Gonality of metric graphs and Catalan-many tropical morphisms to trees

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presented by

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The Dean
Prof. Dr. Zoltán Balogh

Dedicado con amor y admiración
a mi Mamá y a José Luis

Margarita, está linda la mar,
y el viento
lleva esencia sutil de azahar;
yo siento
en el alma una alondra cantar;
tu acento.
Margarita, te voy a contar
un cuento.

Rubén Darío

Pagó el almuerzo, con la exagerada propina de siempre, reconquistó su pieza en la pensión de encima del Berna y después de la siesta, más verdadero, menos notable por haberse aliviado de la valija, se puso a recorrer Santa María, pesado, taconeando sin oírse, paseando ante la gente y puertas y vidrieras de comercios su aire de forastero incurioso. Caminó sobre los cuatro costados y las dos diagonales de la plaza como si estuviera resolviendo el problema de ir desde A hasta B, empleando todos los senderos y sin pisar sus pasos anteriores;

El Astillero
Juan Carlos Onetti

Preface

This thesis consists of two points of view to regard degree-$(g'+1)$ tropical morphisms $\Phi : (\Gamma, w) \to \Delta$ from a genus-$(2g')$ weighted metric graph (Γ, w) to a metric tree Δ, where g' is a positive integer. The first point of view, developed in Part I, is purely combinatorial and constructive. It culminates with an application to bound the gonality of (Γ, w). The second point of view, developed in Part II, incorporates category theory to construct a unified framework under which both Φ and higher dimensional analogues can be understood. These higher dimensional analogues appear in the construction of a moduli space $\mathcal{G}^{\mathrm{trop}}_{g\to 0,d}$ parametrizing the tropical morphisms Φ, and a moduli space $\mathcal{M}^{\mathrm{trop}}_{g}$ parametrizing the (Γ, w). There is a natural projection map $\Pi : \mathcal{G}^{\mathrm{trop}}_{g\to 0,d} \to \mathcal{M}^{\mathrm{trop}}_{g}$ that sends $\Phi : (\Gamma, w) \to \Delta$ to (Γ, w). The strikingly beautiful result is that when $g = 2g'$ and $d = g'+1$, the projection Π itself is an indexed branched cover, thus having the same nature as the maps Φ that are being parametrized. Moreover, fibres of Π have Catalan-many points.

Each part has its own introduction that motivates and describes the problem from its own perspective. Part I and its introduction are based on the articles [DV20] and [DV21], respectively, which are joint work with Jan Draisma. Part II contains material intended to be published as two articles. There is also a layman summary available after this preface.

A summary for very general audience

This non-technical summary discusses with informal metaphors the intuitive ideas behind the objects and the goals of this thesis.

Moduli spaces

Consider the phonebook of a city. Here are two reasons why this physical book is useful:

- Each person has exactly one entry in the phonebook (is a bijection).
- There is an order that makes searching for an entry easy (has a topology).

Consider the following two questions:

(A) Can we compare the population of two cities?

(B) Is the last name of a given person common?

Question (A) is a *global question*, concerning the totality of the object. Since the phonebook is a bijection, the population of a city is proportional to the length of the phonebook. So one just needs to visually inspect which phonebook is bigger. Question (B) is a *local question*, concerning a part of the object close to a point of interest. Assuming alphabetical order, first we locate the last name, and then check if the topological neighbourhood is big: count how many entries before, and after, of the chosen person have the same last name.

A *moduli space* is a phonebook for geometrical objects, useful to solve both global and local questions. Making a phonebook is quite a laborious process; so is constructing a moduli space. It is also very rewarding, hence the study of moduli spaces has been at the forefront of mathematics for the past 150 years.

Tropical geometry

Consider your favourite dinosaur. Our knowledge of it is indirect: we haven't seen it in a natural habitat, but instead have studied fossils. The *deformation process* that transformed the dinosaur into its skeleton lost a great deal of information (e.g. colour, body weight, whether it had feathers). Yet, enough is *preserved* to have paleontology as a field of science.

Tropical geometry is paleontology for mathematical dinosaurs called *algebraic varieties.* These are geometric objects described by polynomial equations, e.g. a circle in the plane is described by $x^2 + y^2 = 1$. The skeletons are *polyhedral complexes.* Here we can picture the shape of a quartz crystal, something with straight edges, straight surfaces, etc. See Figure 1. Tropical geometry is a new field at the intersection of algebraic geometry and combinatorics, with a great development in the last 20 years. Its name honours one of its founding fathers, Brazilian mathematician Imre Simon.

What do we gain from this? Think about how in paleontology skeletons are easier to manage than living creatures. In tropical geometry it is easier to manage polyhedral complexes than algebraic varieties, because the study is mostly combinatorial. Besides studying the skeletons, tropical geometry studies the processes that deform an algebraic variety into a polyhedral complex. The point is to establish *correspondence theorems* that tell us which information is retained, and to *develop efficient methods to compute* polyhedral complexes.

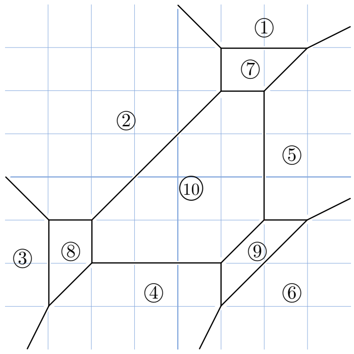


Figure 1: A tropical variety associated to the following optimization problem:

$$\max(-5+2y, -1-x+y, -5-2x, -2-y, -2+x, -7+2x-2y, -2+y, -2-x, -3+x-y, 0)$$

The maximum is attained at $-5+2y$ in region ①, the maximum is attained at $-1-x+y$ in region ②, and so on.

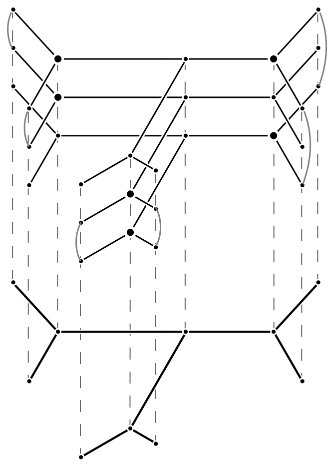


Figure 2: A degree-3 covering that folds the variety from Figure 1.

Coverings

Consider packing a suitcase. All clothes must cover the same space in the suitcase. Clearly, bigger more sophisticated clothes have to be folded more times to fit. So the number of folds for a particular dress encodes, roughly, how complicated this particular dress is. Suppose we are given a folded dress and we are challenged to determine the number of folds without unfolding it. We can take scissors to do a cross-section cut, then count the number of layers to get the answer. This gives a rough idea of how complicated this dress used to be.

A *covering* is a list of three things: the dress, the suitcase, and the specific way the dress was folded. The *degree* of a covering is the number of layers in a cross-section cut. Given a dress and a suitcase, we ask the *folding problem*: What is the *best* way to fold the dress? In quasi-mathematical terms we would say *what is the minimum degree of a covering for a specific dress and suitcase*. This is a fairly difficult problem, driving ongoing research. The reason for this is that essentially we are expressing a complicated geometrical object in terms of a simpler one, a process conceptually akin to expressing a natural number as a product of prime numbers.

Results

If both clothes and suitcase are polyhedral complexes, then we call the covering a *tropical covering.* Do not panic, we still are folding things, just that now we fold the skeletons that tropical geometry cares about. And we still care about finding the most *efficient* way of folding. Efficient means *low-degree tropical coverings.*

This thesis constructs a moduli space, a phone book, for low-degree tropical coverings. Let us name this phonebook $\mathcal{G}^{\mathrm{trop}}_{g\to 0,d}$. We have shown, in a beautiful result, that $\mathcal{G}^{\mathrm{trop}}_{g\to 0,d}$ is a polyhedral complex, and that it can be folded and used to cover another polyhedral complex $\mathcal{M}^{\mathrm{trop}}_{g}$ which is an important moduli space. Please pause a moment and appreciate the self-referential nature of the result: *a moduli space for tropical coverings is itself a tropical covering.* These *meta qualities* are common in moduli spaces and make them, in my opinion, aesthetically appealing. Another beautiful result is that we calculate the degree of the covering that folds $\mathcal{G}^{\mathrm{trop}}_{g\to 0,d}$ and get a *Catalan number.* These numbers have profound combinatorial meanings, but *this note is too narrow to contain them.*

Acknowledgements

I have no words which could fully convey my gratitude. Nor memory that sees far enough to bring everyone to mind throughout five years. Nor understanding to unravel completely the Borges-like labyrinths of who and what influenced my thought. But we are here, with great joy, to attempt nonetheless.

First, I express my admiration and a thousand thanks to my advisor Jan Draisma, for his mentorship, mathematical prowess, optimism, and personal support. Once upon a time, on a summer day, we got together for the farewell of my academic brother. That same summer day, Jan had cycled to the nearest mountain, a good twenty-something kilometres away, ran up, ran down, cycled back, and probably made it on time for lunch with his family. Years later the realization came to me of how apt this was as a metaphor for the mathematical peaks that Jan explores in his everyday life, together with his students and collaborators, before calling it a day and merrily cycling back home to his family. Thank you, for all the fun.

Second, I record a brief chronology of people and papers that were an influence, and events that were turning points. My Master's final project, advised by Jan, was a great starting point for my journey in Tropical Geometry. I enjoyed reading the papers [BN07; BN09; BS13; Bak08; Bac17] on chip-firing and tropical morphisms, and doing some computer experiments. Already then the ultimate peak was the combinatorial Brill-Noether conjecture from [Bak08]. See Conjecture 1.3 on Page 8 of this thesis for a special case. We explored several ideas, but alas, the conjecture did not yield. One idea is particularly relevant in this thesis. Chatting over a coffee break during the BeNeLux mathematical congress 2016, Jan suggested to investigate a deformation argument on tropical morphisms. This avenue would lie dormant until the beginning of my phd, where the initial aim was to solve questions left open in [CD18]. We worked out the main example that would guide our intuition for a constructive proof of the gonality bound, see Section 5.1 on Page 35. This became the first part of this thesis. Getting an idea of how deformation works is straightforward from this main example, and also what should be the multiplicity of morphisms when performing a count, but it was hard to pin down how to formalize the casework required. The inspiration came while I was attending the Stockholm Master Class in Tropical Geometry, where I learned about other directions in tropical geometry and enjoyed a nice community. Completing the casework was a long undertaking, which is to say that I became acquainted with an aspect of mathematical work previously unknown to me: putting a lot of effort and patience into details, plugging in gaps, and rewriting, rewriting, rewriting. The casework was completed in the spring of 2018, while I visited the marvellous tropical environment at the Mittag-Leffler institute during the semester on "Tropical Geometry, Amoebas and Polytopes". It was incredible and inspiring to meet many people that previously I knew from reading their work, and to learn more about the wonderful mathematics they are doing. Later, in the summer, while applying deformation arguments to the chains of loops from [CDPR12], I stumbled upon caterpillars of loops, a crucial example to establish the Catalan count. Progress slowed down towards the end of that year when I had a long hospital stay and two surgeries. I thank the hospital staff, to whom I owe my life, and the friends and people that surrounded me, to whom I owe having light and colour in the darkness. The time immediately afterwards went into recovery and other projects and interests. The original project of tropical morphisms was split in two, the first half investigating metric graph gonality with combinatorial methods, and the second constructing a moduli space with a Catalan count of points in the fibre. The first one was submitted to the conference MEGA: Effective Methods in Algebraic Geometry 2019. Presenting in Madrid was an amazing opportunity, and it appeared in a special volume of the Journal of Symbolic computation [DV20]. Regarding the second one, understanding the mysterious nature of the multiplicity remained one of the main challenges for the second part of the project. Another challenge was laying down a robust framework that takes into account automorphisms of tropical morphisms. As I approached the end of my phd the pandemy started. For the last year, almost two years of pandemy, I wish to thank the Koeniz gang Elia, Helena, Julia, and Livio, for massively cheering up and subsidizing my existence. I must also thank the nature in Switzerland, for many afternoons that I worked in front of a lake or a river. In the summer of 2020, I discovered in [Pay06] and [CCUW20] the seeds to build the missing framework for the second part. Progress was slow during this period of uncertainty, but we are finally here and I wish to thank everyone who has taken the time to read parts of this thesis. Especially, my profound gratitude to the external referee, Erwan Brugallé, for reading the first part of my thesis back when the Pandemy had started, and now later for the rest. And thank you, who reads this, for dedicating a few minutes to these anecdotical recollections, and maybe a few other minutes to the mathematics herein.

With the summary of the journey that this thesis went through, I essentially wish to thank everyone who does mathematics, who disseminates mathematics, who teaches mathematics, who writes wonderful papers, who is open to discussing mathematics, who organizes events and spaces where mathematics may be done, who appreciates mathematics and is supportive of the efforts needed to advance it. Thank you all.

Third, in an attempt of a semi-chronological order to mention names, I wish to thank Jan, Arthur, Almudena C., Sebastian, Löru, Roger, Scott, Annina, Adela, Marcelo, María, Livio L., Marius, Lukas, Valentina, Antoon, Kristina, Kleber, Triin, William, Simona, Carlos, Angela, Jonas, Kyra, Mattias, Kinga, Valeria, Silvia, Rodrigo, Karl, Jorge O., Feli, Sara, Marvin, Madeline, Teo, Borbie, Lucas, José, Plinio, Stephanie, Arturo, Hernando, Levi, Frieder, Leandro, Nafie, Julia, Damaris, Noe, Baraah, Anna, Alice, Jän, Minda, Simon S., Margarita, Almudena P., Chris, Kalina, Andrés, Felipe, Sam, Melody, Spencer, Bo, Farhad, Cristina, Francesco, Regina, Neyah, Olim, Livio F., Eliane, Kirill, Maria, Saraí, Lorenzo, Diana, Laura, Giacomo, Roser, Athénaïs, Tim, Gregory, Helena, Rahel, Elia, Nicolás, Emanuele, Alessandro, Andreas, Geraldine, Flo, Sarah, Matthias, Gari, Joseba, Maggie, Gaofeng, Ulrik, Nicolà, Blendi, Giannis, Ivo, Fernando, Niki, Kevin, Marli, Patrycja, Wiebke, César, Max, and many others.

Last, I mention and thank
Those who gave me light, thus casting away the darkness
Those who extended a hand, and accepted mine.
Those who held me close, so I could continue dreaming.
Today, my voice raises a song,
and this song is of us all.

Además quiero mencionar y agradecer,
Quienes me dieron luz, así ahuyentando la oscuridad
Quienes extendieron su mano, y sostuvieron la mía.
Quienes me abrazaron, y así continué viviendo, soñando.
Hoy alzo mi canto,
y este es el canto de todos.

Contents

Part I

Gonality of metric graphs

Cuando Bergson habla de la creación literaria dice algo muy justo. Afirma que se parte de una intuición muy oscura pero global, que luego se la va desarrollando mediante el análisis y el acercamiento, para llegar finalmente a una intuición última que es infinitamente más rica.

Sabato dialogando con Borges

Cuando éramos niños
los viejos tenían como treinta
un charco era un océano
la muerte lisa y llana
no existía.
(...)
ahora veteranos
ya le dimos alcance a la verdad
el océano es por fin el océano
pero la muerte empieza a ser la
nuestra

Mario Benedetti

Chapter 1

Introduction for Part I

The material in this *introduction for Part I* is largely based on [DV21].

1.1 What is tropical geometry

Tropical geometry is the study of combinatorial objects which arise as skeletons of algebraic varieties. Here, *skeleton* has a precise technical meaning; however, we illustrate the concept with a metaphor borrowing on biology. Animals of the same species posses many physiological features over which they display a lot of variation; yet, deep down, their skeletons (say arrangement of bones), look largely the same. From the skeleton we may recover combinatorial information; for example, number of limbs, metric properties like height, or the dimension of the ambient space that the animal inhabits. The study of skeletons may be conducted as a self contained intellectual pursuit, or with the aim of understanding the connection between the skeleton and other structures of the physiology of the animal.

Tropical geometry accomplishes, roughly speaking, the two goals in the metaphor for algebraic varieties, by embarking on the ambitious task of joining methods from many disparate branches of mathematics. A list of them would include graph theory, polyhedral geometry, polytopes, Berkovich spaces, non-archimedean geometry, moduli spaces, mirror symmetry, optimization, theory of idempotent semirings, etc. Regarding that last item, a prominent example is the semiring with additive monoid $(\mathbb{R}\cup\{\infty\}, \min)$ and multiplicative group $(\mathbb{R}\cup\{\infty\}, +)$. It was first studied by brazillian mathematician Imre Simon. Somehow, to honour him, this structure was later christened the *tropical semiring*. A current line of research seeks to establish a "commutative algebra over the tropical semiring", with the end goal of grounding abstract tropical geometry on it. This might be enough to justify the naming of this peculiar amalgamation of mathematics; plus, what one can only guess are, the possibly paradisiacal images that the word *tropical* evokes on our collective subconscious.

So on the one hand it is possible to engage in pure combinatorics, defining objects and studying problems guided by intuition and motivation mostly coming from algebraic geometry. On the other hand, sophisticated arguments employing whatever firepower is available strive to establish correspondence theorems. Maps that relate a classical object with a combinatorial object are called *tropicalizations*. Here the motivating hope is that the answer of a classical enumerative problem coincides with its corresponding tropical enumerative problem. Completing the two goals for a given problem gives, at the very least, a new proof for a classical result. With some luck, the tropical approach sheds insight into yet uncharted territories. This thesis moves entirely in the realm of the first goal, while the second one remains as an outstanding open question.

1.2 Brill-Noether theory

Recall that a Riemann surface is a connected, complex manifold of complex dimension 1. Arguably the simplest compact Riemann surface is the projective line $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ over the complex numbers, which is topologically a sphere. An arbitrary compact Riemann surface X admits a branched cover to $\mathbb{P}^1$, i.e., a holomorphic map that at every point, in local coordinates, looks like the map $z \mapsto z^e$ for some positive integer e, called the ramification index. The points where $e > 1$ are called the branch points.

For instance, if X is the quotient $\mathbb{C}/(\mathbb{Z}1+\mathbb{Z}\tau)$ of $\mathbb{C}$ by the lattice $\mathbb{Z}1+\mathbb{Z}\tau$, then the Weierstrass $\wp$-function defined as

$$\wp(z) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left(\frac{1}{(z+m+n\tau)^2} + \frac{1}{(m+n\tau)^2} \right)$$

is meromorphic on $\mathbb{C}$ and periodic with periods 1 and τ, so that it factors through a meromorphic function $X \to \mathbb{C}$ of degree 2 and a double pole at 0, which extends to a holomorphic map $X \to \mathbb{P}^1$ with four index-two branch points: the points of order two in the group X.

The lowest degree of a nonconstant, hence surjective holomorphic map from a compact Riemann surface X onto $\mathbb{P}^1$ is called the *gonality* of X. The gonality of $\mathbb{P}^1$ itself is 1, and that of any elliptic curve (the previous example) is 2. Recall that linear series of degree d and dimension r on X are in one-to-one correspondence to morphisms $X \to \mathbb{P}^r$. Thus, gonality may also be defined as the lowest degree of a rank-1 linear series on X.

As a topological space, a compact Riemann surface is uniquely determined by its *genus*, its number of holes formally defined as $g = 1 - \chi/2$ where χ is the Euler characteristic. The fundamental relation between gonality and genus is the following.

Theorem 1.1. *The gonality of a compact Riemann surface X of genus g is at most $1+\lceil g/2\rceil$, with equality if X is sufficiently general. Moreover, if g is even and X is sufficiently general, then the number of holomorphic maps to $\mathbb{P}^1$ of degree $1+g/2$ from X to $\mathbb{P}^1$ (counted up to compositions with elements of the automorphism group $\mathrm{PGL}_2(\mathbb{C})$ of automorphisms of $\mathbb{P}^1$), equals $C_{g/2}$, the $g/2$-th Catalan number.*

The condition "sufficiently general" is understood as follows. The compact genus-g Riemann surfaces (up to isomorphism) correspond to the points in a suitable (non-compact) complex algebraic variety of complex dimension $3g-3$, the *moduli space $\mathcal{M}_g$ of genus-g Riemann surfaces*, and "sufficiently general" means that it holds for all X corresponding to the points in $\mathcal{M}_g$ outside a Zariski-closed subset of positive codimension.

Recall that the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ appear throughout mathematics, and count objects that satisfy the recursion $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$. One example are *ballot sequences*. Suppose that in an election candidates A and B received the same number of votes n. A ballot sequence is an order to count these votes such that candidate B always has at least as many votes as A. The number of ballot sequences in an election where $2n$ votes were cast in total is C_n.

Theorem 1.1 is part of *Brill-Noether theory*, an area of algebraic geometry that has its roots in the late 19th century and is still very active today. The existence of such morphisms was already established by Riemann, and this was later generalised by Kempf [Kem71] and Kleiman-Laksov [KL72]. The Catalan count was established, again in the more general setting of Brill-Noether theory, by Griffiths-Harris [GH80]. Moreover, when g is even, the rank-1 case of the main theorem in [EH87] implies that the Catalan-many maps of Theorem 1.1 can be arranged in a space.

Theorem 1.2. *For even g there is an open set $B \subset \mathcal{M}_g$ of sufficiently general Riemann surfaces and a smooth irreducible family $\pi : \mathfrak{X} \to B$ whose fibre over a point X in B are $C_{g/2}$ points corresponding to the holomorphic maps of X onto $\mathbb{P}^1$.*

1.3 Tropical curves and tropical morphisms

The class of combinatorial objects that we study are called *tropical morphisms*. These are maps between metric graphs with suitable balancing conditions, which make them behave similarly to morphisms between curves; they also have a so-called Riemann-Hurwitz condition, which is a requirement for the realizability of the combinatorial object as the tropicalization of a classical object

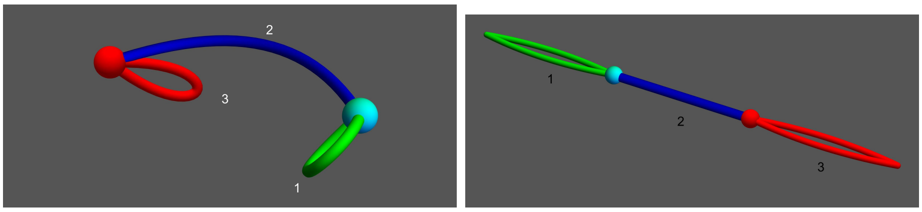


Figure 1.1: Top: On the left a genus-2 metric graph. On the right a picture suggesting a tropical morphism of degree 2 from that graph to a line segment. A choice for the slopes which yields balancing is slope 1 at the loops and slope 2 at the bridge.

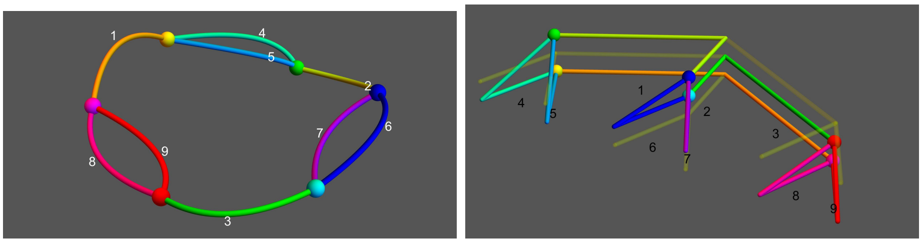


Figure 1.2: On the left a genus-4 metric graph Γ. On the right a tropical morphism of degree 3 from a modification of Γ to a tree whose shape is suggested by the figure. The colours of the edges match, the slope is 1 everywhere, and the dangling edges in the modification are depicted in transparent yellow.

(see [BN09; BBM11; Cha13; Cap14; Mik07]). A striking interplay between graphs and Riemann surfaces/algebraic curves has been discovered over the last two decades. Specialization lemmas and correspondence theorems between both settings are an active field of research at the interface between tropical and non-archimedean geometry (see, e.g., [Bak08; ABBR15a; ABBR15b]).

A tropical morphism is a piecewise linear continuous map between connected metric graphs that satisfies several combinatorial conditions. Roughly speaking, a metric graph Γ is a compact metric space which arises from taking a combinatorial graph G (a model of the metric graph), and "giving lengths" to the edges of G via a length function $\ell : E(G) \to \mathbb{R}_{>0}$. For every point x in Γ there is an ε such that the open ball around x and radius ε is isometric to a k-star (a space that arises by glueing at the origin k copies of the interval $[0, \varepsilon)$). This k is the *valency* of x.

A *harmonic map* Φ from Γ to another metric graph Δ is a map that is continuous, piecewise linear with nonzero positive integral slopes with respect to the metrics on Γ and Δ (allowing slopes equal to zero would have a meaning that we choose to ignore for simplicity of exposition), and which satisfies the following balancing condition: for each $x \in \Gamma$ and a direction d' emanating from $\Phi(x)$, the sum of the slopes of Φ along all directions d that emanate from x and map onto d' is independent of d'; that is, replacing d' by a different direction from $\Phi(x)$ yields the same sum of slopes. We denote this count by $|x|_\Phi$.

The consequences of the balancing condition are in parallel with the theory of Riemann surfaces. Immediately we are able to define a "local degree". By connectedness of Δ and Γ we can extend this to a global degree, namely for any point x in Δ the count with multiplicities $|\cdot|_\Phi$ of the points in $\Phi^{-1}(x)$ is a constant $\deg \Phi$ of Φ which we call *the degree of* Φ. It also implies, given that the slopes of Φ are nonzero positive integers, that Φ is surjective. See Figures 1.1 and 1.2 for examples.

To tighten the analogy with holomorphic maps even further, we introduce the *Riemann-Hurwitz inequality*, which says that at each $x \in \Gamma$ the harmonic map Φ satisfies the condition

$$\operatorname{val} x - 2 \geq |x|_\Phi \cdot (\operatorname{val} \Phi(x) - 2).$$

Since only finitely many points of a metric graph have valency distinct from 2, outside of a finite set of points of Γ both sides of the Riemann-Hurwitz inequality equal zero. The origin of this inequality is that the left-hand side minus the right-hand side plus one is the correct tropical analogue of the ramification index of Φ at x, which should, of course, be positive. A harmonic map Φ with non-zero slopes that satisfies the Riemann-Hurwitz inequality at every point x is called a *tropical morphism.* The map in Figure 1.1 is a tropical morphism; for example, in the red vertex the inequality reads $3 - 1 \geq 2 \cdot (2 - 2)$. So is the map in Figure 1.2.

One tropical ingredient, without a classical counterpart, is *tropical modification.* This consists in iterating the following contruction or its inverse: pick a point x in Γ and a length r, and define $\tilde{\Gamma} := (\Gamma \sqcup [0, r])/\sim$, where $\sim$ is the equivalence relation whose only non-singleton equivalence class is $\{0, x\}$. In short, $\tilde{\Gamma}$ arises from Γ by growing a new line segment of length r at x, and Γ arises from $\tilde{\Gamma}$ by removing this *dangling segment.* A graph $\tilde{\Gamma}'$ that can be obtained from Γ by a sequence of operations consisting of growing new line segments and/or removing dangling segments is called a tropical modification of Γ.

Let Γ be a metric graph. By iteratively deleting line segments ending in monovalent points we get a tropical modification $\tilde{\Gamma}$ of Γ, unique up to isometry, whose minimal valency is 2. Moreover, $\tilde{\Gamma}$ has a model H, unique up to isomorphism, without 2-valent vertices. Thus, the minimal valency of H is at least 3. We define the number $g(H) = \#(()E(H)) - \#(()V(H)) + 1$. Tropicalization maps have the property that they send a genus-g curve to a (metric) graph H such that g and $g(H)$ coincide. Thus, we call $g(H)$ the *genusgenus!of a graph* of H.

If H is trivalent, then $E(H)$ has $3g - 3$ elements; this number coincides with the dimension of $\mathcal{M}_g$. In fact, observe that by fixing a model H and varying the edge-lengths one obtains a cone C_H of metric graphs. By identifying points corresponding to isometric graphs one can glue together all the cones C_H, as H ranges over trivalent genus-g graphs, for a fixed g, and the result is an abstract rational polyhedral cone complex $\mathcal{M}_g^{\text{trop}}$ which is called *the moduli space of genus-g metric graphs.* The two spaces are related by a tropicalization map that realizes $\mathcal{M}_g^{\text{trop}}$ as the tropicalization of $\mathcal{M}_g$ [ACP15].

1.4 Tropicalization

We describe a tropicalization map to motivate our definition of tropical morphisms. Using this tropicalization map one can prove from Theorem 1.1 that an analogous combinatorial version holds. Instead of following the more algebro-geometric approaches in [Bak08; Cap14], we discuss the more hyperbolic-geomeric approach taken in [Lan20].

Let Γ be a metric graph with a trivalent model. We first argue that such a Γ is a tropical limit of a family of compact Riemann surfaces, using Fenchel-Nielsen coordinates. Given three positive reals $\alpha, \beta, \gamma > 0$, up to isometry there exists a unique geodesic hexagon H in the complex upper half-plane, equipped with its hyperbolic metric of constant curvature -1, all of whose angles are $\pi/2$ and of which the edge lengths, in counterclockwise order, are $\alpha, a, \beta, b, \gamma, c > 0$.

In particular, a, b, c are determined by α, β, γ. We denote by q_α (respectively, q_β, q_γ) the vertices of H where the sides of lengths α, a (resepctively, β, b and γ, c) meet. By basic hyperbolic geometry, the area of H (a union of four hyperbolic triangles) is $4\pi - 6\pi/2 = \pi$. So if we let α, β, γ tend to zero, then a, b, c cannot all be bounded from above.

Take two copies H, H' of H and glue them together along the edges of lengths a, b, c; the result is a *pair of pants* $P_{2\alpha,2\beta,2\gamma}$ with geodesic boundary cycles of lengths $2\alpha, 2\beta, 2\gamma$. We orient $P_{2\alpha,2\beta,2\gamma}$ with the unique orientation agreeing with that of the first copy H, and this induces orientations of the boundary cycles. We call the images of $q_\alpha, q_\beta, q_\gamma$ the *special points* on the boundary cycles of $P_{2\alpha,2\beta,2\gamma}$.

Now let G be a graph with all vertices of valency 3 and even genus $g \geq 2$ and let $c(e) > 0$ be a positive real number for each $e \in E(G)$. Then we can construct a compact Riemann surface as follows. For each $v \in V(G)$, incident to $e_1, e_2, e_3 \in E(G)$, take a copy P_v of $P_{c(e_1),c(e_2),c(e_3)}$, and glue all of these copies together along the boundary cycles in the manner prescribed by G: if an edge e of G has v and w as ends, then P_v and P_w are glued along their boundary cycle of circumference $c(e)$ in such a manner that the orientations of the boundary cycles are opposite to each other; if e is a loop, so $v = w$, we glue two legs of P_v to each other. There is one (real) degree of freedom in each of these $3g - 3$ glueings, corresponding to the angle made between the special

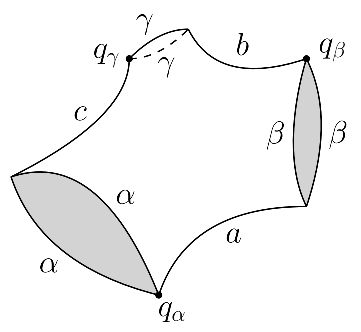


Figure 1.3: A pair of pants from two geodesic hyperbolic hexagons.

points on these two cycles. There is a discrete choice, which we ignore in this exposition, of a cyclic order on the three edges incident to any v in $V(G)$. Together, these $3g-3$ angles and the $3g-3$ lengths $c(e)$ are the *Fenchel-Nielsen coordinates* on Teichmüller space. But for our purposes, where we want to exhibit a tropical morphism to a tree from holomorphic maps to $\mathbb{P}^1$, it suffices to take all $3g-3$ angles to be zero, i.e., to glue the special points onto each other. The resulting structure X_c inherits the structure of a complex manifold from the hexagons, and it is compact, hence a Riemann surface. We call $C_e \subseteq X_c$ the image of the boundary cycle corresponding to e.

Next let $\ell \in \mathbb{R}_{>0}^{E(G)}$ be an edge length function. Define

$$c_t(e) := \frac{2\pi^2}{\ell(e)\log(t)}. \tag{1.1}$$

Then, as we let t tend to ∞, the circumference of the boundary cycles tends to zero and the Riemann surface $X_t := X_{c_t}$ degenerates into a disjoint union of $\mathbb{P}^1$'s, one for each vertex v of G, which is glued together at three distinct points to the $\mathbb{P}^1$s corresponding to the neighbours of v. (Since PGL_2 acts transitively on ordered triples of distinct points on $\mathbb{P}^1$, this is a unique complex algebraic curve up to isomorphism.)

Now let $\psi_t : X_t \to \mathbb{P}^1$ be a holomorphic map of degree $1 + g/2$; this can be chosen to depend continuously on t. In what follows, we take $t \gg 0$ and eventually let t tend to ∞. Then the boundary cycles C_e have disjoint images $\tilde{C}_e := \psi_t(C_e)$ in $\mathbb{P}^1$ for e in $E(G)$. *A priori,* $\tilde{C}_e$ is only an immersed circle in $\mathbb{P}^1$. But it can be shown that after a suitable deformation of the C_e, their images $\tilde{C}_e$ are disjoint, closed circles on the sphere $\mathbb{P}^1$; we assume this from now on. Make a graph T whose vertices are the connected components of $\mathbb{P}^1 \setminus \bigcup_{e\in E(G)} \tilde{C}_e$ and where two are connected by an edge if they have a common $\tilde{C}_e$ in their boundary. The fact that $\mathbb{P}^1$ has genus 0 implies that T is a tree. For each e, $\psi_t^{-1}(\tilde{C}_e)$ is a disjoint union of circles in X_t, one of which is C_e. Cut up X_t along these cycles and construct G' from the connected components of X_t, like we constructed T from the cut-up $\mathbb{P}^1$. Then G' is a refinement of the graph G. Each edge $e' \in E(G')$ corresponds to a circle $C_{e'}$ in X_t mapping to some circle $\tilde{C}_e$; let $s_{e'}$ be the topological degree of the restriction $\psi_t|_{C_{e'}} : C_{e'} \to \tilde{C}_e$.

The precise form of $c_t(e)$ in Equation (1.1) becomes important now: it guarantees that ψ_t converges in a well-defined sense to a tropical morphism φ from a modification of the metric graph (G, ℓ) to the metric tree (T, ℓ_T) for a suitable edge length function ℓ_T [Lan20]. The "dangling trees" in the modification are attached at vertices in $V(G') \setminus V(G)$, and the slopes are precisely the numbers $s_{e'}$ above.

We conclude by explaining the balancing condition and the origin of the Riemann-Hurwitz inequality: any vertex $A \in V(G')$ corresponds to a connected component U of $X_t \setminus \bigcup_{e'\in E(G')} C_{e'}$. For simplicity, we assume that G' has no loops at A, so that $\overline{U}$ is topologically a sphere with k disks removed corresponding to the $C_{e'}$ that form the boundary of U. The map ψ_t is a branched cover from $\overline{U}$ to $\overline{O}$, where O is a connected component of $\mathbb{P}^1 \setminus \bigcup_{e\in E(G)} \tilde{C}_e$ with l cycles $\tilde{C}_e$ in its closure. The map ψ_t sends each of the k circles $C_{e'}$ to one of the l circles $\tilde{C}_e$. The sum of the $s_{e'}$ over all

$C_{e'}$ mapping to one of the l circles $\tilde{C}_e$ in the boundary of O is independent of boundary circle $\tilde{C}_e$, namely, the degree of the branced cover $\overline{U} \to \overline{O}$; this is what we called $|A|_\Phi$. Since removing a disc from a surface reduces the Euler characteristic by 1, $\overline{U}$ has Euler characteristic $2-k$ and $\overline{O}$ has Euler characteristic $2-l$. Now the celebrated Riemann-Hurwitz formula for branched covers says

$$\begin{aligned} 2-k = \chi(\overline{U}) &= |A|_\Phi \, \chi(\overline{V}) - \sum_{x \in U} (e_x - 1) \\ &= |A|_\Phi (2-l) - \sum_{x \in U} (e_x - 1) \end{aligned}$$

where e_x is the ramification index of ψ_t at q. In particular, since $e_x \geq 1$, we find

$$k - 2 = |A|_\Phi \cdot (l-2) + \sum_{x \in U} (e_x - 1) \geq |A|_\Phi (l-2),$$

which was precisely our version of the Riemann-Hurwitz inequality for tropical morphisms.

1.5 Results: Gonality of metric graphs

In the first part of this thesis we investigate the *tree gonality* of metric graphs. This concept has its roots in gonality of algebraic curves. It has a strong relation to the purely combinatorial graph notion of *tree width* [dBG14], to spectral graph theory [Ami14], and distance functions defined on lattices [Man19].

There are two definitions of gonality for (metric) graphs which are relevant to our story. A chip-firing game on a (metric) graph yields a divisor theory (see [BN07; HKN13]). The *divisorial gonality* of a (metric) graph is the minimum degree of a rank-1 divisor. The *tree gonality* of a metric graph is the minimum degree of a tropical morphism from any *tropical modification* of the metric graph to a metric tree.

Divisorial gonality mimics the definition of gonality of X via linear series, while tree gonality mimics the one via non-constant morphisms. Unlike the algebro-geometric setting, in the tropical world divisorial gonality and tree-gonality do not coincide. See Remark 13.8 for a dramatic example. If $\Phi : \Gamma \to \Delta$ is a tropical morphism to a tree, then the divisor $\sum_{x' \in \Phi^{-1}(x)} |x'|_\Phi x'$ has rank at least 1 for any choice of x in Δ (see Section 3.3 for notation). So tree gonality is an upper bound of divisorial gonality.

At the time of writing, perhaps the most famous open question on divisorial gonality is the following:

Conjecture 1.3 (Baker; Conjecture 3.10 in [Bak08])**.** *The divisorial gonality of a finite connected graph G is at most $\lceil g/2 \rceil + 1$, where g is the first Betti number of G.*

This conjecture has attracted a sizeable amount of attention from the community. If one replaces G with a metric graph Γ, then the result is proven in [Bak08, Theorem 3.12]. The proof requires a tropicalization map like the one we outlined in the *Motivation* section, and the corresponding statement for algebraic curves. The algebro-geometric statement requires for its proof the sophisticated machinery of special divisors. There is wide interest for a purely combinatorial proof, as it is hoped it may yield methods better suited to tackle Conjecture 1.3. We give one such purely combinatorial proof:

Theorem 1.4. *The tree gonality of a genus-g metric graph Γ is at most $\lceil g/2 \rceil + 1$.*

In fact, we manage to construct tropical morphisms that witness the bound $\lceil g/2 \rceil + 1$ on tree-gonality, so beyond being combinatorial, the proof is also *effective*. There is some computer code and visualizations available at [Dra]. For even genus, the tropical morphism which realizes the bound belongs to a family of tropical morphisms of dimension $3g-3$ and that has a generically finite-to-one map onto the moduli space of genus-g metric graphs. Our methods focus on the study of such *full-dimensional* families.

1.6 Proof sketch

1.6.1 Main idea

The main idea underpinning this thesis is short, and elegant: we first analyse a favourable case and then continuously deform the result. We need an initial family of graphs Γ such that there is a construction of a tropical morphism $\Phi : \Gamma \to \Delta$ attaining the gonality bound; and we need a procedure to deform Φ into $\bar{\Phi} : \bar{\Gamma} \to \bar{\Delta}$ for any given $\bar{\Gamma}$ via a deformation path in $\mathcal{M}_g^{\text{trop}}$, the moduli space of metric graphs. We argue that for this strategy to work we need that Φ, and all but finitely many of the points in the deformation path to $\bar{\Phi}$, belong to cones of tropical morphisms that we call *full-dimensional*. Intuitively, these morphisms *depend* on the right number of parameters to cover $\mathcal{M}_g^{\text{trop}}$.

To carry out this simple idea we introduce a technical framework which describes and studies properties related to full-dimensional cones, sets up the deformation procedure, and supports the case work required to prove the correctness of deformation. It would be interesting to understand how our work relates to the tropicalization of the space of admissible covers from [CMR16], but this is beyond the scope of our combinatorial methods.

1.6.2 Families of morphisms

Recall that a metric graph Γ has an underlying combinatorial graph G, a model; and that by varying the edge lengths of G we obtain a whole family of metric graphs parametrised by a rational polyhedral structure $\mathbb{R}_{>0}^{E(G)}$. In a similar vein, a tropical morphism $\Phi : \Gamma \to \Delta$ has an underlying *combinatorial type* φ recording its combinatorial information. This consists of models G, T for Γ, Δ; a graph morphism $\varphi : G \to T$; and an index map $|\cdot|_\varphi : E(T) \to \mathbb{Z}_{>0}$ recording the slopes of Φ. If one prescribes edge-lengths ℓ_T for T, the information in φ gives edge-lengths $\ell_G(e) = \ell_T \circ \varphi(e)/|e|_\varphi$ for G. This gives rise to a tropical morphism $(\varphi, \ell_T) : (G, \ell_G) \to (T, \ell_T)$. Importantly, we obtain a family of metric graphs (G, ℓ_G), with ℓ_G in the image of the map $\ell_T \to \ell_G$, that have gonality at most $\deg \varphi$, as witnessed by (φ, ℓ_T).

Since we care about graphs up to tropical modification, and we wish to cover the space of genus-g metric graphs $\mathcal{M}_g^{\text{trop}}$, we iteratively delete valency-1 points, to get a unique modification $(H(\varphi), \ell_{H(\varphi)})$ of the graph (G, ℓ_G), where the minimum valency of $H(\varphi)$ is 3 and $H(\varphi)$ is independent of ℓ_G. The composition of maps $\ell_T \to \ell_G \to \ell_H$ is linear, so it can be extended to a linear map with matrix A_φ. For example, in Figures 1.1,1.2, the matrix A_φ equals

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix},$$

respectively, relative to the labellings of the edges in Γ and Δ indicated there. Note that in the examples the entries of A_φ are rational. This is true in general, and is a consequence of φ having integral slopes. Hence our family of graphs with gonality at most $\deg \varphi$ has the structure of a rational polyhedral cone $A_\varphi(\mathbb{R}_{>0})$, which we denote C_φ, living in $\mathbb{R}^{E(H)}$.

As in metric graphs, there is a notion of tropical modification for tropical morphisms. The cone C_φ is invariant under this operation. We show that there is a unique tropical modification φ' of φ such that $\varphi' : G' \to T'$ satisfies:

$$\dim C_\varphi \leq |E(T')| \leq 2g + 2d - 5, \tag{1.2}$$

where g is the genus of G and d is the degree of φ. We show that equality is attained if and only if the dimension of the column space of A_φ equals $E(T')$ equals $2g + 2d - 5$.

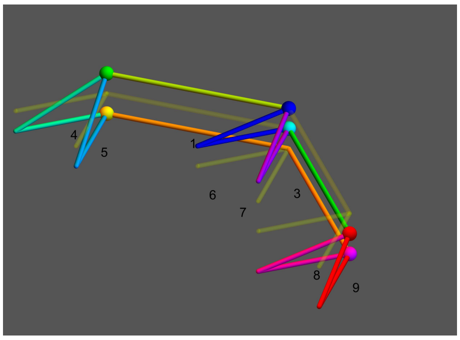


Figure 1.4: The edge labelled 2 in the tree in Figure 1.2 has been contracted to obtain φ_0.

The bound on the number of edges implies there are finitely many cones C_φ. Hence, to cover $\mathcal{M}_g^{\mathrm{trop}}$, which has dimension $3g-3$, we need to have $2g+2d_\phi-5 \geq 3g-3$, namely $\deg\varphi \geq g/2+1$. So when $\deg\varphi = g/2+1$ we have that equality in Equation (1.2) is attained if and only if A_φ is an invertible $(3g-3)\times(3g-3)$-matrix. In this case we call C_φ a *full-dimensional family*. Theorem 1.4 is equivalent to saying that the union of full-dimensional families is dense in $\mathcal{M}_g^{\mathrm{trop}}$. These families had already appeared in an ad-hoc manner in [CD18]. The idea of a parameter count was first mentioned by Mikhalkin in his talk at the *Saarbrücken conference on tropical geometry and computational biology*.

1.6.3 Deformation and its limits

Our proof constructs enough full-dimensional families to cover $\mathcal{M}_g^{\mathrm{trop}}$ in two steps. First, we give an explicit construction for a specific initial metric graph whose families C_φ have nice properties (more on this later); second, we deform the initial families to cover $\mathcal{M}_g^{\mathrm{trop}}$.

Suppose we have a full-dimensional tropical morphism $\Phi_{\mathrm{Start}} : \Gamma_{\mathrm{Start}} \to \Delta_{\mathrm{Start}}$. The idea is to deform it to a tropical morphism $\Gamma \to \Delta$ for a given Γ. Since $A_{\varphi_{\mathrm{Start}}}$ is invertible, the naive approach is to consider a continuous function $\sigma : [0,1] \to \mathcal{M}_g^{\mathrm{trop}}$ with $\sigma(0) = \Gamma_{\mathrm{Start}}$ and $\sigma(1) = \Gamma$. The hope is that $A^{-1}_{\varphi_{\mathrm{Start}}} \circ \sigma$ gives edge-length functions inducing a path of tropical morphisms beginning at Φ_{Start} and finishing at the desired tropical morphism. The problem is that this path might hit one of the facets of the cone $C_{\varphi_{\mathrm{Start}}}$, rendering impossible to continue.

For instance, in Figure 1.2, one facet of the cone C_φ is given by the condition

$$\ell_H(e_2) + \ell_H(e_3) - \ell_H(e_1) > 0, \tag{1.3}$$

expressing that for this combinatorial type the orange edge has length less than the sum of the lengths of the yellow and green edges. So for any Γ where the orange length gets too long, the deformation with σ gets stuck when $A_\phi^{-1} \circ \ell_H$ contains a non-positive edge-length.

At the point where σ hits a facet of C_ϕ we have that $A^{-1}_{\varphi_{\mathrm{Start}}} \circ \sigma$ has some zero-lengths (these zero entries become negative if we are to follow along σ). While we have no way to give sense to negative lengths, we can realize zero-lengths as edge-contractions. In Figure 1.2, if $\ell_H(e_2)+\ell_H(e_3)-\ell_H(e_1) = 0$, then the length of the edge of T labelled by 2 becomes zero. In Figure 1.4 this edge is contracted to obtain φ_0. It is straightforward to check that φ_0 satisfies the axioms of a tropical morphism.

We call a combinatorial type of tropical morphisms arising from contracting an edge a *limit*, since it is both a limit that the deformation σ hits and must overcome, and also an object that informally can be regarded as resulting from an edge length tending to zero. Passing to a limit forgets the information of φ contained above the edge being contracted. So different φ might share the same limit φ_0; we call this set of φ the neighbourhood of φ_0, denoted $\mathrm{neigh}(\varphi_0)$. Another way to regard $\mathrm{neigh}(\varphi_0)$ is as the cones C_φ which share the facet C_{φ_0}. The key question is whether deformation can continue by passing to another combinatorial type in $\mathrm{neigh}(\varphi_0)$; namely we are coming from C_φ, when we hit C_{φ_0}, we prove there is $C_{\varphi'}$ in $\mathrm{neigh}(\varphi_0)$ at the other side.

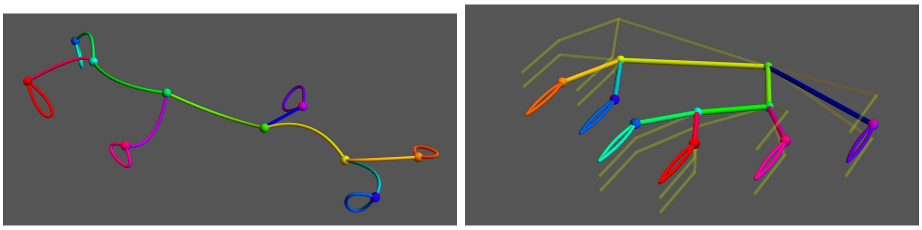


Figure 1.5: Top: a caterpillar of genus 6. Bottom: one of its $C_3 = 5$ tropical morphisms to a metric tree.

We use our framework for full-dimensional families, and an extensive casework, to construct a set of candidates which contains $\mathrm{neigh}(\varphi_0)$. Given a candidate φ', the hard problem to answer in general is whether φ' is full-dimensional, and moreover if C_φ and $C_{\varphi'}$ lie at opposite sides with respect to C_{φ_0}, that is, cover different parts of $\mathcal{M}_g^{\mathrm{trop}}$. The former amounts to showing that $\det A_{\varphi'} \neq 0$. Some manipulations with linear algebra reduce the latter also to a condition on determinants.

Here the story bifurcates. In the first possibility, contracting one edge of T also contracts one edge of $H(\varphi)$. In other words, the limit φ_0 signifies a change in $H(\varphi)$, and also a change of cone of $\mathcal{M}_g^{\mathrm{trop}}$. This is reflected in A_φ by having a row with a single non-zero entry, the one corresponding to the contracted edge. An important observation is that A_φ and $A_{\varphi'}$ agree in all but one column (the column corresponding to the contracted edge). So by a cofactor expansion, $\det A_{\varphi'} = q \det A_\varphi$ for some positive rational number q; in particular, $\det A_{\varphi'} \neq 0$ always.

1.6.4 Balancing

In the second possibility, $H(\varphi_0)$ is trivalent, so the limit σ must overcome happens inside a cone of $\mathcal{M}_g^{\mathrm{trop}}$; this is the case in Equation 1.3. Here C_φ, $C_{\varphi'}$ cover distinct parts of $\mathcal{M}_g^{\mathrm{trop}}$ if and only if $\mathrm{sgn} \det A_\varphi \neq \mathrm{sgn} \det A_{\varphi'}$. Again, this is a consequence of $\det A_\varphi$, $\det A_{\varphi'}$ agreeing in all but one columns. Now, the exceptional insight which allows to “break through the limit” is proving a formula of the form

$$\sum_{\varphi \in \mathrm{neigh}(\varphi_0)} q_\varphi \det A_\varphi = 0,$$

where the q_φ are integers which depend on φ. This way we kill two birds with one stone. At once we prove the existence of another full-dimensional tropical morphism φ', and (crucially!) $\det A_{\varphi'}$ has the opposite sign to $\det A_\varphi$. The inspiration for this equation stems from a common trope in the world of tropical geometry, a *balancing condition*, like the ones satisfied by embedded tropical varieties.

The upshot is that, mimicking the construction of $\mathcal{M}_g^{\mathrm{trop}}$, we can glue together the cones C_φ for all genus-g full-dimensional families and get a space of tropical morphisms together with a projection Π down to $\mathcal{M}_g^{\mathrm{trop}}$.

1.6.5 Walking through the space of tropical morphisms

Coming back to the initial family, the graph Γ_{Start} is a “caterpillar of g loops”: a metric graph whose underlying graph G_{CL} is obtained by taking a path $v_0, e_1, v_1, \ldots, e_{g-1}, v_{g-1}$ of $g-1$ edges and attaching loops to v_0 and v_{g-1}; and lollipops (bridges leading to a loop) to the remaining v_i. See Figure 1.5.

Given a tropical morphism $\varphi : \Gamma_{\mathrm{Start}} \to \Delta$, the underlying graph morphism γ of the restriction $\varphi|_{\Gamma_{\mathrm{Start}}}$ is independent of φ. The slopes at e_i encode a ballot sequence. Conversely, a ballot sequence corresponds to a choice for these slopes. We can extend γ to a tropical morphism by prescribing the slopes and adding trees via tropical modification to satisfy the balancing condition. This gives the bijection with ballot sequences, so there are Catalan-many $C_{g/2}$ such morphisms.

Now we have the ingredients to follow σ. We start with $C_{g/2}$ paths of tropical morphisms by modifying Γ_{Start}. As they trace $C_{g/2}$ paths, when σ hits a limit it might happen that some paths merge. Once some paths have merged, hitting a limit gives the opportunity for these paths to split. We introduce a multiplicity based in the number $q_\varphi \det A_\varphi$ of the balancing condition to reflect this phenomenon of merging and splitting. Then the number of tropical morphisms to trees remains $C_{g/2}$ throughout the walk.

Chapter 2

Background

2.1 Graphs

A graph G is a pair $(V(G), E(G))$ of disjoint sets, the vertex and edge sets respectively, and a map ι_G defined on $E(G)$ encoding incidences, whose values are multisets of two elements of $V(G)$. Given an edge e with $\iota_G(e) = \{A, B\}$ we call A, B the *ends* of e, and say that A and e are *incident*. A *loop* is an edge e such that its two ends are equal. A *subgraph* of G is a graph G' such that $V(G') \subset V(G)$, $E(G') \subset E(G)$, and $\iota_{G'}$ equals the restriction $\iota_G|_{E(G')}$. A subset S of $V(G)$ induces a subgraph by taking for edge set all the edges of G with both ends in S. Likewise, a subset S of $E(G)$ induces a subgraph by taking for vertex set the ends of all the edges in S. A graph G is *finite* if both $V(G)$ and $E(G)$ are finite sets; and *loopless* if it has no loops.

A *path* of G is a sequence $\langle A_0, e_1, A_1, \ldots, e_\mu, A_\mu \rangle$ of alternately vertices and edges, where no element is repeated, such that consecutive elements are incident and $\mu \geq 1$. The *ends* of a path are the vertices A_0, A_μ. The remaining vertices $A_1, \ldots, A_{\mu-1}$ are *interior*. We call μ the *length* of P. We write $x \in P$ to mean that x is an element in the sequence P. If P is a path with ends A, B, and e is an edge that also has ends A, B, then we call the subgraph consisting of e together with the edges in P a *cycle*. Given a vertex A, its *connected!component* is the subgraph induced by the set of ends of all paths with $A \in P$. The vertex set $V(G)$ is partitioned into connected components. A graph is *connected* if it has a single connected component.

Let G be a finite, connected graph. Set $g(G) = |E(G)| - |V(G)| + 1$. Following a convention due to [BN07], we call this number the *genus* of G. If G has genus 0, then we call G a *tree*. Write $x \in G$ for $x \in V(G) \cup E(G)$. Let A be a vertex. Write $E_G(A)$ for the subset of $E(G)$ incident to A. The *valency* $\operatorname{val}_G A$ of A is the number of edges in $E_G(A)$, with loops counting twice. The *minimum valency* of G is the number $\min_{A \in V(G)} \operatorname{val}_G A$. We write $E(A)$ and $\operatorname{val} A$ when G is clear from the context. A vertex is *monovalent*, *divalent*, *trivalent*, or *n-valent* if its valency equals 1, 2, 3, or n, respectively. We call a monovalent vertex of a tree a *leaf*.

Let e be in $E(G)$, and A, B its ends. *Deleting* e induces the graph with edge set $E(G) \setminus \{e\}$. *Contracting* e yields the graph $(V(G)/\sim, E(G) \setminus \{e\})$, where $\sim$ identifies A and B. *Subdividing* e yields the graph $(V(G) \cup \{C\}, (E(G) \setminus \{e\}) \cup \{e_1, e_2\})$, where the ends of e_1 are A, C, and of e_2 are C, B.

Let G, G' be two graphs. A *graph morphism* is a map $\gamma : V(G) \cup E(G) \to V(G') \cup E(G')$ such that: $\gamma(V(G)) \subseteq V(G')$; if $\gamma(e) \in V(G')$ for an edge e in $E(G)$ with ends A and B, then $\gamma(e) = \gamma(A) = \gamma(B)$; if $\gamma(e) \in E(G')$ for an edge e in $E(G)$ with ends A and B, then the ends of $\gamma(e)$ are $\gamma(A)$ and $\gamma(B)$. In essence, a morphism is an incidence preserving map which can contract edges. If $\gamma(E(G)) \subseteq E(G')$, then we call γ a *homomorphism*. An *isomorphism* is a bijective homomorphism; its inverse is then a homomorphism as well.

An edge (resp. vertex) labelling is an injective map λ from $E(G)$ (resp. $V(G)$) to a set S. A total order $\leq_S$ on S induces a total order on $E(G)$ by letting $e \leq_{E(G)} e'$ when $\lambda(e) \leq_S \lambda(e')$. We use $\mathbb{Z}_{>0}$ as labelling set, with its natural order. In the next section we consider elements from the vector space $\mathbb{R}^{E(G)}$ (real valued functions on $E(G)$). By choosing an edge labelling λ this space is identified with $\mathbb{R}^n$, where n equals $\#(()E(G))$. The identification is as follows, we write $e_j \in E(G)$ for $\lambda^{-1}(j)$ (we use this notation whenever a λ has been chosen); we take as ordered basis of $\mathbb{R}^{E(G)}$ the functions ℓ_i which map e_j to 1 if $i = j$, and zero otherwise.

2.2 Metric graphs

Succinctly, metric graphs are graphs whose edges have lengths. A *length function* for G is a map $\ell : E(G) \to \mathbb{R}_{>0}$. The pair (G, ℓ) gives rise to a one-dimensional CW-complex Γ by identifying the endpoints of the intervals in $\bigsqcup_{e \in E(G)}[0, \ell(e)]$ in the manner prescribed by G. We call the first Betti number of the CW-complex Γ the *genus* of Γ, denoted by $g(\Gamma)$. It coincides with $g(G)$. This CW-complex is equipped with the shortest-path metric in which the intervals Γ_e have lengths $\ell(e)$. This metric space is called a *metric graph*. We call G a *model*, and the pair (G, ℓ) a *realization* of Γ.

Let Γ be a metric graph. There are infinitely many realizations of Γ. Indeed, given a realization (G, ℓ_G) of Γ, the set S of points of Γ corresponding to vertices of G has the property that $\Gamma \setminus S$ is a disjoint union of open intervals. We call any finite set S with that property a *vertex set*, since it induces a realization (G_S, ℓ_S) whose vertices are the points of S, the edges are the disjoint intervals of $\Gamma \setminus S$, and the lengths are the lengths of the connected components of $\Gamma \setminus S$. Let $\mathcal{S}$ be the family of all vertex sets of Γ. If $\mathcal{E} = \bigcap_{S \in \mathcal{S}} S$ is non-empty, then $\mathcal{E}$ is a vertex set. We call $G_{\mathcal{E}}$ and $(G_{\mathcal{E}}, \ell_{\mathcal{E}})$, induced by $\mathcal{E}$, the *essential model* and the *essential realization*, respectively. It always exists for the metric graphs of our interest:

Lemma 2.1. *Let Γ be a metric graph, and S_0 the set of points x in Γ such that for all $\varepsilon > 0$ the open ball $B(x, \varepsilon)$ with centre x and radius ε is not isometric to the interval $(-\varepsilon, \varepsilon)$. If S_0 is non empty, then any finite $S \subset \Gamma$ is a vertex set if and only if $S_0 \subset S$.*

We call the elements of S_0 the *essential vertices*. The only family without essential vertices are *metric loops*; *metric loops* they have for model the graph with one vertex and one loop edge. We assume that our metric graphs are never metric loops. Lemma 3.13 shows that $\mathcal{E}$ equals S_0, and that $\mathcal{E}$ is minimal in the sense that all models arise from a sequence of edge subdivisions of $G_{\mathcal{E}}$. So it is well defined to define the valency of a point x in Γ as the valency of x in G_S for S any vertex set containing x.

2.3 Tropical modification

We consider graphs and metric graphs to be equivalent under *tropical modification*, which is an operation that iteratively attaches or removes monovalent points. For this we introduce the important notion of *dangling elements*:

Definition 2.2. Let G be a graph, and $\Gamma = (G, \ell)$ be a metric graph.

- An edge e of G is *dangling* if deleting e produces two connected components and at least one of those is a tree. A vertex A of G is dangling if all e in $E(A)$ are dangling.
- A point x in Γ that corresponds to a vertex v in G is dangling if v is dangling. A point x in Γ in the interior of an edge Γ_e is dangling if e is dangling.

Definition 2.3. Let A be a vertex of G. We denote by nd-$E(A)$ the subset of non-dangling edges in $E(A)$.

It is straighforward to see that the definition of dangling for Γ is independent of the chosen realization. Let $\widetilde{\Gamma}$ be the metric graph obtained from Γ by deleting all dangling points. Let H be the essential model of $\widetilde{\Gamma}$, as in Lemma 3.13. Tropical modification is the equivalence relation on metric graphs generated by identifying Γ with $\widetilde{\Gamma}$, and on graphs by identifying G with H. It follows from construction and by Lemma 3.13 that:

Lemma 2.4. *Let Γ be a metric graph of genus at least 2, and Γ' a tropical modification of Γ, such that Γ' has a model G' with minimum valency at least three. Then Γ' is isometric to $\widetilde{\Gamma}$ and G' is isomorphic to H.* □

So $\widetilde{\Gamma}$ and H are canonical representatives. We call a graph with minimum valency at least 3, or equal to 3, a *combinatorial type*, or a *trivalent combinatorial type*, respectively. We say that H is the combinatorial type of Γ and of G. We call $\widetilde{\Gamma}$ the *deletion of dangling trees* of Γ. The naming is due the fact that $g(\Gamma) = g(\widetilde{\Gamma})$.

2.4 Moduli space of metric graphs

We give a barebones account of the moduli space of genus-g metric graphs $\mathcal{M}_g^{\text{trop}}$. If all the vertices of a graph are trivalent, we call it a *trivalent combinatorial type*, or simply *trivalent*. We denote by $\mathbb{G}_g$ the family of genus-g trivalent combinatorial types. For the construction of $\mathcal{M}_g^{\text{trop}}$ we use the following straightforward lemma:

Lemma 2.5. *Let $g \geq 2$ and H_0 be a genus-g combinatorial type. There exists H in $\mathbb{G}_g$ such that there is a sequence of edge contractions of H yielding a graph isomorphic to H_0.* □

Given a graph G, we denote by C_G the set of length functions defined on G. This set has a cone structure, as it equals the positive orthant $\mathbb{R}_{>0}^{E(G)}$. We identify each point ℓ of C_G with the metric graph (G, ℓ) to obtain the (rational polyhedral) cone C_G of metric graphs with model G.

Motivated by Lemma 2.5 we add some of the boundary points of C_G with the following convention: given a map $\ell : E(G) \to \mathbb{R}_{\geq 0}$, contract all the edges of G for which $\ell(e) = 0$, to get G'; the pair (G, ℓ) then stands for to the metric graph $(G', \ell|_{G'})$. Let $\overline{C}_G \subset {\mathbb{R}_{\geq 0}}^{E(G)}$ be the set of maps ℓ for which the metric graph (G, ℓ) has genus g, that is, all cycles of G have positive length. Glueing together these completed cones gives rise to a space where points bijectively correspond to classes under tropical modification of genus-g metric graphs:

Definition 2.6. The moduli space of genus-g metric graphs is the space

$$\mathcal{M}_g^{\text{trop}} = \left(\bigsqcup_{H \in \mathbb{G}_g} \overline{C}_H \right) / \cong,$$

where $\cong$ identifies points ℓ_H and $\ell_{H'}$ for which (H, ℓ_H) and $(H', \ell_{H'})$ are isometric.

Given a metric graph Γ, by Lemma 2.4 and 2.5 there is a point in $\mathcal{M}_g^{\text{trop}}$ corresponding to a graph $\widetilde{\Gamma}$ which is equivalent to Γ under tropical modification. By construction this point is unique.

Remark 2.7. The reader might find in the literature a different use for the symbol $\mathcal{M}_g^{\text{trop}}$, albeit for constructions related to ours. They differ by imposing additional structure on the metric graph: marked points, legs of infinite length, weights on the vertices, and more. These are not required for our purposes, as they typically arise to encode more algebro-geometric information in a graph. See [Koz09] and [ACP15]. △

Remark 2.8. Identifying isometric points complicates the geometry of $\mathcal{M}_g^{\text{trop}}$. We do not obtain a polyhedral cone complex, as one might expect. This issue is addressed in [ACP15] with the introduction of *abstract polyhedral cone complexes*. See their Section 4.3 for a proof that $\mathcal{M}_g^{\text{trop}}$ is an abstract polyhedral cone complex. △

Remark 2.9 (topology of $\mathcal{M}_g^{\text{trop}}$)**.** We consider $\overline{C}_H$ with the topology induced from the Euclidean space $\mathbb{R}^E(H)$. We consider $\bigsqcup_{H \in \mathbb{G}_g} \overline{C}_H$ with the product topology induced from each $\overline{C}_H$. Finally, we consider $\mathcal{M}_g^{\text{trop}}$ with the quotient topology. △

To close this section, we say a few words about how $\mathcal{M}_g^{\text{trop}}$ is connected through codimension-1. Let H be a combinatorial type. It is a straightforward exercise to show that $\#(()E(H)) \leq$

$3g(H) - 3$, with equality if and only if H is trivalent. Thus, the dimension of $\mathcal{M}_g^{\text{trop}}$ is $3g - 3$, since each point is specified by at most $3g - 3$ parameters. The *codimension* of the cone C_H is the difference

$$\operatorname{codim} C_H = (3g(H) - 3) - \#(()E(H)).$$

We claim that $\mathcal{M}_g^{\text{trop}}$ is *connected through codimension-1*, that is the locus in $\mathcal{M}_g^{\text{trop}}$ of points coming from cones of codimension at most-1 is connected. Specifically, there is a canonical map $C_H \to \mathcal{M}_g^{\text{trop}}$ for any combinatorial type H. We claim that the image of the following map is connected:

$$\bigsqcup_{\operatorname{codim} H \leq 1} \overline{C}_H \to \mathcal{M}_g^{\text{trop}}.$$

Lemma 2.10. *The space $\mathcal{M}_g^{trop}$ is connected through codimension one.*

Proof. At a combinatorial level this result states that given two trivalent graphs H and H', there exists a sequence $H_0 = H$, $H_1, \dots, H_{2n} = H'$ where the graphs with even indices are trivalent, and consecutive graphs are related by contraction of a single edge. An early proof of this result can be found in [HT80]. A study of this property, and other graph linkage properties, and their relation to the connectedness of moduli spaces of metric graphs is done in [Cap12]; in particular, see Proposition 3.3.3 of the cited work. □

2.5 Symmetry in $\mathcal{M}_g^{\text{trop}}$

Let H be a combinatorial type. We can describe explicitly the points in C_H which encode isometric metric graphs by observing the following. Let $\Psi : \Gamma^{(1)} \to \Gamma^{(2)}$ be an isometry. It preserves the valency of a point. Thus, by Lemma 3.13, if S is a vertex set of $\Gamma^{(1)}$, then $\Psi(S)$ is a vertex set of $\Gamma^{(2)}$. So let $\Gamma^{(1)} = (G^{(1)}, \ell^{(1)})$ be induced by S, and $\Gamma^{(2)} = (G^{(2)}, \ell^{(2)})$ by $\Psi(S)$. The connected components of $\Gamma^{(1)} \setminus S$ are mapped one-to-one and isometrically to the connected components of $\Gamma^{(2)} \setminus \Psi(S)$. This means that there is an induced map $\gamma_\Psi : G^{(1)} \to G^{(2)}$ which is an isomorphism; moreover, $\ell^{(2)} = \ell^{(1)} \circ \gamma_\Psi^{-1}$. It is straightforward to verify that this necessary conditions are also enough to specify an isometry.

Lemma 2.11. *Let $\Gamma^{(1)}$, $\Gamma^{(2)}$ be metric graphs, S a vertex set of $\Gamma^{(1)}$, and $\Psi : \Gamma^{(1)} \to \Gamma^{(2)}$ be a continuous map. Then Ψ is an isometry if and only if $\Psi(S)$ is a vertex set of $\Gamma^{(2)}$, the induced map γ_Ψ is an isomorphism, and the length pulls back, so $\ell^{(2)} = \ell^{(1)} \circ \gamma_\Psi^{-1}$.*

Crucially, we get an $\operatorname{Aut} H$-action on C_H by sending ℓ in C_H to $\ell \circ \widetilde{\gamma}^{-1}$ for $\widetilde{\gamma}$ in $\operatorname{Aut} H$. Applying Lemma 2.11, with S equal to the set of essential vertices, gives that two points of C_H encode isometric graphs if and only if they lie in the same $\operatorname{Aut} H$-orbit. By the Orbit-Stabilizer theorem, this means that the number of times that a given $\widetilde{\Gamma} = (H, y)$ is realized in C_H equals $\# \operatorname{Aut} H / \# \operatorname{Stab}_{\operatorname{Aut} H} y$; here $\# \operatorname{Stab}_{\operatorname{Aut} H} y$ is the *stabilizer* of y, that is the subgroup of $\operatorname{Aut} H$ such that $y = y \circ \widetilde{\gamma}^{-1}$.

We say that y in C_H is a *general length function* if all the lengths encoded by y are distinct. In this case the stabilizer is trivial, and we have that (H, y) is realized $\# \operatorname{Aut} H$ times by C_H. Note that the set of general length functions is dense in C_H, in the Euclidean topology.

2.6 Notation

In the remainder of this thesis we use the following letters, and small variations thereof, are used to denote specific objects. Metric graphs: Γ for metric graphs with $g(\Gamma) \geq 2$; Δ for metric trees (due to tree being δέντρο in Greek), that is genus 0 metric graphs; and $\widetilde{\Gamma}$ for the deletion of the dangling elements of Γ. Finite graphs: G for a finite connected graph with $g(G) \geq 2$, A for its vertices and e for its edges; T for a finite tree, v for its vertices and t for its edges; and H for the combinatorial type of G. Morphisms: φ for maps between finite graphs; and Φ for maps between metric graphs. We write $[d]$ for the set $\{1, 2, \dots, d\}$, and $x \in G$ for $x \in V(G) \cup E(G)$.

Chapter 3

Tropical morphisms

3.1 DT-morphisms

We begin with a map encapsulating the combinatorial information of tropical morphisms. These are a special class of graph morphisms, augmented with an index map.

Definition 3.1 (discrete tropical morphism)**.** Let $\varphi : G \to G'$ be a morphism of finite connected graphs, and $|\cdot|_\varphi : G \to \mathbb{Z}_{\geq 0}$ be a map.

(a) $|\cdot|_\varphi$ is an *index-map* for φ if for every e in $E(G)$ we have that $\varphi(e) \in V(G')$ if and only if $|e|_\varphi = 0$.

(b) φ is *harmonic* with index-map $|\cdot|_\varphi$ if for every A in $V(G)$ and e' incident to $\varphi(A)$ following *balancing condition* is satisfied:

$$|A|_\varphi = \sum_{\substack{e \in E(A) \\ \varphi(e) = e'}} |e|_\varphi.$$

Note that this makes the sum independent of the choice of e'.

(c) φ is *non-degenerate* if $|x|_\varphi \geq 1$ for all x in G.

(d) φ satisfies the *Riemann-Hurwitz condition* (RH-condition) if for every A in $V(G)$:

$$r_\varphi(A) = (\operatorname{val} A - 2) - |A|_\varphi(\operatorname{val} \varphi(A) - 2) \geq 0.$$

(e) A *discrete tropical morphism* is a pair $(\varphi, |\cdot|_\varphi)$ consisting of a non-degenerate harmonic morphism φ with index-map $|\cdot|_\varphi$ that satisfies the RH-condition. We write DT-morphismto shorten. We write $|\cdot|$ for $|\cdot|_\varphi$ when it causes no confusion. We call G the *source* and G' the *target* of φ.

Remark 3.2. The non-degeneracy condition ensures that no edge is collapsed. △

Remark 3.3. Definition 3.1 is similar to Definition 2.1 of [Cap14], with the following differences: condition (c) of Definition 2.1 in the cited work is required only for vertices of G (see discussion in Section 1.3 of [CD18]); an alternative formula for r_φ is given on Equation (10) of the cited work (see Lemma 3.5); and we do not have a vertex labelling representing extra genus. △

Let $(\varphi : G \to G', |\cdot|)$ be a DT-morphism. A prevailing philosophy in the field is to regard a graph as a discrete analogue of a Riemann surface. In that context φ would be the analogue of a ramified covering map. Here is one justification for this view:

Lemma 3.4 (degree of φ)**.** *Let $\varphi : G \to G'$ be a DT-morphism. Then the count of elements x in a fibre of φ is constant, where each x is counted with multiplicity $|x|$. Namely, the number*

$$\deg \varphi = \sum_{x \in \varphi^{-1}(x')} |x|$$

is independent of the choice of $x' \in G'$. We call $\deg \varphi$ *the degree of* φ*.*

Proof. Let v' be in $V(G')$. By the balancing condition we have that for all e in $E(v)$:

$$\sum_{A \in \varphi^{-1}(v')} |A| = \sum_{e \in \varphi^{-1}(e')} |e|.$$

Thus, over a path of G' the preimage count is constant. Since G' is connected we get the result. □

Lemma 3.5 (formula for r_φ)**.** *Let* $\varphi : G \to G'$ *be a DT-morphism. Then*

$$r_\varphi(A) = 2(|A| - 1) - \sum_{e \in E(A)} (|e| - 1).$$

Proof. By the balancing condition $|A| \cdot \operatorname{val} \varphi(A) = \sum_{e \in E(A)} |e|$. Thus,

$$\begin{aligned} r_\varphi(A) &= \operatorname{val} A - 2 - |A| \cdot (\operatorname{val} \varphi(A) - 2) \\ &= 2(|A| - 1) + \operatorname{val} A - \sum_{e \in E(A)} |e| \\ &= 2(|A| - 1) - \sum_{e \in E(A)} (|e| - 1). \end{aligned}$$

□

Lemma 3.6 (Riemann-Hurwitz formula)**.** *Let* $\varphi : G \to G'$ *be a DT-morphism. Then*

$$2g(G) - 2 = \deg \varphi \cdot (2g(G') - 2) + \sum_{A \in V(G)} r_\varphi(A).$$

Proof. Recall that in a graph the sum of all the valencies is twice the number of edges:

$$\begin{aligned} \sum_{A \in V(G)} r_\varphi(A) &= \sum_{A \in V(G)} (\operatorname{val} A - 2 - m_\varphi(A) \cdot (\operatorname{val} \varphi(A) - 2)) \\ &= 2|E(G)| - 2|V(G)| - \deg \varphi \cdot (2|E(G')| - 2|V(G')|) \\ &= 2g(G) - 2 - \deg \varphi \cdot (2 \cdot g(G') - 2). \end{aligned}$$

Rearrange terms to obtain the result. □

Remark 3.7. Equations (14) and (15) of [Cap14] calculate a ramification divisor for φ. The divisor

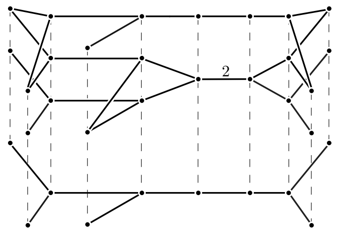
$$R_\varphi = \sum_{A \in V(G)} \left(2|A| - 2 - \sum_{e \in E(A)} (|e| - 1) \right)$$

satisfies that $K_G = \varphi^* K_{G'} + R_\varphi$, where K_G, $K_{G'}$ are the canonical divisors of G and G', respectively, and $\varphi^* K_{G'}$ is the pull-back. We observe two things: the RH-conditions is equivalent to R_φ being effective, and taking degrees on both sides of $K_G = \varphi^* K_{G'} + R_\varphi$ implies Lemma 3.6. △

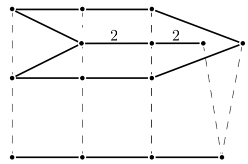
Remark 3.8. By the RH-condition, Lemma 3.6 implies that $g(G) \geq g(G')$, as in the classical setting. △

The index map can be understood as a local degree: for A in $V(G)$ restricting φ to the subgraph G_A induced by $E(A)$ gives a DT-morphismonto the subgraph induced by $\varphi(E(A))$; any fibre of $\varphi|_{G_A}$ has $|A|$ elements, each element counted with multiplicity $|\cdot|_\varphi$.

Example 3.9. The next figures illustrate two DT-morphismsφ_A, φ_B of degrees 3 and 4, respectively. The targets are drawn below, the sources above, and the fibres with dotted lines. Unless noted otherwise, with a number above the edge, the edges have index 1.



φ_A



φ_B ⋆

3.2 Metric graphs

We give a definition for tropical morphism. This definition differs slightly from the ones in the literature (e.g. the one in [CD18]), because highlighting the combinatorial structure of the maps, via DT-morphisms, makes our exposition more transparent. A *tropical morphism* arises by choosing a length function for the target G', in a process analogous to how we obtain a metric graph from a graph G.

Definition 3.10 (tropical morphism)**.** Let $\varphi : G \to G'$ be a DT-morphismand $\ell_{G'}$ in $C_{G'}$. The corresponding *tropical morphism* is the unique map Φ from the source metric graph $\Gamma = (G,\ e \mapsto \ell_{G'}(\varphi(e)) \,/\, |e|\,)$ to the target $\Gamma' = (G', \ell_{G'})$, whose restriction to Γ_e is a bijective linear function to $\Gamma'_{\varphi(e)}$, and that agrees with φ on vertex points.

Let Φ be a tropical morphism arising from $(\varphi, \ell_{G'})$. We call φ a *model* and $(\varphi, \ell_{G'})$ a *realization* of Φ. From the connectedness of G and G' we get that Φ is continuous. The length function for the source is constructed so the linear map $\Phi|_{\Gamma_e}$ has integral slope $|e|$. The count of points in a fibre $\Phi^{-1}(x')$ is done with a multiplicity. If x in $\Phi^{-1}(x')$ is in the interior of an edge, then the multipicity is the slope of Φ at this edge. Otherwise x corresponds to a vertex, so the multiplicity is calculated using the balancing condition. By Lemma 3.4 this count is constant, independent of x', and equals $\deg \varphi$. All of this means that Φ is a tropical morphism in the sense of [CD18].

