

On the signature of positive braids and adjacency for torus knots

Inauguraldissertation

der Philosophisch-naturwissenschaftlichen Fakultät
der Universität Bern

vorgelegt von

Peter Feller

von Köniz

Leiter der Arbeit:

Prof. Dr. Sebastian Baader

Mathematisches Institut

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Introduction

The objects of study in this thesis are knots and links. More precisely, torus knots and links, a subclass of algebraic links, which in turn are a subclass of closures of positive braids. One part of this thesis is concerned with studying how the signature, a classical link invariant, behaves on closures of positive braids. In the other part we study adjacency of algebraic links—in fact, mainly of torus knots—in different contexts; namely in two topological and two algebraic settings.

The text is organized as follows. In the introduction we present the main results of this thesis, while keeping definitions at a minimum. The reader may find definitions in Chapter 1, where we also recall results that are used later on. The signature of closures of positive braids is discussed in Chapter 2. Chapter 3 is on Gordian adjacency of torus knots. In Chapter 4, algebraic notions of adjacency are introduced and their connections to Gordian adjacency and subsurface adjacency are discussed. Finally, subsurface adjacency is inspected in Chapter 5.

To improve readability we restrict the presentation of our results to knots.

Lower bounds on the signature of positive braid knots

We denote the standard generators of the braid group on b strands by a_1, \dots, a_{b-1} . A positive braid is an element in the braid group that can be written as a positive braid word $a_{s_1} a_{s_2} \cdots a_{s_l}$ with $s_i \in \{1, \dots, b-1\}$. Every knot is the closure of a braid by a result of Alexander [Ale23]. Knots that are closures of positive braids are called *positive braid knots*. Torus knots $T(p, q)$, where p and q are coprime positive integers, are examples of positive braid knots.

We compare two classical knot invariants, the genus g and the signature σ , for positive braid knots. The genus of a positive braid knot K is fully understood. If K is the closure of a positive braid β on b strands of length l , then $g(K) = \frac{l-b+1}{2}$ holds by Bennequin's inequality [Ben83]. Our main result on positive braid knots establishes a linear lower bound for the signature in terms of the genus. It is submitted for publication [Fel13].

THEOREM 2.1. *For all positive braid knots K , we have*

$$2g(K) \geq \sigma(K) \geq \frac{g(K)}{50}.$$

For a knot K we denote by $g_t^4(K)$ the *topological 4-ball genus*—the minimal genus of oriented surfaces F that are topologically locally-flat embedded in the 4-ball B^4 with boundary $\partial F = K \subset \partial B^4 = S^3$.

The topological 4-ball genus of positive braid knots is in most cases strictly smaller than their genus. For example, the family of torus knots $T(5, 5n + 1)$ has the property that

$$g_t^4(T(5, 5n + 1)) \leq \frac{9}{10}g(T(5, 5n + 1))$$

for all positive integers n ; see Remark 2.3.

It is therefore interesting to know that g_t^4 is linearly bounded from below in terms of g , which we get as a consequence of Theorem 2.1 by using that $g_t^4(K) \geq \frac{|\sigma(K)|}{2}$ holds for all knots K , a result of Kauffman and Taylor [KT76].

COROLLARY 2.2. *For all positive braid knots K , we have*

$$g_t^4(K) \geq \frac{g(K)}{100}.$$

We emphasize that this is a topological locally-flat result. The smooth 4-ball genus and the genus agree on positive braid knots by Rudolph’s slice-Bennequin inequality [Rud93].

Adjacency

The study of singularities, i.e. the study of the local behavior of polynomial maps $f : \mathbb{C}^n \rightarrow \mathbb{C}$, has a long history going back to the pioneer work of Leibniz, Newton and Oldenburg in the 17th century. After endowing polynomials or holomorphic function germs $f : \mathbb{C}^n \rightarrow \mathbb{C}$ (say up to local biholomorphic coordinate changes around the origin) with a topology, one can ask for a fixed germ f , what classes of germs g can be found arbitrarily close to f ? This is known as the adjacency problem. It is only understood for very restricted classes of singularities even when $n = 2$. For example, Arnol’d described all adjacencies between simple singularities—those corresponding to Dynkin diagrams A_n, D_n, E_6, E_7, E_8 [Arn72].

We introduce two notions of adjacency for knots, results on them, and motivation for their study in purely 3-dimensional-topology terms. However, the name “adjacency” and the main motivation for the study of these 3-dimensional notions in this thesis stem from the connection

to algebraic adjacency notions and the adjacency problem, which is discussed afterward.

3-dimensional notions of adjacency. The *unknotting number* of a knot K , denoted by $u(K)$, is the minimal number of crossing changes needed to get from K to the unknot O . It was already studied by Wendt [Wen37]. Generalizing this, the *Gordian distance* $d_G(K, L)$ of two knots K and L is the minimal number of crossing changes needed to get from K to L . The structure of the discrete metric space given by Gordian distance on the set of isotopy classes of knots is not well understood. We study the following concept for torus knots.

DEFINITION. Let K and L be knots. We say K is *Gordian adjacent* to L , denoted by $K \leq_G L$, if $d_G(K, L) = u(L) - u(K)$.

Our main results on Gordian adjacency for torus knots are the following; they are published in [Fel14].

THEOREM 3.2. *Let (n, m) and (a, b) be pairs of coprime positive integers with $n \leq a$ and $m \leq b$. Then the torus knot $T(n, m)$ is Gordian adjacent to the torus knot $T(a, b)$.*

THEOREM 3.3. *Let n and m be positive integers with n odd and m not a multiple of 3. Then the torus knot $T(2, n)$ is Gordian adjacent to the torus knot $T(3, m)$ if and only if $n \leq \frac{4m+1}{3}$.*

As a consequence of Theorem 3.2, Gordian adjacency and Gordian distance for torus knots of a fixed braid index are completely described. Indeed, if a positive integer $a \geq 2$ is fixed, then

$$T(a, b) \leq_G T(a, c) \text{ if and only if } b \leq c$$

for all b, c coprime to a . Hence,

$$d_G(T(a, b), T(a, c)) = |u(T(a, b)) - u(T(a, c))| = \frac{(a-1)|b-c|}{2},$$

where the second equality follows from the Milnor conjecture, which determines the unknotting number of torus knots; see equation (1) below.

An obvious motivation for finding Gordian adjacencies is that, by definition, every Gordian adjacency determines the Gordian distance of the involved knots. Gordian adjacencies can also lead to good estimates of Gordian distances between non-adjacent torus knots. For example, the adjacencies $T(2, 7) \leq_G T(2, 9)$ and $T(2, 7) \leq_G T(3, 5)$ yield

$$\begin{aligned} d_G(T(2, 9), T(3, 5)) &\leq u(T(2, 9)) - u(T(2, 7)) + u(T(3, 5)) - u(T(2, 7)) \\ &= 4 - 3 + 4 - 3 = 2. \end{aligned}$$

In fact, $d_G(T(2, 9), T(3, 5)) = 2$, which can be proven using signatures. Trying to generalize this example for any two torus knots T_1 and T_2 , we look for the highest unknotting number $u(K)$ realized by a knot K adjacent to both T_1 and T_2 , and ask if $u(T_1) - u(K) + u(T_2) - u(K)$ is close to the Gordian distance $d_G(T_1, T_2)$. An ambitious future goal is to use such Gordian adjacencies to determine Gordian distances between all torus knots up to a constant factor, similarly to what was done for the cobordism distance by Baader [Baa12].

There is a weaker 3-dimensional notion of adjacency for torus knots. One can ask, when can a knot K be obtained from another knot L by applying $2(u(L) - u(K))$ saddle moves. This yields a notion of adjacency that is implied by Gordian adjacency since every crossing change can be realized by two saddle moves, see Remark 1.1. For torus knots, all examples of such adjacencies we know come from a more restrictive notion, which we define now.

DEFINITION. For two knots K and L , K is *subsurface adjacent* to L if a genus-minimizing Seifert surface F_K for K can be obtained from a genus-minimizing Seifert surface F_L for L by removing 1-handles.

Our main results on subsurface adjacency is a complete description of adjacencies of torus knots of braid index 2 to torus knots of braid index less or equal than 4. We also provide examples of torus knots $T(2, n)$ that are adjacent to torus knots $T(m, m + 1)$ with n roughly equal to $\frac{2m^2}{3}$. Such examples are not known in the algebraic case, see below.

A motivation for the study of subsurface adjacency comes from the following question. Given a Seifert surface F_L of a knot, we denote the number of positive and negative eigenvalues of the symmetrized Seifert form on $H_1(F_L)$ by p and n , respectively. When can one find an H_1 -injective subsurface F_K such that the symmetrized Seifert form restricted to $H_1(F_K)$ realizes all the positive (negative) eigenvalues, that is $H_1(F_K)$ has rank p (q) and all eigenvalues of the symmetrized Seifert form are positive (negative)? Our results on subsurface adjacency imply that for the genus-minimizing Seifert surfaces of the torus knots $T(3, 6k + 1)$, $T(3, 6k + 2)$ and $T(4, 4k + 1)$, all positive eigenvalues are realized by a subsurface that is isotopic to the fiber surface of a $T(2, n)$ torus link.

Algebraic notions of adjacency. Torus knots are *algebraic knots*. That is, for a torus knot K there exists a polynomial function $f: \mathbb{C}^2 \rightarrow \mathbb{C}$ such that K is isotopic to $K(f) = V(f) \cap S_\varepsilon^3 \subset S_\varepsilon^3 \cong S^3$ for small enough ε , where $V(f) \subseteq \mathbb{C}^2$ denotes the zero-set of f and S_ε^3 denotes the 3-sphere of radius ε centered at the origin in \mathbb{C}^2 . Indeed, for p, q

coprime the knot $K(f)$ associated with the polynomial $f = y^q - x^p$ is the torus knot $T(p, q)$.

The adjacency problem for polynomial maps from \mathbb{C}^2 to \mathbb{C} yields a notion of adjacency for algebraic knots.

DEFINITION. Let K_1 and K_2 be algebraic knots. We say K_1 is *algebraically adjacent* to K_2 , if there exists a smooth family of square-free polynomials $f_t: \mathbb{C}^2 \rightarrow \mathbb{C}$ such that $K_2 = K(f_0)$ and $K_1 = K(f_t)$ for $t \neq 0$ small.

For a small but fixed $t \neq 0$, the zero-set $V(f_t - \eta)$ for small non-zero η in \mathbb{C} is a smooth algebraic curve F in \mathbb{C}^2 such that K_1 and K_2 are realized as transversal intersection of F with two spheres around the origin of different radii $r_1 < r_2$, i.e.

$$\begin{aligned} K_i &= F \cap \{(x, y) \in \mathbb{C}^2 \mid \|x\|^2 + \|y\|^2 = r_i^2\} \\ &\subset \{(x, y) \in \mathbb{C}^2 \mid \|x\|^2 + \|y\|^2 = r_i^2\} \cong S^3. \end{aligned}$$

Denote by $g_s^4(K)$ the *smooth 4-ball genus* of a knot K —the minimal genus of oriented and smoothly embedded surfaces F in the 4-ball B^4 with $\partial F = K \subset \partial B^4 = S^3$. By the Thom conjecture [KM93], the smooth 4-ball genus of K_i equals the genus of the intersection of F with the ball in \mathbb{C}^2 centered at the origin of radius r_i . Thus, the cobordism

$$F \cap \{(x, y) \in \mathbb{C}^2 \mid r_1^2 \leq \|x\|^2 + \|y\|^2 \leq r_2^2\}$$

in

$$\{(x, y) \in \mathbb{C}^2 \mid r_1^2 \leq \|x\|^2 + \|y\|^2 \leq r_2^2\} \cong S^3 \times [0, 1]$$

has minimal genus $g_s^4(K_2) - g_s^4(K_1)$. By the Milnor conjecture, a consequence of the Thom conjecture, the smooth 4-ball genus and the unknotting number u of torus knots are equal, i.e. one has

$$(1) \quad u(T(n, m)) = g_s^4(T(n, m)) = \frac{(n-1)(m-1)}{2}$$

for all coprime natural numbers n, m .

In summary, we know that u and g_s^4 coincide on torus knots and that Gordian adjacency and algebraic adjacency, which could be thought of as relative versions of u and g_s^4 , respectively, have similar properties. For example, for both notions it holds that if a torus knot K_1 is adjacent to a torus knot K_2 , then $u(K_1) = g_s^4(K_1) \leq u(K_2) = g_s^4(K_2)$ and the cobordism distance—the minimal genus of oriented and smoothly embedded surfaces F in $S^3 \times [0, 1]$ with $\partial F = K \times \{0\} \cup L \times \{1\}$ —equals $u(K_2) - u(K_1) = g_s^4(K_2) - g_s^4(K_1)$. Furthermore, for both notions $T(n, m)$ is adjacent to $T(a, b)$ if $n \leq a$ and $m \leq b$. For Gordian adjacency this is Theorem 3.2. For algebraic adjacency it is immediate;

compare Proposition 4.3. It is then natural to wonder whether the two concepts of adjacency coincide on torus knots. In Chapter 4 we answer in the negative by providing families of algebraic adjacencies that are not Gordian adjacencies; see Proposition 4.5. However, we give a heuristic argument supporting the conjecture that if two torus knots are Gordian adjacent, then they are algebraically adjacent; see Remark 4.7.

On the other hand we show that a more restrictive algebraic notion of adjacency, that is δ -constant adjacency, implies Gordian adjacency, up to a well controlled concordance; and that algebraic adjacency of knots K and L implies that L is obtained from K by $2u(L) - 2u(K)$ saddle moves, again up to a well controlled concordance. Finally, we discuss the suspicion that algebraic adjacency implies subsurface adjacency and that δ -constant adjacency implies Gordian adjacency.

A motivation for the study of algebraic adjacency is provided by the following general question. What topological types of singularities can arise on polynomials $f: \mathbb{C}^2 \rightarrow \mathbb{C}$ of fixed degree d ; see for example [GLS98][GZN00]. The topological type of a singularity f is uniquely determined by $K(f)$. The simplest family of singularities are the A_{n-1} -singularities, which are given by the polynomials $y^2 - x^n$. They correspond to the $T(2, n)$ torus links. Asking what A_{n-1} -singularities can arise on polynomials of fixed degree d is very close to asking for which n the torus link $T(2, n)$ is algebraically adjacent to $T(d, d)$. For example if d is even, the polynomial $y^d - (x^{\frac{d}{2}} - y)^2$ of degree d has an $A_{\frac{d^2}{2}-1}$ -singularity at 0 and it is part of the family of polynomials $y^d - (x^{\frac{d}{2}} - ty)^2$, which show that $T(2, \frac{d^2}{2})$ is algebraically adjacent to $T(d, d)$; see Proposition 4.5 and its proof. Let k be the maximal constant such that for big degrees d one can find polynomials with A_n -singularities with $n \sim kd^2$. The above shows for example that $k \geq \frac{1}{2}$. In fact, $\frac{15}{28} \leq k \leq \frac{3}{4}$ by a result of Gusein-Zade and Nekhoroshev [GZN00]. Finding algebraic adjacencies that are analogs of the subsurface adjacencies of $T(2, n)$ to $T(m, m+1)$ with $n \sim \frac{2m^2}{3}$ could lead to $k \geq \frac{2}{3}$. The upper bound $k \leq \frac{3}{4}$ is the same as one gets for adjacency using signatures, compare Remark 4.6.

CHAPTER 1

Preliminaries

In this chapter we set notations and recall facts that are used later on. All maps and manifolds are assumed to be smooth.

1. Links and braids

A *link* is a 1-dimensional manifold embedded in a 3-dimensional manifold. Links are always oriented and non-empty. A *knot* is a connected link. If nothing else is specified, we study links in the Euclidean 3-space \mathbb{R}^3 or in the 3-sphere $S^3 = \mathbb{R}^3 \cup \{\infty\}$. Mostly, no distinction will be drawn between a link and its smooth isotopy class. We denote the number of components of a link by r and the number of pieces of a link that can be separated by 2-spheres by c . Of course, c is less than or equal to r . A link with $c \neq 1$ is called a *split link*.

Every link is the closure of a braid by a very classical result of Alexander [Ale23]. The *braid index* of a link L is the minimal number of strands among braids with closure L . We recall the definition of braids and their closures. A nice reference for braid theory is provided by Birman [Bir74]. Let D denote the closed unit disc in \mathbb{C} centered at 0. For a positive integer b , fix a set $P \subset [-1, 1] \subset D$ of b points. A *braid* on b strands or *b -braid* β is an embedding of b disjoint closed intervals—the strands—into the cylinder $[0, 1] \times D$ such that the projection of β to the middle line $[0, 1] \times \{0\}$ is a b -fold cover and such that β intersects the 0 and the 1 level of $[0, 1] \times D$ exactly in P . We consider braids up to isotopy fixing $\{0, 1\} \times D$. Braids on b strands form a group, where composition is given by stacking cylinders on top of each other. This group, the *braid group on b strands*, is denoted by B_b . A group presentation with generators a_1, \dots, a_{b-1} and relations

$$a_i a_j = a_j a_i \quad \text{for } |i - j| \geq 2 \quad \text{and} \quad a_i a_j a_i = a_j a_i a_j \quad \text{for } |i - j| = 1,$$

was introduced by Artin [Art25]. The *generator* a_i corresponds to the braid in which strands i and $i + 1$ make a half turn around each other; see Figure 1. When we speak of a “generator” of the braid group we always mean such an a_i . The *closure of a braid* $\beta \subset [0, 1] \times D$ is obtained by identifying $\{0\} \times D$ with $\{1\} \times D$ yielding a link in the solid



FIGURE 1. A generator of the braid group. The “axis” of the cylinder $[0, 1] \times D$, that is the first factor, is drawn vertically in \mathbb{R}^3 .

torus $S^1 \times D$. The closure of a braid is easily understood to be a link in \mathbb{R}^3 or S^3 via a standard embedding of the solid torus $S^1 \times D$ in \mathbb{R}^3 or S^3 . We speak of a *closed braid* when we want to view the closure in $S^1 \times D$ up to isotopy. Two elements of the braid group define the same closed braid if and only if they are conjugate. We orient all strands of braids in the same direction, say upwards, and links that arise as closures of braids are oriented accordingly. The (*algebraic*) *length* of a braid β , denoted by $l(\beta)$, is defined to be the number of occurrences of generators a_i minus the number of occurrences of inverses of generators a_i^{-1} in a braid word for β . This is independent of the choice of braid word.

A *positive braid* on b strands is an element β in B_b that can be given by a positive braid word $a_{s_1} a_{s_2} \cdots a_{s_l}$ with $s_i \in \{1, \dots, b-1\}$. Knots and links that are closures of positive braids are called *positive braid knots* and *positive braid links*, respectively.

Torus links are those links that can be isotoped to lie in the standard torus $S^1 \times S^1 \subset \mathbb{R}^3$ such that all components define the same homology class on the torus. They are uniquely determined by how they wind along the standard meridian and longitude of the torus. Up to reflection they are positive braid links. For two natural numbers $n \geq 2$ and $m \geq 2$, we denote by $T(n, m) = T(m, n)$ the (positive) torus link obtained as the closure of the n -strand positive braid $(a_1 \cdots a_{n-1})^m$ or alternatively as the link of the singularity $x^n - y^m$, see Section 4. The braid index of a torus link $T(n, m)$ is the minimum of n and m , see [FW87, Corollary 2.4]. The torus link $T(n, m)$ is a knot if and only if n and m are coprime.

Let β be a b -braid and γ be a d -braid. The *split union* $\beta \sqcup \gamma$ of β and γ is the $(b+d)$ -braid that is β on the first b strands and γ on the other d strands. The *connected sum* $\beta \sharp \gamma$ of β and γ is the $(b+d)$ -braid given by $a_b(\beta \sqcup \gamma)$, that is, it is the split union with an additional generator that “connects” the two braids. Corresponding notions for links K and L are defined by choosing braids β and γ that close to K

and L , respectively. Note that connected sums of links are only well defined if one specifies which components of the links to connect.

All surfaces in this text are compact and orientable. For a surface F , we denote by $b_1(F)$, $g(F)$ and $\chi(F)$ the first Betti number, the genus and the Euler characteristic of F , respectively. A subsurface F of a surface H is called H_1 -*injective* if the embedding induces an injective map on the first integer homology $H_1(\cdot, \mathbb{Z})$. If H has no closed components, this is equivalent to the fact that F can be obtained from H by removing 1-handles and 0-handles.

The *first Betti number* of a link L , denoted by $b_1(L)$, is defined to be the smallest first Betti number of Seifert surfaces for L . For a braid β , we denote by $b_1(\beta)$ the first Betti number of the closure of β . Here, a *Seifert surface* for a link L in \mathbb{R}^3 is an oriented, embedded surface with oriented boundary L . In Section 3 of this chapter and Chapter 2, Seifert surfaces need not be connected, which allows in a natural way to associate unique Seifert surfaces to positive braid links even if they are split links, see Section 3, and which allows the inclusion of split links in Theorem 2.1 without extra considerations. In the rest of the thesis, Seifert surfaces are assumed to be connected.

Seifert surfaces that realize the first Betti number of their boundary are called *minimal Seifert surfaces*. The *genus* of a knot K , denoted by $g(K)$, is half its first Betti number.

2. Distances of knots and links

Let K and L be knots. Their *Gordian distance* $d_G(K, L)$ is the minimal number of crossing changes needed to get from K to L , see e.g. Murakami [Mur85]. A *crossing change* on a knot K is the following operation. Fix some ball B in \mathbb{R}^3 such that the pair $(B, K \cap B)$ is diffeomorphic to either one of the pairs depicted in Figure 2. Then replace

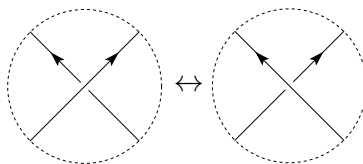


FIGURE 2. A crossing change.

this pair by the other pair in Figure 2. There are two kind of crossing changes. Changing from left to right (right to left) in Figure 2 is called a positive-to-negative crossing change (negative-to-positive crossing change). Accordingly, crossings in a link diagram of a link

in \mathbb{R}^3 —a generic projection of the link to \mathbb{R}^2 with crossing information—are called positive (negative) if they look as depicted in the left (right) of Figure 2.

The *unknotting number* $u(K)$ of a knot K , which was already studied by Wendt [Wen37], is the distance $d_G(K, O)$, where O denotes the unknot—the unique knot with genus 0. The Gordian distance induces a metric on the set of (isotopy classes of) all knots.

The *cobordism distance* d_c between two links K and L is defined to be the minimal $|\chi|$ of all cobordisms between K and L . A *cobordism* between K and L is an oriented and smoothly embedded surface C in $S^3 \times [0, 1]$ with $\partial C = K \times \{0\} \cup L \times \{1\}$ such that every component of C has boundary both in $S^3 \times \{0\}$ and $S^3 \times \{1\}$. We note that defining cobordism distance for knots using the genus instead of $|\chi|$, as is done in the introduction, yields a difference of a factor of 2. The cobordism distance of a link L to the unknot is equal to the *smooth 4-ball first Betti number* $b_1^4(L)$ of L —the minimal first Betti number of all connected, oriented and smoothly embedded surfaces F in B^4 with $\partial F = L \subset \partial B^4 = S^3$. The *smooth 4-ball genus* g_s^4 of a knot is defined to be $\frac{b_1^4}{2}$. Links that have cobordism distance zero are called *concordant*, a cobordism realizing this is called a *concordance*. As an abstract surface a concordance is just a disjoint union of annuli. For two knots K and L we have $d_c(K, L) = 2g_s^4(K\sharp - L)$, where $-L$ denotes the *reflection* of L obtained by reflecting L through a plane and reversing orientation. In particular, for every knot K one has $g_s^4(K\sharp - K) = 0$, i.e. every connected sum of a knot K with his reflection is concordant to the unknot, see for example [Rol90].

A *saddle move* on a link is the operation given by changing the link in a ball as indicated in Figure 3. Every cobordism can be isotoped

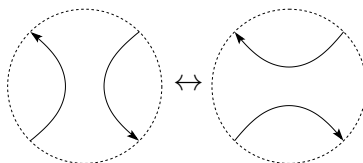


FIGURE 3. A saddle move.

in $S^3 \times [0, 1]$ such that the projection to $[0, 1]$ is a Morse function, i.e. it has a finite number of non-degenerate singular points projecting to different singular values, which correspond to handle attachments. Viewing $[0, 1]$ as time parameter one can see such a cobordism as a movie of S^3 . Such a movie consists of isotopies on the regular open

intervals between the singular values. Going over a singular value an unknotted component appears, a saddle move happens or an unknotted component disappears corresponding to relative minima, saddles and relative maxima in the cobordism, respectively.

Conversely, every such movie defines a cobordism. In particular, a finite sequence of saddle moves changing a link K to a link L defines a cobordism from K to L . For example, if a Seifert surface F_K of K is an H_1 -injective subsurface of a Seifert surface F_L of L , then L can be obtained from K by $b_1(F_L) - b_1(F_K)$ saddle moves since removing a 1-handle in a Seifert surface corresponds to doing a saddle move on the link and to removing a 1-handle (a saddle) in the cobordism, respectively. Of course, not every cobordism is given by saddle moves only as any concordance between two non-isotopic knots shows.

REMARK 1.1. The Gordian distance of two knots is larger than half their cobordism distance. This follows from the fact that a crossing change can be realized by two saddle moves and so by a cobordism of genus 1.

