpha)$ in A_3. So α continues through K_{32} and arrives in B_2. From there, it cannot continue through K_{12} (apply Lemma 4 to K_{22} and an intersection in B_3). If it continues through K_{22}, it cannot go further: K_{23} is impossible because K_{32}, K_{22}, K_{23} are consecutive, K_{24} can be excluded by Lemma 3, applied to A_2, and K_{21} as well as K_{25} can be ruled out by Lemma 4, applied to K_{23} and K_{13} with an intersection occuring in A_1.

Case A.2.2. α continues through K_{33}. This is similar to Case A.2.1. Again there is always a single option to go on, until there is no possibility left after four more steps.

Case A.3. α continues through K_{14}. This is analogous to Case A.1.

Case B. α starts at ∂B_1 between k_{21} and k_{31}. Then, it can only continue through K_{11} by the last item of Remarks 3. From A_1, it can proceed through four distinct bands.

Case B.1. α continues through K_{15}. Since K_{11}, K_{15}, K_{25} are consecutive, it can a priori only continue through K_{35}. But this is impossible as well by Lemma 3, applied to the intersection between α and $\varphi(\alpha)$ occuring in B_1.

Case B.2. α continues through K_{12}. This is analogous to Case B.1.

Case B.3. α continues through K_{14}. Arriving in B_4, it can end between k_{24} and k_{34} (this results in $T(2,3)$ summed with the trefoil component of D_5).

Case B.3.1. α continues through K_{24}. From A_2, it cannot continue through K_{23} because K_{14}, K_{24}, K_{23} are consecutive (Lemma 5). Neither can it go on through K_{22} nor K_{21} (apply Lemma 3 to A_1). Suppose α continues through K_{25}. From B_5, it cannot go on via K_{35} since K_{24}, K_{25}, K_{15} are consecutive. If it proceeds via K_{35}, we can apply Lemma 4 to the band K_{11} with an intersection in B_1 to obtain a contradiction.

Case B.3.2. α continues through K_{34}. From A_3, there are only two options for α to proceed further. Indeed, K_{14}, K_{34}, K_{35} are consecutive, so K_{35} is out of the question. K_{31} can be ruled out by Lemma 3 for A_3. The remaining possibilities are K_{32} and K_{33}.

Case B.3.2.1. α continues through K_{32}. From there, it cannot continue through K_{22} (apply Lemma 4 to K_{32}). So it has to branch off via K_{12} to A_1. From there, it cannot continue through K_{15} since otherwise α would self intersect in A_1. K_{11} is impossible as well, for K_{32}, K_{12}, K_{11} are consecutive. K_{15} can be ruled out using Lemma 3 for A_3. So α can only continue through K_{13}, and from there only through K_{23} (K_{12}, K_{13}, K_{33} are consecutive). From A_2, it cannot go on through any band except K_{25}. Indeed, K_{22} is impossible because K_{13}, K_{23}, K_{22} are consecutive. K_{21} and K_{24} can be ruled out by applying Lemma 4 to (K_{34}, K_{14}) and K_{23} respectively. After passing through K_{25}, α cannot go further: K_{15} is impossible by Lemma 4 (applied to K_{23}, K_{25}) and K_{35} can be ruled out by applying Lemma 4 to K_{34}, K_{14}, K_{11}.

Case B.3.2.2. α continues through K_{33}. Then, K_{13} cannot be next since K_{34}, K_{32}, K_{13} are consecutive. Thus α passes through K_{23}. From A_2, it cannot go on via K_{24}, for K_{33}, K_{23}, K_{24} are consecutive. K_{21} and K_{22} are impossible as well (apply Lemma 4 to K_{14}). So α has to go through K_{25}. Then, it cannot proceed through K_{15} (apply Lemma 4 to K_{25}). It cannot go via K_{35} either (apply Lemma 4 to K_{14}, K_{11}), so α cannot continue at all.

Case B.4. α continues through K_{13}. This is analogous to Case B.3 and finishes the proof. $\square$

Proof of Proposition 2. We will present a case by case analysis for the possible clean arcs α in the fibre surface S of each of E_7 and D_n. The reader interested in studying the proof is advised to follow the arguments along with a pencil and copies of Figures 4.7 and 4.10, top and bottom. As in the proof of Proposition 1 above, we will make extensive use of Lemma 5 to exclude further polygon edges that α might cross on its way from its starting point to its end. In order to keep the proof short, we will usually refer to such situations by just saying "α is trapped", or by saying that an edge "is a trap", meaning that α would traverse too many consecutive bands to be clean.

(E_7) First, let S be the fibre surface of E_7, denote its monodromy φ and let $\alpha \subset S$ be a clean arc. Bring α into normal position with respect to $k_1, \ldots, k_9$. Note that the set of vertices of the hexagons A_1, A_2, A_3 decompose into two orbits under φ, namely the orbit of the vertex of A_1 between k_1 and k_2, and the orbit of the vertex of A_1 between k_2 and k_7. We may therefore assume by Remark 2 that α starts at one of these two vertices.

Case 1. α starts at the vertex of A_1 between k_1 and k_2. Define an involution $\tau : S \to S$ as follows: τ interchanges hexagons A_1 and A_2 and then reflects A_1, A_2, A_3 along the diagonals parallel to k_7, k_8, k_1 respectively, whereby it induces the permutation (13)(49)(58)(67) on the edges $(k_1, \ldots, k_9)$. We have $\varphi \circ \tau \circ \varphi = \tau$, $\tau \circ \varphi$ fixes the vertex of A_1 between k_1 and k_2 and swaps the edges k_4, k_8 as well as the edges k_5, k_7. By Remark 2, we may therefore assume that α continues through k_4 or through k_5.

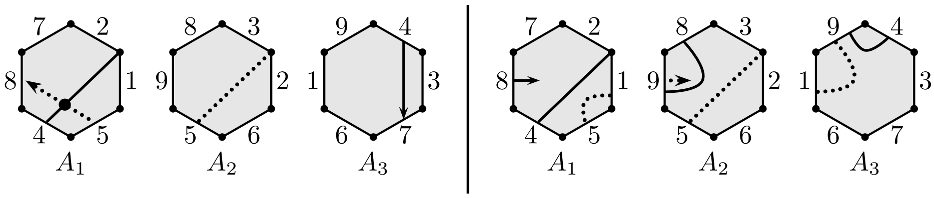

Figure 4.18: Illustration of two of the steps in Case 1.1. The arc α is drawn as solid line, whereas $\varphi(\alpha)$ is shown as a dotted line.

Case 1.1. α continues through k_4. From A_3, it can only choose k_9. Indeed, k_7, k_6 and k_1 would imply an intersection in A_1 (Lemma 3, compare Figure 4.18, left), and k_3 is consecutive to k_4 (Lemma 5, applied to the component with twist length one). Arriving in A_2, k_2 and k_6 would imply an intersection in A_2, so continuation is possible through k_3, k_5, k_8 only. But if α continues through k_5 or k_8, it will be trapped (compare Figure 4.18, right). Therefore it goes through k_3. Arriving in A_3, it has to go through k_4 (k_9 implies an intersection in A_3 and k_1, k_6, k_7 imply intersections in A_1). However, passing through k_4, α is trapped.

Case 1.2. α continues through k_5. From A_2, it can continue through k_3, k_6, k_8 or k_9 (k_2 implies an intersection in A_2). If it passes through k_6 or k_9, it is trapped. So k_8 and k_3 are the only possiblities left.

Case 1.2.1. α continues through k_8. Upon arrival in A_1, it cannot continue through k_2, k_5 (intersection in A_2) nor through k_1 (this would imply an intersection in A_1). But if it continues through either of k_7 or k_4, it is trapped.

Case 1.2.2. α continues through k_3. From A_3, α cannot go on through k_9, k_1 (this would imply an intersection in A_3). If it passes through k_4, it is trapped. Suppose it continues through k_6. Arriving in A_2, it cannot continue through k_9, k_8, k_3 (this would produce an intersection in A_3), nor through k_2 (intersection in A_2). Finally, continuing through k_5, it will be trapped. Therefore α has to continue from A_3 through k_7. Arriving in A_1, it can continue through k_2, k_8 or k_4 (k_1 implies an intersection in A_1 and k_5 implies an intersection in A_2). But all of these are traps.