Given $\Phi : \Gamma \to \Gamma'$ there are infinitely many possible models and realizations, as in the case of metric graphs. One has to be careful though, it is not enough to just choose vertex sets for the source and the target; they must be compatible; Indeed, let S and S' be vertex sets for Γ and Γ', respectively; note that the image of every vertex of Γ under Φ must be a vertex of Γ', and that the preimage of every vertex of Γ' must be a vertex of Γ. A subset $S' \subset \Gamma'$ is a vertex set for Φ if h

Definition 3.11. The models of Φ are in bijective correspondence with

Let Γ be a metric graph. There are infinitely many realizations of Γ. Indeed, given a realization (G, ℓ_G) of Γ, the set S of points of Γ corresponding to vertices of G has the property that $\Gamma \setminus S$ is a disjoint union of open intervals. We call any finite set S with that property a *vertex set*, since it induces a realization (G_S, ℓ_S) whose vertices are the points of S, the edges are the disjoint intervals of $\Gamma \setminus S$, and the lengths are the lengths of the connected components of $\Gamma \setminus S$. Let $\mathcal{S}$ be the family of all vertex sets of Γ. If $\mathcal{E} = \bigcap_{S \in \mathcal{S}} S$ is non-empty, then $\mathcal{E}$ is a vertex set. We call $G_{\mathcal{E}}$ and $(G_{\mathcal{E}}, \ell_{\mathcal{E}})$, induced by $\mathcal{E}$, the *essential model* and the *essential realization*, respectively. It always exists for the metric graphs of our interest:

Lemma 3.12. *Let $\Phi : \Gamma \to \Gamma'$ be a tropical morphism, and $S_0 = \Phi(\mathcal{E}) \cup \mathcal{E}'$, with $\mathcal{E}$ and $\mathcal{E}'$ the sets of essential vertices of Γ and Γ', respectively. Any finite set $S' \subset \Gamma'$ is a vertex set of Φ if and only if $S_0 \subset S$.*

Lemma 3.13. *Let Γ be a metric graph, and S_0 the set of points x in Γ such that for all $\varepsilon > 0$ the open ball $B(x, \varepsilon)$ with centre x and radius ε is not isometric to the interval $(-\varepsilon, \varepsilon)$. If S_0 is non empty, then any finite $S \subset \Gamma$ is a vertex set if and only if $S_0 \subset S$.*

We call the elements of S_0 the *essential vertices*. The only family without essential vertices are *metric loops*; they have for model the graph with one vertex and one loop edge. We assume that our metric graphs are never metric loops. Lemma 3.13 shows that $\mathcal{E}$ equals S_0, and that $\mathcal{E}$

is minimal in the sense that all models arise from a sequence of edge subdivisions of $G_{\mathcal{E}}$. So it is well defined to define the valency of a point x in Γ as the valency of x in G_S for S any vertex set containing x.

3.3 Tropical morphisms

We give a definition for tropical morphism that differs slightly from the ones in the literature (e.g. the one in [CD18]), because highlighting the combinatorial structure of the maps, via DT-morphisms, makes our exposition more transparent. A *tropical morphism* arises by choosing a length function for the target G', in a construction analogous to how we obtain a metric graph from a graph G.

Definition 3.14 (tropical morphism). Let $\varphi : G \to G'$ be a DT-morphismand $\ell_{G'}$ in $C_{G'}$. The corresponding *tropical morphism* is the unique map Φ from the source metric graph $\Gamma = (G,\ e \mapsto \ell_{G'}(\varphi(e)) \,/\, |e|_\varphi\,)$ to the target $\Gamma' = (G', \ell_{G'})$, whose restriction to Γ_e is a bijective linear function to $\Gamma'_{\varphi(e)}$, and that agrees with φ on vertex points.

We call φ and $(\varphi, \ell_{G'})$ a *model* and a *realization* of Φ, respectively. From the connectedness of G and G' we get that Φ is continuous. The length function for the source is constructed so the linear map $\Phi|_{\Gamma_e}$ has integral slope $|e|_\varphi$. The count of points in a fibre $\Phi^{-1}(x')$ is done with a multiplicity: if x is in the interior of an edge, then the multipicity is the slope of Φ at this edge; else x corresponds to a vertex and the multiplicity is calculated using the balancing condition. By Lemma 3.4 this count is independent of x', and equals $\deg \varphi$. All of this means that $\Phi = (\varphi, \ell_{G'})$ is a tropical morphism in the sense of [CD18].

Now, given a tropical morphism $\Phi : \Gamma \to \Delta$ we characterize all its models with a result analogous to Lemma 3.13. The model is given by choosing vertex sets $\mathcal{S}$, $\mathcal{S}'$ for Γ, Γ', respectively. The key observation is that a model of $\Phi : \Gamma \to \Gamma'$ maps vertices of Γ to vertices of Γ'; and that by the non-degeneracy condition all the points in the fibre of a vertex of Γ' are vertices of Γ. Succinctly, $\mathcal{S} = \Phi^{-1}(\mathcal{S}')$. Moreover, $\mathcal{S}$ must contain the essential vertices $\mathcal{E}$ of Γ, and $\mathcal{S}'$ must contain the essential vertices $\mathcal{E}'$ of Γ'. This implies that $\Phi(\mathcal{E}) \cup \mathcal{E}'$ is a subset of $\mathcal{S}'$. We argue that any subset $\mathcal{S}' \subset \Gamma'$ that satisfies this condition induces vertex sets for Φ.

Construction 3.15. Let $\Phi : \Gamma \to \Gamma'$ be a tropical morphism, and $\mathcal{S}' \subset \Gamma'$ be a subset such that $\Phi(\mathcal{E}_\Gamma) \cup \mathcal{E}_{\Gamma'} \subset \mathcal{S}'$. Then by Lemma 3.13 the sets $\mathcal{S} = \Phi^{-1}(\mathcal{S}')$ and $\mathcal{S}'$ are vertex sets which induce realizations $\Gamma = (G_\mathcal{S}, \ell_\mathcal{S})$ and $\Gamma' = (G_{\mathcal{S}'}, \ell_{\mathcal{S}'})$. Define a map $\varphi_{\mathcal{S}'} : G_\mathcal{S} \to G_{\mathcal{S}'}$ by sending a vertex x of Γ to the vertex $\Phi(x)$ of Γ', and for an edge e in $E(G_\mathcal{S})$ we choose a point x in Γ_e, and map e to the edge e' of $E(G_{\mathcal{S}'})$ such that x' is in $\Gamma_{e'}$. As index map we take $|\cdot|_{\varphi_{\mathcal{S}'}} = \ell_{\mathcal{S}'}(\varphi_{\mathcal{S}'}(e))/\ell_\mathcal{S}(e)$. If $\mathcal{S}' = \Phi(\mathcal{E}_\Gamma) \cup \mathcal{E}_{\Gamma'}$ then we denote the induced map from $G_\mathcal{S}$ to $G_{\mathcal{S}'}$ by φ_{ess} and call it the *essential model*.

Lemma 3.16. *In Construction 3.15, the map $\varphi_{\mathcal{S}'}$ is a DT-morphism, the pair $(\varphi_{\mathcal{S}'}, \ell_{\mathcal{S}'})$ realizes Φ, and the essential model φ_{ess} is minimal in the sense that $\varphi_{\mathcal{S}'}$ arises as an edge subdivision of φ_{ess}.*

Proof. It is clear that $\varphi_{\mathcal{S}'}$ is well defined because by properties of Φ and the fact that $\mathcal{S} = \Phi^{-1}(\mathcal{S}')$ we have that each connected component of $\Gamma \setminus \mathcal{S}$ maps linearly and bijectively to a connected component of $\Gamma' \setminus \mathcal{S}'$. Moreover the index map $|\cdot|_{\varphi_{\mathcal{S}'}}$ records the slope of this linear map, so for free we get non-degeneracy, balancing condition, and Riemann-Hurwitz.

That the essential model is minimal, and that every other model arises as an edge subdivision of the essential model is clear from the fact that for any model we must have $\Phi(\mathcal{E}_\Gamma) \cup \mathcal{E}_{\Gamma'} \subset \mathcal{S}'$. □

To summarize, any model of Φ is determined by the vertex set on the target; and any vertex set on the target that contains $\Phi(\mathcal{E}_\Gamma) \cup \mathcal{E}_{\Gamma'}$ induces vertex sets on the source and the target which give rise to a DT-morphism.

3.4 Isomorphisms and tropical modification

Recall that in the space $\mathcal{M}_g^{\mathrm{trop}}$ we consider metric graphs as an equivalence class under isometry and tropical modification. This setup extends naturally to tropical morphisms.

Definition 3.17 (isomorphism of graph morphisms)**.** Let $\varphi_1 : G_1 \to G_1'$ and $\varphi_2 : G_2 \to G_2'$ be graph morphisms. An isomorphism from φ_1 to φ_2 is given by graph isomorphisms (γ, γ') for which Diagram 3.18.a commutes. If additionally φ_1 and φ_2 are DT-morphisms, then (γ, γ') is an isomorphism of DT-morphismsif the index map $|\cdot|_{\varphi_2}$ pulls back to $|\cdot|_{\varphi_1}$, namely Diagram 3.18.b commutes. Let Φ_1 and Φ_2 be tropical morphisms. An isomorphism from Φ_1 to Φ_2 is given by isometries Ψ and Ψ' for which Diagram 3.18.c commutes.

$$\begin{array}{ccc} G_1 & \xrightarrow{\varphi_1} & G_1' \\ \gamma\downarrow & & \downarrow\gamma' \\ G_2 & \xrightarrow{\varphi_2} & G_2' \end{array}$$

(a)

$$\begin{array}{ccc} G_1 & & \\ \gamma\downarrow & \searrow^{|\cdot|_{\varphi_1}} & \\ G_2 & \xrightarrow[|\cdot|_{\varphi_2}]{} & \mathbb{Z}_{\geq 0} \end{array}$$

(b)

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{\Phi_1} & \Gamma_1' \\ \Psi\downarrow & & \downarrow\Psi' \\ \Gamma_2 & \xrightarrow{\Phi_2} & \Gamma_2' \end{array}$$

(c)

Diagram 3.18

We now define equivalence under tropical modification for a DT-morphism $\varphi : G \to G'$. We say that the fibre $\varphi^{-1}(x')$, for x' in G', is *dangling* if all its elements are dangling in G. Let $D \subset E(G')$ be the set of edges with dangling fibres. Consider the subgraph $\hat{G}'$ of G' induced by $E(G') \setminus D$. The preimage $\varphi^{-1}(\hat{G}')$ is not necessarily a connected graph (see Example 3.22), but the definition of dangling implies that there is only one connected component with non-zero genus. Call it $\hat{G}$. Restricting φ to $\hat{G}$ gives a DT-morphism$\hat{\varphi} : \hat{G} \to \hat{G}'$ that we call the *deletion of dangling fibres* of φ. See the Section *Modifications* in [CD18] for some examples. Tropical modification identifies φ with $\hat{\varphi}$.

The same definitions and constructions can be carried out on the metric side for a tropical morphism $\Phi = (\varphi, \ell)$. A fibre $\Phi^{-1}(x)$ is dangling if all its points are dangling, and the deletion of dangling fibres $\hat{\Phi}$ is the restriction of Φ to the connected component with non-zero genus which remains after deleting all points belonging to dangling fibres. One can check that $(\hat{\varphi}, \ell|_{\hat{G}'})$ is a realization of $\hat{\Phi}$. Tropical modification identifies Φ with $\hat{\Phi}$. It is straightforward to see that if Φ_1 and Φ_2 are equivalent under tropical modification, then $\hat{\Phi}_1$ and $\hat{\Phi}_2$ are isomorphic, thus $\hat{\Phi}$ is a canonical representative for the equivalence class of Φ under isomorphism and tropical modification. Thus, it is natural to use a model of $\hat{\Phi}$ as the *combinatorial type* of the class of Φ:

Definition 3.19. Let Φ be a tropical morphism and $\hat{\Phi}$ its deletion of dangling fibres. The *combinatorial type* of Φ is the model $\hat{\varphi}_{\mathrm{ess}}$ of $\hat{\Phi}$ constructed using Construction 3.15.

Alternatively, one can use Construction 3.15 on Φ to get $(\varphi_{\mathrm{ess}}, \ell_{\mathrm{ess}})$ and delete all dangling fibres to get $\widehat{\varphi_{\mathrm{ess}}}$. It is straightforward to see that $\widehat{\varphi_{\mathrm{ess}}}$ is isomorphic to $\hat{\varphi}_{\mathrm{ess}}$. So $(\hat{\varphi}_{\mathrm{ess}}, \ell_{\mathrm{ess}}|_{\hat{G}'_{ess}})$ realizes $\hat{\Phi}$. We call the values of $\ell_{\mathrm{ess}}|_{\hat{G}'_{ess}}$ the *parameters* of Φ. Thus, we say that Φ moves in the cone C_φ with dimension $\dim C_\varphi$ and depends on $|E(\hat{G}'_{ess})|$ many parameters.

3.5 Cone of sources

Given a DT-morphism$\varphi : G \to G'$, let C_φ be the set of sources, modulo tropical modification, of all tropical morphisms with model φ (this set is invariant under isomorphism and tropical modification). If $\widetilde{\Gamma} \in C_\varphi$, we say that φ *realizes* $\widetilde{\Gamma}$. In this Section we give to C_φ the structure of a rational polyhedral cone. To this end, we realize the combinatorial type of G with the following graph:

Construction 3.20. Let $H(\varphi)$ be the graph whose vertices are the elements A in $V(G)$ such that $\text{nd-val}\, A \geq 3$; its edges are the paths of G whose ends are in $V(H(\varphi))$ and whose interior vertices have non-dangling valency equal to 2.

Given a length function $\ell_{G'}$ for the target G' we consider the following length function for $H(\varphi)$:

$$y(h) = \sum_{e \in h} \ell_{G'}(\varphi(e))/|e|.$$

The map $C_{G'} \to C_{H(\varphi)}$ given by $\ell_{G'} \mapsto y$ extends to a linear map $A_\varphi : \mathbb{R}^{E(G')} \to \mathbb{R}^{E(H(\varphi))}$ which we call *the edge-length map.* Using the standard bases on $\mathbb{R}^{E(G')}$, $\mathbb{R}^{E(H(\varphi))}$ we write A_φ as a matrix, the *edge-length matrix*, whose rows are indexed by $E(H(\varphi))$ and columns by $E(G')$. An entry a_{ht} of this matrix is a rational number given by:

$$a_{ht} = \sum_{\substack{e \in h \\ \varphi(e)=t}} \ell_{G'}(e)/|e|,$$

where the sum is zero if the index set is empty.

By construction $(H(\varphi), A_\varphi(\ell_{G'}))$ is isomorphic to the deletion of dangling trees of the source of $(\varphi, \ell_{G'})$. Since $H(\varphi)$ has minimum valency at least 3, by Lemma 2.4 the combinatorial type H of G is isomorphic to $H(\varphi)$. So the punchline is that C_φ is parametrized by the rational polyhedral cone $A_\varphi(\mathbb{R}^{E(G')}_{>0})$. By abuse of notation we identify C_φ with $A_\varphi(\mathbb{R}^{E(G')}_{>0})$.

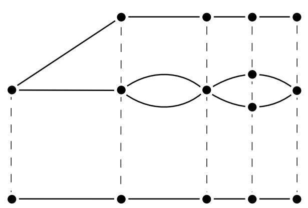
We have a particularly nice situation when the edge-length matrix A_φ is injective. In this case a point ℓ_H in C_φ not only corresponds to a metric graph (H, ℓ_H) appearing as a source, but also to the tropical morphism $(\varphi, A_\varphi^{-1}(\ell_H))$. We say that C_φ, or simply φ, is full-rank when A_φ is injective. For a general DT-morphismthe dimension of the fibre $A_\varphi^{-1}(\ell_H)$ equals the dimension of $\ker A_\varphi$. Thus, if we expect a finite count for the number of *nice enough* tropical morphisms which realize a given Γ, then we must work with full rank DT-morphisms.

3.6 Combinatorial types

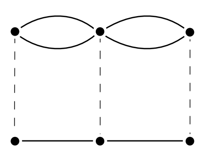
Let $\varphi : G \to G'$ be a DT-morphism, and Φ a tropical morphism with model φ. Recall that the combinatorial type of Φ is the essential model $\hat{\varphi}_{\text{ess}}$ of the deletion of dangling points $\hat{\Phi}$. We give an analogue of Lemma 2.4 for tropical morphisms; namely we give combinatorial conditions that characterize the DT-morphismsthat show up as combinatorial types.

Definition 3.21. A *combinatorial type of DT-morphisms* is a DT-morphism $\varphi : G \to G'$ without dangling fibres and such that $\sum_{A \in \varphi^{-1}(v)} r_\varphi(A) \geq 1$ for every divalent v in $V(G')$.

Example 3.22. The next figures illustrate two DT-morphismsφ_A, φ_B. They are equivalent under tropical modification, and φ_B is a combinatorial type of DT-morphisms. Note that the degree is not invariant.



φ_A



φ_B ⋆

Lemma 3.23. *Let Φ be a tropical morphism, and $\bar{\Phi} : \bar{\Gamma} \to \bar{\Gamma}'$ be a tropical modification of Φ, such that $\bar{\Phi}$ has a model $\bar{\varphi} : \bar{G} \to \bar{G}'$ that is a combinatorial type of DT-morphisms. Then $\bar{\Phi}$ is isomorphic to $\hat{\Phi}$ and $\bar{\varphi}$ is isomorphic to $\hat{\varphi}_{ess}$.*

Proof. By definition $\bar{\varphi}$ has no dangling fibres. So we only need to show that the points of $\bar{\Gamma}'$ corresponding to vertices of $\bar{G}'$ are in the set $\mathcal{T}$, with $\mathcal{T}$ as in Construction 3.15 (the union of the set of essential vertices of $\bar{\Gamma}'$ with the image of the set of essential vertices of $\bar{\Gamma}$). This means proving that for every divalent v in $V(\bar{G}')$ there is a vertex A in $\bar{\varphi}^{-1}(v)$ such that $\operatorname{val} A \geq 3$. Note that for

A in $\bar{\varphi}^{-1}(v)$ we have $r_{\bar{\varphi}}(A) = \operatorname{val} A - 2 - |A|_{\bar{\varphi}}(\operatorname{val} v - 2) = \operatorname{val} A - 2$. By the balancing condition there are no monovalent vertices in $\bar{\varphi}^{-1}(v)$. Thus, $\sum_{A \in \bar{\varphi}^{-1}(v)} r_{\bar{\varphi}}(A) = \sum_{A \in \bar{\varphi}^{-1}(v)} \operatorname{val} A - 2 \geq 1$ if and only if there is at least one vertex A in $\bar{\varphi}^{-1}(v)$ with $\operatorname{val} A \geq 3$. □

Combinatorial types of DT-morphismsare canonical representatives in the equivalence class under tropical modification: the discussion after Definition 3.19 tells us how to construct them, and Lemma 3.23 ensures their uniqueness.

We call the sum $\sum_{A \in \varphi^{-1}(v)} r_{\varphi}(A)$ the *change* at v, and denote it by $\operatorname{ch} v$. We have seen in the proof of Lemma 3.23 that divalent v with $\operatorname{ch} v = 0$ arise as edge-subdivisions of the essential model. Thus, for divalent v if t, t' are the edges incident to v, then morally the fibres of t, t' are equal (each A in $\varphi^{-1}(v)$ has valency 2, and the two edges have the same index $|\cdot|_{\varphi}$). This implies the following result:

Lemma 3.24. *Let v in $V(G')$ be divalent, and $E(A) = \{t, t'\}$. If $\operatorname{ch} v = 0$, then the columns of A_{φ} corresponding to t, t' are equal.*

DT-morphismsthat are full-rank satisfy the combinatorial type conditions.

Lemma 3.25. *If $\varphi : G \to G'$ has full-rank, then φ is a combinatorial type of DT-morphisms.*

Proof. Let H be the combinatorial type of G, and $A_{\varphi} : C_{G'} \to C_H$ be the edge-length map. Suppose t in $E(G')$ has a dangling fibre. Then varying $\ell_{G'}(t)$ does not affect $A_{\varphi}(\ell_{G'})$, contradicting that A_{φ} has full-rank. So φ has no dangling fibres. Moreover, by Lemma 3.24 we have that φ has no divalent vertices with $\operatorname{ch} v = 0$. □

Finally, we prove a combinatorial property that puts the condition $\operatorname{ch} v \geq 1$, for divalent vertices, in a wider context.

Lemma 3.26. *If φ is a combinatorial type, then $\operatorname{ch} v + \operatorname{val} v \geq 3$ for all v in $V(G')$.*

Proof. We only need to show that $\operatorname{ch} v \geq 2$ for monovalent vertices v of G'. Let v be monovalent. Note that a monovalent vertex of G is dangling. Since G does not have dangling fibres, pick A in $\varphi^{-1}(v)$ with $\operatorname{val} A \geq 2$. For such A we have that $r_{\varphi}(A) = \operatorname{val} A - 2 - |A|(\operatorname{val} v - 2) = \operatorname{val} A - 2 + |A|$. Since v is monovalent we have that $|A| \geq \operatorname{val} A$, thus $r_{\varphi}(A) \geq 2$ and by the RH-condition this implies that $\operatorname{ch} v \geq 2$. □

3.7 Dimension Formula

Let $\varphi : G \to G'$ be a DT-morphism. We prove a formula relating the number $|E(G')|$, the degree $d = \deg \varphi$, and the genera of the source and the target of φ. Note that $|E(G')|$ is an upper bound for $\dim C_{\varphi}$. Hence, we call this result *the dimension formula.*

Lemma 3.27 (total change)**.** *Let $\varphi : G \to G'$ be a DT-morphism. Then*

$$\sum_{v \in V(G')} \operatorname{ch} v = 2g(G) - 2g(G') \cdot d + 2d - 2.$$

Proof. This is a consequence of Lemma 3.6 (Riemann-Hurwitz formula). □

Proposition 3.28 (dimension formula)**.** *Let $\varphi : G \to G'$ be a DT-morphism. Then*

$$|E(G')| + \sum_{v \in V(G')} (\operatorname{ch} v + \operatorname{val} v - 3) = 2g(G) - g(G') \cdot (2d - 3) + 2d - 5.$$

Proof. The ingredients are the fact that the sum of valencies gives twice the number of edges, and Lemma 3.27 (total change). We compute:

$$\begin{aligned} |E(G')| + \sum_{v \in V(G')} (\operatorname{ch} v + \operatorname{val} v - 3) &= 3|E(G')| - 3|V(G')| + \sum_{v \in V(G')} \operatorname{ch} v \\ &= 2g(G) - g(G') \cdot (2d - 3) + 2d - 5. \end{aligned}$$

□

We regard the second term on the left hand side in Proposition 3.28 as a *correction term.* Lemma 3.26 gives us a choice in the equivalence class under tropical modification of φ for which the correction term is non-negative. With this we get an upper bound:

Corollary 3.29. *Let $\varphi\colon G \to G'$ be a DT-morphism. Then*

$$\dim C_\varphi \leq 2g(G) - g(G') \cdot (2d-3) + 2d - 5.$$

Proof. Let $\hat{\varphi}\colon \hat{G} \to \hat{G}'$ be a combinatorial type of DT-morphismsthat is equivalent under tropical modification to φ. By Lemma 3.26 and the dimension formula we get

$$\dim C_{\hat{\varphi}} \leq |E(\hat{G}')| \leq 2g(\hat{G}) - g(\hat{G}') \cdot (2d-3) + 2d - 5.$$

By the properties of tropical modification we have that $\dim C_{\hat{\varphi}} = \dim C_\varphi$, $g(\hat{G}) = g(G)$, and $g(\hat{G}') = g(G')$, proving the desired bound. □

3.8 Tree gonality

To close this chapter we use the bound given by the dimension formula to reprove a result on tree gonality that appeared in [CD18].

Definition 3.30. The *tree gonality* of a metric graph Γ is the minimum degree of a tropical morphism from some tropical modification of Γ to some metric tree.

Let $\varphi\colon G \to T$ be a DT-morphismto a tree. Every metric graph Γ in C_φ has tree gonality at most $\deg \varphi$. From now on we only work with DT-morphisms$\varphi : G \to T$ to trees. For such maps Corollary 3.29 gives that $\dim C_\varphi \leq 2g + 2d - 5$. Thus:

Corollary 3.31. *Let g, d be positive integers. The locus of metric graphs in $\mathcal{M}_g^{trop}$ with tree gonality at most d has dimension at most* $\min(2g + 2d - 5, 3g - 3)$.

Proof. Let $\widetilde{\Gamma}$ be in $\mathcal{M}_g^{\mathrm{trop}}$, Γ a tropical modification of $\widetilde{\Gamma}$, and $\Phi : \Gamma \to \Delta$ a tropical morphism to a tree with $\deg \Phi \leq d$. Then $\widetilde{\Gamma} \in C_\varphi$, where $\varphi : G \to T$ is the combinatorial type of Φ. By the preceding discussion and since $\dim \mathcal{M}_g^{\mathrm{trop}} = 3g - 3$ we get $\dim C_\varphi \leq \min(2g + 2d - 5, 3g - 3)$. Note also that $|E(T)| \leq \min(2g + 2d - 5, 3g - 3)$, and that $|E(G)| \leq \deg \varphi \cdot |E(T)| \leq d \cdot |E(T)|$, so there are finitely many possibilities for G, T, and consequently also for φ. This gives the result. □

Since $3g - 3 \leq 2g + 2d - 5$ implies that $\lceil g/2 + 1 \rceil \leq d$, we get that most metric graphs in $\mathcal{M}_g^{\mathrm{trop}}$ have tree gonality at least $\lceil g/2 + 1 \rceil$. This is one of the two inequalities that comprises Theorem 1 in [CD18]. The other inequality, namely that metric graphs have gonality at most $\lceil g/2 + 1 \rceil$, was shown constructively for certain families of graphs, one family for each combinatorial type:

Theorem 3.32. *Let H be a trivalent combinatorial type and $g = g(H)$. Then there exists a DT-morphismφ such that* $\deg \varphi = \lceil g/2 + 1 \rceil$, $H(\varphi) = H$ *and* $\dim C_\varphi = 3g - 3$.

Theorem 3.32 is proven constructively in Section 4 of [CD18]. When g is even, these families have the following property:

Definition 3.33. A DT-morphismφ, with $g = g(H(\varphi))$, is *full dimensional* if

$$\dim C_\varphi = 3g - 3 = 2g + 2d - 5.$$

The pairwise equalities in the definition of full-dimensional have the following consequences:

- $\dim C_\varphi = 3g - 3$ implies that $H(\varphi)$ is trivalent and the tropical morphism moves in a space of the right dimension, a necessary condition to hope to realize all sources in $C_{H(\varphi)}$.
- $\dim C_\varphi = 2g + 2d - 5$, $\dim C_\varphi = 3g - 3$, and the dimension formula together imply that $|E(T)| = 3g - 3$, that φ has full-rank, and $\operatorname{val} v + \operatorname{ch} v = 3$ for all v in $V(T)$.

- $3g-3 = 2g+2d-5$ implies that g is even and $d = g/2+1$.

These observations give a practical criteria to check whether φ is full-dimensional:

Lemma 3.34. *A DT-morphismφ is full-dimensional if and only if A_φ is a nonsingular $(3g-3)\times(3g-3)$ matrix.*

Example 3.35. Lemma 3.34 and the calculation done in the fifth case of Example 5.1 shows that the map φ_A in Example 3.9 is full-dimensional. ⋆

Let H be a trivalent combinatorial type with even genus, and φ the full-dimensional DT-morphismgiven by Theorem 3.32. Our aim here in Part I is to introduce a deformation procedure that when applied to a full-dimensional φ produces full-dimensional DT-morphismswhose cones cover $C_{H(\varphi)}$, showing that all metric graphs with combinatorial type $H(\varphi)$ have gonality at most $g/2+1$. For the remaining we direct our efforts to the construction of this deformation procedure.

Chapter 4

Glueing datums

Let $\varphi : G \to T$ be a DT-morphismto a tree. We regard G as arising from taking $\deg \varphi$ copies of T and identifying together certain vertices and edges between copies. In this view the index map records how many copies of T were glued together in a particular place. We make this notion precise with the combinatorial gadget of a *glueing datum*.

4.1 The glueing datum

Glueing datums are a tool to ease the visualization of DT-morphisms; to do book-keeping on edges of G in the process of deformation of DT-morphisms (see Chapters 6 and 7); and to write computer programs handling DT-morphisms. See [Dra] for computer code.

Definition 4.1 (glueing datum)**.** Let T be a finite tree, d be a positive integer, and $\sim$ be an equivalence relation on $W = T \times [d]$, where $[d] = \{1, \dots, d\}$. Write $\overline{(x,k)}$ for the classes of $W/\sim$, where $x \in V(T) \cup E(T)$. The triple $(T, d, \sim)$ is a *glueing datum* when $\sim$ satisfies these properties:

1. Verticality: If $(x, i) \sim (x', i')$, then $x = x'$. Each x in T defines a relation $\sim_x$ on $[d]$ with $i \sim_x j$ if $(x, i) \sim (x, j)$. We say that $\sim_x$ is the *glueing relation above* x. By verticality these relations determine $\sim$.

2. Refinement: If v is in $V(T)$, and t is incident to v, then $\sim_t$ is a refinement of $\sim_v$.

3. Connectedness: For any two classes $\overline{(x,k)}$ and $\overline{(x',k')}$ there is a sequence

$$\overline{(x_0, i_0)} = \overline{(x,k)}, \overline{(x_1, i_1)}, \dots, \overline{(x_r, i_r)} = \overline{(x', k')}$$

   such that for each q the elements x_q, x_{q+1} are incident (so one of them is a vertex and the other an edge), and there is at least one j such that $(x_q, j) \in \overline{(x_q, i_q)}$ and $(x_{q+1}, j) \in \overline{(x_{q+1}, i_{q+1})}$.

4. Riemann-Hurwitz condition: For v in $V(T)$ and i in $[d]$ let $A \subset [d]$ be the class of i under the relation $\sim_v$. Let $l = \operatorname{val} v$ and $E(v) = \{t_1, \dots, t_l\}$. By the refinement property the relation $\sim_{t_q}$ above t_q partitions A into k_q sets. Then

$$(k_1 + k_2 + \cdots + k_l) - 2 \geq |A| \cdot (l - 2).$$

Let $M = (T, d, \sim)$ be a glueing datum. Write $|\overline{(x,k)}|$ for the cardinality of the equivalence class $\overline{(x,k)}$. We abuse notation and say i in $\overline{(x,k)}$ to mean (x,i) in $\overline{(x,k)}$. By verticality this causes no confusion. A consequence is to regard $\overline{(x,k)}$ as a subset of $[d]$, for the purpose of comparing classes. We do, but to avoid confusion it is pointed out in every such instance of this use that $\overline{(x,k)}$ is momentarily being regarded as a subset of $[d]$, instead of a class of $W/\sim$.

We call T the *base tree* of M. We visualize glueing datums by drawing d copies of T on top of each other, and imagining all of them above T. Curved lines and shaded regions indicate where

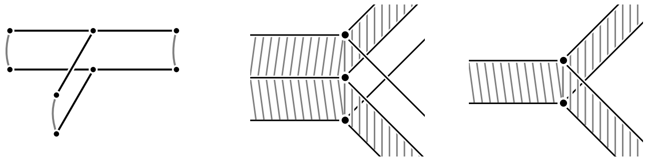


Figure 4.1: On the left, a glueing datum giving rise to the theta graph. In the centre, an allowed set of glueing relations. On the right, a set of glueing relations not allowed because of the RH-condition.

vertices or edges are identified. A class $\overline{(x,k)}$ is said to be *above* x. By the verticality condition, one can only glue elements of W that are above the same element of the base tree, see Figure 4.1.

The quotient $G = W/\sim$ has the following natural structure of a loopless graph: the classes $\overline{(x,k)}$ in $W/\sim$ with x in $V(T)$ are the vertices of G; those with x in $E(T)$ are the edges of G; an edge $\overline{(t,i)}$ in $E(G)$ is incident to a vertex $\overline{(v,k)}$ in $V(G)$ if and only if t and v are incident in T and $i \in \overline{(v,k)}$. This is a well defined graph, by the verticality and the refinement properties. It is connected, by the connectedness property. It has no loops by the verticality property and the fact that T is a tree. Consider the natural map $\varphi_M : G \to T$ given by $\varphi_M\overline{(x,k)} = x$, and the index map $|\overline{(x,k)}|_{\varphi_M} = \#\overline{((),x)k}$.

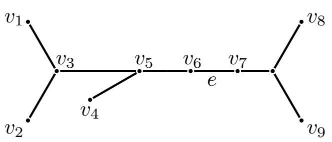
Lemma 4.2 (Prop. 5, [CD18])**.** *For a glueing datum $M = (T, d, \sim)$, the map $\varphi_M\overline{(x,k)} = x$ with index map $|\overline{(x,k)}|_{\varphi_M} = \#\overline{((),x)k}$ is a DT-morphismof degree d.*

Proof. We prove that φ is a DT-morphismby showing the balancing condition; the other requirements are immediate. Let A be a vertex of G. Set $v = \varphi(A)$. Fix t in $E(v)$. Recall that by the refinement property the relation $\sim_t$ above t partitions A. Thus, if $e_1, \dots, e_r$ are the classes in $E(A)$ above t, then $\#(A) = \#(e)_1| + \cdots + \#(e)_r$. Since $|e|_{\varphi_M} = \#(e)$ we get that

$$|A|_{\varphi_M} = \#(A) = \sum_{\substack{e\in E(A)\\ \varphi(e)=t}} |e|_{\varphi_M}. \qquad \square$$

We illustrate the previous concepts in the following example.

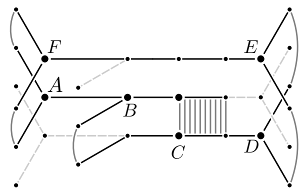
Example 4.3. Let $M = (T, 3, \sim)$ be a glueing datum with T and $\sim$ as follows:



T

$\sim_{v_1}=\sim_{v_2}= \{\{1,2\},\{3\}\}$
$\sim_{v_4}=\sim_{e}= \{\{1\},\{2,3\}\},$
$\sim_{v_1}=\sim_{v_2}= \{\{1,3\},\{2\}\},$
else $\sim_x$ is $\{\{1\},\{2\},\{3\}\}.$

$\sim$



A visualization of M

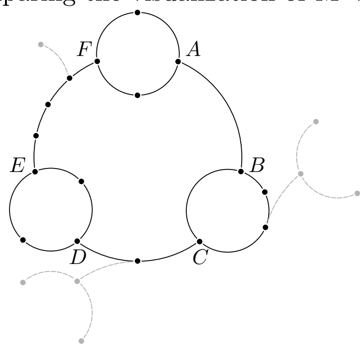
In the visualization we have rendered dangling edges with dotted lines and lighter colour. The copies of T are numbered from top to bottom. We also have labelled with $A, B, \dots, F$ the vertices of G that have non-dangling valency equal to 3. The other non-dangling vertices have non-dangling valency equal to 2. Note that the map φ_M is isomorphic as a DT-morphismto the map φ_A of Example 3.9 (see also Example 4.9). ⋆

We denote by nd-$E(A)$ the set of non-dangling edges of G incident to a vertex A and define the *non-dangling valency* nd-val A as $|$nd-$E(A)|$. Observe that nd-val $A = 0$ if and only if A is dangling, otherwise nd-val $A \geq 2$. In the visualization of M in Example 4.3 the darker coloured edges make up the following graph:

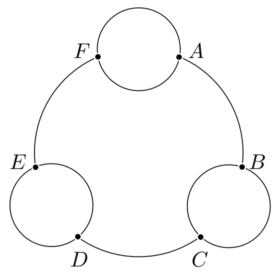
Construction 4.4. Let $H(M)$ be the following graph: $V(H(M))$ is the set of vertices A of G such that nd-val $A \geq 3$; and $E(H(M))$ is the set of paths of G whose ends are in $V(H(M))$ and interior vertices have non-dangling valency equal to 2.

Recall from Construction 3.20 that $H(\varphi_M)$ is the combinatorial type of the source of the DT-morphismφ_M associated to M. It is straightforward to see that $H(M)$ and $H(\varphi_M)$ are isomorphic. Observe also that the edge e of G is either dangling, or there is a unique $h(e)$ in $E(H(M))$ such that $e \in h(e)$. We say that h in $E(H(M))$ *passes through* a class $\overline{(x,k)}$ of G if $\overline{(x,k)} \in h$ and $\overline{(x,k)}$ is not and end of h. If h passes through $\overline{(x,k)}$, then we say that h *passes above* x. This gives a natural way to also visualize $H(M)$ above T, as in Example 4.3.

Example 4.5. The figures below illustrate G and $H(M)$ for the glueing datum M of Example 4.3. In the visualization of M in Example 4.3 we have introduced a labelling for the vertices of G with non-dangling valency greater than 2. We have used that same labelling in the figures below to facilitate comparing the visualization of M with G.



G



H ⋆

Remark 4.6. Unlike our approach here where a glueing datum gives rise to a DT-morphismφ, the definition of glueing datum in [CD18] gives rise to a tropical morphism $\Phi : \Gamma \to \Delta$. To go from a glueing datum as in [CD18] to one as in our setting we need to remove the metric information by taking a model T of Δ where for every edge t of T the relation above each interior point x in Δ_t equals some fixed relation $\sim_t$. In the terminology of the Definition 4 of [CD18] a possibility for such a model is the set of points from the essential model of Δ together with the image under Φ of the set of monovalent points in the metric forest

$$\{u \in \Gamma \mid \exists x \in \Delta \text{ with } \psi_i(x) \sim \psi_j(x) \text{ for distinct } i,j; \text{ and } u = \psi_i(x)/\sim\},$$

where the map ψ_i sents Δ to its i-th copy. △

Construction 4.7. Given a degree-d DT-morphism$\varphi : G \to T$ we construct a glueing datum M with base tree T such that φ is isomorphic to φ_M. We use four steps:

(1) Choose a leaf v of T and a map $\chi_v : \varphi^{-1}(v) \to \mathcal{P}([d])$, where $\mathcal{P}([d])$ is the power set of $\{1,\ldots,d\}$, such that $\#(\chi)_v(A) = |A|_\varphi$ for $A \in \varphi^{-1}(v)$ and $\mathrm{im}(\chi_v)$ is a partition of $[d]$.

(2) Choose incident x, x' in T such that a map χ_x has been chosen for x and no map has been chosen for x' yet.

(3) Choose a map $\chi_{x'} : \varphi^{-1}(x') \to \mathcal{P}([d])$ such that $\#(\chi)_{x'}(X') = |X'|_\varphi$ for $X' \in \varphi^{-1}(x')$, $\mathrm{im}(\chi_{x'})$ is a partition of $[d]$; and for every pair of incident elements (X', X) in $\varphi^{-1}(x') \times \varphi^{-1}(x)$ we have that $\chi_{x'}(X') \subset \chi_x(X)$ if x' is an edge, or $\chi_{x'}(X') \supset \chi_x(X)$ otherwise.

(4) Repeat step 2 and 3 until every x in T has a χ_x.

Take $M = (T, d, \sim)$, where $\sim_x$ is defined by the partition $\mathrm{im}(\chi_x)$.

In Step (3) of Construction 4.7 the choice is possible since φ satisfies the balancing condition. It is straightforward to check that the glueing datum $(T, d, \sim)$ has the properties we desire. This construction is essentially the one given in Proposition 6 of [CD18].

4.2 Isomorphism classes of DT-morphismsand glueing datums

The crucial choices in Construction 4.7 happen at Steps (1) and (3). We identify all the possibilities using a notion of isomorphism of glueing datums. It turns out that with the right notion of isomorphism there is a bijection between isomorphism classes of DT-morphismsand isomorphism classes of glueing datums. This completes the setup to study DT-morphismsvia glueing datums.

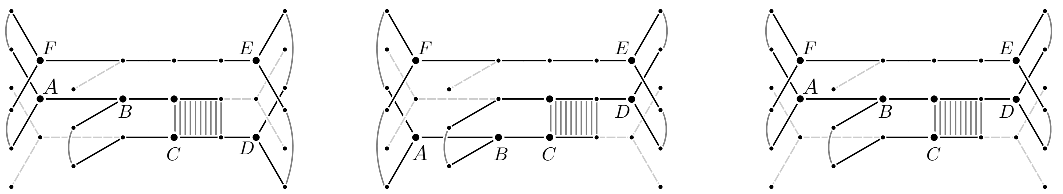
Observe that applying a fixed permutation π of $[d]$ on all relations $\sim_x$ gives a different glueing datum, but does not change the isomorphism type of G, nor the index map for the edges of G, nor which edges of G lie above a particular edge of T. We call this operation a *tree-swap*. It enables us to relate two distinct choices done at Step (1) of Construction 4.7.

Choose a vertex v of T, and let S be the vertex set of one of the connected components of $T \setminus \{v\}$. Let T_{br} be the graph induced by the vertex set $S \cup \{v\}$. We call it a *branch* of T. Choose i and j such that $i \sim_v j$ and swap i with j in all relations above T_{br}. We call this a *branch-swap*. A series of branch-swaps enables us to relate two distinct choices done at Step (3) of Construction 4.7. This motivates the notion of isomorphism of glueing datums. In the following definition we write $\overline{(x,i)}_M$ for a class of M, to avoid confusions, and in general we regard $\overline{(x,i)}_M$ as a subset of $[d]$.

Definition 4.8 (glueing datum isomorphism)**.** An isomorphism from $M = (T, d, \sim)$ to $M' = (T', d, \sim')$ consists of a graph isomorphism $\tau : T \to T'$, and for each x in T a permutation π_x of $[d]$ satisfying two properties:

1. class-preserving: $\pi_x\left(\overline{(x,i)}_M\right)$ and $\overline{(\tau(x), \pi_x(i))}_{M'}$ are equal for all x in T, i in $[d]$.
2. incidence-preserving: for all v in $V(T)$, t in $E(v)$, and i in $[d]$ we have that $\overline{(t,j)}_M \subset \overline{(v,i)}_M$ if and only if $\overline{(\tau(t), \pi_t(j))}_{M'} \subset \overline{(\tau(v), \pi_v(i))}_{M'}$.

Example 4.9. We present three isomorphic glueing datums. On the left, M from Example 4.3. Next, in M' we swapped the middle and the bottom tree. Finally, in M'' we also swapped the middle and the bottom tree, but only above the branch that is right to the vertex C. They all give rise to isomorphic DT-morphisms.



M M' M'' ⋆

Lemma 4.10. *The isomorphism classes of glueing datums are in bijection with the isomorphism classes of DT-morphismsfrom a graph to a tree.*

Proof. Let M, M' be glueing datums giving rise to graphs G, G' respectively, and $\varphi = \varphi_M$, $\varphi' = \varphi_{M'}$. First suppose that τ, $\{\pi_x\}_{x \in T}$ is an isomorphism from M to M'. Consider the natural map $\gamma : G \to G'$ given by $\gamma(\overline{(x,k)}_M) = \overline{(\tau(x), \pi_x(k))}_{M'}$. By the class-preserving property we have that γ is bijective. By the incidence-preserving property we have that both γ, γ^{-1} are graph homomorphisms. We claim that (γ, τ) is an isomorphism from φ to φ'. Diagram 3.18.a commutes because

$$\varphi' \circ \gamma\left(\overline{(x,k)}_M\right) = \varphi'\left(\overline{(\tau(x), \pi_x(k))}_{M'}\right) = \tau(x),$$
$$\tau \circ \varphi\overline{(x,k)} = \tau(x).$$

Diagram 3.18.b commutes because by class-preserving we get that

$$\left|\overline{(x,k)}_M\right|_\varphi = \#(\overline{(),x})k_M = \#(\overline{(),\tau(x)})\pi_x(k)_{M'},$$
$$\left|\gamma\left(\overline{(x,k)}_M\right)\right|_{\varphi'} = \left|\overline{(\tau(x), \pi_x(k))}_{M'}\right|_{\varphi'} = \#(\overline{(),\tau(x)})\pi_x(k)_{M'}.$$

Now suppose that γ, τ is an isomorphism from φ to φ'. Since $|\cdot|_\varphi = |\cdot|_{\varphi'} \circ \gamma$, we get that $|\overline{(x,k)}_M| = |\gamma(\overline{(x,k)}_M)|$. Thus, for each x in T we can choose a permutation π_x of $[d]$ such that $\pi_x(\overline{(x,k)}_M) = \gamma(\overline{(x,k)}_M)$. We claim that τ, $\{\pi_x\}_{x\in T}$ is an isomorphism from M to M'. We have the class-preserving property by construction. Since γ, γ^{-1} are graph homomorphisms, we have that $\overline{(t,j)}_M$, $\overline{(v,i)}_M$ are incident if and only if $\gamma(\overline{(t,j)}_M)$, $\gamma(\overline{(v,i)}_M)$ are incident. Since $\varphi' \circ \gamma = \tau \circ \varphi$, we get that $\gamma(\overline{(x,i)}) = \overline{(\tau(x), \pi_x(i))}$. The previous two facts and the construction of G and G' together imply that $\overline{(t,j)}_M \subset \overline{(v,i)}_M$ if and only if $\overline{(\tau(t), \pi_t(j))}_{M'} \subset \overline{(\tau(v), \pi_v(i))}_{M'}$.

Thus, we have shown that the map that sends the isomorphism class of M to the isomorphism class of φ_M is well defined and injective. Construction 4.7 shows that the map is also surjective. □

Now that the correspondence between DT-morphismsand glueing datums is set up, we use the following notation for convenience:

- $C_M = C_{\varphi_M}$, the cone of sources.
- $A_M = A_{\varphi_M}$, the edge-length map.
- $(M, \ell_T) = (\varphi_M, \ell_T)$, a tropical morphism.
- We say that M has full-rank, or is full-dimensional, or is a combinatorial type if φ_M has full-rank, or is full-dimensional, or is a combinatorial type, respectively.

We also define the *genus of a glueing datum* to be $g(H(M))$. Likewise, the genus of a DT-morphismis the genus of its source.

4.3 Local properties

Let $M = (T, d, \sim)$ be a genus-g glueing datum and φ be $\varphi_M : G \to T$. Ideally, a deformation procedure would modify a local part of M, namely the glueing relations above a particular edge of T and its endpoints. The resulting M' should be full-dimensional if we began with a full-dimensional M. The obstruction to this idea is that a local change in M leads to a change in A_M that may make the rank drop. Controlling the rank of A_M is a global condition, which is harder to check than local conditions.

We defer the hard problem of constructing full-dimensional glueing datums to Chapter 7, and explore a different question here: what conditions are necessary for a glueing datum to be full-dimensional. It turns out that full-dimensional glueing datums have some remarkable combinatorial properties, which are easier to check than the rank. This allows us to easily discard many potential constructions as not full-dimensional. We present these combinatorial properties in Definition 4.12, after giving some auxiliary definitions.

Definition 4.11. Let $M = (T, d, \sim)$ be a glueing datum.

- A vertex v of T is *change-minimal* if $\operatorname{ch} v + \operatorname{val} v = 3$.
- A vertex A of G satisfies the *no-return condition* if there are at least two non-dangling edges in $E(A)$ above different edges of T; in other words $|\varphi(\text{nd-}E(A))|$ is at least 2.
- Let $h = \langle A_0, e_1, \dots, e_\mu, A_\mu \rangle$ be an edge of $H(M)$ (recall that this means that h is a path in G, both ends of h have nd-val value at least 3, and the inner vertices of h have nd-val value equal to 2). We say that h satisfies the *pass-once condition* if φ restricted to the set $\{e_i \in h \mid \varphi(e_i) \text{ is not incident to a leaf}\}$ is injective.

Definition 4.12. Let $M = (T, d, \sim)$ be a glueing datum.

- M is *change-minimal* if all vertices of T are change-minimal.
- M satisfies the *dangling-no-glue* condition if $|\overline{(x,k)}| = 1$ for all dangling $|\overline{(x,k)}|$.

- M satisfies the *no-return* condition if all non-dangling vertices A of G such that $\varphi(A)$ is not monovalent satisfy the no-return condition.
- M satisfies the *pass-once* condition if all edges of $H(M)$ satisfy the pass-once condition.

We proceed to describe informally the intuition behind each property and give an example before beginning with proofs. Being change-minimal is equivalent to saying that the correction term in the dimension formula equals zero, namely that $|E(T)| = 2g + 2d - 5$. By the discussion after Definition 3.33 we have that full-dimensional implies change-minimal. Dangling-no-glue says that no identification is used in parts of G that do not affect $\ell_{H(M)}$. This is a sort of "efficiency" condition. No-return forbids edges of $H(M)$ to change copies of T in any place except above a leaf of T. Pass-once eases the calculation of A_φ (see Proposition 4.25).

Example 4.13. The glueing datum of Example 4.3 is full-dimensional (see the calculation at 5.a of Example 5.1). It can be checked that this glueing datum is change-minimal, satisfies dangling-no-glue, no-return, and pass-once; this agrees with our claim. Now we point out instances in the examples that do not fullfill some of the items in Definition 4.12.

- Consider the map φ_B from Example 3.9. The second vertex from left to right in the target is the only change-minimal vertex, and above it the no-return condition is violated. The pass-once condition is violated by the edge passing above the middle edge of the target. The dangling-no-glue condition is violated by all of the dangling elements.
- Consider the map φ_A from Example 3.22. It does not satisfy dangling-no-glue because of the leftmost vertex in the target, and is not change-minimal. It satisfies no-return and pass-once.

⋆

Instead of showing that full-dimensional implies dangling-no-glue, no-return and pass-once, we derive these conditions from being change-minimal and having full-rank. This opens the possibility of using these results in future work for studying those special graphs that have gonality less than $\lceil g/2 \rceil + 1$. We begin with a generalization of Lemma 3.24.

Lemma 4.14 (zero change)**.** *Let v be in $V(T)$, h an edge of $H(M)$ such that $\varphi(h) \cap E(v)$ contains two edges t_1 and t_2, and $A_1, \ldots, A_r$ the vertices in $h \cap \varphi^{-1}(v)$. If the following conditions are true:*

(a) $r_\varphi(A_q) = 0$ for all $1 \leq q \leq r$,

(b) the ends of h are not above v,

(c) $|e| = 1$ for dangling e in $E(A_q)$ for all $1 \leq q \leq r$,

then in the edge-length matrix we have $a_{ht_1} = a_{ht_2}$.

Before proceeding to the proof, note that the coefficient a_{ht}, for h in $E(H(M))$ and t in $E(T)$, can be calculated as follows:

$$\ell_{H(M)}(h) = \sum_{e \in E(G)\colon e \in h} \ell_G(e) = \sum_{t \in E(T)} \left(\sum_{e \in h\colon \varphi(e) = t} \frac{1}{|e|} \right) \ell_T(t) = \sum_{t \in E(T)} a_{ht} \ell_T(t),$$

where the sum in the parenthesis equals zero if the index set is empty.

Proof. Since A_q is not an end of h we have that nd-val $A_q = 2$. Let $A_q = \overline{(v, k_q)}$, and e,e' be the two edges in nd-$E(A_q)$. Lemma 3.5 and conditions (a) and (c) imply that $2|A_q| = |e| + |e'|$, and since $|A_q| \geq |e|, |e|$ we get that $|A_q| = |e| = |e'|$, which makes the classes $\overline{(t_1, k_q)}$, A_q, $\overline{(t_2, k_q)}$ equal as subsets of $[d]$. Thus, $a_{ht_1} = 1/\overline{(t_1, k_1)} + \cdots + 1/\overline{(t_1, k_r)} = 1/\overline{(t_2, k_1)} + \cdots + 1/\overline{(t_2, k_r)} = a_{ht_2}$. □

Note that Condition (c) of Lemma 4.15 is implied by dangling-no-glue. Recall that if $\operatorname{val} v = 2$, then $r_\varphi(A_q) = 0$ gives that $\operatorname{val} A_q = 2$. Hence, A_q is not an end of h, and there are no dangling edges incident to A_q, fulfilling Conditions (b) and (c). Now we prove dangling-no-glue.

Remark 4.15 (change-minimal leaves)**.** Let v in $V(T)$ be a leaf that is change-minimal, A in $\varphi^{-1}(v)$ non-dangling, and t the edge incident to v. Since A is non-dangling $\operatorname{val} A \geq 2$, and as $\operatorname{val} v = 1$ we get that $|A| \geq \operatorname{val} A$. Thus, $r_\varphi(A) = \operatorname{val} A - 2 - |A|(\operatorname{val} v - 2) \geq 2$. Since $\operatorname{ch} v = 2$ we get $r_\varphi(A) = 2$, $\operatorname{val} A = 2$, and $|A| = 2$. Thus, there is a unique non-dangling vertex above v, $\sim_t$ is trivial, and exactly one edge of $H(M)$ passes above v and t.

This implies that in the edge-length matrix A_M the column corresponding to t has only one non-zero entry; it is a 2 in the row corresponding to the edge of $H(M)$ passing above v. Thus, if M is change-minimal and has full-rank, then each edge of $H(M)$ passes above at most one leaf; for otherwise A_M would have two equal columns, a contradiction. △

Lemma 4.16. *Let M be change-minimal and full-rank, and A in $V(G)$ dangling. Then $r_\varphi(A) = 0$.*

Proof. Let $v = \varphi(A)$. By change-minimal $r_\varphi(A)$ equals 0, 1, or 2. If $r_\varphi(A) = 1$, then Remark 4.15 (change-minimal leaves) implies that $\operatorname{val} v = 2$, and so $\operatorname{ch} v = 1$. Thus, if h in $E(H(M))$ passes through A' above w_0, then $r_\varphi(A') = 0$, because A' is non-dangling, so $A' \neq A$ and $r_\varphi(A') + r_\varphi(A) \leq \operatorname{ch} v = 1$. Hence, Lemma 4.14 (zero change) implies that the columns of A_M corresponding to t_1, t_2 incident to v are equal, contradicting that M has full-rank. If $r_\varphi(A) = 2$, Remark 4.15 implies that A is non-dangling, a contradiction. □

Lemma 4.17. *A change-minimal and full-rank M satisfies the dangling-no-glue condition.*

Proof. Observation I: Let A be a vertex such that $r_\varphi(A) = 0$ and all elements of $E(A)$ have cardinality 1, except possibly one element e. Then by Lemma 3.5 (formula for r_φ) we get that $r_\varphi(A) = 2(|A| - 1) - (|e| - 1)$. As $r_\varphi(A) = 0$ we get $|e| + 1 = 2|A|$. Hence, $|A| = |e| = 1$ since $|A| \geq |e|$.

Now suppose that e_0 is dangling and $|e_0| > 1$. Choose an end A_0 of e_0. If A_0 is dangling then Lemma 4.16 allows us to apply Observation I to choose e_1 in $E(A_0)$, distinct from e_0, such that $|e_1| > 1$. Repeat the previous step with e_1, and continue to construct a sequence of vertices A_0, A_1, and so on; since G is finite, either we arrive to a non-dangling A_r, or a vertex repeats in the sequence. Either way this means that after deleting e_0 the connected component of A_0 has a cycle. This argument can be applied to the other end of e_0 as well, giving a contradiction that e_0 is dangling. Thus, if e is dangling, then $|e| = 1$. If A is dangling, all edges in $E(A)$ are dangling, so they have cardinality 1, and the result follows from another application of Lemma 4.16 and Observation I. □

Now we prove the no-return condition.

Lemma 4.18 (nd. r_φ formula)**.** *Let M satisfy dangling-no-glue, and A be in $V(G)$. If $e_1, e_2, \ldots, e_r$ in $E(G)$ are the non-dangling edges incident to A, then*

$$r_\varphi(A) = \text{nd-val}\, A - 2 + 2|A| - (|e_1| + \ldots + |e_r|).$$

Proof. This is a consequence of Lemma 3.5 and the dangling-no-glue condition. □

Lemma 4.19 (r1 implies no-return)**.** *Let M satisfy dangling-no-glue, and A in $V(G)$ be non-dangling with $\operatorname{val} \varphi(A) \geq 2$. If $r_\varphi(A) \leq 1$, then A satisfies no-return.*

Proof. If $|A| = 1$ the result is clear. Assume that $|A| \geq 2$. Let $e_1, \ldots, e_r$ be the edges in nd-$E(A)$. Note that $r \geq 2$, since A is non-dangling. Assume that all e_q are above the same edge t, so $|e_1| + \ldots + |e_r| \leq |A|$. Lemma 4.18 (nd. r_φ formula) implies that $r_\varphi(A) = r - 2 + 2|A| - (|e_1| + \ldots + |e_r|) \geq r - 2 + |A| \geq 2$, a contradiction. □

Corollary 4.20. *A change-minimal M that satisfies the dangling-no-glue condition also satisfies the no-return condition.*

Proof. Since M is change-minimal by Remark 4.15 if $r_\varphi(A) \geq 2$, then $\varphi(A)$ is a leaf. The corollary follows then from Lemma 4.19. □

By dangling-no-glue, G looks like T locally around dangling vertices. We now give a local description for non-dangling vertices, assuming a change-minimal M. To make coming references to this result easier, we state all the possible cases, though several are straightforward corollaries of previous results.

Proposition 4.21 (local properties)**.** *Let M be change-minimal and satisfying dangling-no-glue, A in $V(G)$ with $\operatorname{nd-val} A$ either 2 or 3, $v = \varphi(A)$, $r = \operatorname{nd-val} A$, and $\{e_1, \ldots, e_r\} = \operatorname{nd-}E(A)$. Then exactly one of the following cases happens:*

- *(r0-nd3): $r_\varphi(A) = 0$ and $\operatorname{nd-val} A = 3$. Then φ restricted to $\operatorname{nd-}E(A)$ is injective; $\operatorname{val} v$ is 3; and $|e_1| + |e_2| + |e_3| = 2|A| + 1$.*
- *(r0-nd2): $r_\varphi(A) = 0$ and $\operatorname{nd-val} A = 2$. Then φ restricted to $\operatorname{nd-}E(A)$ is injective; $\operatorname{val} v$ is 2 or 3; and $|e_1| = |A| = |e_2|$.*
- *(r1-nd3): $r_\varphi(A) = 1$ and $\operatorname{nd-val} A = 3$. Then there is a labelling of the e_i where $\varphi(e_1) = \varphi(e_2) \neq \varphi(e_3)$; $\operatorname{val} v$ is 2; and $|e_1| + |e_2| = |A| = |e_3|$.*
- *(r1-nd2): $r_\varphi(A) = 1$ and $\operatorname{nd-val} A = 2$. Then φ restricted to $\operatorname{nd-}E(A)$ is injective; $\operatorname{val} v$ is 2; and there is a labelling of the e_i where $|e_1| = |A|$ and $|e_2| = |A| - 1$.*
- *(r2-nd2): $r_\varphi(A) = 2$. Then $\varphi(e_1) = \varphi(e_2)$; $\operatorname{val} v$ is 1; $|e_1| = |e_2| = 1$; and $|A| = 2$.*

Proof. Change-minimal implies that $r_\varphi(A)$ equals 0, 1, or 2, and that $\operatorname{val} v$ equals 1, 2, 3 for v in $V(T)$. By Remark 4.15 (change-minimal leaves) $r_\varphi(A) = 2$ and $\operatorname{nd-val} A = 3$ cannot be. So the described cases are all the possibilities. The same remark proves (r2-nd2). Note that M satisfies the no-return condition by Corollary 4.20.

For the case (r0-nd3), no-return implies that either the three non-dangling edges are above different edges of T, or (up to labelling) e_2 and e_3 are above the same edge and e_1 is above another. In the latter case Lemma 4.18 and $|e_1| \leq |A|$ imply $0 = 1+2|A|-|e_1|-|e_2|-|e_3| \geq 1+|A|-|e_2|-|e_3|$, but $|e_2| + |e_3| \leq |A|$, a contradiction. Hence e_1, e_2, e_3 are above distinct edges of T, so $\operatorname{val} v = 3$. Lemma 4.18 implies the last formula.

The remaining follows from change-minimal, dangling-no-glue, Lemma 4.18, and the glueing datum axioms. □

Corollary 4.22 (image of a path)**.** *Let M be change-minimal and satisfy dangling-no-glue. Let $P = \langle A_0, e_1, \ldots, e_\mu, A_\mu \rangle$ be a path of G such that A_i is non-dangling and $\operatorname{val} \varphi(A_i) \geq 2$ for each i. Then $\langle \varphi(A_0), \varphi(e_1), \ldots, \varphi(e_\mu), \varphi(A_\mu) \rangle$ is a path of T.*

Proof. Since φ is a graph morphism, and by the local properties of Proposition 4.21 , we get that consecutive elements of $\varphi(P) := \langle \varphi(A_0), \varphi(e_1), \ldots, \varphi(e_\mu), \varphi(A_\mu) \rangle$ are incident, and distinct. Since consecutive elements of $\varphi(P)$ are distinct, if $\varphi(P)$ repeats an element, then we get a cycle. As T is a tree we conclude that $\varphi(P)$ is a path. □

Now we move on to the pass-once condition.

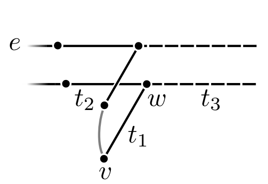
Lemma 4.23. *Let M be change-minimal and have full-rank, u and v be divalent and monovalent vertices of T, respectively, with $E(u) = \{t', t\}$, $E(v) = \{t\}$. Then there is exactly one edge of H above t and one above t'; they are a loop and a bridge, respectively.*

Proof. Swap trees so that $1 \sim_v 2$. Remark 4.15 (change-minimal leaves) yields that $e_1 = \overline{(t,1)}$ and $e_2 = \overline{(t,2)}$ are two distinct edges of G, with $|e_1| = |e_2| = 1$, and are the only non-dangling edges above t. This implies, by the no-return condition, that $\overline{(u,1)}$ and $\overline{(u,2)}$ are the only non-dangling classes above u (it is possible that $\overline{(u,1)} = \overline{(u,2)}$). Let h be the edge of H containing e_1, e_2. If both $\overline{(u,1)}$ and $\overline{(u,2)}$ have non-dangling valency equal to 2, then the only edge of H passing above t, and t' is h. Thus, a_{ht} and $a_{ht'}$ are the only non-zero entries in the columns of A_M corresponding to t and t', respectively, a contradiction. Therefore, assume without loss of generality that $\operatorname{nd-val} \overline{(u,1)} = 3$. As $\operatorname{val} u = 2$ and $|e_1| = 1$, by Case (r1-nd3) of the local properties we must have that $\operatorname{nd-}E(\overline{(u,1)})$ has two edges above t; namely e_1, e_2, that make the loop; and one edge above t', the bridge. □

Lemma 4.24. *A change-minimal and full-rank M satisfies the pass-once condition*

Proof. Let $h = \langle A_0, e_1, \dots, e_\mu, A_\mu \rangle$ be an edge of $H(M)$. If h does not pass above a leaf, then we are done by Corollary 4.22. Assume h passes above a leaf. By Remark 4.15 (change-minimal leaves) there is a single vertex $A_i \in h$ such that $v := \varphi(A_i)$ is a leaf. Thus, h passes at most twice above any edge of T by Corollary 4.22. We also get that $\varphi(e_i) = \varphi(e_{i+1})$ from Remark 4.15. Let $t_1 = \varphi(e_i)$. Suppose that h violates the pass-once condition. That is, for some $j, k \notin \{i, i+1\}$ we have $\varphi(e_j) = \varphi(e_k)$. We get that $\varphi(e_{i-1}) = \varphi(e_{i+2})$, because otherwise $\varphi(\langle A_0, \dots A_i \rangle)$ and $\varphi(\langle A_i, \dots A_\mu \rangle)$ would intersect only at t_1 by Corollary 4.22 and the fact that T is a tree, which contradicts that $\varphi(e_j) = \varphi(e_k)$. Let $t_2 = \varphi(e_{i-1})$.

Label the copies of T such that $1 \sim_v 2$. Label h with 1. Lemma 4.23 implies that $\operatorname{val} w = 3$. Let w be the other end of t_1, and t_3 be the other edge incident to w. We investigate the classes above t_1, t_2 and t_3. Observe that $\overline{(t_3, 1)}$ and $\overline{(t_3, 2)}$ are dangling. Change-minimal and Remark 4.15 imply that $|\overline{(t_1, 1)}| = |\overline{(t_1, 2)}| = 1$. All the vertices of G above w have r_φ-value zero. Thus, case (r0-nd2) of the local properties on $\overline{(w, 1)}$ and $\overline{(w, 2)}$ implies that $|\overline{(t_2, 1)}| = |\overline{(t_2, 2)}| = 1$. Hence, $a_{1,1} = 2, a_{1,2} = 2, a_{1,3} = 0$. As $\overline{(t_1, k)}$ is dangling for $k \geq 3$, the other non-dangling vertices above w have nd-val equal to 2. Lemma 4.14 (zero change) implies that $a_{i,2} = a_{i,3}$ for $i \geq 2$. See the figure below, dangling edges shown dotted.



$a_{1,1} = 2 \qquad a_{1,2} = 2 \qquad a_{1,3} = 0$

$a_{i,1} = 0, \qquad a_{i,2} = a_{i,3},$

for $i \geq 2$.

Let $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{a}_3$ be the first three columns of A_M. Note that $\mathbf{a}_2 = \mathbf{a}_1 + \mathbf{a}_3$, contradicting that M has full-rank. □

Proposition 4.25 (edge-length map is local)**.** *Let $M = (T, d, \sim)$ satisfy that the leaves of T are change-minimal. Then M satisfies the pass-once condition if and only if for all h in $E(H)$, and t in $E(T)$ the entry a_{ht} is:*

(a) 2 if h passes above a leaf v in $V(T)$, and t is incident to v.

(b) $1/|\overline{(t, k)}|$ if h passes through some edge $\overline{(t, k)}$ of G, t not incident to a leaf.

(c) 0 otherwise.

Proof. Part (a) follows from Remark 4.15. Part (b) is true if and only if pass-once is. □

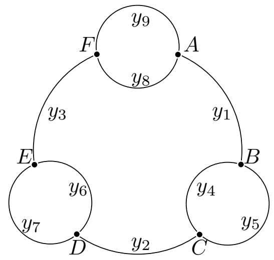
Chapter 5

Example of deformation

5.1 A movie

We now give the motivating example of the deformation process. Most cases involved in deformation can be observed in this example.

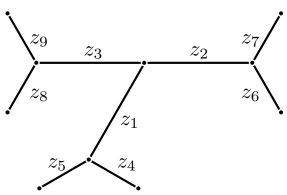
Example 5.1 (3 loops on a loop)**.** Let $\widetilde{\Gamma} = (H, \ell_H)$ be a loop of 3 loops (as in [LPP12]), with H, vertex and edge labellings shown below. Set $y_i = \ell_H(h_i)$, $z_i = \ell_T(t_i)$.



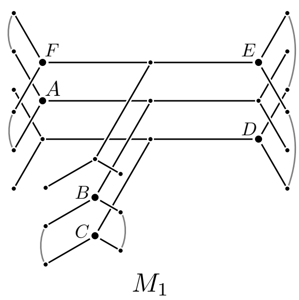
H

Assume without loss of generality that $y_3 = \max(y_1, y_2, y_3)$, $y_4 \leq y_5$, $y_6 \leq y_7$, and $y_8 \leq y_9$. We show 9 cases that depend on how large the length y_3 is compared to several other linear combinations of lengths. Each case depicts: the base tree T with an edge labelling; a glueing datum with combinatorial type H; and its edge-length matrix. The odd cases are full-dimensional, the even cases are not. In two consecutive odd cases the edge-length matrices differ only by one column. We compute $\det A_M$ to show these are indeed full-dimensional.

- First case: $y_3 < y_1 + y_2$.



T_1



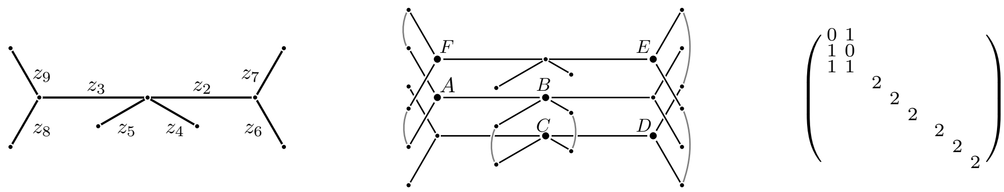
M_1

$$\begin{pmatrix} 1 & 0 & 1 & & & & & & \\ 1 & 1 & 0 & & & & & & \\ 0 & 1 & 1 & & & & & & \\ & & & 2 & & & & & \\ & & & & 2 & & & & \\ & & & & & 2 & & & \\ & & & & & & 2 & & \\ & & & & & & & 2 & \\ & & & & & & & & 2 \end{pmatrix}$$

A_1

$\det A_1 = 128$

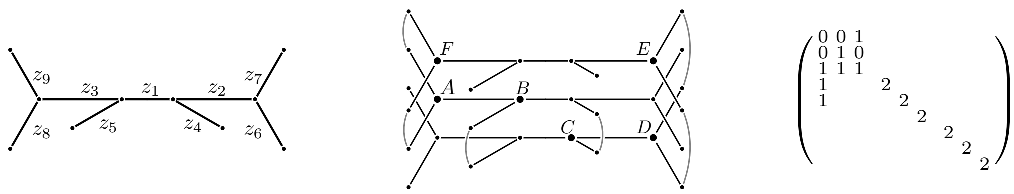
- Second case: $y_3 = y_1 + y_2$. The edge t_1 gets shrunk.



T_2 M_2 A_2

- Third case: $y_1 + y_2 < y_3 < y_1 + y_2 + y_4$.

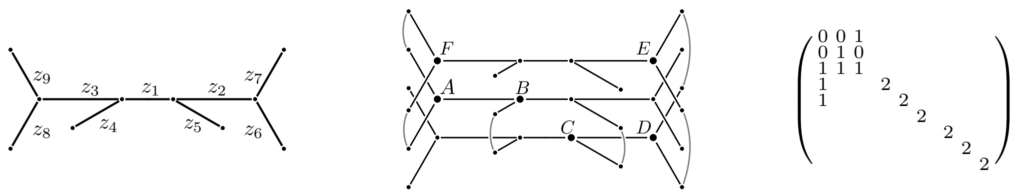
  Case 3.a



$T_3^{(a)}$ M_3 A_3

$\det A_3 = -64$

Case 3.b



$T_3^{(b)}$ M_3 A_3

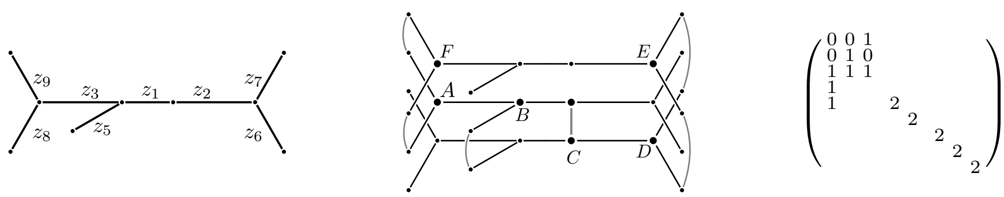
Here $\widetilde{\Gamma}$ is realized by two tropical morphisms (distinct ones in general, see the next paragraph), yet at the combinatorial level the glueing datum is the same. We drew M_3 twice to emphasize where is the longer edge and show the labelling of the base tree. The two length functions $z^{(a)}$, $z^{(b)}$ are related by swapping z_4 and z_5.

We claim that $\Phi^{(a)} = (\varphi_3,\ z^{(a)})$ and $\Phi^{(b)} = (\varphi_3,\ z^{(b)})$ are distinct tropical morphisms if and only if z_4, z_5 are distinct lengths. Set $v = \varphi(B)$. Since z_4, z_5 are distinct, one of the two edges between B, C is longer than the other, and there is a preimage of v under $\Phi^{(a)}$ in this longer edge. This is not true for $\Phi^{(b)}$. Conversely, it is clear that if z_4, z_5 are equal then swapping the lengths has no effect.

This situation persists for the remaining glueing datums in this example.

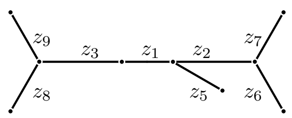
- Fourth case: $y_3 = y_1 + y_2 + y_4$. The edge t_4 gets shrunk. The drawing shows two different glueing datums (which reflects a bifurcation in the shrinking and regrowing sequence so far), but by swapping some branches one can see them to be isomorphic, so we label both by M_4. Same with the rest of the cases.

  Case 4.a

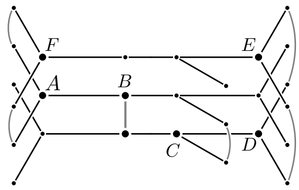


$T_4^{(a)}$ M_4 A_4

Case 4.b



$T_4^{(b)}$

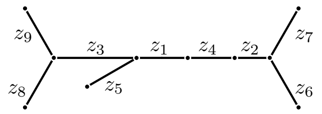


M_4

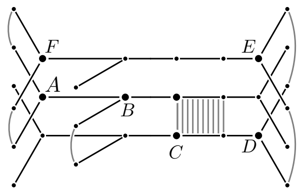
$$A_4 = \begin{pmatrix} 0&0&1&&&&&\\ 0&1&0&&&&&\\ 1&1&1&&&&&\\ 1&&&&&&&\\ 1&&&2&&&&\\ &&&&2&&&\\ &&&&&2&&\\ &&&&&&2&\\ &&&&&&&2 \end{pmatrix}$$

- Fifth case:

  Case 5.a: $y_1 + y_2 + y_4 < y_3 < y_1 + 2y_2 + y_4$.



$T_5^{(a)}$

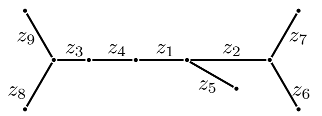


M_5

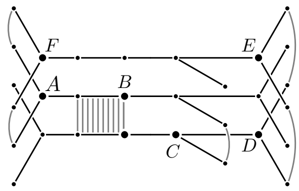
$$A_5 = \begin{pmatrix} 0&0&1&0&&&&&\\ 0&1&0&1/2&&&&&\\ 1&1&1&1&&&&&\\ 1&&&&&&&&\\ 1&&&&2&&&&\\ &&&&&2&&&\\ &&&&&&2&&\\ &&&&&&&2&\\ &&&&&&&&2 \end{pmatrix}$$

$\det A_5 = 16$

Case 5.b: $y_1 + y_2 + y_4 < y_3 < 2y_1 + y_2 + y_4$.



$T_5^{(b)}$

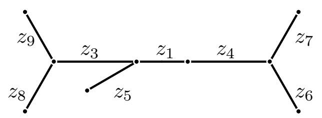


M_5

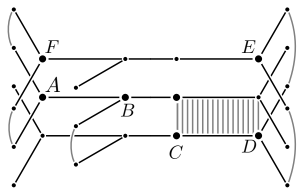
$$A_5 = \begin{pmatrix} 0&0&1&1/2&&&&&\\ 0&1&0&0&&&&&\\ 1&1&1&1&&&&&\\ 1&&&&&&&&\\ 1&&&&2&&&&\\ &&&&&2&&&\\ &&&&&&2&&\\ &&&&&&&2&\\ &&&&&&&&2 \end{pmatrix}$$

- Sixth case:

  Case 6.a: $y_3 = y_1 + 2y_2 + y_4$. The edge t_2 gets shrunk.



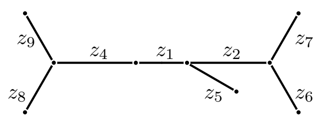
$T_6^{(a)}$



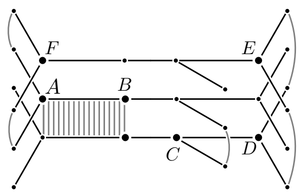
M_6

$$A_6 = \begin{pmatrix} 0&1&0&&&&&\\ 0&0&1/2&&&&&\\ 1&1&1&&&&&\\ 1&&&&&&&\\ 1&&&2&&&&\\ &&&&2&&&\\ &&&&&2&&\\ &&&&&&2&\\ &&&&&&&2 \end{pmatrix}$$

Case 6.b: $y_3 = 2y_1 + y_2 + y_4$. The edge t_3 gets shrunk.



$T_6^{(b)}$

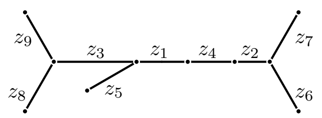


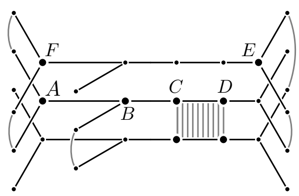
M_6

$$A_6 = \begin{pmatrix} 0&0&1/2&&&&&\\ 0&1&0&&&&&\\ 1&1&1&&&&&\\ 1&&&&&&&\\ 1&&&2&&&&\\ &&&&2&&&\\ &&&&&2&&\\ &&&&&&2&\\ &&&&&&&2 \end{pmatrix}$$

- Seventh case:

  Case 7.a: $y_1 + 2y_2 + y_4 < y_3 < y_1 + 2y_2 + y_4 + y_6$. In 6.a swap 2 and 3 in the branch above t_6. Regrow t_2 in the same place to obtain 7.a.