More generally, even two crossing changes can be realized by a cobordism of genus 1, as long as the two crossing changes are of opposite kind. This can be seen as follows. Let B_1 and B_2 be two disjoint balls where the crossing changes happen. We isotope B_2 along the knot until it sits inside B_1 . Since the two crossing changes are of opposite kind the situation is as in the leftmost in Figure 4. Figure 4 indicates how

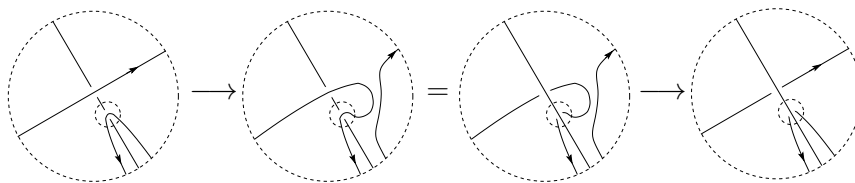


FIGURE 4. The ball B_2 is isotoped into B_1 (left). Two saddle moves (indicated by arrows) realize the crossing changes in B_2 and B_1 .



two saddle moves, which together yield a genus 1 cobordism, achieve the two crossing changes. Finally, one isotopes B_2 back.

3. Minimal Seifert surfaces and fence diagrams for positive braids

In this section we recall how to switch between three ways of viewing positive braids; namely by their braid diagrams, by their associated minimal Seifert surfaces and their fence diagrams.

A braid diagram is a representation of a braid β in $[0, 1] \times D$ as the projection to $[0, 1] \times [-1, 1]$ keeping the crossing information. When drawing braid diagrams for a braid $a_{i_1} a_{i_2} \cdots a_{i_l}$, we start with the left-most generator a_i drawn as in Figure 1, then draw the second generator on top, and so on. For example, the positive 3-braid $a_1 a_2$ is represented

by the diagram , where the generators a_1 and a_2 are represented by

 and $a_2 = | \text{ $, respectively. Since in a diagram associated with a positive braid word all over-crossing strands go from left to right, replacing every crossing in the diagram with a horizontal line still allows one to recover the positive braid. This yields *fence diagrams* of positive braids as used by Rudolph [Rud98]. Let F_β denote the Seifert surface for the closure of β that is given by replacing every vertical line of the fence diagram with a long vertical disc and every horizontal line with a band connecting two discs. The fence diagram of a positive braid β , seen as graph in \mathbb{R}^3 , is a deformation retract of F_β . We feel that all the above becomes very clear by considering an example. Figure 5 provides the braid diagram, the fence diagram and the Seifert surface

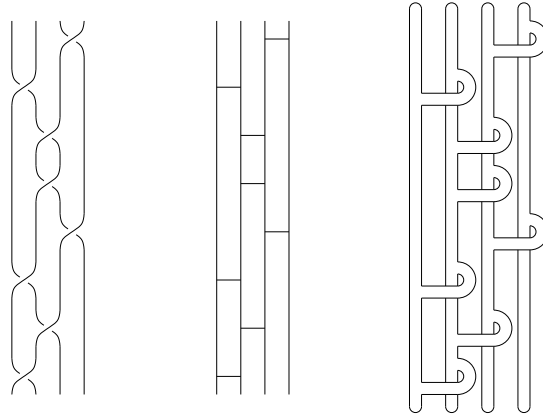


FIGURE 5. The braid diagram, the fence diagram and F_β for the positive 4-braid word $a_1 a_2 a_1 a_3 a_2 a_2 a_1 a_3$. This image is based on a template by Sebastian Baader.

F_β of the closure, for the positive 4-braid $\beta = a_1 a_2 a_1 a_3 a_2 a_2 a_1 a_3$.

In fact, F_β is the unique (up to isotopy) minimal Seifert surface for the closure of the positive braid β . This follows from the fact (due to Stallings [Sta78]) that every component of the surface F_β is a fiber surface, which always is the unique minimal Seifert surface for its boundary.

In particular, the first Betti number of a positive braid is equal to the first Betti number of the corresponding fence diagram. Hence,

$$(2) \quad b_1(\beta) = l(\beta) - b + c \text{ for every positive braid } \beta,$$

where $l(\beta)$ is the length of β , b is the number of strands of β and c equals 1 plus the number of generators a_i that are not used in a positive braid word for β . For positive braids β , this c is equal to the number c of pieces of the closure of β that can be separated by spheres. Equality (2) also follows from Bennequin's inequality [Ben83].

As there are several positive braid words for most positive braids, there are also several fence diagrams. Two fence diagrams for the same positive braid are related by moves corresponding to the braid relations. The relation $a_i a_j = a_j a_i$ for $|i - j| \geq 2$ is incorporated by looking at fence diagrams up to planar isotopy, e.g. $\begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array}$. The braid relation $a_i a_j a_i = a_j a_i a_j$ for $|i - j| = 1$ corresponds to the move $\begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array}$.

4. Algebraic knots and links

We recall the notion of an algebraic knot or link following Milnor [Mil68]. Let $f: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ be a polynomial function or a holomorphic function germ that is square-free in the ring of holomorphic function germs $\mathbb{C}\{x, y\}$ and has an isolated singularity at the origin (in fact, “isolated” follows from “square-free”). We abbreviate this by calling such an f a *singularity*. The transversal intersection of its zero-set $V(f) \subseteq \mathbb{C}^2$ with a sufficiently small sphere around the origin $S_\varepsilon = \{(x, y) \in \mathbb{C}^2 \mid \|x\|^2 + \|y\|^2 = \varepsilon^2\}$ is a well-defined link $K(f)$ in $S_\varepsilon \cong S^3$ called the *link of the singularity* f . Such a sufficiently small sphere and the closed ball B_ε that the sphere encloses are called *Milnor sphere* and *Milnor ball*, respectively. The number of components of $K(f)$ is equal to the number of irreducible factors of f in $\mathbb{C}\{x, y\}$. For example, the torus link $T(n, m)$ is the link of the singularity $x^n - y^m$. In this case the Milnor sphere can be taken to be the standard unit sphere S^3 ; thus,

$$T(n, m) = S^3 \cap \{(x, y) \in \mathbb{C}^2 \mid x^n - y^m = 0\} \subset S^3.$$

Knots and links that arise as links of singularities are called *algebraic*. They are positive braid links, compare Remark 4.16; however, most positive braid links are not algebraic links. An explicit description of algebraic links is, for example, provided in [BK86]. The pair equivalence class of $(B_\varepsilon, f^{-1}\{0\} \cap B_\varepsilon)$ up to homeomorphism of topological pairs is called the *topological type of the singularity*. Two singularities

are of the same topological type if and only if their links are isotopic. The *Milnor number* μ of a singularity f is defined to be the first Betti number of the smooth surface $B_\varepsilon \cap f^{-1}(\eta)$ for small $\eta \neq 0$. In fact, μ is equal to the first Betti number of the link $K(f)$ of the singularity since Milnor showed that $B_\varepsilon \cap f^{-1}(\eta)$ is diffeomorphic to the minimal Seifert surface of $K(f)$ for small $\eta \neq 0$. The *multiplicity* $m \geq 1$ of f is the minimal degree of monomials in the power series f with non-zero coefficient. In particular, $m > 1$ if and only if f is singular, that is $Df(0) = 0$. In fact, the multiplicity of f is equal to the braid index of $K(f)$, which is a consequence of [FW87, Corollary 2.4] as explained in [Wil88].

For a singularity f in $\mathbb{C}\{x, y\}$ with r irreducible factors, one denotes $\frac{\mu+r-1}{2}$ by δ . Roughly speaking this δ -invariant measures how much genus (of the zero-set of f) is “hidden” in the singularity. Note that $\delta = 0$ for non-singular f since $\mu = b_1(O) = 0$ and $r = 1$, and that $\delta = 1$ for the ordinary double point singularity $f = y^2 - x^2$ since $\mu = b_1(T(2, 2)) = 1$ and $r = 2$.

Let f be a singularity that is irreducible in $\mathbb{C}\{x, y\}$. We now describe how f can be “prepared” for study by a small linear coordinate change. This will only be used in the proofs of Proposition 4.9 and Proposition 4.12. References are [BK86] and [GLS07]. If f is non-singular at 0, the zero-set $V(f)$ is locally parametrized as a graph, i.e. after a small linear coordinate change the points of $V(f)$ are locally parametrized by $(x, y(x))$ for a holomorphic function germ $y(x)$ in $\mathbb{C}\{x\}$. If f is singular at 0 with multiplicity m , one still has the following.

LEMMA 1.2. *After a small linear coordinate change, there exists a germ $y(s)$ in $\mathbb{C}\{s\}$ such that locally around the origin $V(f) \cap (D_\varepsilon \times \mathbb{C})$ is the image of*

$$\phi : D_{\frac{1}{\varepsilon^m}} \subset \mathbb{C} \rightarrow \mathbb{C}^2, s \mapsto (s^m, y(s)).$$

The map ϕ is called *normalization* or *resolution* of the singularity. Lemma 1.2 follows from the Weierstrass preparation theorem, which states that after a small linear coordinate change we have

$$f = u(x, y)(y^m + c_{m-1}(x)y^{m-1} + \cdots + c_0(x))$$

where u and c_i are unique germs in $\mathbb{C}\{x, y\}$ and $\mathbb{C}\{x\}$, respectively, with $u(0, 0) \neq 0$ and $c_i(0) = 0$. The germ f can be recovered from $y(s)$ up to the unit u . Indeed,

$$y^m + c_{m-1}(x)y^{m-1} + \cdots + c_0(x) = \prod_{\xi^m=1} \left(y - y(\xi x^{\frac{1}{m}}) \right).$$

The power series with rational powers given by $y(x^{\frac{1}{m}})$ is called *Puiseux expansion* of f .

REMARK 1.3. Two germs f_0 and f_1 have the same link of singularity if and only if there is a family of germs f_t (depending smoothly on $t \in [0, 1]$) such that μ is constant on this family. Indeed, if two germs can be connected by a μ -constant path, then the associated algebraic links are isotopic [TR76]. The converse follows by reducing to the case when the germs f_0 and f_1 are irreducible with Puiseux expansions $y_0(x^{\frac{1}{m}})$ and $y_1(x^{\frac{1}{m}})$ and using that Puiseux expansions for singularities with the same knot can be connected by a family $y_t(x^{\frac{1}{m}})$ that have all the same knot, in particular the same μ ; see e.g. [BK86]. This yields a μ -constant family of germs $f_t = \prod_{\xi^m=1} \left(y - y_t(\xi x^{\frac{1}{m}}) \right) \in \mathbb{C}\{x, y\}$.

REMARK 1.4. Fix a Milnor ball B for an irreducible germ f such that $V(f) \cap B$ is a subset of the image of a normalization

$$\phi: D_{\varepsilon^{\frac{1}{m}}} \rightarrow \mathbb{C}^2, s \mapsto (s^m, y(s)).$$

As f is irreducible we have $\delta = \frac{\mu}{2}$. A small generic deformation of $y(s)$, say $\tilde{y}(s)$, yields an immersion $\tilde{\phi}: D_{\varepsilon^{\frac{1}{m}}} \rightarrow \mathbb{C}^2$ for which all multiple points are transversal double points. This means that for $g = \prod_{\xi^m=1} \left(y - \tilde{y}(\xi x^{\frac{1}{m}}) \right)$, the zero-set $V(g) \cap B$ has only ordinary double point singularities. Also, the number of such double points is equal to δ , which follows from the fact that the zero-set $V(g)$ cannot have genus other than what is “hidden” in its ordinary double points as it is the image of an open disc in \mathbb{C} under $\tilde{\phi}$, see e.g. [GLS07].

From this point of view, ordinary double points are the generic singularities. If f is not irreducible, the above can be done to every factor of f independently such that the irreducible pieces intersect in ordinary double points and the total number of double points is δ .

5. Signatures of links

In this section, we introduce the signature, and more generally the Levine-Tristram signatures. Signatures are present in this thesis in two ways. Firstly, the signature is studied on the class of all positive braid links, see Chapter 2. Secondly, we use signatures as obstructions to adjacency between torus links.

The signature of a link L in \mathbb{R}^3 is defined as follows. Choose any Seifert surface F for L and define a bilinear form $S: H_1(F, \mathbb{Z}) \times H_1(F, \mathbb{Z}) \rightarrow \mathbb{Z}$ on the first homology group of F as follows. The form S assigns to two (classes of sums of) curves γ, δ in F the value of the

linking number of γ with a curve in $\mathbb{R}^3 \setminus F$ that is obtained by moving δ a small amount along the normal vector field of F . Writing S in a basis for $H_1(F, \mathbb{Z})$ yields an integer matrix A , called a *Seifert matrix* for L . The number of positive eigenvalues minus the number of negative eigenvalues of the symmetrization of A is an integer-valued link invariant, called the *signature*, see [Tro62][Mur65]. The signature of a link L is denoted by $\sigma(L)$. The signature of a braid β , denoted by $\sigma(\beta)$, is the signature of its closure. There is an issue with the sign convention for the signature in the literature. We choose sign conventions (for example in the definition of the linking number) such that all (positive) torus knots have positive signature, e.g. $\sigma(T(2, 3)) = 2$ rather than $\sigma(T(2, 3)) = -2$.

More generally, Levine and Tristram defined for every ω in $S^1 \setminus \{1\}$ a notion of signature as follows [Lev69][Tri69].

DEFINITION 1.5. Let A be a Seifert matrix of a link L and let ω be in $S^1 \setminus \{1\} \subset \mathbb{C}$. The ω -signature $\sigma_\omega(L) \in \mathbb{Z}$ is defined to be the number of positive eigenvalues minus the number of negative eigenvalues of the Hermitian matrix $(1 - \omega)A + (1 - \bar{\omega})A^t$.

The ω -signature is also independent of the choice of Seifert matrix, i.e. it is a link invariant. One has $\sigma_\omega = \sigma_{\bar{\omega}}$. Setting $\omega = -1$ one recovers the classical signature $\sigma = \sigma_{-1}$.

For a fixed link L the signature $\sigma_\omega(L)$ is piecewise-constant in ω , “jumping” at a finite number of ω . For a Seifert matrix A of a knot K , if ω is a root of unity of prime order, then $(1 - \omega)A + (1 - \bar{\omega})A^t$ is invertible, and so $\sigma_\omega(K)$ is even and $\sigma_\omega(K)$ does not jump at ω . From now on, every ω is a root of unity of prime order. As roots of unity of prime order are dense in S^1 , one only loses information on “jumping”-points.

We collect some properties of the signatures. The ω -signatures are additive on split union and connected sums¹ of links and reflection of a link changes the sign of the signature.

LEMMA 1.6. *Let L be a link. If L is a split union or a connected sum of links L_1, \dots, L_k , then $\sigma_\omega(L) = \sum_{i=1}^k \sigma_\omega(L_i)$. If $-L$ is the link obtained from L by reflection and changing all (or none) of the orientations, then $\sigma_\omega(-L) = -\sigma_\omega(L)$.*

Lemma 1.6 follows from the fact that the direct sum of Seifert matrices A_i for L_i provides a Seifert matrix $A = \bigoplus_{i=1}^k A_i$ for L and that if A is a Seifert matrix for L , then $-A$ is a Seifert matrix for $-L$.

¹The connected sum L of links with more than one component is not well-defined, but all possible resulting links L have the same signature.

The fact that deleting the first row and the first column of a Hermitian matrix changes its signature by at most ± 1 yields the following Lemma, see e.g. [Tri69].

LEMMA 1.7. *If a Seifert surface F_K for a link K is obtained from a Seifert surface F_L for a link L by adding or deleting a 1-handle, then*

$$\sigma_\omega(K) - 1 \leq \sigma_\omega(L) \leq \sigma_\omega(K) + 1.$$

The following result by Tristram is the basis for signature obstructions to adjacency.

LEMMA 1.8. [Tri69] *Let K and L be links. If there is a cobordism C in $S^3 \times [0, 1]$ from $K \times \{0\}$ to $L \times \{1\}$, then $|\sigma_\omega(L) - \sigma_\omega(K)| \leq |\chi(C)|$. In particular, $|\sigma_\omega(L) - \sigma_\omega(K)| \leq d_c(K, L)$.*

We finish this section by providing results on signatures of torus links, which are rather well understood. The following Lemma provides a combinatorial formula for the Levine-Tristram signatures of torus links; see Gambaudo and Ghys [GG05]. For $\sigma = \sigma_{-1}$ it is originally due to Brieskorn and Hirzebruch [Bri66][Hir95]. We denote the cardinality of a finite set S by $\sharp S$. In what follows cardinality is counted with multiplicity, e.g. $\sharp\{1, 1, 2\} = 3$.

LEMMA 1.9. *Let $n \geq 2$ and $m \geq 2$ be integers and set $S = \{\frac{k}{n} + \frac{l}{m} \mid 1 \leq k \leq n-1 \text{ and } 1 \leq l \leq m-1\} \subset [0, 2]$. Then for $\theta \in [0, 1]$ we have*

$$\sigma_{e^{2\pi i\theta}}(T(n, m)) = \sharp(S \cap [\theta, \theta + 1]) - \sharp(S \setminus (\theta, \theta + 1)).$$

Calculations using this can be tedious, as the reader will experience in Section 3 of Chapter 3. For the classical signature σ , Gordon, Litherland and Murasugi provided the following recursive formulas.

LEMMA 1.10. [GLM81, Theorem 5.2] *Let $n, q > 0$.*

(I) *Suppose $2q < n$.*

If q is odd, then $\sigma(T(n, q)) = \sigma(T(n - 2q, q)) + q^2 - 1$.

If q is even, then $\sigma(T(n, q)) = \sigma(T(n - 2q, q)) + q^2$.

(II) *$\sigma(T(2q, q)) = q^2 - 1$.*

(III) *Suppose $q \leq n < 2q$.*

If q is odd, then $\sigma(T(n, q)) + \sigma(T(2q - n, q)) = q^2 - 1$.

If q is even, then $\sigma(T(n, q)) + \sigma(T(2q - n, q)) = q^2 - 2$.

(IV) *$\sigma(T(n, q)) = \sigma(T(q, n))$, $\sigma(T(n, 1)) = 0$, and $\sigma(T(n, 2)) = n - 1$.*

REMARK 1.11. For a torus link T we have $\sigma(T) = b_1(T)$ if and only if T has braid index less than or equal to 2 or is one of three links $T(3, 3)$, $T(3, 4)$ and $T(3, 5)$. Another family of positive braid links that

we use later on and for which the signature equals the first Betti number is $\{D_n\}_{n \geq 2}$, where D_n is the closure of the 3-braid $a_1^{n-2}a_2a_1a_1a_2$. See, for example, Baader's minor description of positive braid links with $b_1 = \sigma$ [**Baa14**].

CHAPTER 2

The signature of positive braids is linearly bounded by their first Betti number

In this chapter we provide a linear lower bound for the signature of positive braids in terms of the first Betti number. As a corollary we get a linear lower bound on the topological 4-ball genus of positive braid knots, where the *topological 4-ball genus* $g_t^4(K)$ of a knot K in S^3 is the minimal genus of topologically locally-flat embeddings of oriented surfaces F in the 4-ball B^4 with boundary $\partial F = K \subset \partial B^4 = S^3$.

For positive braids, the first Betti number is fully understood since $b_1(\beta) = l(\beta) - b + c$ for every positive braid β , see (2). On the other hand the signature is not well understood for positive braids, even though its calculable for concrete examples.

We relate the first Betti number and the signature for positive braids up to a linear factor. One can restrict considerations to braids with non-split closure (i.e. $c = 1$) because the first Betti number and the signature are additive on split unions of braids, see Lemma 1.6. By the definition of the signature, one has that $-b_1 \leq \sigma \leq b_1$ holds for all links. Rudolph showed that the signature is strictly positive for non-trivial positive braids [Rud82], and Stoimenow provided a monotonically growing function $f: \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ of order $n^{\frac{1}{3}}$ such that $\sigma(\beta) \geq f(b_1(\beta))$ [Sto08]. Here non-triviality of a positive braid β just means $b_1(\beta) > 0$, i.e. the closure of β is not a split union of unknots.

Murasugi's result that half the signature is a lower bound for the smooth 4-ball genus [Mur65], as stated in Lemma 1.8, was generalized to the topological 4-ball genus by Kauffman and Taylor [KT76]. Thus, Stoimenow's result provides a lower bound for the topological 4-ball genus of positive braid knots that grows monotonically to infinity in terms of their genus. In the stronger, smooth setting much more is known. In fact, on positive braids the smooth 4-ball genus and smooth 4-ball first Betti number even agree with the genus and the first Betti number, respectively, by Rudolph's slice-Bennequin inequality [Rud93], which is based on the Thom conjecture as proved by Kronheimer and Mrowka [KM93].

All that follows is motivated by the following (maybe optimistic) conjecture.

CONJECTURE 1. *For all non-trivial positive braids β , the signature is linearly bounded as follows $b_1(\beta) \geq \sigma(\beta) > \frac{1}{2}b_1(\beta)$.*

This would be optimal, as there are families of positive braids on which the ratio $\frac{\sigma}{b_1}$ gets arbitrarily close to $\frac{1}{2}$. For example, positive braids β_n that have the $T(n, n+1)$ torus knots as closures. We provide a linear lower bound as in the conjecture. However, the linear factor is smaller than $\frac{1}{2}$.

THEOREM 2.1. *For all positive braids β , the signature is linearly bounded from below as follows $\sigma(\beta) \geq \frac{b_1(\beta)}{100}$.*

As half the signature is a lower bound for the topological 4-ball genus, Theorem 2.1 has the following immediate corollary.

COROLLARY 2.2. *The topological 4-ball genus of positive braid knots is at least one percent of their genus. I.e. for a knot K that is obtained as the closure of a positive b -braid β , we have*

$$g_4^t(K) \geq \frac{g(K)}{100} = \frac{l(\beta) - b + 1}{200}.$$

Corollary 2.2 complements the following Remark.

REMARK 2.3. It is interesting to notice that the topological 4-ball genus of positive braid knots is generally strictly smaller than their genus. In fact, one can find families of positive braid knots for which the topological 4-ball genus is linearly bounded away from the genus. For example,

$$g_s^4(T(5, 5n + 1)) \leq 9n = \frac{9}{10}g(T(5, 5n + 1))$$

holds for all positive integers n . This is seen as follows. There is a knot K with Alexander polynomial 1 and a minimal Seifert surface F of genus 1 such that the Seifert surface $F_n = F \sharp \cdots \sharp F$ of the n -times connected sum $K_n = K \sharp \cdots \sharp K$ is an H_1 -injective subsurface of the minimal Seifert surface of $T(5, 5n + 1)$. In particular, there is a cobordism of genus $g(T(5, 5n + 1)) - n = 10n - n = 9n$ from $T(5, 5n + 1)$ to K_n . The knot K_n has Alexander polynomial 1 and, therefore, $g_t^4(K_n)$ is 0 as a consequence of Freedman's work [Fre82]. Hence, $g_t^4(T(5, 5n + 1))$ is less than or equal to $9n$. A possible choice for K is the closure of the 6-braid

$$(a_2 a_3 a_4) a_5 (a_2 a_3 a_4)^{-1} a_1 a_2 a_1^{-1} a_5 a_2 a_3 a_2^{-1} a_3 a_4 (a_1 a_2) a_3 (a_1 a_2)^{-1}.$$

Figure 1 shows how a minimal Seifert surface of K is a H_1 -injective subsurface of the minimal Seifert surface of $T(6, 5)$. In a similar way,

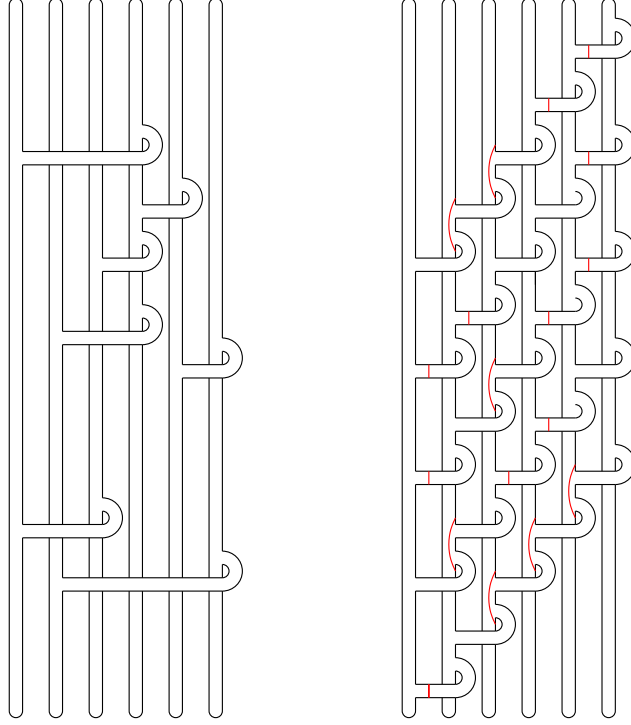


FIGURE 1. The Seifert surface of $T(5, 6)$ (right) with indications what arcs (red) to cut open to obtain a genus 1 Seifert surface for K (left).

the minimal Seifert surface of K_n is seen to be a H_1 -injective subsurface of the minimal Seifert surface of $T(5n + 1, 5)$.

We prove Theorem 2.1 via the study of the asymptotic signature. Gambaudo and Ghys observed that on the b -strand braid group the signature is a quasi-morphism of defect $b - 1$, i.e. for any two b -braids α, β we have

$$|\sigma(\alpha\beta) - \sigma(\alpha) - \sigma(\beta)| \leq b - 1, \text{ see [GG04, Proposition 5.1].}$$

Therefore, the homogenization

$$\tilde{\sigma}(\beta) = \lim_{i \rightarrow \infty} \frac{\sigma(\beta^i)}{i},$$

called the *asymptotic signature* of β , is well-defined. Noting that

$$\lim_{i \rightarrow \infty} \frac{b_1(\beta^i)}{i} = \lim_{i \rightarrow \infty} \frac{l(\beta^i)}{i} = l(\beta),$$

we see that Conjecture 1 implies the following homogenized analog.