<u>*Case 2.*</u> α starts at the vertex of A_1 between k_2 and k_7. Define an involution $\sigma : S \to S$ as follows: σ interchanges A_1 and A_2 and then reflects A_1, A_2, A_3 along the diagonals parallel to k_1, k_2, k_4 respectively, inducing the permutation $(19)(28)(37)(46)$ on the edges. As in Case 1, we have $\varphi \circ \sigma \circ \varphi = \sigma$, and $\sigma \circ \varphi$ fixes the vertex of A_1 between k_2 and k_7, swapping k_4 and k_5 as well as k_1 and k_8. By Remark 2, we may therefore assume that α continues through either k_1 or k_5. However, if α continues through k_1, it is trapped. Therefore it continues through k_5. From A_2, it can continue through k_8 or k_9 (k_6 is a trap and k_2, k_3 imply intersections in A_2).

<u>*Case 2.1.*</u> α continues through k_8. From A_1, it cannot continue through any of k_2, k_1, k_5, because this would produce an intersection in A_2, and k_7 is a trap. Therefore, it continues through k_4 and arrives in A_3. Continuation through k_1 produces an intersection in A_3, and k_6, k_7, k_3 imply intersections in A_1. Finally, k_9 is a trap.

<u>*Case 2.2.*</u> α continues through k_9. Arriving in A_3, it can only continue through k_1 or k_4 (any other continuation produces an intersection in A_1). However, both k_1 and k_4 are traps, ending the proof for E_7.

(D_n, n even) Now, suppose n is even and let α be a clean arc in the fibre surface S of D_n in normal position with respect to $k_1, \ldots, k_{n-1}$, $k'_1, \ldots, k'_{n-1}$. Define an involution $\tau : S \to S$ as follows: τ permutes the disks A_i according to the rule $\tau(A_i) = A_{n-i+2}$ for $i = 1, \ldots, n-1$ and then reflects every A_i on the diagonal that contains the vertex between k_i and k_{i+2} (all indices are to be taken modulo n). Again $\varphi \circ \tau \circ \varphi = \tau$, and $\tau \circ \varphi$ fixes the vertex of A_1 between k'_1 and k'_2 as well as the vertex between k_1 and k_3, and swaps the other two vertices. We may therefore assume that α starts at a vertex of A_1 which is not the vertex between k'_2 and k_3.

<u>*Case 1.*</u> α starts at the vertex of A_1 between k_1 and k'_1. If it continues through k'_2, it is already trapped. So it has to continue through k_3. Arriving in A_3, it can continue through k'_3, k'_4 or k_5.

<u>*Case 1.1.*</u> α continues from A_3 through k'_3. From A_2, it cannot continue through k_2 (otherwise it would intersect with $\varphi(\alpha)$), so it can only proceed through k'_2 or k_4. However, both are traps.

<u>*Case 1.2.*</u> α continues from A_3 through k'_4. This is similar to Case 1.1: arriving in A_4, α can only continue through k'_5 (which is a trap) or k_4. If it goes through k_4, it has to continue from A_2 through k'_3 (k_2 produces an intersection in A_2 and k'_2 produces an intersection in A_3). Then however, it is trapped again.

Case 1.3. α can therefore continue from A_3 through k_5 only. In A_5, the same situation reproduces, except that all indices in consideration are now shifted by $+2$. Therefore the only way for α to continue from A_5 is by passing through the edges $k_7, k_9, k_{11}, \dots$ After at most $n/2$ more steps, α will be trapped.

Case 2. α starts at the vertex of A_1 between k_1' and k_2'. Using τ again, we may assume that it continues through k_1 to A_{n-2}. If it goes through k_{n-1}' next, it is trapped since it is forced to follow the sequence of edges $k_{n-1}, k_{n-2}', k_{n-2}, k_{n-3}', \dots$ If it goes through k_{n-2}' to A_{n-3} instead, it can only continue from there through k_{n-3}' or k_{n-1}, and these are traps again. So it has to continue from A_{n-2} through k_{n-2}. In A_{n-4}, the same situation as one step earlier (where α arrived through k_1 in A_{n-2}) reproduces, except that all indices appearing in the consideration are now shifted by -2. Hence the only way α can continue from A_{n-4} is by going through the sequence of edges $k_{n-4}, k_{n-6}, k_{n-8}, \dots$ After at most $n/2$ steps, α will be trapped.

Case 3. α starts at the vertex of A_1 between k_1 and k_3. Using the involution τ from above, we may assume that it continues through k_2'. From A_2, it cannot go on through k_4, for this would imply an intersection in A_2. However, the two possibilities that remain (k_3' and k_2) are traps, which ends the proof for D_n, n even.

(D_n, n odd) Finally, let n be odd and let S be the fibre surface of D_n. Suppose again we have a clean arc $\alpha \subset S$ in normal position with respect to $k_1, \dots, k_{2n-2}$. Since the monodromy permutes the A_i cyclically and since there are only two orbits of vertices of the A_i, we may assume that α starts in A_1, at the vertex between k_1 and k_2, or at the vertex between k_2 and k_n. As before, we then make use of Remark 2 with the help of the involution $\tau : S \to S$ defined as follows: $\tau(A_i) = A_{n-i+2}$ by translations followed by a reflection on the diagonal of A_i that contains the vertex between k_{n+i-1} and k_{n+i} for $i = 1, 2$ and reflection on the diagonal of A_i that contains the vertex between k_i and k_{i+1} for $i = 3, \dots, n-1$. Applying Remark 2 as before, we may assume that α either starts at the vertex of A_1 between k_1, k_2 and continues through k_n (say), or that it starts at the vertex of A_1 between k_2 and k_n, continuing through k_1 (say). So there are two cases to consider, one being very similar to Case 1 above and the other similar to Case 3. No new arguments are needed. $\square$

Chapter 5

Hopf bands in arborescent Hopf plumbings

The content of this chapter is published as a preprint on the arXiv [Mi2].

For a positive Hopf plumbed arborescent Seifert surface S, we study the set of Hopf bands $H \subset S$, up to homology and up to the action of the monodromy. The classification of Seifert surfaces for which this set is finite is closely related to the classification of finite Coxeter groups.

1 Introduction

Let $S \subset S^3$ be a Seifert surface of a link L, and let q be the quadratic form on $H_1(S,\mathbb{Z})$ associated with the Seifert form. Every oriented simple closed curve $\alpha \subset S$ can be thought of as a framed link in S^3, where the framing is induced by the surface S and encoded by the value q takes on the homology class represented by α. For a fixed integer n, we are interested in the set $\mathscr{C}_n(S)$ of isotopy classes of n-framed unknotted oriented curves $\alpha \subset S$. By Rudolph's work [Ru] on quasipositive surfaces we know for example $\mathscr{C}_n(S) = \varnothing$ whenever $n \leqslant 0$ and S is quasipositive[1]. Here, we focus on positive arborescent (tree-like) Hopf plumbings, where S is a surface obtained by an iterated plumbing of positive Hopf bands according to a finite plane tree T. Positive arborescent Hopf plumbings are particular examples of quasipositive surfaces. At the same time, they are fibre surfaces, i.e. pages of open books with binding $K = \partial S$. In fact every fibre surface in S^3 can be obtained from the standard disc by successively plumbing and deplumbing positive or negative Hopf bands. This results from Giroux' work on open books and contact structures (see the article [GiGo] by Giroux and Goodman). Not much is known about how (non-)unique a presentation of a given fibre surface S as a plumbing of Hopf bands may be. In our previous article [Mi1], we have studied embedded arcs in fibre surfaces cutting along which corresponds to deplumbing a Hopf band, and we gave

[1] For $n = 0$, we exclude the trivial isotopy class in the definition of $\mathscr{C}_n(S)$.

examples showing that the plumbing structure can be highly non-unique. Here, we take a similar, but different approach to understanding the plumbing structure of S by studying the set $\mathscr{C}_n(S)$ in the case $n = 1$, whose elements correspond to Hopf bands that can potentially be deplumbed. The monodromy $\varphi : S \to S$ of the open book acts on the set $\mathscr{C}_n(S)$ as well as on its image $C_n(S)$ in $H_1(S, \mathbb{Z})$, thus providing it with additional structure. Finite trees can be divided into three families named *spherical*, *affine* and *hyperbolic*, according to the classification of Coxeter groups (compare [AC2, Hu]). The spherical trees comprise two families, called A_n and D_n, plus three more trees named E_6, E_7, E_8, whereas the affine trees are denoted $\tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$. Up to these exceptions, all trees fall into the class of hyperbolic trees.