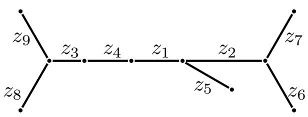


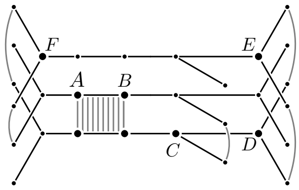
$$\begin{pmatrix} 0&0&1&0&&&&&\\ 0&0&0&1/2&&&&&\\ 1&1&1&1&&&&&\\ 1&&&&&&&&\\ 1&&&&2&&&&\\ &1&&&&2&&&\\ &1&&&&&2&&\\ &&&&&&&2&\\ &&&&&&&&2 \end{pmatrix}$$

$T_7^{(a)}$ M_7 A_7

$\det A_7 = -16$

Case 7.b: $2y_1 + y_2 + y_4 < y_3 < 2y_1 + y_2 + y_4 + y_8$. In 6.b swap 2 and 3 in the branch above t_9. Regrow t_2 in the same place to obtain 7.b.



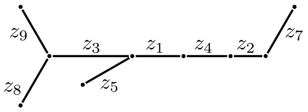


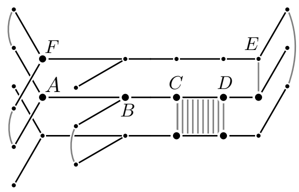
$$\begin{pmatrix} 0&0&0&1/2&&&&& \\ 0&1&0&0&&&&& \\ 1&1&1&1&&&&& \\ 1&&&&&&&& \\ 1&&&&2&&&& \\ &&&&&2&&& \\ &&&&&&2&& \\ &&1&&&&&2& \\ &&1&&&&&&2 \end{pmatrix}$$

$T_7^{(b)}$ M_7 A_7

- Eighth case:

Case 8.a: $y_3 = y_1 + 2y_2 + y_4 + y_6$. The edge t_6 gets shrunk.

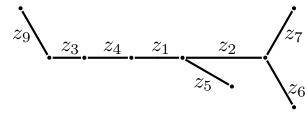


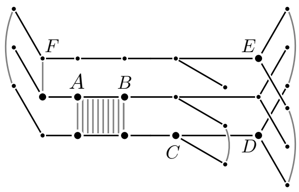


$$\begin{pmatrix} 0&0&1&0&&&& \\ 0&0&0&1/2&&&& \\ 1&1&1&1&&&& \\ 1&&&&&&& \\ 1&&&&2&&& \\ &1&&&&&& \\ &1&&&&2&& \\ &&&&&&2& \\ &&&&&&&2 \end{pmatrix}$$

$T_8^{(a)}$ M_8 A_8

Case 8.b: $y_3 = 2y_1 + y_2 + y_4 + y_8$. The edge t_8 gets shrunk.



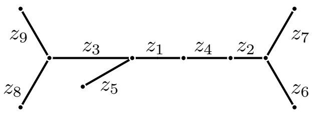


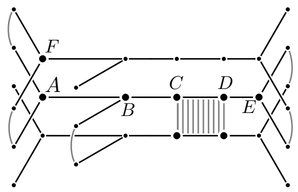
$$\begin{pmatrix} 0&0&0&1/2&&&& \\ 0&1&0&0&&&& \\ 1&1&1&1&&&& \\ 1&&&&&&& \\ 1&&&&2&&& \\ &&&&&2&& \\ &&&&&&2& \\ &&1&&&&& \\ &&1&&&&&2 \end{pmatrix}$$

$T_8^{(b)}$ M_8 A_8

- Ninth case:

Case 9.a: $y_1 + 2y_2 + y_4 + y_6 < y_3$.



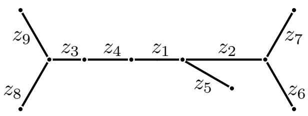


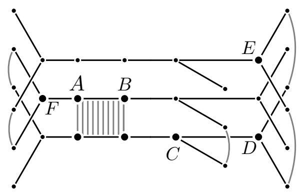
$$\begin{pmatrix} 0&0&1&0&&&&& \\ 0&0&0&1/2&&&&& \\ 1&1&1&1&&2&&& \\ 1&&&&&&&& \\ 1&&&&2&&&& \\ &1&&&&&&& \\ &1&&&&&2&& \\ &&&&&&&2& \\ &&&&&&&&2 \end{pmatrix}$$

$T_9^{(a)}$ M_9 A_9

$\det A_9 = 16$

Case 9.b: $2y_1 + y_2 + y_4 + y_8 < y_3$.





$$\begin{pmatrix} 0&0&0&1/2&&&&& \\ 0&1&0&0&&&&& \\ 1&1&1&1&&&&2& \\ 1&&&&&&&& \\ 1&&&&2&&&& \\ &&&&&2&&& \\ &&&&&&2&& \\ &&1&&&&&& \\ &&1&&&&&&2 \end{pmatrix}$$

$T_9^{(b)}$ M_9 A_9

⋆

Chapter 6

Deformation

We visualize Example 5.1 as a *movie* featuring a continuous deformation of the length function ℓ_H (for an actual movie go here[1]). The deformation path grows the length y_3 while leaving fixed the remaining lengths. The path moves in and through nine different cones C_M. We use the full-rank property of A_M to calculate $\ell_T = A_M^{-1}\ell_H$, which defines the tropical morphism (M, ℓ_T). As we grow the length y_3, some lengths of ℓ_T shrink down to zero. We call the cases where a length of ℓ_T is zero a *limit* in the deformation. Since we do not allow zero lengths in an edge-length function, we contract an edge of T instead.

If we were to walk further past a limit while using the same glueing datum we would get negative entries in $A_M^{-1}\ell_H$, which is not allowed in a length function. In order to go beyond, we pass to a different glueing datum. In Example 5.1 the even cases are *limit glueing datums*, as an edge is contracted. Around these limits there is a change in the glueing relations. In this chapter we show that understanding this change of glueing relations sets up a deformation procedure and gives a proof of Theorem 1.4.

6.1 Limits

We set up a framework to study limits. Let $M = (T, d, \sim)$ be a glueing datum, and t an edge of T.

Definition 6.1 (Limit glueing datum)**.** The *limit* of M by contracting t, in short *the limit at* t, is the glueing datum $M_0 = (T_0, d, \sim_0)$ given by the following data:

- T_0 is obtained by contracting t in T. The ends u and v of t get identified with a vertex w_0 of T_0. Edges and vertices of T different from u, v, and t correspond in a natural manner with edges and vertices of T_0.
- $\sim_{0,w_0}$ equals the finest common coarsening of $\sim_u$ and $\sim_v$; and $\sim_{0,x}$ equals $\sim_x$ for $x \neq w_0$.

For the remainder of this chapter, let $M_0 = (T_0, d, \sim_0)$ be the limit at t of M, giving rise to G_0, and u, v be the ends of t that contract to w_0 in $V(T_0)$. Fix an edge t of T. Let $\varphi_0 : G_0 \to T_0$ be the DT-morphismthat arises from $\varphi_M : G \to T$ by contracting t in T to obtain T_0, and contracting all the edges in $\varphi^{-1}(t)$ to obtain G_0. It is straightforward to see that φ_0 is canonically isomorphic to φ_{M_0}, where M_0 is the limit of M at t. Namely, the classes representing vertices and edges of G *contract* to the classes representing vertices and edges of G_0. Note that a class $\overline{(x, i)}$ of G contracts to $\overline{(x_0, j)}$ of G_0 if and only if $\overline{(x, i)} \subseteq \overline{(x_0, j)}$ and x contracts to x_0.

To prove that M_0 is indeed a glueing datum, we observe that verticality, refinement and connectedness are inherited. We write r_0 for r_{φ_0}. The RH-condition amounts to proving that $r_0(A_0) \geq 0$ for all A_0 in $V(G_0)$. This is true because r_0 is additive under contraction:

[1]https://mathsites.unibe.ch/jdraisma/MovieGenus4.mp4

Proposition 6.2 (r_φ under contraction)**.** *Let A_0 be in $\varphi_0^{-1}(w_0)$, and $A_1, \dots, A_r$ be the vertices of G that contract to A_0. If $g(H(M)) = g(H(M_0))$, then:*

$$r_0(A_0) = r_\varphi(A_1) + \cdots + r_\varphi(A_r),$$
$$\mathrm{ch}_{M_0} w_0 = \mathrm{ch}_M u + \mathrm{ch}_M v.$$

Proof. Let G_A be the subgraph defined by the vertices and edges of G that contract to A_0. Namely, $V(G_A) = \{A_1, \dots, A_r\}$. It is a forest; otherwise we would have $g(H(M_0)) < g(H(M))$; and also connected, hence a tree. Restrict φ to G_A to get a map φ_A onto the graph on $\{u, v\}$, joined by the single edge t. This map is a DT-morphismof degree $|A_0|$. By Lemma 3.27 the total change of φ_A is $\sum_{B \in V(G_A)} r_{\varphi_A}(B) = 2|A_0| - 2$. Using Lemma 3.5 we calculate:

$$\begin{aligned}
\sum_{i=1}^{r} r_\varphi(A_i) &= \sum_{i=1}^{r}(2|A_i| - 2) - \sum_{i=1}^{r} \sum_{\substack{e \in E(A_i) \\ \varphi(e) \neq t}} (|e| - 1) - 2 \cdot \sum_{e \in E(G_A)} (|e| - 1) \\
&= \sum_{B \in V(G_A)} r_{\varphi_A}(B) - \sum_{i=1}^{r} \sum_{\substack{e \in E(A_i) \\ \varphi(e) \neq t}} (|e| - 1) \\
&= 2|A_0| - 2 - \sum_{e \in E(A_0)} (|e| - 1) = r_0(A).
\end{aligned}$$

Apply this formula over all vertices above w_0 to get the second result. □

6.2 Labelling limits

Note that the glueing relations of M_0 agree with those of M outside of w_0. We exploit this fact to compare the edge-length matrices. Recall that a labelling on M is a pair of injective functions $\lambda_T : E(T) \to \mathbb{Z}_{>0}$ and $\lambda_H : E(H(M)) \to \mathbb{Z}_{>0}$; we use them to induce total orders on $E(T)$ and $E(H(M))$; to get ordered bases on $\mathbb{R}^{E(T)}$ and $\mathbb{R}^{E(H(M))}$; and to write down the edge-length matrix A_M. We give a canonical way to induce a labelling on M_0 from a labelling on M, which allows the comparison of A_M and A_{M_0}.

Since G_0 arises from contracting the edges $\varphi^{-1}(t)$ in G, consider the edges h of $H(M)$ such that $h \subset \varphi^{-1}(u) \cup \varphi^{-1}(t) \cup \varphi^{-1}(v)$ and contract them. The resulting graph H_0 is canonically isomorphic to $H(M_0)$ (for h_0 in $E(H_0)$, choose $e \in h_0$ and map h_0 to the edge of $H(M_0)$ that contains the edge corresponding to e). Let ρ_H be the morphism $H(M) \to H_0 \to H(M_0)$, where the first map is the contraction morphism and the second map is the isomorphism we just described.

This gives the canonical embedding $\mathbb{R}^{H(M_0)} \hookrightarrow \mathbb{R}^{H(M)}$ as the injective linear map that sends $y \in C_{H(M_0)}$ to $y \circ \rho_H \in C_{H(M)}$ (for this to work, we extend $\ell \in C_G$ to be a function on $V(G) \cup E(G)$, with the convention that $\ell(v) = 0$ for all v in $V(G)$). Likewise we construct the contraction morphism $\rho_T : T \to T_0$ for the base trees, fulfilling similar properties. Since the glueing relations of M_0 agree with those of M outside of w_0, the following diagram commutes.

$$\begin{array}{ccc}
\mathbb{R}^{E(T_0)} & \hookrightarrow & \mathbb{R}^{E(T)} \\
\downarrow{\scriptstyle A_{M_0}} & & \downarrow{\scriptstyle A_M} \\
\mathbb{R}^{E(H(M_0))} & \hookrightarrow & \mathbb{R}^{E(H(M))}
\end{array}$$

Diagram 6.3

Lemma 6.4. *The canonical embedding $\mathbb{R}^{H(M_0)} \hookrightarrow \mathbb{R}^{H(M)}$*

Thus, if M_0 is the limit of M at t_k

Definition 6.5. Let M be a glueing datum and (λ_T, λ_H) be a labelling on M. The *induced labelling* on a limit M_0 is the labelling $(\lambda_T \circ \rho_T^{-1}, \lambda_H \circ \rho_H^{-1})$.

Lemma 6.6 (limit matrix)**.** *Let M be a glueing datum with labelling (λ_T, λ_H), and M_0 be the limit at t with the labelling induced from M. Then A_{M_0} is equal to the deletion of the k-th column of A_M.*

Proof. Restrict the edge-length map of M to the subspace $\{x_k = 0\}$ to get the edge-length map of M_0. By the choice of order on the bases, this amounts to deleting the k-th column of A_M. □

Lemma 6.6 immediately implies that the following properties, which depend on A_M, are inherited by limits.

Lemma 6.7. *If M has full-rank, then every limit of M has full-rank.*

Lemma 6.8. *If M satisfies pass-once, then every limit of M satisfies pass-once.*

Proof. This is a consequence of Lemma 6.6 and Proposition 4.25. □

6.3 Star of M_0

We also wish to compare $A_{M^{(a)}}$ and $A_{M^{(b)}}$ if $M^{(a)}$ and $M^{(b)}$ have limits isomorphic to M_0; namely, if $M^{(a)}$ and $M^{(b)}$ belong to the following set:

Definition 6.9 (star of M_0)**.** Let M_0 be a full-rank glueing datum. Denote by $\mathrm{Star}(M_0)$ the set of isomorphism classes of full-rank glueing datums that contract to a limit in the isomorphism class of M_0.

Now consider $M^{(a)}$ and $M^{(b)}$ glueing datums such that the limits $M_0^{(a)}$ and $M_0^{(b)}$ at $t^{(a)}$ and $t^{(b)}$, respectively, are isomorphic. Let (γ_0, τ_0) be an isomorphism from $M_0^{(a)}$ to $M_0^{(b)}$. Note that γ_0 descends to an isomorphism $\widetilde{\gamma}_0 : H(M_0^{(a)}) \to H(M_0^{(b)})$. If each of ρ_T, ρ_H contracts at most one edge, then there are unique isomorphisms τ and $\widetilde{\gamma}$, which we call *canonical isomorphisms*, such that the following diagrams commute:

$$\begin{array}{ccc} T^{(a)} & \xrightarrow{\rho_T^{(a)}} & T_0^{(a)} \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\tau_0} \\ T^{(b)} & \xrightarrow{\rho_T^{(b)}} & T_0^{(b)} \end{array} \qquad\qquad \begin{array}{ccc} H^{(a)} & \xrightarrow{\rho_H^{(a)}} & H_0^{(a)} \\ {\scriptstyle\widetilde{\gamma}}\downarrow & & \downarrow{\scriptstyle\widetilde{\gamma}_0} \\ H^{(b)} & \xrightarrow{\rho_H^{(b)}} & H_0^{(b)} \end{array}$$

(a) (b)

Diagram 6.10

Note that if the edge h of $H(M)$ is contracted in the limit, then the h-row of A_M has only one non-zero entry; namely the entry in the t-column. Two such rows would be linearly dependent. Thus,

Lemma 6.11. *If M is full-rank, then at most one edge is contracted by ρ_H.*

So we have canonical isomorphisms τ, $\widetilde{\gamma}$ between any two elements of $\mathrm{Star}(M_0)$. Hence, a labelling $(\lambda_T^{(a)}, \lambda_H^{(a)})$ on $M^{(a)}$ induces the labelling $(\lambda_T^{(a)} \circ \tau^{-1}, \lambda_H^{(a)} \circ \widetilde{\gamma}^{-1})$ on $M^{(b)}$. Note that if the k-th edge is contracted in $M^{(a)}$ to obtain $M_0^{(a)}$, then in the induced labelling on $M^{(b)}$ also the k-th edge is contracted.

Definition 6.12. We say that some labellings on $M^{(a)}$, $M^{(b)}$ are *compatible at t_k* if one is induced by the other upon choosing an isomorphism from the limit $M_0^{(a)}$ of $M^{(a)}$ at t_k to the limit $M_0^{(b)}$ of $M^{(b)}$ at t_k.

Lemma 6.13 (matrices in star)**.** *Let $M^{(a)}$, $M^{(b)}$ be glueing datums with labellings compatible at t_k, and edge-length matrices (a_{ij}), (b_{ij}), respectively. Then $a_{ij} = b_{ij}$ for $j \neq k$.*

Proof. On the one hand, by Lemma 6.6, the edge-length matrices of $M_0^{(a)}$, $M_0^{(b)}$ coincide with the deletion of the k-th column of (a_{ij}), (b_{ij}). On the other hand, since Diagram 6.10 commutes, we can calculate $\lambda_{0,H}^{(b)} = \lambda_H^{(b)} \circ \left(\rho_H^{(b)}\right)^{-1} = \lambda_H^{(a)} \circ \widetilde{\gamma}^{-1} \circ \left(\rho_H^{(b)}\right)^{-1} = \lambda_H^{(a)} \circ \left(\rho_H^{(b)} \circ \widetilde{\gamma}\right)^{-1} = \lambda_H^{(a)} \circ \left(\widetilde{\gamma}_0 \circ \rho_H^{(a)}\right)^{-1} = \lambda_H^{(a)} \circ \left(\rho_H^{(a)}\right)^{-1} \circ \widetilde{\gamma}_0^{-1} = \lambda_{0,H}^{(a)} \circ \widetilde{\gamma}_0^{-1}$. Similarly, $\lambda_{0,T}^{(b)} = \lambda_{0,T}^{(a)} \circ \tau_0^{-1}$. In short, the labellings induced on the limits coincide. So the matrices $M_0^{(a)}$, $M_0^{(b)}$ are equal. □

6.4 Inherited properties

Now we study which other properties of Definition 4.12 are inherited by limits. Let M_0, G_0, u, v, w_0, φ_0 and r_0 be as in Section 6.1.

Lemma 6.14 (dangling in the limit)**.** *If $g(H(M)) = g(H(M_0))$, then a vertex A_0 in $\varphi_0^{-1}(w_0)$ is dangling if and only if all the $A_1, \ldots, A_r$ in $V(G)$ that contract to A_0 are dangling.*

Proof. Since $g(H(M)) = g(H(M_0))$, no cycle of G is contracted in G_0. This implies that A_0 is not contained in a cycle of G_0 if and only if none of $A_1, \ldots, A_r$ are contained in a cycle of G. □

Lemma 6.15. *If $g(H(M)) = g(H(M_0))$ and M satisfies the dangling-no-glue condition, then M_0 satisfies the dangling-no-glue condition.*

Proof. We only need to check that a dangling vertex A_0 above w_0 satisfies $|A_0| = 1$. Since $\sim_{w_0}$ is the finest common coarsening of $\sim_u$ and $\sim_v$, we have $|A_0| > 1$ only if for some A in $V(G)$ contracting to A_0 we have $|A| > 1$. By Lemma 6.14 there is no such A. □

Lemma 6.16 (non-dangling union)**.** *Assume that $g(H(M)) = g(H(M_0))$. Let M satisfy the dangling-no-glue condition, let A_0 in $\varphi_0^{-1}(w_0)$ be non-dangling, and let $A_1, \ldots, A_r$ in $V(G)$ be the non-dangling vertices that contract to A. Then $A_0 = A_1 \cup \cdots \cup A_r$, as subsets of $[d]$.*

Proof. This is equivalent to proving that for every k in A_0 at least one of $B_u = \overline{(u,k)}$, $B_v = \overline{(v,k)}$ in $V(G)$ is non-dangling. Assume not, so by dangling-no-glue $|B_u| = |B_v| = 1$, thus B_u, B_v are not incident to other vertices above u or v. Hence these two dangling classes are the only ones that contract to A_0. By Lemma 6.14 A_0 is dangling, a contradiction. □

Lemma 6.17. *If M satisfies no-return and pass-once, then a limit M_0 satisfies no-return if either w_0 is a leaf or $\operatorname{ch} w_0 \leq 2$.*

Proof. Suppose that A_0 in $V(G_0)$, with $r = \text{nd-val}\, A \geq 2$, violates no-return. Since M satisfies no-return, then A_0 is above w_0. If $\operatorname{val} w_0 = 1$ there is nothing to check, so assume w_0 is not a leaf. If $r_0(A_0) \leq 1$, we get a contradiction with Lemma 4.19 (r1 implies no-return). Since $\operatorname{ch} w_0 \leq 2$, we have that $r_0(A_0) = 2$. As A_0 violates no-return, all the non-dangling edges $e_1, \ldots, e_r$ incident to A_0 are above a single edge of T, thus $|e_1| + \cdots + |e_r| \leq |A|$, so $|A| \geq 2$. Hence, $r_0(A) = 2 = r - 2 + 2|A| - (|e_1| + \cdots + |e_r|) \geq r - 2 + |A|$ gives $|A| = 2$ and $r = 2$. So there is an edge of H_0 passing above w_0 by going through A_0, and twice above t. Since w_0 is not a leaf, M_0 does not satisfy pass-once. On the other hand, Lemma 6.8 implies that M_0 satisfies pass-once, a contradiction. □

The condition of having $\operatorname{ch} w_0 \leq 2$ when w_0 is not a leaf is satisfied when M is change-minimal and has full-rank; in particular it is satisfied by full-dimensional M. Indeed, Proposition 6.2 and the equality $\operatorname{val} w_0 = \operatorname{val} u + \operatorname{val} v - 2$ together imply that

$$\operatorname{ch} w_0 + \operatorname{val} w_0 - 3 = (\operatorname{ch} u + \operatorname{val} u - 3) + (\operatorname{ch} v + \operatorname{val} v - 3) + 1 = 1. \tag{C}$$

This means that $\operatorname{ch} w_0$ is 0,1,2 or 3. The last value implies that w_0 is a leaf. Thus:

Lemma 6.18. *Let M be change-minimal with full-rank, and M_0 be a limit of M with $g(H(M)) = g(H(M_0))$. Then M_0 has full-rank and satisfies dangling-no-glue, no-return, and pass-once.*

Proof. This is a consequence of Lemmas 6.15, 6.7, 6.8, 6.17 and Equation (C). □

6.5 Deformation

We develop the first iteration of the deformation procedure. Let H be a trivalent combinatorial type and σ a path in C_H. A *path* in a topological space is a continuous function with the closed interval $[0,1]$ as domain; here C_H has the topology induced from $\mathbb{R}^{E(H)}$, as per Remark 2.9. We deform a tropical morphism Φ by following σ, and show that upon encountering a limit, as in Example 5.1, it is possible to continue further by changing the combinatorial structure of Φ. This means passing to another glueing datum.

The heavy lifting required to prove this deformation process is deferred to Chapter 7. Here we use Lemma 6.13 to reduce the proof to the problem of constructing full-dimensional glueing datums with a given limit.

Recall that C_{M_0} embeds canonically in C_H Since σ is contained in a cone C_H, namely our walk does not change the combinatorial type of the source, in the following lemma we may assume that $H(M_0)$ and $H(M)$ are isomorphic. This is equivalent to saying that $H(M_0)$ is trivalent. Thus, C_{M_0} is a subset of $C_{H(M)}$. Denote by $d(\ell_1, \ell_2)$ the euclidean distance in $\mathbb{R}^{E(H)}$. We wish to prove:

Proposition 6.19 (deformation I)**.** *Let M_0 be the limit at t_1 of some M in $\mathbb{FD}_g$ such that $H(M_0)$ is trivalent. Let $y_0 \in C_{M_0} \subset C_{H(M_0)}$. There exists a constant $\varepsilon(M_0, y_0) > 0$ such that if $\hat{y}$ is an element of $C_{H(M_0)} \setminus C_{M_0}$ with $d(y_0, \hat{y}) < \varepsilon(M_0, y_0)$, then there is $\hat{M}$ in* $\mathrm{Star}(M_0)$ *with $\hat{y} \in C_{\hat{M}}$.*

The constant $\varepsilon(M_0, y_0)$ is chosen to have the property that if $y \in \mathbb{R}^{E(H(M))}$ is in a ball B centered at y_0 with radius $\varepsilon(M_0, y_0)$ and the labellings of the elements of $\mathrm{Star}(M_0)$ are compatible at t_1, then given $\hat{M} = (\hat{T}, d, \hat{\sim})$ in $\mathrm{Star}(M_0)$ the only obstruction for a $\hat{y}$ in B to also be in $C_{\hat{M}}$ is whether $\hat{y}(t_1)$ is positive. For $\hat{M}$ in $\mathrm{Star}(M_0)$ write $\hat{A}$ for $A_{\hat{M}}$, and let $\hat{z}_0 = \hat{A}^{-1}(y_0) \in \mathbb{R}^{E(\hat{T})}$. Denote by $\min^+ y$ the smallest positive entry of y. Every element of the open ball $B(\hat{z}_0, \min^+ \hat{z}_0) \subset \mathbb{R}^{E(\hat{T})}$ has positive entries, except possibly the one corresponding to t_1. By continuity of $\hat{A}^{-1}$ we can choose a number $\varepsilon(\hat{M}, M_0, y_0) > 0$ such that the image under $\hat{A}^{-1}$ of the open ball with centre y_0 and radius $\varepsilon(\hat{M}, M_0, y_0) > 0$ lies in $B(\hat{z}_0, \min^+ \hat{z}_0)$. We define

$$\varepsilon(M_0, y_0) = \min_{\hat{M} \in \mathrm{Star}(M_0)} \varepsilon(\hat{M}, M_0, y_0). \tag{ε}$$

This is well defined and positive since $\mathbb{FD}_g$ is finite. By construction:

Lemma 6.20. *Let $\hat{M}$ be in* $\mathrm{Star}(M_0)$ *and $\hat{y}$ in C_H. Set $\hat{z} = \hat{A}^{-1}(\hat{y})$. If $d(y_0, \hat{y}) < \varepsilon(M_0, \ell_0)$, then $\hat{y} \in C_{\hat{M}}$ if and only if $\hat{z}(t_1) > 0$.*

Now we compute a formula for $\hat{z}(t_1)$, showing that its sign depends only on M_0, $\hat{y}$, and crucially only on $\det \hat{A}$. Let $\hat{A} = (\hat{a}_{ij})$. Append a column of zeroes at the beginning of the edge-length matrix of M_0 to obtain $A_0 = (a_{ij})$. From Lemma 6.13 (limit matrix) it follows that $\hat{a}_{ij} = a_{ij}$ for all i and $j \geq 2$. Hence, the first row of the adjugate matrix $\mathrm{adj}(\hat{A})$ depends only on M_0. Let $(c_1, c_2, \ldots, c_{3g-3})$ be that first row. Recall that the adjugate matrix satisfies:

$$\mathrm{adj}(\hat{A}) \cdot \hat{A} = \det(\hat{A}) \cdot I = \hat{c} \cdot I,$$

where $\hat{c} = \det(\hat{A})$. This formula holds in general for glueing datums with square edge-length matrices. This gives that:

$$\hat{c} = \sum_{i=1}^{3g-3} c_i \hat{a}_{i1}, \tag{$*$}$$

$$0 = \sum_{i=1}^{3g-3} c_i \hat{a}_{ij} = \sum_{i=1}^{3g-3} c_i a_{ij}, \text{ for } j \geq 2. \tag{$**$}$$

Moreover, since $\hat{M}$ is full-dimensional we have $\hat{c} \neq 0$, so $\hat{A}^{-1} = \frac{1}{\hat{c}} \mathrm{adj}(\hat{A})$. Thus,

$$\hat{z}(t_1) = \frac{1}{\hat{c}} \sum_{i=1}^{3g-3} c_i \, \hat{y}(h_i). \tag{L1}$$

Hence, the sum depends only on $\hat{c}$, $\hat{y}$ and on the c_i (which depend on M_0). Write $c = \det A_M$ as well. Applying Equation (L1) to $z(t_1) = (A_M^{-1}\hat{y})(t_1)$ we get that

$$z(t_1) = \frac{1}{c}\sum_{i=1}^{3g-3} c_i\,\hat{y}(h_i).$$

Using Lemma 6.20 and comparing signs we obtain:

Lemma 6.21. *Assume that $|y_0 - \hat{y}| < \varepsilon(M_0, \ell_0)$ and $\hat{y} \notin C_{M_0}$. If $\operatorname{sgn}\hat{c} = \operatorname{sgn} c$, then $\hat{y} \in C_{\hat{M}}$ if and only if $\hat{y} \in C_M$. If $\operatorname{sgn}\hat{c} \neq \operatorname{sgn} c$, then $\hat{y} \in C_{\hat{M}}$ if and only if $\hat{y} \notin C_M$.*

So we would like to construct full-dimensional glueing datums with prescribed limit at t_1 and a prescribed sign for $\det A_M$.

Lemma 6.22. *If M is in $\operatorname{Star}(M_0)$, the edge labelling on $\operatorname{Star}(M_0)$ is compatible at t_1, and $H(M_0)$ is isomorphic to $H(M)$, then there is $\hat{M}$ in $\operatorname{Star}(M_0)$ with $\operatorname{sgn}\det A_M \neq \operatorname{sgn}\det A_{\hat{M}}$.*

Our constructive proof of Lemma 6.22 is the main achievement of this article. This is carried out in Chapter 7. Proposition 6.19 (deformation I) follows easily from Lemma 6.22:

Proof of Proposition 6.19 (deformation I). If $\hat{y} \in C_M$, then we are done, just take $\hat{M} = M$. Otherwise $\hat{y} \notin C_M$. By Lemma 6.22 there is a glueing datum $\hat{M}$ in $\operatorname{Star}(M_0)$ with $\operatorname{sgn}\hat{c} \neq \operatorname{sgn} c$. By Lemma 6.21 we have that $\hat{y} \in C_{\hat{M}}$. □

6.6 Proof of main result

To close this chapter, we put together the deformation procedure of Proposition 6.19 and the initial families of Theorem 3.32 to derive a proof of Theorem 1.4. Our deformation machinery is designed to work with full-dimensional glueing datums, which implies even genus. The odd genus case follows from the usual trick of attaching a loop. For executing that trick we need a converse to Lemma 4.23.

Lemma 6.23. *(loop and bridge) Let M be change-minimal and full-rank. Let A in $V(H(M))$ be trivalent, incident to a bridge h_b and a loop h_l. Then h_b and h_l are above a path of T of length 2 that leads to a leaf. Moreover, they are the only edges of H above this path.*

Proof. Regard A as a vertex of G. Let $v = \varphi(A)$. Suppose that $r_\varphi(A) = 0$. By Lemma 4.17 we can apply Proposition 4.21 (local properties). By the case (r0-nd3) the three non-dangling edges incident to A are above three distinct edges of T. Since two of the edges of G incident to A are in h_l, we have that h_l passes above at least two leaves of T, contradicting Remark 4.15 (change-minimal leaves). The possibility $r_\varphi(A) = 2$ is ruled out since $\operatorname{nd-val} A = 3$. Thus, $r_\varphi(A) = 1$. The case (r1-nd3) implies that v is divalent and there is an edge t in $E(T)$ incident to v such that there are non-dangling edges e_1 and e_2 in $E(A)$ above t. Both e_1, e_2 are in h_l, for otherwise there is a contradiction again. Hence, h_l passes twice above t, so by pass-once t leads to a leaf. The path is made by t and the other edge incident to v, and now we can apply Lemma 4.23 to conclude uniqueness. □

Let $M = (T, d, \sim)$ be a genus-g glueing datum. We add some boundary points to C_M, in a process akin to that in Section 2.4 where some boundary points are added to C_H to obtain $\overline{C}_H$. For a given ℓ_T in ${\mathbb{R}_{\geq 0}}^{E(T)}$ contract all the edges of T for which $\ell_T(e) = 0$ to get M_0 (this is independent of contraction order); the pair (M, ℓ_T) refers to the tropical morphism $(M_0, \ell|_{T_0})$. Denote by $\overline{C}_M$ the image under A_M of the points ℓ_T in ${\mathbb{R}_{\geq 0}}^{E(T)}$ such that the source of (M, ℓ_T) has genus g; namely points where the genus does not drop.

We now present the proof of Theorem 1.4. The main idea is that to realize a point y_1 in C_H, we choose an initial point y in C_H, which exists by Theorem 3.32, and draw a straight line between y_1 and y. We then walk along this line, applying Proposition 6.19 as needed. To ensure the hypotheses of Proposition 6.19 are met, it might be necessary to wiggle y a bit, replace it with a nearby point using the following standard argument:

Lemma 6.24. *Let $S = \bigcup_{L \in \mathcal{F}} L \subset \mathbb{R}^n$ be a countable union of linear spaces L of codimension at least 2, and y_1, y be in $\mathbb{R}^n$. For every $\varepsilon > 0$ there is a point y^* in the radius-ε ball centered at y such that the interior of the segment between y_1 and y^* does not intersect S.*

Proof. Let $\bar{S} = \bigcup_{L \in \mathcal{F}} \operatorname{span}(y, L) \subset \mathbb{R}^n$. If $y \notin \bar{S}$ we are done, otherwise note that $\operatorname{span}(y, L)$ has codimension at least 1; as $\mathcal{F}$ is countable, this implies that the ball $B(y, \epsilon)$ is not contained in $\bar{S}$. Take y^* in $B(y_1, \varepsilon) \setminus \bar{S}$. □

Proof of Theorem 1.4. Assume that $g = 2g'$. Delete the dangling trees of Γ to obtain the metric graph $\tilde{\Gamma} = (H, y_1)$, with H a combinatorial type. Choose $M = (T, g' + 1, \sim)$ in $\mathbb{FD}_g$ such that $H(M)$ is isomorphic to H. This is possible by Theorem 3.32. Denote by $\mathbb{L}_2 M$ the set of affine spaces in C_H associated to limits of M where at least two edges have been contracted. This means, images under A_M of sets in C_T defined by fixing two coordinates to be null; so they are sets of codimension at least 2. Set $M^{(0)}$ to M, $S^{(0)}$ to $\mathbb{L}_2 M$ and r to 0. By Lemma 6.24 we can choose $y^{(0)}$ in $C_{M^{(0)}} \subset C_H$ such that the interior of the path $\sigma^{(0)}$ between y_1 and $y^{(0)}$ does not intersect $S^{(0)}$.

Walk along the segment $\sigma^{(r)}$ between y_1 and $y^{(r)}$ by iterating the following procedure. If y_1 is in $C_{M^{(r)}}$ we are done. If not, assume that the interior of the segment $\sigma^{(r)}$ between y_1 and $y^{(r)}$ does not intersect $S^{(r)}$. In particular, this means that the point y_0 of intersection between σ and $C_{M^{(r)}}$ is in the interior of a facet of $\overline{C}_{M^{(r)}}$. That means that y_0 is in C_{M_0} with M_0 the limit of $M^{(r)}$ at a single edge t_1. So we can apply Proposition 6.19 to obtain $\hat{M}$ such that σ goes through the interior $C_{\hat{M}}$. Set $M^{(r+1)}$ to $\hat{M}$ and $S^{(r+1)}$ to $S^{(r)} \cup \mathbb{L}_2 \hat{M}$.

We now choose a new point $y^{(r+1)}$ so the assumption at the beginning of the previous paragraph is satisfied. Note that for $q = 0, \ldots, r$ we have that $\sigma^{(r)}$ intersects each $\bar{C}_{M^{(q)}}$ at its interior or the interior of a facet. Hence, there is an $\varepsilon_q > 0$ such that the ball $B(y^{(r)}, \varepsilon_q)$ is a subset of $C_{M^{(0)}}$ and for any $y \in B(y^{(r)}, \varepsilon_r)$ we still have the property that the interior of the segment between y_1 and y intersects $C_{M^{(q)}}$ at either its interior, or the interior of a facet. Take ε to be the mininum of the ε_q, and apply Lemma 6.24 to get a point $y^{(r+1)}$ which satisfies the assumption. Set r to $r + 1$ and iterate again.

Since $\sigma^{(r)}$ is a line and $C_{M^{(q)}}$ is a convex set, the intersection $\sigma^{(r)} \cap C_{M^{(q)}}$ has only one connected component for all q. So the number of cones of $\mathbb{FD}_g$ that $\sigma^{(r)}$ intersects equals r. Since $\mathbb{FD}_g$ is finite, this procedure stops, which means we have realized y_1.

Now assume that g is odd. Choose a point x in Γ and attach to it a loop via a bridge. The lengths of the bridge and loop are not important, hence the construction depends on one parameter (the placement of x). The resulting graph $\Gamma_{\mathrm{b}} = (G_{\mathrm{b}}, \ell_{\mathrm{b}})$ has even genus $g+1$. As g is odd, $\lceil g/2 \rceil + 1$ equals $\lceil (g+1)/2 \rceil + 1$. So there is a tropical morphism $\Phi \colon \Gamma_{\mathrm{b}} \to (T, \ell_T)$ of degree $\lceil g/2 \rceil + 1$. By Lemma 6.23, the bridge and loop added to Γ are above two edges of T, and no other point of Γ_{b} is above them. Make the desired morphism by deleting these two edges and everything above. □

Chapter 7

Constructions

We carry out the constructions necessary to prove deformation. This chapter is motivated by the following paramount observation. The glueing datum M_2 of Example 5.1 is the limit at t_1 of M_1, $M_3^{(a)}$, and $M_3^{(b)}$ with edge labellings λ_1, $\lambda_3^{(a)}$ and $\lambda_3^{(b)}$. There is an isomorphism from $M_3^{(a)}$ to $M_3^{(b)}$, but $\lambda_3^{(a)}$ does not induce $\lambda_3^{(b)}$; moreover $M_3^{(a)}$ and $M_3^{(b)}$ produce in total two non-isomorphic tropical morphisms realizing the same source $\widetilde{\Gamma}$. Let A_1, $A_2^{(a)}$, $A_3^{(b)}$ be the edge-length matrices of these glueing datums. They satisfy:

$$\det A_1 + \det A_3^{(a)} + \det A_3^{(b)} = 0.$$

We view this equation as a balancing condition, something ubiquitous in tropical geometry. We generalize this relation with Equation ($\star$) in Section 7.2.

We construct several candidates to be in $\mathrm{Star}(M_0)$, for M_0 the limit at t_1 of some M in $\mathbb{FD}_g$. We use Equation ($\star$) to prove that at least in one of these candidates the determinant of the edge-length matrix has the sign opposite to that of $\det A_M$; as needed in Lemma 6.22.

7.1 Candidates to being full-dimensional

Let M be in $\mathbb{FD}_g$ and M_0 be the limit at t_1. Recall that giving a construction for $\mathrm{Star}(M_0)$ is hard because having a full-rank edge-length matrix is a global condition. Instead, we pass to a larger class of glueing datums that contains $\mathrm{Star}(M_0)$. In return we get conditions that can be checked locally, namely they only depend on the glueing relations above t_1 and its ends u, v.

Definition 7.1. Let $g = 2g'$ and M be a genus-g glueing datum. We say that M is *possibly full-dimensional* if M is change-minimal, satisfies dangling-no-glue, no-return, and A_M is a $(3g-3)\times(3g-3)$-matrix.

We denote by $\mathrm{PStar}(M_0)$ the set of isomorphism classes of possibly full-dimensional glueing datums that contract to a limit isomorphic to M_0. By Lemma 6.18 we have that $\mathrm{PStar}(M_0)$ contains $\mathrm{Star}(M_0)$. We index the elements of $\mathrm{PStar}(M_0)$ by denoting them $M^{(q)}$, with $q = 1, 2,$ and so on. We write $T^{(q)}$ for the base tree of $M^{(q)}$, $G^{(q)}$ for the graph that arises from $M^{(q)}$, $\varphi^{(q)}$ for $\varphi_{M^{(q)}}$, $r^{(q)}$ for $r_{\varphi^{(q)}}$, $m^{(q)}$ for $m_{\varphi^{(q)}}$, and $A^{(q)} = (a_{ij}^{(q)})$ for $A_{M^{(q)}}$. We also write T_0 for the base tree of M_0, G_0 for the graph that arises from M_0, φ_0 for φ_{M_0}, r_0 for r_{φ_0}, m_0 for m_{φ_0}, and $A_0 = (a_{ij})$ for the matrix we get by inserting the zero column at the beginning of A_{M_0}.

Recall that by Equation (C) on Page 42 we have that w_0 is not change-minimal; more precisely, $\mathrm{ch}\, w_0 + \mathrm{val}\, w_0 = 4$. Our strategy to construct the elements of $\mathrm{PStar}(M_0)$ is to regrow w_0 back to an edge in such a way that the resulting glueing datum is change-minimal and satisfies dangling-no-glue and no-return. This is possible because, as advertised, these conditions depend solely on the glueing relations above t_1 and its ends, u and v.

7.2 A balancing condition

On the other direction, note that since $A^{(q)}$ is a $(3g-3)\times(3g-3)$-matrix we have that $M^{(q)}$ in $\mathrm{PStar}(M_0)$ is full-dimensional if only if $\det A^{(q)} \neq 0$. So we need a relation for the determinants. This is where the balancing condition suggested by Example 5.1 comes in:

Lemma 7.2 (balancing condition). *If M is in $\mathrm{Star}(M_0)$ and $H(M_0)$ is isomorphic to $H(M)$, then there exists a finite sequence of triples $(M^{(q)}, \lambda^{(q)}, K_q)$, indexed by q, where the $M^{(q)}$ are in $\mathrm{PStar}(M_0)$ with edge labellings $\lambda^{(q)}$ compatible at t_1 and the K_q are positive integers, such that every isomorphism class in $\mathrm{PStar}(M_0)$ has a representative in the $M^{(q)}$, and the following equality holds:*

$$\sum_{q=1}^{r} K_q \det(A^{(q)}) = 0. \tag{$\star$}$$

It is straightforward to see that the balancing condition of Lemma 7.2 implies Lemma 6.22.

Proof of Lemma 6.22. Since M is in $\mathrm{Star}(M_0)$, we have that M is in $\mathrm{PStar}(M_0)$. Then by the balancing condition we have some $\hat{M}$ in $\mathrm{PStar}(M_0)$ with $\operatorname{sgn}\det A_{\hat{M}} \neq \operatorname{sgn}\det A_M$ and $\det A_{\hat{M}} \neq 0$. The latter equation implies that $\hat{M}$ is in $\mathrm{Star}(M_0)$, as desired. □

We prove the balancing condition and construct $\mathrm{PStar}(M_0)$, simultaneously, with a case-by-case analysis. For verifying Equation $(\star)$ we rely on formulas that follow from Equations $(*)$ and $(**)$ on Page 43. To shorten, we write $c^{(q)}$ for $\det A^{(q)}$. Consider the adjugate matrix $\operatorname{adj} A^{(q)}$, with columns indexed by edges of $H(M^{(q)})$ and rows indexed by edges of $T^{(q)}$. Let $c_1, \dots, c_{3g-3}$ be the first row of $\operatorname{adj} A^{(q)}$. Recall that the c_i depend only on M_0. For an edge h in $H(M^{(q)})$ or $H(M_0)$ we write c_h for $c_{\lambda_H^{(q)}(h)}$ or $c_{\lambda_{0,H}(h)}$, respectively. Recall that given a non-dangling e in $G^{(q)}$ or G_0 we denote by $h(e)$ the unique edge of $H(M^{(q)})$ or $H(M_0)$, respectively, that contains e. If e is dangling, we let $c_{h(e)}$ be zero. Fix a j in $[3g-3]$. We define:

$$\sigma_0(J, j) = \sum_{e\in J} \frac{c_{h(e)}}{|e|}, \text{ for } J \subseteq {\varphi_{M_0}}^{-1}(t_j),$$
$$\sigma^{(q)}(J, j) = \sum_{e\in J} \frac{c_{h(e)}}{|e|}, \text{ for } J \subseteq {\varphi_{M^{(q)}}}^{-1}(t_j).$$

For convenience we write $\sigma_0(j)$ for $\sigma_0({\varphi_{M_0}}^{-1}(t_j), j)$ and $\sigma^{(q)}(j)$ for $\sigma^{(q)}({\varphi_{M^{(q)}}}^{-1}(t_j), j)$.

Lemma 7.3. *The following equalities hold:*

$$\sigma^{(q)}(j) = c^{(q)} \text{ if } j = 1, \text{ otherwise } \sigma^{(q)}(j) = 0,$$
$$\sigma_0(J, j) = \sigma^{(q)}(J, j) \text{ for } j \neq 1 \text{ and any } J.$$

Proof. The first equality follows from Equations $(*)$ and $(**)$, and the fact that:

$$\sigma^{(q)}(J, j) = \sum_{e\in {\varphi_{M^{(q)}}}^{-1}(t_j)} \frac{c_{h(e)}}{|e|} = \sum_{i=1}^{3g-3} c_i a_{ij}^{(q)}.$$

Note that if $j \neq 1$ the edges in ${\varphi_{M_0}}^{-1}(t_j)$ correspond to those in ${\varphi_{M^{(q)}}}^{-1}(t_j)$, which together with the facts that the c_i depend only on M_0 and $a_{ij}^{(q)} = a_{ij}$ for $j \neq 1$, gives the second equality. □

Additionally, note that if J_1, J_2 are disjoint, then $\sigma_0(J_1, j) + \sigma_0(J_2, j) = \sigma_0(J_1 \cup J_2, j)$.

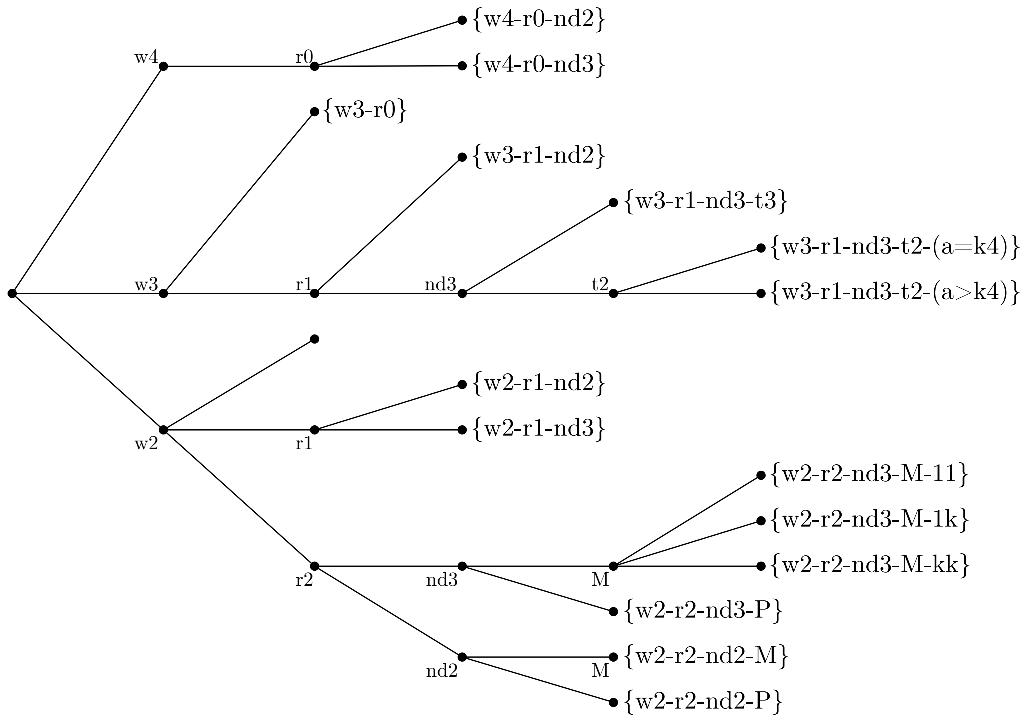


Figure 7.1: Logical flow of cases to regrow w_0.

7.3 Cases for constructing $\mathrm{PStar}(M_0)$

The plan of action to construct $\mathrm{PStar}(M_0)$ has three steps:

(1) Finding the possible base trees that contract to T_0.

(2) Restricting the possibilities for the glueing relations $\sim_u^{(q)}$, $\sim_v^{(q)}$, $\sim_{t_1}^{(q)}$.

(3) Giving a construction of a glueing datum together with edge labellings compatible at t_1 for each of the possibilities found in Step (2).

We deal with Step (1) on Section 7.4. Step (2) is the most delicate one, as it proves that indeed we have representatives for all the elements of $\mathrm{PStar}(M_0)$. We are able to say some useful generalities in Sections 7.5 and 7.6, but the definitive arguments come from assuming certain features of the glueing relations of M_0 above w_0 and edges in $E(w_0)$. Hence, we split the proof in cases. For Step (3) we refer the reader to the illustrations that accompany each case.

We have, by Equation (C) on Page 42, that $\mathrm{ch}\, w_0 + \mathrm{val}\, w_0 = 4$. Thus, $\mathrm{val}\, w_0$ is 1, 2, 3 or 4. In our setting the first option is not possible, because if $\mathrm{val}\, w_0$ were 1, then we would have $\mathrm{val}\, u = 1$ and $\mathrm{val}\, v = 2$ for all glueing datums in $\mathrm{PStar}(M_0)$, including the full-dimensional M. So Lemma 4.23 applied to M would imply that there is a loop above t_1, so the genus of $H(M_0)$ would drop upon contraction of t_1, a contradiction.

Let A_0 be a vertex above w_0. If $H(M_0)$ is isomorphic to $H(M)$, then $H(M_0)$ is trivalent. Hence, $\mathrm{nd\text{-}val}\, A_0$ is 0, 2, or 3. Also, since $\mathrm{ch}\, w_0 \leq 2$ we have that $r_0(A_0)$ is 0, 1 or 2.

Our philosophy of division of cases is similar to the one for proving the local properties: the different cases for calculating $\mathrm{PStar}(M_0)$ are mainly determined by the values of $\mathrm{val}\, w_0$, $r_0(A_0)$, and $\mathrm{nd\text{-}val}\, A_0$. There are other minor case specific factors that come into play. This gives a total of 17 cases. See Figure 7.1 for a map that eases navigating through the 15 main cases. The diagram does not show 2 *auxiliary cases* contained in Case {aux-r0}, which we treat in Subsection 7.7.1.

7.4 Trees contracting to T_0

We describe the possibilities for $T^{(q)}$. Assume that the edges of T_0 incident to w_0 are labelled with $2, \ldots, \deg w_0 + 1$. To construct a tree that contracts to T_0 choose disjoint sets S, S' whose union is $\{2, \ldots, \deg w_0 + 1\}$. At most one of S, S' may be empty. Replace w_0 with an edge t_1 with ends u, v; the edges indexed by S are incident to u, and those indexed by S' are incident to v. We denote the resulting tree by T_S. For ease of notation we omit braces; for example we write $T_{2,3}$ for $T_{\{2,3\}}$, and $T_\varnothing$ for the tree corresponding to empty S. Without loss of generality we assume that $|S| \leq |S'|$, and in case of equality that $2 \in S$. This gives a one-to-one correspondence between unordered partitions of $\{2, \ldots, \deg w_0 + 1\}$ into two parts and trees contracting to T_0. Since $M^{(q)}$ is change-minimal, $\operatorname{val} u \leq 3$ and $\operatorname{val} v \leq 3$, that is $|S|, |S'| \leq 2$.

7.5 The graph nd-$G^{(q)}_{A_0}$

Let nd-$G^{(q)}_{A_0}$ be the subgraph of non-dangling elements of $G^{(q)}$ that contract to A_0. Since $g(H(M)) = g(H(M_0))$, the graph nd-$G^{(q)}_{A_0}$ is a forest; moreover, by Lemma 6.16 it is a tree. By dangling-no-glue, determining the glueing relations $\sim^{(q)}_u$, $\sim^{(q)}_v$, $\sim^{(q)}_{t_1}$ is equivalent to determining $E(A)$ for each A in $V(\text{nd-}G^{(q)}_{A_0})$ and the index $m^{(q)}(x)$ of each element x in nd-$G^{(q)}_{A_0}$.

Lemma 7.4. *Let M be a glueing datum, M_0 the limit at t_1, and A_0 above w_0. If $g(H(M)) = g(H(M_0))$, then:*

$$\text{nd-val}\, A_0 = \sum_{A \in V(\text{nd-}G^{(q)}_{A_0})} (\text{nd-val}\, A - 2) + 2.$$

Proof. Since nd-$G^{(q)}_{A_0}$ is a tree, $|E(\text{nd-}G^{(q)}_{A_0})| = |V(\text{nd-}G^{(q)}_{A_0})| - 1$. So we have that:

$$\begin{aligned} \text{nd-val}\, A_0 &= \sum_{A \in V(\text{nd-}G^{(q)}_{A_0})} \text{nd-val}\, A - 2|E(\text{nd-}G^{(q)}_{A_0})| \\ &= \sum_{A \in V(\text{nd-}G^{(q)}_{A_0})} \text{nd-val}\, A - 2(|V(\text{nd-}G^{(q)}_{A_0})| - 1) \\ &= \sum_{A \in V(\text{nd-}G^{(q)}_{A_0})} (\text{nd-val}\, A - 2) + 2. \end{aligned}$$ □

Let A be in $V(\text{nd-}G^{(q)}_{A_0})$. By Lemma 7.4 if nd-val $A_0 = 2$, then nd-val $A = 2$. If nd-val $A_0 = 3$, then either nd-val $A = 2$ or nd-val $A = 3$; moreover, the latter case occurs exactly once. So when nd-val A is 3 we denote by $A^{(q)}$ the unique vertex of $G^{(q)}$ with non-dangling valency 3 that contracts to A_0. We make two further observations, one on the edge set and one on the vertex set of nd-$G^{(q)}_{A_0}$:

Lemma 7.5. *Let r be the number of vertices of $G^{(q)}$ that contract to A_0 and belong either to Case (r1-nd3) or Case (r2-nd2) of Proposition 4.21 (local properties). Then $|E(\text{nd-}G^{(q)}_{A_0})| \leq r+1 \leq r_0(A_0) + 1$.*

Proof. Let A be in $V(\text{nd-}G^{(q)}_{A_0})$. By the local properties and the fact that all the edges of nd-$G^{(q)}_{A_0}$ are above t_1, we have that $\operatorname{val}_{\text{nd-}G^{(q)}_{A_0}} A$ is either 1 or 2. Thus, nd-$G^{(q)}_{A_0}$ is in fact a path, so the inner vertices have valency 2. Note that valency 2 implies that A belongs either to Case (r1-nd3) or Case (r2-nd2), so the result follows. □

Lemma 7.6. *If $r_0(A_0) \leq 1$, then the vertices of nd-$G^{(q)}_{A_0}$ are the ends above u, v of the edges $e^{(q)}$, where $e^{(q)}$ corresponds to e in nd-$E(A_0)$.*

Proof. Let A be in $V(\text{nd-}G^{(q)}_{A_0})$. Since $r_0(A_0) \leq 1$, we have that $r^{(q)}(A) \leq 1$. Thus, by the local properties A is not above a leaf. So A satisfies the no-return condition, which means that not all of the edges in nd-$E(A)$ are above t_1, so at least one of them corresponds to one of the edges in nd-$E(A_0)$. □

7.6 Combinatorial type

We can obtain $H(M_0)$ from $H(M)$ by contracting a set of edges. If the number of edges contracted is zero, then $H(M_0)$ and $H(M)$ are isomorphic. This is equivalent to having a trivalent $H(M_0)$. This is the setting of Lemma 7.2 (balancing condition). Thus, we assume for the rest of this chapter that $H(M_0)$ and $H(M)$ are isomorphic. Since the edge labelling on $\text{PStar}(M_0)$ is compatible at t_1, the identification is canonical.

The case where $H(M_0)$ and $H(M)$ are not isomorphic is of interest as well, and we deal with it in Part II. It is by studying the $\text{PStar}(M_0)$ of M_0 with non-trivalent $H(M_0)$ that we show how it is possibly to walk between cones C_H with different combinatorial types H. We use this to prove that the space $\mathcal{G}^{\text{trop}}_{g\to 0,d}$ is connected.

7.7 Case-work

Let A_0 be above w_0. Let e in $G^{(q)}$ contract to A_0. Let A_u, A_v be the ends of e, above u, v respectively. If e is dangling, then it does not contribute to $\sigma^{(q)}(1)$. So, by Lemma 6.14 (dangling in the limit) we can assume that A_0 is non-dangling. We denote by J_r the set of those e contracting to A_0 such that $r^{(q)}(A_u) + r^{(q)}(A_v) = r$. By the discussion in Section 7.3 we have that $r_0(A_0) \leq 2$, hence:

$$\det A^{(q)} = \sigma^{(q)}(1) = \sigma^{(q)}(J_0, 1) + \sigma^{(q)}(J_1, 1) + \sigma^{(q)}(J_2, 1).$$

In each of the following cases we calculate $\sigma^{(q)}(J_0, 1)$, $\sigma^{(q)}(J_1, 1)$, and $\sigma^{(q)}(J_2, 1)$ separately. To avoid having several levels of subindices, we write $c(e)$ for $c_{h(e)}$.

7.7.1 Case {aux-r0}

We begin by showing that $T^{(q)} = T_{S^{(q)}}$ determines nd-$G^{(q)}_{A_0}$ when $r_0(A_0)$ is 0. This is an auxiliary case for calculating $\sigma^{(q)}(J_0, 1)$. Since $\operatorname{val} w_0 \geq 2$, by Lemma 4.19 (r1 implies no-return) we have that A_0 satisfies the no-return condition. Thus, if $\text{nd-val}\, A_0 = 2$, then φ_0 is injective on nd-$E(A_0)$. If $\text{nd-val}\, A_0 = 3$, suppose that φ_0 is not injective on nd-$E(A_0)$, so $\sum_{e \in \text{nd-}E(A_0)} |e| \leq 2|A_0|$. Hence, $r_0(A_0) = \text{nd-val}\, A_0 - 2 + 2|A_0| - \sum_{e\in\text{nd-}E(A_0)} |e| \geq 1$, a contradiction.

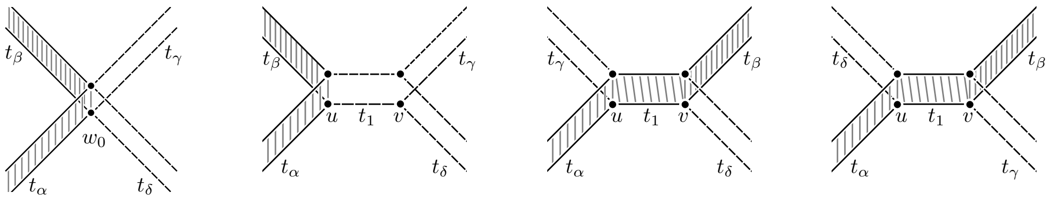
- Case {aux-r0-nd2}: Assume that $\text{nd-val}\, A_0$ is 2. Let e_α, e_β be the edges in nd-$E(A_0)$, above t_α and t_β respectively. By Lemma 7.6 the vertices of nd-$G^{(q)}_{A_0}$ are the ends of $e^{(q)}_\alpha$ and $e^{(q)}_\beta$ above u or v. If $\{\alpha, \beta\} \subset S^{(q)}$, the ends of $e^{(q)}_\alpha$, $e^{(q)}_\beta$ are above u; since $G^{(q)}_{A_0}$ is connected, they equal one vertex A in $G^{(q)}$, which equals A_0 as subsets of $[d]$ by Lemma 6.16. Similarly if $\{\alpha, \beta\} \subset S'^{(q)}$. Otherwise, one end A_u is above u, the other end A_v above v, so they are distinct. By connectivity of nd-$G^{(q)}_{A_0}$ there is one edge e' joining A_u, A_v. By Lemma 7.4 and since $r_0(A_0) = 0$, the vertices A_u and A_v belong to Case (r0-nd2) of the local properties. So, as subsets of $[d]$, the classes $e^{(q)}_\alpha$, A_u, e', A_v, $e^{(q)}_\beta$ are equal; and $h(e^{(q)}_\alpha) = h(e') = h(e^{(q)}_\beta)$.
- Case {aux-r0-nd3}: Assume that $\text{nd-val}\, A_0$ is 3. Let $e_\alpha, e_\beta, e_\gamma$ be the edges in nd-$E(A_0)$, above t_α, t_β, t_γ, respectively. Since $|\varphi_0(\text{nd-}E(A_0))| = 3$ and $\max(|S^{(q)}|, |S'^{(q)}|) \leq 2$, both intersections $S^{(q)} \cap \varphi_0(\text{nd-}E(A_0))$ and $S'^{(q)} \cap \varphi_0(\text{nd-}E(A_0))$ are non-empty. One of these intersections is a singleton. Assume without loss of generality that the singleton is $\{\alpha\}$. By Lemma 7.5 there is at most one edge in nd-$G^{(q)}_{A_0}$, therefore at most two vertices. By Lemma 7.6 the vertices of nd-$G^{(q)}_{A_0}$ are the end A_2 of $e^{(q)}_\alpha$ above u, and the vertex A_3 that is the end of both $e^{(q)}_\beta$ and $e^{(q)}_\gamma$ above v.

So A_2 and A_3 are distinct, joined by a non-dangling edge e', and nd-val $A_2 = 2$, nd-val $A_3 = 3$. By the Case (r0-nd2) of the local properties, as subsets of $[d]$, we get that $e_\alpha^{(q)}$, A_2, and e' are equal, and $h(e_\alpha^{(q)}) = h(e')$. Note that they are also a subset of A_3. Finally Lemma 6.16 implies that $A_0 = A_3 \cup A_2 = A_3$.

7.7.2 Case {w4}

Assume that $\operatorname{val} w_0$ is 4. Then $\operatorname{ch} w_0 = 0$ by Equation (C) on Page 42, so $r_0(A_0) = 0$. Thus, the two Cases {aux-r0-nd2} and {aux-r0-nd3} settle entirely this case. Namely, the elements of $\operatorname{PStar}(M_0)$ are $M^{(1)}$, $M^{(2)}$ and $M^{(3)}$, determined by $T_{2,3}$, $T_{2,4}$, and $T_{2,5}$, respectively.

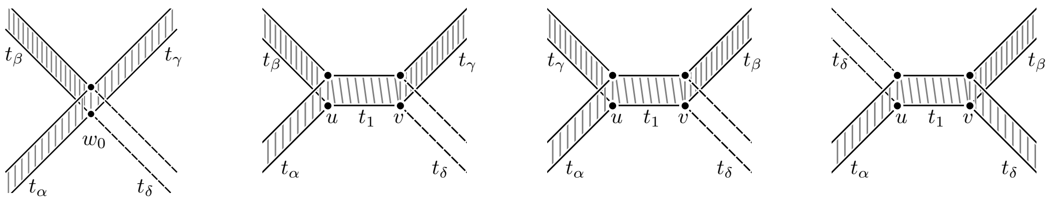
- Case {w4-r0-nd2}: Let nd-$E(A_0) = \{e_\alpha, e_\beta\}$ with $\varphi_0(e_\alpha) = t_\alpha$, $\varphi_0(e_\beta) = t_\beta$. Applying Case {aux-r0-nd2} we get the following diagrams, and a formula.



M_0 $T_{\alpha,\beta}$ $T_{\alpha,\gamma}$ $T_{\alpha,\delta}$

$$\sum_{q=1}^{3} \sum_{e \in E(\text{nd-}G_{A_0}^{(q)})} \frac{c(e)}{|e|} = \frac{c(e_\alpha)}{|e_\alpha|} + \frac{c(e_\beta)}{|e_\beta|}. \tag{w4-nd2}$$

- Case {w4-r0-nd3}: Let nd-$E(A_0) = \{e_\alpha, e_\beta, e_\gamma\}$ with $\varphi_0(e_\alpha) = t_\alpha$, $\varphi_0(e_\beta) = t_\beta$, $\varphi_0(e_\gamma) = t_\gamma$. Applying Case {aux-r0-nd3} we get the following diagrams, and a formula.



M_0 $T_{\alpha,\beta}$ $T_{\alpha,\gamma}$ $T_{\alpha,\delta}$

$$\sum_{q=1}^{3} \sum_{e \in E(\text{nd-}G_{A_0}^{(q)})} \frac{c(e)}{|e|} = \frac{c(e_\alpha)}{|e_\alpha|} + \frac{c(e_\beta)}{|e_\beta|} + \frac{c(e_\gamma)}{|e_\gamma|}. \tag{w4-nd3}$$

- We verify Equation ($\star$) for Case {w4} by putting together Lemma 7.3 and Equations (w4-nd2), (w4-nd3):

$$c^{(1)} + c^{(2)} + c^{(3)} = \sum_{q=1}^{3} \sigma^{(q)}(1) = \sum_{q=1}^{3} \sum_{A_0 \in \varphi_0^{-1}(A)} \sum_{e \in E(\text{nd-}G_{A_0}^{(q)})} \frac{c(e)}{|e|} \tag{7.1}$$

$$= \sum_{j=2}^{5} \sigma^{(q)}(j) = 0.$$

7.7.3 Case {w3}

Assume that $\operatorname{val} w_0$ is 3. Then $\operatorname{ch} w_0 = 1$ by Equation (C). There is exactly one vertex of G_0 with r_0-value 1 above w_0, the others have r_0-value 0. By Section 7.4, the possible base trees contracting to T_0 are T_2, T_3 and T_4; all with $\operatorname{val} u = 2$, $\operatorname{val} v = 3$. Let $\{\alpha\} = S^{(q)}$. We analyse the cases where $r_0(A_0)$ is either 0 or 1.

- Case {w3-r0}: Assume that $r_0(A_0) = 0$, and let A_u in $G^{(q)}_{A_0}$ be non-dangling above u. Then $r^{(q)}(A_u) = 0$. Since $\operatorname{val} u = 2$, only Case (r0-nd2) of the local properties is possible, so nd-val $A = 2$ and as a subset of $[d]$ the class A_u is equal to both of the elements in nd-$E(A_u)$. Since any non-dangling edge in $(\varphi^{(q)})^{-1}(t_\alpha)$ has a non-dangling end above u, we conclude that:
$$\sigma^{(q)}(J_0, 1) = \sigma^{(q)}(J_0, \alpha) = \sigma_0(J_0, \alpha).$$
- Case {w3-r1}: Since H_0 is trivalent either nd-val A_0 is 3 or 2.
- Case {w3-r1-nd3}: Assume that nd-val A_0 is 3. By the no-return condition $\varphi_0(\text{nd-}E(A_0))$ has at least two elements. So let e_2, e_3, e_4 be the edges in nd-$E(A_0)$, labelled such that $|e_4| = \max(|e_2|, |e_3|, |e_4|)$, $\varphi_0(e_3) = t_3$ and $\varphi_0(e_4) = t_4$. Let $k_i = |e_i|$. Lemma 4.18 (nd. r_φ formula) gives $r_0(A_0) = 1 = 3 + 2|A_0| - 2 - (k_2 + k_3 + k_4)$. That is, $2|A_0| = k_2 + k_3 + k_4$. If e_2 were above t_4, then $k_2 + k_4 \leq |A_0|$. Substituting in $k_3 = 2|A_0| - (k_2 + k_4)$ gives $k_3 \geq |A_0| \geq k_2 + k_4 > k_4$, a contradiction. So e_2 is either above t_2 or above t_3.
- Case {w3-r1-nd3-t2}: Assume that e_2 is above t_2. Let $\{\alpha, \beta, \gamma\} = \{2, 3, 4\}$. Recall from Section 7.5 that $A^{(q)}$ is the unique vertex of $G^{(q)}$ with nd-val $A^{(q)} = 3$ that contracts to A_0. We say that $M^{(q)}$ is in Position I or II if $A^{(q)}$ is above u or v, respectively. We prove that $T^{(q)}$ and the position of $A^{(q)}$ determine $M^{(q)}$.

  Position I: Assume that $A^{(q)}$ is above u. Since $\operatorname{val} u = 2$ and nd-val $A^{(q)} = 3$ we have that $A^{(q)}$ belongs to Case (r1-nd3) of the local properties (assuming Case (r0-nd3) contradicts that $\operatorname{val} u = 2$). So there are exactly two edges e', e'' of nd-$E(A^{(q)})$ above t_1; moreover $|e_\alpha| = |A^{(q)}| = |e'| + |e''|$. By Lemma 7.5 these are the two edges of nd-$G^{(q)}_{A_0}$. Hence, nd-$G^{(q)}_{A_0}$ has three vertices. By Lemma 7.6 these are: $A^{(q)}$ incident to $e^{(q)}_\alpha$; and the two ends of $e^{(q)}_\beta$, $e^{(q)}_\gamma$ above v, with non-dangling valency 2 and $r^{(q)}$-value equal to 0. By the Case (r0-nd2) of the local properties we conclude that as subsets of $[d]$, the edges e', $e^{(q)}_\beta$ and their common end are equal, and $h(e') = h(e^{(q)}_\beta)$; similarly with e'' and $e^{(q)}_\gamma$. Thus, $|e_\alpha| = |e_\beta| + |e_\gamma|$ and $|A^{(q)}| = |A_0|$, so this construction is possible only when $\alpha = 4$ and $k_4 = |A_0|$.

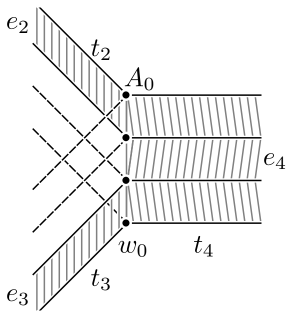
  Position II: Assume that $A^{(q)}$ is above v. Since $\operatorname{val} v = 3$ and nd-val $A^{(q)} = 3$ we have that $A^{(q)}$ belongs to Case (r0-nd3) of the local properties (assuming Case (r1-nd3) contradicts that $\operatorname{val} v = 3$). So there is exactly one edge e' of nd-$E(A^{(q)})$ above t_1. Let A' be the other end of e', which by Lemma 7.4 has non-dangling valency 2. Lemma 7.5 implies that the two vertices of nd-$G^{(q)}_{A_0}$ are $A^{(q)}$ and A'. Hence, by Lemma 7.6 A' is incident to $e^{(q)}_\alpha$; and $h(e^{(q)}_\alpha) = h(e')$. Since $r_0(A_0) = 1$, we have that A' belongs to Case (r1-nd2) of the local properties and this gives two possible cases for the values of $|A'|$, $|e'|$ and $|A^{(q)}|$:

  Position II.a: $|A'| = |e'|$ and $|A'| = |e^{(q)}_\alpha| + 1$. In this case, as subsets of $[d]$, we get that $e^{(q)}_\alpha \subset A'$, that A' and e' are equal, that $e' \subset A^{(q)}$, and therefore, by Lemma 6.16, that $A_0 = A^{(q)} \cup A' = A^{(q)}$. We also get that $|A_0| \geq |A'| > k_\alpha$.

  Position II.b: $|A'| = |e'| + 1$ and $|A'| = |e^{(q)}_\alpha|$. As subsets of $[d]$ we have that $e' \subset A' \cap A^{(q)}$, so $A' \setminus A^{(q)}$ is either empty or a singleton. The former would imply that for $i \in A' \setminus e'$, the classes e' and $\overline{(t_1, i)}_{M^{(q)}}$ are distinct edges of $G^{(q)}$ above t_1 with equal ends, which gives a cycle over t_1, a contradiction. So $|A_0| = |A^{(q)}| + |A'| - |A^{(q)} \cap A'| = |A^{(q)}| + 1$. The ends of $e^{(q)}_\beta$ and $e^{(q)}_\gamma$ are above v, so $|A^{(q)}| \geq k_\beta, k_\gamma$. Hence, $|A_0| > \max(k_\beta, k_\gamma)$, so $k_\alpha \geq 2$.

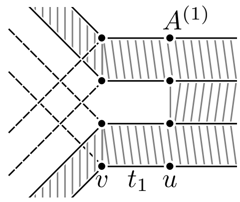
  Since Position I implies $|A_0| = k_4$, we treat two cases: $|A_0| = k_4$ and $|A_0| > k_4$.

- Case {w3-r1-nd3-t2-(a=k4)}: Assume that $|A_0| = k_4$. Then $k_2 + k_3 = |A_0|$, so $k_4 > k_2, k_3$. Let $\alpha = 4$, so $|A_0| = k_4$, which precludes Position II.a. Position I and Position II.b give $M^{(1)}$ and $M^{(2)}$, respectively. Let $\alpha = 2$, which precludes Position I. Then it is not true that $|A_0| > \max(k_\beta, k_\gamma) = k_4$, which precludes Position II.b as well. Position II.a gives $M^{(3)}$. An analogous argument with $\alpha = 3$ gives $M^{(4)}$. See figures and calculations below.



M_0

$$\frac{c(e_4)}{k_2+k_3} + \sigma_0(J_0,4) = 0$$
$$\frac{c(e_2)}{k_2} + \sigma_0(J_0,2) = 0$$
$$\frac{c(e_3)}{k_3} + \sigma_0(J_0,3) = 0$$

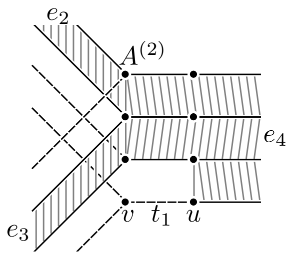


$M^{(1)}$

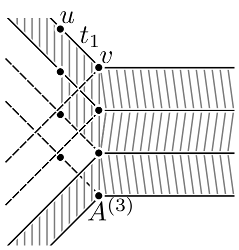
$|e'| = k_2 \qquad |e''| = k_3$

$\sigma^{(q)}(J_0,1) = \sigma_0(J_0,4)$

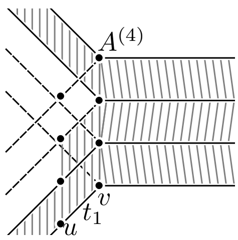
$$c^{(1)} = \frac{c(e_2)}{k_2} + \frac{c(e_3)}{k_3} + \sigma^{(q)}(J_0,1)$$



$M^{(2)}$



$M^{(3)}$



$M^{(4)}$

$|e'| = k_4 - 1 = k_2 + k_3 - 1$

$\sigma^{(2)}(J_0,1) = \sigma_0(J_0,4)$

$$c^{(2)} = \frac{c(e_4)}{k_2+k_3-1} + \sigma^{(2)}(J_0,1)$$

$|e'| = k_2 + 1$

$\sigma^{(3)}(J_0,1) = \sigma_0(J_0,2)$

$$c^{(3)} = \frac{c(e_2)}{k_2+1} + \sigma^{(3)}(J_0,1)$$

$|e'| = k_3 + 1$

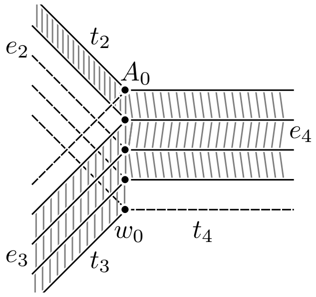
$\sigma^{(4)}(J_0,1) = \sigma_0(J_0,3)$

$$c^{(4)} = \frac{c(e_3)}{k_3+1} + \sigma^{(4)}(J_0,1)$$

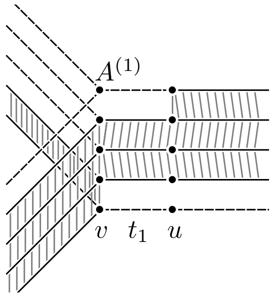
To verify Equation ($\star$) we compute:

$$\begin{aligned} & c^{(1)} + (k_2+k_3-1)c^{(2)} + (k_2+1)c^{(3)} + (k_3+1)c^{(4)} \\ &= \left(\tfrac{(k_2+1)}{k_2}c(e_2) + (k_2+1)\sigma_0(J_0,2)\right) + \left(\tfrac{(k_3+1)}{k_3}c(e_3) + (k_3+1)\sigma_0(J_0,3)\right) \\ &+ (c(e_4) + (k_2+k_3)\sigma_0(J_0,4)) = 0+0+0 = 0. \end{aligned} \tag{7.2}$$

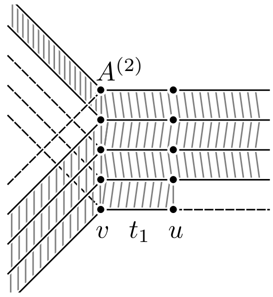
- Case {w3-r1-nd3-t2-(a>k4)}: Assume that $|A_0| > k_4$, which precludes Position I. If $\min(k_2,k_3,k_4)$ were 1, then $k_2 + k_3 + k_4 \leq 2|A_0| - 1$, contradicting $0 = 2|A| - k_2 - k_3 - k_4$. Hence, $\min(k_2,k_3,k_4) \geq 2$, so both cases of Position II are possible. It turns out that they balance each other for each possible α. See figures and calculations below for $\alpha = 4$.