CONJECTURE 2. *For all positive braids β , the homogenization of the signature is linearly bounded as follows $l(\beta) \geq \tilde{\sigma}(\beta) \geq \frac{1}{2}l(\beta)$.*

We provide such a linear bound, but our factor is $\frac{1}{16}$ rather than $\frac{1}{2}$.

THEOREM 2.4. *For every positive braid β , we have $\tilde{\sigma}(\beta) \geq \frac{1}{16}l(\beta)$.*

First, let us recapitulate known linear bounds for the signature of a non-trivial positive braid β in B_b , which of course yield analogs for the homogenization. If $b = 2$, then the closure of β is the torus link $T(2, l(\beta))$ and $\sigma(\beta) = b_1(\beta) = l(\beta) - 1$, see Lemma 1.10. For $b = 3$, Stoimenow (and by different methods Yoshiaki Uchida) has shown $\sigma(\beta) > \frac{1}{2}b_1(\beta)$ [Sto08]. This can also be proven with an argument similar to the proof of Proposition 2.15. The case $b = 4$ is our main concern in this chapter and the following is our main result.

THEOREM 2.5. *For every positive 4-braid β , we have $\tilde{\sigma}(\beta) \geq \frac{5}{12}l(\beta)$.*

Our interest in Theorem 2.5 stems from the fact that it implies Theorem 2.1 and Theorem 2.4. Indeed, Theorem 2.4 follows immediately from Theorem 2.5 and the following observation, which we prove at the end of Section 2.

LEMMA 2.6. *Let C be a positive constant such that $\sigma(\beta) > Cb_1(\beta)$ (respectively $\tilde{\sigma}(\beta) \geq Cl(\beta)$) holds for all non-trivial positive b -braids. Then*

$$\sigma(\beta) > \tilde{C}b_1(\beta) \quad \left(\text{respectively } \tilde{\sigma}(\beta) \geq \tilde{C}l(\beta) \right), \quad \text{where } \tilde{C} = \frac{C(b-1)-1}{b},$$

holds for all non-trivial positive braids β .

REMARK 2.7. Lemma 2.6 remains true when the strict inequalities are replaced by inequalities.

The upshot of Theorem 2.5 is that it provides a bound with a factor that is strictly bigger than $\frac{1}{3}$. If one is only interested in the fact that some linear bound exists for positive 4-braids, Stoimenow provided such a bound with the factor $\frac{2}{11}$ in [Sto08]. In fact, we establish

$$(3) \quad \sigma(\beta) > \frac{1}{3}b_1(\beta) \quad \text{for all non-trivial positive 4-braids } \beta,$$

without relying on Theorem 2.5, see Proposition 2.15. However, these results do not provide linear bounds for general positive braids when

combined with Lemma 2.6. For example, Lemma 2.6 applied to (3) just recovers Rudolph’s positivity result, i.e. $\sigma(\beta) > 0$ for all non-trivial positive braids β .

In Section 1 we use Theorem 2.5 to prove Theorem 2.1. Section 2 contains generalities on the signature of positive braids, which are applied to 4-braids in Section 3 to prove (3) and Theorem 2.5.

We conclude the introductory part of this chapter with some evidence to support the conjectured $\frac{1}{2}b_1$ -bound. For positive 3-braids the conjecture holds. And for positive 4-braids up to 17 crossings it is also checked, compare Stoimenow’s table in [Sto08]. All non-trivial torus links satisfy $\sigma > \frac{1}{2}b_1$, which can be checked using the Gordon-Litherland-Murasugi reduction formulas [GLM81, Theorem 5.2], see Lemma 1.10. More generally, the conjecture holds for all algebraic links. This can be checked using the formula provided by Shinohara, see [Shi71], which calculates the signature of a satellite knot in terms of the signatures of its companion and its pattern. Using Shinohara’s formula one can also check that the conjecture holds for a lot of other families of positive braid knots that are cables of positive braid knots. An improved version of the first inequality of [Baa13, Theorem 3] shows that the conjecture holds for positive braids that are given by a positive braid word in which generators appear with power at least 2, i.e. a braid word of the form $a_{s_1}^{k_1} \cdots a_{s_r}^{k_r}$ with $k_i \geq 2$. This in particular includes “sufficiently complicated” positive braids as studied in [FKP13].

1. From asymptotic signature to signature

In this section we provide consequences of Theorem 2.5 for the (non-homogenized) signature, including a proof of Theorem 2.1. Additionally to Theorem 2.5 this uses Proposition 2.15 and Lemma 2.6, which are proven in Section 2 and Section 3.

Having a linear bound $\sigma \geq Cb_1$ for all positive braids on b or fewer strands yields a bound $\tilde{\sigma} \geq Cl$ for all positive braids on b or fewer strands. The converse is true, if one allows an error of adding an additive constant.

LEMMA 2.8. *Let C be a positive constant such that*

$$\tilde{\sigma}(\beta) = \lim_{i \rightarrow \infty} \frac{\sigma(\beta^i)}{i} \geq Cl(\beta)$$

for all positive b -braids β . Then, for every positive b -braid β , we have

$$\sigma(\beta) \geq Cl(\beta) - b + 1.$$

Lemma 2.8 is an immediate consequence of the fact that the signature and its homogenization stay close. That is, for every β in B_b , we have

$$|\sigma(\beta) - \tilde{\sigma}(\beta)| \leq b - 1,$$

which follows from σ being a quasi-morphism of defect $b - 1$.

For positive 4-braids with non-split closure (i.e. $c = 1$), applying Lemma 2.8 to Theorem 2.5 and using (2) yields $\sigma(\beta) \geq \frac{5}{12}(b_1(\beta)+3)-3$. Therefore, we get the following affine signature bound for positive 4-braids.

COROLLARY 2.9. *If β is a positive 4-braid, then $\sigma(\beta) \geq \frac{5}{12}b_1(\beta) - \frac{7}{4}$.*

Corollary 2.9 can be used to prove the following Proposition.

PROPOSITION 2.10. *If β is a positive 4-braid, then*

$$\sigma(\beta) \geq \left(\frac{1}{3} + \frac{1}{75}\right)b_1(\beta).$$

In turn, Proposition 2.10 implies Theorem 2.1 by Lemma 2.6 since we have

$$\tilde{C} = \frac{C(4-1) - 1}{4} = \frac{1}{100} \quad \text{for } C = \frac{1}{3} + \frac{1}{75}.$$

PROOF OF PROPOSITION 2.10. Set $C = \frac{1}{3} + \frac{1}{75}$. Corollary 2.9 can be written as

$$(4) \quad \sigma(\beta) \geq \frac{5}{12}b_1(\beta) - \frac{7}{4} = Cb_1(\beta) + \left(\frac{5}{12} - C\right)b_1(\beta) - \frac{7}{4}$$

for all positive 4-braids β . The constant C is chosen such that (4) yields $\sigma(\beta) \geq Cb_1(\beta)$ whenever $b_1(\beta) \geq 25$. On the other hand, we have $\sigma(\beta) > \frac{b_1(\beta)}{3}$ for all non-trivial positive 4-braids, see Proposition 2.15, which can be written as $\sigma(\beta) \geq \frac{b_1(\beta)}{3} + \frac{1}{3}$. In particular,

$$\sigma(\beta) \geq \frac{b_1(\beta)}{3} + \frac{1}{75}b_1(\beta)$$

for all positive 4-braids with $b_1 \leq 25$. □

If one were able to strengthen Corollary 2.9 to a linear bound for the signature with factor $\frac{5}{12}$ or even $\frac{1}{2}$, then Theorem 2.1 would follow immediately from Lemma 2.6 with factor $\frac{1}{16}$ or $\frac{1}{8}$, respectively, rather than $\frac{1}{100}$.

Applying Lemma 2.8 to Theorem 2.4 yields the following affine linear bound.

COROLLARY 2.11. *For every positive braid β on at most b strands, we have*

$$\sigma(\beta) \geq \frac{1}{16}b_1(\beta) - \frac{15}{16}(b-1).$$

Corollary 2.11 can also be proved by applying a slight modification of the proof of Lemma 2.6 to Corollary 2.9. With this, the constant $\frac{15}{16}(b-1)$ is improved to be $\frac{7}{4}\lfloor\frac{b}{4}\rfloor$. We remark that Corollary 2.11 shows that the topological 4-ball genus of positive braid knots grows asymptotically at least as fast as $\frac{1}{16}$ of the genus.

2. Signature of positive braids

In this section we discuss properties of the signature of braids and prove Lemma 2.6. Firstly, if we permute braids cyclically, then they have the same closure and thus the same signature. I.e. let $\beta = a_{i_1}^{\varepsilon_1} a_{i_2}^{\varepsilon_2} \cdots a_{i_l}^{\varepsilon_l}$ be a braid, then $a_{i_2}^{\varepsilon_2} \cdots a_{i_l}^{\varepsilon_l} a_{i_1}^{\varepsilon_1}$ has the same closure and thus the same signature as β . If we add or delete a generator in a braid word, then the signature of the corresponding braid changes by at most ± 1 ; this is a consequence of Lemma 1.7. As mentioned above, σ is a quasi-morphism on the b -strand braid group.

LEMMA 2.12. *For any two b -braids α, β , we have*

$$|\sigma(\alpha\beta) - \sigma(\alpha) - \sigma(\beta)| \leq b - 1.$$

Using the fact that cyclic permutations of a braid have the same signature we can state this as follows.

COROLLARY 2.13. *For b -braids α, β, γ , we have*

$$|\sigma(\alpha\gamma\beta) - \sigma(\alpha\beta) - \sigma(\gamma)| \leq b - 1.$$

Lemma 2.12 is based on Lemma 1.7 and the fact that there is a Seifert surface for $\alpha\beta$ that can be obtained from the connected sum of Seifert surfaces for α and β by adding $b-1$ 1-handles, see Gambaudo and Ghys for a proof [GG04, Proof of Proposition 5.1].

REMARK 2.14. If α or β can be written as a braid word without one or several generators a_i , then the statement of Lemma 2.12 is true with defect strictly smaller than $b-1$ by the same proof. In fact,

$$|\sigma(\alpha\beta) - \sigma(\alpha) - \sigma(\beta)| \leq b - \max\{c(\alpha), c(\beta)\},$$

where $c(\alpha)$ and $c(\beta)$ denote 1 plus the number of generators that are not needed in a braid word for α and β , respectively.

In particular, for any α, β in B_b and any integer n we have

$$|\sigma(\alpha a_i^n \beta) - \sigma(\alpha\beta) - \sigma(a_i^n)| \leq 1.$$

Before using the above on positive 4-braids to prove Theorem 2.5, we prove Lemma 2.6.

PROOF OF LEMMA 2.6. We only prove the statement for σ , the proof for $\tilde{\sigma}$ is similar. Let β be a non-trivial positive braid word in some braid group B_n . Without loss of generality $c = 1$, i.e. every generator a_i with $1 \leq i \leq n - 1$ is contained in β at least once.

The idea of the proof is to delete generators in β such that a connected sum of braids on b or fewer strands remains.

For i in $\{1, 2, \dots, b\}$, we denote by $\beta(i)$ the braid obtained from β by deleting all but one (say the leftmost) a_k for all k in $\{i, i + b, i + 2b, i + 3b, \dots\}$. Figure 2 illustrates how $\beta(i)$ is obtained from β . The

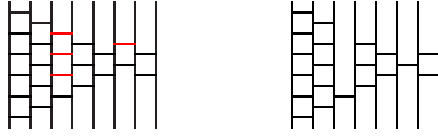


FIGURE 2. A fence diagram of a positive braid β on 8 strands (left) with indications (red) what generators to delete to obtain $\beta(3)$ (right) if $b = 3$. The closure of $\beta(3)$ is a connected sum of the closures of two 3-braids and a 2-braid.

closure of such a $\beta(i)$ is a connected sum of closures of positive braids on b or fewer strands. Since we have $b_1(\beta) = \sum_{k=1}^{n-1} (\#\{a_k \text{ in } \beta\} - 1)$, there is an i such that

$$(5) \quad b_1(\beta(i)) \geq \frac{b-1}{b} b_1(\beta).$$

We fix such an i . Let B_1, \dots, B_l be positive braids on at most b strands such that the closure of $\beta(i)$ is the connected sum of the closures of the B_j . Thus, additivity of the first Betti number and the signature on connected sums (see Lemma 1.6), the assumption $\sigma > C b_1$ for non-trivial positive braids on b strands, and (5) yield

$$\sigma(\beta(i)) = \sum_{j=1}^l \sigma(B_j) > \sum_{j=1}^l C b_1(B_j) = C b_1(\beta(i)) \geq \frac{C(b-1)}{b} b_1(\beta).$$

The braid $\beta(i)$ is obtained from β by deleting $b_1(\beta) - b_1(\beta(i)) \leq \frac{1}{b} b_1(\beta)$ of the generators. Since by Lemma 1.7 deleting one generator changes

the signature by at most ± 1 , we get

$$\begin{aligned}\sigma(\beta) &\geq -\frac{1}{b}b_1(\beta) + \sigma(\beta(i)) \\ &> -\frac{1}{b}b_1(\beta) + \frac{C(b-1)}{b}b_1(\beta) \\ &= \frac{C(b-1)-1}{b}b_1(\beta).\end{aligned}\quad \square$$

3. Signature of positive 4-braids

In this section we provide a $\frac{1}{3}$ -linear bound for the signature of positive 4-braids and we prove Theorem 2.5. Note that positive braids are represented by fence diagrams as described in Section 3 of Chapter 1.

PROPOSITION 2.15. *For all non-trivial positive 4-braids β , we have $\sigma(\beta) > \frac{1}{3}b_1(\beta)$.*

For $b_1 > 21$, Proposition 2.15 follows from Corollary 2.9. We provide a complete proof, which is independent of Corollary 2.9.

PROOF OF PROPOSITION 2.15. Let β be a positive 4-braid and choose a positive braid word $w = a_{i_1}a_{i_2}\cdots a_{i_l}$ for β or cyclic permutations of β such that the number of a_2 in this braid word is minimal among all possible such positive braid words. Without loss of generality we assume that w does contain all three generators a_i at least once. Also, we may assume (by cyclic permutation and using the braid relations) that the first two letters of w are not a_2 , e.g. we consider

$a_1a_1a_3a_2a_2 = \left| \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right|$ instead of $a_2a_1a_1a_3a_2 = \left| \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right|$. Let B_1, \dots, B_n be the blocks of consecutive a_2 and k_i the number of a_i in w . Of course $k_2 \geq n$ holds, and, by the assumption of minimality of the number of a_2 in w , we have at least two generators between two consecutive B_i , which yields $k_1 + k_3 \geq 2n$. Therefore,

$$(6) \quad \frac{k_1 + k_2 + k_3}{3} \geq n.$$

We first show $\sigma(\beta) \geq \frac{1}{3}b_1(\beta)$. Let β' denote the braid obtained from β by removing B_2, B_3, \dots , and B_n . By Remark 2.14 we have

$$\begin{aligned}\sigma(\beta) &\geq \sigma(\beta') + \sum_{i=2}^k \sigma(B_i) - (n-1) \\ &= \sigma(\beta') + \sum_{i=2}^k (l(B_i) - 1) - (n-1).\end{aligned}$$

The closure of β' is a connected sum of the torus links

$$T(2, k_1), T(2, l(B_1)) \quad \text{and} \quad T(2, k_3).$$

This yields $\sigma(\beta') = k_1 - 1 + k_3 - 1 + l(B_1) - 1$ by Lemma 1.6. Therefore,

$$\begin{aligned} \sigma(\beta) &\geq k_1 - 1 + k_3 - 1 + l(B_1) - 1 + \sum_{i=2}^k (l(B_i) - 1) - (n - 1) \\ &= k_1 - 1 + k_3 - 1 + \sum_{i=1}^k (l(B_i) - 1) - (n - 1) \\ &= k_1 - 1 + k_3 - 1 + k_2 - n - (n - 1) \\ &= k_1 + k_2 + k_3 - 2n - 1 \geq \frac{k_1 + k_2 + k_3}{3} - 1 = \frac{b_1(\beta)}{3}, \end{aligned}$$

where in the last line (6) and $b_1(\beta) = k_1 + k_2 + k_3 - 3$ are used.

We observe that if inequality (6) is a strict inequality, then the above calculation proves $\sigma(\beta) > \frac{1}{3}b_1(\beta)$. Thus, it remains to consider w satisfying $\frac{k_1+k_2+k_3}{3} = n$, which implies that the inequalities $k_2 \geq n$ and $k_1 + k_3 \geq 2n$ are equalities. Therefore, the blocks B_i consist of a single a_2 and in w we have exactly two generators between two consecutive B_i . We write w as $a_{i_1}a_{j_1}a_2a_{i_2}a_{j_2}a_2 \cdots a_{i_n}a_{j_n}a_2$, for some i_l, j_l in $\{1, 3\}$.

Since β contains all types of generators and is non-trivial we have $k_2 = n \geq 2$. Removing all but the last two a_2 in w yields a positive 4-braid β'' with

$$l(\beta'') = l(\beta) - (n - 2) = 3n - (n - 2) = 2n + 2.$$

The braid β'' satisfies $\sigma = b_1$, which is seen as follows. The braid β'' equals $a_1^i a_3^j a_2 \gamma a_2$ with $i + j = 2n - 2$, where γ is $a_1 a_3$, a_1^2 or a_3^2 . The closure of $a_1^i a_3^j a_2 a_1 a_3 a_2$ with $i + j = 2n - 2$ is the torus link $T(2, 2n)$, for which $\sigma = b_1$ holds. The closure of $a_1^i a_3^j a_2 a_1 a_1 a_2$ is a connected sum of the torus link $T(2, j)$ and the closure of the 3-braid $a_1^i a_2 a_1 a_1 a_2$, which both satisfy $\sigma = b_1$ by Remark 1.11. Similarly, the closure of $a_1^i a_3^j a_2 a_3 a_3 a_2$ is a connected sum of the torus link $T(2, i)$ and the closure of the 3-braid $a_1^j a_2 a_1 a_1 a_2$.

Using Lemma 1.7, $\sigma(\beta'') = b_1(\beta'')$ and $b_1(\beta) = k_1 + k_2 + k_3 - 3$ we calculate

$$\begin{aligned} \sigma(\beta) &\geq \sigma(\beta'') - (n - 2) = b_1(\beta'') - n + 2 = 2n - 1 - n + 2 \\ &= n + 1 = \frac{k_1 + k_2 + k_3}{3} + 1 = \frac{b_1(\beta)}{3} + 2. \quad \square \end{aligned}$$

The strategy for the proof of Theorem 2.5 is the following. To a braid β we add roughly $\frac{1}{2}l(\beta)$ generators such that the resulting braid $\tilde{\beta}$ is simple enough that one can prove $\tilde{\sigma}(\tilde{\beta}) \geq \frac{2}{3}l(\tilde{\beta}) = l(\beta)$. Since $\tilde{\sigma}(\beta) \geq -\frac{1}{2}l(\beta) + \tilde{\sigma}(\tilde{\beta})$ holds by Lemma 1.7, we conclude that $\tilde{\sigma}(\beta) \geq \frac{1}{2}l(\beta)$ holds. In fact, this only works for a part of the braid (at least for $\frac{2}{3}$ of the braid in terms of length) and for the rest of the braid we are only able to prove $\tilde{\sigma} \geq \frac{1}{4}l$. Combining this yields $\tilde{\sigma}(\beta) \geq \frac{2}{3}\frac{1}{2}l(\beta) + \frac{1}{4}\frac{1}{2}l(\beta) = \frac{5}{12}l(\beta)$.

The braid $\tilde{\beta}$ will be obtained from β using the following Lemma.

LEMMA 2.16. *Let B be a positive 4-braid of length 4. If B is not*

$$a_2a_1a_1a_2 = \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} \quad \text{or} \quad a_2a_3a_3a_2 = \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \end{array},$$

then one can add two generators to B such that it becomes

$$\Delta = a_1a_3a_2a_1a_3a_2 = \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array},$$

$$L = a_1a_2a_3a_1a_2a_3 = \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array}, \quad \text{or} \quad R = a_3a_2a_1a_3a_2a_1 = \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array}.$$

Here, ‘adding a generator to a positive braid β ’ means choosing some positive braid word for β and then adding a generator a_i somewhere in this word.

PROOF. We assume that B is represented as a braid word such that the number of a_2 in B is minimal, e.g. $a_2a_1a_2a_1 = \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \end{array}$ is not considered because it represents the same braid as $a_1a_2a_1a_1 = \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \end{array}$. We group all possible B according to the number of a_2 contained in B and proceed case by case. Cases are only consider up to rotations and reflections. Newly added generators are marked in red.

• The braids of length 4 with no a_2 are $\begin{array}{|c|} \hline \text{---} \\ \hline \end{array}$, $\begin{array}{|c|} \hline \text{---} \\ \hline \end{array}$, and $\begin{array}{|c|} \hline \text{---} \\ \hline \end{array}$. In $\begin{array}{|c|} \hline \text{---} \\ \hline \end{array}$ we first add one a_2 to get $\begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \end{array}$ and then add a a_3 to get $\begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} = L$. In the other cases we add two a_2 as follows.

$$\begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} = L \quad \text{and} \quad \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} = \Delta.$$

- The following are all B with one a_2 . We have always indicated how to add two generators (red) yielding L, R , or Δ .

$$\begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} = L, \quad \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} = L, \quad \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} = L, \quad \text{and} \quad \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} = L$$

- If B contains two a_2 , but is not

$$a_2 a_1 a_1 a_2 = \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} \quad \text{or} \quad a_2 a_3 a_3 a_2 = \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array},$$

then it is one of the following (as before it is indicated in red which generators to add).

$$\begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} = L, \quad \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} = \Delta, \quad \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} = \Delta, \quad \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} = \Delta, \quad \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} = L.$$

- Finally, there are only two B with 3 or 4 a_2 . Namely

$$\begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} = L \quad \text{and} \quad \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} = L. \quad \square$$

PROOF OF THEOREM 2.5. Let $\beta = a_{i_1} a_{i_2} \cdots a_{i_l}$ be a positive 4-braid of length l . We fix a positive integer n that is a multiple of 4 and study β^n , which is a braid of length nl . First, we write β^n as $B_1 B_2 \cdots B_{\frac{nl}{4}}$, where every B_i is a positive braid of length 4. Let k be

the number of $a_2 a_1 a_1 a_2 = \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array}$ and $a_2 a_3 a_3 a_2 = \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array}$ among the B_i .

We may assume that k is less than or equal to $\frac{1}{3} \frac{nl}{4} = \frac{nl}{12}$. For if this were not the case, we switch β^n to one of the cyclic permutations $\beta_1^n = a_{i_1}^{-1} \beta^n a_{i_1}$ or $\beta_2^n = a_{i_2}^{-1} a_{i_1}^{-1} \beta^n a_{i_1} a_{i_2}$, which have the same closure as β^n . It is easy to see that if we decompose β^n, β_1^n , and β_2^n into blocks of length 4 and add up the number of $\begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array}$ and $\begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array}$ in all three decompositions, we get at most $\frac{nl}{4}$; thus, $k \leq \frac{1}{3} \frac{nl}{4}$ for at least one of β^n, β_1^n , and β_2^n .

Now, we apply Lemma 2.16 to change β^n to $\widetilde{\beta}^n = \widetilde{B}_1 \widetilde{B}_2 \cdots \widetilde{B}_{\frac{nl}{4}}$, where the \widetilde{B}_i are braid words of length 4 or 6 that are chosen as follows. If B_i is $\begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array}$ or $\begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array}$, then \widetilde{B}_i is B_i . Otherwise, \widetilde{B}_i is equal to L, R , or Δ such that \widetilde{B}_i can be obtained from B_i by adding 2 generators, which is possible by Lemma 2.16. By Lemma 1.7 we have

$$\sigma(\beta^n) \geq \sigma(\widetilde{\beta}^n) - 2\left(\frac{nl}{4} - k\right).$$

For a braid α , let α^{rot} denote the braid represented by the braid diagram that is obtained by rotating a braid diagram for α by 180 degrees, where braid diagrams are understood to lie in \mathbb{R}^2 and the rotation is the usual 180 degree rotation of \mathbb{R}^2 . The following holds

$$(7) \quad \sigma(\widetilde{\beta^n}(\widetilde{\beta^n})^{\text{rot}}) \geq 2k + \sigma(\Delta^{2(\frac{nl}{4}-k)}) = 2k + 8(\frac{nl}{4} - k) - 1.$$

Before proving (7), we use it to finish the proof. Since $\widetilde{\beta^n}$ and $(\widetilde{\beta^n})^{\text{rot}}$ have the same closure (up to changing the orientation) and σ is a quasi-morphism of defect 3, we get

$$\begin{aligned} 2\sigma(\widetilde{\beta^n}) &= \sigma(\widetilde{\beta^n}) + \sigma(\widetilde{\beta^n}^{\text{rot}}) \geq \sigma(\widetilde{\beta^n}(\widetilde{\beta^n})^{\text{rot}}) - 3 \\ &\stackrel{(7)}{\geq} (2k + 8(\frac{nl}{4} - k) - 1) - 3 = -6k + 2nl - 4. \end{aligned}$$

Therefore,

$$\begin{aligned} \sigma(\beta^n) &\geq \sigma(\widetilde{\beta^n}) - 2(\frac{nl}{4} - k) \geq -3k + nl - 2 - 2(\frac{nl}{4} - k) \\ &= -k + \frac{nl}{2} - 2 \geq -\frac{nl}{12} + \frac{nl}{2} - 2 = \frac{5nl}{12} - 2, \end{aligned}$$

and thus

$$\tilde{\sigma}(\beta) = \lim_{n \rightarrow \infty} \frac{\sigma(\beta^n)}{n} \geq \frac{5l}{12} = \frac{5}{12}l(\beta).$$

It remains to prove (7). For this we use that the full twist on 4 strands

$$LL = RR = \Delta\Delta = \Delta^2$$

commutes with every 4-braid, i.e. for all α in B_4 we have $\alpha\Delta^2 = \Delta^2\alpha$, compare [Gar69].