Theorem 7. *Let T be a finite plane tree and $S \subset S^3$ the corresponding positive arborescent Hopf plumbed surface. Then the set of homology classes of Hopf bands $C_1(S)$ is finite if and only if T is spherical.*
In contrast, if T is hyperbolic and ∂S_T is a knot, $C_1(S)$ consists of infinitely many orbits of the monodromy.

Interestingly, the above correspondence between Coxeter groups and tree-like Hopf plumbings does not seem to be of purely homological nature: in the exceptional cases that correspond to affine Coxeter groups, there are in fact infinitely many φ-orbits of homology classes $a \in H_1(S, \mathbb{Z})$ such that $q(a) = 1$. However, it is conceivable that only finitely many orbits can be realised by honest Hopf bands (that is, by unknotted embedded simple closed curves in S). We prove this for the smallest affine tree $\tilde{D}_4$, where it already suffices to exclude homology classes that are not representable by simple closed curves. In the case $\tilde{E}_6$, for example, infinitely many orbits can be realised by embedded (possibly knotted) simple closed curves. We did not succeed in finding unknotted representatives of these homology classes, though.

In the spherical cases, the set $C_1(S)$ coincides with $q^{-1}(1) \subset H_1(S, \mathbb{Z})$ and consists of the "obvious" Hopf bands only, that is, simple combinations of the ones used in the plumbing construction, and their images under the monodromy. It might be interesting to study $\mathscr{C}_n(S)$ or $C_n(S)$ for other n and other classes of Seifert surfaces, as well as the case where unknotted curves are replaced by curves of a fixed knot type.

Throughout, S denotes a Seifert surface in S^3. The Seifert form $(.\,,.)$ induces a quadratic form $q : H_1(S, \mathbb{Z}) \to \mathbb{Z}$ by $q(a) = -(a, a)$. For a simple closed curve α, the integer $-q(\alpha)$ describes the framing[2] of an annular neighbourhood of α in S. Recall from Chapter 1 that the Seifert matrix V of a fibre surface S with respect to a basis of $H_1(S, \mathbb{Z})$ is invertible and that the matrix M of the homological action of

[2]The sign makes sure that $q = +1$ on positive Hopf bands.

the monodromy with respect to the same basis can be computed using the formula $M = V^{-\top} V$.

The above theorem is a consequence (in fact, a summary) of three propositions. Proposition 4 concerns the spherical trees and is given in Section 4. In Section 5, the affine trees are discussed in Proposition 5. Finally we address the hyperbolic case with Propostion 6 in the last section.

2 Positive arborescent Hopf plumbings

Given a finite plane tree T, construct a fibre surface $S = S_T$ by taking one positive Hopf band for every vertex of T and use plumbing to glue all pairs of Hopf bands that correspond to adjacent vertices in T, respecting the cyclic order of the edges adjacent to each vertex. A Seifert surface S obtained in this way is called a *positive arborescent Hopf plumbing.* This construction is described and studied in greater generality by Bonahon and Siebenmann in their work on arborescent knots [BS]. For example, if T is the tree A_n shown in Figure 5.2, S_T is the standard Seifert surface of the $(2, n+1)$ torus link. For $T = D_4$, we obtain the standard Seifert surface of the $(3, 3)$ torus link. Yet another example is illustrated in Figure 5.1 below. The core curves (with a chosen orientation) of the Hopf bands

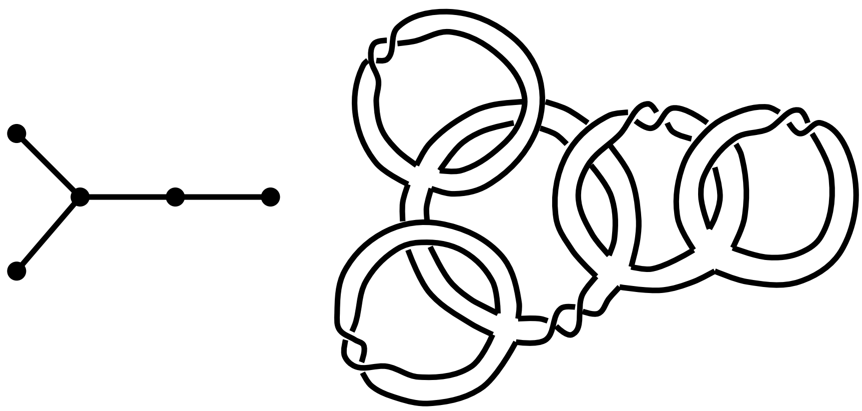

Figure 5.1: The spherical tree D_5 and the corresponding Hopf plumbing S_{D_5}.

used for the construction form a basis of $H_1(S, \mathbb{Z})$. Relative to a basis, the Seifert quadratic form q is a homogeneous polynomial of degree two in r variables, where $r = \operatorname{rank} H_1(S, \mathbb{Z})$ equals the number of vertices of T, or, equivalently, the number of Hopf bands used to construct S_T.

Remark 6 (on different notions of positivity)**.** There are different possible ways to define a Hopf band to be positive or negative. Here, by a *positive Hopf band* we mean an unknotted oriented band whose boundary link with the induced orientation is a *positive* Hopf link, so the core curve of a positive Hopf band has framing -1. Positive braid links are plumbings of positive Hopf bands. The subsequent statements can be translated into statements about negative Hopf plumbings.

3 Quadratic forms and Coxeter-Dynkin trees

Let T be a finite tree with vertices $\{v_1, \ldots, v_n\}$. Use the same symbols v_i to denote the basis of $H_1(S_T, \mathbb{Z}) \cong \mathbb{Z}^n$ consisting of the core curves of the plumbed Hopf bands. We define the matrix A_T to be the symmetric integral $r \times r$-matrix whose diagonal entries are 2's and whose off-diagonal ij-th entry is -1 if v_i and v_j are connected by an edge in T and 0 otherwise. The orientations of the core curves may be chosen such that

$$q_T(x_1 v_1 + \ldots + x_n v_n) = \frac{1}{2} x A_T x^\top, \quad \forall x = (x_1, \ldots, x_n) \in \mathbb{Z}^n.$$

Indeed, two non-adjacent Hopf bands being disjoint corresponds to the zero entries in A_T and positive Hopf bands being (-1)-framed fits the diagonal entries. Finally, the core curves v_i, v_j of two plumbed Hopf bands intersect exactly once and do not link otherwise. Thanks to the arborescent structure of the plumbing, we can choose orientations such that $\{\mathrm{lk}(v_i, v_j^+), \mathrm{lk}(v_j, v_i^+)\} = \{0, 1\}$. In other words,

$$-A_T = V + V^\top,$$

where V is the Seifert matrix of S_T with respect to the basis $v_1, \ldots, v_r$.
The quadratic form q_T is

- positive definite if T corresponds to a spherical Coxeter group,
- positive semidefinite if T corresponds to an affine Coxeter group,
- indefinite otherwise.

Accordingly, we call a tree T *spherical*, *affine*, *hyperbolic.* Compare Figure 5.2 for a list of the spherical trees and Figure 5.4 for the affine trees. Any finite plane tree not appearing in one of these lists is hyperbolic. In fact, the so-called *slalom knots* introduced by A'Campo can be described as arborescent Hopf plumbings, and the slalom knots given by a hyperbolic tree are exactly the ones whose complements admit a complete hyperbolic metric of finite volume [AC2, Theorem 1].