M_0



$M^{(1)}$



$M^{(2)}$

$$\frac{c(e_4)}{k_4} + \sigma_0(J_0, 4) = 0 \qquad \begin{aligned} &|e'| = k_4 - 1 \\ &\sigma^{(1)}(J_0, 1) = \sigma_0(J_0, 4) \\ &c^{(1)} = \frac{c(e_4)}{k_4 - 1} + \sigma^{(1)}(J_0, 1) \end{aligned} \qquad \begin{aligned} &|e'| = k_4 + 1 \\ &\sigma^{(2)}(J_0, 1) = \sigma_0(J_0, 4) \\ &c^{(2)} = \frac{c(e_4)}{k_4 + 1} + \sigma^{(2)}(J_0, 1) \end{aligned}$$

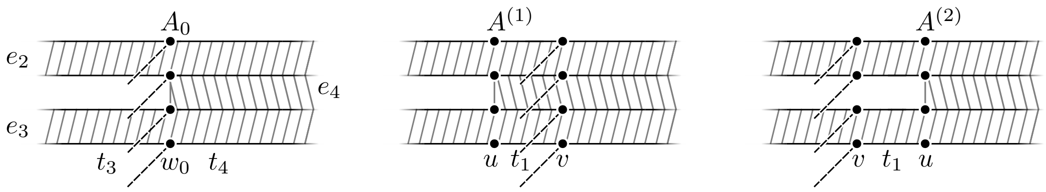
We conclude that:

$$(k_4 - 1)c^{(1)} + (k_4 + 1)c^{(2)} = 2c(e_4) + 2k_4\sigma_0(J_0, 4) = 0.$$

Analogous constructions exist for $\alpha = 2$, which gives $M^{(3)}$, $M^{(4)}$; and $\alpha = 3$, which gives $M^{(5)}$, $M^{(6)}$. We verify Equation ($\star$):

$$\Big((k_4 - 1)c^{(1)} + (k_4 + 1)c^{(2)}\Big) + \Big((k_2 - 1)c^{(3)} + (k_2 + 1)c^{(4)}\Big) + \Big((k_3 - 1)c^{(5)} + (k_3 + 1)c^{(6)}\Big) = 0 + 0 + 0 = 0. \tag{7.3}$$

- Case {w3-r1-nd3-t3}: Assume that e_2 is above t_3. Then $k_2 + k_3 \leq |A_0|$. Recall that $2|A_0| = k_2 + k_3 + k_4$ and $|A_0| \geq k_i$, so $k_4 = |A_0|$, $k_2 + k_3 = |A_0|$ and $|A_0| > k_2, k_3$. We argue that $T^{(q)}$ determines nd-$G^{(q)}_{A_0}$. If α were 2, then by Lemma 7.6 all the vertices of nd-$G^{(q)}_{A_0}$ would be above v, so nd-$G^{(q)}_{A_0}$ would have a single vertex with $r^{(q)}$-value 1 above trivalent v, contradicting that $M^{(q)}$ is change-minimal. For $\alpha = 3, 4$, we get that nd-$G^{(q)}_{A_0}$ has vertices above u and v. The vertices above v belong to the Case (r0-nd2) of the local properties, since Case (r0-nd3) would produce a non-dangling element above t_2. So if $\alpha = 3$, we get $M^{(1)}$ where $e^{(q)}_2$, $e^{(q)}_3$ have ends above v, and by Case (r0-nd2) these determine the two edges e', e'' of nd-$G^{(q)}_{A_0}$; similarly, if $\alpha = 4$ we get $M^{(2)}$ where $e^{(q)}_4$ has an end above v and this determines the edge e' of nd-$G^{(q)}_{A_0}$. So the vertex of nd-$G^{(q)}_{A_0}$ above u is $A^{(q)}$, and it belongs to Case (r1-nd3). So $A^{(q)}$ equals e_4, equals A_0 as subsets of $[d]$. This gives $M^{(1)}$ and $M^{(2)}$ for $\alpha = 3, 4$, respectively. See figures and calculations below.



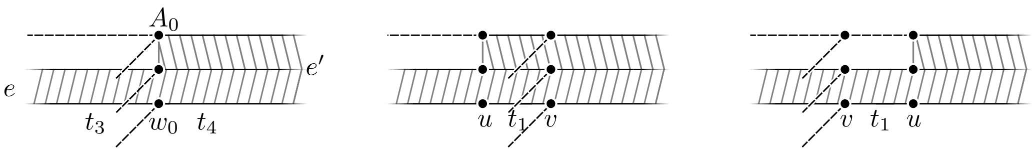
M_0 $\qquad$ $M^{(1)}$ $\qquad$ $M^{(2)}$

$$\begin{aligned} &\frac{c(e_2)}{k_2} + \frac{c(e_3)}{k_3} + \sigma_0(J_0, 3) = 0 \\ &\frac{c(e_4)}{k_2 + k_3} + \sigma_0(J_0, 4) = 0 \end{aligned} \qquad \begin{aligned} &|e'| = k_4 = k_2 + k_3 \\ &\sigma^{(1)}(J_0, 1) = \sigma_0(J_0, 3) \\ &c^{(1)} = \frac{c(e_4)}{k_2 + k_3} + \sigma^{(1)}(J_0, 1) \end{aligned} \qquad \begin{aligned} &|e'| = k_2 \qquad |e''| = k_3 \\ &\sigma^{(2)}(J_0, 1) = \sigma_0(J_0, 4) \\ &c^{(2)} = \frac{c(e_2)}{k_2} + \frac{c(e_3)}{k_3} \\ &\quad + \sigma^{(2)}(J_0, 1) \end{aligned}$$

We verify Equation ($\star$):

$$c^{(1)} + c^{(2)} = \Big(\tfrac{1}{k_2 + k_3}c(e_4) + \sigma_0(J_0, 4)\Big) + \Big(\tfrac{1}{k_2}c(e_2) + \tfrac{1}{k_3}c(e_3) + \sigma_0(J_0, 3)\Big) = 0 + 0 = 0. \tag{7.4}$$

- Case {w3-r1-nd2}: Assume that nd-val A_0 is 2. Let e, e' be in nd-$E(A_0)$. They are in the same edge h of H_0. Assume that $|e| \leq |e'|$. By no-return we may assume that e and e' are above t_3 and t_4 respectively. Lemma 4.18 (nd. r_φ formula) gives $r_\varphi(A_0) = 1 = 2|A_0| - |e'| - |e|$. Thus, if $k = |e|$, then $|A_0| = |e'| = k + 1$. From here, follow a reasoning analogous to case {w3-nd3-t2} to show that $T^{(q)}$ determines $M^{(q)}$, and that the vertices of nd-$G_{A_0}^{(q)}$ above v belong to Case (r0-nd2) of the local properties, and the ones above u to Case (r1-nd2). See figures and calculations below.



M_0 $M^{(1)}$ $M^{(2)}$

$$\frac{c_h}{k} + \sigma_0(J_0, 3) = 0 \qquad \sigma^{(1)}(J_0, 1) = \sigma_0(J_0, 3) \qquad \sigma^{(2)}(J_0, 1) = \sigma_0(J_0, 4)$$

$$\frac{c_h}{k+1} + \sigma_0(J_0, 4) = 0 \qquad c^{(1)} = \frac{c_h}{k+1} + \sigma^{(1)}(J_0, 1) \qquad c^{(2)} = \frac{c_h}{k} + \sigma^{(2)}(J_0, 1)$$

We verify Equation ($\star$):

$$c^{(1)} + c^{(2)} = \left(\tfrac{1}{k} c_h + \sigma_0(J_0, 3)\right) + \left(\tfrac{1}{k+1} c_h + \sigma_0(J_0, 4)\right) = 0 + 0 = 0. \tag{7.5}$$

7.7.4 Case {w2}

Assume that $\operatorname{val} w_0$ is 2. Then $r_0(A_0) = \operatorname{val} A_0 - 2$. So if $r_0(A_0) = 0$, then $\operatorname{val} A_0 = 2$. Hence, $\sigma_0(J_0, 2) = \sigma_0(J_0, 3)$, and we denote this quantity by s. Also, $\operatorname{ch} w_0 = 2$ by Equation (C). The trees contracting to T_0 are $T_\varnothing$ with $\operatorname{val} u = 1$, $\operatorname{val} v = 3$; and T_2 with $\operatorname{val} u = \operatorname{val} v = 2$. If $T^{(q)} = T_\varnothing$, then Remark 4.15 implies that $\sigma^{(q)}(J_0, 1) = \sigma^{(q)}(J_1, 1) = 0$, so $\det A^{(q)} = \sigma^{(q)}(J_2, 1)$. If $T^{(q)} = T_2$, then a reasoning analogous to Case {w3-r0} gives

$$\sigma^{(q)}(J_0, 1) = \sigma^{(q)}(J_0, 2) = s,$$
$$\sigma^{(q)}(J_0, 1) = \sigma^{(q)}(J_0, 3) = s.$$

So we are left with the cases $r_0(A_0) = 2$ and $r_0(A_0) = 1$.

- Case {w2-r2}: Assume that $r_0(A_0) = 2$. Then $\operatorname{val} A_0 = 4$. Since A_0 satisfies no-return, let e_2, e_3 be non-dangling edges in $E(A_0)$, above t_2, t_3, respectively. Let e_1, e_4 be the remaining two edges of $E(A_0)$. We may assume without loss of generality that e_1 is above t_2. By the refinement property:

$$\sum_{\substack{e \in E(A_0) \\ \varphi_0(e) = t_2}} |e| = \sum_{\substack{e \in E(A_0) \\ \varphi_0(e) = t_3}} |e| = |A_0|.$$

  So in particular $\sum_{i=1}^{4} k_i = 2|A_0|$. We explore how $T^{(q)}$ affects nd-$G_{A_0}^{(q)}$:

  Base I: assume that $T^{(q)}$ is $T_\varnothing$. Then, by Remark 4.15 (change-minimal leaves) we have that nd-$G_{A_0}^{(q)}$ has a vertex A_u above u with $|A_u| = 2$, $r^{(q)}(A_u) = 2$, and two incident edges e', e'' with $|e'| = |e''| = 1$. These are all the edges of nd-$G_{A_0}^{(q)}$ by Lemma 7.5 and the fact that $r_0(A_0) = 2$. Thus, A_u and the ends A', A'' of e', e'', respectively, are the vertices of nd-$G_{A_0}^{(q)}$. We may assume that $e_2^{(q)}$ and A' are incident. There are two possibilities for $E(A')$ and $E(A'')$. Either $e_3^{(q)}$ is incident to A' or A''. We call the first case Base I.a. Since $r^{(q)}(A') = r^{(q)}(A'') = 0$, we have that $k_2 = |A'| = k_3$, $k_1 = |A''| = k_4$. We call the second case Base I.b; it implies that k_2 equals the cardinality of the edge above t_3 incident to A'. So either $k_2 = k_4$ and $k_3 = k_1$; or $k_2 = k_1$ and $k_3 = k_4$. In particular, here the two edges of $E(A_0)$ are above t_2, and the other two above t_3. See Case {w2-r2-nd3-M-11} for figures of both possibilities.

  Base II: assume that $T^{(q)}$ is T_2. Then the vertices of nd-$G_{A_0}^{(q)}$ belong to Cases (r0-nd2), (r1-nd2), or (r1-nd3) of the local properties. By Lemma 7.4 at most one vertex of nd-$G_{A_0}^{(q)}$ belongs to Case (r1-nd3). So by Lemma 7.5 we get that nd-$G_{A_0}^{(q)}$ has either one edge e'; or two edges e', e''. We call them Base II.1, Base II.2, respectively.

- Case {w2-r2-nd3}: Assume that nd-val A_0 is 3. We may assume without loss of generality that e_4 is dangling. So $k_4 = 1$ and $2|A_0| - 1 = \sum_{i=1}^{3} k_i$. If e_4 is above t_3, then $k_3 + 1 = |A_0| = k_1 + k_2$. If e_4 is above t_2, then $k_3 = |A_0| = k_1 + k_2 + 1$. We call these two possibilities Cardinality M and Cardinality P.

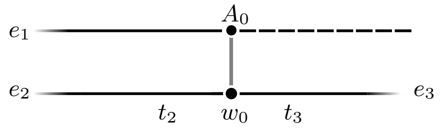
  For Base II we have that exactly one vertex of nd-$G^{(q)}_{A_0}$ belongs to Case (r1-nd3), so the other vertex of nd-$G^{(q)}_{A_0}$ with $r^{(q)}$-value 1 belongs to Case (r1-nd2).

  In Base II.1, Lemma 7.6 implies that the ends of $e^{(q)}_1$, $e^{(q)}_2$ above u are the same vertex of nd-$G^{(q)}_{A_0}$; by no-return this vertex has non-dangling valency 3, so it is $A^{(q)}$. By Case (r1-nd3) we have that $e_1 \cup e_2$, $A^{(q)}$, and e' are equal as subsets of $[d]$; therefore, $|e'| = |A^{(q)}| = k_1 + k_2$. Hence, the other end A' of e' (which is above v, incident to $e^{(q)}_3$) belongs to Case (r1-nd2). Either $|e'| = |A'| = k_3 + 1$, or $|e'| = |A'| - 1 = k_3 - 1$. The first possibility implies $k_1 + k_2 = k_3 + 1$, namely Cardinality M; the second possibility implies $k_1 + k_2 = k_3 - 1$, namely Cardinality P. We call these possibilities Base II.1.M and Base II.1.P, respectively.

  In Base II.2, nd-$G^{(q)}_{A_0}$ has three vertices. By Lemma 7.6 these are the ends A', A'' of $e^{(q)}_1$, $e^{(q)}_2$, respectively, above u, and the end B of $e^{(q)}_3$ above v. By no-return we may assume that e', e'' are incident to A', A'', respectively. Since these are all the edges of nd-$G^{(q)}_{A_0}$, nd-val A' = nd-val $A'' = 2$, and nd-val $B = 3$. So B is $A^{(q)}$ and $h(e^{(q)}_1) = h(e')$, $h(e^{(q)}_2) = h(e'')$. So Case (r1-nd3) gives that $k_3 = |A^{(q)}| = |e'| + |e''|$. One of A', A'' belongs to Case (r1-nd2) and the other to Case (r0-nd2). If A' belongs to Case (r1-nd2) then either $|e'| = |A'| - 1 = k_1 - 1$, so $k_1 > 1$ and $(k_1 - 1) + k_2 = k_3$, that is Cardinality M; or $|e'| = |A'| + 1 = k_1 + 1$, which gives $(k_1 + 1) + k_2 = k_3$, that is Cardinality P. We call these possibilities Base II.2.1.M and Base II.2.1.P, respectively. Analogously, if A'' belongs to Case (r1-nd2) we get Base II.2.2.M and Base II.2.2.P, where the former gives $k_2 > 1$.

- Case {w2-r2-nd3-M}: Assume that e_4 is above t_3, namely Cardinality M. The possibilities are Base I.a, Base I.b, Base II.1.M, Base II.2.1.M, and Base II.2.2.M.

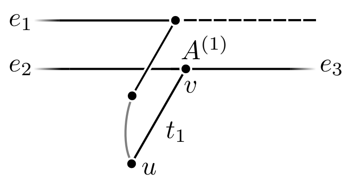
- Case {w2-r2-nd3-M-11}: Assume that $k_1 = 1$ and $k_2 = 1$. Base II.2.1.M, Base II.2.2.M are precluded since $k_1 \not> 1$, $k_2 \not> 1$, respectively. Base I.a, Base I.b, Base II.1.M determine $M^{(1)}$, $M^{(2)}$ and $M^{(3)}$, respectively. See figures and calculations below.



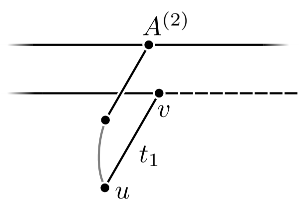
M_0

$c(e_1) + c(e_2) + s = 0$

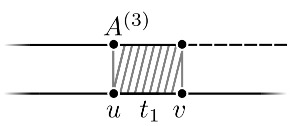
$c(e_1) + s = 0$



$M^{(1)}$



$M^{(2)}$



$M^{(3)}$

$\sigma^{(1)}(J_0, 1) = \sigma^{(1)}(J_1, 1) = 0$

$c^{(1)} = 2c(e_1)$

$\sigma^{(2)}(J_0, 1) = \sigma^{(2)}(J_1, 1) = 0$

$c^{(2)} = 2c(e_2)$

$|e| = 2$

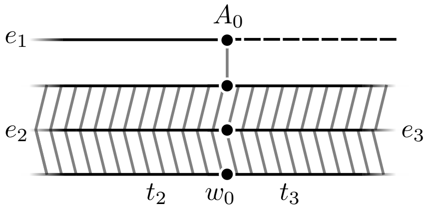
$\sigma^{(3)}(J_0, 1) = s$

$c^{(3)} = \dfrac{c(e_3)}{2} + s$

We verify Equation ($\star$):

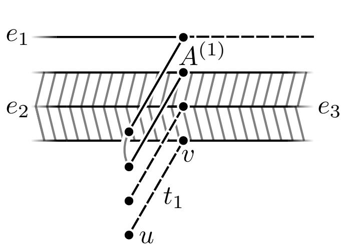
$$c^{(1)} + c^{(2)} + 4c^{(3)} = 2(c(e_1) + c(e_2) + s) + 2(c(e_3) + s) = 2 \cdot 0 + 2 \cdot 0 = 0. \qquad (7.6)$$

- Case {w2-r2-nd3-M-1k}: Assume that $k_1 = 1$ and $k_2 \geq 2$. Let $k = k_2$, so $|A_0| = k+1$ and $|e_3| = k$. Base I.b and Base II.2.1.M are precluded since $|e_2| \neq |e_4|$ and $k_1 \not> 1$, respectively. Base I.a, Base II.2.2.M, Base II.1.M determine $M^{(1)}$, $M^{(2)}$ and $M^{(3)}$, respectively. See figures and calculations below.

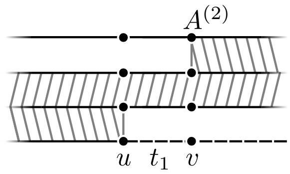


M_0

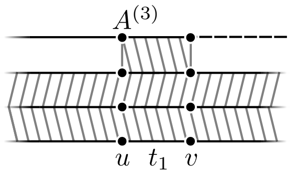
$$c(e_1) + \frac{c(e_2)}{k} + s = 0$$
$$\frac{c(e_3)}{k} + s = 0$$



$M^{(1)}$



$M^{(2)}$



$M^{(3)}$

$\sigma^{(1)}(J_0, 1) = \sigma^{(1)}(J_1, 1) = 0$

$c^{(1)} = 2c(e_1)$

$|e'| = 1 \quad |e''| = k-1$

$\sigma^{(2)}(J_0, 1) = s$

$c^{(2)} = c(e_1) + \frac{c(e_2)}{k-1} + s$

$|e'| = k+1$

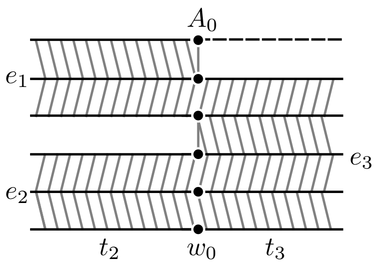
$\sigma^{(3)}(J_0, 1) = s$

$c^{(3)} = \frac{c(e_3)}{k+1} + s$

We verify Equation ($\star$):

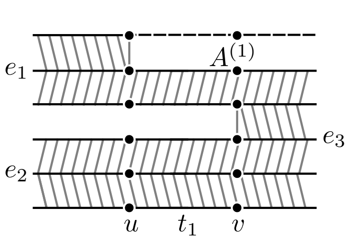
$$\begin{aligned} &c^{(1)} + 2(k-1)c^{(2)} + 2(k+1)c^{(3)} = \\ &2(kc(e_1) + c(e_2) + ks) + (c(e_3) + ks) = 2 \cdot 0 + 0 = 0. \end{aligned} \tag{7.7}$$

- Case {w2-r2-nd3-M-kk}: Assume that $k_1 \geq 2$ and $k_2 \geq 2$. Then $k_3 > k_1, k_2$, so Base I is precluded since $|e_2| \neq |e_3|$ nor $|e_2| \neq |e_4|$. Base II.2.1.M, Base II.2.2.M, and Base II.1.M determine $M^{(1)}$, $M^{(2)}$ and $M^{(3)}$, respectively. See figures and calculations below.

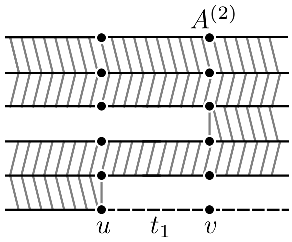


M_0

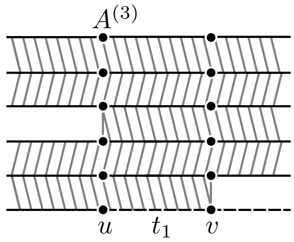
$$\frac{c(e_1)}{k_1} + \frac{c(e_2)}{k_2} + s = 0$$
$$\frac{c(e_3)}{k_1 + k_2 - 1} + s = 0$$



$M^{(1)}$



$M^{(2)}$



$M^{(3)}$

$|e'| = k_1 - 1 \quad |e''| = k_2$

$\sigma^{(1)}(J_0, 1) = s$

$c^{(1)} = \dfrac{c(e_1)}{k_1 - 1} + \dfrac{c(e_2)}{k_2} + s$

$|e'| = k_1 \quad |e''| = k_2 - 1$

$\sigma^{(2)}(J_0, 1) = s$

$c^{(2)} = \dfrac{c(e_1)}{k_1} + \dfrac{c(e_2)}{k_2 - 1} + s$

$|e'| = k_1 + k_2$

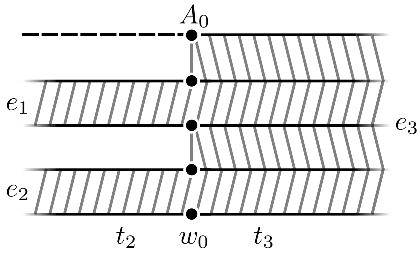
$\sigma^{(3)}(J_0, 1) = s$

$c^{(3)} = \dfrac{c(e_3)}{k_1 + k_2} + s$

We verify Equation ($\star$):

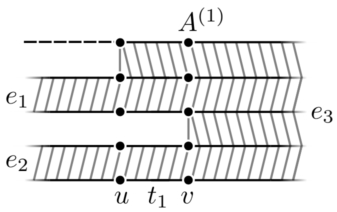
$$(k_1 - 1)c^{(1)} + (k_2 - 1)c^{(2)} + (k_1 + k_2)c^{(3)} = \tag{7.8}$$
$$\left(\tfrac{k_1+k_2-1}{k_1}c(e_1) + \tfrac{k_1+k_2-1}{k_2}c(e_2) + (k_1 + k_2 - 1)s\right) + (c(e_3) + (k_1 + k_2 - 1)s) = 0.$$

- Case {w2-r2-nd3-P}: Assume that e_4 is above t_2, namely Cardinality P. Base I is precluded. Base II.2.1.P, Base II.2.2.P, Base II.1.P determine $M^{(1)}$, $M^{(2)}$ and $M^{(3)}$, respectively. See figures and calculations below.

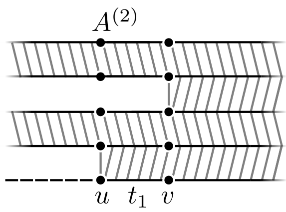


M_0

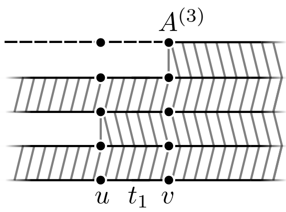
$$\frac{c(e_1)}{k_1} + \frac{c(e_2)}{k_2} + s = 0$$
$$\frac{c(e_3)}{k_1 + k_2 + 1} + s = 0$$



$M^{(1)}$



$M^{(2)}$



$M^{(3)}$

$|e'| = k_1 + 1 \quad |e''| = k_2$

$\sigma^{(q)}(J_0, 1) = s$

$c^{(1)} = \dfrac{c(e_1)}{k_1 + 1} + \dfrac{c(e_2)}{k_2} + s$

$|e'| = k_1 \quad |e''| = k_2 + 1$

$\sigma^{(2)}(J_0, 1) = s$

$c^{(2)} = \dfrac{c(e_1)}{k_1} + \dfrac{c(e_2)}{k_2 + 1} + s$

$|e'| = k_1 + k_2$

$\sigma^{(3)}(J_0, 1) = s$

$c^{(3)} = \dfrac{c(e_3)}{k_1 + k_2} + s$

We verify Equation ($\star$):

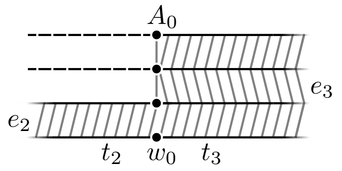
$$(k_1 + 1)c^{(1)} + (k_2 + 1)c^{(2)} + (k_1 + k_2)c^{(3)} = \tag{7.9}$$
$$(k_1 + k_2 + 1)(\tfrac{1}{k_1}c(e_1) + \tfrac{1}{k_2}c(e_2) + s) + (k_1 + k_2 + 1)(\tfrac{1}{k_1+k_2+1}c(e_3) + s) = 0 + 0 = 0.$$

- Case {w2-r2-nd2}: Assume that nd-val A_0 is 2. Then both e_1, e_4 are dangling. Let h be equal to $h(e_2)$ equal to $h(e_3)$.

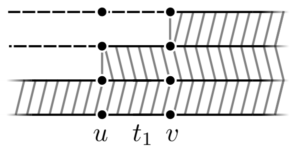
- Case {w2-r2-nd2-M}: Recall that e_1 is above t_2. Assume that e_4 is above t_3. Then $k_2 + 1 = |A_0| = k_3 + 1$. Since $|e_2| = |e_3|$, the columns corresponding to t_2 and t_3 in A_0 are equal. So M_0 is not full-rank, contradicting Lemma 6.7.

- Case {w2-r2-nd2-P}: Assume that e_4 is above t_2. Then $k_3 = |A_0| = k_2 + 2$, and Base I is precluded. In Base II all vertices of nd-$G^{(q)}_{A_0}$ belong to Case (r1-nd2). Thus, nd-$G^{(q)}_{A_0}$ has a single edge $|e'|$.

  The Case (r1-nd2) on the end of e_3 above v implies that $|e'| = k_3 - 1 = k_2 + 1$. This is the only possibility, see diagram below.



M_0



$M^{(1)}$

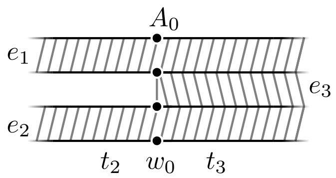
$$s = \sigma_0(J_0, 2) = \sigma_0(J_0, 3)$$
$$\frac{c_h}{k_2} + s = 0, \quad \frac{c_h}{k_2 + 2} + s = 0$$

$$|e'| = k_2 + 1$$
$$\sigma^{(1)}(J_0, 1) = s$$
$$c^{(1)} = \frac{c_h}{k_2 + 1} + s$$

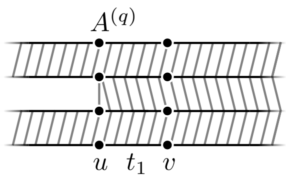
  From $c_h + k_2 s = 0$ and $c_h + (k_2 + 2)s = 0$ it follows that $c^{(1)} = c_h + (k_2 + 1)s = 0$, hence $M^{(1)}$ is not full-dimensional.

- Case {w2-r1}: Assume that $r_0(A_0) = 1$. Then $\operatorname{val} A_0 = 3$. Since A_0 satisfies no-return, let e_2, e_3 be non-dangling edges in $E(A_0)$, above t_2, t_3, respectively. Let e_1 be the remaining edge of $E(A_0)$. We may assume without loss of generality that e_1 is above t_2. By the refinement property $k_1 + k_2 = |A_0| = k_3$. Let $\tilde{A}$ be the unique vertex of nd-$G^{(q)}_{A_0}$ with $r^{(q)}(\tilde{A}) = 1$. If $\tilde{A}$ is above u (resp. v), then the vertices of nd-$G^{(q)}_{A_0}$ above v (resp. u) belong to the Case (r0-nd2) of the local properties (Case (r0-nd3) would contradict that v (resp. u) is divalent). These facts and Lemma 7.6 determine the classes.

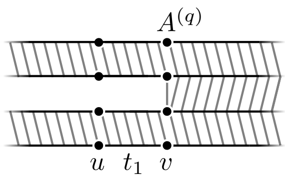
- Case {w2-r1-nd3}: Assume that nd-$\operatorname{val} A_0$ is 3.



local part around A_0



$\varphi^{(1)}(\tilde{A}) = u$



$\varphi^{(2)}(\tilde{A}) = v$

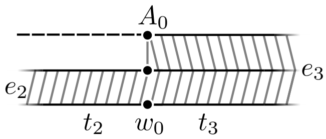
$$\sigma_0(J_{A_0}, 2) = \frac{c(e_1)}{k_1} + \frac{c(e_2)}{k_2}$$
$$\sigma_0(J_{A_0}, 3) = \frac{c(e_3)}{k_1 + k_2}$$

$$|e'| = k_3$$
$$\sigma^{(1)}(J_{A_0}, 1) = \frac{c(e_3)}{k_1 + k_2}$$

$$|e'| = k_1 \quad |e''| = k_2$$
$$\sigma^{(2)}(J_{A_0}, 1) = \frac{c(e_1)}{k_1} + \frac{c(e_2)}{k_2}$$

  Thus,
$$\sigma^{(1)}(J_{A_0}, 1) + \sigma^{(2)}(J_{A_0}, 1) = \sigma_0(J_{A_0}, 2) + \sigma_0(J_{A_0}, 3).$$

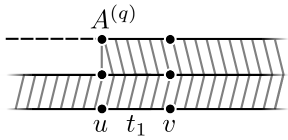
- Case {w2-r1-nd2}: Assume that nd-$\operatorname{val} A_0$ is 2. We may assume that e_1 is dangling. So $k_3 = k_2 + 1$. Let $h = h(e_2) = h(e_3)$.



local part around A_0

$\sigma_0(J_{A_0},2) = \frac{c_h}{k_1}$

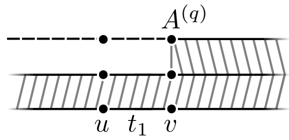
$\sigma_0(J_{A_0},3) = \frac{c_h}{k_1+1}$



$\varphi^{(1)}(\tilde{A}) = u$

$|e'| = k_2 + 1$

$\sigma^{(1)}(J_{A_0},1) = \frac{c_h}{k_2+1}$



$\varphi^{(2)}(\tilde{A}) = v$

$|e'| = k_2$

$\sigma^{(2)}(J_{A_0},1) = \frac{c_h}{k_2}$

Thus,

$$\sigma^{(1)}(J_{A_0},1) + \sigma^{(2)}(J_{A_0},1) = \sigma_0(J_{A_0},2) + \sigma_0(J_{A_0},3).$$

- Proof of ($\star$) for case {w2-r1}: There is another vertex B_0 above w_0 with $r_0(B_0) = 1$. The previous analysis holds for B_0, with notation entirely analogous. Note that in $M^{(q)}$, $\varphi^{(q)}(\tilde{A}) \neq \varphi^{(q)}(\tilde{B})$ because $\operatorname{ch} u = \operatorname{ch} v = 1$. So $\varphi^{(q)}(\tilde{A})$ determines the glueing datum, and it still holds that

$$\sigma^{(1)}(J_{B_0},1) + \sigma^{(2)}(J_{B_0},1) = \sigma_0(J_{B_0},2) + \sigma_0(J_{B_0},3).$$

This gives the following calculation, which verifies Equation ($\star$):

$$c^{(1)} = \sigma^{(1)}(J_{A_0},1) + \sigma^{(1)}(J_{B_0},1) + s, \qquad c^{(2)} = \sigma^{(2)}(J_{A_0},1) + \sigma^{(2)}(J_{B_0},1) + s,$$

$$c^{(1)} + c^{(2)} = \sigma_0(2) + \sigma_0(3) = 0 + 0 = 0. \tag{7.10}$$

7.8 Conclusion

It is remarkable how diverse the arguments of Chapter 7 are. By no means do we stand in front of a construction that has been repeated with subtle variations. The richness and diversity of the behaviour of possibly full-dimensional glueing datums defied many attempts of further consolidation into fewer cases. The end result is exhaustive, so Lemma 7.2 is verified. This finishes the proof of Theorem 1.4. The method is effective as well; see [Dra] for code.

Part II

Catalan-many tropical morphisms to trees

Qué era el hombre? En qué parte de su conversación abierta
entre los almacenes y los silbidos, en cuál de sus movimientos
metálicos
vivía lo indestructible, lo imperecedero, la vida?

Alturas del Macchu Picchu
Pablo Neruda

No, tú no lo negarás
Ahora vamos en el tren
En el tren, en el tren
En el tren a Paysandú
Se va el tren, toma el tren

Hoy me llama aquel rumor
Y a mi corazón le advierte
De cantar con emoción
La canción de Paysandú

Ay, añorado Paysandú
Qué es lo que habremos soñado
O más bien imaginado
Para haber tanto olvidado
Al lejano Paysandú

Los Jaivas

Chapter 8

Introduction to Part II

Our aim is to give a fully combinatorial solution to a combinatorial problem that is inspired by an algebro-geometric result in Brill-Noether theory. In this introduction we first sketch the classical problem; then we motivate how combinatorial methods enter the picture; and finally we outline our results, methods and future directions.

8.1 Representing the abstract

One important class of problems is to consider a category of abstract objects, to choose an ambient space, and to study the structure preserving maps from the abstract objects into the ambient space. These maps *represent* the abstract object by a concrete one; e.g. representation theory studies homomorphisms from abstract groups to groups of matrices. We describe a major theorem of "representation theory for algebraic curves" and the idea behind its proof.

8.1.1 Brill-Noether theory of algebraic curves

An algebraic curve is a 1-dimensional algebraic variety. What we understand with algebraic variety can be: abstract, without an embedding, i.e. a *nice* topological space that locally looks like the spectrum of a commutative ring with the Zariski topology; or can be concrete, embedded in a complex projective space $\mathbb{P}^r$, i.e. the set of solutions to a system of homogeneous polynomials. Given an abstract curve X, Brill-Noether theory studies morphisms

$$\phi : X \to \mathbb{P}^r.$$

A fruitful idea in algebraic geometry is to study families rather than single objects. A natural family containing X is the *moduli space* $\mathcal{M}g$, a geometric object whose points are in bijection with genus-g smooth curves. A natural family for ϕ relates these maps to line bundles. In a favourable setting, e.g. when X is smooth, we get a correspondence between a map $\phi : X \to \mathbb{P}^r$ and a pair of a line bundle L of degree d on X and an $(r+1)$-dimensional basepoint-free vector space of sections $V \subset H^0(X, L)$. Such a pair is called a *linear series on* X, a g^r_d for short. The set of triples (X, L, V) of a smooth genus-g curve X and a linear series (L, V) on X admits a scheme structure $\mathcal{G}^r_d$. This scheme comes with the natural projection π onto $\mathcal{M}g$ sending $(X, L, V) \mapsto X$. The fibre $G^r_d(X) = \pi^{-1}(X)$ turns out to be a projective variety, i.e. there is a map from $G^r_d(X)$ to a projective space that is an isomorphism onto its image.

In [BN74] Alexander von Brill and Max Noether argue heuristically that the expected dimension of $G^r_d(X)$ is

$$\rho(g, r, d) = g - (r+1)(g - d + r). \tag{8.1}$$

With the advent of scheme theory it was proven independently in [Kem71] and [KL72; Kle76] that if $\rho \geq 0$, then $G^r_d(X)$ is non-empty, of dimension at least ρ. The celebrated Brill-Noether theorem, due to Griffiths and Harris [GH80], establishes that the expected dimension count is correct for *almost all curves.*

Theorem 8.1. *There is a dense open subset $\mathcal{BN} \subset \mathcal{M}g$ called the* Brill-Noether general locus*, such that for every $X \in \mathcal{BN}$ we have:*

- *(existence part) If $\rho \geq 0$, then $\dim G^r_d(X) = \rho$.*
- *(non-existence part) If $\rho < 0$, then $G^r_d(X)$ is empty.*

The curve X is *Brill-Noether general* if X is in $\mathcal{BN}$; in this case, other facts are known about the geometry of $G^r_d(X)$. If $\rho > 0$, then $G^r_d(X)$ is an irreducible smooth scheme [Gie82; FL81]. If $\rho = 0$, then by [Kem71; KL72] the number of points in $G^r_d(X)$ is

$$\#G^r_d(X) = g! \prod_0^r \frac{i!}{(g-d+r+i)!}. \tag{8.2}$$

Moreover, the monodromy action on these points is transitive [EH87]. Thus, there is a unique irreducible component of $\mathcal{G}^r_d$ that dominates $\mathcal{M}_g$. See [ACGH13, Chapter VII] for the theory and proofs behind these results.

8.1.2 An argument by deformation

Regarding Theorem 8.1 and its proof, Subsection 0.(c) of [GH80] summarizes the heuristic argument by Brill-Noether. It is straightforward to show that if $\mathcal{BN}$ is non-empty, then it is open and dense, see [Eis83] for an exposition. Thus, the Brill-Noether theorem would be proven if for every genus g we could write down an explicit curve X in $\mathcal{BN}$. But this is remarkably difficult, and has not been achieved for curves of high genus.

The currently known proofs of the BN-theorem, for example [GH80] and [EH86], use a deformation argument. Castelnuovo and Severi observed that for certain genus-g curves $\tilde{X}$ with generic nodal singularities and reducible components isomorphic to $\mathbb{P}^1$, the dimension of $G^r_d(\tilde{X})$ is the expected ρ. They suggested to use a one-parameter deformation, i.e. certain morphism to the affine line $\mathbb{A}^1$ with fibre over 0 equal to $\tilde{X}$ and remaining fibres smooth curves; and to show that a linear series on $X = f^{-1}(\varepsilon)$ induces a linear series on $\tilde{X} = f^{-1}(0)$. This proves that $\dim G^r_d(X) \leq \dim G^r_d(\tilde{X})$, implying the non-existence part and the upper bound for the existence part. See the intro of [GH80] and [HM06, Chapter 5] for details, including more on the history.

8.1.3 Gonality of a curve

Now we look at an invariant called *gonality* [Amo93], to explain the difficulty of writing down a general curve. The classical motivation is to give a rough measure of how close a curve is to being *rational*. Recall that a curve is rational if it is isomorphic to $\mathbb{P}^1$. For later purposes we define gonality in two distinct ways. First,

$$\operatorname{gon}(X) = \min_{\phi: X \to \mathbb{P}^1} \deg \phi, \tag{8.3}$$

where the minimum is over all non-constant morphisms $\phi : X \to \mathbb{P}^1$, and $\deg \phi$ is the degree of ϕ, equal to the count with multiplicity of points in any fibre $\phi^{-1}(y)$. We have that X is rational if and only if $\operatorname{gon}(X) = 1$.

For the second definition, recall that a linear series (L, V) corresponds to a divisor D on X; i.e. an element $D = \sum_{P \in X} a_P(P)$ of the free abelian group $\operatorname{Div}(X)$ on the points of X. The degree $\deg D$ is the image of D under the homomorphism $\deg : \operatorname{Div}(X) \to \mathbb{Z}$ given by $\sum_{P \in X} a_P(P) \mapsto \sum_{P \in X} a_P$. The rank $r(D)$ equals the rank of (L, V), namely $\dim V - 1$. The divisorial gonality is

$$\operatorname{div-gon}(X) = \min_{r(D) \geq 1} \deg D, \tag{8.4}$$

where the minimum is over all divisors $D \in \mathrm{Div}(X)$ with rank at least 1.

The numbers div-gon(X) and gon(X) coincide when X is smooth. The curves X for which we understand $G^r_d(X)$ come from constructions that have a map to $\mathbb{P}^1$ of a fixed degree. For example, hyperelliptic or trigonal curves, have gonality equal to 2 or 3, respectively. On the other hand, substituting $r = 1$ in Equation (8.1) gives that a curve in $\mathcal{BN}$ has gonality $\lceil g/2 \rceil + 1$. So the curves we can write down are close to rational curves, but general curves move away from being rational as g increases. Thus, gon(X) is also a rough measure on the failure of X to be Brill-Noether general; i.e. the *smaller* gon(X) is, the further X is from being Brill-Noether general.

8.2 A voyage to the tropics

We now describe another deformation theory, where instead of nodal curves the targets are certain piecewise-linear objects called *tropical varieties*. This simplifies the geometrical features of the target, and brings combinatorics to the forefront. Thus, problems neatly split into an algebro-geometric half of understanding deformation processes called *tropicalizations*, and a combinatorial half of understanding tropical varieties themselves. We say a few words about tropical varieties in the embedded setting, and then outline how abstract algebraic curves are deformed to tropical curves.

8.2.1 Embedded tropical varieties

We begin with the embedded picture. Chapter 9 of [Stu02] heralded the rise of a *"tropical algebraic geometry"* for investigating systems of polynomial equations. This field combines long established methods in commutative algebra, deformation theory, polyhedral geometry, graph theory, etc. The motto is *algebraic geometry over the tropical semiring* $(\mathbb{R} \cup \{\infty\}, \min, +)$; named so to honour Brazilian computer scientist Imre Simon, see [Sim88; Pin98]. One early triumph is [Mik05], where the count of genus-g degree-d irreducible curves passing through $3d - 1 + g$ points is shown to correspond to the count that uses tropical curves instead; this tropical count is then established by counting certain lattice paths. So the problem is split in two halves following the goals: (I) to establish correspondence theorems in tropicalization maps; and (II) to solve combinatorial problems inspired by algebraic geometry.

For example, to tropicalize an n-dimensional hypersurface $Y = V(f) \subset (\mathbb{C}^*)^m$, one applies in a coordinate-wise manner certain maps called *valuations*, which generalize non-archimedean absolute values. This idea originates from a map $\mathcal{A}$ studied in [Ber71], that applies a logarithm coordinate-wise. The image of $\mathcal{A}$ is called an *amoeba* (a name best understood by looking at the image of $\mathcal{A}$ for $n = 1$, $m = 2$); see [IMS09, Chapter 1] for amoebas in the context of tropical geometry. The amoeba is deformed by varying the base of the logarithm. When the base tends to infinity, the limit object $\operatorname{trop} Y$ is a rational polyhedral fan of pure dimension n [GB84]. On the other hand, one can apply the valuation map to the coefficients of f to get an equation $\operatorname{trop} f$ in the tropical semi-ring, and associate a geometrical object $V^{\text{trop}}(\operatorname{trop} f)$. A major result, commonly referred to as the *fundamental theorem of tropical geometry*, is that the *tropical variety* $V^{\text{trop}}(\operatorname{trop} f)$ coincides with $\operatorname{trop} Y$; see [Kap00; EKL06].

One may ask which polyhedral complexes show up as tropicalizations; i.e. which objects are *realizable*. We seek characterizations in the form of combinatorial *realizability conditions*. One example of such a condition is the above-stated fact that $\Sigma = \operatorname{trop} Y$ is of pure dimension n. For another one, consider an $(n-1)$-dimensional polyhedron $\tau \in \Sigma$ and the primitive integral vectors σ/τ encoding the directions that start at τ and point to a neighbouring polyhedron σ. The *balancing condition at* τ states that the sum of these vectors, each multiplied with a certain *weight* of σ, is 0; see [RST05; Mik07]. Also, Σ is codimension-1 connected when Y is irreducible; i.e. the set $\Sigma \setminus \Sigma^{(n-2)}$ is connected; see [BJSST07; CP12]. Here $\Sigma^{(n-2)}$ is the $(n-2)$-skeleton, the union of all polyhedra σ of Σ with $\dim \sigma \leq n - 2$. See [MS15, Chapter 3] for proofs of all these facts.

8.2.2 Abstract tropical varieties

Tropicalizations in the non-embedded setting transform algebraic varieties into a basic combinatorial object, e.g. a graph or a polyhedral space, that is enriched with extra structure. For example, consider the *dual graph* $G_{\tilde{X}}$ of a nodal abstract curve $\tilde{X}$. The vertices of $G_{\tilde{X}}$ correspond to the irreducible components of $\tilde{X}$, and the edges to their intersections at nodes; see Remark 13.15. Consider as well a map $w : V(G_{\tilde{X}}) \to \mathbb{Z}_{\geq 0}$ such that $w(A)$ is the genus of the irreducible component corresponding to A. The basic combinatorial object is $G_{\tilde{X}}$, the extra information is w, and the pair $(G_{\tilde{X}}, w)$ is called a weighted graph. The first Betti number $g(G_{\tilde{X}}) = \#E(G_{\tilde{X}}) - \#V(G_{\tilde{X}}) + 1$ of $G_{\tilde{X}}$ behaves as a tropical analogue of the genus; e.g. the following identity holds:

$$g(\tilde{X}) = g(G_{\tilde{X}}) + \sum_{A \in V(G_{\tilde{X}})} w(A) =: g(G_{\tilde{X}}, w). \tag{8.5}$$

The number $g(G_{\tilde{X}}, w)$ also plays the role of genus in a Riemann-Roch formula for weighted graphs [BN07; AC13].

Some further enrichment of $G_{\tilde{X}}$ is found in the tropicalization considered in [Bak08]. It captures information from a one-parameter deformation $f : \mathcal{X} \to B$ that takes X to $\tilde{X}$. Since $G_{\tilde{X}}$ only encodes information from $\tilde{X}$, the insight is to consider a map $\ell : E(G) \to \mathbb{R}_{>0}$ where each value $\ell(e)$ is a *deformation parameter* of the singularity corresponding to e. Combinatorially, the map ℓ is interpreted as lengths for the edges of $(G_{\tilde{X}}, w)$. By glueing intervals of length $\ell(e)$ we obtain a metric space $\Gamma_{\tilde{X}}$ called *a weighted metric graph.* Note that this interpretation makes sense from the embedded point of view of Subsection 8.2.1. There an algebraic curve, say in the plane, tropicalizes to a 1-dimensional piece-wise linear subset of $\mathbb{R}^2$; this looks like a metric graph.

8.2.3 Tropicalizing gonality

There are several notions of tropical gonality. This plurality arises because given an algebro-geometric notion, the first guess for a tropical analogue is the original definition verbatim, with the adjective “tropical” inserted at appropriate places. For example, a divisor on an algebraic curve can be regarded as a map $D : X \to \mathbb{Z}$ with finite support, yielding a group $\mathrm{Div}(X)$ with pointwise addition. This definition still works if one replaces X by a graph G or a metric graph Γ. Typically this first guess is good after fixing minor issues, but when there are several equivalent algebraic definitions, as in the case with gonality, tropically maybe only one gives rise to a meaningful notion, or perhaps they give rise to distinct notions.

For tropical Brill-Noether theory, the first ingredient is a theory of divisors on graphs and metric graphs, initiated in [BLN97; BN07; MZ08; GK08]. This theory is closely linked to *chip-firing* [BLS91; Kli18], a combinatorial game played on graphs that has connections with arithmetic geometry, dynamical systems, and tropical geometry. The theory has analogues of many classical results, such as an Abel-Jacobi and a Riemann-Roch theorem. In the latter, the definition of rank of a divisor emulates a combinatorial characterization of the classical rank. One would hope for a tropical rank r^{trop} to arise from a tropical analogue of the space of rational functions associated to a divisor. There is such a space, but it turns out to be a polyhedral complex that is not of pure dimension. Despite being well studied [HMY12], it is not clear how to read off the rank from it; this exemplifies how the obvious candidate for a tropical definition might not work.

With that in mind, the *divisorial gonality* of a metric graph Γ is verbatim given by Equation (8.4), namely the minimum degree of a rank-at-least-one divisor. In the abstract setting of Subsection 8.2.2, let D_X be a divisor on an algebraic curve X. The idea of [Bak08] is to define a map $\mathrm{trop} : \mathrm{Div}(X) \to \mathrm{Div}(\Gamma_{\tilde{X}})$ by first sending D_X to $D_{\tilde{X}}$ in $\mathrm{Div}(\tilde{X})$ using the deformation, and then $D_{\tilde{X}}$ to D in $\mathrm{Div}(\Gamma_{\tilde{X}})$ via a straightforward procedure. The main result, called the *specialization lemma*, shows that

$$r(D_X) \leq r^{\mathrm{trop}}(\mathrm{trop}(D_X)). \tag{8.6}$$

So divisorial gonality can only go up under tropicalization. There are examples with strict inequality.

Equation (8.6) has deep implications, both algebro-geometric and combinatorial. In [CDPR12] a family of genus-g metric graphs Γ is constructed such that if $\rho(g,r,d) < 0$, then all degree-d divisors have rank less than r; combined with Equation (8.6) this yields a new proof of non-existence in Theorem 8.1. On the other hand, Equation (8.6) proves a tropical analogue of Theorem 8.1 (existence part) for metric graphs. This depends on the fact that the tropicalization $X \mapsto \Gamma_{\tilde{X}}$ is surjective, namely the realizability locus consists of all the combinatorial objects, a quite remarkable situation; see [Bak08, Appendix B]. As a corollary, we have:

Theorem 8.2. *Let Γ be a genus-g metric graph. We have that* $\text{div-gon}(\Gamma) \leq \lceil g(\Gamma)/2 \rceil + 1$.

This is a purely combinatorial statement, for which the proof via the specialization lemma means a long detour through deep algebro-geometric results of limit series, and also solving a realizability problem. In contrast to $X \mapsto \Gamma_{\tilde{X}}$, the related map $X \mapsto G_{\tilde{X}}$ is not surjective. Thus, these ideas have been ineffective to tackle a similarly flavoured question, which at the time of writing remains open:

Question 8.3. *Is it true that* $\text{div-gon}(G) \leq \lceil g(G)/2 \rceil + 1$ *for a graph G?*

8.2.4 Tropical morphisms of weighted metric graphs

Defining a tropical gonality via maps involves two subtle issues. First, the natural candidate for tropical morphisms is the class of maps that preserve the structure of polyhedral spaces. But this class is too large, since not all maps come with the properties that their algebraic counterparts have. Namely, let $\phi : X \to Y$ be an algebraic morphism of curves. Recall that the *ramification index* is a map $m_\phi : X \to \mathbb{Z}_{\geq 1}$ given by an algebraic formula and with the geometrical property that the pair (ϕ, m_ϕ) is an *indexed branched cover*. That is, ϕ is locally a homeomorphism over a dense set of Y, whose fibres have $(\deg \phi)$-many points when counted with multiplicity m_ϕ. This fact is not automatic in a morphism of polyhedral spaces.

Moreover, for $\phi : X \to Y$ the expression $R_\phi = \sum_{P \in X}(m_\phi(P) - 1)P$ is an effective divisor called the *ramification divisor*. The Riemann-Hurwitz formula states that

$$K_X \sim \phi^* K_Y + R_\phi, \tag{8.7}$$

where K_X and K_Y are the canonical divisor classes of X and Y respectively, and $\sim$ is linear equivalence of divisors. Taking degrees gives the familiar formula $2g(X) - 2 = (2g(Y) - 2)\deg\phi + \deg R_\phi$ relating the genera of the source and target curve. Tropically, there are canonical divisors for weighted metric graphs Γ and Δ [AC13]. An indexed branched cover $\Phi : \Gamma \to \Delta$ pulls back equivalent divisors in $\mathrm{Div}(\Delta)$ to equivalent divisors in $\mathrm{Div}(\Gamma)$. So there is a tropical Equation (8.7), but $K_\Gamma - \Phi^* K_\Delta$ might not be effective.

A *tropical morphism* is an indexed branched cover of weighted metric graphs with index map given by a certain algebraic expression, such that the ramification divisor is effective. For practical reasons, instead of the latter we require an equivalent inequality called the Riemann-Hurwitz inequality; see Equation (RH). The extra structures imposed on morphisms of metric graphs can be regarded as realizability conditions. This definition is equivalent to the maps studied in [Mik07; BN09; BBM11; Cha13; Cap14; CMR16; CD18; DV20]. These maps appear when tropicalizing algebraic morphisms [Cap14; Lan20], and the realizability locus has been studied [ABBR15a; ABBR15b].

The next question is what should be the target space of the maps. Since $\mathbb{P}^1$ has genus 0, the natural candidates are genus-0 graphs, called metric trees because they have no cycles. The second subtle issue is that there are Γ that admit no maps to any metric tree Δ. A rough intuition to explain this phenomenon is that Γ could be singular. Singularities of algebraic curves are resolved by a sequence of operations called blow-ups and blow-downs, which yield birational equivalence. The tropical analogue to blow-ups and blow-downs is *tropical modification*, an operation that retracts or attaches edges ending in a monovalent vertex. Tropical modification makes all metric trees equivalent. So, as in the classical setting, the target space is the unique tropical space with the simplest geometry. See [Kal15] for a detailed survey on tropical modifications.

With those two subtle issues in mind, for a weighted metric graph $\widetilde{\Gamma}$ we define

$$\mathrm{gon}(\widetilde{\Gamma}) = \min \deg \Phi : \Gamma \to \Delta,$$

where the minimum ranges over tropical morphisms Φ, metric trees Δ, and weighted metric graphs Γ which are a *tropical modification* of $\widetilde{\Gamma}$. Recall that an indexed branched cover Φ induces a pull back of divisors that preserves linear equivalence, and a degree-1 divisor on Δ has rank-1. This implies

$$\text{div-gon}(\widetilde{\Gamma}) \leq \mathrm{gon}(\widetilde{\Gamma}),$$

an inequality that can be strict, see Remark 13.8, so these tropical gonalities are distinct.

8.3 A combinatorial solution to a combinatorial problem

We now discuss our methods, results and future directions. We give a brief summary of the Part I, proceed to describe the sections in this Part II, and close with outstanding questions.

8.3.1 Constructing tropical morphisms witnessing the gonality bound

In Part I we take a metric graph $\widetilde{\Gamma}$ and a metric tree Δ, and construct tropical morphisms $\Phi : \Gamma \to \Delta$ with degree $\lceil g(\widetilde{\Gamma})/2 \rceil + 1$, where Γ is a tropical modification of $\widetilde{\Gamma}$. This gives a combinatorial proof of Theorem 8.2, and the motivation for this undertaking is the hope that the methods involved can help tackle Question 8.3.

We studied metric graphs Γ as pairs (G, y) of a graph G and a *length function* $y : E(G) \to \mathbb{R}_{\geq 0}$. The idea is to separate combinatorial from metric information. This, applied to tropical morphisms, lead us to work with the conept of *discrete tropical morphism*, DT-morphism to shorten, which is a graph morphism φ and an index map m_φ such that (φ, m_φ) is an indexed branched cover and m_φ satisfies the Riemann-Hurwitz inequality; see Equation (RH). Therefore, we defined tropical morphisms Φ as pairs (φ, z) of a discrete tropical morphism $\varphi : G \to T$ and a length function $z : T \to \mathbb{R}_{\geq 0}$. Taking $y' : G \to \mathbb{R}_{\geq 0}$ given by $e \mapsto z(\varphi(e))/m_\varphi(e)$, the data of φ gives rise to a tropical morphism $(G, y') \to (T, z)$.

Fix G and $\varphi : G \to T$. The families C_G and C_T of length functions for G and T, respectively, can be identified with rational polyhedral cones: the positive orthants of $\mathbb{R}^{E(G)}$ and $\mathbb{R}^{E(T)}$, seen as $\mathbb{R}$-vector spaces. The collection of cones $\mathcal{M}_g^{\mathrm{trop}}(3g-3) = \{C_H\}_H$ indexed by trivalent H with genus g constitutes the top-dimensional cones of a tropical moduli space $\mathcal{M}_g^{\mathrm{trop}}$ of dimension $3g-3$, analogous to $\mathcal{M}_g$. The family C_φ of graphs in $\mathcal{M}_g^{\mathrm{trop}}$ that appear as the domain of some map (φ, z) is a rational polyhedral cone as well, since there is a linear map $A_\varphi : C_T \to C_\varphi$, called the *edge-length map*, whose coefficients as a matrix are rational. There is a notion of tropical modification also for φ, leaving the cone C_φ invariant, and yielding a representative such that

$$\dim C_\varphi \leq \#(E(T)) \leq \min(2g + 2d - 5, 3g - 3), \tag{8.8}$$

with equality in the first relation if and only if A_φ has full rank. In that case, A_φ is invertible and a point y in C_φ corresponds to the tropical morphism $(\varphi, A_\varphi^{-1} y)$. The collection of cones $\mathcal{G}_{g\to 0,d}^{\mathrm{trop}}(2g + 2d - 5) = \{C_\varphi\}_\varphi$ indexed by degree-d genus-g full-rank DT-morphisms to trees, i.e. $\varphi : G \to T$ such that G has genus g, the graph T is a tree, $\deg \varphi$ equals d, and A_φ has full rank, constitutes the top-dimensional cones of a tropical moduli space $\mathcal{G}_{g\to 0,d}^{\mathrm{trop}}$ analogous to $\mathcal{G}_r^1$. See Sections 11 and 12 for constructions of these spaces.

We showed that the cones in $\mathcal{G}_{2g'\to 0,g'+1}^{\mathrm{trop}}(6g'-3)$ cover $\mathcal{M}_{2g'}^{\mathrm{trop}}$, hence proving Theorem 8.2 for even genus; odd genus follows from a trick of attaching a loop to the graph, see for example [DV20, Subsection 5.4]. We built upon an earlier construction from [CD18] that for each C_H in $\mathcal{M}_g^{\mathrm{trop}}(3(2g')-3)$ gives a cone C_φ in $\mathcal{G}_{2g'\to 0,g'+1}^{\mathrm{trop}}(6g'-3)$ such that $C_\varphi \subseteq C_H$. We developed a deformation argument that from this initial C_φ produces enough cones to cover C_H. Here, *deforming* $\varphi : G \to T$ means to choose an edge t_1 of T, contract it, and contract all the edges of G that map to t_1. This gives rise to a DT-morphism φ_0 with the property that its edge-length map A_0 is equal to deleting from A_φ the column corresponding to t_1. This fact is useful in computations.

Given a full-rank DT-morphism φ that contracts to φ_0, it is relatively straightforward to list all other $\varphi^{(q)}$ that contract to φ_0. For this we have used a one-to-one correspondence between DT-morphisms and a combinatorial gadget introduced in [CD18] called a *glueing datum*. What is not so straightforward is to prove that among the $\varphi^{(q)}$ that contract to φ_0, there is at least one that covers a part of C_H not yet covered by C_φ.

By Equation (8.8), for C_φ in $\mathcal{G}^{\mathrm{trop}}_{2g'\to 0,g'+1}(6g'-3)$ we get $\dim C_\varphi = 3g-3$ and $\dim C_0 = 3g-4$, where $g = 2g'$ and C_0 is the cone of φ_0. When the span of C_0 cuts C_H in two halves, we have C_φ in one half, and we need a $\varphi^{(q)}$ whose cone $C^{(q)}$ is in the other half. This follows from the existence of positive integers K_{φ/φ_0} such that

$$\sum K_{\varphi/\varphi_0} v_{\varphi/\varphi_0} \equiv 0 \mod \operatorname{span} C_0, \tag{8.9}$$

where v_{φ/φ_0} is the column of $A^{(q)}$ corresponding to the contracted edge t_1, and the sum runs over a multiset that contains all $\varphi^{(q)}$ that contract to φ_0. This is reminiscent to a kind of equation known as *wall-relation* in the context of toric varieties.

The proof of Equation (8.9) is rather involved, resting on two main components. First, among all the $\varphi^{(q)}$ that contract to φ_0, if $\varphi^{(q)}$ does not have full rank, then v_{φ/φ_0} is in $\operatorname{span} C_0$, thus not counting in Equation (8.9). With some effort we characterized the combinatorial behaviour of full-rank DT-morphisms in order to exclude many $\varphi^{(q)}$. So we circumvent the calculation of $\operatorname{rk} A^{(q)}$, a global property, and study instead the local combinatorics above the edge t_1 that gets contracted. We reproduce several of these results in Sections 11, 12 and 13. Second, we did a case analysis for possibilities of φ_0, and on each case proved Equation (8.9). The cases are summarized in Appendix 14.4, and Figure 14.1 illustrates the logical flow. We are working on an implementation of our deformation method[1]. See also [DV21] for an expository paper.

8.3.2 Abstract spaces of polyhedra and their indexed branched covers

The first sections of this Part II are foundational. The aim is to take the above-described machinery which combinatorially studies metric graphs and tropical morphisms using graphs and DT-morphisms, and generalize it to higher dimensions to study abstract spaces of polyhedra and their covers using partially ordered sets and combinatorial morphisms.

In Section 9, we introduce a category PolySpace which contains both metric graphs and spaces that arise by glueing together cones, like $\mathcal{M}_g^{\mathrm{trop}}$ or $\mathcal{G}^{\mathrm{trop}}_{g\to 0,d}$. In general, the objects of PolySpace can be thought as topological spaces that locally have an integral polyhedral structure; the morphisms are piece-wise linear continuous maps preserving this structure. Our main result is that indexed branched covers in PolySpace can be studied combinatorially. Moreover, we show that under a connectivity assumption, a map Φ in PolySpace is an indexed branched cover if and only if it satisfies a balancing condition on the codimension-1 cones of the domain.

To make these notions precise, observe that metric graphs may be regarded as collections of one-dimensional bounded polyhedra, glued at faces in a manner prescribed by a graph G. Going to higher dimensions, we wish to glue polyhedra. Instead of using a graph, we keep track of the to-be-glued polyhedra with a functor σ from a finite category Σ to a category of polyhedra $\mathrm{Poly}_{\mathbb{Z}}$. The objects of $\mathrm{Poly}_{\mathbb{Z}}$ record both the polyhedron and its ambient space, so there is no globally defined ambient space; the morphisms are certain affine maps.

We require the functor σ to satisfy several *lifting conditions* that make Σ capture all the combinatorial information we need; see Definition 9.4 and Remark 9.6. The objects of PolySpace are these functors, and the morphisms are natural transformations. Given $\sigma : \Sigma \to \mathrm{Poly}_{\mathbb{Z}}$ in PolySpace, taking a colimit yields a topological realization $|\Sigma|$ in the category Top of topological spaces with continuous maps. To go back from topology to the category, we define a map $\mathrm{poly}_\Sigma : |\Sigma| \to \Sigma$ that takes a point x in the topological realization to the unique cone of Σ containing x in its topological interior. Several results of this section are summarized by:

[1]Project page: https://github.com/AV-2/Tropical

Theorem A. *Let $\Phi : \Sigma \to \Delta$ be a morphism of polyhedral spaces. Under* mild conditions *we have that the square*

$$\begin{array}{ccc} |\Sigma| & \xrightarrow{\text{poly}_\Sigma} & \Sigma \\ {\scriptstyle |\Phi|}\downarrow & & \downarrow{\scriptstyle \varphi} \\ |\Delta| & \xrightarrow{\text{poly}_\Delta} & \Delta \end{array}$$

is a fibre product in TOP*; and if one of the pairs (φ, m_φ) or $(|\Phi|, \text{poly}_\Sigma \circ m_\varphi)$ is an indexed branched cover, then so is the other pair.*

In Theorem A we have that φ is the map of underlying categories, and the topology on Σ is that a set V is open if $\alpha \in V$ and $\text{Hom}(\alpha, \gamma) \neq \varnothing$ imply that $\gamma \in V$. The mild conditions alluded to are that both Σ and Δ are posets, and that the map $|\Phi|$ restricts to an homeomorphism for each cone of Σ. When the former condition is satisfied, we call Σ and Δ *polyhedral complexes*; and when the latter is satisfied we call Φ a *combinatorial morphism.* These conditions are mild because we show at the end of the section that by using barycentric subdivisions and certain subdivisions induced by Φ, we can always attain them:

Theorem B. *Let $\Phi : \Sigma \to \Delta$ be a morphism of polyhedral spaces. There are morphisms of polyhedral spaces r_Σ, r_Δ, such that their topological realizations are homeomorphisms, the following diagram commutes*

$$\begin{array}{ccc} \Sigma & \xrightarrow{\Phi} & \Delta \\ {\scriptstyle r_\Sigma}\downarrow & & \downarrow{\scriptstyle r_\Delta} \\ \Sigma'' & \xrightarrow{\Phi''} & \Delta'', \end{array} .$$

and Φ'' is a combinatorial morphism of polyhedral complexes.

We close the section by studying how combinatorial morphisms allows to bridge indexed branched covers with the balancing condition, and exploring ways that an index map can be extended in a map of posets.

The idea to use the poset topology, combinatorial morphisms, and to relate $|\Phi|$ with φ via a fibre product square is due [Pay09, Section 2]. The lifting conditions of σ are closely modelled after [CCUW20, Section 2], who go a step further and consider categories fibred in groupoids. It is noted in [ACP15, Section 2] that a barycentric subdivision refines Σ into a poset. In toric and toroidal geometry it is enough to consider categories of cones. But our desire to produce a theory that can also handle tropical morphisms pushes us to generalize. We believe that putting together all these foundational results can be of interest to the tropical community, to combinatorially study other maps between tropical moduli spaces, or general tropical varieties that have both bounded and unbounded parts.

On a historical note, the development of abstract polyhedral spaces of cones as combinatorial gadgets that reflect the properties of an algebraic variety, begins with *polyhedral cone complexes* introduced in Chapter 2 of [MKKS73]. These spaces and slight generalizations have a long history in toric geometry; see [Pay09] for an account. They are now a staple of tropical geometry; e.g. [BMV11; Cha12; ACP15; CMR16; CCUW20] in the study of moduli spaces.

8.3.3 Parametrizing metric graphs with a space of cones $\mathcal{M}_g^{\mathbf{trop}}$

Section 11 and 12 construct $\mathcal{M}_g^{\text{trop}}$ and $\mathcal{G}_{g\to 0,d}^{\text{trop}}$ as polyhedral spaces of cones, i.e. a functor to the subcategory $\text{CONE}_{\mathbb{Z}}^f$ of cones in $\text{POLY}_{\mathbb{Z}}^f$, where $g \geq 2$ and $d \leq \lceil g/2 \rceil + 1$. We begin with $\mathcal{M}_g^{\text{trop}}$. To get a feeling of the category and the functor to $\text{CONE}_{\mathbb{Z}}^f$, observe that for a given graph G and length function y in C_G the metric graph (G, y) is isometric to $(G/S, y_0)$, where S is the subset of e in $E(G)$ such that $y(e) = 0$, the map $\rho : G \to G/S$ is a contraction of edges, and $y_0 = \rho^*(y) = y \circ \rho$ is the pull-back of the length map. The genus of G may drop under edge contractions, so we work in the category WG_g of genus-g connected weighted graphs and specialization morphisms, which are edge-contractions that keep track of contracted cycles using the vertex weight, as in Equation (8.5).

The correspondence $G \mapsto C_G$ and $[\rho : G \to G/S] \mapsto [\rho^* : C_{G/S} \to C_G]$ is a contravariant functor C. The polyhedral space of cones $\mathcal{M}_g^{\mathrm{trop}}$ is the restriction of C to the subcategory of ${\mathrm{WG}_g}^{\mathrm{op}}$ generated by $\mathcal{M}_g^{\mathrm{trop}}(3g-3)$. The points of the topological realization $|\mathcal{M}_g^{\mathrm{trop}}|$ are in one-to-one correspondence with equivalence classes under tropical modification of genus-g weighted metric graphs.

8.3.4 Parametrizing tropical morphisms with a space of cones $\mathcal{G}_{g\to 0,d}^{\mathbf{trop}}$

Similarly, for a given DT-morphism $\varphi : G \to T$ and a length function z in C_T, the tropical morphism (φ, z) is isometric to (φ_0, z_0), where we have a specialization $\rho : \varphi \to \varphi_0$ given by specializations ρ_G, ρ_T that satisfy the commutative square

$$\begin{array}{ccc} G & \xrightarrow{\rho_G} & G_0 \\ \varphi\big\downarrow & & \big\downarrow\varphi_0 \\ T & \xrightarrow{\rho_T} & T_0, \end{array}$$

and $z_0 = z \circ \rho_T$. This is nothing more than contracting a subset S_T of $E(T)$, and all the edges of G that map to S_T. So we let $\mathrm{DTM}_{g\to 0}^d$ be the category with objects the degree-d genus-g DT-morphisms, and morphisms the specialization morphisms. The correspondence $\varphi \mapsto C_\varphi$ and $[\rho : \varphi \to \varphi_0] \mapsto [\rho^* : C_{\varphi_0} \to C_\varphi]$ is a contravariant functor. The polyhedral space of cones $\mathcal{G}_{g\to 0,d}^{\mathrm{trop}}$ is the restriction of C to the subcategory of $\left(\mathrm{DTM}_{g\to 0}^d\right)^{\mathrm{op}}$ generated by $\mathcal{G}_{g\to 0,d}^{\mathrm{trop}}(2g+2d-5)$. The points of the topological realization $|\mathcal{G}_{g\to 0,d}^{\mathrm{trop}}|$ are in one-to-one correspondence with equivalence classes under tropical modification of degree-d genus-g tropical morphisms which move in cones whose dimension achieve the upper bound $\#(E(T))$.

There is a projection morphism $\Pi : \mathcal{G}_{g\to 0,d}^{\mathrm{trop}} \to \mathcal{M}_g^{\mathrm{trop}}$ given by a functor π that sends $\varphi : G \to T$ to the combinatorial type $H(\varphi)$ of G, and the family $\{\Pi_\varphi\}_\varphi$ of inclusions of C_φ in $C_{H(\varphi)}$. Topologically, $|\Pi|$ sends a tropical morphism $\Phi : \Gamma \to \Delta$ in $\mathcal{G}_{g\to 0,d}^{\mathrm{trop}}$ to the equivalence class of Γ in $\mathcal{M}_g^{\mathrm{trop}}$.

8.3.5 Properties of the projection $\Pi : \mathcal{G}_{g\to 0,d}^{\mathbf{trop}} \to \mathcal{M}_g^{\mathbf{trop}}$

Now we fix $g = 2g'$ and $d = g'+1$ in order to study the gonality bound for metric graphs. The main goals in Section 13 are to introduce an index map m_π for the cones of $\mathcal{G}_{g\to 0,d}^{\mathrm{trop}}$, show that (π, m_π) is an indexed branched cover, hence so is $(|\Pi|, m_\pi \circ \mathrm{poly})$, and calculate its degree.

Recall from toric geometry, that the multiplicity of a rational polyhedral cone (N, σ), with primitive generators $\theta_1, \ldots, \theta_s$ for its rays, is equal to the index

$$\mathrm{mult}(\sigma) = [N_\sigma : (\mathbb{Z}\theta_1 + \cdots + \mathbb{Z}\theta_s)], \tag{8.10}$$

where $N_\sigma = N \cap \operatorname{span}\sigma$. While it would be natural to set $m_\pi(\varphi) = \mathrm{mult}(C_\varphi)$, for a top-dimensional φ in $\mathcal{G}_{g\to 0,d}^{\mathrm{trop}}$, what turns out to work in our setting is the multiplicity of the cone associated to the transpose matrix $A_\varphi^\top$; i.e. $m_\pi(\varphi) = \mathrm{mult}(\operatorname{span}_{\mathbb{R}_{\geq 0}} A_\varphi^\top)$ on top dimensional cones. This hints at a dualization process ocurring under the hood, which would be worthy to investigate further. We arrive then to a beautiful outcome, that we envision as a rank-1 tropical version of Theorem 1 in [EH87].

Theorem C. *Let g' be a positive integer, and $g = 2g'$ and $d = g'+1$. The projection $\Pi : \mathcal{G}_{g\to 0,d}^{trop} \to \mathcal{M}_g^{trop}$ given by $[\Phi : \Gamma \to \Delta] \mapsto \Gamma$, with index map $m_\pi(\varphi) = \mathrm{mult}(\widehat{A_\varphi^\top})$ as above, is a surjective indexed branched cover of cone spaces, and* $\deg \Pi$ *equals the g'-th Catalan number.*

The theory from Section 10 reduces Theorem C to showing that m_π satisfies the balancing condition in codimension-1 for certain refinements of $\mathcal{M}_g^{\mathrm{trop}}$ and $\mathcal{G}_{g\to 0,d}^{\mathrm{trop}}$. This uses the fact that $\mathcal{M}_g^{\mathrm{trop}}$ is codimension-1 connected [HT80; Cap12]. We reduce the balancing condition to studying the coefficients of the wall-relation Equation (8.9), and of a similar expression for when $H(\varphi_0)$ is not trivalent. The proof of the latter is deferred to Section 14. To accomplish this, we recall several combinatorial conditions that can be checked locally and are satisfied by the elements of $\mathcal{G}^{\text{trop}}_{g\to 0,d}$. These are necessary, but not sufficient, and lead to the notion of quasi full-rank DT-morphisms. We also need to explore several local structures that can ocurr in quasi full-rank DT-morphisms, but not in full-rank ones.

It remains to calculate $\deg \Pi$. Using the combinatorial conditions on quasi full-rank DT-morphisms we calculate the fibre $\Pi^{-1}(\Gamma)$ for a particular family of Γ. This family consists of a path and loops attached to it via bridges, see Figure 13.1. In Section 10 we also prove that if $\Phi : \Sigma \to \Delta$ is an indexed branched cover, and Δ is codimension-1 connected, then Σ is as well, which gives:

Theorem D. *The cone space $\mathcal{G}^{trop}_{2g'\to 0,g'+1}$ is codimension-1 connected.*

Currently, we do not know if $\mathcal{G}^{\text{trop}}_{g\to 0,d}$ is codimension-1 connected for general g and d. The case of $\mathcal{G}^{\text{trop}}_{g\to 0,2}$ is straightforward. Note that $\Pi(\mathcal{G}^{\text{trop}}_{g\to 0,2})$ gives a locus of hyperelliptic graphs in $\mathcal{M}^{\text{trop}}_g$.

8.3.6 Constructions of tropical morphisms specializing to a given φ_0

Section 14 proves, with the same notation as in Equation (8.9), that in the case where $H(\varphi_0)$ is non-trivalent we have integers K_{φ/φ_0} such that:

$$\sum K_{\varphi/\varphi_0} v_{\varphi/\varphi_0} \equiv 0 \mod \operatorname{span} C_{H_0}. \tag{8.11}$$

The ambient space of Equation (8.11) is not obvious, since generically there are three maximal cones C_1, C_2, C_3 of $\mathcal{M}^{\text{trop}}_g$ with $C_{H(\varphi_0)}$ as a face, and each v_{φ/φ_0} lives in exactly one of these cones. Following foundational work from [Gro18] on tropical cycles in polyhedral space of cones, Equation (8.11) is calculated inside

$$\langle C_{H(\varphi_0)}, h_1^{(1)}, h_1^{(2)}, h_1^{(3)}\rangle / \langle h_1^{(1)} + h_1^{(2)} + h_1^{(3)}\rangle,$$

where $h_1^{(i)}$ is the primitive generator of the ray of C_i not in $C_{H(\varphi_0)}$.

The methods in this section are the closest in spirit to the Part I of this series. Given a top-dimensional $\varphi : G \to T$ in $\mathcal{G}^{\text{trop}}_{g\to 0,d}$, we complete the combinatorial classification, initiated in Section 7 of Part I, of the maps $\varphi_0 : G_0 \to T_0$ that arise by contracting one edge t_1 of $E(T)$ with a specialization morphism $\rho : \varphi \to \varphi_0$. Case-by-case we construct the set star-quasi(φ_0) of quasi full-rank DT-morphisms $\varphi^{(q)}$ such that there is a specialization morphism $\rho_q : \varphi^{(q)} \to \varphi_0$. The cases depend on the valency of $\rho_q(t_1)$ in $V(T_0)$ and the combinatorics of the local part around the non-trivalent vertex A_0. We show that all $\varphi^{(q)}$ in star-quasi(φ_0) have the same multiplicity $m_\pi(\varphi^{(q)})$, and that having the same combinatorial type $H(\varphi^{(q)})$ partitions the elements of star-quasi(φ_0) in sets of equal size.

8.3.7 Future directions

In conclusion, our combinatorial study of $\Pi : \mathcal{G}^{\text{trop}}_{g\to 0,d} \to \mathcal{M}^{\text{trop}}_g$ has uncovered the behaviour of a tropical object. This is one face of the coin, it remains to relate Π to an algebro-geometric object. One possibility is to find a toroidal variety X associated to $\mathcal{G}^{\text{trop}}_{g\to 0,d}$, generalizing the toric setup; i.e. with an open dense set $U \subset X$ that *locally analytically* looks like a torus, and the cones of $\mathcal{G}^{\text{trop}}_{g\to 0,d}$ describe the boundary strata of $X \setminus U$; see [MKKS73, Chapter 2]. Equations (8.9) and (8.11) bear resemblance to expressions in intersection theory of toric varieties; see e.g. [CLS11, Equation (6.4.4)]. We wonder if there is a short geometric argument for them, circumventing our lengthy case work, and extending to $\mathcal{G}^{\text{trop}}_{g\to 0,d}$ for general $g \geq 2$ and $d \leq \lceil g/2 \rceil + 1$. While our case work applies to these $\mathcal{G}^{\text{trop}}_{g\to 0,d}$, additional cases would be needed to completely describe the wall-relations, since for $\mathcal{G}^{\text{trop}}_{2g'\to 0,g'+1}$ we use the fact that $H(\varphi)$ is trivalent for top-dimensional φ. The latter is not true in general, in Example 12.46 we exhibit a top-dimensional cone C_φ of the space of genus-3 hyperelliptic graphs $\mathcal{G}^{\text{trop}}_{3\to 0,2}$ such that $H(\varphi)$ is non-trivalent. We foresee that for such argument one would explore whether the wall-relations correspond to tropical cycles, per the theory of [Gro18].

We would like a better understanding of the index map m_π, both from the combinatorial and the geometrical side. Given a tropical morphism $\Phi : \Gamma \to \Delta$ in $\mathcal{G}^{\text{trop}}_{2g' \to 0, g'+1}$ with underlying DT-morphism φ, there are three natural multiplicities:

1. The multiplicity $m_\pi(\varphi)$ defined on the top-dimensional cones, satisfying the balancing condition.

2. The multiplicity $\operatorname{mult}(\operatorname{span}_{\mathbb{R}_{\geq 0}} A^\top_\varphi)$.

3. The number of maps $\phi : X \to \mathbb{P}^1$ that tropicalize to Φ, for a fixed curve X such that its tropicalization Γ_X is equivalent to Γ in $\mathcal{M}^{\text{trop}}_g$.