Let us study $\widetilde{\beta^n}(\widetilde{\beta^n})^{\text{rot}} = \widetilde{B_1} \widetilde{B_2} \cdots \widetilde{B_{\frac{nl}{4}-1}} \widetilde{B_{\frac{nl}{4}}}^{\text{rot}} \cdots \widetilde{B_1}^{\text{rot}}$. The braids L, R , and Δ are equal to their rotation by 180 degrees, i.e.

$$L = \begin{array}{|c|c|c|c|} \hline \text{---} & \text{---} & \text{---} & \text{---} \\ \hline \text{---} & \text{---} & \text{---} & \text{---} \\ \hline \end{array} = L^{\text{rot}}, \quad R = \begin{array}{|c|c|c|c|} \hline \text{---} & \text{---} & \text{---} & \text{---} \\ \hline \text{---} & \text{---} & \text{---} & \text{---} \\ \hline \end{array} = R^{\text{rot}} \quad \text{and} \quad \Delta = \begin{array}{|c|c|c|c|} \hline \text{---} & \text{---} & \text{---} & \text{---} \\ \hline \text{---} & \text{---} & \text{---} & \text{---} \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline \text{---} & \text{---} & \text{---} & \text{---} \\ \hline \text{---} & \text{---} & \text{---} & \text{---} \\ \hline \end{array} = \Delta^{\text{rot}}.$$

Therefore, if $\widetilde{B_{\frac{nl}{4}}}$ is L, R , or Δ , then $\widetilde{B_{\frac{nl}{4}}}(\widetilde{B_{\frac{nl}{4}}})^{\text{rot}} = \Delta^2$; and thus,

$$\begin{aligned} \widetilde{\beta^n}(\widetilde{\beta^n})^{\text{rot}} &= \widetilde{B_1} \widetilde{B_2} \cdots \widetilde{B_{\frac{nl}{4}-1}} \widetilde{B_{\frac{nl}{4}}}^{\text{rot}} \cdots \widetilde{B_1}^{\text{rot}} \\ &= \Delta^2 \widetilde{B_1} \widetilde{B_2} \cdots \widetilde{B_{\frac{nl}{4}-1}} \widetilde{B_{\frac{nl}{4}-1}}^{\text{rot}} \cdots \widetilde{B_1}^{\text{rot}}. \end{aligned}$$

Otherwise, i.e. if $\widetilde{B_{\frac{nl}{4}}}$ is $\left| \begin{array}{c} \text{H} \\ \text{H} \end{array} \right|$ or $\left| \begin{array}{c} \text{H} \\ \text{H} \end{array} \right|$, we apply Corollary 2.13 to get

$$\sigma(\widetilde{\beta^n}(\widetilde{\beta^n})^{\text{rot}}) \geq -3 + \sigma(\widetilde{B_{\frac{nl}{4}}}\widetilde{B_{\frac{nl}{4}}}^{\text{rot}}) + \sigma(\widetilde{B_1}\widetilde{B_2}\cdots\widetilde{B_{\frac{nl}{4}-1}}\widetilde{B_{\frac{nl}{4}-1}}^{\text{rot}}\cdots\widetilde{B_1}^{\text{rot}}),$$

where

$$\sigma(\widetilde{B_{\frac{nl}{4}}}\widetilde{B_{\frac{nl}{4}}}^{\text{rot}}) = b_1(\widetilde{B_{\frac{nl}{4}}}\widetilde{B_{\frac{nl}{4}}}^{\text{rot}}) = 5$$

since the closure of $\widetilde{B_{\frac{nl}{4}}}\widetilde{B_{\frac{nl}{4}}}^{\text{rot}}$ is a connected sum of two $T(2, 2)$ and one $T(2, 4)$. Therefore, we have

$$\sigma(\widetilde{\beta^n}(\widetilde{\beta^n})^{\text{rot}}) \geq 2 + \sigma(\widetilde{B_1}\widetilde{B_2}\cdots\widetilde{B_{\frac{nl}{4}-1}}\widetilde{B_{\frac{nl}{4}-1}}^{\text{rot}}\cdots\widetilde{B_1}^{\text{rot}}).$$

Applying the same argument to $\widetilde{B_1}\widetilde{B_2}\cdots\widetilde{B_{\frac{nl}{4}-1}}\widetilde{B_{\frac{nl}{4}-1}}^{\text{rot}}\cdots\widetilde{B_1}^{\text{rot}}$ or $\Delta^2\widetilde{B_1}\widetilde{B_2}\cdots\widetilde{B_{\frac{nl}{4}-1}}\widetilde{B_{\frac{nl}{4}-1}}^{\text{rot}}\cdots\widetilde{B_1}^{\text{rot}}$, respectively, and continuing inductively, we get

$$\sigma(\widetilde{\beta^n}(\widetilde{\beta^n})^{\text{rot}}) \geq 2k + \sigma(\Delta^{2(\frac{nl}{4}-k)}).$$

Now (7) follows from Murasugi's formula for the signature of torus links of braid index 4, which implies that $\sigma(\Delta^{2j}) = \sigma(T(4, 4j)) = 8j - 1$ holds for all positive integers j , see [Mur74, Proposition 9.2] or use Lemma 1.10. \square

CHAPTER 3

Gordian Adjacency

In this chapter we study Gordian adjacency for knots, which can be defined via Gordian distance of knots. The structure of the discrete metric space given by Gordian distance on the set of isotopy classes of knots is badly understood. However, it is known to be “big” globally and locally. For example, for every positive integer n the lattice \mathbb{Z}^n embeds quasi-isometrically into it [GG05], and infinitely many knots are between any two knots of distance two [Baa06].

We restrict our study of this metric space to the subset consisting of torus knots, asking the following question. ‘When is the triangle inequality $d_G(K, L) \geq d_G(L, O) - d_G(K, O)$ an equality?’

DEFINITION 3.1. Let K and L be knots. We say K is *Gordian adjacent* to L , denoted by $K \leq_G L$, if $d_G(K, L) = u(L) - u(K)$.

Equivalently, a knot K is Gordian adjacent to another knot L if L can be unknotted via K , that is, if there exists a minimal unknotting sequence for L that contains K . A *minimal unknotting sequence* for a knot L is a sequence of $u(L) + 1$ knots starting with L and ending with the unknot O such that any two consecutive knots are related by a crossing change, see Baader [Baa10]. The name ‘Gordian adjacency’ is motivated by the connection to algebraic adjacency; compare Chapter 4. Gordian adjacency is a partial order.

We prove the following results on Gordian adjacency for torus knots.

THEOREM 3.2. *Let (n, m) and (a, b) be pairs of coprime positive integers with $n \leq a$ and $m \leq b$. Then the torus knot $T(n, m)$ is Gordian adjacent to the torus knot $T(a, b)$.*

THEOREM 3.3. *Let n and m be positive integers with n odd and m not a multiple of 3. Then the torus knot $T(2, n)$ is Gordian adjacent to $T(3, m)$ if and only if $n \leq \frac{4m+1}{3}$.*

The core of the proof of Theorem 3.2, given in Section 2, is a generalization to knots in $S^1 \times S^1 \times \mathbb{R}$ of the following elementary fact. If a knot K in \mathbb{R}^3 has a knot diagram with n crossings, then $u(K) \leq \frac{n-1}{2}$. The proof of Theorem 3.3 relies on explicit constructions of the required

adjacencies, given in Section 1, and on Levine-Tristram signatures as obstructions to Gordian adjacency, see Section 3.

As a consequence of Theorem 3.2, Gordian adjacency and Gordian distance for torus knots of a fixed braid index are completely described. More precisely, if a positive integer a is fixed, then

$$T(a, b) \leq_G T(a, c) \text{ if and only if } b \leq c$$

for all b, c coprime to a . Hence,

$$d_G(T(a, b), T(a, c)) = |u(T(a, b)) - u(T(a, c))| = \frac{(a-1)|b-c|}{2},$$

where the second equation follows from the Milnor conjecture,

$$(8) \quad u(T(n, m)) = g_s^4(T(n, m)) = \frac{(n-1)(m-1)}{2}.$$

Note that to decide whether a knot is Gordian adjacent to another knot, the unknotting numbers of the involved knots should certainly be known; thus, even ignoring the connection to algebraic adjacency discussed in the introduction, equality (8) is relevant to the study of Gordian adjacency for torus knots. It is used throughout this chapter.

For torus knots $T(a, b)$ and $T(c, d)$ of different braid indices, it is in general not clear how Gordian adjacency is characterized in terms of a, b, c , and d . Theorem 3.3 provides such a characterization for the case of braid index 2 and 3.

REMARK 3.4. To completely determine Gordian adjacency for torus knots of braid index 2 and 3, additionally to Theorem 3.3, one has to show that no torus knot of braid index 3 is adjacent to a torus knot of braid index 2. More generally, Borodzik and Livingston show that a torus knot cannot be Gordian adjacent to a torus knot of strictly smaller braid index [BL13]. For this, they use a semicontinuity property that they prove using the Heegaard Floer correction term d —a Spin^c -3-manifold invariant which was defined by Ozsváth and Szabó [OS03]. Using signature obstructions one can only partially prove this result, see Section 3.

We calculate the signature obstruction to Gordian adjacencies between torus knots of higher braid indices, see Section 4, which (at least asymptotically) yields that Gordian adjacencies are algebraic adjacencies, see Remark 4.7.

1. Examples of Gordian adjacencies.

By definition, the unknot O is adjacent to every knot K . Let k be a positive integer. The unknotting number of the torus knot $T(2, 2k+1)$

is k . A minimal unknotting sequence of $T(2, 2k + 1)$ is provided by

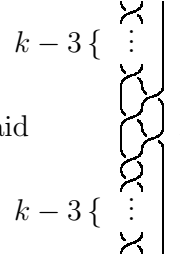
$$T(2, 2k + 1) \rightarrow T(2, 2k - 1) \rightarrow \dots \rightarrow T(2, 5) \rightarrow T(2, 3) \rightarrow O.$$

Consequently, $T(2, 2l + 1) \leq_G T(2, 2k + 1)$ for all $l \leq k$, a simple instance of Theorem 3.2. We now construct explicit examples of Gordian adjacencies that are not provided by Theorem 3.2. Let $\lfloor \cdot \rfloor$ denote the integer part of a real number.

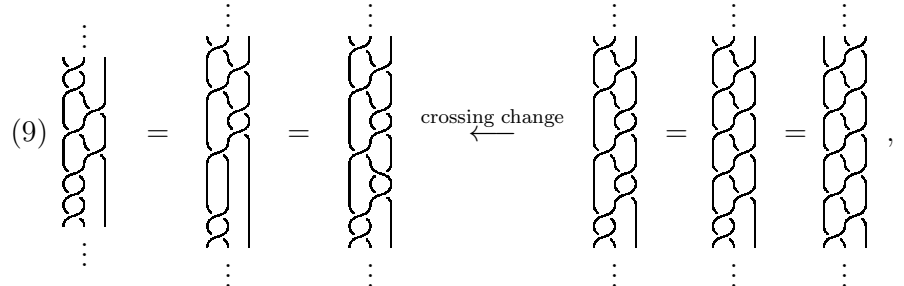
PROPOSITION 3.5. *For every natural number k , we have*

$$T(2, 2k + 1) \leq_G T(3, \lfloor \frac{3}{2}k + 1 \rfloor).$$

PROOF. The knot $T(2, 2k+1)$ is the closure of the braid



where $k - 3$ denotes the number of the crossings not drawn. We introduce a crossing change for knots containing a part that looks (in an appropriate braid diagram) like the above $T(2, 2k + 1)$.



where the first and the two last equalities are obtained by applying the braid relation

$$a_2 a_1 a_2 = \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} = a_1 a_2 a_1.$$

First consider the case when k is odd. We use (9) inductively.

$$\begin{array}{ccc}
 & \begin{array}{c} k-5 \{ \vdots \} \\ \left| \right. \\ \left. \right\} \\ k-5 \{ \vdots \} \end{array} & = \begin{array}{c} k-5 \{ \vdots \} \\ \left| \right. \\ \left. \right\} \\ k-5 \{ \vdots \} \end{array} \\
 T(2, 2k+1) \longleftarrow & & \\
 & \begin{array}{c} k-7 \{ \vdots \} \\ \left| \right. \\ \left. \right\} \\ k-7 \{ \vdots \} \end{array} & = \begin{array}{c} k-7 \{ \vdots \} \\ \left| \right. \\ \left. \right\} \\ k-7 \{ \vdots \} \end{array} \\
 \longleftarrow & & \\
 & \underbrace{\longleftarrow \cdots \longleftarrow}_{\frac{k-5}{2} \text{ crossing changes}} T(3, 3\frac{k-1}{2} + 2), &
 \end{array}$$

where every arrow indicates a crossing change as in (9) and the equalities are obtained by using that the full twist



commutes with every

3-braid. Thus,

$$\begin{aligned} d_G(T(2, 2k+1), T(3, 3\frac{k-1}{2} + 2)) &\leq \frac{k-1}{2} = (3\frac{k-1}{2} + 1) - k \\ &\stackrel{(8)}{=} u(T(3, 3\frac{k-1}{2} + 2)) - u(T(2, 2k+1)). \end{aligned}$$

The case when k is even has essentially the same proof except that the last crossing change does not use (9) but a slight variation of it. \square

2. Unknotting on the torus and proof of Theorem 3.2

Knots in \mathbb{R}^3 can be studied via knot diagrams on \mathbb{R}^2 up to Reidemeister equivalence. Similarly, for a surface F knots in $F \times \mathbb{R}$ can be studied via knot diagrams on F .

In a knot diagram on \mathbb{R}^2 with n crossings one needs to change at most $\lfloor \frac{n-1}{2} \rfloor$ of the crossings to get the unknot. This is easily proved geometrically by drawing a knot in \mathbb{R}^3 that projects to the curve on \mathbb{R}^2 given by the diagram and that descends (or ascends) monotonically except over one point in the diagram, see Figure 1, and remarking that

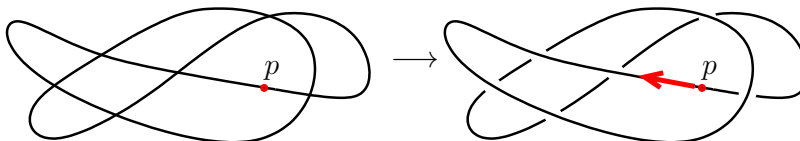


FIGURE 1. Any curve c in \mathbb{R}^2 is the projection of the unknot in \mathbb{R}^3 given by starting at any point p in \mathbb{R}^3 that projects to c and then descending while following c .

such a knot is the unknot. To prove Theorem 3.2, which is a statement entirely about knots in \mathbb{R}^3 , one is surprisingly led to ask whether a similar fact holds for knots in $S^1 \times S^1 \times \mathbb{R}$. We provide such a result, which we then use to prove Theorem 3.2.

Let F be a surface. In what follows a closed smooth curve $c: [0, 1] \rightarrow F$ is called *presimple* if its lift $\tilde{c}: \mathbb{R} \rightarrow \tilde{F}$ to the universal cover \tilde{F} of F is injective and if c is homotopic to a simple closed curve. A knot in $F \times \mathbb{R}$ that is isotopic to a knot that projects to a simple closed curve on F is called *unknotted*.

REMARK 3.6. There is at most one unknot (up to isotopy) in every homotopy class of closed curves in $F \times \mathbb{R}$. This follows from the fact that homotopic simple closed curves in surfaces are isotopic, see Epstein [Eps66].

In the case of the torus we can be more precise. A homotopy class of closed curves in $S^1 \times S^1 \times \mathbb{R}$ contains an unknot, which is unique up to isotopy, if and only if (via the usual identification of $\pi_1(S^1 \times S^1) \cong \pi_1(S^1 \times S^1 \times \mathbb{R})$ with \mathbb{Z}^2) the corresponding element in \mathbb{Z}^2 has coprime entries or is $(0, 0)$. This is a reformulation of the classification of simple closed curves in $S^1 \times S^1$, written, for example, in Rolfsen's textbook [Rol90].

LEMMA 3.7. *For every presimple curve c in $S^1 \times S^1$ there is a knot O in $S^1 \times S^1 \times \mathbb{R}$ that projects to c on $S^1 \times S^1$ and that is unknotted.*

REMARK 3.8. In terms of knot diagrams Lemma 3.7 means that if a knot K in $S^1 \times S^1 \times \mathbb{R}$ projects to a presimple diagram with n crossings on $S^1 \times S^1$, then one can get the diagram of the unknot by changing at most $\lfloor \frac{n}{2} \rfloor$ of the n crossings.

To prove this, we use Lemma 3.7 to get the unknot O with the same diagram as K , except it differs in the choice of crossings. If this new diagram differs from the original one in less than half of the crossings, we are done. Otherwise, we switch all crossings in the diagram of O yielding a knot diagram of a knot \overline{O} . The knot \overline{O} is also unknotted, as the following shows. Let H_t be an isotopy that changes O to a knot that projects to a simple closed curve on $S^1 \times S^1$. Then parametrize \overline{O} in $S^1 \times S^1 \times \mathbb{R}$ exactly the same way as O , except changing the sign in the \mathbb{R} coordinate. The same isotopy H_t as for O (with a change of sign in the last coordinate) shows that \overline{O} is unknotted.

Clearly the assumption that c is homotopic to a simple closed curve is necessary in Lemma 3.7. We conjecture that Lemma 3.7 holds for all curves c that are homotopic to a simple closed curve and, furthermore, that Lemma 3.7 generalizes to all surfaces.

PROOF OF LEMMA 3.7. Denote $S^1 \times S^1$ by F . Our strategy is to construct a presimple homotopy h_t of c (meaning that h_t is presimple for every $t \in [0, 1]$) to a simple closed curve and then to find an isotopy H_t of knots in $F \times \mathbb{R}$ that has h_t as projection.

We first lift the curve c to a mapping $\tilde{c}: \mathbb{R} \rightarrow \tilde{F}$, where $\varphi: \tilde{F} \rightarrow F$ denotes the universal covering map. Since c is presimple, $\tilde{c}: \mathbb{R} \rightarrow \tilde{F}$ is injective and there exists a simple closed curve $g: [0, 1] \rightarrow F$ that is homotopic to c . We take g such that $g(0) = g(1) = c(0) = c(1)$ and denote by $\tilde{g}: \mathbb{R} \rightarrow \tilde{F}$ its lift to \tilde{F} with $\tilde{g}(k) = \tilde{c}(k)$ for all $k \in \mathbb{Z}$. Let $\tilde{h}_t: \mathbb{R} \rightarrow \tilde{F}$ be an equivariant¹ isotopy between \tilde{c} and \tilde{g} that is constant on \mathbb{Z} , see Figure 2. Of course $h_t = \varphi \circ \tilde{h}_t: [0, 1] \rightarrow F$ is a presimple

¹I.e. $\tilde{h}_t(s+1) = D(\tilde{h}_t(s))$ for all s in \mathbb{R} , where D denotes the unique deck transformation sending $\tilde{c}(0)$ to $\tilde{c}(1)$.

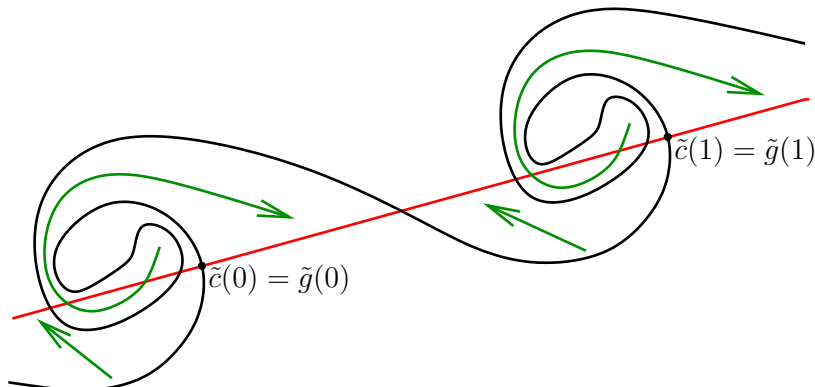


FIGURE 2. An equivariant isotopy (green) of \tilde{c} (black) to \tilde{g} (red) is indicated.

homotopy.

The idea for building H_t is to measure how far away from g points $p = h_t(s)$ are and then to put this distance $d(p)$ in the second coordinate of H_t . We need a metric to make this precise and the distance will actually be measured in the universal cover. Put a Riemannian metric on F with constant curvature 0 such that g is a simple closed geodesic of length 1. The universal cover \tilde{F} is identified with the Euclidean plane \mathbb{R}^2 such that $\varphi: \tilde{F} \rightarrow F$ is locally an isometry. Let $d: \tilde{F} \rightarrow \mathbb{R}$ denote the oriented distance to the straight line \tilde{g} .² We claim that the homotopy

$$H_t: [0, 1] \rightarrow F \times \mathbb{R}, s \mapsto (h_t(s), d(\tilde{h}_t(s))),$$

which projects to the homotopy h_t on F , is an isotopy. This claim implies that $H_0: [0, 1] \rightarrow F \times \mathbb{R}$ is an unknot O that projects to $h_0 = c$; therefore, it finishes the proof.

In order to prove that H_t is an isotopy, we assume towards a contradiction that H_t is not injective for some fixed t . Without loss of generality we assume $t = 0$, i.e. $\tilde{h}_t = \tilde{c}$. If there exist $s \neq r \in [0, 1]$ such that $H_0(s) = H_0(r)$, then, by definition of H_0 , the points $\tilde{p}_1 = \tilde{c}(s)$ and $\tilde{p}_2 = \tilde{c}(r)$ in \tilde{F} satisfy

$$\varphi(\tilde{p}_1) = \varphi(\tilde{p}_2) \quad \text{and} \quad d(\tilde{p}_1) = d(\tilde{p}_2).$$

As $d(\tilde{p}_1) = d(\tilde{p}_2)$, there is a geodesic segment parallel to \tilde{g} from \tilde{p}_1 to \tilde{p}_2 . The length of this segment is an integer k since $\varphi(\tilde{p}_1) = \varphi(\tilde{p}_2)$. It follows that $\tilde{p}_2 = \tilde{c}(k + s)$ if the sign of k is chosen correctly. This

²Ordinary Euclidean distance of points in $\tilde{F} = \mathbb{R}^2$ to the straight line \tilde{g} with a sign depending on whether the point is on the left or the right of \tilde{g} .

is seen by lifting c to \tilde{F} such that the lift starts at $\tilde{g}(k) = \tilde{c}(k)$, see Figure 3 for a case with $k = 1$. However, $\tilde{c}(r) = \tilde{c}(k + s)$ and $k + s \neq r$

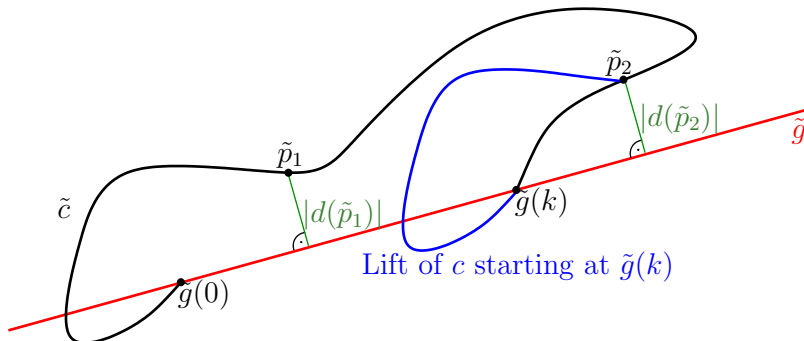


FIGURE 3. The curve $\tilde{c}_{|[0,1]}$ (black) intersects $\tilde{c}_{|[k,k+s]}$ (blue) in \tilde{p}_2 .

contradict the injectivity of $\tilde{h}_t = \tilde{c}$.

□

Let us shortly introduce notations and the general strategy for the proof of Theorem 3.2. In the following $S^1 \times S^1$ denotes the standard torus in \mathbb{R}^3 and $N(S^1 \times S^1)$ a tubular neighborhood of $S^1 \times S^1$. Also, we denote the curve obtained by projecting a knot K in $N(S^1 \times S^1)$ to $S^1 \times S^1$ by $\pi(K)$. Such a curve $\pi(K)$ (together with crossing information) provides a knot diagram on $S^1 \times S^1$ for the knot K in $N(S^1 \times S^1) \cong S^1 \times S^1 \times \mathbb{R}$.

To show the adjacency $K_1 \leq_G K_2$ for the knots $K_2 = T(a, b)$ and $K_1 = T(n, m)$, i.e. to show that $d_G(K_2, K_1)$ is less than or equal (and thus equal) to $u(K_2) - u(K_1)$, we proceed as follows. We isotope K_2 and K_1 into $N(S^1 \times S^1)$ in such a way that

- (I) $\pi(K_1)$ is simple closed (thus, K_1 is unknotted in $N(S^1 \times S^1)$),
- (II) K_2 is homotopic to K_1 in $N(S^1 \times S^1)$,
- (III) and $\pi(K_2)$ has $2(u(K_2) - u(K_1))$ crossings.