As we already suggested in the introduction, there is a bijective correspondence between positive Hopf bands H embedded in S_T (up to isotopy in S_T) and (-1)-framed unknotted simple closed curves $\alpha \subset S_T$ (up to isotopy in S_T). The correspondence is given in one direction by assigning to a Hopf band H its core curve α, and in the other direction by setting H to be a regular neighbourhood of α in S_T. Passing to homology classes, we can think of an element $x \in H_1(S_T, \mathbb{Z})$ as the homology class of a positive Hopf band if and only if x can be realised as the homology class of an unknotted simple closed curve in S_T and $q_T(x) = 1$.

Definition 8. We denote by $C_1(S_T)$ the set of homology classes of positive Hopf bands in S_T. Note that $C_1(S_T) \subset q_T^{-1}(1) \subset H_1(S_T, \mathbb{Z})$.

We complete this section with a few remarks that concern the above definition and which are important for the rest of the article.

Remark 7. If ∂S_T is a knot, $x \in H_1(S_T, \mathbb{Z})$ is representable by a simple closed curve if and only if it is *primitive*, i.e., if it cannot be written as a multiple of another vector (see [FM, Proposition 6.2] for a proof in the closed case). In particular, any $x \in q_T^{-1}(1)$ can be realised by a (possibly knotted) simple closed curve if ∂S_T is a knot.

Remark 8. Let T be a finite plane tree, S_T the corresponding surface and $\varphi : S_T \to S_T$ the monodromy. If $w \in H_1(S_T, \mathbb{Z})$ is a homology class represented by an unknotted simple closed curve α, then $\varphi(\alpha)$ is again an unknotted simple closed curve, since the flow of the monodromy vector field describes an isotopy from α to $\varphi(\alpha)$ in S^3. In addition, $\varphi(\alpha)$ represents the homology class $\varphi_*(w)$, whose framing equals the framing of w. In particular, $w \in C_1(S_T)$ iff $(\varphi_*)^n(w) \in C_1(S_T)$, $\forall n \in \mathbb{Z}$.

Remark 9. If T' is a subtree of a tree T (that is, T' is obtained from T by contracting edges), $S_{T'}$ can be viewed as a subsurface of S_T in such a way that the map on homology induced by the inclusion is injective. In particular, $C_1(S_{T'})$ can be identified with a subset of $C_1(S_T)$.

Remark 10. We immediately spot a certain number of Hopf bands in an arborescent Hopf plumbing S_T, such as the Hopf bands corresponding to the vertices v_i of T that were used in the Hopf plumbing construction. Based on this observation we say that $x \in C_1(S_T)$ is a *standard Hopf band* if $x \in C_1(S_{T'})$ for some subtree $T' \subset T$ of type A_n. See Figure 5.3 for an example and the paragraph after Proposition 4 for a complete description of all standard Hopf bands in S_{A_n}.

4 The spherical Coxeter-Dynkin trees

If T is one of the spherical trees A_n, D_n, E_6, E_7 or E_8 depicted in Figure 5.2, the quadratic form q_T is positive definite. Therefore, the equation $q_T(x) = k$ has only finitely many integral solutions, for any fixed k and in particular for $k = 1$. In the rest of this section, we explicitely determine all solutions to $q_T(x) = 1$ for each of the spherical trees. These were already studied and classified in the context of Lie algebra theory and Coxeter groups; see for example the book by Humphreys [Hu]. The following proposition summarises the results of this section (see Remark 10 for the definition of a standard Hopf band).

Proposition 4. *If T is a spherical tree, then the set of integral solutions to $q_T(x) = 1$ is finite. Moreover, every solution is contained in the orbit of a standard Hopf band under the monodromy and is therefore realisable as an unknotted simple closed curve in S_T. In particular, $C_1(S_T) = q_T^{-1}(1)$.*

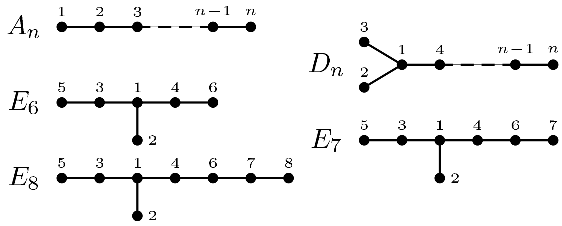

Figure 5.2: The simply laced spherical Coxeter trees correspond to finite Coxeter groups. The numbers indicate the order of the chosen homology basis vectors.

First let $T = A_n$. The associated quadratic form q then takes the following form with respect to the basis of $H_1(S_T, \mathbb{Z}) \cong \mathbb{Z}^n$ described above:

$$\begin{aligned} q(x) &= x_1^2 + \ldots + x_n^2 \; - x_1x_2 - x_2x_3 - \ldots - x_{n-1}x_n \\ &= \frac{1}{2}((x_1 - x_2)^2 + \ldots + (x_{n-1} - x_n)^2 + x_1^2 + x_n^2) \end{aligned}$$

For any integral solution x to $q(x) = 1$, necessarily $|x_1|, |x_n| \leqslant 1$. Therefore, we obtain

$$C_1(S_{A_n}) = \{\pm(\mathbf{0}_r, \mathbf{1}_s, \mathbf{0}_t) \in \mathbb{Z}^n \mid r, t \geqslant 0, s \geqslant 1\},$$

where $\mathbf{c}_\nu$ stands for ν consecutive occurences of the number c. It is easily seen that all $n(n+1)$ elements of the above set can be represented by unknotted simple closed curves in S_{A_n}, see Figure 5.3 for an example.

For $T = D_n$, one finds:

$$\begin{aligned} q(x) &= x_1^2 + \ldots + x_n^2 \; - x_1(x_2 + x_3 + x_4) - x_4x_5 - \ldots - x_{n-1}x_n \\ &= (\frac{1}{2}x_1 - x_2)^2 + (\frac{1}{2}x_1 - x_3)^2 + \frac{1}{2}(x_1 - x_4)^2 \\ &\quad + \frac{1}{2}((x_4 - x_5)^2 + \ldots + (x_{n-1} - x_n)^2) + \frac{1}{2}x_n^2 \end{aligned}$$

Let $x \in \mathbb{Z}^n$ be a solution of $q(x) = 1$. Then $|x_n| \leqslant 1$, since otherwise the last summand would already be larger than one. If $x_n = 0$, then $x \in C_1(S_{D_{n-1}})$.

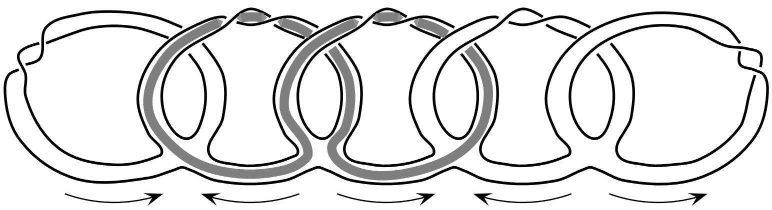

Figure 5.3: A standard Hopf band (in grey) in S_{A_5}, representing the homology class $(0, 1, 1, 0, 0)$ with respect to the homology basis described above. The arrows indicate the chosen orientations of the homology basis vectors.

Otherwise, we may assume $x_n = 1$, up to changing the sign of x. If $x_1 = 0$, then x_2, x_3 cannot be both nonzero, so we can view $x \in C_1(S_{A_{n-1}})$. Hence we may assume $x_1 \neq 0$. If x_1 is odd, then the first two squares are in $\frac{1}{4}\mathbb{N} \setminus \mathbb{N}$ while the last summand is equal to $\frac{1}{2}$. This implies that the rest vanishes, i.e., $x_1 = x_4 = x_5 = \ldots = x_n = 1$, and hence $x_2, x_3 \in \{0, 1\}$. If x_1 is even, the first two squares are in $\mathbb{N}$ and must therefore vanish, while

$$(x_1 - x_4)^2 + (x_4 - x_5)^2 + \ldots + (x_{n-2} - x_{n-1})^2 + (x_{n-1} - 1)^2 = 1.$$

This implies that exactly one of these squares equals one and the rest vanishes, so we obtain the following solutions (recall that $x_1 \neq 0$ by assumption).

$$x = (2, 1, 1, \mathbf{2}_r, \mathbf{1}_s) \in \mathbb{Z}^n, \quad r \geqslant 0, \; s \geqslant 1$$