We have shown that (1) satisfies a balancing condition, while (2) is easier to calculate, and (3) provides a connection with tropicalizations and could be applied to enumerative problems. Also, (2) and (3) generalize immediately to $\mathcal{G}^{\text{trop}}_{g \to 0,d}$ for general $g \geq 2$ and $d \leq \lceil g/2 \rceil + 1$, whereas generalizing (1) is asking whether $\Pi : \mathcal{G}^{\text{trop}}_{g \to 0,d} \to \mathcal{M}^{\text{trop}}_g$ is an indexed branched cover as well. By definition (1) and (2) coincide when φ is top-dimensional. Ideally all three would coincide for all cones of φ of $\mathcal{G}^{\text{trop}}_{g \to 0,d}$. Also, Example 13.27 suggests that DT-morphisms with high multiplicity produce graphs that are interesting for probing Question 8.3. Currently we do not know if $m_\pi(\varphi)$ can be arbitrarily big, the biggest we have found is $m_\pi(\varphi) = 4$ via a computer search.

Heading towards more speculative territory, inspired by the work of [Thu07; ACP15; CMR16] we wonder if there is a toroidal variety $U \hookrightarrow X$, a retraction $\mathbf{p}_X : X^{\text{an}} \to X^{\text{an}}$ from the Berkovich analytification of X to the skeleton $\Sigma(X)$, a morphism of polyhedral spaces of cones $\operatorname{trop} : \Sigma(X) \to \mathcal{G}^{\text{trop}}_{g \to 0,d}$, such that U corresponds to a moduli problem and X to a nice compactification; e.g. maps to $\mathbb{P}^1$ and a compactification via admissible covers. We would hope this to render tropicalization as a functorial relation, potentially improving the understanding between classical Brill-Noether and the combinatorial spaces $\mathcal{G}^{\text{trop}}_{g \to 0,d}$ and loci $\Pi(\mathcal{G}^{\text{trop}}_{g \to 0,d})$ of d-gonal metric graphs. There is also much work left to do on the higher rank case; i.e. to introduce and study maps $\varphi : \Gamma \to \Sigma$ from a metric graph Γ to a tropical analogue Σ of $\mathbb{P}^r$ such that the pullback $\varphi^*(x)$ of a point $x \in \Sigma$ is a divisor of rank r and degree $\deg \varphi$.

Chapter 9

Polyhedral spaces

We embark into a journey foundational results. In Subsection 9.1 we begin with a reminder of polyhedral geometry done in real vector spaces with a choice of a lattice to act as an integral structure, and introduce the abstract notion of polyhedral spaces to generalize to a non-embedded setting.

9.1 Polyhedral spaces

These first subsections describe a category-theoretical framework for glueing together rational polyhedra. We draw much inspiration from [ACP15] and [CCUW20], who glue together rational cones. The aim of our generalization is to produce a framework that unifies the treatment of metric graphs Γ, tropical morphisms $\varphi : \Gamma \to \Delta$, the tropical moduli spaces $\mathcal{G}^{\mathrm{trop}}_{g\to 0,d}$ and $\mathcal{M}^{\mathrm{trop}}_{g}$, and the projection $\Pi : \mathcal{G}^{\mathrm{trop}}_{g\to 0,d} \to \mathcal{M}^{\mathrm{trop}}_{g}$.

We begin with a quick tour through the embedded version of the story. Let V be a real vector space of finite dimension, with the Euclidean topology. Given a functional u in the dual space $V^* = \mathrm{Hom}(V, \mathbb{R})$ and a constant $c \in \mathbb{R}$ we consider three spaces: the closed upper half-space $H^+(u,c) = \{x \in V \colon\ u(x) \geq c\}$, the closed lower half-space $H^-(u,c) = \{x \in V \colon\ u(x) \leq c\}$, and the hyperplane $H(u,c) = H^+(u,c) \cap H^-(u,c)$. A *polyhedron* in V is a non-empty intersection of finitely-many closed upper half-spaces. Note that a polyhedron is a convex set, and closed in the Euclidean topology of V.

An *integral structure* on V is a choice of a full-rank lattice N, i.e. a subgroup of V such that N is a discrete subset and $V = \mathrm{span}_{\mathbb{R}} N$. In tropical geometry, finite dimensional real vector spaces with integral structures arise naturally from field valuations. In toric geometry, one usually begins with a free abelian group N of finite rank, and takes as vector space the tensor product $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$. The elements of N are the *integral points*, and the elements of $N^* = \mathrm{Hom}(N, \mathbb{Z})$ are the *integral functionals*. Given $u \in N^*$, we linearly extend the domain from N to $\mathrm{span}_{\mathbb{R}} N = V$ to get a functional on V. A half-space is *rational* if u is in N^* and c is in $\mathbb{Z}$, and a polyhedron is rational if its defining half-spaces are rational. An integrally affine map $f : (V, N) \to (V', N')$ is an affine map $f : V \to V'$ such that $f(N) \subset N'$; i.e. $f(x) = y + L(x)$ with $y \in N'$ and $L \in \mathrm{Hom}(N, N')$ extended to a map from V to V'. Integrally affine maps send rational polyhedra to rational polyhedra.

Definition 9.1. Let Poly be the category of pairs (N, σ) of a finite-rank free abelian group N and a polyhedron σ in $N_{\mathbb{R}}$, with morphisms $f : (N, \sigma) \to (N', \sigma')$ given by integrally affine maps such that $f(\sigma) \subset \sigma'$.

Example 9.2. A non-empty subset C of a real vector space V is a *cone* if it is closed under multiplication by a non-negative real number. Let $\sigma = \bigcap H(u_i, c_i)$. The polyhedron σ is a cone if and only if $c_i = 0$ for all indices i. In this case we call σ a *polyhedral cone*. ⋆

Given a polyhedron σ in $N_{\mathbb{R}}$, a hyperplane $H(u,c)$ is *supporting* if $\sigma \cap H(u,c)$ is non-empty and $\sigma \subset H^+(u,c)$. A *face* of σ is a polyhedron τ of the form $\sigma \cap H(u,c)$, with $H(u,c)$ a supporting hyperplane of σ. The dimension $\dim \sigma$ of σ is the dimension of its affine span aff-span σ, i.e. the smallest set that contains σ and is a translate of a linear subspace of V. A k-face of σ is a face of dimension k. The 0-faces are called *vertices*. If σ is rational with vertices in N, then for all faces τ of σ we have $\text{aff-span}\,\tau = \text{aff-span}(N \cap \text{aff-span}\,\tau)$. We let $N^\tau = N \cap \text{aff-span}\,\tau$. A translation taking a point of N^τ to 0 makes this set a lattice, so (N^τ, τ) is a polyhedron as well. Thus, the following category is closed under taking faces:

Definition 9.3. Let $\text{POLY}_{\mathbb{Z}}$ be the subcategory of POLY induced by the pairs (N, σ) such that σ is rational, all the vertices of σ are in N, and $N = N^\sigma$.

A *face morphism* $f : (N, \sigma) \to (N', \sigma')$ is a morphism of polyhedra such that $f(\sigma) = \sigma'$ and f maps N bijectively to $(\text{aff-span}\, f(N)) \cap N'$. In particular, the inclusion $(N^\tau, \tau) \to (N^\sigma, \sigma)$ is a face morphism. We say that f is proper if $f(\sigma)$ is a proper face of σ'. We denote by $\text{POLY}^f_{\mathbb{Z}}$ the subcategory of $\text{POLY}_{\mathbb{Z}}$ restricted to face morphisms. The non-proper face morphisms are precisely the isomorphisms in $\text{POLY}^f_{\mathbb{Z}}$. A polyhedral space is a collection of polyhedra glued by face morphisms.

Definition 9.4. Let Σ be a finite category, and $\sigma : \Sigma \to \text{POLY}^f_{\mathbb{Z}}$ a functor $\alpha \mapsto (N^\alpha, \sigma_\alpha)$. We slightly abuse notation and shorten $(N^\alpha, \sigma_\alpha)$ to just σ_α. We say that σ is a *polyhedral space* if the following *lifting conditions* are satisfied:

(a) For each α in Σ, and each proper face inclusion $f_{\tau\sigma_\alpha} : (N, \tau) \to (N^\alpha, \sigma_\alpha)$ in $\text{POLY}^f_{\mathbb{Z}}$, there is a morphism f in Σ such that $\sigma(f) = f_{\tau\sigma_\alpha}$.

(b) For any two morphisms $g : \gamma \to \alpha$ and $h : \beta \to \alpha$ in Σ that give proper face morphisms $\sigma(g)$ and $\sigma(h)$, there is a bijection of diagrams in Σ and in $\text{POLY}^f_{\mathbb{Z}}$ as shown in Diagram 9.5.

Diagram 9.5

In other words, for every face morphism $f_{\sigma_\alpha\sigma_\beta} : \sigma_\alpha \to \sigma_\beta$ for which the diagram on the right commutes, there is exactly one face morphism f such that the diagram on the left commutes and $\sigma(f) = f_{\sigma_\alpha\sigma_\beta}$.

(c) The only isomorphisms in Σ are self-maps.

Remark 9.6. Definition 9.4 follows closely Definition 2.15 of [CCUW20], but we do not pursue 2-categorical aspects. Condition (a) implies that Σ captures information on all the faces in the polyhedral space. Condition (b) is only used to prove unicity at the crucial Lemma 9.16. Condition (c) is for the convenience of working with a skeleton category, as done in [ACP15], and plays a role in endowing Σ with the structure of a partially ordered set later on. △

Definition 9.7. A morphism $\Phi : [\sigma : \Sigma \to \text{POLY}^f_{\mathbb{Z}}] \to [\delta : \Delta \to \text{POLY}^f_{\mathbb{Z}}]$ is a pair $(\varphi, \{\Phi_\alpha\}_{\alpha\in\Sigma})$ of a functor $\varphi : \Sigma \to \Delta$ and a natural transformation $\{\Phi_\alpha : \sigma_\alpha \to \delta_{\varphi(\alpha)}\}_{\alpha\in\Sigma}$ from σ to $\delta \circ \varphi$ such that the image of Φ_α is not contained in a proper face of $\delta_{\varphi(\alpha)}$. That is, for every $f : \alpha \to \beta$ in Σ we have that Diagram 9.8 in $\text{POLY}^f_{\mathbb{Z}}$ commutes.

$$\begin{array}{ccc} \sigma_\alpha & \xrightarrow{\sigma(f)} & \sigma_\beta \\ \Phi_\alpha \downarrow & & \downarrow \Phi_\beta \\ \delta_{\varphi(\alpha)} & \xrightarrow{\delta(\varphi(f))} & \delta_{\varphi(\beta)} \end{array}$$

Diagram 9.8

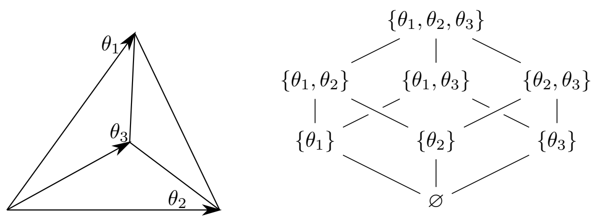


Figure 9.1: On the left, a simplicial cone σ of dimension 3. On the right, the face diagram of σ when embedded in PolySpace.

We denote by PolySpace the category of polyhedral spaces with morphisms given by Definition 9.7. When it causes no confusion we write Σ for $\sigma : \Sigma \to \text{Poly}^f_{\mathbb{Z}}$. There is a faithful embedding $\text{Poly}_{\mathbb{Z}} \to \text{PolySpace}$ by associating to a polyhedron σ the polyhedral space of all proper and non-proper faces of σ.

Example 9.9. We say that an n-dimensional polyhedral cone (N, σ) is *simplicial* if its set of 1-faces is linearly independent. This is equivalent to requiring that σ has n-many 1-faces. In this case, any subset of 1-faces generates a face. Thus, the face diagram of σ is isomorphic to the power set of $[n]$, with morphisms given by the containment relation. See Figure 9.1 for an example. $\star$

Let $\sigma : \Sigma \to \text{Poly}^f_{\mathbb{Z}}$ be a polyhedral space. A polyhedron σ_α is *top-dimensional* if there is no morphism $\alpha \to \beta$ in Σ such that $\dim \sigma_\alpha < \dim \sigma_\beta$. By Condition (c) of Definition 9.4 we have that σ_α is top-dimensional if and only if all the morphisms in Σ with domain α are self-maps. If all the top-dimensional polyhedra have the same dimension n then we say that Σ is of *pure dimension* n.

9.2 Topological realization

Now we glue the polyhedra $\{\sigma_\alpha\}_{\alpha \in \Sigma}$ to obtain a topological space $|\Sigma|$. Let Top be the category with objects the topological spaces and morphisms the continuous functions. Recall that all finite colimits exist in Top. There is a faithful topological realization functor $|\cdot| : \text{Poly} \to \text{Top}$ mapping (N, σ) to $|\sigma|$, the topological space with underlying set σ and topology induced from $N_{\mathbb{R}}$; and mapping a morphism $f : (N', \sigma') \to (N, \sigma)$ to the map $f : |\sigma'| \to |\sigma|$, which is continuous since f is affine. As noted in Remark 2.2.1 of [ACP15], this extends to a faithful topological realization functor for PolySpace, which sends Σ to the colimit

$$|\Sigma| = \operatorname*{colim}_{\alpha \in \Sigma} |\sigma_\alpha|$$

in Top. The universal maps of this colimit are $p_\alpha : |\sigma_\alpha| \to |\Sigma|$, which satisfy $p_\alpha = p_\beta \circ \sigma(f)$ for any $f : \alpha \to \beta$ in Σ. Recall that $|\Sigma|$ carries the final topology, namely the finest topology such that p_α is continuous for all α.

Remark 9.10. Let $\Phi : \Sigma \to \Delta$ be a morphism of polyhedral spaces. The topological realization of Φ is a continuous function $|\Phi| : |\Sigma| \to |\Delta|$ obtained by glueing all the $|\Phi_\alpha|$, such that for all $\alpha \in \Sigma$ we have that Diagram 9.11 commutes. $\triangle$

$$\begin{array}{ccc} |\sigma_\alpha| & \xrightarrow{p_\alpha} & |\Sigma| \\ {\scriptstyle |\Phi_\alpha|}\downarrow & & \downarrow{\scriptstyle |\Phi|} \\ |\delta_{\varphi(\alpha)}| & \xrightarrow{p_{\varphi(\alpha)}} & |\Delta| \end{array}$$

Diagram 9.11

Remark 9.12. We have not used any of the lifting conditions from Definition 9.4. In fact, any functor $\sigma : \Sigma \to \text{Poly}^f_{\mathbb{Z}}$ and any morphism $(\varphi, \{\Phi_\alpha\}_{\alpha \in \Sigma})$ of functors has a topological realization. This fact is relevant in Subsection 11.2. $\triangle$

9.3 Relative interior

Let σ be a polyhedron in V, with aff-span σ not necessarily equal to V. Recall that a point x in σ is *interior* if for any supporting hyperplane $H(u,c)$ of σ such that $u(x)=c$ we have that $\sigma \subset H(u,c)$. The *relative interior* σ° of σ is the set of interior points of σ. It equals the complement in $|\sigma|$ of the union of all the proper faces of σ. It is also the topological interior of $|\sigma|$ as a subspace of aff-span σ, which motivates the naming. It commutes with affine maps, namely $f(\sigma^\circ) = f(\sigma)^\circ$. Given x in σ, there is a unique face τ of σ such that x is in τ°. See [Zie12, Section 2.3] for proofs of these facts. We explore how the relative interior interacts with affine maps and with face morphisms.

Lemma 9.13. *Let $f : (N_\sigma, \sigma) \to (N_\delta, \delta)$ be an affine map in* Poly. *If $f(\sigma)$ is not contained in a proper face of δ, then $f(\sigma^\circ) \subseteq \delta^\circ$.*

Proof. Since f is an affine map, $f(\sigma)$ is a polyhedron in aff-span N_δ. Suppose there is a point $x \in f(\sigma)^\circ$ contained in a proper face τ of δ. Let $H(u,c)$ be a supporting hyperplane of δ such that $\tau = H(u,c) \cap \delta$. So x is in $H(u,c)$ and since $f(\sigma) \subset \delta \subset H^+(u,c)$, we have that $H^+(u,c)$ is a supporting hyperplane of $f(\sigma)$. Since x is in $f(\sigma)^\circ$, the definition of interior point gives that $f(\sigma) \subset H(u,c)$, thus $f(\sigma) \subset \tau$, a contradiction. Hence, $f(\sigma)^\circ$ is disjoint from all proper faces, which means that $f(\sigma)^\circ \subset \delta^\circ$. We are done since $f(\sigma^\circ) = f(\sigma)^\circ$. □

Let $(\varphi, \{\Phi_\alpha\}_{\alpha\in\Sigma})$ be a morphism of polyhedral spaces. Recall from Definition 9.7 that $\operatorname{im}\Phi_\alpha$ is not contained in a proper face of $\delta_{\varphi(\alpha)}$, so the conditions of Lemma 9.13 are satisfied. If we deal with a face morphism, more can be said:

Lemma 9.14. *Let Σ be a polyhedral space, α and β elements in Σ, and $p_\alpha : |\sigma_\alpha| \to |\Sigma|$ and $p_\beta : |\sigma_\beta| \to |\Sigma|$ the universal maps. There is a morphism $\alpha \to \beta$ if and only if $p_\alpha(\sigma_\alpha^\circ) \cap \operatorname{im} p_\beta \neq \varnothing$.*

Proof. For notational convenience, in this proof we write σf instead of $\sigma(f)$. If there is a morphism $f_{\alpha\beta} : \alpha \to \beta$, then by the universal property of $|\Sigma|$ we have $p_\alpha = p_\beta \circ \sigma f_{\alpha\beta}$. So not only is the intersection $p_\alpha(\sigma_\alpha^\circ) \cap \operatorname{im} p_\beta$ non-empty, in fact it equals $p_\alpha(\sigma_\alpha^\circ)$.

Now suppose there is $z \in p_\alpha(\sigma_\alpha^\circ) \cap \operatorname{im} p_\beta$. Let $x \in \sigma_\alpha^\circ$ and $y \in \sigma_\beta$ be such that $p_\alpha(x) = p_\beta(y) = z$, and τ the unique face of σ_β such that $y \in \tau^\circ$. By Condition (a) of Definition 9.4 there is a morphism $\iota : \alpha' \to \beta$ in Σ that maps $\sigma_{\alpha'}$ isomorphically to τ in σ_β. Let x' in $\sigma_{\alpha'}$ be the preimage of y under $\sigma\iota$.

Note that x' is in $\sigma_{\alpha'}^\circ$ and $p_{\alpha'}(x') = z$. This means that x and x' are connected by a sequence of face morphisms; e.g. $\alpha \xrightarrow{f_1} \alpha_1 \xleftarrow{f_2} \alpha_2 \xleftarrow{f_3} \dots \xrightarrow{f_n} \alpha'$ with $x' = \sigma f_n(\dots \sigma f_3^{-1}(\sigma f_2^{-1}(\sigma f_1(x))))$. The strategy is to replace the first two morphisms in such a sequence by a single morphism. Iterating this step gives a length-1 sequence, which we argue is an isomorphism, so $\alpha = \alpha'$ by Condition (c) of Definition 9.4, which makes $\iota : \alpha' \to \beta$ fulfill the conditions.

There are 4 cases for the first 2 morphisms; namely, $\alpha \xrightarrow{f_1} \alpha_1 \xrightarrow{f_2} \alpha_2$, $\alpha \xleftarrow{f_1} \alpha_1 \xleftarrow{f_2} \alpha_2$, $\alpha \xleftarrow{f_1} \alpha_1 \xrightarrow{f_2} \alpha_2$, and $\alpha \xrightarrow{f_1} \alpha_1 \xleftarrow{f_2} \alpha_2$. In the first two we compose the morphisms. In the third case, we have that $\sigma f_1(\sigma_{\alpha_1}) \subset \sigma_\alpha$, so $\dim \alpha_1 \leq \dim \alpha$. Note that x is in $\sigma f_1(\sigma_{\alpha_1})$, so the inequality cannot be strict because if it were then $\sigma f_1(\sigma_{\alpha_1})$ would be a proper face of σ_α, contradicting that x is in σ_α°. Thus, $\dim \alpha_1 = \dim \alpha$, so f_1 is an isomorphism and we can reverse the arrow to fall into the first case.

The fourth case gives Diagram 9.15. There are points $x_2 \in \sigma_{\alpha_2}$ and $x_1 \in \sigma_{\alpha_1}$, and face morphisms $f_1 : N^\alpha \to N^{\alpha_1}$ and $f_2 : N^{\alpha_2} \to N^{\alpha_1}$, such that $x_1 = \sigma f_1(x) = \sigma f_2(x_2)$. Observe that $\operatorname{im} \sigma f_1$ and $\operatorname{im} \sigma f_2$ are faces of σ_{α_1}, and that the latter contains an interior point of the former, namely $x_1 = \sigma f_1(x)$. Thus, $\operatorname{im} \sigma f_2$ contains $\operatorname{im} \sigma f_1$. Since both maps are face morphisms, they are injective and affine, so $(\sigma f_2)^{-1} \circ \sigma f_1$ restricts to a face morphism $\sigma_\alpha \to \sigma_{\alpha_2}$. By Condition (b) of Definition 9.4, there exists a unique lift $\tilde{f} : \alpha \to \alpha_2$. Note that $x_2 = \sigma\tilde{f}(x)$, so we replace $\alpha \xrightarrow{f_1} \alpha_1 \xleftarrow{f_2} \alpha_2$ with $\alpha \xrightarrow{\tilde{f}} \alpha_2$.

Diagram 9.15

Iterating, we get a length-1 sequence $f : \alpha \to \alpha'$ with $\sigma f(x) = x'$. If $\alpha \neq \alpha'$, then Condition (c) of Definition 9.4 implies that $\dim \alpha \neq \dim \alpha'$, but then $\sigma f(x)$ is contained in a proper face of $\sigma_{\alpha'}$, contradicting that $x' = \sigma f(x)$ is in $\sigma_{\alpha'}^\circ$. ☐

Lemma 9.14 implies a dichotomy, namely $p_\alpha(\sigma_\alpha^\circ) \cap \operatorname{im} p_\beta$ is either empty, or equal to $p_\alpha(\sigma_\alpha^\circ)$, for all α and β in Σ.

9.4 Decomposing the topological realization

The structure of Σ induces a stratification that decomposes the topological space $|\Sigma|$ as a disjoint union of the relative interiors of the polyhedra σ_α, modulo automorphisms. This is a straightforward generalization of Proposition 2.6.2 from [ACP15].

Lemma 9.16. *Let $\sigma : \Sigma \to \mathrm{POLY}^f_{\mathbb{Z}}$ be in* POLYSPACE. *We have that*

$$|\Sigma| = \bigsqcup_{\alpha \in \Sigma} p_\alpha(\sigma_\alpha^\circ).$$

Proof. Recall that the underlying set of $|\Sigma| = \operatorname{colim}_{\alpha\in\Sigma}|\sigma_\alpha|$ is the disjoint union $\bigsqcup_{\alpha\in\Sigma}|\sigma_\alpha|$ modulo the equivalence relation $x \sim x'$ identifying those $x \in |\sigma_\alpha|$ and $x' \in |\sigma_{\alpha'}|$ for which there is a morphism $f : \alpha \to \alpha'$ with $\sigma f(x) = x'$. The universal map p_α equals the composition $|\sigma_\alpha| \hookrightarrow \bigsqcup_{\alpha\in\Sigma}|\sigma_\alpha| \mapsto \bigsqcup_{\alpha\in\Sigma}|\sigma_\alpha|/\sim$. Given x in $|\Sigma|$, there is γ in Σ with Choose $\hat{x}$ in the fibre $p_\gamma^{-1}(x)$, and τ a face of σ_γ with $\hat{x}$ in τ°. By Property (a) of Definition 9.4 there is a morphism $f : \alpha \to \gamma$ in Σ such that σf maps σ_α to τ. Thus, $\hat{x}$ is in $\tau^\circ = \sigma f(\sigma_\alpha)^\circ = \sigma f(\sigma_\alpha^\circ)$, and $x = p_\gamma(\hat{x})$ is in $(p_\gamma \circ \sigma f)(\sigma_\alpha^\circ) = p_\alpha(\sigma_\alpha^\circ)$, so indeed $|\Sigma| = \bigcup p_\alpha(\alpha_\alpha^\circ)$. By Lemma 9.14 this union is disjoint: if there were a point x in $p_\alpha(\sigma_\alpha^\circ)$ and $p_{\alpha'}(\sigma_{\alpha'}^\circ)$, we would have morphisms $\alpha \to \alpha'$ and $\alpha' \to \alpha$, so α and α' would be isomorphic, hence equal by Property (c) of Definition 9.4. ☐

Putting Lemmas 9.14 and 9.16 together, we can describe intersections $\operatorname{im} p_\alpha \cap \operatorname{im} p_\beta$.

Lemma 9.17. *Let $\sigma : \Sigma \to \mathrm{POLY}^f_{\mathbb{Z}}$ be in* POLYSPACE*, and α, β in Σ. We have that*

$$\operatorname{im} p_\alpha \cap \operatorname{im} p_\beta = \bigcup_\eta \operatorname{im} p_\eta,$$

where η ranges over the domains of morphisms in $\operatorname{Hom}(-,\alpha) \cap \operatorname{Hom}(-,\beta)$.

Proof. If x is in $\operatorname{im} p_\alpha \cap \operatorname{im} p_\beta$, then Lemma 9.16 gives an $\eta \in \Sigma$ with $x \in p_\eta(\sigma_\eta^\circ)$, so Lemma 9.14 gives morphisms $\eta \to \alpha$ and $\eta \to \beta$. Conversely, if there are morphisms $f : \eta \to \alpha$ and $g : \eta \to \beta$, we have $p_\eta = p_\alpha \circ \sigma f$ and $p_\eta = p_\beta \circ \sigma g$, so $\operatorname{im} p_\eta \subset \operatorname{im} p_\alpha \cap \operatorname{im} p_\beta$. ☐

Since all the self-maps are isomorphisms, we denote $\operatorname{Hom}_\Sigma(\alpha,\alpha)$ by $\operatorname{Aut}\alpha$. Note that $\operatorname{Aut}\alpha$ acts on $|\sigma_\alpha|$ and that p_α factors through the projection to the quotient space $|\sigma_\alpha|/\operatorname{Aut}\alpha$ that identifies points in the same orbit. We get a map $\overline{p}_\alpha$ that is an homeomorphism onto its image; see Diagram 9.18. Thus, by Lemma 9.16, $|\Sigma|$ decomposes as a disjoint union of interiors of polyhedra modulo automorphisms.

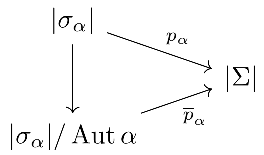


Diagram 9.18

9.5 Polyhedral complexes

If $\operatorname{Aut}\alpha$ is trivial for all $\alpha \in \Sigma$, then the p_α in Diagram 9.18 are injective, hence $|\Sigma|$ is locally polyhedral in a straightforward manner. This happens, for example, if $\operatorname{Hom}_\Sigma(\alpha, \beta)$ has at most one element for all $\alpha, \beta \in \Sigma$.

Definition 9.19. A *rational polyhedral complex*, or polyhedral complex for short, is a polyhedral space $\sigma : \Sigma \to \textsc{Poly}^f_\mathbb{Z}$ such that $\#(\operatorname{Hom}_\Sigma(\alpha, \beta)) \leq 1$. We write $f_{\alpha\beta}$ for the unique element in $\operatorname{Hom}_\Sigma(\alpha, \beta)$, if there is one, and PolyComplex for the full subcategory of PolySpace induced by the polyhedral complexes.

While non-trivial groups of automorphisms frequently arise in the construction of moduli spaces, we work first in the more favourable subcategory PolyComplex; later, in Subsection 9.8 we develop tools to study $\Pi : \mathcal{G}^{\text{trop}}_{g\to 0,d} \to \mathcal{M}^{\text{trop}}_g$ inside this subcategory.

Example 9.20. Let (V, N) be a vector space with an integral structure, and Σ a finite family of integral polyhedra in (V, N). We call Σ an *embedded polyhedral complex* if the following two conditions are satisfied:

(a) If σ is in Σ and τ is a face of σ, then τ is in Σ.

(b) If σ_1, σ_2 are in Σ and $\sigma_1 \cap \sigma_2 \neq \varnothing$, then $\sigma_1 \cap \sigma_2$ is a face of σ_1 and a face of σ_2.

If we consider Σ as a category by regarding set inclusions as morphisms, and take σ to be the identity functor, we get a polyhedral complex per Definition 9.4. We have

$$|\Sigma| \sim \bigcup_{\sigma\in\Sigma} \sigma, \tag{9.1}$$

the topological realization generalizes the support of the embedded polyhedral complex. ⋆

Since an embedded polyhedral complex is a family of subsets of a vector space V, the containment relation $\subseteq$ induces a natural poset structure. That is, a set with a partial order, namely a relation which is reflexive, transitive and antisymmetric. This poset is an important combinatorial invariant, so we argue that there is also a natural poset structure for a polyhedral space.

Remark 9.21. Given objects α, β in an arbitrary category Σ, we write $\alpha \preceq \beta$ if there is an arrow $\alpha \to \beta$. This relation is reflexive because $\mathrm{id}_\alpha : \alpha \to \alpha$ is in $\operatorname{Hom}_\Sigma(\alpha, \alpha)$, and transitive because arrows compose. A reflexive transitive relation is called a *preorder*. We get symmetry, hence that $\preceq$ is a partial order, if for example Σ is a skeleton category and we prove that $\alpha \preceq \beta$ together with $\beta \preceq \alpha$ implies that α and β are isomorphic. △

Lemma 9.22. *If $\sigma : \Sigma \to \textsc{Poly}^f_\mathbb{Z}$ is a polyhedral space, then $(\Sigma, \preceq)$ is a poset.*

Proof. If $\alpha \preceq \beta$ and $\beta \preceq \alpha$, we have face morphisms $f : \alpha \to \beta$ and $g : \beta \to \alpha$. Hence $\dim \sigma_\alpha \leq \dim \sigma_\beta$ and $\dim \sigma_\beta \leq \dim \sigma_\alpha$. So $\dim \sigma_\alpha = \dim \sigma_\beta$, which means that the face morphism f is an isomorphism. We are done since Σ is a skeleton category by Condition (c) of Definition 9.4. □

We call $(\Sigma, \preceq)$ the *face poset*. When Σ is a polyhedral complex, the poset structure on Σ fully captures the structure of Σ as a category since $\#(\operatorname{Hom}_\Sigma(\alpha, \beta)) \leq 1$. In general, the face poset forgets how many morphisms there are between objects of Σ.

9.6 Partially ordered sets

Before proceeding, we pause for an intermission on posets, which mostly follows [Sta11, Chapter 3]. Let $(\mathcal{C}, \leq)$ be a finite partially ordered set, or *poset* for short. The relation $\leq$ induces a partial order on any subset of $\mathcal{C}$ by restriction. Given x in $\mathcal{C}$, the up-set and down-set generated by x are the posets

$$\uparrow_{\mathcal{C}} x = \{y \in \mathcal{C} : \ x \leq y\}, \qquad \downarrow_{\mathcal{C}} x = \{y \in \mathcal{C} : \ y \leq x\}.$$

If the context allows, we simply write $\uparrow x$ and $\downarrow x$. A subset S of $\mathcal{C}$ is a *down-set* if $x \in S$ implies that $\downarrow x \subseteq S$. One defines an up-set likewise.

Arbitrary unions and intersections of up-sets produce an up-set. Thus, they can be taken as the open sets of a topology, which we call the *poset topology* on $\mathcal{C}$. The complement of an up-set is a down-set. A map $f : \mathcal{C} \to \mathcal{D}$ is order preserving if $x \leq y$ implies $f(x) \leq f(y)$ for all $x, y \in \mathcal{C}$. The posets, with order preserving maps as morphisms, make a category Poset. Moreover, endowing $\mathcal{C}$ with the poset topology gives a fully faithful functor from Poset to Top, because a map $f : \mathcal{C} \to \mathcal{D}$ is continuous in the poset topology if and only if f is order preserving.

Remark 9.23. The topology where the up-sets of a preorder, not necessarily finite, are the open sets is also known as Alexandrov topology, Alexandrov-discrete space or finitely generated space. This topology satisfies the separation axiom T_0, i.e. all points are topologically distinguishable, if and only if the preorder that induces it is a partial order. This topology satisfies T_1 if and only if the space is discrete. Hence, all interesting examples happen to be non-metrizable. △

If $\mathcal{C}$ is a poset, we say that y in $\mathcal{C}$ is *minimal* if $x \leq y$ implies that $x = y$, and *maximal* if $x \geq y$ implies that $x = y$. We denote by $\min \mathcal{C}$ the set of minimal elements of $\mathcal{C}$, and $\max \mathcal{C}$ the set of maximal elements. An element y *covers* x if y is minimal in $(\uparrow x) \setminus \{x\}$. Two elements x and y are *comparable* if either $x \leq y$ or $y \leq x$. An order is *total* if all pairs of elements are comparable. A chain L is a subset of $\mathcal{C}$ where all elements are pairwise comparable. A chain is *maximal* if it is inclusion-wise maximal in the set of chains of $\mathcal{C}$. The length $\operatorname{length}(L)$ of a chain L is $\#(L) - 1$. The length of $\mathcal{C}$ is the maximum length of a chain of $\mathcal{C}$. If all maximal chains have equal length then we say that $\mathcal{C}$ is graded. A *rank function* $r : \mathcal{C} \to \mathbb{Z}$ is a map such that y covers x if and only if $r(y) = r(x) + 1$. Whenever we work with a rank function r, we assume it is zero on minimal elements, clearly this determines r. An example of a class of posets that admit a rank function are graded posets.

Remark 9.24. For Σ in PolyComplex we have that $\dim \sigma_\alpha = \operatorname{length}(\downarrow \alpha)$. Thus, Σ is pure-dimensional if and only if $(\Sigma, \preceq)$ is a graded poset; in this case the rank function is the dimension function. △

It is straightforward to define products in Poset. Quotients are slightly more subtle.

Example 9.25. Let Σ and Σ' be posets, and $\sim$ an equivalence relation on Σ.

- The *product of posets* is just the set theoretical product $\Sigma \times \Sigma'$ with the relation $(\alpha_1, \beta_1) \preceq_{\Sigma \times \Sigma'} (\alpha_2, \beta_2)$ if and only if$\alpha_1 \preceq_\Sigma \alpha_2$ and $\beta_1 \preceq_{\Sigma'} \beta_2$.
- The quotient $\Sigma / \sim$ is a preorder with the relation $\overline{\alpha} \preceq_\sim \overline{\beta}$ if and only if there are $\alpha \in \overline{\alpha}$ and $\beta \in \overline{\beta}$ such that $\alpha \preceq \beta$.

The quotient is not always a poset, for example $\Sigma = [3]$ with $1 \preceq 2 \preceq 3$ and $\sim$ equals $\{\{1, 3\}, \{2\}\}$, gives $\Sigma / \sim$ which is a preorder but not a poset. Under some conditions $\Sigma / \sim$ is straightaway a poset, like for example if Σ admits a rank function r and $\sim$ only identifies elements of equal rank. ⋆

9.7 Relating the poset and the final topology

The poset topology on $(\Sigma, \preceq)$ and the final topology on $|\Sigma|$ are related by considering the set $\{\gamma \in \Sigma \colon\ x \in \operatorname{im} p_\gamma\}$ for a point $x \in |\Sigma|$. By Lemmas 9.14 and 9.16 this set is an up-set generated by the element α in Σ such that $x \in p_\alpha(\sigma_\alpha^\circ)$. So the following map is well defined:

Definition 9.26. Let Σ be in PolySpace. We define $\text{poly}_\Sigma : |\Sigma| \to \Sigma$ as the map sending a point $x \in |\Sigma|$ to the minimal polyhedron in Σ that contains x:

$$\text{poly}_\Sigma(x) = \min \{\gamma \in (\Sigma, \preceq) \colon\ x \in \operatorname{im} p_\gamma\}.$$

Now we assume that Σ is a polyhedral complex, so the universal maps $p_\alpha : \sigma_\alpha \to |\Sigma|$ have inverses.

Lemma 9.27. *Let $\sigma : \Sigma \to \text{Poly}_{\mathbb{Z}}^f$ be in PolyComplex. The map $\text{poly}_\Sigma : |\Sigma| \to \Sigma$ is surjective, open, and continuous.*

Proof. Since $\text{poly}_\Sigma(x)$ equals α for any $x \in p_\alpha(\sigma_\alpha^\circ)$, we have surjectivity.

Recall that $V \subseteq |\Sigma|$ is open in the final topology if and only if $p_\alpha^{-1}(V)$ is open in $|\sigma_\alpha|$ for all $\alpha \in \Sigma$. To see that poly_Σ is an open map, let $V \subset |\Sigma|$ be an open set, $\alpha \in \text{poly}_\Sigma(V)$, and β in Σ such that $\alpha \preceq \beta$. There is x in V such that $x \in p_\alpha(\sigma_\alpha^\circ)$. Since $\alpha \preceq \beta$, there is $f_{\alpha\beta} : \alpha \to \beta$. By the definition of colimit $p_\alpha = p_\beta \circ \sigma(f_{\alpha\beta})$. Thus, $x \in \operatorname{im} p_\beta$, so $p_\beta^{-1}(V)$ is an open neighbourhood of $p_\beta^{-1}(x)$ in $|\sigma_\beta|$. There is a point y in $p_\beta^{-1}(V) \cap \sigma_\beta^\circ$, because any neighbourhood of a point in $|\sigma_\beta|$ intersects σ_β°. This gives that $p_\beta(y) \in V$ and $\text{poly}_\Sigma(p_\beta(y)) = \beta$. Hence $\text{poly}_\Sigma(V)$ is an up-set, namely open in the poset topology.

Finally, since $\{\uparrow\alpha \colon\ \alpha \in \Sigma\}$ generates the poset topology, poly_Σ is continuous if

$$\text{poly}_\Sigma{}^{-1}(\uparrow\alpha) = \bigcup_{\alpha \preceq \gamma} p_\gamma(\sigma_\gamma^\circ) \tag{9.2}$$

is open for all $\alpha \in \Sigma$. So we must show that $p_\beta^{-1}(\text{poly}_\Sigma{}^{-1}(\uparrow\alpha))$ is open for all β in Σ. By Lemma 9.14 we have that $p_\beta(\sigma_\beta)$ intersects $p_\gamma(\sigma_\gamma^\circ)$ if and only if then $\gamma \preceq \beta$. Moreover, if $p_\beta(\sigma_\beta)$ intersects $p_\gamma(\sigma_\gamma^\circ)$, then the intersection equals $p_\gamma(\sigma_\gamma^\circ)$ and $p_\gamma = p_\beta \circ \sigma(f_{\gamma\beta})$. This implies:

$$\begin{aligned} \text{poly}_\Sigma{}^{-1}(\uparrow\alpha) \cap p_\beta(\sigma_\beta) &= \bigcup_{\alpha \preceq \gamma} p_\gamma(\sigma_\gamma^\circ) \cap p_\beta(\sigma_\beta) = \bigcup_{\alpha \preceq \gamma \preceq \beta} p_\gamma(\sigma_\gamma^\circ) \\ &= \bigcup_{\alpha \preceq \gamma \preceq \beta} p_\beta \circ \sigma(f_{\gamma\beta})(\sigma_\gamma^\circ) = p_\beta\Big(\bigcup_{\alpha \preceq \gamma \preceq \beta} \sigma(f_{\gamma\beta})(\sigma_\gamma^\circ)\Big). \end{aligned}$$

Applying p_β^{-1} on both sides we are left with showing that

$$p_\beta^{-1}(\text{poly}_\Sigma{}^{-1}(\uparrow\alpha)) = \bigcup_{\alpha \preceq \gamma \preceq \beta} \sigma(f_{\gamma\beta})(\sigma_\gamma^\circ)$$

is open for all $\alpha, \beta \in \Sigma$ with $\alpha \preceq \beta$. We claim that

$$|\sigma_\beta| = \bigcup_{\alpha \preceq \gamma \preceq \beta} \sigma(f_{\gamma\beta})(\sigma_\gamma^\circ) \sqcup \bigcup_{\substack{\eta \preceq \beta \\ \alpha \not\preceq \eta}} \sigma(f_{\eta\beta})(\sigma_\eta) \tag{9.3}$$

is a partition. Indeed, for $x \in |\sigma_\beta|$ we let σ_μ be the face of σ_β such that x is in σ_μ°. Note that either $\alpha \preceq \mu$, so x is in the union on the left, or $\alpha \not\preceq \mu$, so x is in the union on the right of Equation (9.3). The two unions are disjoint, since if x is in some $\sigma(f_{\gamma\beta})(\sigma_\gamma^\circ)$ on the left, then by Lemma 9.14 we have $\gamma \preceq \mu$, so by transitivity $\alpha \preceq \mu$ and so μ is not in the union on the right. Since the union on the right is a finite union of faces of σ_β, and each face is a closed set, the complement in $|\sigma_\beta|$ is an open set, so we are done. $\square$

Lemma 9.28. *Let* $\Phi : \Sigma \to \Delta$ *be a morphism in* PolyComplex*, and* $|\Phi| : |\Sigma| \to |\Delta|$ *the induced map on the topological realizations. We have that Diagram 9.29 commutes.*

$$\begin{array}{ccc} |\Sigma| & \xrightarrow{\text{poly}_\Sigma} & \Sigma \\ {\scriptstyle |\Phi|}\downarrow & & \downarrow{\scriptstyle \varphi} \\ |\Delta| & \xrightarrow{\text{poly}_\Delta} & \Delta \end{array}$$

Diagram 9.29

Proof. We write $(\varphi, \{\Phi_\alpha\}_{\alpha\in\Sigma})$ for the pair defining Φ. Let x be in $|\Sigma|$ and $\alpha = \text{poly}_\Sigma(x)$. On the one hand, $\varphi \circ \text{poly}_\Sigma(x) = \varphi(\alpha)$. On the other hand, by Lemma 9.13, and the discussion succeeding it, we have that $\Phi_\alpha(\sigma_\alpha^\circ) \subseteq \delta^\circ_{\varphi(\alpha)}$. Thus, $\Phi_\alpha(p_\alpha^{-1}(x))$ is in $\delta^\circ_{\varphi(\alpha)}$. This means that $\text{poly}_\Delta \circ p_{\varphi(\alpha)} \circ \Phi_\alpha \circ p_\alpha^{-1}(x) = \varphi(\alpha)$. By Remark 9.10 we have $p_{\varphi(\alpha)} \circ \Phi_\alpha \circ p_\alpha^{-1}(x) = |\Phi|(x)$, so we are done. □

Remark 9.30. Note that φ is an order preserving map, because $\alpha \preceq \beta$ implies the existence of $f_{\alpha\beta}$ in Σ. So $\varphi(f_{\alpha\beta}) : \varphi(\alpha) \to \varphi(\beta)$ is in Δ, namely $\varphi(\alpha) \preceq \varphi(\beta)$. Thus, Diagram 9.29 is a commutative square in the Top category. △

From Diagram 9.29 we get, for every $x \in |\Delta|$, a containment of fibres

$$\text{poly}_\Sigma(|\Phi|^{-1}(x)) \subset \varphi^{-1}(\text{poly}_\Delta(x)). \tag{9.4}$$

This is an intermediate result as we work towards reducing the count of points of the fibre $|\Pi|^{-1}(\widetilde{\Gamma})$ in Theorem C to a combinatorial question. i

9.8 Refinements

Now we go back to the general setting of objects $\sigma : \Sigma \to \text{Poly}^f_\mathbb{Z}$ and morphisms Φ in PolySpace. We study refinements, a tool that modifies the polyhedral structure of a space, while keeping the topological and the integral structure unchanged. Our first goal is to obtain polyhedral complexes out of polyhedral spaces of cones.

Definition 9.31. A *refinement* of a polyhedral space $\Sigma \in$ PolySpace is a morphism $\Phi : \widetilde{\Sigma} \to \Sigma \in$ PolySpace such that the topological realization $|\Phi| : |\Sigma'| \to |\Sigma|$ is a homeomorphism and induces a bijection on integral points, i.e. $|\Phi|(N') = N$.

9.9 Stellar subdivisions

We first look at refinements produced by *stellar subdivisions*. We treat only the case of polyhedral cones and polyhedral spaces of cones. This simplifies the exposition, and is enough for our intended application.

Example 9.32. Let σ be a rational polyhedral cone in (V, N) and x a point in σ. Let $\text{star}_\sigma(x)$ be the set of faces of σ that contain x. The stellar subdivision

$$\text{sd}_\sigma(x) = \left\{\text{span}_{\mathbb{R}_{\geq 0}}(\tau, x) \colon \ \tau \text{ face of } \sigma,\, \tau \notin \text{star}_\sigma(x)\right\}$$

is an embedded polyhedral complex and the morphism $\text{sd}_\sigma(x) \to \sigma$ given by inclusion is a refinement. See Figure 9.2 for a subdivision of a 4-dimensional simplicial cone. ⋆

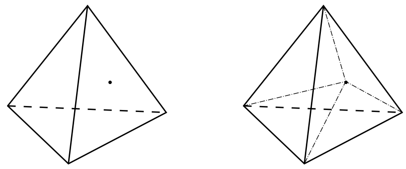


Figure 9.2: On the left, the intersection of the positive orthant of $\mathbb{R}^4$ with the hyperplane $\sum x_i = 1$, and $p = (1/3, 1/3, 1/3, 0)$. On the right, the stellar subdivision by p, which adds 1 cone of dimension 1, 4 cones of dimension 2, 6 cones of dimension 3 and 3 cones of dimension 4.

Example 9.33. Consider an embedded cone complex Σ in (V, N), and a point $x \in |\Sigma|$. We get a refinement $\mathrm{sd}_\Sigma(x) \to \Sigma$ by performing a stellar subdivision on each cone containing x. Concretely, if we let $\delta \in \Sigma$ be the unique cone with $x \in \delta^\circ$, the cones that are refined are precisely those in the set $\mathrm{star}_\Sigma(\delta) = \{\sigma \in \Sigma\colon \delta \subset \sigma\}$. The refinement deletes the cones in $\mathrm{star}_\Sigma(\delta)$, and adds the cones $\mathrm{span}_{\mathbb{R}_{\geq 0}}(x, \tau)$ with τ ranging over cones that do not contain x and are a face of a cone σ in $\mathrm{star}_\Sigma(\delta)$. That is,

$$\mathrm{sd}_\Sigma(x) = (\Sigma \setminus \mathrm{star}_\Sigma(\delta)) \cup \left\{\mathrm{span}_{\mathbb{R}_{\geq 0}}(\tau, x)\colon \tau \notin \mathrm{star}_\Sigma(\delta) \text{ and } \tau \subset \sigma \text{ for some } \sigma \in \mathrm{star}_\Sigma(\delta)\right\}.$$

See [Ewa12, Definition 2.1] for more information, including the generalization to the cell complex case. See [Koz07, Definition 2.22] for the abstract simplicial complex case. ⋆

Now the goal is to define the stellar subdivision in our setting. As there is no ambient space, we deal with the morphisms between the polyhedra in Σ.

Definition 9.34. Let Σ be the face poset of a polyhedral space and $\alpha \in \Sigma$. We define the *star* and the *closed star* of α as

$$\begin{aligned} \mathrm{star}_\Sigma(\alpha) &= \mathrm{Cat}_\Sigma(\uparrow_\Sigma \alpha), \\ \overline{\mathrm{star}}_\Sigma(\alpha) &= \mathrm{Cat}_\Sigma(\downarrow_\Sigma(\uparrow_\Sigma \alpha)); \end{aligned}$$

where $\mathrm{Cat}_\Sigma(-)$ denotes the full subcategory of Σ induced by a set of objects.

In our setting, to read off the faces of σ_α we look at the images of the morphisms in $\mathrm{Hom}_\Sigma(-, \alpha)$. This means that the objects of the finite category indexing the stellar subdivision are both objects and morphisms of Σ. We first construct this category and then construct the functor to $\mathrm{C{\small ONE}}^f_{\mathbb{Z}}$ that corresponds to the stellar subdivision.

Construction 9.35 (Combinatorial stellar subdivision)**.** Given a finite category Σ and α in $\mathrm{Obj}(\Sigma)$, the *combinatorial stellar subdivision* $\mathrm{sd}_\Sigma(\alpha)$*combinatorial stellar subdivision* is a category whose objects are

$$\{a \in \mathrm{Obj}(\Sigma)\colon a \notin \mathrm{star}_\Sigma(\alpha)\} \sqcup \left\{a : \mu \to \nu \in \mathrm{Mor}(\Sigma)\colon \mu \in \overline{\mathrm{star}}_\Sigma(\alpha), \nu \in \mathrm{star}_\Sigma(\alpha)\right\},$$

and whose morphisms are of the following four types:

(I) if $a, b \in \mathrm{Obj}(\Sigma)$, then $\mathrm{Hom}_{\mathrm{sd}_\Sigma(\alpha)}(a, b) = \mathrm{Hom}_\Sigma(a, b)$.

(II) if $a \in \mathrm{Obj}(\Sigma)$ and $b : \mu \to \nu \in \mathrm{Mor}(\Sigma)$, then $\mathrm{Hom}_{\mathrm{sd}_\Sigma(\alpha)}(a, b) = \mathrm{Hom}_\Sigma(a, \mu)$, i.e. those morphisms f that give a sequence $a \xrightarrow{f} \mu \xrightarrow{b} \nu$.

(III) if $a \in \mathrm{Mor}(\Sigma)$ and $b \in \mathrm{Obj}(\Sigma)$, then $\mathrm{Hom}_{\mathrm{sd}_\Sigma(\alpha)}(a, b) = \varnothing$.

(IV) if $a : \mu_1 \to \nu_1, b : \mu_2 \to \nu_2 \in \mathrm{Mor}(\Sigma)$, then $\mathrm{Hom}_{\mathrm{sd}_\Sigma(\alpha)}(a, b)$ equals the set of pairs $(f : \mu_1 \to \mu_2, g : \nu_1 \to \nu_2)$ such that Diagram 9.36 commutes.

$$\begin{array}{ccc} \mu_1 & \xrightarrow{a} & \nu_1 \\ f\downarrow & & \downarrow g \\ \mu_2 & \xrightarrow{b} & \nu_2 \end{array}$$

Diagram 9.36

Construction 9.37 (Stellar subdivision of a polyhedral space)**.** Let $\sigma : \Sigma \to \mathrm{CONE}^f_{\mathbb{Z}}$ be a polyhedral space of cones, $x \in |\Sigma|$ such that $\#(p_\gamma^{-1}(x)) \leq 1$ for all $\gamma \in \Sigma$, and $\alpha = \mathrm{poly}_\Sigma(x)$. The stellar subdivision of σ by x is a functor $\mathrm{sd}_\sigma(x) : \mathrm{sd}_\Sigma(\alpha) \to \mathrm{CONE}^f_{\mathbb{Z}}$ defined as follows:

1. If $a \in \mathrm{Obj}(\mathrm{sd}_\Sigma(\alpha))$ is an object of Σ, then a is mapped to $\sigma(a)$.
2. If $a \in \mathrm{Obj}(\mathrm{sd}_\Sigma(\alpha))$ is a morphism $a : \mu \to \nu$ of Σ, then $\sigma(a)$ is a face morphism $(N^\mu, \sigma_\mu) \to (N^\nu, \sigma_\nu)$ and $\mathrm{span}_{\mathbb{R}_{\geq 0}}({p_\nu}^{-1}(x), \mathrm{im}\,\sigma(a)) \subset \sigma_\nu$ is a cone because ν is in $\uparrow_\Sigma\alpha$, so ${p_\nu}^{-1}(x)$ is one point. We set
$$(\mathrm{sd}_\sigma(x))(a) = (N^\nu, \mathrm{span}_{\mathbb{R}_{\geq 0}}({p_\nu}^{-1}(x), \mathrm{im}\,\sigma(a))).$$
3. If $f \in \mathrm{Mor}(\mathrm{sd}_\Sigma(\alpha))$ is a type (I) morphism, namely f is in $\mathrm{Mor}(\Sigma)$, then it is mapped to $\sigma(f)$.
4. If $f \in \mathrm{Mor}(\mathrm{sd}_\Sigma(\alpha))$ is a type (II) morphism corresponding to $f : a \to \mu \in \mathrm{Mor}(\Sigma)$ sending $a \in \mathrm{Obj}(\Sigma)$ to $b : \mu \to \nu \in \mathrm{Mor}(\Sigma)$, then it is mapped to the composition
$$\sigma_\alpha \xrightarrow{\sigma(f)} \sigma_\mu \xrightarrow{\sigma(b)} \mathrm{span}_{\mathbb{R}_{\geq 0}}({p_\nu}^{-1}(x), \mathrm{im}\,\sigma(b)) = (\mathrm{sd}_\sigma(x))(b).$$
The image equals $\mathrm{span}_{\mathbb{R}_{\geq 0}}({p_\nu}^{-1}(x), \mathrm{im}\,\sigma(b \circ f))$, which is a face of $(\mathrm{sd}_\sigma(x))(b)$ by Example 9.32, because $\mathrm{im}\,\sigma(b \circ f)$ is a face of σ_ν that does not contain ${p_\nu}^{-1}(x)$ since the domain of f is not in $\uparrow_\Sigma\alpha$.
5. Finally, assume $(f : \mu_1 \to \mu_2, g : \nu_1 \to \nu_2) \in \mathrm{Mor}(\mathrm{sd}_\Sigma(\alpha))$ is a type (IV) morphism, namely sending $a : \mu_1 \to \nu_1 \in \mathrm{Mor}(\Sigma)$ to $b : \mu_2 \to \nu_2 \in \mathrm{Mor}(\Sigma)$ and verifying $g \circ a = b \circ f$. We set
$$(\mathrm{sd}_\sigma(x))((f, g)) = \sigma(g)|_{(\mathrm{sd}_\sigma(x))(a)}.$$
This is a face morphism because $\sigma(g)$ is an affine map so
$$\begin{aligned} \sigma(g)((\mathrm{sd}_\sigma(x))(a)) &= \sigma(g)(\mathrm{span}_{\mathbb{R}_{\geq 0}}({p_{\nu_1}}^{-1}(x), \mathrm{im}\,\sigma(a))) \\ &= \mathrm{span}_{\mathbb{R}_{\geq 0}}(\sigma(g)({p_{\nu_1}}^{-1}(x)), \sigma(g)(\mathrm{im}\,\sigma(a))) \\ &= \mathrm{span}_{\mathbb{R}_{\geq 0}}({p_{\nu_2}}^{-1}(x), \mathrm{im}\,\sigma(g \circ a)) \\ &= \mathrm{span}_{\mathbb{R}_{\geq 0}}({p_{\nu_2}}^{-1}(x), \mathrm{im}\,\sigma(b \circ f)). \end{aligned}$$
The last line is a face of $\mathrm{span}_{\mathbb{R}_{\geq 0}}({p_{\nu_2}}^{-1}(x), \mathrm{im}\,\sigma(b)) = (\mathrm{sd}_\sigma(x))(b)$ for reasons similar to Item 4.

Lemma 9.38. *Let $\sigma : \Sigma \to \mathrm{CONE}^f_{\mathbb{Z}}$ be a polyhedral space of cones, and x be a point in $|\Sigma|$ such that $\#(p_\gamma^{-1}(x)) \leq 1$ for all $\gamma \in \Sigma$. The skeleton of $\mathrm{sd}_\sigma(x)$ is a polyhedral space of cones, and the following map Φ is a morphism of polyhedral spaces*

$$\begin{array}{rl} \varphi : \mathrm{sd}_\Sigma(\alpha) \to \Sigma & \\ a \in \mathrm{Obj}(\Sigma) \mapsto a & \Phi_a : (\mathrm{sd}_\sigma(x))(a) = \sigma_a \hookrightarrow \sigma_{\varphi(a)} = \sigma_a \\ b : \mu \to \nu \in \mathrm{Mor}(\Sigma) \mapsto \nu & \Phi_b : (\mathrm{sd}_\sigma(x))(b) = \mathrm{span}_{\mathbb{R}_{\geq 0}}({p_\nu}^{-1}(x), \mathrm{im}\,\sigma(b)) \hookrightarrow \sigma_{\varphi(b)} = \sigma_\nu. \end{array}$$

Moreover, $|\Phi| : |\mathrm{sd}_\sigma(x)| \to |\Sigma|$ is a homeomorphism.

Remark 9.39. In general, stellar subdivisions are not commutative. See Figure 9.3 for an example. If x_1, x_2 are points in $|\Sigma|$ such that $\uparrow_\Sigma \mathrm{poly}_\Sigma(x_1) \cap \uparrow_\Sigma \mathrm{poly}_\Sigma(x_1) \neq \varnothing$, then the operations commute. △

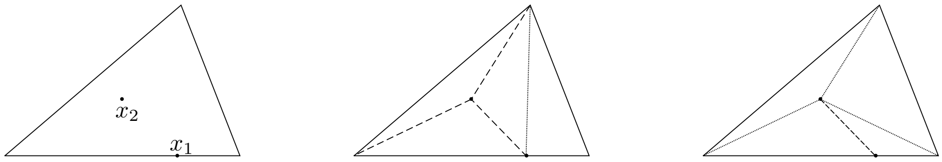


Figure 9.3: On the left, the intersection of a simplicial cone of dimension 3, with the hyperplane $\sum x_i = 1$. On the middle, first a stellar subdivision by x_1 and then by x_2; on the right first by x_2 and then x_1.

9.10 Barycentric subdivision

The condition $\#(p_\gamma^{-1}(x)) \leq 1$ for all $\gamma \in \Sigma$ is equivalent to saying that $p_\gamma{}^{-1}(x)$ is a fixed point of $\{\sigma(f)\colon f \in \operatorname{Aut}\gamma\}$ for all $\gamma \in {\uparrow_\Sigma} \operatorname{poly}_\Sigma(x)$. The barycentre of σ_γ is a fixed point, regardless of the structure of $\operatorname{Aut}\gamma$. Also repeated subdivision by barycentres resolves away the automorphisms of Σ.

Let $\sigma \subset (V, N)$ be a rational cone; for a moment we do not assume that $\operatorname{span}\sigma = V$. Consider a ray θ of σ, i.e. a 1-face. As σ is rational, the set $\theta \cap N$ is a monoid generated by one element v_θ called the *primitive vector* of θ. An automorphism $f \in \operatorname{Aut}\sigma$ restricts to an isomorphism from the ray (N_θ, θ) to the ray $(N_{f(\theta)}, f(\theta))$, so $f(v_\theta) = v_{f(\theta)}$. Hence, f maps the set of rays $\{\theta \preceq \sigma\colon \dim\theta = 1\}$ of σ bijectively onto itself. Thus, the *barycentre* $\bar\beta(\sigma)$ given by

$$\bar\beta(\sigma) = \sum_{0 \lessdot \theta \preceq \sigma} v_\theta$$

is a fixed point of σ for all $f \in \operatorname{Aut}\sigma$. In general, we get that if $f : \sigma \to \delta$ is a face morphism, then $f(\bar\beta(\sigma)) = \bar\beta(f(\sigma))$; i.e. the barycentre of σ is mapped to the barycentre of the face $f(\sigma)$ of δ.

Definition 9.40 (Barycentric subdivision)**.** Let $\sigma : \Sigma \to \textsc{Cone}^f_{\mathbb{Z}}$ be a polyhedral space of cones, L a total order extending $\preceq$ on Σ, $\alpha_1, \alpha_2, \ldots, \alpha_{\#(\Sigma)}$ the cones of Σ in $\preceq_L$-decreasing order. Write s_i for the stellar subdivision by the barycentre b_i of the cone α_i. The following iterated stellar subdivision, if it exists, is called the *barycentric subdivision* of Σ .

$$\operatorname{bcs}(\sigma : \Sigma \to \textsc{Cone}^f_{\mathbb{Z}}, L) = s_{\#(\Sigma)} \circ \cdots \circ s_2 \circ s_1(\sigma : \Sigma \to \textsc{Cone}^f_{\mathbb{Z}}).$$

Example 9.41. Let Σ be an embedded polyhedral cone complex, as in Example 9.20. Recall that a chain of a poset is a subset whose elements are all pairwise comparable. The set $L(\Sigma)$ of all chains of $(\Sigma, \subseteq)$ is a poset with the set inclusion relation, and $L(\Sigma)$ indexes the cones of $\operatorname{bcs}(\Sigma)$. That is, given a chain $L : \sigma_1 \subsetneq \sigma_2 \subsetneq \cdots \subsetneq \sigma_l$ we have

$$\sigma_L = \operatorname{span}_{\mathbb{R}_{\geq 0}}(\bar\beta(\sigma_1), \bar\beta(\sigma_2), \ldots, \bar\beta(\sigma_l))$$
$$\operatorname{bcs}(\Sigma) = \{\sigma_L\colon L \text{ is a chain of } \Sigma\}.$$

If L is a subchain of a chain M, then σ_L is a face of σ_M. ⋆

The following construction and lemma generalizes a claim made in the proof of [Koz07, Proposition 2.23]. From it follows that the barycentric subdivision is always defined and is independent of the chosen L extending $\preceq$.

Construction 9.42. Let $\sigma : \Sigma \to \textsc{Cone}^f_{\mathbb{Z}}$ be a polyhedral space of cones, L a total order extending $\preceq$ on Σ, and $\eta \in \Sigma$. The category $\Sigma_{\eta,L}$ has as objects the sequences of morphisms

$$S : \alpha_0 \xrightarrow{f_{01}} \alpha_1 \xrightarrow{f_{12}} \ldots \xrightarrow{f_{(l-1)l}} \alpha_l$$

of Σ such that $\alpha_0 \prec_L \eta \preceq_L \alpha_1$, and the dimension of the cones is strictly increasing, i.e. $\dim\alpha_i < \dim\alpha_{i+1}$. A morphism from the length-l sequence S to the length-m sequence $T : \gamma_0 \xrightarrow{g_{01}} \ldots \xrightarrow{g_{(m-1)m}} \gamma_m$ is a tuple of morphisms

$$(\hat h : \alpha_0 \to \gamma_0, h_1 : \alpha_1 \to \gamma_{\iota(1)}, \ldots, h_l : \alpha_l \to \gamma_{\iota(l)})$$

such that the h_i are isomorphisms and Diagram 9.43 commutes.

$$\begin{array}{ccccccccccccc}
\alpha_0 & \longrightarrow & & \alpha_1 & \longrightarrow & & \alpha_2 & \longrightarrow \dots \longrightarrow & \alpha_l & & & & \\
\downarrow g & & & \downarrow h_1 & & & \downarrow h_2 & & \downarrow h_l & & & & \\
\gamma_0 & \longrightarrow \dots \longrightarrow & & \gamma_{\iota(1)} & \longrightarrow \dots \longrightarrow & & \gamma_{\iota(2)} & \longrightarrow \dots \longrightarrow & \gamma_{\iota(l)} & \longrightarrow \dots \longrightarrow & \gamma_m & &
\end{array}$$

Diagram 9.43

Given a sequence $S \in \Sigma_{\eta,L}$, we write f_{ij} for the composition $f_{(j-1)j} \circ \cdots \circ f_{i(i+1)}$. The functor $\sigma_{\eta,L} : \Sigma_{\eta,L} \to \textsc{Cone}^f_{\mathbb{Z}}$ is given by

$$\begin{aligned}
\sigma_{\eta,L}(S) &= (N^{\alpha_l}, \operatorname{span}_{\mathbb{R}_{\geq 0}} \{\sigma(f_{0l})(\sigma(\alpha_0)),\, \sigma(f_{1l})(\bar{\beta}(\alpha_1)),\, \dots, \sigma(f_{(l-1)l})(\bar{\beta}(\alpha_{l-1})),\, \bar{\beta}(\alpha_l)\}) \\
&= (N^{\alpha_l}, \operatorname{span}_{\mathbb{R}_{\geq 0}} \{\sigma(f_{0l})(\sigma(\alpha_0)),\, \bar{\beta}(\sigma(f_{1l})(\alpha_1)),\, \dots, \bar{\beta}(\sigma(f_{(l-1)l})(\alpha_{l-1})),\, \bar{\beta}(\alpha_l)\}) \\
\sigma_{\eta,L}(S \to T) &= \sigma(h_l)
\end{aligned}$$

We see that $\sigma_{\eta,L}(S \to T)$ is a face morphism by calculating

$$\begin{aligned}
&\sigma(h_l)(\sigma_{\eta,L}(S)) = \\
&\sigma(h_l)(\operatorname{span}_{\mathbb{R}_{\geq 0}} \{\sigma(f_{0l})(\sigma(\alpha_0)),\, \sigma(f_{1l})(\bar{\beta}(\alpha_1)),\, \dots, \sigma(f_{(l-1)l})(\bar{\beta}(\alpha_{l-1})),\, \bar{\beta}(\alpha_l)\}) = \\
&\operatorname{span}_{\mathbb{R}_{\geq 0}} \{\sigma(h_l) \circ \sigma(f_{0l})(\sigma(\alpha_0)),\, \dots, \sigma(h_l) \circ \sigma(f_{(l-1)l})(\bar{\beta}(\alpha_{l-1})),\, \sigma(h_l) \circ \bar{\beta}(\alpha_l)\}) = \\
&\operatorname{span}_{\mathbb{R}_{\geq 0}} \{\sigma(h_l \circ f_{0l})(\sigma(\alpha_0)),\, \dots, \bar{\beta}(\sigma(h_l \circ f_{(l-1)l})(\alpha_{l-1})),\, \bar{\beta}(\sigma(h_l)(\alpha_l))\}) = \\
&\operatorname{span}_{\mathbb{R}_{\geq 0}} \{\sigma(g_{0m} \circ h_0)(\sigma(\alpha_0)),\, \dots, \bar{\beta}(\sigma(g_{\iota(l-1)m} \circ h_{l-1})(\alpha_{l-1})),\, \bar{\beta}(\sigma(h_l)(\alpha_l))\}).
\end{aligned}$$

By Example 9.41, the last line is a face of

$$\sigma_{\eta,L}(T) = (N^{\gamma_m}, \operatorname{span}_{\mathbb{R}_{\geq 0}} \{\sigma(g_{0m})(\sigma(\gamma_0)),\, \dots, \bar{\beta}(\sigma(g_{(m-1)m})(\gamma_{m-1})),\, \bar{\beta}(\gamma_l)\}).$$

Lemma 9.44. *Let $\sigma : \Sigma \to \textsc{Cone}^f_{\mathbb{Z}}$ be a polyhedral space of cones, and η in Σ be the q-th cone in $\preceq_L$-decreasing order. The functor $\sigma_{\eta,L} : \Sigma_{\eta,L} \to \textsc{Cone}^f_{\mathbb{Z}}$ is isomorphic to the functor $s_q \circ \cdots \circ s_2 \circ s_1(\sigma : \Sigma \to \textsc{Cone}^f_{\mathbb{Z}})$.*

Proof. Straightforward generalization of the proof of [Koz07, Proposition 2.23]. □

Remark 9.45. The $\preceq_L$-minimal element of Σ is the apex $(\{0\}, \{0\})$ of Σ. Let θ be the element that covers $(\{0\}, \{0\})$. Observe that $\sigma_{\theta,L} : \Sigma_{\theta,L} \to \textsc{Cone}^f_{\mathbb{Z}}$ is independent of L; i.e. the objects are sequences of Σ that begin with $(\{0\}, \{0\})$. Moreover, Lemma 9.44 implies that $\operatorname{bcs}(\sigma : \Sigma \to \textsc{Cone}^f_{\mathbb{Z}}, L)$ is isomorphic to $\sigma_{\theta,L} : \Sigma_{\theta,L} \to \textsc{Cone}^f_{\mathbb{Z}}$. Thus, we simply write $\operatorname{bcs}(\sigma : \Sigma \to \textsc{Cone}^f_{\mathbb{Z}})$. △

Conjecture 9.46. *Let $\sigma : \Sigma \to \textsc{Cone}^f_{\mathbb{Z}}$ be a polyhedral space of cones. The barycentric subdivision* $\operatorname{bcs}(\sigma)$ *is a polyhedral complex of simplicial cones.*

Chapter 10

Indexed branched covers

10.1 Indexed branched covers

Let $F : X \to Y$ be a continuous map. We call the cardinality of the fibre $F^{-1}(y)$ the *degree* of F at $y \in Y$. We now study certain maps F for which the degree is a constant $\deg F$ independent of the chosen $y \in Y$, leading up to the notion of indexed branched cover. We begin with:

Definition 10.1. A continuous map $F : X \to Y$ is a *cover* if for every $y \in Y$ there exists an open neighbourhood U of y such that $F^{-1}(U)$ is a union of disjoint open sets $V_i \subset X$, and F maps each V_i homeomorphically to U.

See Figure 10.1 (a) for an example. If $F : X \to Y$ is a cover, then the degree at $y \in Y$ is constant over each *connected component* of Y, i.e. over each subset of Y that is inclusion-wise maximal among connected sets. Recall that a set $S \subset Y$ is connected if there is no pair U_1, U_2 of open sets such that $(U_1 \cap S) \cup (U_2 \cap S) = S$ and $(U_1 \cap S) \cap (U_2 \cap S) = \varnothing$. Thus, if Y is connected, all fibres have the same cardinality.

Remark 10.2. Note that in Definition 10.1 we do not require $F : X \to Y$ to be surjective. In practice we get it from F being a cover plus some other mild condition. For example, when X is non-empty and Y is connected, surjectivity follows from the above observation that all fibres are equipotent. △

In algebraic geometry it is usual to deal with maps that are almost a cover, except that some $U \subset Y$ have copies V_i in $F^{-1}(U)$ that glue together at a roughly small set, like in Example 10.8.

Definition 10.3. A pair (F, B) of a continuous map $F : X \to Y$ and a closed subset $B \subset Y$ is a *branched cover* if $Y \setminus B$ is open and dense, and F restricted to $F^{-1}(Y \setminus B)$ is a cover.

Remark 10.4. Many times we omit writing the pair and say that F is a branched cover with *branch locus* $B \subset Y$. We also say that F is a branched cover *unramified* over $U = Y \setminus B$. There is

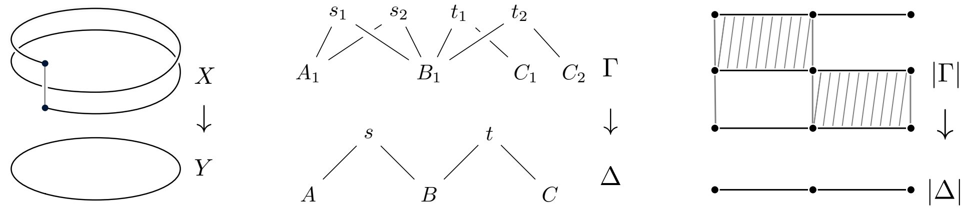


Figure 10.1: On the left, a degree-2 cover of a circle, where the vertical grey line indicates identification of two points. In the centre, a branched covering of posets. On the right, a branched covering of a segment subdivided in two pieces. Both branched coverings admit an index map that makes them a degree-3 indexed branched covering.

no widespread agreement on what a general branched cover should be, so Definition 10.3 simply expresses some of the bare minimums typically imposed. In our setting we are concerned with branched covers of polyhedral complexes and of posets. △

Example 10.5. Let $\varphi : \Sigma \to \Delta$ be a morphism of posets. The closure of a set $\mathcal{U} \subset \Delta$ is $\downarrow_\Delta \mathcal{U}$, so a subset of Δ is dense if and only if it contains $\max \Delta$. An element $\beta \in \Delta$ is maximal if and only if the singleton $\{\beta\}$ is an open set. So, the fibre $\varphi^{-1}(\{\beta\})$ of an open $\{\beta\}$ is a disjoint union of open sets homeomorphic to $\{\beta\}$ if and only if every α in $\varphi^{-1}(\beta)$ is maximal. Thus, if φ is a branched cover unramified over $\mathcal{U} \subset \Delta$, then

1. If β is in $\max \Delta$, then $\varphi^{-1}(\beta) \subset \max \Sigma$.
2. We have that $\max \Delta \subset \mathcal{U}$.

Moreover, if φ fulfills (1), then φ is a branched cover unramified over $\max \Delta$. ⋆

If $U = Y \setminus B$ is not connected, we may have different values for the degree at different connected components of $Y \setminus B$. And even if all values happen to coincide, it is dissatisfying to not have a count for fibres above B. When all the fibres of F are finite, we remedy this situation by introducing a map that indicates a positive integral multiplicity for counting the points: an *index map* for X is a map from X to $\mathbb{Z}_{\geq 1}$. The freedom provided by this relaxation of the problem is counterbalanced by a stronger requirement, a local preimage count that must be constant:

Definition 10.6 (local degree)**.** Let $F : X \to Y$ be a map such that for all $y \in Y$ the set $F^{-1}(y)$ is finite, $m_X : X \to \mathbb{Z}_{\geq 1}$ be an index map, and $V \subset X$. We define the *local degree* function $\deg(F, m_X, V) : Y \to \mathbb{Z}_{\geq 0}$ as

$$\deg(F, m_X, V)(y) = \sum_{\substack{x \in F^{-1}(y) \\ x \in V}} m_X(x). \tag{10.1}$$

From now on we assume that F has finite fibres and Y is connected. The convention for empty sums in Equation (10.1) is that they evaluate to 0. The reason for having the domain of $\deg(F, m_X, V)$ be Y, and not just $U = F(V)$, is to make the following observation: for any pair of sets $V_1, V_2 \subset X$ we have that

$$\deg(F, m_X, V_1) + \deg(F, m_X, V_2) = \deg(F, m_X, V_1 \cup V_2) + \deg(F, m_X, V_1 \cap V_2). \tag{10.2}$$

Moreover, if $F^{-1}(y) \cap V_1 = F^{-1}(y) \cap V_2$, we have the transition equality

$$\deg(F, m_X, V_1)(y) = \deg(F, m_X, V_2)(y). \tag{10.3}$$

Finally, we say that $\deg(F, m_X, V)$ is constant if it is constant over $F(V)$. We now come to the main definition of this subsection:

Definition 10.7. A pair (F, m_X) of a branched covering $F : X \to Y$, and an index map $m_X : X \to \mathbb{Z}_{\geq 1}$, is an *indexed branched cover* if for every connected open set $U \subset Y$ and connected component V of $F^{-1}(U)$ the local degree $\deg(F, m_X, V)$ is constant.

In particular, since Y is open and connected, this means that the count with multiplicity m_X of the points in the fibre $F^{-1}(y)$ is a constant $\deg(F, m_X)$ over Y.

Example 10.8. Let $f : X \to Y$ be a non-constant holomorphic map of compact Riemann surfaces, and x be a point in X. Recall that there are charts for X and Y that express f in a neighbourhood U of x simply as $f = z^k$. The integer k is independent of how the charts are chosen, so the map $m_X : X \to \mathbb{Z}_{\geq 1}$ sending $x \mapsto k$ is well defined. This is called the *ramification index* of x. A central fact in the theory of Riemann surfaces is that the pair (f, m_X) is an indexed branched cover, with a branch locus that consists of finitely-many points of Y. See [CM16, Section 4] for an accessible and streamlined exposition. ⋆

Example 10.9. Consider the map $\Gamma \to \Delta$ from Figure 10.1 (b). The following table specifies an index map m_Γ which makes $(\Gamma \to \Delta, m_\Gamma)$ a degree-3 indexed branched cover with branch locus equal to $\{A, B\}$.

x	$m_\Gamma(x)$	x	$m_\Gamma(x)$
A_1	3	s_1	2
B_1	3	s_2	1
C_1	1	t_1	1
C_2	2	t_2	2

Note that $\{s, t\} \subset \{s, t, C\}$ indeed is a dense set, since its closure is $\downarrow s \cup \downarrow t = \Delta$. ⋆

Remark 10.10. Definition 10.7 is inspired by, and synthesizes together, Definitions 2.17 and 2.23 from [Pay09]. △

10.2 Morphisms of branched covers

We now make a few considerations on what properties should a morphism of branched covers preserve, and propose a definition.

Let $F : X \to Y$ and $\overline{F} : \overline{X} \to \overline{Y}$ be branched covers with branch loci B and $\overline{B}$. Consider a pair (g, h) of continuous maps as in Diagram 10.15, which we want to be *structure preserving*. The question is, *which structure*. On the one hand, the topology of $X \setminus F^{-1}(B)$ is locally homeomorphic to that of $Y \setminus B$. So f should be a local homeomorphism from $X \setminus F^{-1}(B)$ to $\overline{X} \setminus \overline{F}^{-1}(\overline{B})$, and likewise g a local homeomorphism from $Y \setminus B$ to $\overline{Y} \setminus \overline{B}$.

On the other hand, we have not imposed conditions on the branch locus, besides being closed with dense complement. Thus, we cannot say much about how $F^{-1}(B)$ relates to B. It would be too rigid to preserve all properties of $F^{-1}(B) \to B$ when mapping to $\overline{F}^{-1}(\overline{B}) \to \overline{B}$, hence we focus on the most relevant to us, namely connectivity. First, recall some topological definitions and facts; see e.g. [Mun00].

Definition 10.11. Let $f : X \to Y$ be a map of topological spaces.