In all our cases $\pi(K_2)$ will have an injective lift to the universal cover \mathbb{R}^2 . This together with (I) and (II) yields that $\pi(K_2)$ is a presimple curve in $S^1 \times S^1$. Hence, Remark 3.8 applies and, because of (III), guaranties the existence of $u(K_2) - u(K_1)$ crossing changes in $N(S^1 \times S^1) \cong S^1 \times S^1 \times \mathbb{R}$ changing K_2 to the unknot. This unknot is homotopic to K_1 by (II) and thus isotopic to K_1 by Remark 3.6.

Before giving a proof of Theorem 3.2, we apply this strategy in a concrete example.

EXAMPLE 3.9. We show that $T(3, 5)$ is Gordian adjacent to $T(3, 7)$. As $u(T(3, 7)) - u(T(3, 5)) = 2$ we need to show that we can change $T(3, 7)$ to $T(3, 5)$ via 2 crossing changes. First we isotope $T(3, 7)$ into $N(S^1 \times S^1)$ as shown on the left-hand side of Figure 4. Projecting

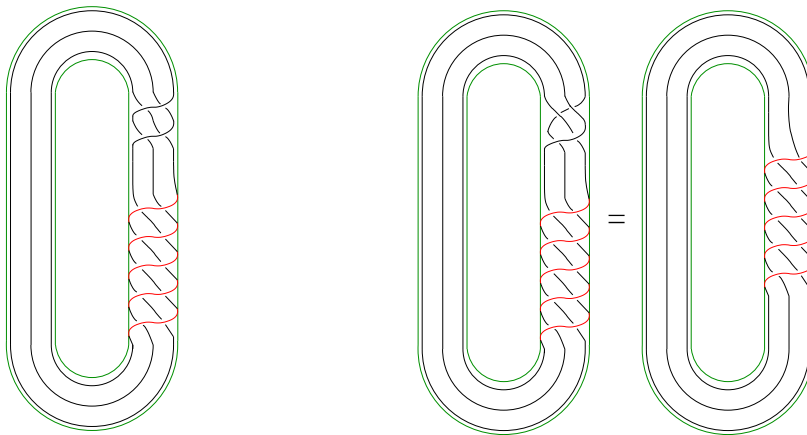


FIGURE 4. Knots contained in a tubular neighborhood of the standard torus (green) that are homotopic in this neighborhood. Five arcs (red) are on the upper half of the torus, the rest of the knots (black) lie on the lower half. Left: The knot $T(3, 7)$ with 4 crossings when projected on to the torus. Right: Two isotopic (in a neighborhood of the torus) occurrences of the knot $T(3, 5)$, one of them without crossings.

this $T(3, 7)$ to $S^1 \times S^1$ yields a curve $\pi(T(3, 7))$ with 4 crossings. The curve $\pi(T(3, 7))$ is presimple since it has an injective lift to \mathbb{R}^2 and is homotopic to the standard embedding of the torus knot $T(3, 5)$. Thus, by Remark 3.8 changing 2 of the crossings suffices to produce a knot K in $N(S^1 \times S^1)$ that is unknotted. As the knot K and the standard $T(3, 5)$ are homotopic unknots in $N(S^1 \times S^1)$ they are isotopic in $N(S^1 \times S^1)$ by Remark 3.6. In particular, K and $T(3, 5)$ are isotopic as knots in \mathbb{R}^3 ; thus, $d_G(T(3, 5), T(3, 7)) = 2$. In this example with only 4 crossings one can quickly exhibit the knot K explicitly. E.g. the right-hand side of Figure 4 provides a knot K that is obtained from the knot on the left-hand side of Figure 4 by performing two crossing changes in $N(S^1 \times S^1)$ and that is isotopic to the standard $T(3, 5)$ as predicted by Remark 3.8. This last isotopy can be seen by applying braid relations (similarly as in the proof of Proposition 3.5) and checking that these can be realized while staying within $N(S^1 \times S^1)$.

PROOF OF THEOREM 3.2. By assumption we have pairs of coprime positive integers (a, b) and (n, m) such that $n \leq a$ and $m \leq b$. Without loss of generality we suppose that $a < b$ and $n < m$.

Let us first consider the case $n = a$, for which we proceed as in Example 3.9. We need to show that $d_G(T(a, b), T(n, m))$ is equal to

$$\begin{aligned} u(T(a, b)) - u(T(n, m)) &= \frac{(b-1)(a-1)}{2} - \frac{(m-1)(n-1)}{2} \\ &= \frac{(b-m)(a-1)}{2}. \end{aligned}$$

We consider the knot $T(a, b)$ as the closure of the braid $(a_1 a_2 \cdots a_{a-1})^b$ and isotope it into a neighborhood $N(S^1 \times S^1)$ of the standard torus $S^1 \times S^1$ in \mathbb{R}^3 . More precisely, we isotope m arcs on the upper half of the torus and the rest of $T(a, b)$ on the lower half of the torus, in such a way that the curve $\pi(T(a, b))$ winds m times around the core of $S^1 \times S^1$ and $n = a$ times in the direction of the core of $S^1 \times S^1$, see left-hand side of Figure 4. Since n and m are coprime, there is a simple closed curve in $S^1 \times S^1$ that is homotopic to $\pi(T(a, b))$ by the second part of Remark 3.6, namely the standard embedding of the torus knot $T(n, m)$ in $S^1 \times S^1$. Also, $\pi(T(a, b))$ lifts injectively to the universal cover \mathbb{R}^2 ; thus, $\pi(T(a, b))$ is presimple. The m arcs do not intersect the rest of the curve $\pi(T(a, b))$ on the torus, so $\pi(T(a, b))$ has $(b-m)(a-1)$ crossings on the torus. By Remark 3.8 we need to change at most $\frac{(b-m)(a-1)}{2}$ crossings in the diagram on the torus (which correspond to crossing changes in $N(S^1 \times S^1) \cong S^1 \times S^1 \times \mathbb{R}$) to get an unknot K in $N(S^1 \times S^1)$. As the unknotted K and the standard $T(n, m)$ are homotopic in $N(S^1 \times S^1)$ they are also isotopic by Remark 3.6. Of course K is isotopic to $T(n, m)$ in \mathbb{R}^3 via the same isotopy as in $N(S^1 \times S^1)$. Therefore,

$$d_G(T(a, b), T(n, m)) \leq \frac{(b-m)(a-1)}{2}$$

as we wanted. The same argument works if $m = b$ or $a = m$.

This leaves the case $n < a$ and $m < b$. In the first case we interpreted $T(a, b)$ as the closure of a braid on a strands, in the following we see $T(a, b) = T(b, a)$ as a braid on b strands. We may assume $m > b-a$, otherwise we replace (inductively) a, b by $a, b-a$ (respectively by $b-a, a$ if $b-a < a$) since by the first case $T(a, b-a) \leq_G T(a, b)$. To apply the same idea as before we reduce the braid on b strands to one on m strands. More precisely, the representation of $T(a, b)$ as the closure of the b -strand braid

$$(10) \quad (a_1 \cdots a_{b-1})^a = a_a \cdots a_1 (a_2 \cdots a_{b-1})^a,$$

has the same closure as the $b - 1$ -strand braid

$$\tau_{b-1} = a_{a-1} \cdots a_1 (a_1 \cdots a_{b-2})^a,$$

see Figure 5. If $m = b - 1$, we isotope $T(a, b)$ (seen as the closure of

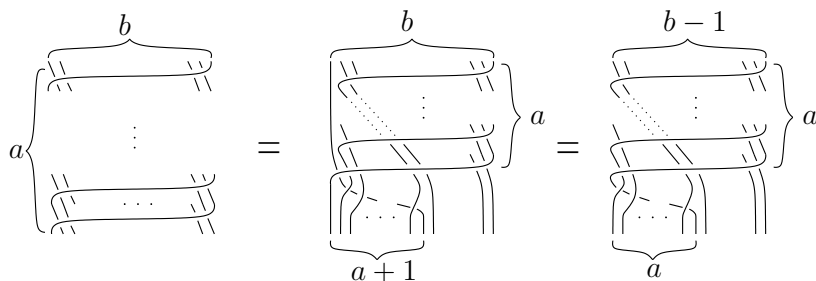


FIGURE 5. The first equality is the pictorial version of equation (10). The second equality is meant to hold for the closures only.

τ_{b-1}) into $N(S^1 \times S^1)$ such that n of the a over-passing arcs in the right part of Figure 5 project to the upper half of the torus and the rest of $\pi(T(a, b))$, including $a - 1 + (a - n)(b - 2)$ crossings, lies on the lower half. The curve $\pi(T(a, b))$ is presimple since it winds n respectively m times around the torus, i.e. it is homotopic in $N(S^1 \times S^1)$ to the standard embedding of the knot $T(n, m)$, and $\pi(T(a, b))$ lifts injectively to \mathbb{R}^2 . Therefore, we can use Remark 3.8 to get $T(n, m)$ by at most $\frac{a-1+(a-n)(b-2)}{2}$ crossing changes. Thus, $d_G(T(n, m), T(a, b))$ is less than or equal to

$$\begin{aligned} \frac{a - 1 + (a - n)(b - 2)}{2} &= \frac{(a - 1)(b - 1)}{2} - \frac{(n - 1)(b - 2)}{2} \\ &= u(T(a, b)) - u(T(n, m)). \end{aligned}$$

Suppose now $m < b - 1$. We no longer isotope $T(a, b)$ into $N(S^1 \times S^1)$. We first apply some crossing changes in \mathbb{R}^3 and then isotope the result into $N(S^1 \times S^1)$. More precisely, we apply a crossing change to τ_{b-1} that yields

$$(11) \quad a_{a-1} \cdots a_2 a_1^{-1} (a_1 \cdots a_{b-2})^a = a_{a-1} \cdots a_2 a_2 \cdots a_{b-2} (a_1 \cdots a_{b-2})^{a-1}.$$

Then, we replace the part $(a_1 \cdots a_{b-2})^{a-1}$ on the left-hand side of (11) by $a_{a-1} \cdots a_1 (a_2 \cdots a_{b-2})^{a-1}$ as in (10), which has the same closure as the $b - 2$ braid

$$\tau_{b-2} = (a_{a-2} \cdots a_1 a_1 \cdots a_{b-3})^2 (a_1 \cdots a_{b-3})^{a-2},$$

see Figure 6. If $m = b - 2$, we isotope the closure of τ_{b-2} into $N(S^1 \times S^1)$

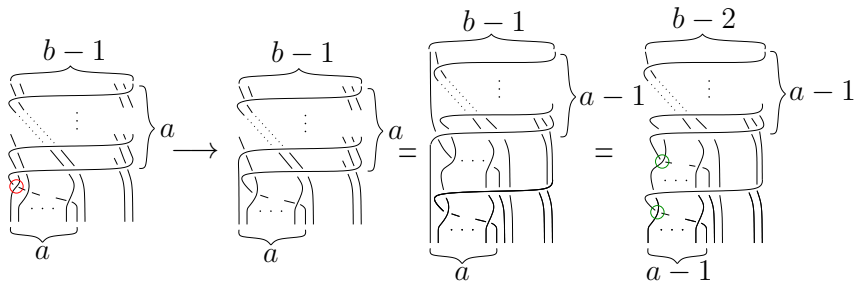


FIGURE 6. The arrow \longrightarrow indicates the changing of the marked (red) crossing. The two equalities are seen as in Figure 5. The two marked (green) crossings on the right side indicate the crossing changes that are necessary to obtain τ_{b-3} from τ_{b-2} , which is needed when $m < b - 2$.

in such away that it is homotopic to $T(n, m)$, namely such that n of the a over-passing arcs get to lie on the upper part of the torus and the reminding part (including $2(a - 2) + (a - n)(b - 3)$ crossings) lies on the lower part. Therefore, Remark 3.8 implies that $T(n, m)$ can be obtained from the closure of τ_{b-2} by changing $\frac{2(a-2)+(a-n)(b-3)}{2}$ crossings. Thus, $d_G(T(n, m), T(a, b))$ is less than or equal to

$$\begin{aligned}
 1 + \frac{2(a-2) + (a-n)(b-3)}{2} &= \frac{2a-2 + (a-n)(b-3)}{2} \\
 &= \frac{2a-2 + (a-1)(b-3)}{2} \\
 &\quad - \frac{(n-1)(b-3)}{2} \\
 &= \frac{(a-1)(b-1)}{2} - \frac{(n-1)(b-3)}{2} \\
 &= u(T(a, b)) - u(T(n, m)).
 \end{aligned}$$

For general $m > b - a$ it follows similarly that we need to change

$$1 + 2 + \cdots + (b - m - 1) = \frac{(b - m)(b - m - 1)}{2}$$

crossings of $T(a, b)$ to get the closure of the m braid

$$\tau_m = (a_{a-(b-m)} \cdots a_1 a_1 \cdots a_{m-1})^{b-m} (a_1 \cdots a_{m-1})^{a-(b-m)},$$

see Figure 7. For example, the closure of τ_{b-3} is obtain from the closure of τ_{b-2} by the two crossing changes that are indicated (green) in Figure 6. We isotope the closure of τ_m into $N(S^1 \times S^1)$ such that n of the a over-passing arcs lie on the upper half of the torus and

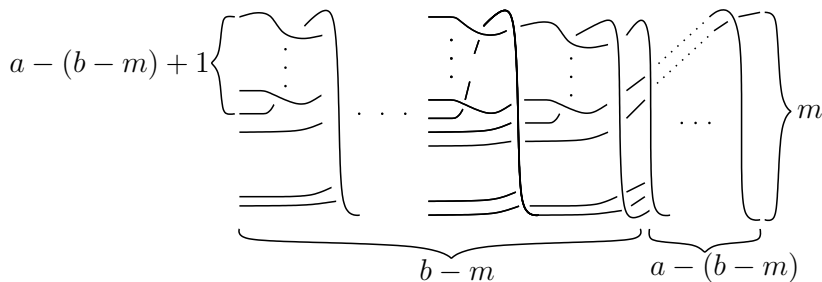


FIGURE 7. The braid τ_m , which can be obtained from $T(a, b)$ by $1 + 2 + \cdots + (b - m - 1)$ crossing changes.

$(b-m)(a-(b-m)) + (a-n)(m-1)$ crossings on the lower half. Therefore, we get $T(n, m)$ from τ_m by changing $\frac{(b-m)(a-(b-m)) + (a-n)(m-1)}{2}$ crossings by Remark 3.8. Combined we have that $d_G(T(n, m), T(a, b))$ is less than or equal to

$$\frac{(b-m)(b-m-1) + (b-m)(a-(b-m)) + (a-n)(m-1)}{2},$$

which is equal to $u(T(a, b)) - u(T(n, m))$. \square

3. Signatures as obstructions to Gordian adjacency

The goal of this section is to describe how Levine-Tristram signatures [Lev69][Tri69] yield obstructions to Gordian adjacency and to use this to prove Theorem 3.3.

The Gordian distance of two knots is greater than or equal to twice their cobordism distance since every crossing change can be realized by a cobordism of genus one, see Remark 1.1. Therefore, Lemma 1.8 yields the following obstruction to Gordian adjacency.

COROLLARY 3.10. *For knots K and L , we have that*

$$\left| \frac{\sigma_\omega(L) - \sigma_\omega(K)}{2} \right| \leq d_G(K, L).$$

In particular, if K is Gordian adjacent to L , then

$$\left| \frac{\sigma_\omega(L) - \sigma_\omega(K)}{2} \right| \leq u(L) - u(K).$$

As a consequence of Corollary 3.10 we prove that most torus knots are not adjacent to torus knots of braid index two as claimed in Remark 3.4. For braid index two torus knots the signature equals twice the unknotting number, that is $\frac{\sigma(T(2, n))}{2} = u(T(2, n)) = \frac{n-1}{2}$; see Lemma 1.10. This is also true for $T(3, 4)$ and $T(3, 5)$, but for all other

torus knots T there is a signature defect, i.e. $u(T) > \frac{\sigma(T)}{2}$. Thus, by Corollary 3.10, we have

$$d_G(T(2, n), T) \geq \frac{\sigma(T(2, n))}{2} - \frac{\sigma(T)}{2} > u(T(2, n)) - u(T)$$

for all torus knots T not equal to $T(3, 4)$, $T(3, 5)$ or some $T(2, m)$.

As Corollary 3.10 is a consequence of Lemma 1.8, similar results will hold for other adjacency notions as well. We now provide a signature obstruction to Gordian adjacency, which fails for algebraic adjacency and subsurface adjacency, compare Proposition 4.5 and Proposition 5.4. Let us denote by $s(K)$ the Rasmussen invariant of a knot K [Ras10]. The next lemma shows how ω -signatures and s behave with respect to crossing changes.

LEMMA 3.11. *If K_- is obtained from K_+ via one positive-to-negative crossing change, then*

$$\sigma_\omega(K_-) \in \{\sigma_\omega(K_+), \sigma_\omega(K_+) - 2\}.$$

The same holds for the Rasmussen invariant.

Rasmussen used an observation by Livingston [Liv04, Corollary 2 and 3] to prove Lemma 3.11 for s [Ras10]. Of course Corollary 3.10 is an immediate consequence of Lemma 3.11. In the literature one finds direct proofs for Corollary 3.10, see [Kaw96, Theorem 11.2.1] and [GG05], but we found no proof for the ω -signatures statement of Lemma 3.11. At the end of this section we provide a proof of Lemma 3.11 using a variation of Livingston's observation. The following proposition explains how Lemma 3.11 yields another obstruction to Gordian adjacency of torus knots, which is often better than Corollary 3.10.

PROPOSITION 3.12. *Let $K \leq_G L$ be a Gordian adjacency of positive braid knots. Then $\sigma_\omega(K) \leq \sigma_\omega(L)$ holds.*

PROOF. For positive braid knots we have $\frac{s}{2} = u$ [Ras10]. Thus, Lemma 3.11 yields that a minimal unknotting sequence of any positive braid knot involves only positive-to-negative crossing changes since s has to drop by 2 with every crossing change. Let

$$L = K_{u(L)} \rightarrow K_{u(L)-1} \rightarrow \cdots \rightarrow K \rightarrow \cdots \rightarrow K_1 \rightarrow K_0 = O$$

be a minimal unknotting sequence for L that contains K . As it involves only positive-to-negative crossing changes we have

$$\sigma_\omega(L) \geq \sigma_\omega(K_{u(L)-1}) \geq \cdots \geq \sigma_\omega(K) \geq \cdots \geq \sigma_\omega(O) = 0$$

by Lemma 3.11. Therefore, $\sigma_\omega(K) \leq \sigma_\omega(L)$ holds. □

REMARK 3.13. By the above proof, Proposition 3.12 remains true for any knot K with $\frac{s(K)}{2} = u(K)$. Instead of half the Rasmussen invariant we could have used any other knot invariant that satisfies the conditions of Lemma 3.15 and that is equal to the unknotting number on positive braid knots, compare [Liv04].

Proposition 3.12 provides obstructions that are strong enough to prove Theorem 3.3.

PROOF OF THEOREM 3.3. Fix $n = 2k + 1$ and note that $m = \lfloor \frac{3}{2}k + 1 \rfloor$ is minimal with $n \leq \frac{4}{3}m + \frac{1}{3}$. By Proposition 3.5 we have

$$T(2, 2k + 1) \leq_G T(3, \lfloor \frac{3}{2}k + 1 \rfloor).$$

Together with

$$T(3, \lfloor \frac{3}{2}k + 1 \rfloor) \leq_G T(3, m) \text{ for all } m \geq \lfloor \frac{3}{2}k + 1 \rfloor,$$

an easy instance of Theorem 3.2, we conclude that

$$T(2, 2k + 1) \leq_G T(3, m) \text{ for all } m \geq \lfloor \frac{3}{2}k + 1 \rfloor.$$

For the other direction we let $n = 2k + 1$ be any odd number and write $m = \lceil \frac{3}{2}k - 1 \rceil$, which is the largest m that does not satisfy $n \leq \frac{4}{3}m + \frac{1}{3}$. Thus, we have to show that $T(2, 2k + 1) \not\leq_G T(3, m)$. For $k \leq 4$ calculating unknotting numbers yields

$$T(2, 5) \not\leq_G T(3, 2), T(2, 7) \not\leq_G T(3, 4), \quad \text{and} \quad T(2, 9) \not\leq_G T(3, 5).$$

If $k \geq 5$ we distinguish two cases. Either, k equals 1 or 2 modulo 4, or k equals 3 or 4 modulo 4.

For $k = 1 + 4l, 2 + 4l$ with $l \geq 1$, Murasugi's formula for torus knots of braid index 3, see [Mur74, Proposition 9.1] or use Lemma 1.10, provides

$$\sigma(T(3, m)) = 2k - 2,$$

which is strictly less than

$$\sigma(T(2, 2k + 1)) = 2k.$$

Thus, Proposition 3.12 yields $T(2, 2k + 1) \not\leq_G T(3, m)$.

For $k = 3 + 4l, 4 + 4l$ with $l \geq 1$, Murasugi's formula gives

$$(12) \quad \sigma(T(3, m)) = 2k = \sigma(T(2, 2k + 1)).$$

In this case σ does not suffice as obstruction directly, but we use (12) to calculate $\sigma_\omega(T(3, m))$ for ω close to -1 , which yields the desired

obstruction. More precisely, set

$$(13) \quad \omega = e^{2\pi i \theta} \text{ with } \begin{cases} \theta \in (\frac{1}{2} - \frac{2}{3m}, \frac{1}{2} - \frac{1}{3m}) & \text{for } m \text{ even, i.e. } k = 3 + 4l \\ \theta \in (\frac{1}{2} - \frac{3}{6m}, \frac{1}{2} - \frac{1}{6m}) & \text{for } m \text{ odd, i.e. } k = 4 + 4l \end{cases}.$$

By Lemma 1.9 the value of $\sigma_\omega(T(3, m))$ is the same for all these ω .

CLAIM 3.14. For all $m = 4 + 6l, 5 + 6l$ with $l \geq 1$ and ω as in (13), we have $\sigma_\omega(T(3, m)) = \sigma(T(3, m)) - 2$.

Lemma 1.9 shows that the above ω can be chosen such that

$$\sigma(T(2, 2k + 1)) = \sigma_\omega(T(2, 2k + 1)).$$

Hence, Claim 3.14 and (12) yield

$$\begin{aligned} \sigma_\omega(T(3, m)) &= \sigma(T(3, m)) - 2 < \sigma(T(3, m)) \\ &= \sigma(T(2, 2k + 1)) = \sigma_\omega(T(2, 2k + 1)). \end{aligned}$$

Therefore, $T(2, 2k + 1) \not\leq_G T(3, m)$ by Proposition 3.12. It remains to prove Claim 3.14.

For the case when m is even, Lemma 1.9 applied to the knot

$$T = T(3, m) = T(3, 4 + 6l)$$

yields that $\sigma(T)$ is

$$\#(S \cap [\frac{1}{2} - \frac{1}{3m} + \varepsilon, \frac{3}{2} - \frac{1}{3m} + \varepsilon]) - \#(S \setminus (\frac{1}{2} - \frac{1}{3m} + \varepsilon, \frac{3}{2} - \frac{1}{3m} + \varepsilon))$$

and that $\sigma_\omega(T)$ is

$$\#(S \cap [\frac{1}{2} - \frac{1}{3m} - \varepsilon, \frac{3}{2} - \frac{1}{3m} - \varepsilon]) - \#(S \setminus (\frac{1}{2} - \frac{1}{3m} - \varepsilon, \frac{3}{2} - \frac{1}{3m} - \varepsilon))$$

for ε small enough. Observe that

$$\begin{aligned} \frac{3}{2} - \frac{1}{3m} &= \frac{2}{3} + \frac{5m - 2}{6m} = \frac{2}{3} + \frac{5(4 + 6l) - 2}{6(4 + 6l)} \\ &= \frac{2}{3} + \frac{3 + 5l}{4 + 6l} = \frac{2}{3} + \frac{3 + 5l}{m} \in S \end{aligned}$$

$$\text{and } \frac{1}{2} - \frac{1}{3m} = \dots = \frac{1}{3} + \frac{l}{m} + \frac{1}{3m} \notin S.$$

This means

$$(S \cap [\frac{1}{2} - \frac{1}{3m} - \varepsilon, \frac{3}{2} - \frac{1}{3m} - \varepsilon]) \dot{\cup} \{\frac{3}{2} - \frac{1}{3m}\} = S \cap [\frac{1}{2} - \frac{1}{3m} + \varepsilon, \frac{3}{2} - \frac{1}{3m} + \varepsilon]$$

and

$$S \setminus (\frac{1}{2} - \frac{1}{3m} - \varepsilon, \frac{3}{2} - \frac{1}{3m} - \varepsilon) = (S \setminus (\frac{1}{2} - \frac{1}{3m} + \varepsilon, \frac{3}{2} - \frac{1}{3m} + \varepsilon)) \dot{\cup} \{\frac{3}{2} - \frac{1}{3m}\}.$$

Therefore, $\sigma(T) = 2 + \sigma_\omega(T)$ holds.

If m is odd, we have $m = 5 + 6l$. Similarly to the even case, we get

$$\frac{3}{2} - \frac{1}{6m} = \frac{2}{3} + \frac{4 + 5l}{m} \in S, \text{ but } \frac{1}{2} - \frac{1}{6m} \notin S.$$

The rest of the argument is as in the case when m is even. \square

In the remainder of the section we prove Lemma 3.11 using the following variant of an observation by Livingston.