If M denotes the matrix of the homological action of the monodromy with respect to the basis $v_1, \ldots, v_n$, the following relations hold for $r \geqslant 0$, $s \geqslant 1$.

$$\begin{aligned} (\mathbf{1}_{r+3}, \mathbf{0}_{n-r-3})^\top &= (-1)^{r+1} M^{r+2} (\mathbf{0}_{n-r-1}, \mathbf{1}_{r+1})^\top \\ (2, 1, 1, \mathbf{2}_r, \mathbf{1}_s, \mathbf{0}_{n-r-s-3})^\top &= (-1)^{r+1} M^{r+1} (1, 0, 0, \mathbf{1}_{s-1}, \mathbf{0}_{n-s-2})^\top \end{aligned}$$

Hence all solutions to $q(x) = 1$ are contained in orbits of standard Hopf bands under the monodromy and are therefore realisable by unknotted simple closed curves in S_T. In total, we obtain

$$\begin{aligned} C_1(S_{D_n}) &= \{\pm(2,1,1,\mathbf{2}_r,\mathbf{1}_s,\mathbf{0}_t) \in \mathbb{Z}^n \mid r,t \geqslant 0,\ s \geqslant 1\} \\ &\cup \{\pm(1,x_2,x_3,\mathbf{1}_r,\mathbf{0}_s) \in \mathbb{Z}^n \mid x_2,x_3 \in \{0,1\},\ r,s \geqslant 0\} \\ &\cup \{\pm(\mathbf{0}_r,\mathbf{1}_s,\mathbf{0}_t) \in \mathbb{Z}^n \mid r \geqslant 3,\ s \geqslant 1\} \ \cup\ \{\pm v_2, \pm v_3\}. \end{aligned}$$

A combinatorial calculation shows that $\#C_1(S_{D_n}) = 2n(n-1)$.
For $T = E_6$, the quadratic form q takes the following form.

$$\begin{aligned} q(x) &= x_1^2 + \ldots + x_6^2 \ - x_1(x_2+x_3+x_4) - x_3x_5 - x_4x_6 \\ &= (\frac{1}{2}x_1 - x_2)^2 + \frac{1}{3}((x_1 - \frac{3}{2}x_3)^2 + (x_1 - \frac{3}{2}x_4)^2) \\ &\quad + (\frac{1}{2}x_3 - x_5)^2 + (\frac{1}{2}x_4 - x_6)^2 + \frac{1}{12}x_1^2 \end{aligned}$$

E_6 contains A_5 and D_5 as subtrees, whose sets of Hopf bands we know. Let x be a solution of $q(x) = 1$ different from these (in particular, $x_2, x_5, x_6 \neq 0$). From the condition $\frac{1}{12}x_1^2 \leqslant 1$ we obtain $|x_1| \leqslant 3$. If $x_1 = 0$, at most one of x_2, x_3, x_4 can be nonzero. It follows that x is supported in one of the arms of the tree, which we excluded. Therefore we may assume $x_1 \in \{1,2,3\}$ (up to changing the sign of x). If $x_1 = 3$, then $\frac{1}{12}x_1^2 = \frac{3}{4}$ and $(\frac{1}{2}x_1 - x_2)^2 \in \frac{1}{4}\mathbb{N}\setminus\mathbb{N}$. So $x_2 \in \{1,2\}$ and $x_3 = x_4 = 2$, $x_5 = x_6 = 1$, hence

$$x = (3, x_2, 2, 2, 1, 1), \quad x_2 \in \{1,2\}.$$

If $x_1 = 2$, then $x_2 = 1$ (otherwise $(\frac{1}{2}x_1 - x_2)^2 + \frac{1}{12}x_1^2 > 1$), and similarly $x_3, x_4 \in \{1,2\}$, which implies $x_5 = x_6 = 1$ (remember we excluded $x_5 = 0$ or $x_6 = 0$). This yields the four possibilities

$$x = (2, 1, x_3, x_4, 1, 1), \quad x_3, x_4 \in \{1,2\}.$$

Finally, if $x_1 = 1$, we can successively deduce $|x_i| \leqslant 1$ for $i = 2, \ldots, 6$, hence

$$x = (1,1,1,1,1,1).$$

It turns out that all of these homology classes can be realised by unknotted simple closed curves. In fact, they all lie, up to sign, in the orbits of $v_1, v_1 + v_2, v_1 + v_3, v_1 + v_4$ under the monodromy. In total, we have 14 vectors plus $2 \cdot (\#C_1(S_{D_5})) - \#C_1(S_{D_4}) + 2 = 58$ vectors coming from the subtrees D_5 and A_5, so $\#C_1(S_{E_6}) = 72$.
For E_7, we obtain

$$\begin{aligned} q(x) &= x_1^2 + \ldots + x_7^2 \ - x_1(x_2+x_3+x_4) - x_3x_5 - x_4x_6 - x_6x_7 \\ &= (\frac{1}{2}x_1 - x_2)^2 + \frac{1}{3}((x_1 - \frac{3}{2}x_3)^2 + (x_4 - \frac{3}{2}x_6)^2) \\ &\quad + \frac{2}{3}(\frac{3}{4}x_1 - x_4)^2 + (\frac{1}{2}x_3 - x_5)^2 + (\frac{1}{2}x_6 - x_7)^2 + \frac{1}{24}x_1^2 \end{aligned}$$

The vectors x satisfying $q(x) = 1$ that are not supported on the D_6 or E_6 subtrees are (up to sign)

$$\begin{array}{lll} (4,2,3,3,x_5,2,1), & (3,x_2,2,x_4,1,2,1), & (2,1,x_3,2,1,x_6,1), \\ (4,2,2,3,1,2,1), & (3,x_2,2,2,1,1,1), & (2,1,x_3,1,1,1,1), \\ (1,1,1,1,1,1,1), & & \end{array}$$

where $x_2, x_3, x_5, x_6 \in \{1,2\}$ and $x_4 \in \{2,3\}$ can be freely chosen. As before, these homology classes are contained in the orbits of the Hopf bands v_1, v_2, v_3, $v_1 + v_3$, $v_1 + v_4$, $v_4 + v_6$, $v_1 + v_4 + v_6$ under the monodromy, hence realisable by unknotted simple closed curves. A count of elements yields $\#C_1(S_{E_7}) = 126$.
Finally, for E_8

$$\begin{aligned} q(x) &= x_1^2 + \ldots + x_8^2 \ - x_1(x_2 + x_3 + x_4) - x_3x_5 - x_4x_6 - x_6x_7 - x_7x_8 \\ &= (\frac{1}{2}x_1 - x_2)^2 + \frac{1}{3}((x_1 - \frac{3}{2}x_3)^2 + (x_6 - \frac{3}{2}x_7)^2) + \frac{2}{5}(x_1 - \frac{5}{4}x_4)^2 \\ &\quad + \frac{3}{8}(x_4 - \frac{4}{3}x_6)^2 + (\frac{1}{2}x_3 - x_5)^2 + (\frac{1}{2}x_7 - x_8)^2 + \frac{1}{60}x_1^2 \end{aligned}$$

The vectors not supported on the D_7 or E_7 subtrees are

$$\begin{array}{lll} (6,3,4,5,2,4,3,x_8), & (6,3,4,5,2,x_5,2,1), & (6,3,4,4,2,3,2,1), \\ (5,x_2,4,4,2,3,2,1), & (5,x_2,3,4,y_5,3,2,1), & (4,2,3,4,y_5,3,2,1), \\ (4,2,2,4,1,3,2,1), & (4,2,3,3,y_5,3,2,1), & (4,2,3,3,y_5,2,x_7,1), \\ (4,2,2,3,1,x_6,2,1), & (4,2,2,3,1,2,1,1), & (3,y_2,2,3,1,3,2,1), \\ (3,y_2,2,3,1,2,x_7,1), & (3,y_2,2,2,1,2,x_7,1), & (3,y_2,2,2,1,1,1,1), \\ (2,1,x_3,2,1,2,x_7,1), & (2,1,x_3,x_4,1,1,1,1), & (1,1,1,1,1,1,1,1), \end{array}$$

where $x_3, x_4, x_7, x_8, y_2, y_5 \in \{1,2\}$, $x_2, x_6 \in \{2,3\}$, $x_5 \in \{3,4\}$. Again, these vectors all lie in the orbits of v_1, v_2, v_3, v_4, v_1+v_2, v_1+v_3, v_1+v_4, v_6+v_7, v_7+v_8, $v_1+v_4+v_6$ under the monodromy and are therefore represented by unknotted simple closed curves. Together with the vectors supported on the D_7 and E_7 subtrees, they add up to a total count of $\#C_1(S_{E_8}) = 240$.