- A subset $U \subset X$ is *saturated* if $f^{-1}(y) \cap U \neq \varnothing$ implies that $f^{-1}(y) \subset U$, for all $y \in Y$; i.e. for some $V \subset Y$ we have that $U = f^{-1}(V)$.
- The map f is a *quotient map* if f is continuous, surjective, and every saturated open set is mapped to an open set.

Remark 10.12. If $f : X \to Y$ is a quotient map, then Y is homeomorphic to $X/\sim_f$ with the quotient topology, where $\sim_f$ is the equivalence relation on X given by $x \sim_f x'$ if $f(x) = f(x')$ △

Example 10.13. Any continuous map that is surjective and open is a quotient map. Thus, given a polyhedral space $\sigma : \Sigma \to \mathrm{POLY}^f_{\mathbb{Z}}$, Lemma 9.27 implies that the map poly from Definition 9.26 is a quotient map. The associated equivalence relation identifies two points if they belong to the same relative interior. ⋆

Definition 10.14. Let $(F : X \to Y, B)$ and $(\overline{F} : \overline{X} \to \overline{Y}, \overline{B})$ be branched covers. A pair of quotient maps $(g : X \to \overline{X}, h : Y \to \overline{Y})$ such that $h \circ F = \overline{F} \circ g$, i.e. such that Diagram 10.15 commutes, is a *morphism of branched covers* if $h(B) \subset \overline{B}$ and for every connected $\overline{U} \subset \overline{Y}$ and every $\overline{V} \in \pi_0(\overline{F}^{-1}(\overline{U}))$ we have that $\pi_0(g^{-1}(\overline{V})) \subset \pi_0(F^{-1}(h^{-1}(\overline{U})))$.

$$\begin{array}{ccc} X & \xrightarrow{g} & \overline{X} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \overline{F}} \\ Y & \xrightarrow{h} & \overline{Y} \end{array}$$

Diagram 10.15

In Theorem 10.46 and in Definition 12.16 we deal with quotient maps $g : X \to \overline{X}$ that have *connected fibres*, i.e. $f^{-1}(y)$ is connected for all $y \in Y$. We explore some consequences of this condition.

Lemma 10.16. *Let $g : X \to \overline{X}$ be a quotient map. If g has connected fibres, X is locally path connected, and $\overline{X}$ is connected, then X is connected.*

Proof. That g is surjective with connected fibres, implies that the set $\overline{P}$ given by $\{g(V) \colon V \in \pi_0(X)\}$ is a partition of $\overline{X}$. It also implies that each $V \in \pi_0(X)$ is saturated. Moreover, each $V \in \pi_0(X)$ is open because X is locally path connected. Since g is a quotient map, $\overline{P}$ is a partition of $\overline{X}$ into open connected sets. Hence, $\overline{P}$ has only one element, because $\overline{X}$ is connected. Note that the map $\pi_0(X_0) \to \overline{P}$ induced by g is injective, so we are done. □

10.3 Morphisms of indexed branched covers

We now propose a particular kind of branched covers that interacts well with index maps.

Definition 10.17. Let $(F : X \to Y, m_X)$ and $(\overline{F} : \overline{X} \to \overline{Y}, \overline{m})$ be two indexed branched covers, and (g, h) a morphism from F to $\overline{F}$ as branched covers. We say that (g, h) is a morphism of indexed branched covers if for any connected open set $\overline{U} \subset \overline{Y}$ and every $\overline{V} \in \pi_0(\overline{F}^{-1}(\overline{U}))$ we have

$$\deg(F, m_X, g^{-1}(\overline{V})) = \deg(\overline{F}, \overline{m}, \overline{V}).$$

Lemma 10.18. *Let the pair (g, h) be a morphism of indexed branched covers $(F : X \to Y, m) \to (\overline{F} : \overline{X} \to \overline{Y}, \overline{m})$. If for any $\overline{y}$ in $\overline{Y}$ the induced topology on $\overline{F}^{-1}(\overline{y}) \subset \overline{X}$ is discrete, then*

$$\overline{m}(\overline{x}) = \deg(F, g^{-1}(\overline{x}), m) = \sum_{x \in g^{-1}(\overline{x})} m(x).$$

We have a converse of sorts, which enables us to induce an indexed branched cover structure on $\overline{F}$ given an indexed branched cover F and a morphism $(g, h) : F \to \overline{F}$ of branched covers.

Lemma 10.19. *Let the pair (g, h) be a morphism of branched covers $(F : X \to Y, B) \to (\overline{F} : \overline{X} \to \overline{Y}, \overline{B})$. If $m : X \to \mathbb{Z}_{>0}$ is an index map that makes (F, m) and indexed branched cover, then*

$$\overline{m}(\overline{x}) = \deg(F, g^{-1}(\overline{x}), m) = \sum_{x \in g^{-1}(\overline{x})} m(x)$$

makes $(\overline{F}, \overline{m})$ and indexed branched cover.

Remark 10.20. If $\overline{F} : \overline{X} \to \overline{Y}$ has finite fibres, and $\overline{X}$ satisfies the T_1 separation axiom, then any fibre $\overline{F}^{-1}(y)$ has the discrete topology, so the condition on fibres from Lemma 10.18 is satisfied. Recall that posets with the poset topology are not T_1, unless the partial order is trivial. In Lemma 10.40 we prove this condition for the posets that interest us. △

10.4 A criterion for being an indexed branched cover

Having constant local degrees is a desirable realizability condition, but proving it is a difficult task. Thus, we derive equivalent conditions that are easier to prove. As a first result, we show that it is enough to consider a basis with good properties for the topology of Y. We let $\pi_0(X)$ denote the set of connected components of X.

Remark 10.21. Typically in the literature $\pi_0(X)$ denotes the set of *path-connected components*; i.e. inclusion-wise maximal elements in the family of path-connected subsets S of X. A *path* connecting $y, z \in X$ is a continuous map $f : [0,1] \to X$ with $f(0) = y$ and $f(1) = z$. A subset $S \subset X$ is path-connected if every pair of points in S is connected by a path. All the spaces that we consider are *locally path-connected*, i.e. there is a basis consisting of path-connected sets, see Propositions 10.24 and 10.29. In this case the connected components and the path-connected components coincide [Mun00, Theorem 25.5]. △

Lemma 10.22. *Let $F : X \to Y$ be a branched cover, $m_X : X \to \mathbb{Z}_{\geq 1}$, and $U^{(1)}$, $U^{(2)}$ connected open sets in Y with $\deg(F, m_X, V^{(q)})$ constant for $V^{(q)} \in \pi_0(F^{-1}(U^{(q)}))$, for $q = 1$ and 2. The degree $\deg(F, m_X, V)$ is constant for $V \in \pi_0(F^{-1}(U^{(1)} \cup U^{(2)}))$.*

Proof. We are done if $U^{(1)}$ and $U^{(2)}$ are disjoint, so assume that $U^{(1)} \cap U^{(2)} \neq \varnothing$. Let V be in $\pi_0(F^{-1}(U^{(1)} \cup U^{(2)}))$. If $\left\{V_j^{(q)}\right\}_{j\in J}$ is the family of connected components of $F^{-1}(U^{(q)})$ that intersect V, for $q = 1$ and 2, we claim that

$$V = \bigcup_{j\in J} V_j^{(1)} \cup \bigcup_{k\in K} V_k^{(2)}.$$

Indeed, it is clear that the set on the right contains the set on the left, and the other containment follows from the observation that if S is a connected set with $S \cap V \neq \varnothing$, then $S \cup V$ is connected, so $S \subset V$. Let $y_0 \in U^{(1)} \cap U^{(2)}$, $y_1 \in U^{(1)}$, and $y_2 \in U^{(2)}$. By Equations (10.2) and (10.3) we have

$$\begin{aligned} \deg\left(F, m_X, \bigcup_{j\in J} V_j^{(1)}\right)(y_1) &= \sum_{j\in J} \deg\left(F, m_X, V_j^{(1)}\right)(y_1) \\ &= \sum_{j\in J} \deg\left(F, m_X, V_j^{(1)}\right)(y_0) \\ &= \deg\left(F, m_X, \bigcup_{j\in J} V_j^{(1)}\right)(y_0). \end{aligned} \tag{10.4}$$

Likewise, we derive

$$\deg\left(F, m_X, \bigcup_{k\in K} V_k^{(2)}\right)(y_2) = \deg\left(F, m_X, \bigcup_{k\in K} V_k^{(2)}\right)(y_0). \tag{10.5}$$

Since y_0 is in $U^{(1)}$ and $\left\{V_j^{(1)}\right\}_{j\in J}$ is the family of connected components of $F^{-1}(U^{(1)})$ that intersects V, we have that

$$F^{-1}(y_0) \cap \bigcup_{j\in J} V_j^{(1)} = F^{-1}(y_0) \cap V.$$

Likewise, we obtain a similar expression in relation to $F^{-1}(U^{(2)})$, and conclude

$$F^{-1}(y_0) \cap \bigcup_{j\in J} V_j^{(1)} = F^{-1}(y_0) \cap V = F^{-1}(y_0) \cap \bigcup_{k\in K} V_k^{(2)}.$$

This implies, by Equation (10.3), that the right hand sides of Equations (10.4) and (10.5) are equal. Thus, so are the left hand sides, as desired. □

Under mild conditions for the topological spaces X and Y, we get a countable version of Lemma 10.22.

Lemma 10.23. *Let $F : X \to Y$ be a branched cover and $m_X : X \to \mathbb{Z}_{\geq 1}$. If X is locally path-connected, Y has a countable basis $\mathcal{U}$ of connected sets, and $\deg(F, m_X, V)$ is constant for all $V \in \pi_0(F^{-1}(U))$ and $U \in \mathcal{U}$, then (F, m_X) is an indexed branched cover.*

Proof. Let U be a connected open in Y. By asumption, we can write U as a countable union $U = \bigcup_{q=0}^{\infty} U_q$ with $U_q \in \mathcal{U}$. Suppose there is $V \in \pi_0(F^{-1}(U))$ such that $\deg(F, m_X, V)$ is not constant; i.e. there are y and z in Y such that $\deg(F, m_X, V)(y) \neq \deg(F, m_X, V)(z)$. Let $\{y_i\}_{i\in I}$ and $\{z_j\}_{j\in J}$ be $F^{-1}(y) \cap V$ and $F^{-1}(z) \cap V$, respectively, so we get

$$\deg(F, m_X, V)(y) = \sum_{i\in I} m_X(y_i) \neq \sum_{j\in J} m_X(z_j) = \deg(F, m_X, V)(z). \tag{10.6}$$

Since U is open and F continuous, we have that $F^{-1}(U)$ is an open subspace of X. Being locally path-connected is inherited to open subspaces, so we have that $F^{-1}(U)$ is locally path-connected. Thus, the connected components and the path-connected components of $F^{-1}(U)$ coincide, so $V \in \pi_0(F^{-1}(U))$ is path-connected. So for each pair $i \in I, j \in J$ we can choose a path $P_{ij} : [0, 1] \to X$ connecting y_i with z_j. As $F \circ P_{ij}$ is a continuous map, the image $\operatorname{im} F \circ P_{ij}$ is a compact set in Y. Since $\{U_q\}$ is a cover of $\operatorname{im} F \circ P_{ij}$, we can choose a finite subcover and let r_{ij} be the highest index of the U_q in this finite subcover.

Let $r = \max r_{ij}$ and $U_r = \bigcup_{q=1}^{r} U_q$. Note that $\operatorname{im} P_{ij} \subset F^{-1}(U_r)$ for all pairs i, j. So the connected component $\tilde{V}$ of $F^{-1}(U_r)$ that contains y_1, also contains all y_i, z_j since $\tilde{V}$ is a path-connected component as well. Thus, by applying finitely-many times Lemma 10.22, we get that $\deg(F, m_X, \tilde{V})(y) = \deg(F, m_X, \tilde{V})(z)$, which in particular implies

$$\sum_{i\in I} m_X(y_i) = \sum_{j\in J} m_X(z_j),$$

contradicting Equation (10.6). Thus, $\deg(F, m_X, V)$ is constant, as desired. □

10.5 Bases of path-connected sets for Σ and $|\Sigma|$

Given a polyhedral complex we describe bases for its poset and topological realization that satisfy the conditions of Lemmas 10.22 and 10.23. First, recall that a *principal open set* in the poset topology is a set of the form $\uparrow_\Sigma \alpha$. We have:

Proposition 10.24. *Let Σ be a finite poset. The family of principal open sets of Σ is a finite basis of path-connected sets for the poset topology on Σ.*

Proof. The family $\{\uparrow\alpha\}_{\alpha\in\Sigma}$ is a finite basis for the poset topology. To see that $\uparrow\alpha$ is path-connected, note that given γ_1, γ_2 in $\uparrow_\Sigma \alpha$ the following map $f : [0, 1] \to \Sigma$ with $f(0) = \gamma_1$ and $f(1) = \gamma_2$ is continuous:

$$f(t) = \begin{cases} \gamma_1 & \text{for } 0 \leq t < \frac{1}{3}, \\ \alpha & \text{for } \frac{1}{3} \leq t \leq \frac{2}{3}, \\ \gamma_2 & \text{for } \frac{2}{3} < t \leq 1. \end{cases} \tag{10.7}$$

Thus, $\uparrow_\Sigma \alpha$ is path-connected, as desired. □

Second, let N be a finite-rank free abelian group, and $\mathcal{S}$ a choice of a basis. We take $\mathcal{S}$ to be an orthonormal basis for $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$; this gives rise to the Euclidean norm $d : N_\mathbb{R} \times N_\mathbb{R} \to \mathbb{R}$. We denote by $B(N_\mathbb{R}, x, \varepsilon) = \{y \in N_\mathbb{R} \colon d(x, y) < \varepsilon\}$ the open ball in $N_\mathbb{R}$ centred at x with radius ε. When x is in $N_\mathbb{Q} = N \otimes \mathbb{Q}$ and ε is in $\mathbb{Q}$ we say that $B(N_\mathbb{R}, x, \varepsilon)$ is *rational.* Since $\mathbb{Q}$ is dense in $\mathbb{R}$, the family of rational balls is a countable basis for the Euclidean topology of $N_\mathbb{R}$; moreover each ball is connected.

The goal now is, given a polyhedral complex Σ, to glue balls in several $N^\alpha_{\mathbb{R}}$ to obtain a topological basis of connected sets for $|\Sigma|$. We say that a family $\{\mathcal{S}_\alpha\}_{\alpha\in\Sigma}$ of generating sets for $\{N^\alpha\}_{\alpha\in\Sigma}$ is a *family of compatible bases* if

$$\sigma(f_{\alpha\beta})(\mathcal{S}_\alpha) - \sigma(f_{\alpha\beta})(0) \subset \mathcal{S}_\beta$$

for all $\alpha \to \beta$ in Σ, i.e. the linear part of the integrally affine map $\sigma(f_{\alpha\beta})$ sends the set $\mathcal{S}_\alpha$ into $\mathcal{S}_\beta$. This implies that the map $\sigma(f_{\alpha\beta}) : N^\alpha_{\mathbb{R}} \to N^\beta_{\mathbb{R}}$ is an isometric embedding, so the Euclidean norms d_α and d_β induced by $\mathcal{S}_\alpha$ and $\mathcal{S}_\beta$ satisfy that $d_\alpha = d_\beta \circ \sigma(f_{\alpha\beta})$. For a point x in $|\Sigma|$ we define the *principal ball centred at x with radius ε* as

$$\uparrow B(x,\varepsilon) = \bigcup_{\beta \succeq \mathrm{poly}_\Sigma(x)} p_\beta(B(N^\beta_{\mathbb{R}}, p_\beta^{-1}(x), \varepsilon) \cap \sigma_\beta). \tag{10.8}$$

For an open set $U \subset |\Sigma|$ with $x \in U$, we show that for small enough ε the set $\uparrow B(x,\varepsilon)$ is an open neighbourhood. This follows from two lemmas. First, a consequence of the fact that the morphisms f in Σ are isometries when a family of compatible bases is chosen.

Lemma 10.25. *Let Σ be a polyhedral complex with a chosen family of compatible bases, x a point in $|\Sigma|$, and γ in $\uparrow_\Sigma \mathrm{poly}_\Sigma x$. We have that*

$$p_\gamma^{-1}(\uparrow B(x,\varepsilon)) = B(N^\gamma_{\mathbb{R}}, p_\gamma^{-1}(x), \varepsilon) \cap \sigma_\gamma. \tag{10.9}$$

Proof. The set on the left contains the right one because of the assumption that $\gamma \succeq \mathrm{poly}_\Sigma(x)$. For the other containment, let $\hat{y}$ be in $p_\gamma^{-1}(\uparrow B(x,\varepsilon))$, namely for some β in Σ with $\beta \succeq \mathrm{poly}_\Sigma(x)$ we have that

$$p_\gamma(\hat{y}) \in p_\beta(B(N^\beta_{\mathbb{R}}, p_\beta^{-1}(x), \varepsilon) \cap \sigma_\beta).$$

Let y equal $p_\gamma(\hat{y})$, and note that this point is in $\mathrm{im}\, p_\gamma \cap p_\beta$, so by Lemma 9.17 there exists η in Σ with morphisms $f : \eta \to \beta$ and $g : \eta \to \gamma$, such that y is in $\mathrm{im}\, p_\eta$. Recall that f, g are isometries, and that $f = p_\beta^{-1} \circ p_\eta$, $g = p_\gamma^{-1} \circ p_\eta$, so we calculate

$$\begin{aligned} d_\gamma(\hat{y}, p_\gamma^{-1}(x)) &= d_\gamma(p_\gamma^{-1}(y), p_\gamma^{-1}(x)) = d_\gamma(p_\gamma^{-1} \circ p_\eta \circ p_\eta^{-1}(y), p_\gamma^{-1} \circ p_\eta \circ p_\eta^{-1}(x)) \\ &= d_\gamma(g \circ p_\eta^{-1}(y), g \circ p_\eta^{-1}(x)) = d_\eta(p_\eta^{-1}(y), p_\eta^{-1}(x)) \\ &= d_\beta(p_\beta^{-1}(y), p_\beta^{-1}(x)) \leq \varepsilon, \end{aligned}$$

and conclude that $\hat{y}$ is in $B(N^\gamma_{\mathbb{R}}, p_\gamma^{-1}(x), \varepsilon)$, as desired. ☐

Second, we express containment in terms of the universal maps p_α. Denote by $\mathrm{cl}(-)$ the closure of a set.

Lemma 10.26. *Let Σ be a polyhedral complex, U and V subsets of $|\Sigma|$, and $\mathcal{U} \subset \Sigma$ such that $\mathrm{poly}_\Sigma U \subset \mathrm{cl}(\mathcal{U})$. If $p_\beta^{-1}(U) \subset p_\beta^{-1}(V)$ for all β in $\mathcal{U}$, then $U \subset V$.*

Proof. Let x be in U, set $\alpha = \mathrm{poly}_\Sigma x$, and $\hat{x} = p_\alpha^{-1}(x)$. Since $\mathrm{poly}_\Sigma U \subset \mathrm{cl}(\mathcal{U})$, there is a β in $\mathcal{U}$ and a morphism $f : \alpha \to \beta$ in Σ. Hence, $x = p_\alpha(\hat{x}) = p_\beta \circ f(\hat{x})$ gives that $f(\hat{x}) \in p_\beta^{-1}(U)$, so by assumption also in $p_\beta^{-1}(V)$, hence x is in V as desired. ☐

Putting both ingredients together, we get:

Lemma 10.27. *Let Σ be a polyhedral complex with a chosen family of compatible bases, $V \subset |\Sigma|$ an open set, and x in V. There is a constant K such that if $\varepsilon < K$ we have*

$$\uparrow B(x,\varepsilon) \subset V.$$

Proof. Let $\alpha = \mathrm{poly}_\Sigma x$, set $U = \uparrow B(x,\varepsilon)$ and $\mathcal{U} = \uparrow_\Sigma \alpha$. By Equation 10.8 we have that $\mathrm{poly}_\Sigma U \subset \mathrm{cl}(\mathcal{U})$. Since V is open, for every $\beta \in \mathcal{U}$ we have that $p_\beta^{-1}(V)$ is open. Moreover, note that $p_\beta^{-1}(x) \in p_\beta^{-1}(V)$. Thus, we can choose ε_β such that

$$B(N^\beta_{\mathbb{R}}, p_\beta^{-1}(x), \varepsilon_\beta) \cap \sigma_\beta \subset p_\beta^{-1}(V).$$

By Lemma 10.25 the left hand side equals $p_\beta^{-1}(\uparrow B(x,\varepsilon_\beta))$. Hence, if we take $K = \min_{\beta\in\mathcal{U}} \varepsilon_\beta$, we are done by Lemma 10.26. ☐

Lemma 10.28. *Let Σ be a polyhedral complex with a chosen family of compatible bases, and x in $|\Sigma|$. There is a constant K such that if $\varepsilon < K$ we have that $\uparrow B(x, \varepsilon)$ is open.*

Proof. Let $\alpha = \text{poly}_\Sigma$. Applying Lemma 10.27 to $\text{poly}_\Sigma^{-1}(\uparrow_\Sigma \alpha)$, which is open because poly_Σ is continuous, we get a K such that if $\varepsilon < K$, then $\uparrow B(x, \varepsilon) \subset \text{poly}_\Sigma^{-1}(\uparrow_\Sigma \alpha)$. So $\text{poly}_\Sigma(\uparrow B(x, \varepsilon)) = \uparrow_\Sigma \alpha$, and we are done by Lemma 10.25. □

Note that any point y in $\uparrow B(x, \varepsilon)$ is connected by a path to x. Also, the set of rational points $\Sigma_\mathbb{Q} = \bigcup_{\alpha \in \Sigma} p_\alpha(N^\alpha \otimes_\mathbb{Z} \mathbb{Q} \cap \sigma_\alpha)$ of Σ is countable and dense. So we arrive to:

Proposition 10.29. *Let Σ be a polyhedral complex with a chosen family of compatible bases. The family of principal balls of Σ that are rational and open, i.e. $\uparrow B(x, \epsilon)$ with $x \in \Sigma_\mathbb{Q}$ and rational ε small enough, is a countable basis of path-connected sets for $|\Sigma|$.* □

Example 10.30. If Γ is a polyhedral complex of dimension 1, i.e. a metric graph, then the requirement of a family of compatible bases is immediate. In higher dimensions, the constructions of the moduli spaces $\mathcal{M}_g^{\text{trop}}$ and $\mathcal{G}_{g \to 0, d}^{\text{trop}}$ come with obvious choices for a family of compatible bases. ⋆

10.6 Combinatorial morphisms

Let $\Phi : [\sigma : \Sigma \to \text{POLY}_\mathbb{Z}^f] \to [\delta : \Delta \to \text{POLY}_\mathbb{Z}^f]$ be a morphism in PolyComplex, and $|\Phi| : |\Sigma| \to |\Delta|$ its topological realization. We give a condition which implies that $|\Sigma|$ arises as several copies of $|\Delta|$ glued together in a manner prescribed solely by $\varphi : \Sigma \to \Delta$. This strengthens Equation (9.4) to a bijection, which is crucial to relate the count of points in the fibres of $|\Phi|$ and φ.

Definition 10.31. We say that $\Phi = (\varphi, \{\Phi_\alpha\}_{\alpha \in \Sigma})$ is *combinatorial* if for all $\alpha \in \Sigma$ we have that $\Phi_\alpha(\sigma_\alpha) = \delta_{\varphi(\alpha)}$ and $\dim \sigma_\alpha = \dim \delta_{\varphi(\alpha)}$.

The condition $\Phi_\alpha(\sigma_\alpha) = \delta_{\varphi(\alpha)}$ implies that $\Phi_\alpha : N_\mathbb{R}^\alpha \to N_\mathbb{R}^{\varphi(\alpha)}$ is a surjective linear map. Combined with the dimension condition, we get that the linear map Φ_α is bijective, hence a homeomorphism. In fact, it is straightforward to see that Φ is combinatorial if and only if Φ_α is injective and $\Phi_\alpha(\sigma_\alpha) = \delta_{\varphi(\alpha)}$ for all α in Σ.

Remark 10.32. In [Pay06] a combinatorial morphism of cone complexes satisfies that Φ_α maps $(N^\alpha, \sigma_\alpha)$ isomorphically to $(N^{\varphi(\alpha)}, \delta_{\varphi(\alpha)})$. This is equivalent to $\Phi_\alpha : N_\mathbb{R}^\alpha \to N_\mathbb{R}^{\varphi(\alpha)}$ being injective, $\Phi_\alpha(\sigma_\alpha) = \delta_{\varphi(\alpha)}$, and $\Phi_\alpha(N^\alpha) = N^{\varphi(\alpha)}$. We have omitted the last condition, since in our setting the index $[N_{\varphi(\alpha)} : \Phi_\alpha(N_\alpha)]$ is relevant. △

We give another characterization in terms of the map φ.

Lemma 10.33 (a combinatorial characterization)**.** *A morphism $(\varphi : \Sigma \to \Delta, \{\Phi_\alpha\}_{\alpha \in \Sigma})$ of polyhedral complexes is combinatorial if and only if φ maps $\downarrow_\Sigma \alpha$ isomorphically to $\downarrow_\Delta \varphi(\alpha)$ for all $\alpha \in \Sigma$.*

Proof. Suppose that $(\varphi, \{\Phi_\alpha\}_{\alpha \in \Sigma})$ is combinatorial, so $\Phi_\alpha : N_\mathbb{R}^\alpha \to N_\mathbb{R}^{\varphi(\alpha)}$ is bijective. As Φ_α is a bijective linear map, we have that the polyhedral structure of σ_α is isomorphic to that of $\Phi_\alpha(\sigma_\alpha)$. By supposition $\Phi_\alpha(\sigma_\alpha) = \delta_{\varphi(\alpha)}$, so their polyhedral structure is also isomorphic.

For the converse, suppose that φ maps $\downarrow_\Sigma \alpha$ isomorphically to $\downarrow_\Delta \varphi(\alpha)$ for all $\alpha \in \Sigma$. On the one hand, by Remark 9.24 we have that $\dim \sigma_\alpha = \text{length}(\downarrow_\Sigma \alpha) = \text{length}(\downarrow_\Delta \varphi(\alpha)) = \dim \delta_{\varphi(\alpha)}$. On the other hand, since $\varphi(\downarrow_\Sigma \alpha) = \downarrow_\Delta \varphi(\alpha)$, we have that all the 1-faces of $\delta_{\varphi(\alpha)}$ are in $\Phi_\alpha(\sigma_\alpha)$. Since $\delta_{\varphi(\alpha)}$ is the convex hull of its 1-faces, we have that $\delta_{\varphi(\alpha)} \subset \Phi_\alpha(\sigma_\alpha)$. Thus, $\delta_{\varphi(\alpha)} = \Phi_\alpha(\sigma_\alpha)$, so we are done. □

Thus, we define:

Definition 10.34. We call a morphism of posets $\varphi : \Sigma \to \Delta$ *combinatorial* if and only if Σ and Δ are finite and φ maps $\downarrow_\Sigma \alpha$ isomorphically to $\downarrow_\Delta \varphi(\alpha)$ for all $\alpha \in \Sigma$.

So Lemma 10.33 says that a morphism $\Phi = (\varphi : \Sigma \to \Delta, \{\Phi_\alpha\}_{\alpha\in\Sigma})$ in PolyComplex is combinatorial if and only if the underlying morphism of categories $\varphi : \Sigma \to \Delta$ is combinatorial.

Now we set our sights into proving that $|\Sigma|$ is homeomorphic to the fibre product $|\Delta| \times_\Delta \Sigma$ under the maps poly_Δ and φ. Recall that given $f\colon X \to Z$ and $g\colon Y \to Z$ in Top, the fibre product $X \times_Z Y$ is the limit of the diagram $X \to Z \leftarrow Y$. This is homeomorphic to the subspace

$$X \times_Z Y = \{(x, y)\colon\ f(x) = g(y)\}$$

of the product topology $X \times Y$. The universal property states that for any W with maps $p\colon W \to X$ and $q\colon W \to Y$, there is a unique map $\phi : W \to X \times_Z Y$ that gives factorizations $p = \pi_X \circ \phi$ and $q = \pi_Y \circ \phi$, with π_X and π_Y the canonical projections. Componentwise the map $\phi : W \to X \times_Z Y$ is $x \mapsto (p(x), q(x))$. The following generalizes [Pay06, Proposition 3.23] to the setting of polyhedral complexes.

Lemma 10.35. *If $\Phi : \Sigma \to \Delta$ in* PolyComplex *is combinatorial, then the following map*

$$\begin{aligned}\phi\colon |\Sigma| &\to |\Delta| \times \Sigma\\ x &\mapsto (|\Phi|(x),\ \text{poly}_\Sigma(x))\end{aligned}$$

is a bijection onto $|\Delta| \times_\Delta \Sigma$.

Proof. That the image of ϕ is contained in $|\Delta| \times_\Delta \Sigma$ follows from Lemma 9.28. Let (y, γ) be such that $\text{poly}_\Delta(y) = \varphi(\gamma)$, so $y \in p_{\varphi(\gamma)}(\delta^\circ_{\varphi(\gamma)})$. Since $(\varphi, \{\Phi_\alpha\}_{\alpha\in\Sigma})$ is combinatorial, the map $\Phi_\gamma : |\sigma_\gamma| \to |\delta_{\varphi(\gamma)}|$ is a homeomorphism. Thus, there is exactly one x in $p_\gamma(\sigma^\circ_\gamma) = \text{poly}_\Sigma^{-1}(\gamma)$ such that $|\Phi|(x) = y$. This shows that ϕ is bijective. □

For a point y in $|\Delta|$ the fibre ${\pi_{|\Delta|}}^{-1}(y)$ is equal to $\{(y,\ \alpha)\colon\ \varphi(\alpha) = \text{poly}_\Delta(y)\}$. Thus, Lemma 10.35 strengthens Equation (9.4) to an equality:

Lemma 10.36. *If $(\varphi, \{\Phi_\alpha\}_{\alpha\in\Sigma}) : \Sigma \to \Delta$ in* PolyComplex *is combinatorial, then for any y in $|\Delta|$ the map poly_Σ induces a bijection*

$$|\Phi|^{-1}(y) \to \varphi^{-1}(\text{poly}_\Delta(y)).$$ □

Lemma 10.35 implies that ϕ restricts to a bijective continuous map from $|\Sigma|$ to $|\Delta| \times_\Delta \Sigma$. It is left to prove that ϕ is an open map. Given an open set V of $|\Sigma|$, we give an open set U of $|\Delta| \times \Sigma$ such that $\phi(V) = U \cap (|\Delta| \times_\Delta \Sigma)$. Since poly_Σ is open, $\text{poly}_\Sigma(V)$ is open. Also, the maps $\{\Phi_\alpha\}_{\alpha\in\Sigma}$ which glue together to form $|\Phi|$ are open. Thus, a natural candidate for U is $|\Phi|(V) \times \text{poly}_\Sigma(V)$, which is explored in the following example.

Example 10.37. Let $(\varphi, \{\Phi_\alpha\}_{\alpha\in\Sigma})$ be a combinatorial morphism in PolyComplex mapping a complex with three cones down to a complex with two cones. Figure 10.2 displays the topological realization $|\Phi|$ and the order preserving map φ. We consider the open subset U as shown in the figure, and V one of the connected components of $|\Phi|^{-1}(U)$ in the cone β_2. Observe that $\text{poly}_\Sigma(V) = \{B_2, \beta_2\}$. Hence, the set $\text{poly}_\Delta(\Phi(V)) = \varphi(\text{poly}_\Sigma(V)) = \{B, \beta\}$ is not open, since $\uparrow_\Delta B = \{B, \alpha, \beta\}$. As poly_Δ is open, we get that neither $|\Phi|(V)$ nor $|\Phi|(V) \times \text{poly}_\Sigma(V)$ are open. As $\text{poly}_\Sigma(V)$ is open, we get that φ is not an open map. ⋆

So the proof that ϕ is open, which implies that ϕ is a homeomorphism, is slightly more involved.

Lemma 10.38. *Let $\Phi : \Sigma \to \Delta$ in* PolyComplex *be combinatorial. Assume that Σ and Δ have chosen families of compatible bases. The map ϕ given by $x \mapsto (|\Phi|(x), \text{poly}_\Sigma(x))$ is an open map.*

Proof. Let V be an open subset of $|\Sigma|$, choose any point x in V, set $y = |\Phi|(x)$, $\alpha = \text{poly}_\Sigma(x)$ and $\beta = \varphi(\alpha) = \text{poly}_\Delta(y)$. We are done if we exhibit an open neighbourhood of $\phi(x) = (y, \alpha)$ contained in $\phi(V)$. Note that by Lemma 10.35 we have

$$\phi(V) = (|\Phi|(V) \times \text{poly}_\Sigma(V)) \cap (|\Delta| \times_\Delta \Sigma). \tag{10.10}$$

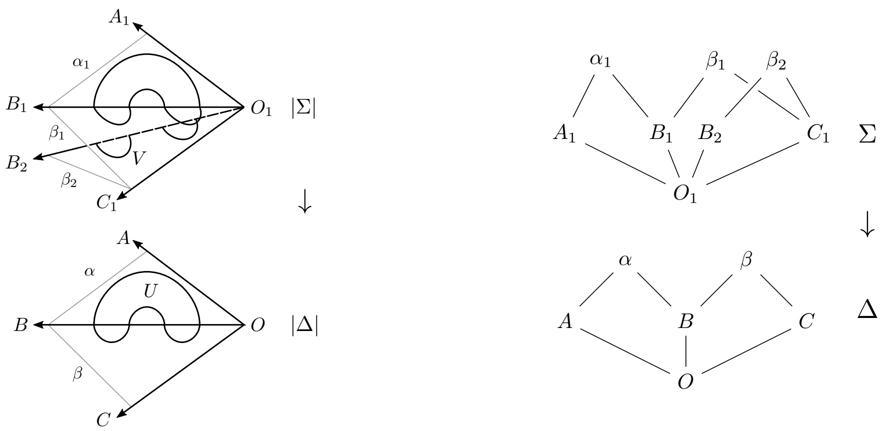


Figure 10.2: On the left, the topological realization $|\Phi|$ of a combinatorial morphism in PolyComplex, an open subset U of the codomain $|\Delta|$, and the fibre $|\Phi|^{-1}(U)$ above U. On the right the map $\varphi : \Sigma \to \Delta$ of face posets.

We claim that for small enough ε, the neighbourhood of (y, α) given by

$$U_\varepsilon = (\uparrow B(y, \varepsilon) \times \uparrow_\Sigma \alpha) \cap (|\Delta| \times_\Delta \Sigma),$$

is an open and contained in $\phi(V)$.

By Proposition 10.29 there is $\varepsilon_x > 0$ such that $\uparrow B(x, \varepsilon_x)$ is open and contained in V; there is also K_x such that if $0 < \varepsilon < K_x$ the set $\uparrow B(y, \varepsilon)$ is open. Let (z, η) be in U_ε. Since (z, η) is in $|\Delta| \times_\Delta \Sigma$, we have that $\gamma = \varphi(\eta)$ equals $\text{poly}_\Delta(z)$. We also have that η is in $\uparrow_\Sigma \alpha$, so Lemma 10.25 gives

$$p_\eta^{-1}(\uparrow B(x, \varepsilon_x)) = B(N_\mathbb{R}^\eta, p_\eta^{-1}(x), \varepsilon_x) \cap \sigma_\eta.$$

Since Φ is combinatorial, Φ_η maps σ_η homeomorphically to δ_γ. As $B(N_\mathbb{R}^\eta, p_\eta^{-1}(x), \varepsilon_x) \cap \sigma_\eta$ is open in σ_η, the set $\Phi_\eta(B(N_\mathbb{R}^\eta, p_\eta^{-1}(x), \varepsilon_x) \cap \sigma_\eta)$ is open in δ_γ, so there is ε_η such that

$$B(N_\mathbb{R}^\gamma, \Phi_\eta \circ p_\eta^{-1}(x), \varepsilon_\eta) \cap \delta_\gamma \subset \Phi_\eta(B(N_\mathbb{R}^\eta, p_\eta^{-1}(x), \varepsilon) \cap \sigma_\eta) \subset p_\gamma^{-1}(|\Phi|(\uparrow B(x, \varepsilon_x))).$$

The second containment above follows from Diagram 9.8; from the same diagram we also get that $\Phi_\eta \circ p_\eta^{-1}(x) = p_\gamma^{-1}(y)$. So if $\varepsilon < \varepsilon_\eta$, we get

$$z \in p_\gamma(B(N_\mathbb{R}^\gamma, p_\gamma^{-1}(y), \varepsilon) \cap \delta_\gamma) \subset |\Phi|(\uparrow B(x, \varepsilon_x)). \tag{10.11}$$

If we choose an ε_η for each η in $\uparrow_\Sigma \alpha$ as above, and set ε to be the minimum of the ε_η and K_x, we have by Equation (10.11) that

$$(\uparrow B(y, \varepsilon) \times \uparrow_\Sigma \alpha) \cap (|\Delta| \times_\Delta \Sigma) \subset (|\Phi|(\uparrow B(x, \varepsilon_x)) \times \uparrow_\Sigma \alpha) \cap (|\Delta| \times_\Delta \Sigma).$$

Note that $|\Phi|(\uparrow B(x, \varepsilon_x))$ is contained in $|\Phi|(V)$. Also, since V is open, $\text{poly}_\Sigma(V)$ is an up-set, so $\uparrow_\Sigma \alpha$ is contained in it. Comparing with Equation (10.10) we see that $U_\varepsilon \subset \phi(V)$; and as both $\uparrow B(y, \varepsilon)$ and $\uparrow_\Sigma \alpha$ are open, we are done. □

Remark 10.39. In Subsection 10.11 we show that if Φ_α is injective and $\dim \sigma_\alpha = \dim \delta_{\varphi(\alpha)}$ for all α in Σ, then it is possible to construct Δ' in PolyComplex as a *subdivision* of Δ such that the topological realizations $|\Delta|$ and $|\Delta'|$ are homeomorphic, and which gives an induced map $(\varphi', \{\Phi'_\alpha\}_{\alpha \in \Sigma}) : \Sigma \to \Delta'$ that is combinatorial. △

10.7 Relating indexed branched covers

Let Σ and Δ be polyhedral complexes, $\Phi : \Sigma \to \Delta$ a combinatorial morphism and $|\Phi|$ its topological realization. This subsection studies the connected components of the fibres in Σ over principal open sets $\uparrow_\Delta \beta$ in Δ, and of fibres in $|\Sigma|$ over principal balls in $|\Delta|$ that are open. Using Lemma 10.23 this study culminates the proof of Theorem A.

Lemma 10.40. *Let $\varphi : \Sigma \to \Delta$ be a combinatorial morphism of posets, and β be in Δ. We have that*

$$\pi_0(\varphi^{-1}(\uparrow_\Delta \beta)) = \{\uparrow_\Sigma \alpha\}_{\alpha \in \varphi^{-1}(\beta)}\,.$$

Proof. First observe that for any $\gamma \in \varphi^{-1}(\uparrow_\Delta \beta)$ we have that $\beta \in \downarrow_\Delta \varphi(\gamma)$. Since φ is a combinatorial morphism, it maps $\downarrow_\Sigma \gamma$ bijectively to $\downarrow_\Delta \varphi(\gamma)$. Thus, there is an $\alpha \in \Sigma$ such that $\alpha \preceq \gamma$ and $\varphi(\alpha) = \beta$, which gives

$$\varphi^{-1}(\uparrow_\Delta \beta) = \bigcup_{\alpha \in \varphi^{-1}(\beta)} \uparrow_\Sigma \alpha. \tag{10.12}$$

This union is disjoint, because if there were α, α' in $\varphi^{-1}(\beta)$ such that $\uparrow_\Sigma \alpha \cap \uparrow_\Sigma \alpha'$ contained an element γ, then φ would map $\downarrow_\Sigma \gamma$ isomorphically to $\downarrow_\Delta \varphi(\gamma)$. Since $\alpha, \alpha' \in \downarrow_\Sigma \gamma$, this gives $\varphi(\alpha) \neq \varphi(\alpha')$, a contradiction. Finally, any open set that contains α must contain $\uparrow_\Sigma \alpha$, so each $\uparrow_\Sigma \alpha$ in Equation (10.12) is connected. □

The following is a version of Lemma 10.36 for connected components.

Proposition 10.41. *Let $U \subset |\Delta|$ be a principal ball that is open, and $\mathcal{U} = \operatorname{poly}_\Delta U$. If Φ is combinatorial and Σ and Δ have chosen families of compatible bases, then $\operatorname{poly}_\Sigma$ induces a bijection of connected components of the fibres*

$$\pi_0(|\Phi|^{-1}(U)) \to \pi_0(\varphi^{-1}(\mathcal{U})).$$

Proof. Let V be a component in $\pi_0(|\Phi|^{-1}(U))$. We first show that $\operatorname{poly}_\Sigma(V)$ is a component in $\pi_0(\varphi^{-1}(\mathcal{U}))$. Let $y \in |\Delta|$ and $\varepsilon > 0$ be such that $U = \uparrow B(y, \varepsilon)$. Set $\beta = \operatorname{poly}_\Delta y$; since U is open we have that $\mathcal{U} = \operatorname{poly}_\Delta U = \uparrow_\Delta \beta$. Assume there is a point x in V such that $|\Phi|(x) = y$ and set $\alpha = \operatorname{poly}_\Sigma(x)$. Since $\operatorname{poly}_\Sigma$ is a continuous open map, we have that $\operatorname{poly}_\Sigma(V)$ is a connected open set containing $\uparrow_\Sigma \alpha$. Lemma 10.40 then implies that $\operatorname{poly}_\Sigma(V)$ is a connected component of $\pi_0(\varphi^{-1}(\mathcal{U})) = \pi_0(\varphi^{-1}(\uparrow_\Delta \beta))$, since α is in $\varphi^{-1}(\beta)$.

To prove there is a point x in V such that $|\Phi|(x) = y$, choose any z in V and set $\gamma = \operatorname{poly}_\Sigma(z)$. We claim that $\beta \preceq \varphi(\gamma)$. Indeed, from $|\Phi|(V) \subset U$ we get $\operatorname{poly}_\Delta \circ |\Phi|(V) \subset \operatorname{poly}_\Delta(U) = \mathcal{U}$; and from $\operatorname{poly}_\Delta \circ |\Phi|(V) = \varphi \circ \operatorname{poly}_\Sigma(V)$ we get $\varphi(\gamma) \in \mathcal{U} = \uparrow_\Delta \beta$. So there is a morphism $f_{\beta\varphi(\gamma)} : \delta_\beta \to \delta_{\varphi(\gamma)}$, and we have $p_\beta = p_{\varphi(\gamma)} \circ f_{\beta\varphi(\gamma)}$. Since y is in $\operatorname{im} p_\beta$, we have y in $\operatorname{im} p_{\varphi(\gamma)}$, and we may set $\hat{y} = p^{-1}_{\varphi(\gamma)}(y)$.

As Φ is combinatorial, the affine map Φ_γ sends σ_γ homeomorphically to $\delta_{\varphi(\gamma)}$. So $\hat{y}$ is in $\operatorname{im} \Phi_\gamma$, and we may set $\hat{x} = \Phi^{-1}_\gamma(\hat{y})$. By Diagram 9.11, we have that $|\Phi| \circ p_\gamma(\hat{x}) = p_{\varphi(\gamma)} \circ \Phi_\gamma(\hat{x}) = p_{\varphi(\gamma)}(\hat{y}) = y$. Set $x = p_\gamma(\hat{x})$, so $|\Phi|(x) = y$. Note that x is in $p_\gamma(\Phi^{-1}_\gamma(p^{-1}_{\varphi(\gamma)}(U)))$. We claim the latter set is contained in V.

Since $\operatorname{poly}_\Delta y = \beta \preceq \varphi(\gamma)$, we may apply Lemma 10.25 to get that

$$p^{-1}_{\varphi(\gamma)}(U) = B(N^{\varphi(\gamma)}_{\mathbb{R}}, p^{-1}_{\varphi(\gamma)}(y), \varepsilon) \cap \delta_{\varphi(\gamma)},$$

which in particular means that $p^{-1}_{\varphi(\gamma)}(U)$ is a convex set, hence connected. As Φ_γ is a homeomorphism, $\Phi^{-1}_\gamma(p^{-1}_{\varphi(\gamma)}(U))$ is connected as well, and so is $p_\gamma(\Phi_\gamma(p^{-1}_{\varphi(\gamma)}(U)))$. Finally, recall we have chosen a point z in V and set $\gamma = \operatorname{poly}_\Sigma(z)$. Since $|\Phi|(z) \in |\Phi|(V) \subset U$ and $\operatorname{poly}_\Sigma(z) = \gamma$, by Diagram 9.11 we have that z is in $p_\gamma(\Phi^{-1}_\gamma(p^{-1}_{\varphi(\gamma)}(U)))$. So $p_\gamma(\Phi^{-1}_\gamma(p^{-1}_{\varphi(\gamma)}(U))) \subset |\Phi|^{-1}(U)$ is connected, and intersects the connected component V, so we conclude the desired inclusion.

To conclude, we have proven that each component of $|\Phi|^{-1}(U)$ intersects $|\Phi|^{-1}(y)$, for y the centre of the ball U. Moreover, we have proven that for x in $|\Phi|^{-1}(y)$ and V the component containing x, we get $\operatorname{poly}_\Sigma(V) = \uparrow_\Sigma(\operatorname{poly}_\Sigma(x))$. These two facts, together with Lemma 10.36 and Lemma 10.40 imply surjectivity and injectivity. □

Remark 10.42. In Figure 10.2 the set U is connected, and the set $\mathcal{U} = \mathrm{poly}_\Delta U$ is principal, yet $\pi_0(|\Phi|^{-1}(U))$ has three elements, while $\pi_0(\varphi^{-1}(\mathcal{U}))$ has only two. This highlights the need to impose restrictions on U in Proposition 10.41. △

Now, we show that if one of the vertical maps in Diagram 9.29 is an indexed branched cover, then so is the other. Consider an index map $m_\Phi : |\Sigma| \to \mathbb{Z}_{\geq 1}$, a connected open $U \subset |\Delta|$, a connected component V in $\pi_0(|\Phi|^{-1}(U))$, and a point y in U. We have that

$$\deg(|\Phi|, m_\Phi, V)(y) = \sum_{\substack{x \in |\Phi|^{-1}(y) \\ x \in V}} m_\Phi(x).$$

Likewise, for an index map $m_\varphi : \Sigma \to \mathbb{Z}_{\geq 1}$, a connected open $\uparrow_\Delta \beta \subset \Delta$, a connected component $\mathcal{V}$ in $\pi_0(|\Phi|^{-1}(\uparrow_\Delta \beta))$, and a point γ in $\uparrow_\Delta \beta$. We have that

$$\deg(\varphi, m_\varphi, \mathcal{V})(\gamma) = \sum_{\substack{\alpha \in \varphi^{-1}(\gamma) \\ \alpha \in \mathcal{V}}} m_\varphi(\alpha).$$

Our aim is to relate both degrees. The first step compares the index sets $|\Phi|^{-1}(y) \cap V$ and $\varphi^{-1}(\gamma) \cap \mathcal{V}$ by putting Lemmas 10.36 and Proposition 10.41 together.

Lemma 10.43. *Let $U \subset |\Delta|$ be a principal ball that is open, V in $\pi_0(|\Phi|^{-1}(U))$, and y in U. If Φ is combinatorial and Σ and Δ have chosen families of compatible bases, then* poly_Σ *induces a bijection*

$$V \cap |\Phi|^{-1}(y) \to \mathrm{poly}_\Sigma(V) \cap \varphi^{-1}(\mathrm{poly}_\Delta(y)). \tag{10.13}$$

Proof. Since Φ is combinatorial, by Lemma 10.36 the map from Equation (10.13) is injective; and for α in $\mathrm{poly}_\Sigma(V) \cap \varphi^{-1}(\mathrm{poly}_\Delta(y))$, there is x in $|\Phi|^{-1}(y)$ such that $\mathrm{poly}_\Sigma(x) = \alpha$. Suppose that x is not in V. Let V' be the connected component of x in $|\Phi|^{-1}(U)$. By Proposition 10.41, we have that $\mathrm{poly}_\Sigma(V')$ is a connected component distinct from $\mathrm{poly}_\Sigma(V)$. But α is both in $\mathrm{poly}_\Sigma(V)$ and $\mathrm{poly}_\Sigma(V')$, a contradiction. Hence, x is in V, as desired. □

Having related both index sets, it is straightforward to prove the following:

Proposition 10.44. *Assume that Φ is combinatorial and that Σ and Δ have chosen families of compatible bases. Let $m_\varphi : \Sigma \to \mathbb{Z}_{\geq 1}$ be an index map. We have that:*

1. *If (φ, m_φ) is an indexed branched cover unramified over $\mathcal{W} \subset \Delta$, then $(|\Phi|, m_\varphi \circ \mathrm{poly}_\Sigma)$ is an indexed branched cover unramified over* $\mathrm{poly}_\Delta^{-1}\mathcal{W}$.
2. *If $(|\Phi|, m_\varphi \circ \mathrm{poly}_\Sigma)$ is an indexed branched cover unramified over $W \subset |\Delta|$, then (φ, m_φ) is an indexed branched cover unramified over* $\mathrm{poly}_\Delta(W)$.

Proof. We first make two observations, and then proceed to prove each item.

Observation I: Let $y \in |\Delta|$ and $\beta \in \Delta$ be such that $\mathrm{poly}_\Delta(y) = \beta$. By Lemma 10.28, if ε is small enough, then $U_\varepsilon = \uparrow B(y, \varepsilon)$ is open, hence $\mathrm{poly}_\Delta(U_\varepsilon) = \uparrow_\Delta \beta$. Let V be a connected component in $\pi_0(|\Phi|^{-1}(U_\varepsilon)))$. By Proposition 10.41 we have that $\mathrm{poly}_\Sigma(V)$ equals $\uparrow_\Sigma \alpha$ for some α in $\varphi^{-1}(\beta)$, and is a connected component of $\varphi^{-1}(\uparrow_\Delta \beta)$. So Diagram 9.29 restricts to

$$\begin{array}{ccc} V & \xrightarrow{\mathrm{poly}_\Sigma} & \uparrow_\Sigma \alpha \\ {\scriptstyle |\Phi|}\big\downarrow & & \big\downarrow{\scriptstyle \varphi} \\ U_\varepsilon & \xrightarrow{\mathrm{poly}_\Delta} & \uparrow_\Delta \beta. \end{array}$$

Diagram 10.45

Both vertical maps are continuous. Since poly_Σ and poly_Δ are open maps, if one of the vertical maps is an open map, so is the other one. Finally, Lemma 10.43 impies that if one of the vertical maps is bijective, so is the other one. Hence, if one of the vertical maps is a homeomorphism, so is the other one.

Observation II: Let $U \subset |\Delta|$ be a principal ball that is open, V in $\pi_0(|\Phi|^{-1}(U))$, and y a point in U. Set $\mathcal{U} = \mathrm{poly}_\Sigma(U)$. Lemma 10.43 then gives

$$\begin{aligned} \deg(|\Phi|, m_\varphi \circ \mathrm{poly}_\Sigma, V)(y) &= \sum_{\substack{x \in |\Phi|^{-1}(y) \\ x \in V}} m_\varphi \circ \mathrm{poly}_\Sigma(x) \\ &= \sum_{\substack{\gamma \in \varphi^{-1}(\mathrm{poly}_\Delta(y)) \\ \gamma \in \mathrm{poly}_\Sigma(V)}} m_\varphi(\gamma) \\ &= \deg(\varphi, m_\varphi, \mathrm{poly}_\Sigma(V))(\mathrm{poly}_\Delta(y)). \end{aligned} \tag{10.14}$$

Proof of Item (1): Assume that (φ, m_φ) is an indexed branched cover that is unramified over $\mathcal{W} \subset \Delta$. Since poly_Δ is open and continouous, the set $W = \mathrm{poly}_\Delta^{-1}(\mathcal{W})$ is open and dense. To see that $|\Phi|$ is a branched cover that is unramified over W, let y be in W, so $\beta = \mathrm{poly}_\Delta\, y$ is in $\mathcal{W}$. Consider a principal ball $U_\varepsilon = \uparrow B(y, \varepsilon) \subset |\Delta|$ that is open, so $\mathrm{poly}_\Delta U_\varepsilon = \uparrow_\Delta \beta$. Since φ is a branched cover that is unramified over $\mathcal{W}$ and $\uparrow_\Delta \beta$ is the smallest open neighbourhood that contains β, we have that $\varphi^{-1}(\uparrow_\Delta \beta)$ comprises several disjoint sets, each mapped to $\uparrow_\Delta \beta$ homeomorphically by φ. Since $\uparrow_\Delta \beta$ is connected, these disjoint sets are the elements of $\pi_0(\varphi^{-1}(\uparrow_\Delta \beta))$. By Observation I and Proposition 10.41 this means that each element in $\pi_0(|\Phi|^{-1}(U_\varepsilon))$ is mapped homeomorphically to U_ε by $|\Phi|$, as desired.

Since $|\Phi|$ is a branched cover, to see that $(|\Phi|, m_\varphi \circ \mathrm{poly}_\Sigma)$ is an indexed branched cover, by Lemma 10.23 and Proposition 10.29, it is enough to consider a principal ball $U = \uparrow B(y, \varepsilon) \subset |\Delta|$ that is open. Let V be in $\pi_0(|\Phi|^{-1}(U))$. By Proposition 10.41 the open set $\mathrm{poly}_\Sigma(V)$ is a connected component of $\varphi^{-1}(\mathcal{U})$. Since (φ, m_φ) is an indexed branched cover, $\deg(\varphi, m_\varphi, \mathrm{poly}_\Sigma(V))$ is constant over $\uparrow_\Delta \beta$, so Equation (10.14) implies that $\deg(|\Phi|, m_\varphi \circ \mathrm{poly}_\Sigma, V)(y)$ is constant over U.

Proof of Item (2): Assume that $(|\Phi|, m_\varphi \circ \mathrm{poly}_\Sigma)$ is an indexed branched cover unramified over $W \subset |\Delta|$. Since poly_Δ is open and surjective, the set $\mathcal{W} = \mathrm{poly}_\Delta W$ is open and dense. To see that φ is a branched cover that is unramified over $\mathcal{W}$, let β be in $\mathcal{W}$. Since $\mathcal{W}$ is open, it is an up-set, we have $\uparrow_\Delta \beta \subset \mathcal{W}$, so the set $U = \mathrm{poly}_\Delta^{-1}(\uparrow_\Delta \beta)$ is non-empty and contained in W. So once again, we observe that U is connected, so each element in $\pi_0(|\Phi|^{-1}(U))$ gets mapped homeomorphically to U by $|\Phi|$. We conclude again by Observation I and Proposition 10.41.

Since φ is a branched cover, to see that (φ, m_φ) is an indexed branched cover, by Lemma 10.22 it is enough to consider an upset $\uparrow_\Delta \beta$. Let $\mathcal{V}$ be in $\pi_0(\varphi^{-1}(\uparrow_\Delta \beta))$. Since poly_Δ is surjective, we can choose $y \in |\Delta|$ such that $\mathrm{poly}_\Delta(y) = \beta$, and by Proposition 10.29 we can choose ε such that $U = \uparrow B(y, \varepsilon)$ is open. So $\mathrm{poly}_\Delta(U) = \uparrow_\Delta \beta$. By Proposition 10.41 there is V in $\pi_0(|\Phi|^{-1}(U))$ such that $\mathrm{poly}_\Sigma(V) = \mathcal{V}$. Since $(|\Phi|, m_\varphi \circ \mathrm{poly}_\Sigma)$ is an indexed branched cover, $\deg(|\Phi|, m_\varphi \circ \mathrm{poly}_\Sigma, V)(y)$ is constant over U. so Equation (10.14) implies that $\deg(\varphi, m_\varphi, \mathrm{poly}_\Sigma(V))$ is constant over $\uparrow_\Delta \beta$. □

Putting everything together we get Theorem A.

Theorem 10.46. *Let $\Phi : \Sigma \to \Delta$ in* PolyComplex *be a combinatorial morphism such that Σ and Δ have chosen families of compatible bases. The diagram*

$$\begin{array}{ccc} |\Sigma| & \xrightarrow{\mathrm{poly}_\Sigma} & \Sigma \\ {\scriptstyle |\Phi|}\downarrow & & \downarrow{\scriptstyle \varphi} \\ |\Delta| & \xrightarrow{\mathrm{poly}_\Delta} & \Delta \end{array}$$

is a fibre product in Top*, and if one of the pairs (φ, m_φ) or $(|\Phi|, \mathrm{poly}_\Sigma \circ m_\varphi)$ is an indexed branched cover, then so is the other pair.*

Proof. The square commutes by Lemma 9.28. It is a fibre product by Lemmas 10.35 and 10.38. Finally Proposition 10.44 gives that if one of the vertical maps is an indexed branched cover, so is the other. □

Remark 10.47. By Theorem A, if $\Phi : \Sigma \to \Delta$ is a morphism in PolyComplex, and we have an indexed branched cover pair $(|\Phi| : |\Sigma| \to |\Delta|, m : |\Sigma| \to \mathbb{Z}_{\geq 1})$ such that m is constant on the interiors of cones of $|\Sigma|$, i.e. on each subset of the disjoint union from Lemma 9.16 then we get an indexed branched cover $(\varphi : \Sigma \to \Delta, m \circ \mathrm{poly}_\Sigma^{-1})$ reflecting many properties of $|\Phi|$. So we reduce our study to a combinatorial object. That is, φ is a higher dimensional analogue of DT-morphismsas studied in Part I. △

10.8 The balancing condition

In the case of posets, asking $\deg(\varphi : \Sigma \to \Delta, m_\varphi, \mathcal{V})$ to be constant imposes relations on the values of m_φ. We explore one such relation and show that it is equivalent to Φ being an indexed branched cover of polyhedral complexes. This gives, together with the results of the next subsection, a criteria to simplify the work of showing that a morphism φ is an indexed branched cover.

As motivation, we recall the 1-dimensional case. Let Γ and Δ be polyhedral complexes of dimension 1, i.e. $\dim \sigma_\alpha \leq 1$ for all α in Σ. Consider an indexed branched cover $\Phi : \Gamma \to \Delta$ with index map m_φ and topological realization $|\Phi| : |\Gamma| \to |\Delta|$. If Γ and Δ are connected, and Φ satisfies that for every non-maximal element A of Γ the inequality

$$r_\varphi(A) = 2(m_\varphi(A) - 1) - \sum_{A \lessdot e} (m_\varphi(e) - 1) \geq 0. \tag{RH}$$

is true, then we say that Φ is a *tropical morphism*. Recall that $A \lessdot e$ means that e covers A, i.e. that e is in $\min(\uparrow A \setminus \{A\})$. The inequality from Equation (RH) is called *the Riemann-Hurwitz inequality.*

One intuition behind Equation (RH) is that the value $r_\varphi(A) + 1$ is a tropical analogue of the ramification index, so it should be non-negative. This is seen in a tropical analogue of the Riemann-Hurwitz formula, see Lemma 12.2 on Page 122. Another way to look at Equation (RH) is as a realizability condition; see [DV21] for an exposition. This definition of tropical morphism differs from the usual one that includes a balancing condition, e.g. the one from Part I. These two definitions are in fact equivalent by the results of this section, i.e. that being an indexed branched cover of posets is equivalent to satisfying the balancing condition, defined as:

Definition 10.48. Let $\varphi : \Sigma \to \Delta$ be a morphism of finite posets, $\mathcal{V} \subset \Sigma$ an up-set, and $m_\mathcal{V} : \mathcal{V} \to \mathbb{Z}_{\geq 1}$ a map. We say that $(\varphi, m_\mathcal{V})$ is *balanced*, if for any α in $\mathcal{V}$ and any choice of β in Δ such that $\varphi(\alpha) \lessdot \beta$ the following equation, called *the balancing condition*, holds:

$$m_\mathcal{V}(\alpha) = \sum_{\substack{\gamma \in \varphi^{-1}(\beta) \\ \alpha \lessdot \gamma}} m_\mathcal{V}(\gamma). \tag{10.15}$$

Example 10.49 (discrete tropical morphisms)**.** Recall that using the index map from Example 10.9, the map $\Gamma \to \Delta$ of posets from Figure 10.1 (b) becomes an indexed branched cover. One can verify the balancing condition, and moreover the RH-inequality, thus this map is an example of a tropical morphism.

By Theorem A, the topological realization $|\Phi| : |\Gamma| \to |\Delta|$ is an indexed branched cover of metric graphs. Moreover, Γ can be regarded as several copies of $|\Delta|$ that have been glued at certain regions. This is pictured in Figure 10.1 (c), where the dashed lines represent identification of points, as done in Part I. ⋆

If φ is clear from context, we simply say that $m_\mathcal{V}$ is a *balanced map.* Given a morphism of finite posets $\varphi : \Sigma \to \Delta$ and an index map $m_\mathcal{V}$, the balancing condition is automatically verified for those $\alpha \in \Sigma$ for which there is no β in Δ such that $\varphi(\alpha) \lessdot \beta$. Note that if $m_\mathcal{V}(\alpha)$ is balanced, and α is in $\max \mathcal{V}$, then the fact that $m_{\mathcal{V}}(\alpha) \geq 1$ implies that $\varphi(\alpha)$ is maximal in Δ, for otherwise the sum on the right of Equation (10.15) would be an empty sum, and this evaluates to zero. Moreover, if φ is combinatorial we get:

Lemma 10.50. *Let $\varphi : \Sigma \to \Delta$ be a combinatorial morphism of posets, $\mathcal{V} \subset \Sigma$ an up-set, and $m_{\mathcal{V}} : \mathcal{V} \to \mathbb{Z}_{\geq 1}$ a balanced map. The restriction $\varphi|_{\mathcal{V}} : \mathcal{V} \to \Delta$ is a branched cover unramified over $\max \Delta$.*

Proof. We check the two conditions from Example 10.5. To check the first condition, let β be in $\max \varphi(\mathcal{V})$. Suppose there is $\alpha \in \varphi^{-1}(\beta)$ such that α is not maximal, i.e. there is γ in Σ such that $\alpha < \gamma$. Since φ is combinatorial, it maps $\downarrow_\Sigma \gamma$ isomorphically to $\downarrow_\Delta \varphi(\gamma)$, thus $\varphi(\gamma)$ is strictly greater than β, contradicting that β is maximal. Hence, $\varphi^{-1}(\beta) \subset \max \Sigma$. The second condition is true by definition. □

The main technical idea behind the proofs of this subsection is to traverse sequences $\beta_0 \lessdot \beta_1 \lessdot \cdots \lessdot \beta_n$ in Δ and to use the balancing condition to show that the local degrees are constant. Thus, given two elements $\mu \lessdot \nu$, we investigate the fibres above them.

Lemma 10.51. *Let $\varphi : \Sigma \to \Delta$ be a combinatorial morphism of posets. For any α in Σ, and μ, ν in $\uparrow_\Delta \varphi(\alpha)$ such that $\mu \lessdot \nu$, we have the following partition:*

$$\varphi^{-1}(\nu) \cap \uparrow_\Sigma \alpha = \varphi^{-1}(\nu) \cap \bigsqcup_{\substack{\gamma \in \varphi^{-1}(\mu) \\ \gamma \in \uparrow_\Sigma \alpha}} \{\eta \in \Sigma \colon \gamma \lessdot \eta\}. \tag{10.16}$$

Proof. Consider the family $\{\uparrow_\Sigma \gamma\}$ indexed by $\gamma \in \varphi^{-1}(\mu) \cap \uparrow_\Sigma \alpha$. Each member of $\{\uparrow_\Sigma \gamma\}$ is a subset of $\uparrow_\Sigma \alpha$, and by Lemma 10.40 they are pairwise disjoint. Since $\{\eta \in \Sigma \colon \gamma \lessdot \eta\} \subseteq \uparrow_\Sigma \gamma$, we are done if we show that the right hand side of Equation (10.16) contains the left hand side. Namely, for η in $\varphi^{-1}(\nu) \cap \uparrow_\Sigma \alpha$, show there is $\gamma \in \varphi^{-1}(\mu) \cap \uparrow_\Sigma \alpha$ such that $\gamma \lessdot \eta$. Since φ is combinatorial, it maps $\downarrow_\Sigma \eta$ isomorphically to $\downarrow_\Delta \varphi(\eta)$, so there is a unique γ such that γ is in $\downarrow_\Sigma \eta$ and $\varphi(\gamma) = \mu$. Since $\varphi(\eta) = \nu$ and $\mu \lessdot \nu$, we have that $\gamma \lessdot \eta$. Moreover, $\uparrow_\Sigma \gamma$ is a connected set that intersects at η the set $\uparrow_\Sigma \alpha$. The latter is a connected component of the space $\varphi^{-1}(\uparrow_\Delta \varphi(\alpha))$, thus $\uparrow_\Sigma \gamma \subset \uparrow_\Sigma \alpha$, so $\gamma \in \uparrow_\Sigma \alpha$ as desired. □

As a corollary of Lemma 10.51 we get two crucial formulas that bridge together the situations where (φ, m_φ) is balanced and where (φ, m_φ) has constant local degrees.

Lemma 10.52. *Let $\varphi : \Sigma \to \Delta$ be a combinatorial morphism of posets, $\mathcal{V}$ an up-set, and $m_{\mathcal{V}} : \mathcal{V} \to \mathbb{Z}_{\geq 1}$ a balanced map. For any α in Σ, and μ, ν in $\uparrow_\Delta \varphi(\alpha)$ such that $\mu \lessdot \nu$ and $\varphi^{-1}(\mu) \cap \uparrow_\Sigma \alpha$ is contained in $\mathcal{V}$, we have:*

$$\sum_{\substack{\gamma \in \varphi^{-1}(\mu) \\ \gamma \in \uparrow_\Sigma \alpha}} m_{\mathcal{V}}(\gamma) = \sum_{\substack{\gamma \in \varphi^{-1}(\mu) \\ \gamma \in \uparrow_\Sigma \alpha}} \sum_{\substack{\eta \in \varphi^{-1}(\nu) \\ \gamma \lessdot \eta}} m_{\mathcal{V}}(\eta) = \sum_{\substack{\eta \in \varphi^{-1}(\nu) \\ \eta \in \uparrow_\Sigma \alpha}} m_{\mathcal{V}}(\eta) \tag{10.17}$$

In particular, if $\beta \in \Delta$ is such that $\varphi(\alpha) \lessdot \beta$, we have:

$$\sum_{\substack{\eta \in \varphi^{-1}(\beta) \\ \eta \in \uparrow_\Sigma \alpha}} m_{\mathcal{V}}(\eta) = \sum_{\substack{\eta \in \varphi^{-1}(\beta) \\ \alpha \lessdot \eta}} m_{\mathcal{V}}(\eta). \tag{10.18}$$

Proof. The first equality in Equation (10.17) is implied by the balancing condition. By Lemma 10.51 the index sets of the sums in the middle constitute a partition of the index sets of the sums on the right hand side, so the second equality folows. We obtain Equation (10.18) from Equation (10.17) by setting $\mu = \varphi(\alpha)$, $\nu = \beta$ and noting that the index set for the sum on the left becomes $\varphi^{-1}(\alpha) \cap \uparrow_\Sigma(\alpha) = \{\alpha\}$ a singleton, because φ is combinatorial. □

Remark 10.53. Note that by Equation (10.1), the left hand side in Equation (10.17) equals $\deg(\varphi, m_\varphi, \uparrow_\Sigma \alpha)(\mu)$ and the right hand side equals $\deg(\varphi, m_\varphi, \uparrow_\Sigma \alpha)(\nu)$. △

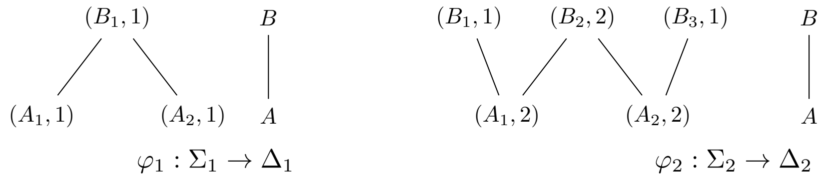


Figure 10.3: Two morphisms of posets φ_1, φ_2 and maps $m_1 : \Sigma_1 \to \mathbb{Z}_{\geq 1}, m_2 : \Sigma_2 \to \mathbb{Z}_{\geq 1}$ given by the second numbers in the pairs in the diagrams; (φ_1, m_1) is balanced but not an indexed branched cover, (φ_2, m_2) is an indexed branched cover but not balanced.

Applying the previous formulas, we arrive to a characterization of indexed branched covers of posets.

Proposition 10.54. *Let $\varphi : \Sigma \to \Delta$ be a combinatorial morphism of posets, and $m_\varphi : \Sigma \to \mathbb{Z}_{\geq 1}$ an index map. The pair (φ, m_φ) is an indexed branched cover unramified over* $\max \Delta$ *if and only if (φ, m_φ) is balanced.*

Proof. Let α be in Σ, β in Δ with $\varphi(\alpha) \lessdot \beta$, and $\mathcal{U} = \uparrow_\Delta \varphi(\alpha)$. By Lemma 10.40 the connected component $\mathcal{V}$ of $\varphi^{-1}(\mathcal{U})$ that contains α is the upper-set $\uparrow_\Sigma \alpha$. So if (φ, m_φ) is an indexed branched cover, by Equation (10.18) we have

$$m_\varphi(\alpha) = \sum_{\substack{\eta \in \varphi^{-1}(\beta) \\ \eta \in \uparrow_\Sigma \alpha}} m_\varphi(\eta) = \sum_{\substack{\eta \in \varphi^{-1}(\beta) \\ \alpha \lessdot \eta}} m_\varphi(\eta).$$

Now assume that (φ, m_φ) is balanced. By Lemma 10.50 the map φ is a branched cover unramified over $\max \Delta$. By Lemmas 10.22 and 10.40, we show that $\deg(\varphi, m_\varphi, \uparrow_\Sigma \alpha)$ is constant over $\uparrow_\Delta \varphi(\alpha)$. This is clear from Equation (10.17) since the left hand side is $\deg(\varphi, m_\varphi, \uparrow_\Sigma \alpha)(\mu)$ and the right hand side is $\deg(\varphi, m_\varphi, \uparrow_\Sigma \alpha)(\nu)$. So we are done by applying over a sequence that goes from $\varphi(\alpha)$ up to any desired ν in $\uparrow_\Delta \varphi(\alpha)$, such that each element covers the previous one. ☐

Remark 10.55. In Proposition 10.54, the condition on φ being combinatorial is crucial. Figure 10.3 shows two possible failures when this condition is absent. On the left we have that (φ_1, m_1) is balanced but is not an indexed branched cover, since the count of points in ${\varphi_1}^{-1}(A)$ is 2 and in ${\varphi_1}^{-1}(B)$ is 1. On the right, the preimage count is right, but neither A_1 nor A_2 satisfy the balancing condition, e.g. the value of $m_2(A_1)$ would need to be 3. △

10.9 Connectivity of posets

Given a morphism of posets $\varphi : \Sigma \to \Delta$, we study how to lift paths from Δ to Σ. This is a common question in the theory of topological covers, that allows to transport connectivity properties from Δ to Σ. Our main result here is that if $(\varphi, m_\mathcal{V})$ is balanced, $\varphi|_\mathcal{V}$ is combinatorial, $\varphi(\mathcal{V})$ is connected, and there is at least one fibre $\varphi^{-1}(\beta)$ such that the elements of $\varphi^{-1}(\beta) \cap \mathcal{V}$ are pairwise connected by paths in $\mathcal{V}$, then $\mathcal{V}$ is connected. Towards the end we introduce two versions of connectivity for posets, which play a role in proving Theorem C. We begin by associating a graph to Σ.

Definition 10.56. Given a poset Σ, its *comparability graph* G_Σ has as vertices the elements of Σ, and two vertices are joined by an edge if and only if they are comparable.

Another way to get G_Σ is to regard Σ as a category and forget the direction of the arrows of its diagram. Note that G_Σ is a simple graph, so a path in G_Σ is a sequence $\langle \gamma_0, \gamma_1, \ldots, \gamma_q \rangle$ of elements in Σ such that consecutive elements are comparable, i.e. $\gamma_{i-1} \leq \gamma_i$ or $\gamma_{i-1} \geq \gamma_i$ for $1 \leq i \leq q$. A graph is connected if its vertices are pairwise connected by a path.

Lemma 10.57. *Let Σ be a finite poset with the poset topology. The poset Σ is connected as a topological space if and only if the comparability graph G_Σ is connected as a graph.*

Proof. Assume that Σ is connected. Let $\alpha_1, \ldots, \alpha_k$ be the minimal elements of Σ. Consider any non-empty $I \subsetneq \{1, \ldots, k\}$, and the open sets $\mathcal{U} = \bigcup_{i \in I} \uparrow\alpha_i$ and $\mathcal{V} = \bigcup_{j \notin I} \uparrow\alpha_j$. If $\mathcal{U}$ and $\mathcal{V}$ were disjoint, this would contradict that Σ is connected. Thus, they intersect at an element γ, so there are $q \in I$ and $r \notin I$ connected by a sequence $\langle \alpha_q, \gamma, \alpha_r \rangle$. Iterating this argument, starting with I a singleton and adding one element at a time, shows that $\min \Sigma$ is connected in G_Σ. To conclude, every element of Σ is comparable to some $\alpha \in \min \Sigma$.

For the converse, observe that a path $P = \langle \gamma_0, \gamma_1, \ldots, \gamma_q \rangle$ gives rise to a topological path, via a concatenation of functions of the form given in Equation (10.7). So if G_Σ is connected, Σ is path-connected, hence connected. □

Remark 10.58. If $\mathcal{V} \subset \Sigma$ is connected, we can consider the induced poset on $\mathcal{V}$ and apply Lemma 10.57 to conclude the existence of a sequence inside of $\mathcal{V}$ with consecutive elements that are comparable, between any two elements of $\mathcal{V}$. △

We now use the balancing condition to lift paths. Given a path $P = \langle \beta_0, \beta_1, \ldots, \beta_q \rangle$ in Δ, a *lift* is a path $\tilde{P} = \langle \gamma_0, \gamma_1, \ldots, \gamma_q \rangle$ in Σ such that $\varphi(\gamma_i) = \beta_i$ for $i \in \{0, \ldots, q\}$.

Lemma 10.59. *Let $\varphi : \Sigma \to \Delta$ be a morphism of posets, $\mathcal{V} \subset \Sigma$ an up-set, and α in $\mathcal{V}$. Assume there is a balanced map $m_\mathcal{V}$. Any upwards path $\beta_0 < \beta_1 < \cdots < \beta_k$ in $\uparrow_\Delta \varphi(\mathcal{V})$ with $\beta_0 = \varphi(\alpha)$ lifts to a path $\tilde{P} = \langle \gamma_0, \gamma_1, \ldots, \gamma_q \rangle$ in $\mathcal{V}$ with $\gamma_0 = \alpha$.*

Proof. Proceeding by induction, suppose that β_i lifts to γ_i in $\mathcal{V}$. We search a lift $\gamma_{i+1} \in \mathcal{V}$ of β_{i+1}, that is comparable to γ_i. Since $\beta_i \le \beta_{i+1}$ and Δ is finite, we can choose a sequence $\mu_0 \lessdot \mu_1 \lessdot \cdots \lessdot \mu_k$ in Δ with $\mu_0 = \beta_i$ and $\mu_k = \beta_{i+1}$. Note that γ_i is a lift of $\mu_0 = \beta_i$, and that γ_i is in $\mathcal{V}$ so $m_\mathcal{V}(\gamma_i) \ge 1$, Hence, succesive applications of the balancing condition give lifts $\eta_0 \lessdot \eta_1 \lessdot \cdots \lessdot \eta_k = \beta_{i+1}$ of the μ_j, and we set $\gamma_{i+1} = \mu_k$. Since $\gamma_i \le \gamma_{i+1}$ and $\mathcal{V}$ is an up-set, we have $\gamma_{i+1} \in \mathcal{V}$. □

As a corollary of Lemma 10.59 we get that $\varphi(\mathcal{V}) = \uparrow_\Delta \varphi(\mathcal{V})$, i.e. $\varphi(\mathcal{V})$ is open. Indeed, if β is in $\uparrow_\Delta \varphi(\mathcal{V})$, then there is $\mu = \varphi(\alpha)$ in $\varphi(\mathcal{V})$ such that $\mu \le \beta$. If $\mu = \beta$ we are done, otherwise we can lift the upwards path $\mu < \beta$ to a path $\alpha < \gamma$ with γ in $\mathcal{V}$ and $\beta = \varphi(\gamma)$, showing that β is in $\varphi(\mathcal{V})$. In particular, we get the following corollary:

Lemma 10.60. *Let $\varphi : \Sigma \to \Delta$ be a morphism of posets. If there exists a balanced map $m_\Sigma : \Sigma \to \mathbb{Z}_{\ge 1}$, then φ is an open map.*

Proof. Let $\mathcal{V}$ be an up-set. The restriction $m_\mathcal{V} = m_\Sigma|_\mathcal{V}$ is a balanced map, because for any element $\alpha \in \mathcal{V}$ the elements that cover α are also in $\mathcal{V}$. By Lemma 10.59 and the discussion after it, $\varphi(\mathcal{V})$ is an upset. Hence φ is an open map. □

Remark 10.61. Combining Lemma 10.60 and Proposition 10.54 we get that if $\varphi : \Sigma \to \Delta$ is a combinatorial morphism of posets, then φ being an open map is a necessary condition for the existence of an index map m_φ such that (φ, m_φ) is an indexed branched cover. This criterion rules out the existence of an index map making the morphism from Example 10.37 an indexed branched cover. △

To go beyond upward paths we add an additional condition:

Lemma 10.62. *Let $\varphi : \Sigma \to \Delta$ be a morphism, $\mathcal{V} \subset \Sigma$ an up-set, $m_\mathcal{V}$ a balanced map, α in $\mathcal{V}$ an element, and $P = \langle \beta_0, \beta_1, \ldots, \beta_k \rangle \subset \varphi(\mathcal{V})$ a path with $\beta_0 = \varphi(\alpha)$. If $\psi : \mathcal{V} \to \varphi(\mathcal{V})$ given by $\gamma \mapsto \varphi(\gamma)$ is combinatorial, there is a lift $\tilde{P} = \langle \gamma_0, \gamma_1, \ldots, \gamma_q \rangle$ in $\mathcal{V}$ with $\gamma_0 = \alpha$.*

Proof. Proceeding by induction, suppose that β_i lifts to γ_i in $\mathcal{V}$. We find a lift $\gamma_{i+1} \in \mathcal{V}$ of β_{i+1}, that is comparable to γ_i. First assume that $\beta_i \le \beta_{i+1}$. In that case, Lemma 10.59 gives a lift. Now assume that $\beta_i \ge \beta_{i+1}$. Since ψ is combinatorial, the down-set $\downarrow_\mathcal{V} \gamma_i$ is mapped isomorphically to $\downarrow_{\varphi(\mathcal{V})} \varphi(\gamma_i) = \downarrow_{\varphi(\mathcal{V})} \beta_i$. As β_{i+1} is in $\downarrow_\Delta \beta_i$ and in $\varphi(\mathcal{V})$, it is in $\downarrow_{\varphi(\mathcal{V})} \beta_i$ as well, so there is γ_{i+1} in $\downarrow_\mathcal{V} \gamma_i \subset \mathcal{V}$ such that $\varphi(\gamma_{i+1}) = \beta_{i+1}$. □

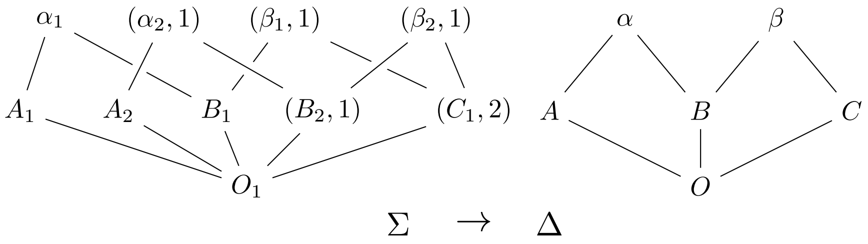


Figure 10.4: A combinatorial morphism $\varphi : \Sigma \to \Delta$, and a balanced map $m_{\mathcal{V}} : \mathcal{V} \to \mathbb{Z}_{\geq 1}$ with $\mathcal{V} = \uparrow_\Sigma \{B_2, C_1\}$ such that φ restricted to $\mathcal{V}$ is not an indexed branched cover.