LEMMA 3.15. *Let τ be a integer valued knot invariant satisfying*

- $\tau(K_1 \# K_2) = \tau(K_1) + \tau(K_2)$ and $\tau(-K_1) = -\tau(K_1)$ for all knots K_1 and K_2 ,
- $\tau(K) \leq g_s^4(K)$ for all knots K ,
- *there exists a knot K with $\tau(K) = 1$ that can be transformed to the unknot O by a positive-to-negative crossing change.*

Then τ is a concordance invariant, $|\tau(K)| \leq g_s^4(K)$ for all knots K , and

$$0 \leq \tau(K_+) - \tau(K_-) \leq 1$$

whenever K_- is a knot obtained from K_+ by a positive-to-negative crossing change.

Lemma 3.15 is a variation of Corollaries 2 and 3 in [Liv04]. The first two assertions are given in [Liv04, Corollary 2]. The proof of the third assertion given in [Liv04] needs to be modified as follows to yield a proof Lemma 3.15.

PROOF OF THE THIRD ASSERTION. Let K_+ and K_- differ by a positive-to-negative crossing change. Therefore, we have

$$g_s^4(K_+ \# (-K_-)) = \frac{d_c(K_+, K_-)}{2} \leq 1$$

since a crossing change can be realized by a cobordism of genus 1. Choose a knot K such that $\tau(K) = 1$ and such that K can be unknotted by a positive-to-negative crossing change. Now we take $K_+ \# -K_+$ and do a negative-to-positive crossing change (in the $-K_+$ -part) and a positive-to-negative crossing change such that we get $K_+ \# -K_- \# -K$. These two crossing changes can be realized by a cobordism of genus 1 since they are of opposite kind, compare Remark 1.1. This together with $g_s^4(K_+ \# -K_+) = 0$ yields $g_s^4(K_+ \# -K_- \# -K) \leq 1$. Thus, we have

$$|\tau(K_+) - \tau(K_-)| = |\tau(K_+ \# -K_-)| \leq g_s^4(K_+ \# -K_-) \leq 1$$

and

$$\begin{aligned} |\tau(K_+) - \tau(K_-) - 1| &= |\tau(K_+ \# -K_- \# -K)| \\ &\leq g_s^4(K_+ \# -K_- \# -K) \leq 1, \end{aligned}$$

which can be combined to give

$$0 \leq \tau(K_+) - \tau(K_-) \leq 1.$$

□

This proof is from [Liv04, Corollary 3], except that the knot $T(2, 3)$ was replaced by a knot K with $\tau(K) = 1$ that can be unknotted by a positive-to-negative crossing change. This is necessary since we do not want to assume that $\tau(T(2, 3)) = 1$.

PROOF OF LEMMA 3.11. Rasmussen proves all conditions of Lemma 3.15 for $\tau = \frac{s}{2}$ in [Ras10] (note that $\frac{s(T(2,3))}{2} = 1$).

For ω -signatures the first condition is provided by Lemma 1.6. The second is contained in Lemma 1.8. If $\omega = -1$, we can choose K to be $T(2, 3)$ for the third condition since $\sigma(T(2, 3)) = 2$. In general, fix a root of unity ω of prime order. For a positive integer k let $T(2k - 1)$ be the positive twist knot with $2k - 1$ half-twists, see Figure 8. These

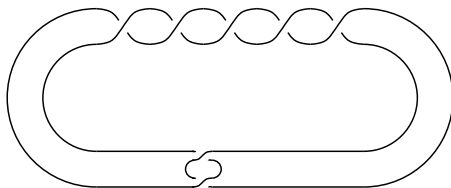


FIGURE 8. The twist knot $T(5)$.

knots can be unknotted by a positive-to-negative crossing change and

$$A = \begin{bmatrix} k & 1 \\ 0 & 1 \end{bmatrix}$$

is a Seifert matrix for $T(2k - 1)$. For sufficiently large k both eigenvalues of the Hermitian matrix $(1 - \omega)A + (1 - \bar{\omega})A^t$ are positive. Thus, choosing K to be $T(2k - 1)$ for a sufficiently large k provides the third condition. □

4. A bound on Gordian adjacency for torus knots of higher braid indices

This section is concerned with the question, when is $T(a, n) \leq_G T(b, m)$ for fixed $a < b$ and n, m large? Concretely, we study the numbers

$$\underline{k}(a, b) = \liminf_{m \rightarrow \infty} \frac{n(m)}{m} \quad \text{and} \quad \bar{k}(a, b) = \limsup_{m \rightarrow \infty} \frac{n(m)}{m},$$

where $n(m)$ denotes the largest integer such that

$$T(a, n(m)) \leq_G T(b, m).$$

We suspect, but cannot prove, that

$$\underline{k}(a, b) = \overline{k}(a, b) \text{ for all } a < b \in \mathbb{N}.$$

Certainly, we have $\underline{k}(2, 3) = \overline{k}(2, 3) = \frac{4}{3}$ by Theorem 3.3. We also note that $1 \leq \underline{k}(a, b)$ by Theorem 3.2 and $\overline{k}(a, b) \leq \frac{b-1}{a-1}$ since

$$\frac{(a-1)(n(m)-1)}{2} = u(T(a, n(m))) \leq u(T(b, m)) = \frac{(b-1)(m-1)}{2}.$$

Using ω -signatures we get an upper bound for $\overline{k}(a, b)$ that is strictly better than $\frac{b-1}{a-1}$.

PROPOSITION 3.16. *If $a \leq b \in \mathbb{N}$, then*

$$\overline{k}(a, b) \leq \frac{a \lceil \frac{b}{a} \rceil^2 - (a+2b) \lceil \frac{b}{a} \rceil + b(b+1)}{(a-1)b} \leq \frac{b}{a}.$$

A calculation shows that $\frac{a \lceil \frac{b}{a} \rceil^2 - (a+2b) \lceil \frac{b}{a} \rceil + b(b+1)}{(a-1)b} = \frac{b}{a}$ if and only if a divides b . If for example $b-a$ equals 1, Proposition 3.16 yields

$$\overline{k}(a, a+1) \leq \frac{a+2}{a+1}.$$

This is better than $\frac{b}{a} = \frac{a+1}{a}$ or even $\frac{b-1}{a-1}$, but we only know it to be optimal for $a=2$, namely $\underline{k}(2, 3) = \overline{k}(2, 3) = \frac{4}{3}$. Note that Proposition 3.16 is strictly better than what one gets using the classical signature. For example, the signature provides only $\overline{k}(3, 4) \leq \frac{3}{2}$ since $\sigma(T(3, n)) \sim \frac{4}{3}n$ and $\sigma(T(4, m)) \sim 2m$ by Lemma 1.10, which is the same factor one gets when using that the unknotting number has to decrease.

PROOF. We use the following approximation by Gambaudo and Ghys [GG05, Proposition 5.2]. Let l be a positive integer and θ a real number with $\frac{l-1}{b} < \theta \leq \frac{l}{b}$, then

$$(14) \quad \left| \sigma_{e^{2\pi i \theta}}(T(b, m)) - m \left(2(b - (2l-1)\theta) + \frac{2l(l-1)}{b} \right) \right| \leq 2b.$$

In fact, the complicated looking part on the left-hand side is the asymptotic $e^{2\pi i \theta}$ -signature of $T(b, m)$. This means that (14) states precisely that the asymptotic signature and the signature stay close, compare Chapter 2.

Proposition 3.12 yields $\sigma_\omega(T(b, m)) - \sigma_\omega(T(a, n(m))) \geq 0$. By the approximation we get

$$(15) \quad m \left(2(b - (2l - 1))\theta + \frac{2l(l - 1)}{b} \right) - n(m) \left(2(a - (2l' - 1))\theta + \frac{2l'(l' - 1)}{a} \right) \geq -2(a + b),$$

where l and l' are positive integers with $\frac{l-1}{b} < \theta \leq \frac{l}{b}$ and $\frac{l'-1}{a} < \theta \leq \frac{l'}{a}$, respectively. Choosing $\theta = \frac{1}{a}$ inequality (15) becomes

$$m \left(2 \frac{(b - (2\lceil \frac{b}{a} \rceil - 1))}{a} + 2 \frac{\lceil \frac{b}{a} \rceil (\lceil \frac{b}{a} \rceil - 1)}{b} \right) - n(m) 2 \frac{a - 1}{a} \geq -2(a + b)$$

or equivalently

$$(16) \quad \frac{n(m)}{m} \leq \frac{a \lceil \frac{b}{a} \rceil^2 - (a + 2b) \lceil \frac{b}{a} \rceil + b(b + 1)}{(a - 1)b} + \frac{a(a + b)}{m(a - 1)}.$$

This proves the first inequality.³ The second inequality can be checked by a calculation. \square

REMARK 3.17. Our choice $\theta = \frac{1}{a}$ is the best possible and yields the optimal bound for $\bar{k}(a, b)$ that can be achieved using the properties of signatures from Lemma 3.11. This can be checked using the above approximation from [GG05].

In order to determine $\underline{k}(a, b)$ and $\bar{k}(a, b)$ for $(a, b) \neq (2, 3)$, we now wish to find geometric constructions in the spirit of Section 1 that at least for some a and b yield a lower bound for $\underline{k}(a, b)$ that is equal to the upper bound given by Proposition 3.16. So far we have only found constructions giving lower bounds that do not coincide with the upper bounds, e.g. $\frac{5}{3} \leq \underline{k}(2, 4) \leq \bar{k}(2, 4) \leq 2$ and $\frac{9}{8} \leq \underline{k}(3, 4) \leq \bar{k}(3, 4) \leq \frac{5}{4}$.

³Note the following technical point. If $\omega = e^{2\pi i \frac{1}{a}}$ is not a root of unity of prime order, then Lemma 3.11 cannot be applied as above. Instead one chooses a sequence of θ_k tending to $\frac{1}{a}$, such that every $e^{2\pi i \theta_k}$ is a root of unity of prime order.

CHAPTER 4

Algebraic adjacency notions

In this chapter we study algebraic adjacency. The relation between algebraic and Gordian adjacency is discussed. In particular, Proposition 4.5 provides an infinite family of examples of algebraically adjacent torus knots that are not Gordian adjacent. On the other hand, we prove that δ -constant adjacency—a more restrictive notion than algebraic adjacency—implies Gordian adjacency, up to well-behaved concordances.

1. Algebraic adjacency

Arnol'd studied adjacency of singular function germs [Arn72, Definition 2.1], see also [Sie74]. As we are interested in knots and links, we study singular function germs only up to topological type, i.e. up to the isotopy class of their links of singularity. Thus, we use the following version of adjacency.

A *deformation* of a singularity f in $\mathbb{C}\{x, y\}$ is a family f_t of singularities with $f_0 = f$ depending smoothly on a positive real parameter $t \geq 0$.

DEFINITION 4.1. Let K and L be algebraic links. We say K is *algebraically adjacent* to L , denoted by $K \leq_a L$, if there exists a singularity f in $\mathbb{C}\{x, y\}$ with L as link of the singularity and a deformation f_t of f , such that for small non-zero t the germ f_t has K as link of the singularity.

In the classical algebraic or holomorphic setting the deformations often depend holomorphically on t and singularities are studied up to local biholomorphic base changes. All adjacencies in this stronger sense are in particular adjacencies as defined in Definition 4.1; and so, consequences of algebraic adjacency in this chapter hold for these notions as well.

REMARK 4.2. Algebraic links can be identified canonically with μ -constant-homotopy classes of square-free germs in $\mathbb{C}\{x, y\}$, see Remark 1.3. With this identification algebraic adjacency for algebraic

links corresponds to the concept of μ -adjacency studied by Siersma in [Sie74].

As described in the introduction, algebraic adjacency has similar properties as Gordian adjacency. For example, Theorem 3.2 is known and easy to show for \leq_a instead of \leq_G .

PROPOSITION 4.3. *If $n \leq a$ and $m \leq b$, then $T(n, m) \leq_a T(a, b)$.*

PROOF. The torus link $T(a, b)$ is the link of the singularity $y^a - x^b$. We choose as deformation $f_t(x, y) = y^a - x^b + t(y^n - x^m)$. For t small (but fixed) we perform, in a small chart around the origin, a biholomorphic coordinate change, which does not change the topological type of the singularity, such that $f_t = y^n(t + y^{a-n}) - x^m(t + x^{b-m})$ becomes $y^n - x^m$. To be explicit, the coordinate change is given as the inverse of the holomorphic map $(x, y) \mapsto (x \sqrt[m]{t + x^{b-m}}, y \sqrt[n]{t + y^{a-n}})$. \square

The obstruction to Gordian adjacency given in Corollary 3.10 also holds for algebraic adjacency.

LEMMA 4.4. *Let K and L be algebraic links. If there exists an adjacency $K \leq_a L$, then*

$$|\sigma_\omega(L) - \sigma_\omega(K)| \leq b_1(L) - b_1(K).$$

This follows from Lemma 1.8 and the fact that every adjacency $K \leq_a L$ yields a cobordism C in $S^3 \times [0, 1]$ between K and L with $|\chi(C)| = b_1(L) - b_1(K)$, which is seen as follows. Let f_t be the deformation that realizes the algebraic adjacency $K \leq_a L$ and let S_L be a Milnor sphere for L , i.e. an arbitrarily small sphere with $L = S_L \cap V(f_0)$. Then, by transversality, t can be chosen small enough such that $S_L \cap V(f_t)$ is still L and $K = S_K \cap V(f_t)$ for a Milnor sphere S_K for K . By a small perturbation of f_t the zero-set $V(f_t)$ becomes a smooth algebraic curve F with $L = S_L \cap F$ and $K = \overline{S_K \cap F}$. The cobordism between K and L that is given by $C = F \cap B_L \setminus B_K$ satisfies $|\chi(C)| = b_1(L) - b_1(K)$ and this realizes the cobordism distance by the Thom conjecture as proved by Kronheimer and Mrowka [KM93]. More precisely, by the Thom conjecture the first Betti numbers of $F \cap B_K$ and $F \cap B_L$ realize $b_1^4(K) = b_1(K)$ and $b_1^4(L) = b_1(L)$, respectively. In particular, $|\chi(C)| = b_1(L) - b_1(K)$. There cannot exist a cobordism C' with $|\chi(C')| < b_1(L) - b_1(K)$ as otherwise $F \cap B_K$ glued together with C' would yield a surface F' bounding L in B^4 with $b_1(F') < b_1^4(L)$, which is impossible by the definition of b_1^4 .

Despite similarities Gordian adjacency and algebraic adjacency do not agree for algebraic knots or even torus knots. The obstruction given in Proposition 3.12 does not hold for algebraic adjacency. Concretely,

we have $T(2, 15) \not\leq_G T(3, 10)$ by Theorem 3.3, but $T(2, 15) \leq_a T(3, 10)$, which we show now. The next proposition generalizes the algebraic adjacency $T(2, 6) \leq_a T(3, 4)$ calculated by Arnol'd [Arn72, $A_5 \leftarrow E_6$]. This gives a large class of examples of algebraic adjacencies of torus links, including $T(2, 15) \leq_a T(3, 10)$, which are not covered by Proposition 4.3.

PROPOSITION 4.5. *Let a, b, c be positive integers with $a \leq b$. Then $T(a, bc) \leq_a T(b, ac)$.*

In particular, $T(2, 3c) \leq_a T(3, 2c)$ for all positive integers c and $T(2, \frac{d^2}{2}) \leq_a T(d, d)$ for all even positive integers d .

REMARK 4.6. By the signature obstructions given in Lemma 4.4 much more would be “allowed”. For example, if $T(2, n) \leq_a T(d, d)$, Lemma 4.4 only yields $n \leq \frac{3d^2}{4}$. This is seen as follows. By Lemma 1.10 the signature of $T(d, d)$ is $\lfloor \frac{d^2-1}{2} \rfloor$. Therefore, $\sigma(T(2, n)) - \sigma(T(d, d)) \leq b_1(T(d, d)) - b_1(T(2, n))$ is equivalent to $n - 1 - \lfloor \frac{d^2-1}{2} \rfloor \leq (d-1)^2 - n - 1$, which implies $2n \leq \frac{3}{2}d^2$.

PROOF OF PROPOSITION 4.5. Suppose $a < b$, and choose $f_t = y^b - (x^c - ty)^a$ as deformation. Since $T(b, ac)$ is the link of the singularity $f_0 = y^b - x^{ac}$, it remains to show that for small $t \neq 0$, the link of the singularity f_t is $T(a, bc)$. We fix an arbitrary $t > 0$ and change coordinates locally around the origin by $(x, y) \mapsto (x, \frac{x^c - y}{t})$. With this f_t becomes $(\frac{x^c - y}{t})^b - y^a$. For all monomials of f_t except $-y^a$, the bi-degree—the tuple of integers consisting of the x degree and y degree of a monomial—lies on the line in \mathbb{Z}^2 that goes through $(bc, 0)$ and $(0, b)$. This shows that f_t and $x^{bc} - y^a$ have the same two monomials with bi-degree on the line $(bc, 0)$ and $(0, a)$ and all other bi-degrees lie strictly above this line. Therefore, they have the same link of singularity by a result of Kouchnirenko [Kou76, Corollaire 1.22]. \square

Proposition 4.5 gives an algebraic proof of an observation by Baader (see also Proposition 5.4), which states that the cobordism distance of $T(a, bc)$ and $T(b, ac)$ is equal to

$$bc + a - ac - b = (b - a)(c - 1)$$

and which is a key proposition in [Baa12].

REMARK 4.7. Proposition 4.5 shows that if we define an algebraic counterpart of $k(a, b)$ in Section 4, it is larger or equal to $\frac{b}{a}$, whereas in the Gordian setting $k(a, b)$ is smaller or equal to $\frac{b}{a}$ by Proposition 3.16. Thus, asymptotically, whenever $T(a, n) \leq_G T(b, m)$ for $a \leq b$ we get

roughly $n \leq \frac{b}{a}m$ and, therefore, $T(a, n) \leq_a T(b, m)$. We take this as evidence to conjecture that for torus knots Gordian adjacency implies algebraic adjacency.

There is a more restrictive algebraic notion of adjacency. A deformation f_t of f is called δ -constant if for every small enough Milnor ball B for f , there is a $t_0 > 0$ such that for every $t \leq t_0$ the sum

$$\sum_{z \in V(f) \cap B} \delta_z$$

is constant. Here δ_z denotes the δ -invariant of z in $V(f)$, which is zero if z is a smooth point of $V(f)$.

This notion arises naturally. Let f be irreducible in $\mathbb{C}\{x, y\}$. If $V(f)$ is locally parametrized by a normalization ϕ as described in Lemma 1.2, then a natural notion of deformation would be to study smooth families ϕ_t instead of smooth families f_t . Such a deformation ϕ_t can be used to yield a deformation f_t , but then f_t is necessarily δ -constant; compare Remark 1.4. In fact, if f_t is holomorphic in the t -variable, then f_t is δ -constant if and only if there is a parametric deformation ϕ_t yielding f_t , see e.g. [GLS07].

DEFINITION 4.8. Let K and L be algebraic links. We say K is δ -constant adjacent to L , denoted by $K \leq_\delta L$, if there exists a singularity f in $\mathbb{C}\{x, y\}$ with L as link of the singularity and a δ -constant deformation f_t of f such that for small non-zero t the germ f_t has K as link of the singularity.

For example one has $T(2, 2n) \leq_\delta T(2, 2n+1)$. Indeed, the deformation $f_t = y^2 - x^{2n+1} - tx^{2n}$ is δ -constant since for all t the zero-set $V(f_t)$ is non-singular outside the origin and the δ -invariant at the origin is n . In fact, if $t = 0$, then the link of the singularity at the origin is the knot $T(2, 2n+1)$, which yields $\mu = b_1(T(2, 2n+1)) = 2n$, and so $\delta = \frac{2n+1-1}{2} = n$. If $t \neq 0$, then the link of the singularity at the origin is the link $T(2, 2n)$, which has $\mu = 2n - 1$, and so $\delta = \frac{(2n-1)+2-1}{2} = n$.

Our motivation to study δ -constant adjacency is the connection to Gordian adjacency discussed in the next section.

2. From algebraic adjacency to 3-dimensional notions

Topological (but in some sense 4-dimensional) consequences of algebraic adjacency and δ -constant adjacency are known. Algebraically adjacent links $K \leq_a L$ have cobordism distance $b_1(L) - b_1(K)$ by the Thom conjecture, as elaborated in Section 1. For the stronger condition δ -constant adjacency, one has a stronger conclusion. In fact, Borodzik

and Livingston proved that if a knot K is δ -constant adjacent to a knot L , then L can be obtained from K by a so called positively self-intersecting concordance between the involved knots. This corresponds to changing $u(L) - u(K)$ crossings and some concordances [BL13].

In this section we describe in braid theoretic terms, how algebraic and Gordian adjacencies look like.

PROPOSITION 4.9. *Let $K \leq_a L$ be an algebraic adjacency of links K and L with multiplicities m_K and m_L , respectively. Then there exist positive m_L -braid words β_L and β_K such that the following holds. The closure of β_L is L ; the closure of β_K is the split union of K and $m_L - m_K$ unknots; and, up to conjugation, β_L can be obtained by adding*

$$l(\beta_L) - l(\beta_K) = m_L - m_K + b_1(L) - b_1(K)$$

conjugates of generators to β_K .

We illustrate this lengthy proposition with an example.

EXAMPLE 4.10. The link $T(2, 6)$ of the singularity $y^2 - x^6$ with multiplicity 2 is adjacent to the knot $T(3, 4)$ of the singularity $y^3 - x^4$ of multiplicity 3 via the deformation $f_t = y^2 - (x^2 + ty)^2$; see Proposition 4.5. One can add two generators (marked red below) to the 3-braid

$$\beta_{T(2,6)} = a_1^6 = \left| \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right| \quad \text{such that the result is}$$

$$a_1^2 a_2 a_1^3 a_2 a_1 = \left| \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right| = \left| \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right| = a_1 a_2 a_1 a_2 a_1 a_2 a_1 a_2 = \beta_{T(3,4)}.$$

The 3-braid $\beta_{T(2,6)}$ closes to the split union of $T(2, 6)$ and an unknot. The 3-braid $\beta_{T(3,4)}$ closes to $T(3, 4)$. Therefore, we checked that Proposition 4.9 is true for this concrete adjacency. In fact, the proof of Proposition 4.9 produces precisely these braids when applied to the deformation f_t .

Note that one can first add one generator to get $\widetilde{\beta_{T(2,6)}} = a_1^2 a_2 a_1^4$, which closes to $T(2, 6)$, and then add the second generator to get $a_1^2 a_2 a_1^3 a_2 a_1 = \beta_{T(3,4)}$.

REMARK 4.11. In Proposition 4.9 the addition of generators can be organized as follows. First $m_L - m_K$ conjugates of generators are added such that in the closure one has saddle moves connecting the $m_L - m_K$ unknots to the rest of the link. The result is a braid with closure \tilde{K} that is obtained from K by first adding $m_L - m_K$ unknotted components and then connecting them to K , i.e. a concordance from K to \tilde{K} . Hence, L is obtained from K by first doing a concordance

(with $m_L - m_K$ minima and $m_L - m_K$ saddles) and then a cobordism with $|\chi| = b_1(L) - b_1(K)$ given by saddle moves only. In particular, no maxima are needed.

If in addition the adjacency is δ -constant and the involved links are knots, we get a stronger conclusion.

PROPOSITION 4.12. *Notations are as in Proposition 4.9. If $K \leq_\delta L$ is a δ -constant adjacency of knots, then the addition of conjugates in Proposition 4.9 can be arranged as follows. First, $m_L - m_K$ conjugates of generators are added to β_K . Then $u(L) - u(K)$ conjugates of squares of generators are added yielding a conjugate of β_L .*

Note that adding conjugates of squares corresponds to crossing changes. Hence, we get (as in Remark 4.11) that L is obtained from K by first doing a rather well understood concordance (with $m_L - m_K$ minima and $m_L - m_K$ saddles) to a knot \tilde{K} and then $u(L) - u(K)$ crossing changes.

As a consequence we have that a δ -constant adjacency of knots K and L with the same multiplicity implies the existence of a Gordian adjacency. However, one is more interested in adjacencies that change the multiplicity as, for example, Gordian adjacency of torus knots of the same braid index (which corresponds to multiplicity) is fully understood by Theorem 3.2.

REMARK 4.13. Proposition 4.9 and Proposition 4.12 give a more precise description than the statement that there is a cobordism with $|\chi| = b_1(L) - b_1(K)$ or the existence of a “positively self-intersecting concordance” in the sense of [BL13]. However, our main interest in these two propositions comes from the suspicion that the addition of the first $m_L - m_K$ conjugates of generators can be done such that $K = \tilde{K}$, as it is the case in Example 4.10. In particular, this would yield that δ -constant adjacency implies Gordian adjacency.

The hope is that, instead of adding conjugates of (squares) of generators in Proposition 4.9 (Proposition 4.12), one finds positive braid words β_K and β_L such that β_L can be obtained from β_K by adding (squares) of generators. This would imply that if $K \leq_a L$, then the minimal Seifert surface of K can be obtained from the minimal Seifert surface F_{β_L} of L by removing 1-handles corresponding to generators. In particular, $K \leq_a L$ would imply that K is subsurface adjacent to L , compare Chapter 5. Also, a δ -constant adjacency $K \leq_\delta L$ would imply that K is obtained from L by $u(L) - u(K)$ crossing changes that are seen on F_{β_L} as deplumbings of the minimal Seifert surface for the trefoil knot.

The main motivation for the above propositions are of a speculative nature and their proofs are rather long. The next section is devoted to these proofs.