5 The affine Coxeter-Dynkin trees

Proposition 5. *If T is an affine tree, the set of solutions to $q_T(x) = 1$ contains at least one infinite orbit of a standard Hopf band under the monodromy. In particular, the set $C_1(S_T)$ is infinite. More precisely, there exist vectors $u, w_1, \dots, w_d \in H_1(S_T, \mathbb{Z})$, such that every solution x to $q_T(x) = 1$ is of the form $x = w_i + ku$ for some $k \in \mathbb{Z}$, $i \in \{1, \dots, d\}$.*

Let T be an affine tree and let n be the number of vertices of T. Observe that T is obtained by adding one edge and one vertex v_n to a suitable spherical subtree T'.

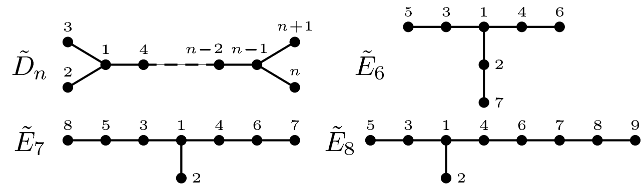

Figure 5.4: The simply laced affine Coxeter trees.

In terms of the associated quadratic forms q_T, $q_{T'}$ this means: there exists a subspace of codimension one on which q_T has a positive definite restriction, i.e., the radical $\ker A_T$ of $q_T(x) = \frac{1}{2} x A_T x^\top$ is one-dimensional. Let $u \in \mathbb{Z}^n$ be such that $\mathbb{Z}u = \ker A_T$. Observe that

$$q_T(w + ku) = q_T(w) + k\, w A_T u^\top + k^2 q_T(u) = q_T(w),$$

for all $w \in \mathbb{Z}^n$, $k \in \mathbb{Z}$. Let V be the Seifert matrix of S_T with respect to the basis $v_1, \dots, v_n$ of $H_1(S_T, \mathbb{Z})$ and denote by M the matrix of the monodromy of S_T with respect to the same basis. As mentioned in Chapter 1 and in Section 3 respectively, the Seifert matrix V is invertible and the following relations hold.

$$M = V^{-\top} V, \quad -A_T = V + V^\top$$

Since $u \in \ker A_T$, we have $Vu = -V^\top u$ and therefore $Mu = -u$. Moreover, we will see that the last coordinate of u (corresponding to the vertex v_n of T) equals ± 1. Therefore, $k \in \mathbb{Z}$ can be chosen such that $w + ku$ is supported in T', so

$$q_T^{-1}(1) = \{w + ku \mid w \in q_{T'}^{-1}(1) \times \{0\},\ k \in \mathbb{Z}\}.$$

To simplify notation, we identify the vector $(x_1, \dots, x_{n-1}) \in \mathbb{Z}^{n-1}$ with the vector $(x_1, \dots, x_{n-1}, 0) \in \mathbb{Z}^n$. It should be clear from the context when a last entry equal to zero is to be deleted from a vector and when a zero should be appended to a vector.

The smallest example of an affine tree T is the the "X"-shaped tree $\tilde{D}_4$ with five vertices $v_1, \ldots, v_5$ where v_1 has degree four and $v_2, \ldots, v_5$ have degree one (compare Figure 5.4). In that case, $q(x) = q_T(x)$ can be written as a sum of four squares:

$$\begin{aligned} q(x) &= x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 \; - x_1(x_2 + x_3 + x_4 + x_5) \\ &= (\frac{1}{2}x_1 - x_2)^2 + (\frac{1}{2}x_1 - x_3)^2 + (\frac{1}{2}x_1 - x_4)^2 + (\frac{1}{2}x_1 - x_5)^2 \end{aligned}$$

One easily finds that the vector $u = (2, 1, 1, 1, 1)$ spans the radical. The subtree whose vertices are $v_1, \ldots, v_4$ is of type D_4. Therefore, all solutions x to $q(x) = 1$ are of the form $x = w + ku$, where $k \in \mathbb{Z}$ and $w \in C_1(S_{D_4}) \subset \mathbb{Z}^4$. We know from the previous section on spherical trees that $C_1(S_{D_4})$ consists of the following vectors, up to sign and up to permutation of the last three entries:

$$\begin{aligned} w_1 &= (2, 1, 1, 1), \quad & w_2 &= (1, 1, 1, 1), \quad & w_3 &= (1, 1, 1, 0), \\ w_4 &= (1, 1, 0, 0), \quad & w_5 &= (1, 0, 0, 0), \quad & w_6 &= (0, 1, 0, 0). \end{aligned}$$

The matrix M of the (homological) monodromy with respect to the basis $v_1, \ldots, v_5$ of $H_1(S_T, \mathbb{Z})$ can be computed using the formula in terms of the Seifert matrix V mentioned above. Concretely:

$$M = \begin{pmatrix} -3 & 1 & 1 & 1 & 1 \\ -1 & 1 & & & \\ -1 & & 1 & & \\ -1 & & & 1 & \\ -1 & & & & 1 \end{pmatrix}$$

The following relations hold:

$$\begin{aligned} M^2(w_1) &= w_1 + u, \quad & M^2(w_2) &= w_2 - u, \quad & &\boxed{M^2(w_3) = w_3} \\ M^2(w_4) &= w_4 + u, \quad & M^2(w_5) &= w_5 + 2u, \quad & M^2(w_6) &= w_6 - u \end{aligned}$$

We will now deduce from these relations that the set of homology classes of Hopf bands $C_1(S_T)$ decomposes into finitely many orbits under the action of the monodromy M. Indeed, the above relations imply that the families $\{w_i + ku\}_{k \in \mathbb{Z}}$, $i \neq 3$, fall into finitely many orbits. On the other hand, each of the vectors $w^{(k)} := w_3 + ku$, $k \in \mathbb{Z}$, is fixed by M^2, so they cannot be contained in a finite union of M-orbits. However, $w^{(k)} \notin C_1(S_T)$ for $k \neq 0, -1$, since these homology classes cannot be represented by a simple closed curve in S_T, as we demonstrate now. S_T is a surface of genus one with four boundary components. If we forget about the embedding of $S_T \subset S^3$, we can abstractly glue three disks to cap off all but one boundary component. The result is an abstract (non-embedded) once punctured torus. Therein, a nonzero first homology class c can be represented by a simple closed curve if and only if c is a primitive vector, that is, if $c = \lambda c'$ implies $|\lambda| = 1$ for $\lambda \in \mathbb{Z}$ (compare Remark 7). To make use of this criterion, we change the basis $(v_1, \ldots, v_5)$ of $H_1(S_T, \mathbb{Z})$ to the basis $(v_1, v_2, v_2 - v_3, v_3 - v_4, v_4 - v_5)$. The last three elements of this new basis can be represented by three of the four boundary curves of S_T. Therefore, capping off these three boundary components has the effect of deleting the last three entries of the corresponding coordinate vectors. Rewriting $w^{(k)}$ in the new coordinates yields the vector

$$(2k+1, 4k+2, -(3k+1), -2k, -k).$$

Under the inclusion of S_T into the capped-off surface, we obtain the vector

$$(2k+1, 4k+2) = (2k+1) \cdot (1, 2),$$

which is primitive for $k \in \{0, -1\}$ only. Hence $w^{(k)} \notin C_1(S_T)$ for $k \neq 0, -1$ and the set $C_1(S_T)$ decomposes into finitely many orbits under M.
For the other members of the $\tilde{D}_n$ family ($n \geqslant 5$), we obtain