Example 10.63. We wonder if in Lemma 10.62, and also in the upcoming Lemma 10.70, one can ommit the condition of having a combinatorial ψ. It could, for example, follow from a weaker condition such as requiring φ to be combinatorial. This is not the case, e.g. in Figure 10.4 let $\mathcal{V} = \uparrow_\Sigma \{B_2, C_1\} = \{B_2, C_1, \beta_1, \beta_2, \alpha_2\}$. We have that $\beta_1 \in \mathcal{V}$, and $\downarrow_{\mathcal{V}} \beta_1 = \{\beta_1, C_1\}$, but $\downarrow_{\varphi\mathcal{V}} \psi(\beta_1) = \{\beta, C, B\}$, so ψ is not combinatorial, despite φ being. Moreover, this example has a path that cannot be lifted. Indeed, consider the path $\langle \beta, B\rangle \subset \varphi(\mathcal{V})$, there is no lift in $\mathcal{V}$ starting with β_1. ⋆

We now use the results on lifting paths to give a criterion for the connectedness of $\mathcal{V}$.

Lemma 10.64. *Let $\varphi : \Sigma \to \Delta$ be a morphism, $\mathcal{V} \subset \Sigma$ an up-set such that $\psi : \mathcal{V} \to \varphi(\mathcal{V})$ is combinatorial, and $m_{\mathcal{V}} : \mathcal{V} \to \mathbb{Z}_{\geq 1}$ a balanced map. If $\varphi(\mathcal{V})$ is connected and there is β in $\varphi(\mathcal{V})$ such that $\varphi^{-1}(\beta)$ is connected in $\mathcal{V}$, then $\mathcal{V}$ is connected.*

Proof. Consider $\alpha \in \mathcal{V}$. There is a path $P = \langle \varphi(\alpha), \beta_1, \dots, \beta\rangle$ connecting $\varphi(\alpha)$ and β, since both are in $\varphi(\mathcal{V})$. By Lemma 10.57, there is a lift $\tilde{P} = \langle \alpha, \nu_1, \dots, \nu_{k-1}, \nu_k\rangle$ connecting α with fibre $\varphi^{-1}(\beta)$. Since the fibre $\varphi^{-1}(\beta)$ is connected, we are done. □

Finally, we give an up-set $\mathcal{V}$ such that the restriction $\psi : \mathcal{V} \to \varphi(\mathcal{V})$ of any combinatorial φ is combinatorial as well. Recall that a poset is graded if all the maximal chains have the same length. A graded poset has a rank function, i.e. the rank of $x \in \Sigma$ is the length of any maximal chain in $\downarrow_\Sigma x$. The set of all the rank-k elements of Σ is denoted $\Sigma(k)$.

Lemma 10.65. *Let Σ and Δ be graded posets and $\mathcal{V} = \uparrow_\Sigma \Sigma(k)$ the up-set of elements of rank at least k. If $\varphi : \Sigma \to \Delta$ is a combinatorial morphism, then the restriction $\psi : \mathcal{V} \to \varphi(\mathcal{V})$ given by $\gamma \mapsto \varphi(\gamma)$ is combinatorial as well.*

Proof. Since φ is a combinatorial morphism, we have that $\downarrow_{\mathcal{V}} \alpha = \downarrow_\Sigma \alpha \cap \mathcal{V}$ is mapped by ψ isomorphically to $\psi(\downarrow_\Sigma \alpha \cap \mathcal{V})$, and we must show this equals $\downarrow_{\varphi(\mathcal{V})} \psi(\alpha) = \downarrow_\Delta \psi(\alpha) \cap \varphi(\mathcal{V})$. Note that the rank is preserved by a combinatorial morphism, i.e. $\mathrm{rk}_\Sigma\, \alpha = \mathrm{rk}_\Delta\, \varphi(\alpha)$. Let β be in $\downarrow_\Delta \psi(\alpha) \cap \varphi(\mathcal{V})$. Since β is in $\varphi(\mathcal{V})$, there is $\gamma_1 \in \mathcal{V}$ such that $\varphi(\gamma_1) = \beta$, and so $\mathrm{rk}_\Delta\, \beta = \mathrm{rk}_\Sigma\, \gamma_1 \geq k$. On the other hand, β is in $\downarrow_\Delta \psi(\alpha) = \downarrow_\Delta \varphi(\alpha)$, so there is γ_2 in $\downarrow_\Sigma \alpha$ such that $\varphi(\gamma_2) = \beta$. Note that $\mathrm{rk}_\Sigma\, \gamma_2 = \mathrm{rk}_\Delta\, \beta \geq k$, so γ_2 is in $\mathcal{V}$, which implies $\psi(\downarrow_\Sigma \alpha \cap \mathcal{V}) \supset \downarrow_\Delta \psi(\alpha) \cap \varphi(\mathcal{V})$. Since $\psi(\downarrow_\Sigma \alpha \cap \mathcal{V}) \subset \downarrow_\Delta \psi(\alpha) \cap \varphi(\mathcal{V})$, we are done. □

To conclude our study of connectivity for posets, we look at some stronger notions.

Definition 10.66. Let Σ be a graded poset of dimension d. We say that Σ is

- *connected in codimension-k* if the up-set $\uparrow_\Sigma \Sigma(d-k)$ is connected, where k is in $\{0, \dots, d\}$.
- *strongly connected* if Σ is connected and for every $\alpha \in \Sigma \setminus \uparrow_\Sigma \Sigma(d-1)$ the up-set $(\uparrow_\Sigma \alpha) \setminus \{\alpha\}$ is connected.

Let $k, l \in \{0, \dots, d\}$. Observe that if $k \leq l$ and Σ is connected in codimension-k, then Σ is connected in codimension-l. Moreover, Σ is connected in codimension-d if and only if Σ is connected. So being connected in codimension-k is stronger than being connected, and in the following we see that being strongly connected is even stronger.

Lemma 10.67. *Let Σ be a graded poset of dimension d. If Σ is strongly connected, then Σ is connected in codimension-1.*

Proof. Let α and β be elements in $\uparrow_\Sigma \Sigma(d-1)$. Since Σ is strongly connected, we have that Σ is connected, so there is a sequence $P_0 = \langle \gamma_0, \ldots, \gamma_q \rangle$ whose consecutive elements are comparable and with $\gamma_0 = \alpha$ and $\gamma_q = \beta$. Now, let $k = \min_{\gamma \in P_0} \operatorname{rk} \gamma$. If $k \geq d-1$, then α and β are already connected in codimension-1. So assume that $k < d-1$, and let $\gamma_i \in P_0$ be an element with $\operatorname{rk} \gamma_i = k$. We have that either $\gamma_{i-1} \geq \gamma_i$ or $\gamma_{i-1} \leq \gamma_i$, but the latter would imply $\operatorname{rk} \gamma_{i-1} = k-1$, so it is excluded. The same holds for γ_{i+1}, so γ_{i-1} and γ_{i+1} are in $\uparrow_\Sigma \gamma_i \setminus \{\gamma_i\}$, and since Σ is strongly connected we connect γ_{i-1} with γ_{i+1} via a path Q whose minimal rank is $k+1$. Thus, we replace γ_i in P_0 with the path Q to obtain P_1. Iterating this procedure elliminates all elements in P_0 with rank k, rank $k+1$, and so on until all have either rank $d-1$ or d, as desired. □

Remark 10.68. The poset Σ shown in Figure 10.5 is connected in codimension-1, but is not strongly connected. △

Refinement preserves connectivity properties. Also, later, we see it also preserves balancing condition.

Lemma 10.69. *Let $\Phi : \widetilde{\Sigma} \to \Sigma \in$ PolySpace be a refinement. If Σ is strongly connected, then so is $\widetilde{\Sigma}$.*

10.10 Extending balanced maps

As the index map in Proposition 10.54 is defined over the whole domain, this raises the question of what can be said when dealing with a balanced map $m_\mathcal{V}$ defined over a proper subset $\mathcal{V} \subsetneq \Sigma$. Specifically, whether $m_\mathcal{V}$ induces an indexed branched cover by restriction, whether the domain of the function can be extended.

Lemma 10.70. *Let $\varphi : \Sigma \to \Delta$ be a combinatorial morphism of posets, $\mathcal{V} \subset \Sigma$ an open set, $m_\mathcal{V} : \mathcal{V} \to \mathbb{Z}_{\geq 1}$ a balanced map, and $\psi : \mathcal{V} \to \varphi(\mathcal{V})$ given by $\gamma \mapsto \varphi(\gamma)$. If ψ is combinatorial, then $(\psi, m_\mathcal{V})$ is an indexed branched cover.*

Proof. Since φ is a combinatorial morphism, by Lemma 10.50 the restriction $\varphi|_\mathcal{V}$ is a branched cover, which remains true if we restrict the codomain to $\varphi(\mathcal{V})$, giving rise to ψ. Since ψ is combinatorial and $m_\mathcal{V}$ is defined over all the domain of ψ, we apply Proposition 10.54 to conclude that $(\psi, m_\mathcal{V})$ is an indexed branched cover. □

Example 10.71. Figure 10.4 again exemplifies what can go wrong when ψ is not combinatorial. Setting $m_\mathcal{V}(C_1) = 2$ and $m_\mathcal{V}$-value 1 for the remaining elements in $\mathcal{V}$, we get a balanced map $m_\mathcal{V}$, yet $(\psi, m_\mathcal{V})$ is not an indexed branched cover, since the count for the fibre over C is 2, and for the fibre over B is 1. ⋆

Thus, the answer to the question of whether a balanced map restricts to an indexed branched cover is negative in general by Example 10.71, and by Lemma 10.70 it is possible when a technical condition is fulfilled. We now see this is also the case for the question of whether $m_\mathcal{V}$ can be extended to the whole domain Σ.

Example 10.72. It is straightforward to construct balanced maps $m_\mathcal{V}$ that cannot be extended when $\mathcal{V}$ is disconnected; e.g. consider the poset $\alpha \leftarrow A \leftarrow O \to B \to \beta$, the combinatorial morphism $\varphi = \mathrm{id}$, the set $\mathcal{V} = \uparrow\{A, B\}$, the balanced map $m_\mathcal{V}(\alpha) = m_\mathcal{V}(A) = 2$ and $m_\mathcal{V}(\beta) = m_\mathcal{V}(B) = 1$. It is not possible to extend $m_\mathcal{V}$ to O.

Moreover, we give an example where $\mathcal{V}$ is connected, yet extension is not possible. Figure 10.5 shows a combinatorial morphism $\varphi : \Sigma \to \Delta$ given by $\varphi(A_i) = A$, $\varphi(\beta_i) = \beta$, etc. and a map $m_\mathcal{V} : \mathcal{V} \to \Delta$ with $\mathcal{V} = \Sigma \setminus \left\{O_1, \tilde{O}_1, \tilde{O}_2\right\}$ and whose values are given by the second numbers of the

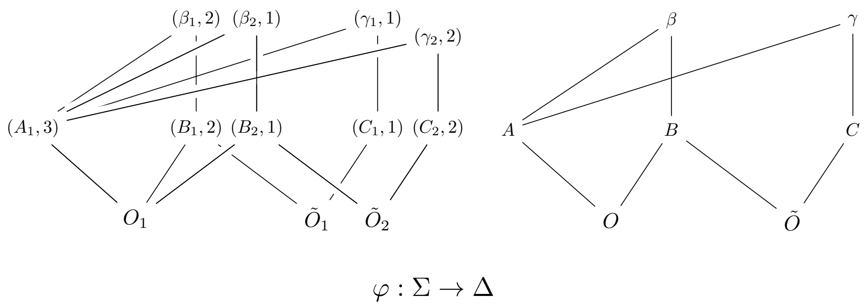


Figure 10.5: On the left, a poset Σ and a map $m_1 : \mathcal{V} \to \mathbb{Z}_{\geq 1}$ with $\mathcal{V} = \Sigma \setminus \left\{O_1, \tilde{O}_1, \tilde{O}_2\right\}$; right, a poset Δ; together, a combinatorial morphism of posets φ such that m_1 can be extended to O_1 but not to $\tilde{O}_1$ nor $\tilde{O}_2$.

pairs in the diagram. If we consider $\tilde{O}_1$, we have $\varphi(\tilde{O}_1) = \tilde{O}$, and both B and C cover $\tilde{O}$. Note that

$$\sum_{\substack{\eta \in \varphi^{-1}(B) \\ \tilde{O}_1 \lessdot \eta}} m_{\mathcal{V}}(\eta) = 2 \neq 1 = \sum_{\substack{\eta \in \varphi^{-1}(C) \\ \tilde{O}_1 \lessdot \eta}} m_{\mathcal{V}}(\eta).$$

Thus, there is no possible value for $\tilde{O}_1$ to fulfill the balancing condition. Same with $\tilde{O}_2$. On the other hand, O_1 can be given the value 3, and this satisfies the balancing condition. ⋆

The following result sheds light on why in Example 10.72 it was possible to extend $m_{\mathcal{V}}$ to O_1 but not to $\tilde{O}_1$ nor $\tilde{O}_2$.

Proposition 10.73. *Let $\varphi : \Sigma \to \Delta$ be a combinatorial morphism and $\mathcal{V} \subset \mathcal{W} \subset \Sigma$ open sets. If $\mathcal{U}_\alpha = (\uparrow_\Delta \varphi(\alpha)) \setminus \{\varphi(\alpha)\}$ is connected and $\varphi^{-1}(\mathcal{U}_\alpha) \subset \mathcal{V}$ for all α in $\mathcal{W} \setminus \mathcal{V}$, then any balanced map $m_{\mathcal{V}} : \mathcal{V} \to \mathbb{Z}_{\geq 1}$ extends to a balanced map $m_{\mathcal{W}} : \mathcal{W} \to \mathbb{Z}_{\geq 1}$ by setting*

$$m_{\mathcal{W}}(\alpha) = \sum_{\substack{\gamma \in \varphi^{-1}(\beta) \\ \alpha \lessdot \gamma}} m_{\mathcal{V}}(\gamma) \tag{10.19}$$

for $\alpha \in \mathcal{W} \setminus \mathcal{V}$ and β covering $\varphi(\alpha)$. The value is independent of the choice of β.

Proof. We show that the value in Equation (10.19) is independent of the choice of β; this also shows the balancing condition. Let β_1, β_2 be elements that cover $\varphi(\alpha)$, so they are in $(\uparrow_\Delta \varphi(\alpha)) \setminus \{\varphi(\alpha)\} = \mathcal{U}_\alpha$. Since $\mathcal{U}_\alpha$ is connected, there is a path from β_1 to β_2, which can be completed to a path $P = \langle \mu_0, \mu_1, \ldots, \mu_{k-1}, \mu_k \rangle \subset (\uparrow_\Delta \varphi(\alpha)) \setminus \{\varphi(\alpha)\}$ such that $\mu_0 = \beta_1$, $\mu_k = \beta_2$ and either $\mu_{i-1} \lessdot \mu_i$ or $\mu_{i-1} \gtrdot \mu_i$. Since $\varphi^{-1}(\mathcal{U}_\alpha) \subset \mathcal{V}$, we have that $\varphi^{-1}(\mu_i) \subset \mathcal{V}$. Moreover, μ_{i-1} and μ_i are in $\uparrow_\Delta \varphi(\alpha)$, so we can apply Lemma 10.52 to consecutive elements of P to obtain the result. □

Finally, we discuss a situation where the technical conditions of Proposition 10.73 are met. This situation is relevant when proving Theorem C.

Lemma 10.74. *Let Σ and Δ be graded posets, $\varphi : \Sigma \to \Delta$ a combinatorial morphism, $\mathcal{V} = \uparrow_\Sigma \Sigma(k+1)$ and $\mathcal{W} = \uparrow_\Sigma \Sigma(k)$, where k is in $\{0, \ldots, \operatorname{rk} \Sigma - 2\}$. For all $\alpha \in \mathcal{W} \setminus \mathcal{V}$ we have that $\varphi^{-1}((\uparrow_\Delta \varphi(\alpha)) \setminus \{\varphi(\alpha)\}) \subset \mathcal{V}$*

Proof. Note that $\operatorname{rk} \alpha = k$, because α is in $\mathcal{W} \setminus \mathcal{V}$, and that φ preserves rank because φ is combinatorial. Hence, $\operatorname{rk} \varphi(\alpha) = k$, all elements of $(\uparrow_\Delta \varphi(\alpha)) \setminus \{\varphi(\alpha)\}$ have rank at least $k+1$, and so do all elements of $\varphi^{-1}((\uparrow_\Delta \varphi(\alpha)) \setminus \{\varphi(\alpha)\})$. Therefore, the latter set is in $\mathcal{V} = \uparrow_\Sigma \Sigma(k+1)$, as desired. □

Remark 10.75. Let Σ and Δ be graded posets, and $\varphi : \Sigma \to \Delta$ a combinatorial morphism. If Δ is strongly connected, then by Lemma 10.74 any balanced map $m_{\mathcal{U}}$ defined on codimension-q of Σ extends to a balanced map $m_{\mathcal{W}}$ defined on codimension-$(q+1)$. △

10.11 Refinement induced by a morphism

Now we refine polyhedral complexes. Our aim is to take a morphism $\Phi : \Sigma \to \Delta$ of polyhedral complexes and simultaneously refine the domain and codomain to obtain a combinatorial morphism. We use this construction in two different situations, In the first, we have a map $\Phi : \Gamma \to \Delta$ of 1-dimensional polyhedral complexes, and the construction gives a combinatorial morphism, which encodes a graph morphism $\varphi : G \to T$. In the second situation, we take the map $\Pi : \mathcal{G}^{\text{trop}}_{g\to 0,d} \to \mathcal{M}^{\text{trop}}_g$, first we subdivide domain and codomain with the barycentric subdivision, and once we have graph complexes we subdivide again to obtain $\widetilde{\Pi} : \widetilde{\mathcal{G}}^{\text{trop}}_{g\to 0,d} \to \widetilde{\mathcal{M}}^{\text{trop}}_g$, a combinatorial morphism to which Theorem 10.46 can be applied. The following sketches some properties we want from this subdivision, and how we would go around doing it.

Example 10.76. Let σ be a rational polyhedron in (V, N), again we assume for a moment that $\operatorname{span}\sigma$ is not necessarily the whole V. Given a rational functional $u \in N^*$ we get a subdivision of σ by considering the two polyhedra $\sigma^+ = \sigma \cap H^+(u, c)$ and $\sigma^- = \sigma \cap H^-(u, c)$, where H^+ and H^- are the half-spaces introduced in Subsection 9.1.

It is relevant for us to describe the face poset $\widetilde{\Sigma}$ of $\{\sigma^+, \sigma^-\}$. Let Σ be the face poset of σ. The subdivision partitions Σ into four sets:

- Σ_0 the set of those β that are not contained in either $H^+(u, c)$ or in $H^-(u, c)$.
- $\Sigma^\pm$ the set of those γ that are contained in $H(u, c)$.
- Σ^+ the set of those μ not in $\Sigma^\pm$ and contained in $H^+(u, c)$.
- Σ^- the set of those ν not in $\Sigma^\pm$ and contained in $H^-(u, c)$.

The elements of $\widetilde{\Sigma}$ consist of one copy of Σ^+, Σ^-, and $\Sigma^\pm$, plus three copies of Σ_0. The three copies are because if τ is in Σ_0, then we obtain three polyhedra out of it: first $\tau^+ = \tau \cap H^+(u, c)$ which we put in Σ_0^+, second $\tau^\pm = \tau \cap H(u, c)$ which we put in $\Sigma_0^\pm$, and third $\tau^- = \tau \cap H^-(u, c)$ which we put in Σ_0^-. These three polyhedra are new and were not present originally in Σ.

Likewise, the morphisms in $\widetilde{\Sigma}$ can be described in terms of those in Σ. Let $f : \eta \to \tau$ be a morphism in Σ. If neither η nor τ are contained in Σ_0, then we put f in $\widetilde{\Sigma}$. If η is in Σ_0, then τ is in Σ_0 as well, and we put three copies of f in $\widetilde{\Sigma}$, namely $f : \eta^+ \to \tau^+$ and $f : \eta^- \to \tau^-$ and $f^\pm : \eta^\pm \to \tau^\pm$. Finally, if τ is in Σ_0 and η is not, we have three options: if η is in Σ^+ then we put a morphism $f : \eta \to \tau^+$; if η is in Σ^- then we put a morphism $f : \eta \to \tau^-$; if η is in $\Sigma^\pm$ then we put three morphisms: $f : \eta \to \tau^\pm$, and $f : \eta \to \tau^+$, and lastly $f : \eta \to \tau^-$.

In a more succint manner, consider $\operatorname{poly}_\Sigma : |\sigma| \to \Sigma$ from Definition 9.26. We have that

$$\Sigma_0 = \operatorname{poly}_\Sigma(|\sigma| \setminus H(u, c)) \cap \operatorname{poly}_\Sigma(H(u, c)).$$

We have argued that Σ_0 is an up-set. Consider the categories induced by Σ on $\Sigma \setminus \Sigma_0$ and on Σ_0. WWe have argued that $\widetilde{\Sigma}$ is the sum of the categories $\Sigma \setminus \Sigma_0$, Σ_0^+, Σ_0^-, $\Sigma_0^\pm$ to which we add morphisms in the sets $\operatorname{Hom}(\operatorname{poly}_\Sigma(H^+(u, c)), \Sigma_0^+)$, $\operatorname{Hom}(\operatorname{poly}_\Sigma(H^-(u, c)), \Sigma_0^-)$, and $\operatorname{Hom}(\operatorname{poly}_\Sigma(H^\pm(u, c)), \Sigma_0^\pm)$. ⋆

So let $\Phi : [\sigma : \Sigma \to \mathrm{P{\scriptsize OLY}}^f_\mathbb{Z}] \to [\delta : \Delta \to \mathrm{P{\scriptsize OLY}}^f_\mathbb{Z}]$ be a morphism of polyhedral complexes given by the pair $(\varphi, \{\Phi_\alpha\}_{\alpha\in\Sigma})$. Recall that face posets of polyhedral spaces are ranked by $\operatorname{rk}\alpha = \dim \sigma_\alpha$. Fix $\alpha \in \Sigma$ such that $\operatorname{rk}\alpha = \operatorname{rk}\varphi(\alpha)$. Recall that $\Phi_\alpha : \sigma_\alpha \to \delta_{\varphi(\alpha)}$ is not necessarily a face morphism, in fact it needs not even be injective! So suppose that Φ_α is injective, hence σ_α and its face poset is mapped isomorphically onto $\Phi_\alpha(\sigma_\alpha) \subset \delta_{\varphi(\alpha)}$. Since $\dim \sigma_\alpha = \dim \delta_{\varphi(\alpha)}$, we have that if τ is a facet of σ_α then $\Phi_\alpha(\tau)$ spans a hyperplane in $N^\alpha_\mathbb{R}$. The idea is to consider all such hyperplanes $\{\operatorname{span}_\mathbb{R} \Phi_\alpha(\tau) \colon \tau \lessdot \alpha\}$ and refine $\delta_{\varphi(\alpha)}$ with respect to them. It is straightforward to show that the result is independent of the order of how each succesive refinement is done.

Now the problem is that such subdivision of $\delta_{\varphi(\alpha)}$ also might subdivide a cone δ_γ with $\gamma \preceq \varphi(\alpha)$. Now such δ_γ is contained in δ_η with $\gamma \preceq \eta$, and δ_η might not be subdvided yet, and it needs to be, to accomodate for the subdivision of δ_γ. But a codimension-1 space in δ_γ is not codimension-1 in δ_η. So the fix would be to consider faces of δ_η which do not contain δ_γ, in a move similar to constructing the star subdivision.

That is, a subdivision of $\delta_{\varphi(\alpha)}$ by a hyperplane requires to throw away everything in $\uparrow_\sigma\downarrow_\Sigma\varphi(\alpha)$ and replace it by subdivided polyhedra. Moreover, then these subdivisions performed on Δ have to be pulled back to Σ.

The challenges are to argue that this process of subdivision eventually stops, that at each step we get a polyhedral space and a morphism of polyhedral spaces, and that at the end the resulting morphism is combinatorial.

Conjecture 10.77. *Let $\Phi : \sigma \to \delta$ be a morphism of polyhedral spaces. Assume there are refinements σ' and δ' of σ and δ, respectively, which are either polyhedral complexes of simplicial cones or polyhedral complexes of simplices. There exists refinements $\widetilde{\sigma} : \widetilde{\Sigma} \to \mathrm{POLY}^f_{\mathbb{Z}}$ and $\widetilde{\delta} : \widetilde{\Delta} \to \mathrm{POLY}^f_{\mathbb{Z}}$ of σ and δ respectively, such that Π induces a morphism $\widetilde{\Pi} : \widetilde{\sigma} \to \widetilde{\delta}$ that is a combinatorial morphism of polyhedral complexes.*

10.12 Gluing datums

In Part I gluing datums are introduced as a tool to ease the visualization of DT-morphisms, to ease the book-keeping in the process of deformation of a DT-morphism, and to write computer programs. We sketch how this tool can be brought to our now more general setting.

Definition 10.78 (gluing datum)**.** Let Δ be a finite poset admitting a rank function, d a positive integer, and $\sim$ an equivalence relation on $\Delta \times [d]$, where $[d] = \{1, \dots, d\}$. We write $\overline{(\alpha, k)}$ for the classes of $(\Delta \times [d])/\sim$. The triple $(\Delta, d, \sim)$ is a *gluing datum* if $\sim$ satisfies these properties:

1. Verticality: If $(\alpha, i) \sim (\beta, j)$, then $\alpha = \beta$. So each α in Δ defines a relation $\sim_\alpha$ on $[d]$ with $i \sim_\alpha j$ if $(x, i) \sim (x, j)$.
2. Refinement: For any α and γ in Δ such that $\gamma \in \downarrow_\Delta \alpha$, the relation $\sim_\gamma$ is a coarsening of $\sim_\alpha$.

Let $M = (\Delta, d, \sim)$ be a glueing datum. We call Δ the *base poset*, d the degree, and $\sim$ the gluing relations. We consider $[d]$ with the trivial partial order, i.e. $x \le y$ if and only if $x = y$. Now recall Example 9.25. We get that the product $\Delta \times [d]$ has a poset structure, and $r_M : \Delta \times [d] \to \mathbb{Z}$ given by $r_M(\overline{(\alpha, i)}) = r(\alpha)$ is a rank function. Also by the verticality property, we are identifying together elements of the same rank, thus $\Sigma_M = (\Delta \times [d])/\sim$ is a poset that admits a rank function. The map $\varphi_M : \Sigma_M \to \Delta$ is a morphism of posets, and the refinement property of M implies that in fact φ is combinatorial. Finally, if we set $m_{\varphi_M}(\overline{(\alpha, i)}) = \#(\overline{(\alpha, i)})$ it is straightforward to prove that $(\varphi_M, m_{\varphi_M})$ is an indexed branched cover. Moreover, we conjecture this correspondonce to be one-to-one

Conjecture 10.79. *Fix a poset Δ that admits a rank function, and $d \in \mathbb{Z}_{\ge 1}$. The set of degree-d indexed branched covers $(\varphi : \Sigma \to \Delta, m_\varphi)$ such that φ is combinatorial is in one-to-one correspondence to gluing datums $(\Delta, d, \sim)$.*

Sketch of proof. It is quite likely that a constructive proof can be derived by modifying Construction 4.7 from Part I accordingly. Yet, it feels that a conceptual proof can be achieved by fleshing out better the concept of morphisms of indexed of branched covers. So given a degree-d combinatorial indexed branched cover $\varphi : \Sigma \to \Delta$ of posets, the point would be to factor the projection $p : \Delta \times [d] \to \Delta$ through φ, giving us a diagram

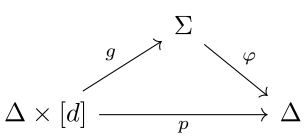


Diagram 10.80

And so we can say something to the efect that $\Sigma = \Delta \times [d]/\ker g$. $\square$

Remark 10.81. Whenever we represent graphically a DT-morphism, this representation follows the philosophy of gluing datums, as done in Part I as well. Namely, for the graph case $\varphi : G \to T$, we regard G as the graph resulting from taking $\deg \varphi$ copies of T, and identifying together certain vertices and edges between copies. In this view the index map m_φ records how many copies of T were glued together in a particular place. Such places of gluing are represented with dashed lines. See for example the figure in Example 12.68. △

Chapter 11

Parametrizing metric graphs

This section describes a polyhedral space of cones $\mathcal{M}_g^{\text{trop}} \to \text{POLY}_{\mathbb{Z}}^f$ such that the points of the topological realization $|\mathcal{M}_g^{\text{trop}}|$ are in one-to-one correspondence with equivalence classes under isometry of genus-g connected weighted metric graphs. While several sources in the literature do a similar account, e.g see [Koz09; Cha12; ACP15], we undertake working through all the details to have a template for the description of $\mathcal{G}_{g\to 0,d}^{\text{trop}}$.

Metric graphs are introduced as topological realizations of 1-dimensional polyhedral spaces with bounded polyhedra. For this end, we outline how to frame graphs and graph morphisms in category theoretical terms.

11.1 Weighted graphs

This subsection recalls some basics of graph theory, framed in category-theoretical concepts. This is a non-conventional approach, but it suits well our aim of studying a polyhedral space $\Gamma : G^{\text{op}} \to \text{POLY}_{\mathbb{Z}}^f$, where G is a graph, using the theory developed in Section 10.

Definition 11.1 (finite graph)**.** Let $(G, V(G), E(G))$ be a triple with G a finite category, and $\{V(G), E(G)\}$ a partition of $\text{Obj}(G)$. These partition sets are called the *vertex set* and the *edge set*. This triple is a *graph* if it satisfies the following two conditions:

(G1) For every A in $V(G)$ we have $\text{Hom}(A, -) = \{\text{id}_A\}$.

(G2) For every e in $E(G)$ we have $\text{Hom}(e, -) = \{\text{id}_e, i_1 : e \to A, i_2 : e \to B\}$, where A and B are in $V(G)$, and not necessarily distinct.

Two distinct elements x_1, x_2 of G are *incident* if there is a morphism between them. We write $E(A)$ for the set of edges incident to a vertex A. The *ends* of e are $i_1(e)$ and $i_2(e)$. Two vertices are *adjacent* if they are the ends of some edge. If $i_1(e) = i_2(e)$, we call e a *loop*. The *valency* $\text{val}\, A$ of a vertex A is the cardinality of the set $\text{Hom}(-, A) \setminus \{\text{id}_A\}$. The *min-valency* and *max-valency* of G are $\text{min-val}(G) = \min_{A\in V(G)} \text{val}(A)$ and $\text{max-val}(G) = \max_{A\in V(G)} \text{val}(A)$, respectively. A monovalent, divalent, trivalent, n-valent vertex is a vertex of valency 1, 2, 3 and n, respectively.

A *subgraph* of G is a subcategory H and a partition $V(H) \subset V(G)$, $E(H) \subset E(G)$ of H, satisfying properties (G1) and (G2). A subset $V(H) \subset V(G)$ induces a subgraph H, with $E(H)$ equal to the edges of G with both ends in $V(H)$. Likewise, a subset $E(H) \subset E(G)$ induces a subgraph H, with $V(H)$ equal to those vertices that appear as an end of some edge in $E(H)$.

A *graph morphism* $\gamma : G \to G_0$ is an incidence preserving map. Formally,

Definition 11.2 (morphism of graphs)**.** A *graph morphism* is a functor $\gamma : G \to G_0$ such that the restriction of γ to $\text{Hom}(x, -)$ is surjective onto $\text{Hom}(\gamma(x), -)$ for all x in $\text{Obj}(G)$.

The condition that a morphism of graphs $\gamma : G \to G_0$ maps $\text{Hom}(x, -)$ surjectively onto $\text{Hom}(\gamma(x), -)$ ensures that for every subgraph H of G the image $\gamma(H)$ is a subgraph of G_0. The map γ indeed preserves incidences; given e in $E(G)$ if $\gamma(e)$ is in $E(G_0)$, then the ends of e map to the ends of $\gamma(e)$; otherwise, e and the ends of e map to the same vertex $\gamma(e)$ in $V(G)$. Also, $\gamma(V(G)) \subset V(G_0)$. We denote by $\operatorname{contr}(\gamma)$ the set of edges of G that map to a vertex of G_0, that is $\operatorname{contr}(\gamma) = \gamma^{-1}(V(G_0)) \cap E(G) = E(G) \setminus \gamma^{-1}(E(G_0))$.

We consider graphs with the topology generated by the sets $\{e\}$ and $\{A\} \cup E(A)$, for e in $E(G)$ and A in $V(G)$. This coincides with the preorder topology of the opposite category G^{op}. A subset $H \subseteq G$ is closed if and only if H is a subgraph. Hence, a graph morphism is continuous and a closed map. A subgraph P of G that is connected under the graph topology and has max-valency at most 2 is a *path*. The 2-valent vertices of P are the *interior vertices*, and the 1-valent vertices are the *endpoints*. If P has no endpoints then we call P a *cycle*. By Lemma 10.57, the graph G is connected in the graph topology if and only if for every pair of vertices A and B there is a path P with endpoints A and B, that is if and only if there is a sequence beginning in A and ending in B of elements of G such that consecutive elements are incident.

Remark 11.3. We phrase some properties in the language of Section 9. A graph G is loopless if and only if G is a poset, i.e. $\#(\operatorname{Hom}(x_1, x_2)) \leq 1$ for all $x_1, x_2 \in G$. Given graphs G_1 and G_2, a map of objects $\gamma : \operatorname{Obj}(G_1) \to \operatorname{Obj}(G_2)$ extends to a functor if and only if γ is order preserving with the order from Lemma 9.22, if and only if γ is continuous. If the graphs are loopless, this extension to a functor is unique. Not all order preserving maps induce graph morphisms, e.g the constant map that sends everything to an edge of G_2 is not a graph morphism. An order preserving map γ induces a graph morphism if and only if $\gamma(V(G_1)) \subset V(G_2)$. A graph morphism γ satisfies that $\operatorname{contr}(\gamma) = \varnothing$ if and only if γ is a combinatorial morphism. If G is connected and $E(G) \neq \varnothing$, then $\max G^{\mathrm{op}} \subset E(G)$. △

Now we give two important constructions. Fix a subset $S \subset E(G)$. The *deletion* $G \setminus S$ of S is the subcategory of G restricted to the objects $V(G) \sqcup (E(G) \setminus S)$. Moreover, let $\hat{G}$ be the category generated by $G \setminus S$ plus the family of isomorphisms $\{\psi_e : i_1(e) \to i_2(e) \colon\ e \in S\}$; note that ψ_e identifies together the vertices that are ends of e, thereby *contracting* e. The *contraction* G/S of S is the skeleton category of $\hat{G}$. That is, its objects are equivalence classes of isomorphic objects of $\hat{G}$. There are graph morphisms $G \to G \setminus S$ and $G \to G/S$.

Finally, per Subsection 8.2.2 we are interested in studying graphs that have weights on the vertices. A *weighted graph* is a pair of a graph G and a weight map $w : V(G) \to \mathbb{Z}_{\geq 0}$. A subgraph $H \subset G$ induces a weighted subgraph $(H, w|_H)$, where $w|_H$ equals the restriction of $w : V(G) \to \mathbb{Z}_{\geq 0}$ to $V(H)$. A morphism of weighted graphs is just a morphism of the underlying graphs, without restrictions on the weights. Recall from Equation (8.5) that the weight map encodes extra genus on the graph, i.e. we define:

$$g(G, w) = g(G) + \sum_{A \in V(G)} w(A). \tag{11.1}$$

Intuitively, the weight $w(A)$ records loops that were contracted to A; so one could picture them as *infinitely small loops*. This leads to define a *weighted valency* as

$$\operatorname{wtval} A = \operatorname{val} A + 2w(A). \tag{11.2}$$

This notation streamlines the exposition at several points. We denote by min-wtval(G, w) the minimum weighted valency of (G, w), that is $\min_{A \in V(G,w)} \operatorname{wtval} A$.

11.2 Weighted metric graphs

This subsection looks at weighted metric graphs, gives a recipe to construct them from combinatorial and metric data, and characterizes metric spaces that can be given a metric graph structure.

Definition 11.4. A *metric graph with polyhedral structure* is a polyhedral space

$$\Gamma : G^{\mathrm{op}} \to \mathrm{P{\scriptsize OLY}}^f_{\mathbb{Z}},$$

where G is a graph, and the 1-dimensional polyhedra are bounded with ambient space $\mathbb{R}$.

Let Γ be a metric graph with polyhedral structure. We make some observations on the edges of Γ. Let (N, σ) in $\text{POLY}^f_{\mathbb{Z}}$ be a 1-dimensional bounded polyhedra with $\sigma \subset \mathbb{R}$. From the boundedness, we get that σ is a closed interval $[x_1, x_2]$. We denote by $\text{length}(\sigma) = x_2 - x_1$ the length of σ. Since polyhedra in $\text{POLY}^f_{\mathbb{Z}}$ have integral vertices, x_1 and x_2 are in N, so there is $n_\sigma \in \mathbb{Z}_{\geq 1}$ such that $N = (c/n_\sigma)\mathbb{Z}$. We call n_σ the *integral-length of* σ.

We write $|\Gamma|$ for the topological realization of Γ. This deviates from the convention of Section 9, because later we consider families C_G of those Γ with a fixed underlying graph G. Each polyhedron of Γ inherits the metric from $\mathbb{R}$. This metric extends globally to a shortest-path metric, making $|\Gamma|$ a metric space. A weight map w_G on $V(G)$ extends to a map $|w| : |\Gamma| \to \mathbb{Z}_{\geq 0}$ with finite support. Thus, the pair $(|\Gamma|, w_G)$ is an example of the following kind of spaces:

Definition 11.5. A *weighted metric space* is a pair (X, w) of a metric space X and a weight map $w : X \to \mathbb{Z}_{\geq 0}$ with finite support.

Two weighted metric spaces $(X, w : X \to \mathbb{Z}_{\geq 0})$, $(X', w' : X' \to \mathbb{Z}_{\geq 0})$ are isometric if there is an isometry $\Psi : X' \to X$ such that $w' = w \circ \Psi$. We denote this by $(X, w) \equiv (X', w')$. The genus of the graph G, without weights, coincides with the first Betti number of $|\Gamma|$. Thus, we define the genus of (Γ, w_G) to be $g(G, w_G)$. Note that the genus is invariant under isometry.

Definition 11.6. A *weighted metric graph* is a weighted metric space isometric to some $(|\Gamma|, w)$, with (Γ, w_G) a weighted metric graph with polyhedral structure. We call $(\Gamma : G \to \text{POLY}^f_{\mathbb{Z}}, w_G)$ a *model* of $(|\Gamma|, w)$.

Now, given a weighted graph (G, w_G), we construct a family of weighted metric graphs with polyhedral structure. Consider the following functions, which specify metric information:

Definition 11.7. A *length function* is a map ℓ in $\mathbb{R}^G_{\geq 0}$ such that $\ell(A) = 0$ for all A in $V(G)$.

In other words, vertices have zero length, the support of ℓ is contained in $E(G)$. In Part I the domain of a length function was restricted to $E(G)$. We have enlarged the domain for the sake of a cleaner exposition. The family C_G of length functions of a fixed graph G is naturally identified with a polyhedral cone:

$$C_G = \left\{\ell \in \mathbb{R}^G_{\geq 0} \colon \ \ell(A) = 0 \text{ for all } A \in V(G)\right\}. \tag{11.3}$$

For a weighted graph (G, w) we set $C_{(G,w)} = C_G$. Each point ℓ in $C_{(G,w)}$ corresponds to a weighted metric graph with polyhedral structure via the following construction:

Construction 11.8. To the quadruple (G, w, ℓ, n) of a graph G, a weight map $w : V(G) \to \mathbb{Z}_{\geq 0}$, a length function $\ell : G \to \mathbb{Z}_{\geq 0}$, and an integral-length function $n : G \to \mathbb{Z}_{\geq 1}$, we associate a functor $\Gamma : G^{\text{op}} \to \text{POLY}^f_{\mathbb{Z}}$ that sends:

- The object $x \in G$ to $(N^x, [0, \ell(x)])$ and $N^x = (\ell(x)/n(x))\mathbb{Z}$.
- The morphism $i_1 : e \to A \in \text{Hom}(e, -)$ to $\Gamma(A) \to \Gamma(e)$ given by $0 \mapsto 0$.
- The morphism $i_2 : e \to B \in \text{Hom}(e, -)$ to $\Gamma(B) \to \Gamma(e)$ given by $0 \mapsto \ell(e)$.

Let $\mathbf{1}_G : G \to \mathbb{Z}_{\geq 1}$ be the map given by $x \mapsto 1$. Per Subsection 9.2 there is a topological realization $|\Gamma|$ with a metric structure and $w : G \to \mathbb{Z}_{\geq 0}$ extends to it, so we write $|(G, w, \ell, n)| = (|\Gamma|, w)$. We also write (G, w, ℓ) for $(G, w, \ell, \mathbf{1}_G)$ in Construction 11.8, and associate $|(G, w, \ell)|$ to $\ell \in C_{(G,w)}$.

The aim is to give a polyhedral structure to $|\Gamma|$, which is not immediate because the Γ from Construction 11.8 might not be a polyhedral space. Indeed, consider $e \in E(G)$ and $i_1 : A \to e \in G$ such that $\ell(e) = 0$. Both Γ_A and Γ_e are isomorphic to the point $(\{0\}, \{0\})$ in $\text{POLY}^f_{\mathbb{Z}}$, giving raise to the first triangle in Diagram 11.9. If Γ were a polyhedral space, by the second triangle and Condition (b) from Definition 9.4 we would have a morphism $e \to A$ in G^{op}, i.e. a morphism $A \to e$ in G, a contradiction.

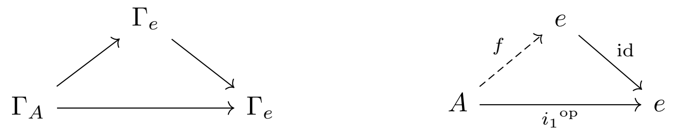


Diagram 11.9

The solution is to consider the set $S = \{e \in E(G) \colon \ell(e) = 0\}$ of edges of length 0, and the contraction G/S. The realization $|(G/S, w|_S, \ell_S)|$ is isometric to the realization $|(G, w, \ell)|$; this is a key fact used in Part I. Moreover, it is straightforward to prove that if $\operatorname{supp} \ell = E(G)$, then Γ is a polyhedral space. Thus, $(G/S, w|_S, \ell|_S)$ gives the desired polyhedral structure, showing that $C_{(G,w)}$ is a family of metric graphs.

Now, given a weighted metric space (X, w), we look closer to the question of when it can be endowed with a metric graph structure. A *vertex set* is a finite subset $S \subset X$ such that $\operatorname{supp} w \subset S$ and each element in $\pi_0(X \setminus S)$ is isometric to a bounded open interval of $\mathbb{R}$. The existence of a vertex set gives rise to a weighted metric graph: let G_S be the graph with vertices $V(G_S) = S$, edges $E(G_S) = \{\mathrm{cl}(U) \colon U \in \pi_0(X \setminus S)\}$, and incidence relations given by inclusion; ℓ_S be the length function recording the length of the closure $\mathrm{cl}(U)$; and $w|_S$ as weight function. We have that $|(G_S, w|_S, \ell_S)|$ is isometric to (X, w).

Adding a finite number of points to a vertex set S produces yet another vertex set. So the set of $|(G_S, w|_S, \ell_S)|$ isometric to (X, w) is infinite. We characterize vertex sets and describe a canonical representative. Let $\mathcal{S}$ be the family of all vertex sets of (X, w). If both $\mathcal{S}$ and $\mathcal{E} = \bigcap_{S \in \mathcal{S}} S$ are non-empty, then $\mathcal{E}$ is a vertex set. We call $\mathcal{E}$ and $(G_{\mathcal{E}}, w_{\mathcal{E}}, \ell_{\mathcal{E}})$ the set of *essential vertices* and the *essential model*, respectively.

Lemma 11.10. *Let (X, w) be a weighted metric space, and S_0 the set of $x \in X$ such that either $w(x) \geq 1$ or for all $\varepsilon > 0$ the open ball $B(x, \varepsilon)$ is not isometric to $(-\varepsilon, \varepsilon) \subset \mathbb{R}$. If S_0 is non-empty and finite, any finite $S \subset X$ is a vertex set if and only if $S_0 \subset S$.* □

The family of *metric loops* is the set of weighted metric graphs homeomorphic to a circle and with weight function identically zero. If (Γ, w) is a weighted metric graph, then the set S_0 from Lemma 11.10 is empty if and only if (Γ, w) is a metric loop. For every other metric graph, Lemma 11.10 shows that the set of essential vertices $\mathcal{E}$ equals S_0, and that $\mathcal{E}$ is minimal in the sense that all models arise from a sequence of edge subdivisions of $G_{\mathcal{E}}$. Thus, we can define the *valency* of a point x in Γ as the valency of x in G_S for S any vertex set containing x. Similarly for the weighted valency.

Remark 11.11. In terms of the weighted valency, the set of essential vertices of $(|\Gamma|, w)$ equals the points x such that $\operatorname{wtval} x \neq 2$. △

11.3 Contraction and specialization of graphs

The discussion after Construction 11.8 motivates us to do a closer study on contraction of edges in a weighted graph. This is an important step in making the correspondence $(G, w) \to C_{(G,w)}$ functorial.

Definition 11.12. A *contraction* is a graph morphism $\rho : G \to G_0$ such that the restriction $\rho^{-1}(E(G_0)) \xrightarrow{\rho} E(G_0)$ is a bijection, and $\rho^{-1}(A_0)$ is a connected subgraph of G for all A_0 in $V(G_0)$. A contraction of weighted graphs is a contraction of the underlying graphs.

Remark 11.13. Since ρ is continuous, if P is a path in G, then $\rho(P)$ is a path in G_0; and because the fibres of ρ are connected, any path P_0 in G_0 has a lift, i.e. a path $\tilde{P}$ with $\rho(\tilde{P}) = P_0$. Thus, if $\rho : G \to G_0$ is a contraction morphism and at least one of G or G_0 is connected, then both G and G_0 are connected. △

Recall that $\mathrm{contr}(\rho)$ are the edges of G that the morphism ρ maps to vertices. If $\rho : G \to G_0$ is a contraction morphism, there is a well defined inverse map $\rho^{-1} : E(G_0) \to E(G)$. Thus, we have that

$$\#(\mathrm{contr}(\rho)) = \#(E(G)) - \#(E(G_0)). \tag{11.4}$$

Given x_0 in G_0, we say that the subgraph $\rho^{-1}(x_0)$ of G *gets contracted to* x_0. The genus of G and of G_0 are related:

Lemma 11.14. *Let $\rho : G \to G_0$ be a contraction morphism of graphs. We have that*

$$g(G) = g(G_0) + \sum_{A_0 \in V(G_0)} g(\rho^{-1}(A_0)).$$

Proof. The result follows from the fact that $\rho^{-1}(A_0)$ is connected, and two counts:

$$\begin{aligned}
\#(E(G)) &= \#(\rho^{-1}(E(G_0))) + \#(\rho^{-1}(V(G_0)) \cap E(G)) \\
&= \#(E(G_0)) + \sum_{A_0 \in V(G_0)} \#(E(\rho^{-1}(A_0))), \\
\#(V(G)) &= \#(\rho^{-1}(V(G_0)) \cap V(G)) \\
&= \sum_{A_0 \in V(G_0)} \#(V(\rho^{-1}(A_0))) \\
&= \#(V(G_0)) + \sum_{A_0 \in V(G_0)} (\#(V(\rho^{-1}(A_0))) - 1).
\end{aligned}$$

□

Consider the set WG_g of genus-g connected weighted graphs. If $\rho : G \to G_0$ is a contraction morphism, then $g(G) \geq g(G_0)$. When the inequality is strict, we say that ρ *contracts cycles* of G. To remain inside WG_g, we consider a kind of contraction morphisms that keep track of contracted cycles using the vertex weight.

Definition 11.15. A *specialization morphism* is a contraction morphism $\rho : (G, w) \to (G_0, w_0)$ of weighted graphs such that the weight of A_0 in $V(G_0)$ equals the genus of the weighted graph that contracts to A_0, namely

$$w_0(A_0) = g(\rho^{-1}(A_0)) + \sum_{A \in V(\rho^{-1}(A_0))} w(A) = g(\rho^{-1}(A_0), w|_{\rho^{-1}(A_0)}). \tag{11.5}$$

Remark 11.16. If a specialization morphism $\rho : G \to G_0$ contracts zero edges, then $E(G)$ is mapped one-to-one to $E(G_0)$, and the condition that $\rho^{-1}(A_0)$ is connected gives $\#(V(G)) = \#(V(G_0))$. Thus, ρ is an isomorphism △

By setting $H = G$ and $H_0 = G_0$ in the following result, we obtain that specialization morphisms preserve the genus.

Lemma 11.17. *Let $\rho : (G, w) \to (G_0, w_0)$ be a specialization morphism, $(H_0, w_0|_{H_0}) \subset (G_0, w_0)$ a weighted subgraph, and $(H, w|_H) \subset (G, w)$ a weighted subgraph such that $H = \rho^{-1}(H_0)$. The map $\rho|_H : (H, w|_H) \to (H_0, w_0|_{H_0})$ is a specialization morphism and*

$$g(H, w|_H) = g(\rho(H), w_0|_{\rho(H)}) = g(H_0, w_0|_{H_0}).$$

Proof. Since ρ is a contraction, $\rho^{-1}(E(G_0)) \xrightarrow{\rho} E(G_0)$ is a bijection, so ${\rho|_H}^{-1}(E(H_0)) \xrightarrow{\rho|_H} E(H_0)$ is a bijection as well; also $\rho^{-1}(A_0)$ is connected, and since $H = \rho^{-1}(H_0)$ we have that ${\rho|_H}^{-1}(A_0) = \rho^{-1}(A_0)$. Moreover, it is straightforward to verify Equation (11.5) for $\rho|_H$ from the fact that ρ is a specialization morphism. Thus, $\rho|_H$ is a specialization morphism. The result on the genera follows from calculating:

$$\begin{aligned}
g(\rho(H), w_0|_{\rho(H)}) &= g(\rho(H)) + \sum_{A_0 \in V(\rho(H))} w_0(A_0) \\
&= g(\rho(H)) + \sum_{A_0 \in V(\rho(H))} g(\rho^{-1}(A_0)) + \sum_{A_0 \in V(\rho(H))} \sum_{A \in V(\rho^{-1}(A_0))} w(A) \\
&= g(H) + \sum_{A \in V(H)} w(A) = g(H, w|_H).
\end{aligned}$$

We applied Lemma 11.14 in the second line because $\rho|_H$ is a specialization morphism, and in the last line we used the fact that $H = \rho^{-1}(H_0)$. □

Another consequence of Lemma 11.14 is that composition of specialization morphisms is well behaved.

Lemma 11.18. *Let $\rho_1 : (G_1, w_1) \to (G_0, w_0)$ and $\rho_2 : (G_2, w_2) \to (G_1, w_1)$ be specialization morphisms. The composition $\rho = \rho_1 \circ \rho_2$ is a specialization morphism as well.*

Proof. Note that $\rho^{-1}(E(G_0)) \xrightarrow{\rho} E(G_0)$ equals

$$\rho_2{}^{-1} \circ \rho_1{}^{-1}(E(G_0)) \xrightarrow{\rho_2} \rho_1{}^{-1}(E(G_0)) \xrightarrow{\rho_1} E(G_0),$$

and that both of these arrows are bijections since ρ_1 and ρ_2 are specialization morphisms. Moreover, since $\rho_1{}^{-1}(A_0)$ is connected, so is $\rho^{-1}(A_0) = \rho_2{}^{-1}(\rho_1{}^{-1}(A_0))$. Finally, let $H_2 = \rho^{-1}(A_0)$ and $H_1 = \rho_1{}^{-1}(A_0)$. Note that $\rho_1{}^{-1}(H_1) = \rho_1{}^{-1} \circ \rho_2{}^{-1}(A_0) = \rho^{-1}(A_0) = H_2$. So applying Lemma 11.17 we get that Equation (11.5) is fulfilled:

$$\begin{aligned} w_0(A_0) &= g(\rho_1{}^{-1}(A_0), w_1|_{\rho_1{}^{-1}(A_0)}) = g(H_1, w_1|_{H_1}) \\ &= g(H_2, w_2|_{H_2}) = g(\rho^{-1}(A_0), w_2|_{\rho^{-1}(A_0)}). \end{aligned}$$ □

Thus, by Remark 11.13 and Lemmas 11.17 and 11.17 we obtain a category:

Definition 11.19. The category WG_g has as objects genus-g connected weighted graphs, and as morphisms the specialization morphisms.

Now we focus in some constructive aspects, in order to contract a subset $S \subset E(G)$.

Construction 11.20 (weighted contraction)**.** Let (G, w) be a weighted graph and S a subset of $E(G)$. The contraction morphism $G \to G/S$ becomes a specialization morphism $\rho_S : (G, w) \to (G/S, w_S)$ by setting

$$w_S(A_0) = g(\rho^{-1}(A_0)) + \sum_{A \in V(\rho^{-1}(A_0))} w(A) = g(\rho^{-1}(A_0), w|_{\rho^{-1}(A_0)}).$$

It turns out that all specialization morphisms arise from Construction 11.20. We prove this with the following two lemmas.

Lemma 11.21. *Let $\gamma : (\overline{G}, \bar{w}) \to (G, w)$ be a graph morphism, and $\bar{\rho} : (\overline{G}, \bar{w}) \to (\overline{G}_0, \bar{w}_0)$, $\rho : (G, w) \to (G_0, w_0)$ specialization morphisms. If* $\mathrm{contr}(\bar{\rho})$ *is in* $\gamma^{-1}(\mathrm{contr}(\rho))$*, then there exists a unique $\gamma_0 : (\overline{G}_0, \bar{w}_0) \to (G_0, w_0)$ such that Diagram 11.22 commutes.*

$$\begin{array}{ccc} (\overline{G}, \bar{w}) & \xrightarrow{\gamma} & (G, w) \\ \bar{\rho}\big\downarrow & & \big\downarrow\rho \\ (\overline{G}_0, \bar{w}_0) & \xrightarrow{\exists! \gamma_0} & (G_0, w_0). \end{array}$$

Diagram 11.22

Proof. Let $\overline{A}_0$ be in $V(\overline{G}_0)$. Note that ρ is a contraction and $\mathrm{contr}(\bar{\rho}) \subset \gamma^{-1}(\mathrm{contr}(\rho))$. So, if $e \in E(\bar{\rho}^{-1}(\overline{A}_0))$, then $\rho(\gamma(e))$ equals a vertex A_0 of G_0. Moreover, by connectedness of $\bar{\rho}^{-1}(\overline{A}_0)$, this vertex A_0 is independent of e; i.e. $\rho(\gamma(\rho_S{}^{-1}(\overline{A}_0))) = \{A_0\}$. So the only possibility is the map $\gamma_0 : (\overline{G}_0, \bar{w}_0) \to (G_0, w_0)$ given by $\overline{A}_0 \mapsto A_0$ for a vertex $\overline{A}_0$, and $\bar{e} \mapsto \rho(\gamma(\bar{\rho}^{-1}(\bar{e})))$ for an edge $\bar{e}$. We have argued that the map γ_0 is well defined and satisfies Diagram 11.22.

Note that γ_0 is a graph morphism, because if $\bar{e}$ is incident to $\overline{A}_0$ in $\overline{G}_0$, then $\bar{\rho}^{-1}(\bar{e})$ is incident to some vertex in $\bar{\rho}^{-1}(\overline{A}_0)$, hence $\gamma_0(\bar{e}) = \rho(\gamma(\bar{\rho}^{-1}(\bar{e})))$ and $\gamma(\overline{A}_0) = \rho(\gamma(\bar{\rho}^{-1}(\overline{A}_0)))$ are incident. □

Remark 11.23. In general, given graph morphisms $\gamma, \gamma_0, \bar{\rho}$ and ρ that commute, say $\gamma_0 \circ \bar{\rho} = \rho \circ \gamma$ as in Diagram 11.22, in the set $E(\overline{G})$ we have that

$$\mathrm{contr}(\bar{\rho}) \sqcup \bar{\rho}^{-1}(\mathrm{contr}(\gamma_0)) = \mathrm{contr}(\gamma) \sqcup \gamma^{-1}(\mathrm{contr}(\rho)).$$ △

Lemma 11.24. *Let $\rho : (G, w) \to (G_0, w_0)$ be a specialization morphism. Choose any subset $S \subset \operatorname{contr}(\rho)$. There exists a unique contraction morphism $\tilde{\rho}$ such that Diagram 11.25 commutes.*

$$\begin{array}{ccc} (G,w) & \xrightarrow{\ \rho\ } & (G_0,w_0) \\ & \underset{\rho_S}{\searrow} & \uparrow{\scriptstyle \exists! \widetilde{\rho}} \\ & & (G/S, w_S). \end{array}$$

Diagram 11.25

Moreover, $\operatorname{contr}(\rho) = \operatorname{contr}(\rho_S) \sqcup {\rho_S}^{-1}(\operatorname{contr}(\widetilde{\rho}))$ *and* $\operatorname{contr}(\widetilde{\rho}) = \rho_S(\operatorname{contr}(\rho) \setminus \operatorname{contr}(\rho_S))$.

Proof. Lemma 11.21 with $\gamma = \mathrm{id}_{(G,w)}$ and $(\overline{G}_0, \bar{w}_0) = (G/S, w_s)$ gives the only possible candidate morphism $\widetilde{\rho}$, and it remains to show that $\widetilde{\rho}$ is a specialization morphism. We first show that $\widetilde{\rho}$ is a contraction. Note that the restriction $\widetilde{\rho}^{-1}(E(G_0)) \xrightarrow{\widetilde{\rho}} E(G_0)$ is bijective. This because it equals $(\rho \circ {\rho_S}^{-1})^{-1} = \rho_S \circ \rho^{-1}$; and since ρ and ρ_S are contractions, ρ^{-1} is injective into $E(G)$, and ρ_S is injective on $E(G)$. Moreover, let A_0 be a vertex of G_0. The fibre $\rho^{-1}(A_0)$ is connected, and since ρ_S is continuous in the graph topology, we get that $\rho_S(\rho^{-1}(A_0)) = \widetilde{\rho}^{-1}(A_0)$ is connected as well. To prove Equation (11.5), let $A_0 \in V(G_0)$, and set $H = \rho^{-1}(A_0)$ and $\tilde{H} = \rho_S(H)$. We calculate:

$$\begin{aligned} \widetilde{\rho}^{-1}(A_0) &= (\rho \circ {\rho_S}^{-1})^{-1}(A_0) = \rho_S \circ \rho^{-1}(A_0) = \rho_S(H) = \tilde{H}, \\ H = \rho^{-1}(A_0) &= (\widetilde{\rho} \circ \rho_S)^{-1}(A_0) = {\rho_S}^{-1} \circ \widetilde{\rho}^{-1}(A_0) = {\rho_S}^{-1}(\tilde{H}). \end{aligned}$$

As $H = {\rho_S}^{-1}(\tilde{H})$ and ρ and ρ_S are specialization morphisms, we apply Lemma 11.17 on $\tilde{H}$:

$$\begin{aligned} w(A_0) &= g(\rho^{-1}(A_0), w|_{\rho^{-1}(A_0)}) = g(H, w|_H) \\ &= g(\rho_S(H), w_S|_{\rho_S(H)}) = g(\tilde{H}, w_S|_{\tilde{H}}) = g(\widetilde{\rho}^{-1}(A_0), w_S|_{\widetilde{\rho}^{-1}(A_0)}). \end{aligned}$$

Finally, $\widetilde{e}$ is in $\operatorname{contr}(\widetilde{\rho})$ if and only if $\widetilde{\rho}(\widetilde{e}) = \rho \circ {\rho_S}^{-1}(\widetilde{e})$ is a vertex of G_0, if and only if ${\rho_S}^{-1}(\widetilde{e})$ is in $\operatorname{contr}(\rho)$. Clearly ${\rho_S}^{-1}(\widetilde{e})$ is not in $\operatorname{contr}(\rho_S)$, so we obtain the claim on $\operatorname{contr}(\widetilde{\rho})$. □

Remark 11.26. In the setting of Lemma 11.24, if we have that $S = \operatorname{contr}(\rho)$, then $\operatorname{contr}(\widetilde{\rho}) = \varnothing$, which by Remark 11.16 means that $\widetilde{\rho}$ is an isomorphism. Thus, all specialization morphisms ρ are isomorphic to ρ_S with $S = \operatorname{contr}(\rho)$, as claimed. △

Example 11.27 (Final object in WG_g)**.** For any graph G, if we consider the specialization morphism $\rho_{E(G)}$ contracting all edges, we get a map to the graph $G_\mathrm{f} \in \mathrm{WG}_g$ that has one vertex A of weight g and no edges. This is the final object in WG_g. ⋆

Remark 11.28. Note that the opposite of Diagram 11.25 is

$$\begin{array}{ccc} (G_0,w_0) & \xrightarrow{\ \rho^{\mathrm{op}}\ } & (G,w) \\ {\scriptstyle \exists! \widetilde{\rho}^{\mathrm{op}}}\downarrow & \underset{{\rho_S}^{\mathrm{op}}}{\nearrow} & \\ (G/S, w_S) & & . \end{array}$$

Diagram 11.29

Later we use this fact to prove that a functor $C : {\mathrm{WG}_g}^{\mathrm{op}} \mapsto \mathrm{P{\scriptstyle OLY}}^f_{\mathbb{Z}}$ via $(G, w) \mapsto C_{(G,w)}$ satisfies Condition (b) from Definition 9.4. △

11.4 Tropical modification for graphs

Recall that Subsection 8.2.4 describes tropical modification as an operation that iteratively attaches or removes monovalent points; such elements we call *dangling*. We study how this operation interacts with specialization morphisms.

Definition 11.30. Let (G, w) be a weighted graph, $e \in E(G)$ and $A \in V(G)$. We have that

- e is *dangling* if deleting e gives two connected components (G_1, w_1),(G_2, w_2) and at least one of those is a tree, i.e. $\min(g(G_1, w_1), g(G_2, w_2))$ is 0.
- A is dangling if $w(A) = 0$ and all e in $E(A)$ are dangling.

An important fact is that being dangling commutes with specialization morphisms.

Lemma 11.31. *Let $\rho : (G, w) \to (G_0, w_0)$ be a specialization morphism, and x_0 in G_0. The element x_0 is dangling if and only if all elements of $\rho^{-1}(x_0)$ are dangling in G.*

Proof. Observation I: For any $S_0 \subset E(G_0)$, we have that ρ induces a genus preserving bijection $\pi_0(G \setminus \rho^{-1}(S_0)) \to \pi_0(G_0 \setminus S_0)$. This follows from the fact that the fibres of ρ are connected, thus a path in $G_0 \setminus S_0$ lifts to a path in $G \setminus \rho^{-1}(S_0)$.

Observation II: Let T be a subgraph of G such that $(T, w|_T)$ is a tree, and S be the subset of $E(G) \setminus E(T)$ of edges with at least one end in $V(T)$. We claim that all the elements of T are dangling if and only if all the edges in S are dangling. For one direction, note that if all vertices of T are dangling then so are all the edges incident to them; this includes S as a subset. Conversely, suppose there is a non-dangling x_0 in G_0. So there is another non-dangling element x_0' incident to x_0. Iterating, we get a sequence $y_0 = x_0, y_1 = x_0'$, and so on, until one element repeats, giving us a cycle. Since T is a tree, the sequence cannot be contained in $E(T)$, so it has an edge of S. So one edge of S is contained in a cycle, thus it is non-dangling, a contradiction.

Now, if x_0 from the statement of the lemma is an edge, then the result follows from Observation I applied to $S = \{x_0\}$; if it is a vertex then apply first Observation II and then Observation I. □

Thus, we come naturally to the following definition.

Definition 11.32. An *elementary tropical modification* $\rho : (G, w) \to (G_0, w_0)$ is a specialization morphism that contracts only non-dangling edges.

We write $\cong_{\text{trop}}$ for the equivalence relation on WG_g generated by declaring (G, w) and $(\tilde{G}, \widetilde{w})$ equivalent if there is an elementary tropical modification $\rho : (G, w) \to (\tilde{G}, \widetilde{w})$. We denote by $\mathrm{nd}(G, w)$ the weighted subgraph of (G, w) induced by the non-dangling elements.

Lemma 11.33. *Let (G, w) be a connected weighted graph, and $\rho : (G, w) \to (\tilde{G}, \widetilde{w})$ an elementary tropical modification. We have that ρ maps $\mathrm{nd}(G, w)$ isomorphically to $\mathrm{nd}(\tilde{G}, \widetilde{w})$. Moreover, $\mathrm{nd}(G, w)$ is connected.*

Proof. Lemma 11.31 implies that $\rho(\mathrm{nd}(G, w))$ is a subgraph of $\mathrm{nd}(\tilde{G}, \widetilde{w})$. On the other hand, if $\tilde{A}$ is a vertex of $\mathrm{nd}(\tilde{G}, \widetilde{w})$, by Lemma 11.31 all the edges of $\rho^{-1}(\tilde{A})$ are non-dangling, but ρ is an elementary tropical modification, it cannot contract non-dangling edges, so $\rho^{-1}(\tilde{A})$ is a single vertex that is in $\mathrm{nd}(G, w)$. Since $\rho^{-1}(\widetilde{e})$ is a single edge that is in $\mathrm{nd}(G, w)$, for all $\widetilde{e}$ in $E(\mathrm{nd}(\tilde{G}, \widetilde{w}))$, we get the first claim. The second claim follows from considering the elementary tropical modification $\tilde{\rho}$ that contracts all dangling edges. Since (G, w) is connected, the image of $\tilde{\rho}$, which is isomorphic to $\mathrm{nd}(G, w)$, is connected as well. □

Hence, $\mathrm{nd}(G, w)$ is a canonical representative for the class $[(G, w)]_{\cong_{\text{trop}}}$. It is straightforward to see that $\text{min-wtval}(\mathrm{nd}(G, w)) \geq 2$ and that $\mathrm{nd}(G, w)$ is the unique graph in $[(G, w)]_{\cong_{\text{trop}}}$ with this property. Next, we extend the previous definitions to metric graphs and use the previous observation to give a canonical representative for $[\Gamma]_{\cong_{\text{trop}}}$.

Definition 11.34. Let $(|\Gamma|, w)$ be a weighted metric graph. A point x in $|\Gamma|$ is dangling if for some model $G \to \mathrm{POLY}^f_{\mathbb{Z}}$ we have that $\mathrm{poly}_G\, x$ is dangling

Observe that whether a point x in $(|\Gamma|, w)$ is dangling or non-dangling is independent of the chosen model. Let $(|\widetilde{\Gamma}|, \widetilde{w})$ be the weighted metric graph obtained from $(|\Gamma|, w)$ by deleting all dangling points. By our previous observation, $\text{min-wtval}(\widetilde{\Gamma}, \widetilde{w}) \geq 2$. Moreover, by Remark 11.11 the essential model (H, w_H, ℓ_H) of $(\widetilde{\Gamma}, w)$ given in Lemma 11.10 satisfies

$$\mathrm{wtval}\, A = \mathrm{val}\, A + 2w(A) \geq 3 \text{ for all } A \in V(H). \tag{11.6}$$

By Lemmas 11.10 and 11.31 we have the following.

Lemma 11.35. *Let* $(|\Gamma|, w)$ *be a weighted metric graph of genus at least 2, and* $(|\Gamma'|, w')$ *a tropical modification that has a model* $G' \to \mathrm{POLY}_{\mathbb{Z}}^{f}$ *with* min-wtval(G') *at least three. The metric graph* $(|\Gamma'|, w')$ *is isometric to the deletion of all dangling points* $(|\widetilde{\Gamma}|, w)$ *and* G' *is isomorphic to the essential model* H. □

So $(\widetilde{\Gamma}, \widetilde{w})$ and (H, w_H) are canonical representatives of the equivalence classes under tropical modification of (Γ, w) and (G, w). We call a weighted graph that satisfies Equation (11.6) a *combinatorial type*. If the weight map is identically zero and all valencies are equal to 3, we have a *trivalent combinatorial type*. We say that the weighted graph (H, w_H) is the combinatorial type of both (Γ, w) and of (G, w). Being a combinatorial type is preserved by specialization morphisms.

Lemma 11.36. *Let* $\rho : (G, w) \to (G_0, w_0)$ *be a specialization morphism. For any* A_0 *in* $V(G_0)$ *we have that*

$$\operatorname{wtval} A_0 = 2 + \sum_{A \in \rho^{-1}(A_0)} (\operatorname{wtval} A - 2).$$

Proof. Let $B_0 \in V(G_0)$, and consider an incidence $i_0 : e_0 \to B_0$. Since ρ is a graph morphism and a contraction, there is a unique $[i : \rho^{-1}(e_0) \to B] \in \operatorname{Hom}(\rho^{-1}(e_0), -)$ such that $\rho(i) = i_0$. Note that $\rho(B) = B_0$ and $\rho(E(B)) \subset E(B_0) \cup \{B_0\}$. Thus,

$$\operatorname{val}_{G_0} B_0 = \sum_{B \in \rho^{-1}(B_0)} \operatorname{val}_G B - \operatorname{val}_{\rho^{-1}(B_0)} B. \tag{11.7}$$

So let $H = \rho^{-1}(A_0)$ and calculate

$$\begin{aligned}
\operatorname{wtval} A_0 &= \operatorname{val} A_0 + 2w(A_0) \\
&= \sum_{A \in V(H)} (\operatorname{val}_G A - \operatorname{val}_H A) + 2g(H, w_H) \\
&= \sum_{A \in V(H)} \operatorname{val}_G A + 2w(A) - 2\#(E(H))) \\
&\quad + 2(\#(E(H)) - \#(V(H)) + 1) \\
&= 2 + \sum_{A \in V(H))} (\operatorname{wtval} A - 2).
\end{aligned}$$
□

Lemma 11.37. *Let* $\rho : (H, w) \to (H_0, w_0)$ *be a specialization morphism. If* (H, w) *is a combinatirial type, then so is* (H_0, w_0).

Proof. For any vertex A_0 of G_0 we have that

$$\begin{aligned}
\operatorname{wtval} A_0 &= \sum_{A \in \rho^{-1}(A_0)} \operatorname{wtval} A - 2(\#(V(\rho^{-1}(A_0))) - 1) \\
&\geq 3\#(V(\rho^{-1}(A_0))) - 2(\#(V(\rho^{-1}(A_0))) - 1) = \#(V(\rho^{-1}(A_0))) + 2 \geq 3.
\end{aligned}$$
□

11.5 A polyhedral space $\mathcal{M}_g^{\mathrm{trop}}$ parametrizing weighted metric graphs

We finish the description of the functor $C_- : \mathrm{WG}_g{}^{\mathrm{op}} \to \mathrm{CONE}_{\mathbb{Z}}^{f}$, and define a polyhedral space parametrizing weighted metric graphs by restricting C_- to a subcategory $\mathcal{M}_g^{\mathrm{trop}}$ of trivalent graphs. We argue that $\mathcal{M}_g^{\mathrm{trop}} \to \mathrm{CONE}_{\mathbb{Z}}^{f}$ is a polyhedral space.

Definition 11.38. Given a specialization morphism $\rho : (G, w) \to (G_0, w_0)$, the *pullback map* $\rho^* : \operatorname{span}_{\mathbb{R}} C_{(G_0, w_0)} \to \operatorname{span}_{\mathbb{R}} C_{(G,w)}$ sends $y_0 \mapsto y_0 \circ \rho$; and the *push-forward map* $\rho_* : \operatorname{span}_{\mathbb{R}} C_{(G,w)} \to \operatorname{span}_{\mathbb{R}} C_{(G_0, w_0)}$ sends $y \mapsto y \circ \rho^{-1}$.

Definition 11.39. Given a weighted graph (G, w) we define $N^{(G,w)} \subset \operatorname{span}_{\mathbb{R}} C_{(G,w)}$ to be the set of functions with integral values. It is a lattice, and $N^{(G,w)} \cap C_{(G,w)}$ corresponds to metric graphs with integral lengths.

First, we show that ρ^* is a face morphism, which implies that the assignment

$$\begin{aligned} (G,w) \in \mathrm{Obj}(\mathrm{WG}_g) &\to (N^{(G,w)}, C_{(G,w)}) \\ \rho \in \mathrm{Hom}((G,w),(G_0,w_0)) &\to [\rho^* : (N^{(G_0,w_0)}, C_{(G_0,w_0)}) \to (N^{(G,w)}, C_{(G,w)})] \end{aligned} \quad (11.8)$$

is a contravariant functor $C : \mathrm{WG}_g{}^{\mathrm{op}} \to \mathrm{CONE}^f_{\mathbb{Z}}$.

Lemma 11.40. *For $\rho \in \mathrm{Hom}((G,w),(G_0,w_0))$ the pullback ρ^* is a face morphism.*

Proof. We have that $\rho_* \circ \rho^*$ is the identity on $\operatorname{span}_{\mathbb{R}} C_{(G_0,w_0)}$. Also, $\rho^* \circ \rho_*$ is the canonical projection that sencs a vector in $\operatorname{span}_{\mathbb{R}} C_{(G,w)}$ to the codimension-k linear space $\rho^*(\operatorname{span}_{\mathbb{R}} C_{(G_0,w_0)})$ of those maps y that are zero on $\mathrm{contr}(\rho)$, where $k = \#(\mathrm{contr}(\rho))$. In particular, $\rho^* \circ \rho_*$ is the identity on $\rho^*(\operatorname{span}_{\mathbb{R}} C_{(G_0,w_0)})$. Thus, ρ^* restricts to a face morphism $\rho^*|_{C_{(G_0,w_0)}} : C_{(G_0,w_0)} \to C_{(G,w)}$. □

By Lemma 11.37 the subset $\mathcal{M}_g^{\mathrm{trop}}$ of combinatorial types in WG_g is a subcategory. The results of Subsection 11.4 suggest that the following space parametrizes equivalence classes of weighted metric graphs under tropical modification:

Definition 11.41. Let $\mathcal{M}_g^{\mathrm{trop}}$ be the subcategory of WG_g of combinatorial types. The *tropical moduli space of genus-g connected weighted metric graphs* is the restriction of the functor C to $(\mathcal{M}_g^{\mathrm{trop}})^{\mathrm{op}} \to \mathrm{CONE}^f_{\mathbb{Z}}$.