3. Proof of Propositions 4.9 and 4.12

We now prepare the notions needed for the proofs of Proposition 4.9 and Proposition 4.12. The main tool is Rudolph's description of the intersection of the zero-set $V(f)$ of a polynomial $f: \mathbb{C}^2 \rightarrow \mathbb{C}$ with $\partial Z = \gamma \times \mathbb{C}$ as a closed (quasi-positive) braid, where γ is a simple closed curve in \mathbb{C} . The beautiful original source is [Rud83].

Let $f: D_\varepsilon \times \mathbb{C} \rightarrow \mathbb{C}$ be a square-free holomorphic map of the form

$$f = y^m + c_{m-1}(x)y^{m-1} + \cdots + c_0(x),$$

where $c_k(x)$ are holomorphic on a neighborhood of the disc $D_\varepsilon \subset \mathbb{C}$ of radius ε .

We study the zero-set of f in the cylinder $Z_\varepsilon = D_\varepsilon \times \mathbb{C}$. For a given x in D_ε the equation $f = 0$ (as a polynomial equation in y) has m solutions when counted with multiplicity. Following Rudolph we study certain subsets of \mathring{D}_ε . Note that Rudolph named these sets B and B^+ . We change the notation from “ B ” to “ S ” since 4-balls in \mathbb{C}^2 are already denoted by B . The subsets are:

- The finite set S of all x such that some of the m solutions y_1, \dots, y_m of $f(x, y) = 0$ coincide.
- The semi real-analytic set S^+ of all x such that the m solutions of $f(x, y) = 0$ do not have m distinct real parts.
- S_{gen} the semi real-analytic open subset of S^+ given by those x that are not in S and have precisely $m - 1$ different real parts among the real parts of y_1, \dots, y_m .

By definition we have $S_{gen} \subseteq S^+$ and $S \subseteq S^+ \setminus S_{gen}$. The set S is the image of the projection onto the x -coordinate of the finite set C of all points in Z_ε where $V(f)$ is singular or has a vertical tangent. By a small affine change of coordinates we can make sure that the following conditions are satisfied, compare also [Ore96, Appendix].

- (I) No two points of C lie over the same point of S .
- (II) For a point (x_0, y_0) in C that is a singularity of $V(f)$ with multiplicity m_0 , we have locally

$$f = u(x, y) \left((y - y_0)^{m_0} + c_{m_0-1}(x)(y - y_0)^{m_0-1} + \cdots + c_0(x) \right)$$

with $u(x_0, y_0) \neq 0$ and $c_i(x_0) = 0$, compare Lemma 1.2.

- (III) At a smooth point $z_0 = (x_0, y_0)$ in $V(f)$ that is in C , i.e. that has a vertical tangent, the vertical tangent is simple, i.e. we have

locally $f = u(x, y) (c(x, y)(y - y_0)^2 - (x - x_0))$ with $u(x_0, y_0) \neq 0 \neq c(x_0, y_0)$.

- (IV) The set $S^+ \setminus S_{\text{gen}}$ is finite.
- (V) In a small neighborhood of a point x in S , no two different local solutions y_i, y_j have the same real part, except when they both tend to the point $z \in C$ above x , i.e. $y_i(x) = y_j(x) = z$.

In fact, changing f only slightly one could even achieve that there are no singularities on $V(f)$, rendering condition (II) obsolete. This is what Rudolph does. We do not do this since we are actually interested in what happens around a singularity.

As examples we consider the polynomials $f = y^2 + x$, $g = y^2 - x^2$ and $h = y^3 - x^4$ on $D \times \mathbb{C}$, where D is the unit disc centered at $0 \in \mathbb{C}$. Their sets S^+ are depicted in Figure 1.

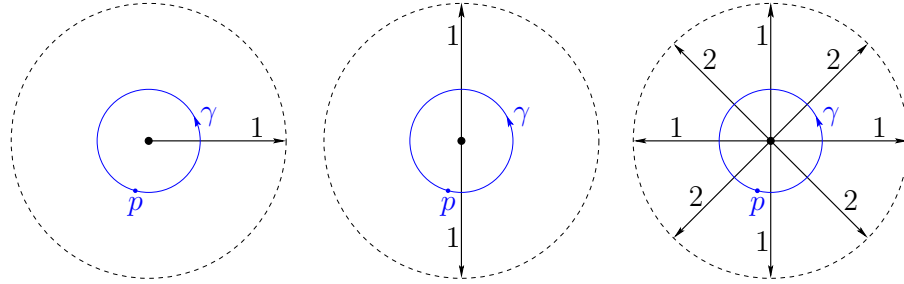


FIGURE 1. The set S^+ (black) for $y^2 + x$ (left), $y^2 - x^2$ (middle) and $y^3 - x^4$ (right). For all three polynomials the subset S consists of a single point.

For every closed oriented curve γ in D_ϵ , the intersection $V(f) \cap (\gamma \times \mathbb{C})$ is a closed m -braid in $S^1 \times D$ via the identification $\gamma \times \mathbb{C} \cong S^1 \times \overset{\circ}{D}$. Similarly, for every oriented arc α in $D_\epsilon \setminus S$ with endpoints in $D_\epsilon \setminus S^+$ (which guarantees that at endpoints the m -solutions have different real parts), the intersection $V(f) \cap (\alpha \times \mathbb{C}) \subset \alpha \times \mathbb{C}$ is a m -braid by identifying $\alpha \times \mathbb{C}$ with $[0, 1] \times \overset{\circ}{D}$. Note that for this to be well-defined, the identification should preserve the order of the real parts in the second factor. An endpoint-fixing homotopy of two arcs and a homotopy of two closed curves in $D_\epsilon \setminus S$ correspond to an isotopy of braids and an isotopy of closed braids, respectively.

Rudolph shows that S^+ is a graph and that its edges can be oriented and labeled meaningfully. The set of vertices is $S^+ \setminus S_{\text{gen}}$ and the components of S_{gen} are the edges. Note that this allows edges that do not end in a vertex if the edge tends to ∂D_ϵ . For x in S_{gen} we label

the unique edge containing x by $1, \dots, m-1$ according to which of the $m-1$ real parts of the $y_i(x)$ (ordered as real numbers) has multiplicity 2. These labels have the following interpretation. If a short oriented arc α intersects an edge with label i once transversely, then the corresponding m -braid $V(f) \cap (\alpha \times \mathbb{C})$ is a_i or a_i^{-1} . An edge labeled i is oriented such that the edge orientation followed by the orientation of an oriented arc α intersecting it transversely provides the standard orientation of \mathbb{C} if and only if the corresponding braid is a_i rather than a_i^{-1} .

For a curve γ in $D_\varepsilon \setminus S$ that hits S^+ only in S_{gen} and only transversely, one obtains a braid word for a braid that closes to $V(f) \cap (\gamma \times \mathbb{C})$ by following γ and reading off the labels of the edges of S^+ crossed by γ . More precisely, if γ crosses an edge labeled i such that the orientation of the edge followed by the orientation of γ gives the standard orientation of \mathbb{C} , we write a_i in the braid word, otherwise we write a_i^{-1} .

We describe how to read off braid words by following curves in $D_\varepsilon = D$ for the three examples f, g and h from above.

EXAMPLE 4.14. For the polynomials $f = y^2 + x$, $g = y^2 - x^2$, and $h = y^3 - x^4$ on $D \times \mathbb{C}$ we consider the curve γ that goes counter-clockwise around the origin starting and ending at the point p (defined in Figure 1). We read off the 2-braids $\text{H} = a_1$, $\text{H} = a_1^2$ and the

3-braid $\begin{array}{c} \text{H} \\ \text{H} \\ \text{H} \end{array} = a_1 a_2 a_1 a_2 a_1 a_2 a_1 a_2$, respectively. The closures of these braids are the unknot, $T(2, 2)$ and $T(3, 4)$, respectively, which are the corresponding links of the singularities f, g and h .

REMARK 4.15. We now discuss how S^+ looks in a small neighborhood of points x_0 in S . If x_0 corresponds to a z_0 in C that is a smooth point of $V(f)$, then S^+ consists of a single edge pointing out of x_0 as for the standard example $y^2 + x$, compare Figure 1 (left). This follows from condition (III). If x_0 corresponds to a z_0 in C that is a singular point of $V(f)$, then S^+ consists of several edges pointing out of x_0 , compare for example Figure 1 (middle and right). Note that this is not discussed by Rudolph as he assumes that $V(f)$ is smooth. We indicate how to establish this.

Let $z_0 = (x_0, y_0)$ be a singular point in $V(f)$ with multiplicity $n > 1$. We assume z_0 is the origin and $f(z_0) = 0$, in particular $x_0 = 0$, and that the germ in $\mathbb{C}\{x, y\}$ defined by f is irreducible. Locally around $(0, 0)$ we can parametrize $V(f)$ by $\phi: D_{\frac{1}{\varepsilon^n}} \rightarrow \mathbb{C}^2, (s \mapsto s^n, y(s))$ for some holomorphic $y: D_{\frac{1}{\varepsilon^n}} \subset \mathbb{C} \rightarrow \mathbb{C}$ with $y(0) = 0$; see Lemma 1.2. This means that for $x \neq 0$ in D_ε the n solutions y_i of $f = 0$ are locally

given by the Puiseux expansion $y(x^{\frac{1}{n}})$. In particular, $y_j - y_i$ can be understood as a holomorphic function $w(x^{\frac{1}{n}}) = y(\xi x^{\frac{1}{n}}) - y(x^{\frac{1}{n}})$ in x with $w(0) = 0$, where ξ is a n th-root of unity and $x^{\frac{1}{n}}$ is a local n -root of x . The set S^+ is given by those x for which $w(x^{\frac{1}{n}})$ is purely imaginary. Letting x move along an arc α that goes counter clockwise around 0 on a very small circle, will make $w(x^{\frac{1}{n}})$ turn counter-clockwise around 0 as well. This follows by expanding $w(s) = a_k s^k + a_{k+1} s^{k+1} + \dots = a_k s^k + o(s^{k+1})$. Therefore, close to the origin, S^+ is a union of lines going out of the origin. In fact, this means that any two solutions $y_j(x)$ and $y_i(x)$ of any singularity spiral counter-clockwise around each other, when following a counter-clockwise arc α . And so all lines pointing out of the origin are (as edges of the graph) oriented in the same direction; in fact, they are all oriented outward. This means that (after shrinking D_ε) S^+ consist of one vertex x_0 with edges pointing out of it.

This type of argument (local parametrization by Puiseux expansions) can be used to fully understand what types of links can arise as links of singularities, see Brieskorn and Knörrer for a detailed account of this [BK86].

REMARK 4.16. Generalizing the examples given in Figure 1, we describe the situation for Milnor balls of singularities. Let f in $\mathbb{C}\{x, y\}$ be a singularity. Let L be the link of singularity, m the multiplicity. Here we are (a priori) not in Rudolph's setting.

After a small linear change of coordinates we have $f = u(x, y)(y^m + c_{m-1}(x)y^{m-1} + \dots + c_0(x))$ on a Milnor ball $B_{\varepsilon'}$ by the Weierstrass preparation Theorem, where $u(x, y)$ is a nowhere vanishing holomorphic function on $B_{\varepsilon'}$, $c_k(x)$ are defined on a disc in \mathbb{C} of radius bigger than ε' , and $c_k(0) = 0$; see Lemma 1.2. As we wish to study the zero-set $V(f)$ on the Milnor ball we may assume that $u \equiv 1$, i.e.

$$f = y^m + c_{m-1}(x)y^{m-1} + \dots + c_0(x).$$

For small enough $\varepsilon < \varepsilon'$, we have that if x is in D_ε , then the m (counted with multiplicity) solutions y_1, \dots, y_m of $f = 0$ are in $B_{\varepsilon'}$; see Figure 2. So, studying $V(f)$ intersected with $\widetilde{Z}_\varepsilon = (D_\varepsilon \times \mathbb{C}) \cap B_{\varepsilon'}$ is the same as intersected with $Z_\varepsilon = D_\varepsilon \times \mathbb{C}$ and, therefore, we are in Rudolph's setting and can study S^+ for f .

After possible shrinking ε , $S^+ \subset D_\varepsilon$ consists of one vertex with edges pointing outwards by Remark 4.15. In particular, the m -braid β_L obtained following ∂D_ε counter-clockwise around 0 is given by a positive braid word. The closure of β_L is L . This is true since the bounded cylinder $\widetilde{Z}_\varepsilon$ is a Milnor ball, i.e. the pair $(\widetilde{Z}_\varepsilon, \widetilde{Z}_\varepsilon \cap V(f))$ is homeomorphic to $(B_\varepsilon, B_\varepsilon \cap V(f))$; see e.g. [BK86].

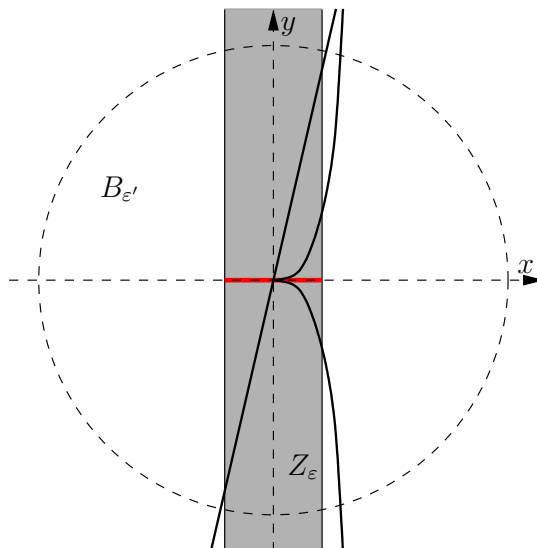


FIGURE 2. In the cylinder $Z_\varepsilon = D_\varepsilon \times \mathbb{C}$ over the disk D_ε (red) the zero-set $V(f)$ (black) is contained in $B_{\varepsilon'}$.

PROOF OF PROPOSITION 4.9. Let f_t be a deformation realizing the algebraic adjacency $K \leq_a L$ of links K and L with multiplicity m_K and m_L , respectively. We choose a Milnor ball $B_{\varepsilon'}$ and a disk D_ε for f_0 such that after a small change of coordinates $f_0 = u_0(x, y)(y^m + c_{0,m-1}(x)y^{m-1} + \cdots + c_0(x))$ as in Remark 4.16. For small enough $t > 0$ we have $f_t = u_t(x, y)(y^m + c_{t,m-1}(x)y^{m-1} + \cdots + c_{t,0}(x))$, where some $c_{t,k}(0)$ might be non-zero. Fix such a small $t > 0$. After a small coordinate change to guarantee conditions (I) to (V) if necessary, we study $S^+ \subset D_\varepsilon$ for $f_t = y^m + c_{t,m-1}(x) + \cdots + c_{t,0}(x)$. Close to the boundary of D_ε the set S^+ looks the same for f_0 and f_t . Figure 3 depicts S^+ qualitatively for $f_0 = y^3 - x^4$ and $f_t = y^3 - (x^2 - ty)^2$. Let $B_{\eta'}$ be a Milnor ball for the singularity at 0 of f_t . Note that by imposing conditions (I) to (V), we guaranteed that for a small enough disc D_η around 0 the situation in $B_{\eta'}$ is as in Remark 4.16.

We denote by γ_K the curve given by counter-clockwise parametrization of ∂D_η . Similarly, γ_L denotes the curve given by counter-clockwise parametrization of ∂D_ε . Additionally, we fix starting points p_K and p_L on the curves γ_K and γ_L . Figure 4 illustrates this for $f_t = y^3 - (x^2 - ty)^2$. This allows us to read off positive m_L -braid words β_K and β_L , which have as corresponding closed braids $V(f_t) \cap (\gamma_K \times \mathbb{C})$ and $V(f_t) \cap (\gamma_L \times \mathbb{C})$, respectively. Since close to γ_L the graph S^+ looks the same for f_t and f_0 , the braid β_L is the same for both f_0 and f_t . Its closure is L .

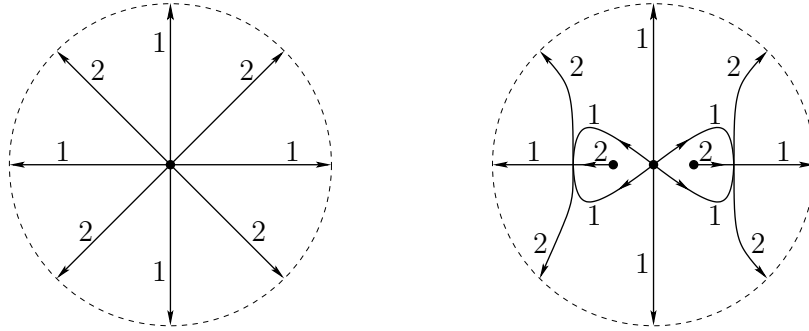


FIGURE 3. The set S^+ for $y^3 - x^4$ (left) and $y^3 - (x^2 - ty)^2$ (right). For $y^3 - (x^2 - ty)^2$ the set S consists of 3 points and there are two vertices in S^+ that are not in S or S_{gen} .

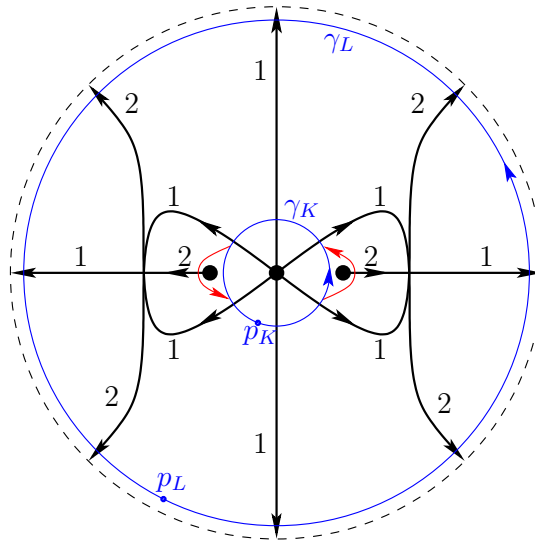


FIGURE 4. The set S^+ (black) and the curves γ_K and γ_L (blue) for $y^3 - (x^2 - ty)^2$. The curve γ that is obtained as a modification (red) of γ_K is homotopic to γ_L in $D_\epsilon \setminus S$.

Similarly, $V(f_t) \cap (\gamma_K \times \mathbb{C})$ intersected with $B_{\eta'}$ is an m_K -braid $\widetilde{\beta}_K$ with closure K . However, in the intersection of $V(f_t) \cap (\gamma_K \times \mathbb{C})$ with B_ϵ , there are $m_L - m_K$ additional local solutions y_i yielding the m_L -braid β_K . By condition (V), these extra $m_L - m_K$ local solutions have pairwise different real parts and their real part is bounded away from the real part of the m_K solutions that are in $B_{\eta'}$. Therefore, the m_L -braid β_K is given by the m_K -braid $\widetilde{\beta}_K$ with $m_L - m_K$ additional braid strands

that never cross another strand. In particular, the closure of β_K is a split union of K and $m_L - m_K$ unknots.

We choose disjoint paths from ∂D_η to the points of $S \setminus \{0\}$ and change γ_K as follows yielding a closed curve γ . The curve γ is equal to γ_K except that it makes detours along the paths to the points of $S \setminus \{0\}$ as indicated in Figure 5. The braid word β that we read off

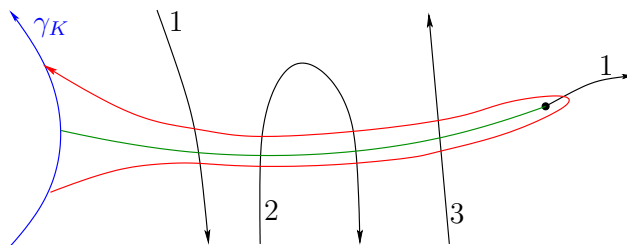


FIGURE 5. The curve γ_K (blue) is modified (red) to follow paths (green) to the points in $S \setminus \{0\}$. In this example β_K is changed by inserting a conjugate of a_1 ; namely by inserting $(a_1 a_2^{-1} a_2 a_3^{-1}) a_1 (a_3 a_2^{-1} a_2 a_1^{-1})$.

following γ starting at p_K is β_K with a conjugate of a positive braid word inserted for every path-detour γ makes.

The curve γ is homotopic to γ_L in the complement of S since it incloses all of S , see Figure 4. Therefore, the closed braids obtained as closures of β and β_L are the same and so β and β_L are conjugate. Adding a conjugate of a generator changes the length of a braid by one. Of course, a conjugate of a positive braid word $\omega a_{i_1} \cdots a_{i_l} \omega^{-1}$ can be written as a product of conjugates of generators $\omega a_{i_1} \omega^{-1} \cdots \omega a_{i_l} \omega^{-1}$. Thus, β is obtained from β_K by adding

$$l(\beta) - l(\beta_K) = l(\beta_L) - l(\beta_K) = b_1(L) + m_L - 1 - (b_1(K) + m_K - 1)$$

conjugates of generators since

$$l(\beta_L) = b_1(L) + m_L - 1 \text{ and } l(\beta_K) = l(\widetilde{\beta_K}) = b_1(K) + m_K - 1, \text{ by (2).}$$

□

For the proof of Proposition 4.12 we need the following.

REMARK 4.17. Fix a Milnor ball $B_{\varepsilon'}$ for a singularity of some f . By Remark 1.4 one can change f slightly to another holomorphic function g such that f and g stay close in a neighborhood of $\partial B_{\varepsilon'}$ and such that g has $\delta(f)$ ordinary double points. Double point singularities are $y^2 - x^2$ up to topological type, i.e. have $T(2, 2)$ as link of singularity. In particular, they have $\delta = 1$ and multiplicity 2. In terms of Rudolph's

S^+ -graph this means the following (a small coordinate change to guarantee conditions (I) to (V) is done if necessary). Close to the boundary of D_ε the set S^+ for g is the same as for f (up to a small isotopy). In neighborhoods of points x in S that correspond to singularities of the zero-set $V(g)$, the graph consists of one vertex x with valency 2 with two edges with the same label pointing away from x since all singularities are double points. The other points of S have valency 1 since they correspond to smooth points of $V(g)$ with a vertical tangent. For $f = y^3 - x^4$, which has $\delta = 3$, Figure 6 indicates qualitatively how S^+ might look like for g .

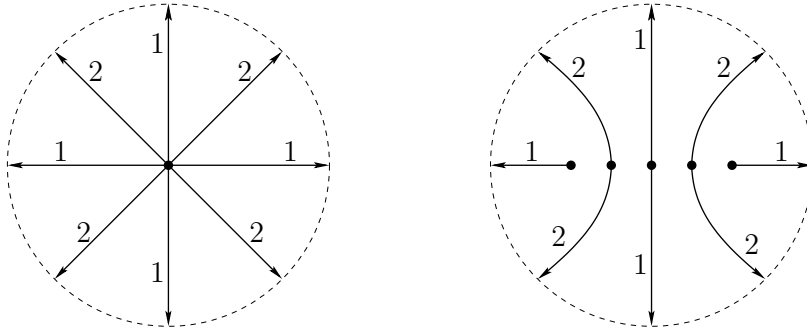


FIGURE 6. The set S^+ for the singularity $y^3 - x^4$ (left) changes to have 3 points in S that correspond to an ordinary double point and 2 points that correspond to non-singular points on $V(g)$ that have a vertical tangent (right).

PROOF OF PROPOSITION 4.12. We use the notions of the proof for Proposition 4.9. Now K and L are knots. This means the δ -invariants of the singularities are half their Milnor number, i.e. $\delta_K = \frac{b_1(K)}{2}$ and $\delta_L = \frac{b_1(L)}{2}$. Furthermore, we assume that the deformation f_t is δ -constant, i.e. the sum of the δ of the singularities of $V(f_t) \setminus \{0\}$ is $\delta_L - \delta_K$. Choose Milnor balls B_1, \dots, B_k around every singularity of $V(f_t) \setminus \{0\}$ and small discs D_1, \dots, D_k in D_ε around the corresponding points in S . We use Remark 4.17 to change the graph S^+ . Inside a small disc D_j we replace S^+ with what we obtain by changing f_t to a g_j that has only double points as described in Remark 4.17. This new S^+ has the following properties. All points of $S \setminus \{0\}$ that correspond to singularities are ordinary double points. The number of such points is $\delta_L - \delta_K$ since f_t is δ -constant. The set S_s of points in S corresponding to smooth points that have a vertical tangent has precisely $m_L - m_K$

elements, which is seen as follows. Points in S_s have one edge pointing out of them, i.e. they contribute one conjugate of a single generator to the m_L -braid β obtained by following γ . Every double point contributes one conjugate of the square of a generator. This means that the length of β is $l(\beta_K) + \#S_s + 2(\delta_L - \delta_K)$. As β is conjugate to β_L , they have the same length. Using $l(\beta_L) = b_1(L) + m_L - 1$ and $l(\beta_K) = b_1(K) + m_K - 1$, see (2), we conclude that $\#S_s = m_L - m_K$.

Adding a conjugate of the square of a generator in a braid word does not change the number of components of the closure. Therefore, changing the m_L -braid word β_K by adding all conjugates of single generators corresponding to points in S_s yields a braid word $\widetilde{\beta}_K$, which closes to a knot. This follows since adding conjugates of squares to $\widetilde{\beta}_K$ gives β , which closes to the knot L . As the closure of β_K has $1 + m_L - m_K$ components, all of the $m_L - m_K$ conjugates of single generators that are added to β_K to yield $\widetilde{\beta}_K$ have to connect different components of the closure of β_K . \square

CHAPTER 5

Subsurfaces of Seifert surfaces

In this chapter we discuss an adjacency notion for links given by studying H_1 -injective subsurfaces of Seifert surfaces.