$$\begin{aligned} q(x) &= (\frac{1}{2}x_1 - x_2)^2 + (\frac{1}{2}x_1 - x_3)^2 \\ &\quad + \frac{1}{2}\left((x_1 - x_4)^2 + (x_4 - x_5)^2 + \ldots + (x_{n-2} - x_{n-1})^2\right) \\ &\quad + (\frac{1}{2}x_{n-1} - x_n)^2 + (\frac{1}{2}x_{n-1} - x_{n+1})^2 \end{aligned}$$

whose radical is spanned by $u = (2, 1, 1, \mathbf{2}_{n-4}, 1, 1)$. As for $\tilde{D}_4$, the solutions to $q(x) = 1$ are the vectors of the form $w_i + ku$, where $k \in \mathbb{Z}$ and $w_i \in C_1(S_{D_n})$, $i \in \{1, \ldots, 2n(n-1)\}$, are the homology classes of Hopf bands supported in the D_n subtree of $\tilde{D}_n$. A calculation with the monodromy matrix M and the first standard Hopf band v_1 shows that $M^{n-2}v_1 = (-1)^n(v_1 + 2u)$. Since $Mu = -u$, this implies

$$M^{(n-2)k}v_1 = (-1)^{nk}(v_1 + 2ku), \quad \forall k \in \mathbb{Z}.$$

Therefore, there is at least one infinite orbit of homology classes of Hopf bands in $C_1(S_T)$. In contrast, there do exist w_i such that the family $\{w_i + ku\}_{k \in \mathbb{Z}}$ does not fall into finitely many orbits under the monodromy and still consists of homology classes of simple closed curves. For example, the vectors $w = (\mathbf{0}_r, \mathbf{1}_s, \mathbf{0}_t) \in \mathbb{Z}^{n+1}$, $3 \leqslant r \leqslant n-1$, $1 \leqslant s \leqslant n-r-1$, are in fact all fixed by $M^{2(n-2)}$, and $w + ku$

is realisable as a simple closed curve, for every $k \in \mathbb{Z}$. However, we do not know whether it can be realised as an *unknotted* simple closed curve for $k \notin \{0, -1\}$. The same situation occurs for $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$. For $T = \tilde{E}_6$, the corresponding quadratic form q can be written as follows.

$$\begin{aligned} q(x) &= \frac{1}{3}\left((x_1 - \frac{3}{2}x_2)^2 + (x_1 - \frac{3}{2}x_3)^2 + (x_1 - \frac{3}{2}x_4)^2\right) \\ &\quad + (\frac{1}{2}x_2 - x_7)^2 + (\frac{1}{2}x_3 - x_5)^2 + (\frac{1}{2}x_4 - x_6)^2 \end{aligned}$$

The radical of q is spanned by the vector $u = (3, 2, 2, 2, 1, 1, 1)$, and $M^2 v_1 = v_1 + u$, where M and v_1 denote again the monodromy and the first standard Hopf band, respectively. This implies that the orbit of v_1 under the monodromy is infinite. For the tree $\tilde{E}_7$, we have:

$$\begin{aligned} q(x) &= (\frac{1}{2}x_1 - x_2)^2 + (\frac{1}{2}x_5 - x_8)^2 + (\frac{1}{2}x_6 - x_7)^2 \\ &\quad + \frac{2}{3}\left((\frac{3}{4}x_1 - x_3)^2 + (\frac{3}{4}x_1 - x_4)^2\right) \\ &\quad + \frac{1}{3}\left((x_3 - \frac{3}{2}x_5)^2 + (x_4 - \frac{3}{2}x_6)^2\right) \end{aligned}$$

The radical is generated by $u = (4, 2, 3, 3, 2, 2, 2, 1, 1)$ and one verifies the relation $M^3 v_1 = -v_1 - u$. Finally, for $\tilde{E}_8$, we obtain:

$$\begin{aligned} q(x) &= (\frac{1}{2}x_1 - x_2)^2 + (\frac{1}{2}x_3 - x_5)^2 + (\frac{1}{2}x_8 - x_9)^2 \\ &\quad + \frac{3}{5}(\frac{5}{6}x_1 - x_4)^2 + \frac{2}{5}(x_4 - \frac{5}{4}x_6)^2 + \frac{2}{3}(\frac{3}{4}x_6 - x_7)^2 \\ &\quad + \frac{1}{3}\left((x_1 - \frac{3}{2}x_3)^2 + (x_7 - \frac{3}{2}x_8)^2\right) \end{aligned}$$

The radical is the span of $u = (6, 3, 4, 5, 2, 4, 3, 2, 1)$, $M^5 v_1 = -v_1 - u$.

In summary, every solution x to the equation $q_T(x) = 1$ (T affine) is of the form $x = w_i + ku$ for some $k \in \mathbb{Z}$, where u generates the radical of q_T and the w_i are finitely many homology classes of Hopf bands contained in $S_{T'}$ for a spherical subtree $T' \subset T$. For certain i, the family $\{w_i + ku\}_{k \in \mathbb{Z}}$ is contained in finitely many orbits under the monodromy M of S_T, while the members of the remaining families are fixed by some power M^d. Among the latter, there are homology classes that cannot be realised by simple closed curves, and there are such families whose members are realisable by simple closed curves.

Question. Can these homology classes be realised by *unknotted* simple closed curves? In other words, does $C_1(S_T)$ decompose into finitely many orbits under the monodromy, for any affine tree T?

6 Infinite sets of orbits for hyperbolic trees

As described above, $C_1(S_T)$ is finite for the spherical trees and infinite for the affine trees. However, $C_1(S_T)$ could still decompose into finitely many orbits under the monodromy. We claim this is not anymore true for hyperbolic trees, at least when ∂S_T is a knot.

Proposition 6. *Let T be a hyperbolic tree, let S_T be the corresponding fibre surface and denote the monodromy by φ. If ∂S_T is a knot, then the set $C_1(S_T)$ of homology classes of Hopf bands consists of infinitely many φ_*-orbits.*

The key idea for proving the proposition is to take an affine subtree $T' \subset T$ and to compare the action of the monodromy φ'_* on $H_1(S_{T'}, \mathbb{Z})$ with the action of φ_* on $H_1(S_T, \mathbb{Z})$. In the previous section, we found φ'_*-orbits of Hopf bands consisting of vectors $v_k \in H_1(S_{T'}, \mathbb{Z})$ that grow linearly in $k \in \mathbb{Z}$ with respect to any norm. This was possible because the Jordan normal form of φ'_* has Jordan blocks of size two to eigenvalues which are roots of unity. However, it turns out that the monodromy φ_* of the larger surface S_T does not have such Jordan blocks. Therefore, the family v_k cannot be a union of finitely many orbits under φ_* since the "gaps" between consecutive members of an orbit must either stay bounded or grow exponentially. More specifically, the proposition will follow from the two subsequent lemmas.

Lemma 6. *Let T be a hyperbolic tree such that ∂S_T is a knot, and denote the corresponding monodromy by $\varphi : S_T \to S_T$. The Jordan normal form of $\varphi_* : H_1(S_T, \mathbb{Z}) \to H_1(S_T, \mathbb{Z})$ cannot contain any Jordan block of size greater than one to an eigenvalue of modulus one.*

Proof. The main ideas for the proof are contained in an appendix by Feller and Liechti to an article of Liechti [Li1]; see also [GL]. However, we choose to reformulate them here for the reader's convenience. Let A be a Seifert matrix for S_T with respect to some basis of $H_1(S_T, \mathbb{Z})$. Then, the monodromy φ_* has matrix $A^{-\top}A$ with respect to that basis. Given an eigenvalue ω of φ_*, we denote the algebraic and geometric multiplicities of ω by $m_{alg}(\omega)$ and $m_{geom}(\omega)$, respectively. Thus, $m_{alg}(\omega)$ is the multiplicity of the zero ω of the characteristic polynomial of φ_*, while $m_{geom}(\omega)$ is the number of Jordan blocks to ω in the Jordan normal form of φ_*. Our goal is to prove $m_{geom}(\omega) = m_{alg}(\omega)$ for every eigenvalue $\omega \in S^1$ of φ_*, or, equivalently, of φ_*^{-1}. Suppose to this end that $\omega \in S^1$ is an eigenvalue of $\varphi_*^{-1} = A^{-1}A^\top$. Then we have

$$0 = \det(A)\det(\omega I - \varphi_*^{-1}) = \det(\omega A - A^\top) = \Delta_K(\omega),$$

where $\Delta_K(t)$ denotes the Alexander polynomial of the knot $K = \partial S_T$ and I denotes the identity map on $H_1(S_T, \mathbb{Z})$. For $\omega \in S^1$, the ω-signature (after Levine and Tristram [Tr]) is defined to be the signature $\sigma_\omega \in \mathbb{Z}$ of the Hermitian matrix

$$M_\omega = (1-\omega)A + (1-\bar{\omega})A^\top = -(1-\bar{\omega})(\omega A - A^\top),$$

that is, the number of positive eigenvalues minus the number of negative eigenvalues of M_ω. As $\omega = e^{\pi i t}$ traverses one half of the unit circle for $t \in (0,1]$, the ω-signature σ_ω stays constant except at zeros ω_0 of Δ_K (by the first equation above), where it may jump by some even amount $2j_{\omega_0}$. In addition, if $\omega \in S^1$ is an eigenvalue of φ_*^{-1},

$$|j_\omega| \leqslant \mathrm{null}(M_\omega) = \mathrm{null}(\omega I - \varphi_*^{-1}) = m_{geom}(\omega) \leqslant m_{alg}(\omega),$$

where $\mathrm{null}(B)$ denotes the nullity of a matrix B, that is, the geometric multiplicity of the eigenvalue 0 of B. For $\omega \in S^1$ near 1 we have $\sigma_\omega = 0$ (see Feller and Liechti's appendix of [Li1]), whereas for $\omega = -1$, σ_ω takes the value of the classical signature invariant for knots, $\sigma(K)$. This implies

$$\sigma(K) = \sigma_{-1}(K) \leqslant 2 \cdot \sum_{\omega \in S^1_+} |j_\omega| \leqslant 2 \cdot \sum_{\omega \in S^1_+} m_{alg}(\omega) = \sigma(K),$$

where $S^1_+ = \{e^{\pi i t} \mid t \in (0,1]\}$ denotes the upper half of the unit circle. The last equality follows from the fact that the number of zeros of Δ_L on S^1 (counted with multiplicity) equals $\sigma(L) + \mathrm{null}(L)$ for any tree-like Hopf plumbing L, where $\mathrm{null}(L) = \mathrm{null}(M_{-1})$ equals zero if L is a knot. This is proven in a preprint by Liechti [Li2]. Therefore the above inequalities are in fact equalities, which implies $m_{geom}(\omega) = m_{alg}(\omega)$ for all zeros $\omega \in S^1_+$ of Δ_K. By the symmetry of Alexander polynomials, the same holds for the zeros ω of $\Delta_K(t)$ such that $-\omega \in S^1_+$. Finally, $\Delta_K(1) \neq 0$ because K is a knot. This ends the proof. □

Lemma 7. *Let Φ be a matrix of size $n \times n$ with coefficients in $\mathbb{C}$, and let $v \in \mathbb{C}^n$ be any vector. Suppose that the Jordan normal form of Φ does not contain any Jordan block of size greater than one to an eigenvalue of modulus one. Then the sequence $\{\Phi^k(v)\}_{k \in \mathbb{N}}$ is either bounded or grows exponentially (with respect to any norm $\|.\|$ on $\mathbb{C}^n$).*

By exponential growth of a sequence $\{v_k\}_{k \in \mathbb{N}} \subset \mathbb{C}^n$ with respect to a norm $\|.\|$ we mean the existence of constants $h > 1$, $c > 0$, $d \geqslant 0$, such that $\|v_k\| \geqslant ch^k - d$ for all $k \in \mathbb{N}$.

Proof of Lemma 7. We may first assume that Φ is already in Jordan normal form, and second that Φ consists of just one Jordan block to some eigenvalue λ. If $v = 0$, the conclusion is clear, so we assume $v \neq 0$. If $|\lambda| < 1$, the sequence $\Phi^k(v)$ is bounded, if $|\lambda| > 1$, it grows exponentially and if $|\lambda| = 1$, Φ is of size one, so $\Phi^k(v)$ is bounded. $\square$

Proof of Proposition 6. Let T be a hyperbolic tree. Since T contains an affine subtree T', we have at least one infinite family $\{v_k\}_{k\in\mathbb{N}}$ of elements $v_k \in C_1(S_{T'}) \subset C_1(S_T)$ which grow linearly in k when seen as a sequence of vectors in $H_1(S_T, \mathbb{Z}) \subset H_1(S_T, \mathbb{C}) \cong \mathbb{C}^n$. More precisely, the v_k are pairwise distinct, and there exist constants $a > 0, b \geqslant 0$, such that

$$\|v_k\| \leqslant ak + b, \quad \forall k \in \mathbb{N}.$$

Indeed, we found such families for every affine tree in the preceding section. Assume now that ∂S_T is a knot. While the family $\{v_k\}$ could still decompose into finitely many orbits under the monodromy φ'_* of the smaller surface $S_{T'}$, we will show this cannot be the case for orbits of φ_*, the monodromy of S_T. Namely, assume to the contrary that there were r indices $k_1, \dots, k_r \in \mathbb{N}$ such that the φ_*-orbits of $v_{k_1}, \dots, v_{k_r}$ covered the whole sequence $\{v_k\}_{k\in\mathbb{N}}$. By Lemma 6 and Lemma 7 (applied to $\Phi = \varphi_*$), we obtain the following, for every $i \in \{1, \dots, r\}$. Either the φ_*-orbit of v_{k_i} is bounded and thus finite, or there exist $h_i > 1$, $c_i > 0$, $d_i \geqslant 0$, such that $\|\varphi_*^k(v_{k_i})\| \geqslant c_i h_i^k - d_i$ for all $k \in \mathbb{N}$. Replacing h_i, c_i, d_i by minimal or maximal values $h > 1, c > 0, d \geqslant 0$ respectively, we have

$$\|\varphi_*^k(v_{k_i})\| \geqslant ch^k - d, \quad \forall k \in \mathbb{N}.$$

Now, let $K \in \mathbb{N}$ be large (to be specified later) and set $R := aK + b$. We would like to compare the numbers

$$\begin{aligned} p &:= \#\{k \in \mathbb{N} \mid \|v_k\| \leqslant R\}, \\ q &:= \#\{(k,i) \in \mathbb{N} \times \{1, \dots, r\} \mid \|\varphi_*^k(v_{k_i})\| \leqslant R\}. \end{aligned}$$

First, $p \geqslant K$, since $\|v_k\| \leqslant aK + b = R$ for all $k \leqslant K$, and the v_k are pairwise distinct. Second, taking $N := \lfloor \frac{K}{r} \rfloor - 1$, we have $K > rN$, and

$$ch^N - d = ch^{\lfloor \frac{K}{r} \rfloor - 1} - d \geqslant aK + b = R,$$

for large enough K, since $h, c > 1$. Then, we have $q \leqslant rN$, since $\|\varphi_*^k(v_{k_i})\| \geqslant ch^N - d \geqslant R$ as soon as $k \geqslant N$. Thus $p \geqslant K > rN \geqslant q$, contradicting our assumption on the sequence v_k being covered by the φ_*-orbits of its members $v_{k_1}, \dots, v_{k_r}$. This finishes the proof. $\square$

Question (by Pierre Dehornoy)**.** Can every embedded Hopf band in S_T be obtained from one of the Hopf bands v_i by successively applying the monodromies of $S_{T'}$, where T' ranges over suitable subtrees of T?

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Erklärung

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LeiterIn der Arbeit: Prof. Dr. Sebastian Baader

Ich erkläre hiermit, dass ich diese Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen benutzt habe. Alle Stellen, die wörtlich oder sinngemäss aus Quellen entnommen wurden, habe ich als solche gekennzeichnet. Mir ist bekannt, dass andernfalls der Senat gemäss Artikel 36 Absatz 1 Buchstabe r des Gesetzes vom 5. September 1996 über die Universität zum Entzug des auf Grund dieser Arbeit verliehenen Titels berechtigt ist. Ich gewähre hiermit Einsicht in diese Arbeit.

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