DEFINITION 5.1. For links K and L , we say K is *subsurface adjacent* to L , denoted by $K \leq_s L$, if a minimal Seifert surface F_K of K is isotopic to a H_1 -injective subsurface of a minimal Seifert surface F_L of L .

The number of 1-handles that are removed from F_L to get F_K is $b_1(F_L) - b_1(F_K)$ and, therefore, Lemma 1.7 yields the following obstruction to subsurface adjacency.

LEMMA 5.2. *Let K and L be links. If there exists an adjacency $K \leq_s L$, then*

$$|\sigma_\omega(L) - \sigma_\omega(K)| \leq b_1(L) - b_1(K) = b_1(F_L) - b_1(F_K),$$

where F_L and F_K denote minimal Seifert surfaces of L and K , respectively.

Again we restrict our studies to torus links and we denote by $F(a, b)$ the unique minimal Seifert surface of $T(a, b)$. All the examples of such adjacencies will be provided by starting with a positive braid that closes to L and then removing generators—corresponding to removing 1-handles in the minimal Seifert surface—until we reach a positive braid that closes to K .

In the introduction we motivated the study of subsurface adjacency by a question about geometric realization of eigenvalues of the symmetrized Seifert form. Another motivation for this study is our suspicion that if two links K, L are algebraically adjacent, then they are also subsurface adjacent, see Remark 4.13. At least something weaker is true. Every algebraic adjacency $K \leq_a L$ yields a cobordism with $|\chi|$ equal to $b_1(L) - b_1(K)$, i.e. $d_c(K, L) = b_1(L) - b_1(K)$. It is therefore interesting to study for which pairs of algebraic links, or more specifically torus links, we have $d_c(K, L) = b_1(L) - b_1(K)$. And then to ask whether this can be realized by an algebraic adjacency. All examples

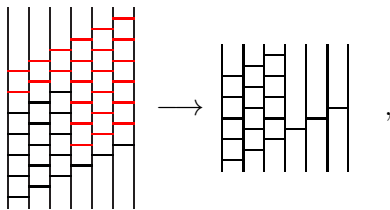
of torus links K and L with $d_c(K, L) = b_1(L) - b_1(K)$ we know of, come in fact from an adjacency $K \leq_s L$.

1. First examples

As a first example, we notice that the analog of Theorem 3.2 and Proposition 4.3 is immediate.

PROPOSITION 5.3. *Let n, m, a, b be positive integers. If $n \leq a$ and $m \leq b$, then $T(n, m) \leq_s T(a, b)$.*

This is proved by deleting generators in the positive a -braid word $(a_1 \cdots a_{a-1})^b$, which has closure $T(a, b)$; until one reaches a positive braid word with closure $T(n, m)$. We illustrate this for the adjacency $T(4, 5) \leq_s T(7, 7)$.



where the arrow indicates the removal of the generators marked in red. As in the case of Gordian adjacency Proposition 5.3 describes \leq_s for all pairs of torus knots of the same braid index and the simplest questions for different indices is, what torus links of braid index 2 are adjacent to a given torus link of braid index $a > 2$. Section 2 gives a precise answer for small a . A proposition due to Baader provides examples of subsurface adjacencies.

PROPOSITION 5.4. [**Baa12**] *Let a, b, c be natural numbers with $a \leq b$. Then $T(a, bc) \leq_s T(b, ac)$ holds.*

This proposition led to the suspicion that a similar statement could hold for algebraic adjacency, which led to Proposition 4.5.

2. Subsurface adjacencies for torus links of braid index 3 and 4

We provide examples of $T(2, n)$ that are subsurface adjacent to $T(3, m)$ and $T(4, m)$, respectively, and that do not follow from Proposition 5.3 or Proposition 5.4. Concretely, we show that $T(2, n) \leq_s T(3, m)$ and $T(2, n) \leq_s T(4, m)$ if roughly $n \leq \frac{5}{3}m$ and $n \leq \frac{5}{2}m$, respectively. In fact, the factors $\frac{5}{3}$ and $\frac{5}{2}$ are optimal, which is shown using the adjacency obstruction given in Lemma 5.2.

PROPOSITION 5.5. *Let n and m be positive integers. If $n \leq \frac{5m-1}{3}$, then the surface $F(2, n)$ is an H_1 -injective subsurface of $F(3, m)$.*

PROPOSITION 5.6. *Let n and m be positive integers. If $n \leq \frac{5m-3}{2}$, then $F(2, n)$ is an H_1 -injective subsurface of $F(4, m)$.*

The two propositions above are optimal, at least when $T(3, m)$ and $T(4, m)$, respectively, are knots.

THEOREM 5.7. *Let n, m be positive integers. If m is not a multiple of 3, then $T(2, n)$ is subsurface adjacent to $T(3, m)$ if and only if $n \leq \frac{5m-1}{3}$. Similarly, if m is odd, then $T(2, n)$ is subsurface adjacent to $T(4, m)$ if and only if $n \leq \frac{5m-3}{2}$.*

PROOF OF PROPOSITION 5.5. We denote the 3-strand full twist $(a_1a_2a_1)^2$ by Δ^2 . The full twist commutes with every other 3-braid.

Let us first consider the case where $m = 3l$ for some positive integer l . The torus link $T(3, 3l)$ is the closure of Δ^{2l} . Note that

$$\Delta^2\Delta^2 = a_1a_2a_1a_1a_2a_1\Delta^2 = a_1a_2a_1a_1a_2\Delta^2a_1.$$

Adding another full twist yields

$$\Delta^2\Delta^2\Delta^2 = a_1a_2a_1a_1a_2\Delta^2\Delta^2a_1 = a_1a_2a_1a_1a_2(a_1a_2a_1a_1a_2\Delta^2a_1)a_1$$

and inductively we get $\Delta^{2l} = (a_1a_2a_1a_1a_2)^l(a_1)^l$. The subword $a_2a_1a_2$ occurs $l-1$ times in $(a_1a_2a_1a_1a_2)^l(a_1)^l$. Applying $l-1$ time the braid relation $a_2a_1a_2 = a_1a_2a_1$ produces the braid word

$$w = a_1a_2a_1a_1(a_1a_2a_1a_1a_1)^{l-1}a_2(a_1)^l.$$

Deleting all but the first a_2 in w yields $a_1a_2a_1^{5l-2}$, which has $T(2, 5l-1)$ as closure. Since deleting generators in a positive braid word corresponds to removing 1-handles in its minimal Seifert surface, we conclude that $F(2, 5l-1)$ is an H_1 -injective subsurface of $F(3, 3l)$.

For the case $m \equiv 1 \pmod{3}$ we write $T(3, 3l+1)$ as the closure of

$$\begin{aligned} a_1a_2\Delta^{2l} &= a_1a_2w = a_1a_2(a_1a_2a_1a_1(a_1a_2a_1a_1a_1)^{l-1}a_2(a_1)^l) \\ &= a_1a_1a_2a_1a_1(a_1a_2a_1a_1a_1)^{l-1}a_2(a_1)^l \end{aligned}$$

Deleting all but the first a_2 yields $a_1a_1a_2(a_1)^{5l-1}$, which has closure $T(2, 5l+1)$.

Finally, for $m \equiv 2 \pmod{3}$, we write the closure of $T(3, 3l+2)$ as

$$\begin{aligned} a_1a_2a_1a_2\Delta^{2l} &= a_1a_1a_2a_1\Delta^{2l} = a_1(a_1a_2\Delta^{2l})a_1 \\ &= a_1(a_1a_1a_2a_1a_1a_1(a_1a_2a_1a_1a_1)^{l-1}a_2(a_1)^l)a_1. \end{aligned}$$

Deleting all but the first a_2 yields $a_1a_1a_1a_2(a_1)^{5l}$, which has $T(2, 5l+3)$ as closure. \square

PROOF OF PROPOSITION 5.6. We view $T(4, 2l + 1)$ as the closure of the 4-braid $(a_1 a_3 a_2)^{2l+1}$. Using the fact that the half twist on 4 strands

$$\Delta = a_1 a_3 a_2 a_1 a_3 a_2 = a_1 a_2 a_1 a_3 a_2 a_1$$

anti-commutes with every other 4-braid, i.e. $a_1 \Delta = \Delta a_3$, $a_3 \Delta = \Delta a_1$, and $a_1 \Delta = \Delta a_3$, we have that

$$\begin{aligned} \Delta^k &= a_1 a_2 a_1 a_3 a_2 a_1 \Delta^{k-1} = a_1 a_2 a_1 a_3 a_2 \Delta^{k-1} a_{2+(-1)^{k-1}} \\ &= a_1 a_2 a_1 a_3 a_2 \left(a_1 a_2 a_1 a_3 a_2 \Delta^{k-2} a_{2+(-1)^{k-2}} \right) a_{2+(-1)^{k-1}} \\ &= (a_1 a_2 a_1 a_3 a_2)^2 \Delta^{k-2} a_1 a_3 = \cdots = (a_1 a_2 a_1 a_3 a_2)^k a_1^{\lfloor \frac{k+1}{2} \rfloor} a_3^{\lfloor \frac{k}{2} \rfloor}. \end{aligned}$$

With this we can write $(a_1 a_3 a_2)^{2l+1}$ as follows.

$$\begin{aligned} (a_1 a_3 a_2)^{2l+1} &= (a_1 a_3 a_2)^2 a_1 a_3 a_2 (a_1 a_3 a_2)^{2(l-1)} \\ &= (a_1 a_3 a_2)^2 a_1 a_3 a_2 (a_1 a_2 a_1 a_3 a_2)^{l-1} a_1^{\lfloor \frac{l}{2} \rfloor} a_3^{\lfloor \frac{l-1}{2} \rfloor} \\ &= (a_1 a_3 a_2)^2 a_1 a_3 (a_2 a_1 a_2 a_1 a_3)^{l-1} a_2 a_1^{\lfloor \frac{l}{2} \rfloor} a_3^{\lfloor \frac{l-1}{2} \rfloor} \\ &= (a_1 a_3 a_2)^2 a_1 a_3 (a_1 a_2 a_1 a_1 a_3)^{l-1} a_2 a_1^{\lfloor \frac{l}{2} \rfloor} a_3^{\lfloor \frac{l-1}{2} \rfloor}. \end{aligned}$$

Deleting the last l occurrences of a_2 in this braid word gives

$$(a_1 a_3 a_2)^2 a_1 a_3 (a_1 a_1 a_1 a_3)^{l-1} a_1^{\lfloor \frac{l}{2} \rfloor} a_3^{\lfloor \frac{l-1}{2} \rfloor} = (a_1 a_3 a_2)^2 a_1^{3l-2+\lfloor \frac{l}{2} \rfloor} a_3^{l+\lfloor \frac{l-1}{2} \rfloor},$$

which has closure

$$T \left(2, 4 + (3l - 2 + \lfloor \frac{l}{2} \rfloor) + (l + \lfloor \frac{l-1}{2} \rfloor) \right) = T(2, 5l + 1).$$

Since deleting generators in a positive braid word corresponds to removing 1-handles in its minimal Seifert surface, we conclude that $F(2, 5l+1)$ is an H_1 -injective subsurface of $F(4, 2l + 1)$.

Similarly, the torus link $T(4, 2l + 2)$ is the closure of

$$(a_1 a_3 a_2)^{2l+2} = (a_1 a_3 a_2)^2 a_1 a_3 a_2 a_1 a_3 (a_1 a_2 a_1 a_1 a_3)^{l-1} a_2 a_1^{\lfloor \frac{l}{2} \rfloor} a_3^{\lfloor \frac{l-1}{2} \rfloor}.$$

Deleting the last $l + 1$ occurrences of a_2 yields a braid word that has closure $T(2, 5l + 3)$. \square

PROOF OF THEOREM 5.7. By Proposition 5.5 and Proposition 5.6 it remains to show that $F(2, 5l+2)$ is neither an H_1 -injective subsurface of $F(3, 3l + 1)$ nor of $F(4, 2l + 1)$, and that $F(2, 5l + 4)$ is not an H_1 -injective subsurface of $F(3, 3l + 2)$.

First, we consider the case that $l = 2k$ is even. By Murasugi's calculation of the signature for torus knots of braid index 3 and 4, see [Mur74, Proposition 9.1 and Proposition 9.2], we have

$$\begin{aligned}\sigma(T(3, 3l + 1)) &= 8k = 4l, \\ \sigma(T(3, 3l + 2)) &= 8k + 2 = 4l + 2, \\ \text{and } \sigma(T(4, 2l + 1)) &= 8k = 4l;\end{aligned}$$

compare also Lemma 1.10. Therefore we get the following.

$$\begin{aligned}\sigma(T(2, 5l + 2)) - \sigma(T(3, 3l + 1)) &= b_1(T(2, 5l + 2)) - \sigma(T(3, 3l + 1)) \\ &= 5l + 1 - 4l > l - 1 \\ &= b_1(T(3, 3l + 1)) - b_1(T(2, 5l + 2)), \\ \sigma(T(2, 5l + 4)) - \sigma(T(3, 3l + 2)) &= b_1(T(2, 5l + 4)) - \sigma(T(3, 3l + 2)) \\ &= 5l + 3 - (5l - 2) > l - 1 \\ &= b_1(T(3, 3l + 2)) - b_1(T(2, 5l + 4)), \\ \sigma(T(2, 5l + 2)) - \sigma(T(4, 2l + 1)) &= b_1(T(2, 5l + 2)) - \sigma(T(4, 2l + 1)) \\ &= 5l + 1 - 4l > l - 1 \\ &= b_1(T(4, 2l + 1)) - b_1(T(2, 5l + 2)).\end{aligned}$$

Thus, $F(2, 10k + 2) = F(2, 5l + 2)$ is not an H_1 -injective subsurface of $F(3, 6k + 1) = F(3, 3l + 1)$ or $F(4, 4k + 1) = F(4, 2l + 1)$ and $F(2, 10k + 4) = F(2, 5l + 4)$ is not an H_1 -injective subsurface of $F(3, 6k + 2) = F(3, 3l + 2)$ by Lemma 5.2.

Now for the case that $l = 2k + 1$ is odd. Murasugi's calculation of the signature for torus knots of braid index 3 and 4, see [Mur74, Proposition 9.1 and Proposition 9.2] or Lemma 1.10, provides

$$\begin{aligned}\sigma(T(3, 3l + 1)) &= 8k - 2 = 4l - 2, \\ \sigma(T(3, 3l + 2)) &= 8k = 4l, \\ \sigma(T(4, 2l + 1)) &= 8k - 2 = 4l - 2.\end{aligned}$$

These signature calculations do not suffice as obstruction. However, they can be used to show the following. There exist ω in $S^1 \setminus \{1\}$ such that

$$(17) \quad \sigma_\omega(T(3, 3l + 1)) = 8k = 4l \text{ and } \sigma_\omega(T(2, 5l + 2)) = \sigma(T(2, 5l + 2)),$$

$$(18) \quad \sigma_\omega(T(3, 3l + 2)) = 8k + 2 = 4l + 2 \text{ and } \sigma_\omega(T(2, 5l + 4)) = \sigma(T(2, 5l + 4)),$$

and

$$(19) \quad \sigma_\omega(T(4, 2l + 1)) = 8k = 4l \text{ and } \sigma_\omega(T(2, 5l + 2)) = \sigma(T(2, 5l + 2))$$

hold, respectively. From this we can conclude as in the first case using σ_ω instead of σ . We now specify how to choose ω such that (17), (18), and (19) hold, respectively. For

$$\omega = e^{2\pi i\theta} \quad \text{with} \quad \begin{cases} \theta \in (\frac{1}{2} - \frac{2}{3m}, \frac{1}{2} - \frac{1}{3m}) & \text{for } m = 3l + 1 = 6k + 4 \\ \theta \in (\frac{1}{2} - \frac{3}{6m}, \frac{1}{2} - \frac{1}{6m}) & \text{for } m = 3l + 2 = 6k + 5 \end{cases},$$

we have

$$\sigma_\omega(T(3, 3l + 1)) - 2 = \sigma(T(3, 3l + 1))$$

$$\text{and } \sigma_\omega(T(3, 3l + 1)) - 2 = \sigma(T(3, 3l + 1)),$$

see Claim 3.14. Additionally, by choosing the above θ close to

$$\frac{1}{2} - \frac{1}{3m} \quad \text{and} \quad \frac{1}{2} - \frac{1}{6m}, \text{ respectively,}$$

we get that

$$\sigma_\omega(T(2, 5l + 2)) = \sigma(T(2, 5l + 2))$$

$$\text{and } \sigma_\omega(T(2, 5l + 4)) = \sigma(T(2, 5l + 4)),$$

respectively, by Lemma 1.9. Thus, (17) and (18) hold for this choice of ω . For

$$\omega = e^{2\pi i\theta} \quad \text{with } \theta \in (\frac{1}{2} - \frac{3}{8m}, \frac{1}{2} - \frac{1}{8m}),$$

where $m = 2l + 1 = 4k + 1$, we have $\sigma_\omega(T(4, 2l + 1)) - 2 = \sigma(T(4, 2l + 1))$. This can be proved using Lemma 1.9 in a similar way as in the proof of Claim 3.14. Choosing θ close to $\frac{1}{2} - \frac{1}{8m}$ yields $\sigma_\omega(T(2, 5l + 2)) = \sigma(T(2, 5l + 2))$. Therefore, (19) holds for this choice of ω . \square

3. Subsurface adjacency for the torus link $T(m, m)$

In this section, we study which $T(2, n)$ is subsurface adjacent to $T(m, m)$. The result is roughly that when ever $n \leq \frac{2m^2}{3}$, then $T(2, n) \leq_s T(m, m)$. Using Proposition 5.4 one gets $T(2, n) \leq_s T(m, m)$ only for $n \leq \frac{m^2}{2}$.

PROPOSITION 5.8. *Let m and n be positive integers. If $n \leq \frac{2m^2+4}{3} - m$, then $T(2, n)$ is subsurface adjacent to $T(m, m)$.*

We do not know whether the factor $\frac{2}{3}$ is optimal. If it is, the signature obstructions are not strong enough to show this since Lemma 5.2 only yields $n \leq \frac{3}{4}m^2$, compare Remark 4.6. Our interest in Proposition 5.8 is that it is better than what is known to exist in the algebraic adjacency setting.

PROOF. We denote by Δ_m the half twist on m strands

$$(a_1 a_2 \cdots a_{m-1})(a_1 a_2 \cdots a_{m-2}) \cdots (a_1 a_2) a_1.$$

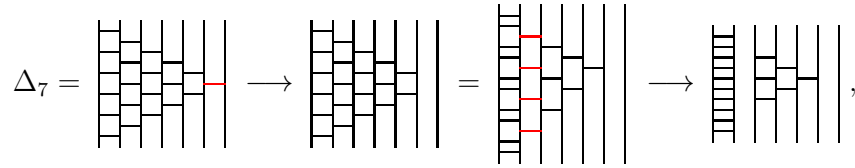
The torus link $T(m, m)$ is the closure of the full twist on m strands Δ_m^2 .

The main step in the proof consists of deleting generators in Δ_m yielding a split union of positive 2-braids for which the first Betti number is roughly $\frac{2}{3}$ of the first Betti number of Δ_m . More precisely, we delete the generator a_{m-1} in Δ_m and then apply braid relations to get the positive braid word

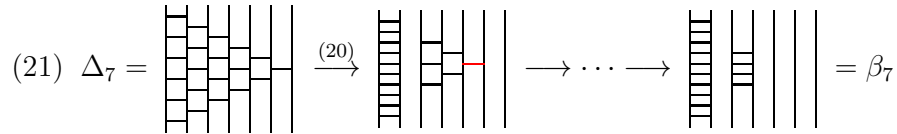
$$(a_1^2 a_2 \cdots a_{m-2}) \cdots (a_1^2 a_2) a_1^2 \text{ in } B_m.$$

Then, we delete all a_2 yielding a split union of $a_1^{2(m-2)}$ on strands 1 and 2, a half twist on the strands 3 to $m-1$, and strand m . We illustrate this for $m = 7$.

(20)



where arrows indicate the deletion of the generators marked in red. To the remaining half twist, which we readily identify with Δ_{m-3} , we apply the same procedure. And we do this inductively until the remaining half twist is Δ_3, Δ_2 , or Δ_1 , where Δ_1 is just the trivial 1-strand braid. Applying the procedure to Δ_3 just yields the split union of a_1^2 and one strand. On $\Delta_2 = a_1^2$ and Δ_1 it does not do anything. This inductive procedure yields a braid β_m , which closes to split union of $T(2, k)$ torus links. As before we illustrate this for $m = 7$, again the generators that are removed are marked in red.



The length $l(\beta_m)$ of β_m is described by the following formula, where we write m as $3l, 3l + 1$, or $3l + 2$, respectively.

$$l(\beta_m) = 2(m - 2) + 2(m - 5) + 2(m - 8) + \cdots$$

$$= \begin{cases} (3l - 1)l & \text{for } m = 3l \\ (3l + 1)l & \text{for } m = 3l + 1 \\ (3l + 3)l + 1 & \text{for } m = 3l + 2 \end{cases}.$$

We use the above to obtain a braid γ_m that closes to a $T(2, k)$ by deleting generators in Δ_m^2 , which shows that $F(2, k)$ is an H_1 -injective subsurface of $F(m, m)$. For this we write Δ_m^2 as

$$\Delta_m \Delta_m = (a_1 a_2 \cdots a_{m-1})(a_1 a_2 \cdots a_{m-2}) \widetilde{\Delta_{m-2}} \Delta_m,$$

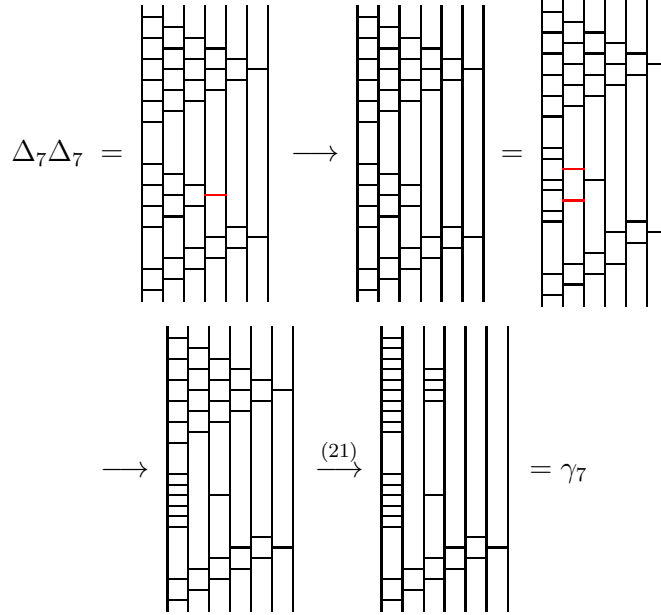
where $\widetilde{\Delta_{m-2}}$ is a half twist on the first $m - 2$ strands. Now, we apply the above deleting algorithm to $\widetilde{\Delta_{m-2}}$, which is seen as Δ_{m-2} , and Δ_m yielding

$$\gamma_m = (a_1 a_2 \cdots a_{m-1})(a_1 a_2 \cdots a_{m-2}) \widetilde{\beta_{m-2}} \beta_m,$$

where $\widetilde{\beta_{m-2}}$ is the m strand braid which is obtained by having β_{m-2} on the first $m - 2$ strands. The braid γ_m is of the form

$$\gamma_m = (a_1 a_2 \cdots a_{m-1})(a_1 a_2 \cdots a_{m-2}) a_1^{\alpha_1} a_3^{\alpha_3} \cdots a_{2k-1}^{\alpha_{2k-1}},$$

where $k = \lfloor \frac{m}{2} \rfloor$ and α_k are positive integers. As above we illustrate this for $m = 7$, again the generators that are removed are marked in red.



The closure of γ_m is a torus link $T(2, n)$. This follows from observing that the closure of $(a_1 a_2 \cdots a_{m-1})(a_1 a_2 \cdots a_{m-2})$ is $T(2, m - 1)$. Since

$$l(\gamma_m) - l((a_1 a_2 \cdots a_{m-1})(a_1 a_2 \cdots a_{m-2})) = l(\beta_m) + l(\beta_{m-2})$$

we see that $n = m - 1 + l(\beta_{m-2}) + l(\beta_m)$, i.e. the closure of γ_m is

$$T(2, n) = T(2, m - 1 + l(\beta_{m-2}) + l(\beta_m)).$$

With the above calculations for $l(\beta_m)$ we get

$$n = 3l - 1 + (3l - 2)(l - 1) + (3l - 1)l = 6l^2 - 3l + 1,$$

$$n = (3l + 1 - 1) + (3l)(l - 1) + 1 + (3l + 1)l = 6l^2 + l + 1,$$

$$n = (3l + 2 - 1) + (3l - 1)l + (3l + 3)l + 1 = 6l^2 + 5l + 2,$$

for $m = 3l$, $m = 3l + 1$, and $m = 3l + 2$, respectively. This finishes the proof since n is the largest integer with $n \leq \frac{2m^2+4}{3} - m$. \square

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Bern, 2.6.2014

Peter Feller

Lebenslauf

- 1986** Geboren am 28. Juli in Bern
- 1993–1999** Primarschule in Bern
- 1999–2001** Sekundarschule in Bern
- 2001–2005** Mathematisch-Naturwissenschaftliches Gymnasium
Bern-Neufeld
Schwerpunkt: Physik und Anwendungen der Mathematik
- 2005–2008** Bachelorstudium in Mathematik an der Universität Bern
Minors: Physik und Philosophie
- 2008–2010** Masterstudium in Mathematik an der Universität Bern
- 2010–2014** Doktorat an der Universität Bern