Lemma 11.42. *The* tropical moduli space of genus-g connected weighted metric graphs C : $\mathcal{M}_g^{trop} \to \mathrm{POLY}^f_{\mathbb{Z}}$ *is a polyhedral space of cones.*

Proof. We check the three conditions from Definition 9.4. Let (H, w) be in $\mathcal{M}_g^{\mathrm{trop}}$, and τ be a face of $C_{(H,w)}$. Condition (a) is true because all the faces of $C_{(H,w)}$ arise by prescribing a subset $S \subset E(H)$ to have zero lengths, thus $\rho_S : (H, w) \to (H/S, w_S)$ provides the desired face inclusion. Condition (b) follows from Lemma 11.24 and Remark 11.28. Condition (c) is true because WG_g is already a skeleton category. □

Remark 11.43 (dimension of $\mathcal{M}_g^{\mathrm{trop}}$)**.** Recall that $\dim C_{(H,w)} = \#(E(H))$. The maximum value of $\#(E(H))$ under the constrain that $\text{min-wtval}(H) \geq 3$ is $3g-3$, and is attained at trivalent combinatorial types. It is straightforward to prove that for every (H_0, w_0) there exists a specialization morphism $\rho : (H, w) \to (H_0, w_0)$ with $\#(E(H)) = 3g - 3$. Thus, $\mathcal{M}_g^{\mathrm{trop}}$ is pure-dimensional of dimension $3g - 3$. △

Remark 11.44. The rays of $\mathcal{M}_g^{\mathrm{trop}}$ correspond to genus-g connected weighted graphs H with $\#(E)(H) = 1$. There are $\lceil g/2 \rceil$ such graphs that are loopless, and 1 with a loop. Thus, $\mathcal{M}_g^{\mathrm{trop}}$ has $\lceil g/2 \rceil + 1$ rays. The number of trivalent graphs grows exponentially with respect to g [Bol82]. See OEIS A002851 for the first few terms. This gives a feeling of how far $\mathcal{M}_g^{\mathrm{trop}}$ is from being a poset, as the number of maximal cones of a cone complex with pure dimension is polynomial on the number of rays. △

11.6 Specialization with a given target

Now we say a few words about the inverse problem of fixing a combinatorial type H_0 and constructing specializations with target H_0. As an application, we get that the poset $\mathcal{M}_g^{\mathrm{trop}}$ is strongly connected, per Definition 10.66. For this we need some notions on marked graphs.

Definition 11.45 (marked weighted graphs)**.** Let (G, w) be a weighted graph.

- An *n-marking* is a map $m : [n] \to V(G)$.

- An n-marked weighted graph is *stable* if for all $A \in V(G)$ we have that
$$\operatorname{wtval} A + \#(\rho^{-1}(A)) \geq 3.$$
- A specialization of n-marked weighted graphs (G_1, w_1, m_1) and (G_2, w_2, m_2) is a specialization $\rho : (G_1, w_1) \to (G_2, w_2)$ of weighted graphs such that ${m_1}^{-1}(A) \subset {m_2}^{-1}(\rho(A))$.
- We denote by $\mathcal{M}^{\mathrm{trop}}_{g,n}$ the skeleton of the category whose objects are n-marked weighted graphs and whose morphisms are specializations.

In the literature, marked weighted graphs are intuitively regarded as weighted graphs *with legs*. Having said that, now we have:

Lemma 11.46. *Let (H_0, w_0) be a combinatorial type, and $A_1, \dots, A_q$ an enumeration of the set $\{A \in E(H_0)\colon \operatorname{wtval} A > 3\}$. The isomorphism classes of specializations $\rho : H \to H_0$ are in bijection with the product*
$$\mathfrak{H} = \prod_{i=1}^{q} \mathcal{M}^{trop}_{w_0(A_i), \operatorname{val}(A_i)}.$$

Proof. Fix a labelling $\lambda_i : \{1, \dots, \operatorname{val} A_i\} \to \operatorname{Hom}(-, A_i)$, for each i. We construct a graph $H_{\mathfrak{h}}$ from a given element $\mathfrak{h} = ((H_1, w_1, m_1), \dots, (H_k, w_k, m_k)) \in \mathfrak{H}$, by taking the category associated to H_0 and for each $i \in [q]$ we replace A_i with H_i, and for each k in $\{1, \dots, \operatorname{val} A_i\}$ we let $e_{i,k} = \operatorname{domain}({\lambda_i}^{-1}(k))$ and $B_{i,k} = {m_i}^{-1}(k)$, and insert a morphism $e_{i,k} \to B_{i,k}$. It is clear how to construct the specialization $\rho_{\mathfrak{h}} : H_{\mathfrak{h}} \to H_0$. Going the other direction, given a specialization $\rho : H \to H_0$, we get an element $\mathfrak{h}$ of $\mathfrak{H}$ by considering $((\rho^{-1}(A_1), \operatorname{codomain}(\rho^{-1} \circ \lambda_1)), \dots, (\rho^{-1}(A_q), \operatorname{codomain}(\rho^{-1} \circ \lambda_q)) \in \mathbb{H}$. We leave to the reader the verification that these two operations are inverses to each other, proving that we have a bijection. □

11.7 Connectivity of $\mathcal{M}^{\mathrm{trop}}_g$

We prove that $\mathcal{M}^{\mathrm{trop}}_g$ is strongly connected, setting up the stage for using Proposition 10.73 on a nice enough map with target $\mathcal{M}^{\mathrm{trop}}_g$, e.g. the map $\Pi : \mathcal{G}^{\mathrm{trop}}_{g \to 0,d} \to \mathcal{M}^{\mathrm{trop}}_g$. We first check that $(\mathcal{M}^{\mathrm{trop}}_g, \preceq)$, with the relation $\preceq$ from Remark 9.21 is a poset, and that boils down to proving:

Lemma 11.47. *Let H and H_0 be two graphs in $\mathcal{M}^{trop}_g$. If there are specializations $\rho_1 : H \to H_0$ and $\rho_2 : H_0 \to H$, then H and H_0 are isomorphic.*

Proof. Let $\rho : H \to H$ be the specialization $\rho = \rho_2 \circ \rho_1$. On the one hand, $\#(\operatorname{contr}(\rho)) = \#(E(H)) - \#(E(H)) = 0$ by Equation (11.4) on Page 114. On the other hand, $\#(\operatorname{contr}(\rho)) = \#(\operatorname{contr}(\rho)_1) + \#(\operatorname{contr}(\rho)_2)$ by Lemma 11.24. Thus, $\#(\operatorname{contr}(\rho)_1) = \#(\operatorname{contr}(\rho)_2) = 0$, which by Remark 11.16 implies that ρ_1 and ρ_2 are isomorphisms. □

In [Cap12, Proposition 3.3.3] it is proven that a so-called space of pointed tropical curves is connected in codimension-1. It is a fun exercise to get, out of this fact, that $\mathcal{M}^{\mathrm{trop}}_{g,n}$ is connected in codimension-1. Anyway, this leads us to:

Lemma 11.48. *The poset $(\mathcal{M}^{trop}_g, \preceq)$ is codimension-1 connected.*

Proof. By Lemma 10.57 we need to prove that if H, H' have $3g - 3$ edges, then there is a sequence $H_0, \tilde{H}_1, H_1, \tilde{H}_2, \dots, H_r$ with $\tilde{H}_i$ arising from both H_{i-1} and H_i by contracting one edge, and $H_0 = H$, $H_r = H'$. For a topological proof see [HT80]. For one in the context of tropical geometry see [Cap12]. □

Proposition 11.49. *The poset $(\mathcal{M}^{trop}_g, \preceq)$ is strongly connected.*

Proof. A high level proof: Let $H_0 \in \mathcal{M}^{\mathrm{trop}}_g$. We have to prove that $(\uparrow H_0) \setminus \{H_0\}$ is connected. The idea is to use Lemma 11.46. So we would see that $\uparrow H_0$, as a poset, is isomorphic to a product of $\mathcal{M}^{\mathrm{trop}}_{g_i,n_i}$. Each of the terms of this product is connected through codimension-1, and then some general result on products of posets that are connected in codimension-1 giving a strongly connected poset should suffice. □

11.8 The points of $|\mathcal{M}_g^{\mathbf{trop}}|$

We now show that points in $|\mathcal{M}_g^{\mathrm{trop}}|$ are in bijection with isomorphism classes of genus-g weighted metric graphs without points that have weighted valency equal to 1, i.e. equivalence classes under tropical modification of genus-g weighted metric graphs. This is a baby example of the steps for showing a similar result for $\mathcal{G}_{g\to 0,d}^{\mathrm{trop}}$.

Let $\Psi : (\Gamma^{(1)}, w^{(1)}) \to (\Gamma^{(2)}, w^{(2)})$ be an isometry. It preserves the valency and weight of a point. Thus, by Lemma 11.10, if S is a vertex set of $(\Gamma^{(1)}, w^{(1)})$, then $\Psi(S)$ is a vertex set of $(\Gamma^{(2)}, w^{(2)})$. So let $\Gamma^{(1)} = (G^{(1)}, w^{(1)}, \ell^{(1)})$ be induced by S, and $\Gamma^{(2)} = (G^{(2)}, w^{(2)}, \ell^{(2)})$ by $\Psi(S)$. The connected components of $\Gamma^{(1)} \setminus S$ are mapped one-to-one and isometrically to the connected components of $\Gamma^{(2)} \setminus \Psi(S)$. So there is an induced map $\gamma_\Psi : G^{(1)} \to G^{(2)}$ which is an isomorphism; moreover, $w^{(1)} = w^{(2)} \circ \gamma_\Psi$ and $\ell^{(1)} = \ell^{(2)} \circ \gamma_\Psi$. It is straightforward to verify that these necessary conditions are also enough to specify an isometry.

Lemma 11.50. *Let $(\Gamma^{(1)}, w^{(1)})$, $(\Gamma^{(2)}, w^{(2)})$ be weighted metric graphs, S a vertex set of $(\Gamma^{(1)}, w^{(1)})$, and $\Psi : \Gamma^{(1)} \to \Gamma^{(2)}$ be a continuous map. The map Ψ is an isomorphism if and only if $\Psi(S)$ is a vertex set of $\Gamma^{(2)}$, the induced map γ_Ψ is an isomorphism, and $w^{(1)} = w^{(2)} \circ \gamma_\Psi$, $\ell^{(1)} = \ell^{(2)} \circ \gamma_\Psi$.*

Thus, isomorphisms $(\Gamma^{(1)}, w^{(1)}) \to (\Gamma^{(2)}, w^{(2)})$ are in one-to-one correspondence with isomorphisms $(G^{(1)}, w^{(1)}) \to (G^{(2)}, w^{(2)})$, hence with isomorphisms of the cone $C_{(G^{(1)},w^{(1)})}$ to the cone $C_{(G^{(2)},w^{(2)})}$. So by construction of $\mathcal{M}_g^{\mathrm{trop}}$, each point corresponds to one of the desired isomorphism classes.

Chapter 12

Parametrizing tropical morphisms

In this section we study tropical morphisms, how to parametrize them, and a special class of tropical morphisms that maximize the dimension of the space parametrizing them and minimize their degree as an indexed branched cover of topological spaces. Subsection 12.1 studies the combinatorial structure of tropical morphisms. We include results from Part I, slightly generalized to accomodate weights at vertices.

12.1 The combinatorial structure behind a tropical morphism

We begin by looking at the underlying combinatorial structure behind tropical morphisms: a category of graph morphisms that we call *discrete tropical morphisms*, or DT-morphisms for short.

Definition 12.1. Let $\varphi : (G, w_G) \to (T, w_T)$ be a morphism of weighted graphs, and $m_\varphi : G \to \mathbb{Z}_{\geq 1}$ an index map. The *weighted RH-number* $r_\varphi(A)$ of $A \in V(G)$ equals

$$= 2(m_\varphi(A) + w_G(A) - 1) - \left[2m_\varphi(A) \cdot w_T(\varphi(A)) + \sum_{A \lessdot e} (m_\varphi(e) - 1) \right]. \tag{12.1}$$

Note that Equation (12.1) reduces to Equation (RH) in the case of zero weights. In Lemma 3.5 of Part I the following formula is proven when both w_G and w_T are identically zero, and the proof extends in a straightforward manner to consider non-trivial weights.

$$r_\varphi(A) = (\operatorname{wtval}_G A - 2) - m_\varphi(A)(\operatorname{wtval}_T \varphi(A) - 2). \tag{12.2}$$

The right hand side of Equation (12.2) is the coefficient of A in the *tropical ramification divisor*

$$R_\varphi = K_{(G,w_G)} - \varphi^*(K_{(T,w_T)}), \tag{12.3}$$

where $K_{(G,w_G)}$ and $K_{(T,w_T)}$ are the canonical divisors of (G, w_G) and (T, w_T), as defined in [AC13], and $\varphi^* : \operatorname{Div}(T, w_T) \to \operatorname{Div}(G, w_G)$ is the pullback under φ. Namely,

$$K_{(G,w_G)} = \sum_{A \in V(G)} (\operatorname{wtval} A - 2)A,$$

$$\varphi^*(D) = \sum_{A \in V(G)} (m_\varphi(A) \cdot D(\varphi(A)))A.$$

We have $\deg K_{(G,w)} = 2g(G, w) - 2$, and $\deg \varphi^*(D) = \deg \varphi \cdot \deg D$ because φ is an indexed branched cover. So taking degrees on both sides of Equation (12.3) yields:

Lemma 12.2 (tropical Riemann-Hurwitz). *Let $\varphi : (G, w_G) \to (T, w_T)$ be an indexed branched cover of weighted graphs with index map m_φ. We have that*

$$2g(G, w_G) - 2 = \deg \varphi \cdot (2g(T, w_T) - 2) + \sum_{A \in V(G)} r_\varphi(A). \quad \square$$

If $r_\varphi(A) \geq 0$ for all $A \in V(G)$, then Lemma 12.2 implies that $g(G, w_G) \geq g(T, w_T)$. This fact parallels the algebro-geometric setting. Other consequences of Lemma 12.2 are Lemma 12.26, about the behaviour of r_φ under edge contractions, and Proposition 12.15, about the dimension of the family of tropical morphisms with underlying combinatorial structure φ. The condition $r_\varphi(A) \geq 0$ for all $A \in V(G)$ is central in our work.

Definition 12.3. A DT-morphism is an indexed branched cover φ of loopless connected weighted graphs such that $\operatorname{contr}(\varphi) = \varnothing$ and the weighted RH-number is non-negative for all vertices $A \in V(G)$.

Remark 12.4. By the observations of Remark 11.3, a DT-morphism is an indexed branched cover (φ, m_φ) of connected weighted graphs, such that φ is a combinatorial morphism of posets and has non-negative weighted RH-number for all $A \in V(G)$. △

Lemma 12.5. *If the pair (φ, m_φ) is a DT-morphism, then the map φ is a graph morphism.*

Proof. By Remark 11.3, since φ is order preserving, φ is a graph morphism if and only if $\varphi(V(G)) \subset V(T)$. Note that G is connected and φ is combinatorial, so $E(G) = \varnothing$ if and only if $E(T) = \varnothing$. If $E(T) = \varnothing$, we are done. Otherwise, since T is connected we have that $\max T^{\mathrm{op}} = E(T)$. By Example 10.5 we get $\varphi^{-1}(\max T^{\mathrm{op}}) \subset \max G^{\mathrm{op}}$. Since $E(T) \neq \varnothing$ implies that $E(G) \neq \varnothing$, and G is connected, we have $\max G^{\mathrm{op}} = E(G)$. Thus, $\varphi^{-1}(E(T)) \subset E(G)$, which gives $\varphi(V(G)) \subset V(T)$. □

12.2 Tropical morphisms

For a fixed DT-morphism (φ, m_φ) we describe a construction that takes metric data on the target of φ and gives rise to a tropical morphism. The space of all possible metric data is the family of tropical morphisms associated φ.

Definition 12.6. A *tropical morphism with polyhedral structure* is a pair (Φ, m_φ) of a morphism $\Phi = (\varphi, \{\Phi_x\}_{x \in G})$ of weighted metric graphs with polyhedral structure $\Gamma : G^{\mathrm{op}} \to \mathrm{POLY}^f_{\mathbb{Z}}$ and $\Delta : T^{\mathrm{op}} \to \mathrm{POLY}^f_{\mathbb{Z}}$, and an index map $m_\varphi : G \to \mathbb{Z}_{\geq 1}$, such that (φ, m_φ) is a DT-morphism and for all $e \in E(G)$ we have that

$$m_\varphi(e) = [\Phi_e(N_e) : N_{\varphi(e)}] = \operatorname{length}(\Delta_{\varphi(e)}) / \operatorname{length}(\Gamma_e) \text{ for all } e \in E(G). \tag{12.4}$$

Remark 12.7. By Remark 12.4 we can apply Theorem 10.46 to get that $(\Phi, m_\varphi \circ \operatorname{poly}_G)$ is an indexed branched cover. △

Remark 12.8. A *tropical morphism* is a pair isometric to some $(|\Phi|, m_\varphi \circ \operatorname{poly}_G)$, with (Φ, m_φ) a tropical morphism with polyhedral structure. △

Remark 12.9. The first equality in Equation (12.4) indicates a multiplicity given by a determinant, as in Equation (8.10) on Page 71. The second equality gives that the map Φ_e is linear with slope $m_\varphi(e)$ for every e in $E(G)$. Hence, the topological realization $|\Phi|$ satisfies the usual definition of tropical morphism; e.g. the one in Part I. That is, a tropical morphism is a piecewise-linear continuous map between weighted metric graphs, with integral positive slopes that satisfy the balancing condition and the Riemann-Hurwitz inequality. For the other direction, Construction 12.11 shows how to give a polyhedral structure to a topological realization $|\Phi|$. △

Let $\varphi : (G, w_G) \to (T, w_T)$ be a DT-morphism with index map m_φ. Equation (12.4) suggests that choosing a length function and an integral-length function for (T, w_T) uniquely determines length and integral-length functions for (G, w_G) such that the resulting morphism of polyhedral complexes is a tropical morphism. Indeed, it is straightforward to verify that the following construction gives a tropical morphism with polyhedral structure.

Construction 12.10. Given a triple (φ, m_φ, z) of a DT-morphism $\varphi : (G, w_G) \to (T, w_T)$ with index map m_φ and a length function $z \in C_{(T,w_T)}$, we consider the weighted metric graphs with polyhedral structure $\Gamma = (G, w_G,\, z \circ \varphi/m_\varphi,\, (\deg\varphi \cdot \mathbf{1})/m_\varphi)$ and $\Delta = (T, w_T,\, z,\, \deg\varphi \cdot \mathbf{1})$ by Construction 11.8. To (φ, m_φ, z) we associate the morphism of polyhedral complexes given by the pair $(\varphi, \{\Phi_x\}_{x\in G})$ and with index map m_φ, where Φ_e is a linear map with slope $m_\varphi(e)$ for $e \in E(G)$, and Φ_A is the unique map from $N^A_{\mathbb{R}}$ to $N^{\varphi(A)}_{\mathbb{R}}$.

We write $|(\varphi, m_\varphi, z)|$ for the topological realization of (φ, m_φ, z).

Next, following a line of thought similar to Lemma 11.10, given a tropical morphism $\Phi : (\Gamma, w_G) \to (\Delta, w_T)$ we characterize the choices of vertex sets $\mathcal{G}$ and $\mathcal{T}$ for $|\Gamma|$ and $|\Delta|$, respectively, that give rise to a polyhedral structure for $|\Phi|$. We show that any polyhedral structure for $|\Phi|$ is determined by a choice of a vertex set of $|\Delta|$ satisfying certain properties.

Construction 12.11. Fix a tropical morphism $|\Phi| : (|\Gamma|, w_G) \to (|\Delta|, w_T)$. Let $\mathcal{E}_{(|\Delta|,w_T)}$ and $\mathcal{E}_{(|\Gamma|,w_G)}$ be the essential vertices of $(|\Delta|, w_T)$ and $(|\Gamma|, w_G)$, respectively. Choose a subset $\mathcal{T} \subset |\Delta|$ that contains $\mathcal{E}_{(|\Delta|,w_T)} \cup |\Phi|(\mathcal{E}_{(|\Gamma|,w_G)})$. We construct the following:

- The set $\mathcal{G} = |\Phi|^{-1}(\mathcal{T}) \subset |\Gamma|$.
- The graphs $T_{\mathcal{T}}$ and $G_{\mathcal{G}}$ induced by $\mathcal{T}$ and $\mathcal{G}$, respectively, via Construction 11.8.
- The function $z_{\mathcal{T}}$ recording the lengths of $E(T_{\mathcal{T}}) = \{\mathrm{cl}(U) \colon U \in \pi_0(|\Delta| \setminus \mathcal{T})\}$.
- The map $\varphi_{\mathcal{T}} : G_{\mathcal{G}} \to T_{\mathcal{T}}$ that sends $A \in V(G_{\mathcal{G}}) = \mathcal{G}$ to $\Phi(A) \in V(T_{\mathcal{T}}) = \mathcal{T}$, and $e \in E(G_{\mathcal{G}})$ to $\Phi(e) \in E(T_{\mathcal{T}})$.
- The index map $m_{\varphi_{\mathcal{T}}}(e) = \mathrm{length}(\varphi_{\mathcal{T}}(e))/\,\mathrm{length}(e)$ for edges, and for vertices the value is given by the balancing condition.

Remark 12.12. If in Construction 12.11 we take $\mathcal{T} = \mathcal{E}_{(|\Delta|,w_T)} \cup \Phi(\mathcal{E}_{(|\Gamma|,w_G)})$ then we denote the triple $(\varphi_{\mathcal{T}}, m_{\varphi_{\mathcal{T}}}, z_{\mathcal{T}})$ by $(\varphi_{\mathrm{ess}}, m_{\varphi_{\mathrm{ess}}}, z_{\mathrm{ess}})$ and we call it the *essential model.* △

Lemma 12.13. *For any choice of $\mathcal{T}$ in Construction 12.11, the pair $(\varphi_{\mathcal{T}}, m_{\varphi_{\mathcal{T}}})$ is a DT-morphism, $|\Phi|$ is isometric to $|(\varphi_{\mathcal{T}}, m_{\varphi_{\mathcal{T}}}, z_{\mathcal{T}})|$, and the essential model is minimal in the sense that $\varphi_{\mathcal{T}}$ arises from an edge subdivision of φ_{ess}.*

Proof. Note that $\mathcal{T}$ contains $\mathcal{E}_{(|\Delta|,w_T)}$ and $\mathcal{G}$ contains $\mathcal{E}_{(|\Gamma|,w_G)}$, thus by Lemma 11.10 they are vertex sets. One can prove that the property $\mathcal{E}_{(|\Delta|,w_T)} \cup |\Phi|(\mathcal{E}_{(|\Gamma|,w_G)}) \subset \mathcal{T}$ and that taking $\mathcal{G} = |\Phi|^{-1}(\mathcal{T})$ are necessary and sufficient conditions for getting that $\Phi(V(G_{\mathcal{G}})) = \Phi(\mathcal{G}) = \mathcal{T} = \Phi(V(T_{\mathcal{T}}))$ and that each connected component of $|\Gamma|\setminus\mathcal{G}$ maps linearly and bijectively to a connected component of $|\Delta|\setminus\mathcal{T}$. In turn, this is necessary and sufficient to get that $\varphi_{\mathcal{T}}$ is well defined and is a DT-morphism with index map $m_{\varphi_{\mathcal{T}}}$. It is straightforward to see that $|\Phi|$ is isomorphic to $|(\varphi_{\mathcal{T}}, m_{\varphi_{\mathcal{T}}}, z_{\mathcal{T}})|$.

The essential model is minimal, and every other model arises as an edge subdivision of φ_{ess}, because we always have $\mathcal{E}_{(|\Delta|,w_T)} \cup \Phi(\mathcal{E}_{(|\Gamma|,w_G)}) \subset \mathcal{T}$. □

12.3 Dimension of families of tropical morphisms

Given a DT-morphism $\varphi : (G, w_G) \to (T, w_T)$ with index map m_φ, we denote by C^{tm}_φ the family of tropical morphisms isomorphic to some $|(\varphi, m_\varphi, z)|$, and by C^{src}_φ the family of equivalence classes under tropical modification of weighted metric graphs that appear as source of some map in C^{tm}_φ. In other words, $C^{\mathrm{src}}_\varphi = \Pi(C^{\mathrm{tm}}_\varphi)$, where Π is the map introduced in Subsection 8.3.3. We identify C^{tm}_φ with $C_{(T,w_T)}$, so it has a polyhedral structure, and endow C^{src}_φ with a polyhedral structure as well. Also, we discuss how to regard C^{src}_φ as a locus in $\mathcal{M}^{\mathrm{trop}}_g$, where $g = g(G, w_G)$, and give an upper bound for the dimension of C^{src}_φ.

First, observe that C^{src}_φ can be regarded as a subset of C_G. In general this inclusion is proper; e.g. see the families in Section 3 of Part I. To relate C^{src}_φ with $\mathcal{M}^{\mathrm{trop}}_g$, the following construction gives the combinatorial type of G, as defined in Subsection 11.4.

Construction 12.14. Let $(H(G), w_{H(G)})$ be a weighted graph whose vertex set is the subset of those A in nd-$V(G)$ satisfying Equation (11.6), i.e. nd-val $A + 2w(A) \geq 3$; the edge set is given by the paths of G with ends in $V(H(G))$ and interior vertices in $V(G) \setminus V(H(G))$; and the weight function $w_{H(G)}$ is induced by w_G.

Given a length function $z \in C_T$, we consider the following length function for $H(\varphi)$:

$$y(h) = \sum_{e \in h} z(\varphi(e))/m_\varphi(e). \tag{12.5}$$

The map $C_T \to C_{H(\varphi)}$ sending $z \mapsto y$ extends to a linear map $A_\varphi : \operatorname{span}_{\mathbb{R}} C_T \to \operatorname{span}_{\mathbb{R}} C_{H(\varphi)}$ called *the edge-length map*. The weighted metric graph $|(H(\varphi), w_{H(\varphi)}, A_\varphi(z))|$ is isomorphic to the deletion of dangling trees $|(H, w_H, \ell)|$ of the source Γ of (φ, m_φ, z). Since min-wtval$(H(\varphi)) \geq 3$, by Lemma 11.35 the combinatorial type H of G is isomorphic to $H(\varphi)$. The punchline is that C^{src}_φ is parametrized by the rational polyhedral cone $A_\varphi(C_{(T,w_T)})$.

Now we bound the dimension of C^{src}_φ by using the fact that $\dim C^{\text{src}}_\varphi \leq \#(E(T))$. The following is a consequence of the Riemann-Hurwitz formula.

Proposition 12.15 (dimension formula)**.** *Let $\varphi : (G, w_G) \to (T, w_T)$ be a degree-d DT-morphism with index map m_φ. Set $g = g(G, w_G)$ and $h = g(T, w_T)$. We have that*

$$\#(E(T)) + \sum_{v \in V(T)} (\varphi_* R_\varphi(v) + \operatorname{val} v + 3w(v) - 3) = 2g - h \cdot (2d - 3) + 2d - 5, \tag{12.6}$$

where $\varphi_* R_\varphi(v) = \sum_{A \in \varphi^{-1}(v)} r_\varphi(A)$ is the push-forward of the ramification divisor R_φ.

Proof. The degree of a divisor is preserved under push-forward, so $\deg \varphi_* R_\varphi$ equals $\deg R_\varphi$. The latter is given by the tropical Riemann-Hurwitz. Thus,

$$\begin{aligned} \#(E(T)) + \sum_{v \in V(T)} (\operatorname{val} v + 3w(v) - 3) + \deg \varphi_* R_\varphi &= \\ 3\#(E(T)) - 3\#(V(T)) + \sum_{v \in V(T)} 3w(v) + \deg R_\varphi &= \\ 3g(T, w_T) - 3 + 2g(G, w_G) - 2 - \deg \varphi \cdot (2g(T, w_T) - 2) &= \\ 2g(G, w_G) - g(T, w_T)(2 \deg \varphi - 3) + 2 \deg \varphi - 5. & \end{aligned}$$ □

From now on, we focus on DT-morphisms whose target is a tree, which we denote by $\varphi : (G, w) \to T$. To shorten, and for backwards compatibility with Part I, we write $\operatorname{ch} v$ for $\varphi_* R_\varphi(v)$. The intuition is that $\operatorname{ch} v$ detects *change* above v; e.g. if v is divalent, $\operatorname{ch} v$ says whether *above* v there are vertices A with nd-val $A \geq 3$, or a vertex where the slope of $|\Phi|$ changes. Therefore, Equation (12.6) reduces to:

$$\#(E(T)) + \sum_{v \in V(T)} (\operatorname{ch}(v) + \operatorname{val} v - 3) = 2g + 2d - 5. \tag{12.7}$$

In Subsection 12.5 we introduce an equivalence $\cong_{\text{trop}}$ by tropical modifications for DT-morphisms, argue that the cone C^{src}_φ is invariant under $\cong_{\text{trop}}$, and give a canonical representative of $[\varphi]_{\cong_{\text{trop}}}$ such that $\operatorname{ch} v + \operatorname{val} v - 3 \geq 0$. This implies the following important bound:

$$\dim C^{\text{src}}_\varphi \leq \#(E(T)) \leq 2g + 2d - 5. \tag{12.8}$$

12.4 Specialization of tropical morphisms

Now we introduce morphisms that give us a category whose objects are DT-morphisms.

Definition 12.16. Let $(\varphi : (G, w) \to T, m_\varphi)$ and $(\varphi_0 : (G_0, w_0) \to T_0, m_0)$ be DT-morphisms. A *specialization of DT-morphisms* is a pair $\rho = (\rho_G, \rho_T)$ of specialization morphisms that give a morphism of indexed branched covers.

Lemma 12.17. *If $(\rho_G, \rho_T) : \varphi \to \varphi_0$ is a specialization of DT-morphisms, then* $\mathrm{contr}(\rho_G) = \varphi^{-1}(\mathrm{contr}(\rho_T))$.

Proof. By Remark 11.23, we have

$$\mathrm{contr}(\varphi) \sqcup \varphi^{-1}(\mathrm{contr}(\rho_T)) = \mathrm{contr}(\rho_G) \sqcup \rho_G^{-1}(\mathrm{contr}(\varphi_0)).$$

We are done, since by definition of DT-morphism $\mathrm{contr}(\varphi)$ and $\mathrm{contr}(\varphi_0)$ are empty. □

Lemma 12.18. *Let $(\varphi : G \to T, m_\varphi)$ and $(\varphi_0 : G_0 \to T_0, m_0)$ be DT-morphisms, and (ρ_G, ρ_T) a pair of specialization morphisms such that Diagram 12.19 (a) commutes. The pair (ρ_G, ρ_T) is a specialization of DT-morphisms if and only if Diagram 12.19 (b) commutes.*

$$\begin{array}{ccc} G & \xrightarrow{\rho_G} & G_0 \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi_0} \\ T & \xrightarrow{\rho_T} & T_0 \end{array} \qquad\qquad \begin{array}{ccc} E(G_0) & & \\ & \searrow^{m_0} & \\ {\scriptstyle\rho_G^{-1}}\downarrow & & \mathbb{Z}_{\geq 0} \\ & \nearrow_{m_\varphi} & \\ E(G) & & \end{array}$$

(a) *(b)*

Diagram 12.19

Remark 12.20. By Remark 11.16, if both ρ_G and ρ_T contract zero edges, then Definition 12.16 reduces to φ being isomorphic to φ_0. By Lemma 12.17 we have that $\mathrm{contr}(\rho_G) = \varphi^{-1}(\mathrm{contr}(\rho_T))$, so if $\mathrm{contr}(\rho_T) = \varnothing$ then $\mathrm{contr}(\rho_G) = \varnothing$. Also, since φ is surjective, $\varphi^{-1}(\mathrm{contr}(\rho_T))$ is empty only if ρ_T is empty, hence it is enough that either ρ_G or ρ_T contracts zero edges to conclude that we have an isomorphism. △

Lemma 12.21. *A composition of specializations of DT-morphisms is a specialization of DT-morphisms as well.*

Proof. This follows from two facts. First, by Lemma 11.18 a composition of specialization morphisms is a specialization morphism. Second, Diagram 12.19 (a) and (b) behave well under composition. □

Definition 12.22. The category $\mathrm{DTM}^d_{g\to 0\to 0}$ has as objects degree-d DT-morphisms from genus-g connected weighted graphs to trees, and as morphisms the specializations of DT-morphisms.

Now, given $\varphi : (G, w) \to T$ in $\mathrm{DTM}^d_{g\to 0\to 0}$ we choose a subset of edges of T to contract.

Construction 12.23. Given a DT-morphism $\varphi : (G, w) \to T$ with index map m_φ and a subset S of $E(T)$, the contraction $\varphi/S : (G/\varphi^{-1}(S), w_{\varphi^{-1}(S)}) \to T/S$ is the unique graph morphism such that Diagram 11.22 commutes. Such morphism exists by Lemma 11.21. To φ/S we associate the index map $m_0 = m_\varphi \circ \rho_G^{-1}$. Thus, m_0 satisfies Diagram 12.19 (b). So by Lemma 12.18 the pair $(\varphi/S, m_0)$ is an indexed branched cover.

$$\begin{array}{ccc} (G, w) & \xrightarrow{\quad\varphi\quad} & T \\ {\scriptstyle\rho_{\varphi^{-1}(S)}}\downarrow & & \downarrow{\scriptstyle\rho_S} \\ (G/\varphi^{-1}(S), w_{\varphi^{-1}(S)}) & \xrightarrow{\varphi/S} & T/S. \end{array}$$

Diagram 12.24

Remark 12.25. In Part I the indexed branched cover φ/S was called *limit of φ at S*, based on the geometrical picture of the deformation result proved there. △

It remains to show that the RH-inequality is satisfied to conclude that $(\varphi/S, m_0)$ is a DT-morphism. This is implied by the following result.

Lemma 12.26. *Let $(\rho_G, \rho_T) : [\varphi : (G, w) \to T] \to [\varphi_0 : (G_0, w_0) \to T_0]$ be a specialization of DT-morphisms. For every A_0 in $V(G_0)$ we have that*

$$r_{\varphi_0}(A_0) = \sum_{A \in V(\rho_G^{-1}(A_0))} r_\varphi(A).$$

Proof. Let $A_1, A_2, \ldots, A_q$ be the vertices of $\rho_G^{-1}(A_0)$, and set $w_0 = \varphi_0(A_0)$. Restrict φ to $\rho_{\widetilde{S}}^{-1}(A_0)$ to get an indexed branched cover $\psi : \rho_{\widetilde{S}}^{-1}(A_0) \to \rho_S^{-1}(w_0)$. By (the proof of the previous lemma?) $\deg \psi = m_0(A_0)$. By Lemma 12.2 we have

$$\begin{aligned}\sum_{q=1}^{r} r_\psi(A_i) &= 2(\deg \psi + g(\rho_G^{-1}(A_0)) - 1 - \deg(\psi) \cdot g(\rho_T^{-1}(A_0))) \\ &= 2(m_0(A_0) + w_G(A_0) - 1 - m_0(A_0) \cdot w_T(\varphi(A_0))).\end{aligned}$$

We calculate:

$$\begin{aligned}\sum_{i=1}^{r} r_\varphi(A_i) &= \sum_{i=1}^{r} 2(m_\varphi(A_i) + w_G(A_i) - 1 - m_\varphi(A_i) \cdot w_T(\varphi(A_i))) \\ &- \sum_{i=1}^{r} \sum_{\substack{A_i \lessdot e \\ e \in \rho_G^{-1}(A_0)}} (m_\varphi(e) - 1) - \sum_{i=1}^{r} \sum_{\substack{A_i \lessdot e \\ e \notin \rho_G^{-1}(A_0)}} (m_\varphi(e) - 1) \\ &= \sum_{q=1}^{r} r_\psi(A_i) - \sum_{i=1}^{r} \sum_{\substack{A_i \lessdot e \\ e \notin \rho_G^{-1}(A_0)}} (m_\varphi(e) - 1) \\ &= 2(m_0(A_0) + w_G(A_0) - 1 - m_0(A_0) \cdot w_T(\varphi(A_0))) \\ &- \sum_{A_0 \lessdot e_0} (m_\varphi(e_0) - 1) \\ &= r_0(A).\end{aligned}$$

□

Once again, as in Subsection 11.3, Construction 12.23 gives all the morphisms in $\mathrm{DTM}^d_{g \to 0 \to 0}$. This follows from an analogue to Lemma 11.24.

Lemma 12.27. *Let $(\rho_G, \rho_T) : \varphi \to \varphi_0$ be a specialization of DT-morphisms. Choose any $S \subset \mathrm{contr}(\rho_T)$, and set $Q = \varphi^{-1}(S_T)$. There exists specializations $\widetilde{\rho}_G, \widetilde{\rho}_T$ such that Diagram 12.28 (a) and (b) commute.*

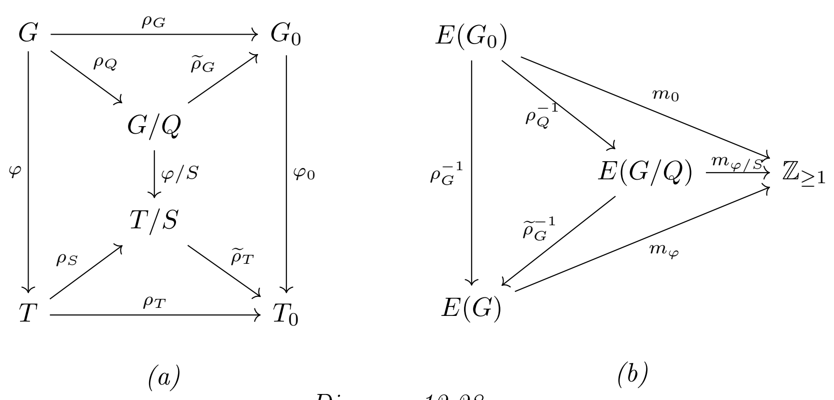


Diagram 12.28

Proof. That $\widetilde{\rho}_G$ and $\widetilde{\rho}_T$ exist and are unique follows from Lemma 11.24. The map φ/S is given by Construction 12.23. Thus, we know that the outer square, the top and bottom triangle, and the left trapezoid in Diagram 12.28 (a) commute. A standard diagram chase gives that $\widetilde{\rho}_T \circ \varphi/S \circ \rho_Q = \varphi_0 \circ \widetilde{\rho}_G \circ \rho_Q$. Since ρ_Q is surjective, we get that $\widetilde{\rho}_T \circ \varphi/S = \varphi_0 \circ \widetilde{\rho}_G$, so Diagram 12.28 (a) commutes. Similarly in Diagram 12.28 (b), we know by construction that the outer, the top, and the left triangles commute. This gives, via a diagram chase, that $m_{\varphi/S} \circ \rho_Q^{-1} = m_\varphi \circ \widetilde{\rho}_G^{-1} \circ \rho_Q^{-1}$, hence $m_{\varphi/S} = m_\varphi \circ \widetilde{\rho}_G^{-1}$ and the diagram commutes. □

If we set $S = \operatorname{contr}(\rho_T)$, then Lemma 12.27 implies that a specialization of DT-morphisms $\rho : \varphi \to \varphi_0$ is isomorphic to $\varphi/\operatorname{contr}(\rho_T)$.

Example 12.29 (Final object in $\mathrm{DTM}^d_{g\to 0\to 0}$)**.** Consider $d \in \mathbb{Z}_{\geq 1}$ and the final objects G_f and T_f of WG_g and WG_0, respectively; see Example 11.27. The pair of maps $(\varphi_\mathrm{f}, m_\mathrm{f})$ sending $A \mapsto v$ and $A \mapsto d$, respectively, is a degree-d DT-morphism that is the final object of $\mathrm{DTM}^d_{g\to 0\to 0}$. For any $\varphi : G \to T$ the pair $(\rho_{E(G)}, \rho_{E(T)})$ is a morphism to φ_f. ⋆

12.5 Tropical modification for tropical morphisms

Now we consider *tropical modification* of a DT-morphism $\varphi : (G, w) \to T$.

Definition 12.30. Let (φ, m_φ) be a DT-morphism. A fibre $\varphi^{-1}(x)$ is *dangling* if all its elements are dangling.

We delete the dangling fibres by considering the subset $S \subset E(T)$ of those e such that $\varphi^{-1}(e)$ is dangling.

given by the $e' \in E(G')$ such that $\varphi^{-1}(e')$ is not a dangling fibre. Let $\hat{G}'$ be the subgraph of G' induced by $\hat{E}$. Let $\hat{G}$ be the unique connected component of $\varphi^{-1}(\hat{G}')$ that has non-zero genus. The *deletion of dangling fibres* $\hat{\varphi}$ of φ is the restriction $\varphi|_{\hat{G}} : \hat{G} \to \hat{G}'$. We identify φ with $\hat{\varphi}_{\text{ess}}$ under tropical modification. This means that tropical modification allows to attach or delete dangling fibres to φ, and subdivide the model; or undo a subdivision. For $\Phi = (\varphi, \ell)$ the deletion of dangling fibres is $\hat{\Phi} = (\hat{\varphi}, \ell|_{\hat{G}'})$. Now we give an analogue of Lemma 11.35 for tropical morphisms

We delete the dangling fibres by considering the subset $S \subset E(T)$ given by the $e' \in E(G')$ such that $\varphi^{-1}(e')$ is not a dangling fibre. Let $\hat{G}'$ be the subgraph of G' induced by $\hat{E}$. Let $\hat{G}$ be the unique connected component of $\varphi^{-1}(\hat{G}')$ that has non-zero genus. The *deletion of dangling fibres* $\hat{\varphi}$ of φ is the restriction $\varphi|_{\hat{G}} : \hat{G} \to \hat{G}'$. We identify φ with $\hat{\varphi}_{\text{ess}}$ under tropical modification. This means that tropical modification allows to attach or delete dangling fibres to φ, and subdivide the model; or undo a subdivision. For $\Phi = (\varphi, \ell)$ the deletion of dangling fibres is $\hat{\Phi} = (\hat{\varphi}, \ell|_{\hat{G}'})$. Now we give an analogue of Lemma 11.35 for tropical morphisms.

Definition 12.31. Let Φ be a tropical morphism and $\hat{\Phi}$ its deletion of dangling fibres. The *combinatorial type* of Φ is the model $\hat{\varphi}_{\text{ess}}$ of $\hat{\Phi}$ constructed using Construction 12.10.

Definition 12.32. A *combinatorial type of DT-morphisms* is a DT-morphism $\varphi : G \to G'$ without dangling fibres and such that $\sum_{A \in \varphi^{-1}(v)} r_\varphi(A) \geq 1$ for every divalent v in $V(G')$.

Lemma 12.33 (Lemma 3.23 in Part I). *Let Φ be a tropical morphism, and $\bar{\Phi} : \bar{\Gamma} \to \bar{\Gamma}'$ be a tropical modification of Φ, such that $\bar{\Phi}$ has a model $\bar{\varphi} : \bar{G} \to \bar{G}'$ which is a combinatorial type of DT-morphisms. The map $\bar{\Phi}$ is isomorphic to $\hat{\Phi}$ and $\bar{\varphi}$ is isomorphic to $\hat{\varphi}_{ess}$.* □

Combinatorial types of DT-morphismsare canonical representatives in the equivalence class under tropical modification: the discussion before Definition 12.31 tells us how to construct them, and Lemma 12.33 ensures their uniqueness.

12.6 The edge-length matrix

We associate a matrix to the edge-length map A_φ, and study how it changes under specialization morphisms. These observations enable the calculations in Section 14.

Definition 12.34. Let (G, w) be a weighted graph. The *standard basis for* $\operatorname{span}_{\mathbb{R}} C_{(G,w)}$ is the set $\{y_e \colon e \in E(G)\}$, where $y_e : G \to \mathbb{R}_{\geq 0}$ is a map with $y_e(x) = 1$ when $x = e$, and $y_e(x) = 0$ otherwise.

Using the standard bases on $\operatorname{span}_{\mathbb{R}} C_T$ and $\operatorname{span}_{\mathbb{R}} C_{H(\varphi)}$ we write A_φ as a matrix whose rows are indexed by $E(H(\varphi))$ and columns by $E(T)$. An entry a_{ht} of this matrix is a rational number given by:

$$a_{ht} = \sum_{\substack{e \in h \\ \varphi(e)=t}} 1/m_\varphi(e), \tag{12.9}$$

where the sum is zero if the index set is empty. So we have

$$y(h) = \sum_{t \in E(T)} a_{ht} z(t). \tag{12.10}$$

For writing down the edge-length matrix we need an ordering on the rows and columns. We are quite careful when choosing these orders, because we need to keep compatibility between distinct DT-morphisms, as later on we take a determinant of the matrix.

Definition 12.35. Let $\varphi : (G, w) \to T$ be a DT-morphism, and $H(\varphi)$ the combinatorial type from Construction 12.14. A *labelling* λ^φ of φ is a pair of injective maps $\lambda^\varphi_T : E(T) \to \mathbb{N}$ and $\lambda^\varphi_H : H(\varphi) \to \mathbb{N}$.

Remark 12.36. In Definition 12.35 the key property we wish for the codomain of a labelling is to have a total order, with this we induce a total order on the domains via pullback, i.e. $e \leq e'$ if $\lambda(e) \leq \lambda(e')$. We have chosen $\mathbb{N}$ for simplicity of exposition. △

Definition 12.37. Let $\mathcal{U}$ be a set of DT-morphisms. We say that $\mathcal{U}$ is compatibly labelled by $(\lambda^\varphi_T : E(T) \to \mathbb{N}, \lambda^\varphi_H : E(H(\varphi)) \to \mathbb{N})\colon \varphi \in$ if the following two conditions are satisfied:

(a) Given a specialization morphism $\rho : \varphi \to \varphi_0$ we have that $\lambda^{\varphi_0} = \lambda^\varphi \circ \rho$.

(b) Given two specializations $\rho_1 : \rho_1 \to \rho_0$ and $\rho_2 : \rho_2 \to \rho_0$, such that $H(\varphi_1)$ has the same number of edges as $H(\varphi_2)$, and T_1 has the same number of edges as T_2, we have that the orders of ρ_1 are isomorphic to those of ρ_2.

Lemma 12.38. *Let $\mathcal{U}$ be a graded poset of DT-morphisms. If $\mathcal{U}$ is strongly connected and φ is maximal in $\mathcal{U}$, then a choice of labelling for φ induces a family that compatibly labells $\mathcal{U}$.*

Lemma 12.39. *Let $\mathcal{U}$ be a set of DT-morphisms that is compatibly labelled. If $\rho : \varphi \to \varphi_0$ is a specialization, then A_0 is obtained from A_φ by deleting the rows indexed by* $\operatorname{contr}(\rho_H)$*, and the columns indexed by* $\operatorname{contr}(\rho_T)$.

12.7 Full rank and change-minimal DT-morphisms

Recall that given a combinatorial type H, our goal is to find, and eventually enumerate, DT-morphisms $\varphi : (G, w) \to T$ such that $H(\varphi)$ is isomorphic to H, while simultaneously minimizing $\deg \varphi$ and maximizing $\dim C^{\mathrm{src}}_{\varphi}$. Equation (12.8) says that $\dim C^{\mathrm{src}}_{\varphi}$ is bounded above by a linear polynomial in $\deg \varphi$ and $g(\varphi)$. We study those φ that attain the bound.

Definition 12.40. Let $\varphi : (G, w) \to T$ be a map in $\mathrm{DTM}^d_{g\to 0\to 0}$, and set $d = \deg \varphi$ and $g = g(G, w)$. We say that φ is *top-dimensional* if

$$\dim C^{\mathrm{src}}_{\varphi} = \#(E(T)) = 2g + 2d - 5. \tag{12.11}$$

To study Equation (12.11) we split it into two conditions. The first equality is equivalent to saying that φ has full rank, the second that φ is change-minimal, defined as follows:

Definition 12.41. Let $\varphi : (G, w) \to T$ be a map in $\mathrm{DTM}^d_{g\to 0\to 0}$.

- We say that φ has *full rank* if the edge-length map A_φ has full-rank.
- A vertex v of T is *change-minimal* if $\operatorname{ch} v + \operatorname{val} v = 3$; and φ is *change-minimal* if all vertices of T are change-minimal.

The study of the above defined properties is the focus of [DV20, Section 4]. We summarize several of those results, since they are used again extensively in Section 14.

Remark 12.42 (full rank)**.** When A_φ has full rank a point $y \in C^{\mathrm{src}}_{\varphi}$ not only corresponds to the metric graph $((H(\varphi), w_H), y)$, but also to the tropical morphism $(\varphi, A^{-1}_{\varphi}(y))$. Since the dimension of the fibre $A^{-1}_{\varphi}(y)$ equals $\dim \ker A_\varphi$, if we expect a finite count for the number of *nice enough* tropical morphisms which realize a given $[(\Gamma, w)]_{\cong_{\mathrm{trop}}}$, then we must work with full-rank DT-morphisms. △

On the other hand, being change-minimal also is a natural condition, and implies several properties for φ.

Lemma 12.43. *Let $(\rho_G, \rho_T) : [\varphi : (G, w) \to T] \to [\varphi_0 : (G_0, w_0) \to T_0]$ be a specialization of DT-morphisms. For w_0 in $V(T_0)$ we have that*

$$\operatorname{ch} w_0 + \operatorname{val} w_0 - 3 = (\#(V(\rho_T^{-1}(w_0))) - 1) + \sum_{v \in V(\rho_T^{-1}(w_0))} (\operatorname{ch} v + \operatorname{val} v - 3) \tag{12.12}$$

Proof. By Lemmas 11.36 and 12.26 we have that

$$\begin{aligned} \operatorname{ch} w_0 + \operatorname{val} w_0 - 3 &= \sum_{v \in V(\rho_T^{-1}(w_0))} \operatorname{ch} v + \sum_{v \in V(\rho_T^{-1}(w_0))} \operatorname{val} v - 2(\#(V(\rho_T^{-1}(w_0))) - 1) - 3 \\ &= \sum_{v \in V(\rho_T^{-1}(w_0))} (\operatorname{ch} v + \operatorname{val} v - 3) + (\#(V(\rho_T^{-1}(w_0))) - 1). \end{aligned}$$ □

By Equation (12.12) if t is in $\operatorname{contr}(\rho_T)$, then $\operatorname{ch} \rho_T(t) + \operatorname{val} \rho_T(t) - 3 \geq 1$. Therefore, change-minimal DT-morphisms cannot be the target of non-trivial specialization morphisms, meaning they are maximal elements in $(\mathrm{DTM}^d_{g\to 0\to 0})^{\mathrm{op}}$. We also have:

Lemma 12.44. *Let $(\rho_G, \rho_T) : [\varphi : (G, w) \to T] \to [\varphi_0 : (G_0, w_0) \to T_0]$ be a specialization of DT-morphisms. If φ is top-dimensional and T is a tree, then we have that*

$$\sum_{w_0 \in V(T_0)} (\operatorname{ch} w_0 + \operatorname{val} w_0 - 3) = \#(\operatorname{contr}(\rho_T)). \tag{12.13}$$

Proof. Consider Equation (12.12). Since φ is top-dimensional, we have that $(\operatorname{ch} v + \operatorname{val} v - 3)$ equals 0. Since T is a tree, $\#(V(\rho_T^{-1}(w_0))) - 1 = \#(E(\rho_T^{-1}(w_0)))$. The result follows now from summing over all $w_0 \in V(T_0)$. □

Lemma 12.45. *Let $\varphi : (G, w) \to T$ be in $\mathrm{DTM}^d_{g\to 0\to 0}$. If φ is change-minimal, then*

$$\text{max-val}(T) \leq 3.$$ □

Example 12.46. One might wonder whether max-val$(H)(\varphi) \leq 3$ for a change-minimal φ. For $g' \geq 2$, a top-dimensional φ in $\mathcal{G}^{\mathrm{trop}}_{2g'\to 0,g'+1}$ satisfies that

$$\dim C^{\mathrm{src}}_{\varphi} = 2g + 2d - 5 = 3g - 3. \tag{12.14}$$

Equation (12.14) implies that the combinatorial type $H(\varphi)$ is indeed trivalent. This conclusion is not necessarily true when $2g + 2d - 5 \neq 3g - 3$. Figure 12.1 shows a DT-morphism that has full rank, is change-minimal, hence is a top-dimensional cone of $\mathcal{G}^{\mathrm{trop}}_{3\to 0,2}$, but its combinatorial type is not trivalent. Thus, φ being maximal in $(\mathrm{DTM}^d_{g\to 0\to 0})^{\mathrm{op}}$ does not necessarily imply that $H(\varphi)$ is maximal in $(\mathcal{M}^{\mathrm{trop}}_g)^{\mathrm{op}}$. ⋆

Figure 12.1: A top-dimensional DT-morphism in $\mathcal{G}^{\mathrm{trop}}_{3\to 0,2}$ with non-trivalent combinatorial type.

Remark 12.47 (deformation procedure)**.** The introduction of [DV20, Section 6] regards the main example of Part I as a movie that starts with a metric graph $\widetilde{\Gamma}$ and a tropical morphism $\Phi = ((\varphi, m_\varphi), A_\varphi^{-1}(y))$ whose source is in $[\widetilde{\Gamma}]_{\cong_{\mathrm{trop}}}$. The movie features how Φ is deformed when one edge length of y is increased and how the combinatorial structure captured by $\varphi : G \to T$ changes nine times. The changes are *local*, they happen at a small subgraph $\mathcal{U}$ of T and at the fibres of φ above $\mathcal{U}$. A new DT-morphism φ' is produced via this process and the main technical difficulty is to check that φ' has full rank. This is resolved with an argument that relies on a balancing condition and a lengthy case analysis. The case analysis itself would be too big if we didn't have conditions that can be checked locally, such as being change minimal, that help to construct the candidates for having full rank. △

The combination of being change-minimal and having full rank implies the following combinatorial conditions, which can be checked locally.

Definition 12.48. Let $\varphi : (G, w) \to T$ be in $\mathrm{DTM}^d_{g\to 0\to 0}$.

- A vertex A of G satisfies the *no-return* condition if there are at least two non-dangling edges in $E(A)$ above different edges of T, i.e. $\#(\varphi(\text{nd-}E(A))) \geq 2$. The map φ satisfies the *no-return* condition if all $A \in \{B \in \text{nd-}V(G) \colon \operatorname{val}\varphi(B) \neq 1\}$ satisfy no-return.
- An edge $h = \langle A_0, e_1, \ldots, e_\mu, A_\mu\rangle$ of $H(\varphi)$ satisfies the *pass-once* condition if φ restricted to the set $\{e_i \in h \colon \varphi(e_i) \text{ not incident to a leaf}\}$ is injective. The map φ satisfies the *pass-once* condition if all edges of $H(\varphi)$ satisfy pass-once.
- The map φ satisfies the *dangling-no-glue* condition if $m_\varphi(x) = 1$ for all dangling x in G.

Lemma 12.49. *Let φ be a DT-morphism. If φ has full rank and is change-minimal, then φ satisfies the dangling-no-glue, no-return, and pass-once conditions.* □

Proof. The proof follows closely the lines of those for Lemmas 4.17, 4.20, 4.24 in Part I. A few extra considerations must be made now that vertex weights are allowed. □

We have a combinatorial description of the fibres of a change-minimal φ that satisfies dangling-no-glue.

Lemma 12.50 (nd. r_φ formula)**.** *Let φ be a DT-morphism that satisfies dangling-no-glue. We have that*

$$r_\varphi(A) = \text{nd-wtval}\, A - 2 + 2m_\varphi(A) - \sum_{e \in \text{nd-}E(A)} m_\varphi(e).$$

Proof. Similar proof to [DV20, Lemma 4.18] □

Proposition 12.51 (local properties)**.** *Let $\varphi : (G, w) \to T$ be a change-minimal DT-morphism that satisfies the dangling-no-glue condition. For A in* nd-$V(G)$ *such that* nd-val $A \leq 3$ *and $w(A) = 0$ exactly one of the following cases happens:*

(r0) If $r_\varphi(A) = 0$, then $\varphi|_{\text{nd-}E(A)}$ is injective and nd-val $A \leq$ val $\varphi(A)$.

(r1) If $r_\varphi(A) = 1$, then $\varphi(A)$ is divalent, val $A = 3$, *and* nd-val A *is 2 or 3. Moreover, if we write $E(A) = \{e, e', e''\}$ with $m_\varphi(e) \geq m_\varphi(e') \geq m_\varphi(e'')$, then $m_\varphi(e) = m_\varphi(A) = m_\varphi(e') + m_\varphi(e'')$.*

(r2) If $r_\varphi(A) = 2$, then $\varphi(A)$ is monovalent, nd-val $A = 2$, *and* val $A = 2$. *Moreover, if we write $E(A) = \{e, e'\}$, then $m_\varphi(e) = m_\varphi(e') = 1$.*

Moreover, if G_{dan} is a connected subgraph of G such that all edges are dangling in G, then φ restricted to G_{dan} is injective.

Proof. Since $w(A) = 0$ the setting reduces to that of Proposition 4.21 in Part I. □

Moreover, for DT-morphisms satisfying some extra conditions we can characterize the edge-length matrix from Subsection 12.6.

Lemma 12.52. *Let $\varphi : (G, w) \to T$ be a DT-morphism such that if v is a leaf then v is change-minimal and any vertex in the fibre $\varphi^{-1}(v)$ has weight 0. The map φ satisfies the pass-once condition if and only if for all h in $E(H(\varphi))$ and t in $\varphi(h)$ the entry a_{ht} is*

(a) 2 if $\varphi(h)$ contains a leaf of T.

(b) $1/m_\varphi(e)$ for $e \in h \cap \varphi^{-1}(t)$ otherwise.

Proof. See Proposition 4.25 in Part I. □

Finally, specialization morphisms preserve the properties of having full rank, and satisfying the pass-once and the dangling-no-glue conditions.

Lemma 12.53. *Specialization morphisms preserve having full rank, and satisfying dangling-no-glue and pass-once.*

Therefore, we define the following category. All its objects have full rank, and satisfy pass-once and dangling-no-glue. Moreover the maximal elements of its opposite category are change-minimal and satisfy no-return.

Definition 12.54. Let $\mathcal{G}^{\text{trop}}_{g \to 0, d}$ denote the subcategory of $\text{DTM}^d_{g \to 0 \to 0}$ that is the closure of the set of top-dimensional DT-morphisms, i.e. all DT-morphisms that arise as a specialization of some degree-d top-dimensional DT-morphisms with genus-g connected source and target a tree.

12.8 A space parametrizing tropical morphisms

We now describe a contravariant functor $C_-^{\mathrm{src}} : (\mathrm{DTM}^d_{g\to 0})^{\mathrm{op}} \to \mathrm{POLY}^f_{\mathbb{Z}}$, define our main tropical moduli space as a restriction of C_-^{src} to $\mathcal{G}^{\mathrm{trop}}_{g\to 0,d}$, and argue that $\mathcal{G}^{\mathrm{trop}}_{g\to 0,d} \to \mathrm{POLY}^f_{\mathbb{Z}}$ is a polyhedral space.

As a first step, given a specialization $(\rho_G, \rho_T) : \varphi \to \varphi_0$ we need a face morphism $C^{\mathrm{src}}_{\varphi_0} \to C^{\mathrm{src}}_{\varphi}$, so we construct a specialization morphism $\rho_H : H(\varphi) \to H(\varphi_0)$.

Construction 12.55. Let $\rho : (G, w) \to (G_0, w_0)$ be a specialization morphism. We construct the map $\rho_H : (H(G, w), w_H) \to (H(G_0, w_0), w_0)$ given on vertices by $A \mapsto \rho_G(A)$ and on edges by

$$\langle B_0, e_1, B_1, \dots, e_\mu, B_\mu\rangle \mapsto \begin{cases} \rho_G(B_0) & \text{if } \rho(B_i) = \rho(e_j) \text{ for all } i, j, \\ \langle \rho_G(B_0), \dots, \rho_G(B_\mu)\rangle & \text{otherwise.} \end{cases}$$

The following two results show that the map ρ_H has good properties.

Lemma 12.56. *Let $\rho : (G, w) \to (G_0, w_0)$ be a specialization morphism. For any A_0 in $V(G_0)$ we have*

$$\text{nd-wtval}\, A_0 = 2 + \sum_{A \in \rho^{-1}(A_0)} (\text{nd-wtval}\, A - 2). \tag{12.15}$$

Proof. By Lemma 11.31 an edge e_0 in $E(G_0)$ is dangling if and only if $\rho^{-1}(e_0)$ in $E(G)$ is dangling. Using this fact, and following the first lines of the proof of Lemma 11.36, we get a non-dangling version of Equation (11.7), namely

$$\text{nd-val}_{G_0} B_0 = \sum_{B \in \rho^{-1}(B_0)} \text{nd-val}_G B - \text{nd-val}_{\rho^{-1}(B_0)} B. \tag{12.16}$$

The remaining of the proof proceeds as for Lemma 11.36, and using the fact that tropical modification preserves genus. □

Lemma 12.57. *Let $\rho : (G, w) \to (G_0, w_0)$ be a specialization morphism. The map ρ_H from Construction 12.55 is well defined and is a specialization morphism.*

Proof. If A is a vertex of $H(G, w)$, then by Lemma 12.56 we have that

$$\text{nd-wtval}_{G_0} \rho(A) = 2 + \sum_{B \in \rho^{-1}(\rho(A))} (\text{nd-wtval}\, B - 2) \geq 2 + \text{nd-wtval}\, A - 2 \geq 3,$$

so $\rho(A)$ is a vertex of $H(G_0, w_0)$ and the map ρ_H is well defined. Moreover, ρ_H restricted to $\rho_H^{-1}(E(H(G_0, w_0)))$ is injective and surjective because for an edge h_0 of $H(G_0, w_0)$, and an edge e_0 in h_0, there is exactly one edge h of $H(G, w)$ such that $\rho^{-1}(e_0) \in h$. This h is independent of the choice of e_0 in h_0, that is it is the unique edge in $H(G, w)$ such that $\rho_H(h) = h_0$. Also, we have that $\rho_H(A_0)$ is connected since $\rho(A_0)$ is connected. Hence ρ_H is a contraction. Finally, we note that $\mathrm{nd}(\rho^{-1}(A_0))$ is an edge subdivision of $\rho_H^{-1}(A_0)$. Since ρ is a specialization morphism we have that

$$\begin{aligned} w_0(A_0) &= g(\rho^{-1}(A_0), w|_{V(\rho^{-1}(A_0))}) \\ &= g(\mathrm{nd}(\rho^{-1}(A_0)), w|_{V(\mathrm{nd}(\rho_H^{-1}(A_0)))}) \\ &= g(\rho_H^{-1}(A_0), w|_{V(\rho_H^{-1}(A_0))}). \end{aligned}$$

Therefore ρ_H is a specialization. □

Remark 12.58. Lemma 12.57 implies that an isomorphism $\gamma : (G, w) \to (G_0, w_0)$ descends to an isomorphism $\widetilde{\gamma} : H(G, w) \to H(G_0, w_0)$. On the other hand, the examples in Section 4 of Part I show there are specializations $(\rho_G, \rho_T) : \varphi \to \varphi_0$ that contract edges, yet the specialization ρ_H associated to ρ_G is still an isomorphism. △

Next, we relate the edge-length maps of φ and φ_0 via their entries of the edge-length matrix.

Lemma 12.59. *Let $(\rho_G, \rho_T) : (\varphi, m_\varphi) \to (\varphi_0, m_0)$ be a specialization of DT-morphisms. For any pair (h_0, t_0), whre $h_0 \in E(H(\varphi_0))$ and $t_0 \in E(T_0)$, we have that*

$$a_{h_0 t_0} = a_{\rho_H^{-1}(h_0)\rho_T^{-1}(t_0)},$$

where $a_{h_0 t_0}$ is given by Equation (12.9) on Page 130.

Proof. Set $h = \rho_H^{-1}(h_0)$ and $t = \rho_T^{-1}(t_0)$. By Equation (12.9) we calculate:

$$\begin{aligned} a_{ht} &= \sum_{\substack{e \in h \\ \varphi(e)=t}} \frac{1}{m_\varphi(e)} = \sum_{\substack{\rho_G^{-1}(e_0) \in h \\ \varphi(\rho_G^{-1}(e_0))=t}} \frac{1}{m_\varphi \circ \rho_G^{-1}(e_0)} \\ &= \sum_{\substack{\rho_G^{-1}(e_0) \in \rho_H^{-1}(h_0) \\ \rho_T^{-1}(\varphi_0(e_0))=\rho_T^{-1}(t_0)}} \frac{1}{m_0(e_0)} = \sum_{\substack{e_0 \in h_0 \\ \varphi_0(e_0)=t_0}} \frac{1}{m_0(e_0)} = a_{h_0 t_0}. \end{aligned}$$

We have used that, if $e \in h$ is such that $\varphi(e) = \rho_T^{-1}(t_0)$, then there is a unique e_0 in $E(G_0)$ such that $e = \rho_G^{-1}(e_0)$. This is true because $\varphi(e) = \rho_T^{-1}(t_0)$ implies $t_0 = \rho_T(\varphi(e)) = \varphi_0(\rho_G(e))$, so $e_0 = \rho_G(e)$ is in $\varphi_0^{-1}(t_0)$, hence e_0 is an edge because $\varphi_0^{-1}(t_0) \subset E(G_0)$. ☐

Recall that in Definition 11.38 we introduced the pullback ρ^* and the push-forward ρ_* associated to a specialization morphism ρ. From Lemma 12.59 we get commutative diagrams that relate the pullback and the push-forward of ρ_T and ρ_H.

Lemma 12.60. *Let $(\rho_G, \rho_T) : \varphi \to \varphi_0$ be a specialization of DT-morphisms. Both squares in Diagram 12.61 commute.*

$$\begin{array}{ccc} \rho_T^*(\operatorname{span}_{\mathbb{R}} C_T) & \xrightarrow{\ \rho_{T*}\ } & \operatorname{span}_{\mathbb{R}} C_{T_0} \\ \downarrow{\scriptstyle A_\varphi} & & \downarrow{\scriptstyle A_{\varphi_0}} \\ \operatorname{span}_{\mathbb{R}} C_{(H(\varphi),w_H)} & \xrightarrow{\ \rho_{H*}\ } & \operatorname{span}_{\mathbb{R}} C_{(H(\varphi_0),w_0)} \end{array}$$

(a)

$$\begin{array}{ccc} \operatorname{span}_{\mathbb{R}} C_T & \xleftarrow{\ \rho_T^*\ } & \operatorname{span}_{\mathbb{R}} C_{T_0} \\ \downarrow{\scriptstyle A_\varphi} & & \downarrow{\scriptstyle A_{\varphi_0}} \\ \operatorname{span}_{\mathbb{R}} C_{(H(\varphi),w_H)} & \xleftarrow{\ \rho_H^*\ } & \operatorname{span}_{\mathbb{R}} C_{(H(\varphi_0),w_0)} \end{array}$$

(b)

Diagram 12.61

Proof. Let $z \in \operatorname{span}_{\mathbb{R}} C_T$ and $h_0 \in E(H(\varphi_0))$. By definition of the push-forward and Equation (12.10) on Page 130 we have that

$$\begin{aligned} (\rho_{H*}(A_\varphi(z)))(h_0) &= (A_\varphi(z) \circ \rho_H^{-1})(h_0) \\ &= \sum_{t \in E(T)} a_{\rho_H^{-1}(h_0)t} z(t) \\ &= \sum_{t \in \rho_T^{-1}(E(T_0))} a_{\rho_H^{-1}(h_0)t} z(t) + \sum_{\substack{t \in \rho_T^{-1}(V(T_0)) \\ t \in E(T)}} a_{\rho_H^{-1}(h_0)t} z(t). \end{aligned}$$

If z is in $\rho_T^*(\operatorname{span}_{\mathbb{R}} C_{T_0}) \subset \operatorname{span}_{\mathbb{R}} C_T$, then we get 0 for the second sum in the last line of the previous calculation. By Lemma 12.59 we have:

$$\begin{aligned} (A_{\varphi_0}(\rho_{T*}(z)))(h_0) &= \sum_{t_0 \in E(T_0)} a_{h_0 t_0} z \circ (\rho_T^{-1}(t_0)) \\ &= \sum_{t_0 \in E(T_0)} a_{\rho_H^{-1}(h_0)\rho_T^{-1}(t_0)} z(\rho_T^{-1}(t_0)) = \sum_{t \in \rho_T^{-1}(E(T_0))} a_{\rho_H^{-1}(h_0)t} z(t). \end{aligned}$$

Hence, Diagram 12.61 (a) commutes. Namely, $A_\varphi \circ \rho_{T*} = \rho_{H*} \circ A_{\varphi_0}$. Now, recall from the proof of Lemma 11.40 that $\rho_{T*} \circ \rho_T^*$ is the identity on $\operatorname{span}_\mathbb{R} C_T$, and $\rho_H^* \circ \rho_{H*}$ is the identity on $\rho_H^*(\operatorname{span}_\mathbb{R} C_{(H(\varphi),w_H)}) \supset A_{\varphi_0}(\operatorname{span}_\mathbb{R} C_{T_0})$. Thus, composing on the right by ρ_T^* and on the left by ρ_H^* gives that $\rho_H^* \circ A_\varphi = A_{\varphi_0} \circ \rho_T^*$, as desired. □

As a consequence, we get that ρ_H^* induces a face morphism. We consider C_φ^{src} with the following integral structure.

Definition 12.62. Given a DT-morphism φ we define $N^\varphi \subset \operatorname{span}_\mathbb{R} C_\varphi^{\mathrm{src}}$ to be the set of functions with integral values. It is a lattice, and $N^\varphi \cap C_\varphi^{\mathrm{src}}$ corresponds to DT-morphisms whose source have a deletion of dangling elements $((H(\varphi), m_H), A_\varphi(z))$ with integral lengths.

Lemma 12.63. *The map ρ_H^* induces a face morphism from $C_{\varphi_0}^{src}$ to C_φ^{src}.*

Proof. We have that $\rho_H^*(C_{\varphi_0}^{\mathrm{src}}) = \rho_H^*(A_{\varphi_0}(C_{T_0})) = A_\varphi(\rho_T^*(C_{T_0}))$. Since ρ_T^* is a face morphism, $\rho_T^*(C_{T_0})$ is a face of C_T. Since A_φ is a linear map, $A_\varphi(\rho_T^*(C_{T_0})) = \rho_H^*(C_{\varphi_0}^{\mathrm{src}})$ is a face of $A_\varphi(C_T) = C_\varphi^{\mathrm{src}}$. □

Definition 12.64. The *tropical moduli space of top-dimensional DT-morphisms and their specializations* is the functor $(\mathcal{G}_{g\to 0,d}^{\mathrm{trop}})^{\mathrm{op}} \to \mathrm{P{\small OLY}}_\mathbb{Z}^f$ obtained by restricting C_-^{src}.

Lemma 12.65. *The functor $C_-^{src} : (\mathcal{G}_{g\to 0,d}^{trop})^{op} \to \mathrm{P{\small OLY}}_\mathbb{Z}^f$ is a polyhedral space of cones.*

Proof. We check the three conditions from Definition 9.4. Let φ be in $\mathcal{G}_{g\to 0,d}^{\mathrm{trop}}$. Condition (a) is true because all the faces of C_φ^{src} prescribe a subset $S \subset E(T)$ to have zero lengths. This gives rise to the specialization $(\rho_G, \rho_T) : \varphi \to \varphi/S$. The specialization $\rho_H : H(\varphi) \to H(\varphi_0)$ associated to ρ_G gives the desired face inclusion. Condition (b) follows from Lemma 12.27 and an observation similar to Remark 11.28. Condition (c) is true because $\mathrm{DTM}_{g\to 0\to 0}^d$ is already a skeleton category. □

By Equation (12.11) we have that $C_-^{\mathrm{src}} : (\mathcal{G}_{g\to 0,d}^{\mathrm{trop}})^{\mathrm{op}} \to \mathrm{P{\small OLY}}_\mathbb{Z}^f$ has pure dimension $2g + 2d - 5$. Let $\rho : \varphi \to \varphi_0$ be in $\mathcal{G}_{g\to 0,d}^{\mathrm{trop}}$ such that φ is top-dimensional. The dimension of $C_{\varphi_0}^{\mathrm{src}}$ equals $\#(T_0) = 2g + 2d - 5 - \#(\operatorname{contr}(\rho_T))$. Hence, the codimension of φ_0 equals $\#(\operatorname{contr}(\rho_T))$, which by Lemma 12.44 equals the sum of $(\operatorname{ch} w_0 + \operatorname{val} w_0 - 3)$ over all $w_0 \in V(T_0)$.

12.9 The points of $|\mathcal{G}_{g\to 0,d}^{\mathbf{trop}}|$

Following similar steps to those taken in Subsection 11.8, we show that distinct points of $\mathcal{G}_{g\to 0,d}^{\mathrm{trop}}$ correspond to distinct isomorphism classes of weighted tropical morphisms.

Let $\varphi : (G, w) \to T$ be a DT-morphism that is full-rank. We can describe explicitly the points in C_φ which encode isometric DT-morphisms by observing the following. Let $\Phi^{(1)}$ to $\Phi^{(2)}$ be weighted tropical morphisms. Recall that an isomorphism from $\Phi^{(1)}$ to $\Phi^{(2)}$ is a pair of isometries Ψ, Υ such that $\Phi^{(2)} \circ \Psi = \Upsilon \circ \Phi^{(1)}$ and $w^{(1)} = w^{(2)} \circ \Psi$. We have that (Ψ, Υ) is compatible with any choice of vertex set structure for Φ:

Lemma 12.66. *Let $\Phi^{(q)} : (\Gamma^{(q)}, w^{(q)}) \to \Delta^{(q)}$, with $q = 1, 2$, be tropical morphisms, (Ψ, Υ) an isomorphism, and $\mathcal{T}^{(q)} = \Phi^{(q)}(\mathcal{E}_{\Gamma^{(q)}}) \cup \mathcal{E}_{\Delta^{(q)}}$, with $\mathcal{E}_{\Gamma^{(q)}}$, $\mathcal{E}_{\Delta^{(q)}}$ the set of essential vertices of $\Gamma^{(q)}$ and $\Delta^{(q)}$, respectively. We have that:*

1. $\mathcal{T}^{(2)} = \Upsilon(\mathcal{T}^{(1)})$.
2. $(\Phi^{(2)})^{-1}(\Upsilon(\mathcal{S}')) = \Psi((\Phi^{(1)})^{-1}(\mathcal{S}'))$ *for any* $\mathcal{S}' \subset \Delta^{(1)}$.

Proof. Since Ψ, Υ are isometries we have that $\mathcal{E}_{\Gamma^{(2)}} = \Psi(\mathcal{E}_{\Gamma^{(1)}})$ and $\mathcal{E}_{\Delta^{(2)}} = \Upsilon(\mathcal{E}_{\Delta^{(1)}})$. We calculate $\Upsilon(\mathcal{T}^{(1)}) = \Upsilon(\Phi^{(1)}(\mathcal{E}_{\Gamma^{(1)}})) \cup \Upsilon(\mathcal{E}_{\Delta^{(1)}}) = \Phi^{(2)}(\Psi(\mathcal{E}_{\Gamma^{(1)}})) \cup \mathcal{E}_{\Delta^{(2)}} = \mathcal{T}^{(2)}$.

Next, note that $\Phi^{(2)}(\Psi((\Phi^{(1)})^{-1}(\mathcal{S}'))) = \Upsilon(\Phi^{(1)}((\Phi^{(1)})^{-1}(\mathcal{S}))) = \Upsilon(\mathcal{S}')$. We conclude that $\Psi((\Phi^{(1)})^{-1}(\mathcal{S}')) \subseteq (\Phi^{(2)})^{-1}(\Upsilon(\mathcal{S}'))$. The same calculation using $(\Psi^{-1}, \Upsilon^{-1})$ as an isomorphism gives $\Psi^{-1}((\Phi^{(2)})^{-1}(\Upsilon(\mathcal{S}'))) \subseteq (\Phi^{(1)})^{-1}(\mathcal{S}')$, so we are done by applying Ψ on both sides. □

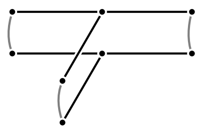
Thus, let $\mathcal{S}' \subset \Delta^{(1)}$ be such that $\mathcal{T}^{(1)} \subset \mathcal{S}'$. Lemma 12.13 gives that $\mathcal{S}'$ is a vertex set for $\Delta^{(1)}$, and $\mathcal{S} = \left(\Phi^{(1)}\right)^{-1}(\mathcal{S}')$ is one for $\Gamma^{(1)}$; they give rise to a realization $\Phi^{(1)} = (\varphi_{\mathcal{S}'} : G_{\mathcal{S}} \to T_{\mathcal{S}'}, z_{\mathcal{S}'})$. By Lemma 12.66 we have that $\mathcal{T}^{(2)} \subset \Upsilon(\mathcal{S}')$, and $\Psi(\mathcal{S}) = \left(\Phi^{(2)}\right)^{-1}(\Upsilon(\mathcal{S}'))$. Thus, again by Lemma 12.13, we have that $\Upsilon(\mathcal{S}')$ is a vertex set for $\Delta^{(2)}$, and $\Psi(\mathcal{S})$ is one for $\Gamma^{(2)}$; they give rise to a realization $\Phi^{(2)} = (\varphi_{\Upsilon(\mathcal{S}')} : G_{\Psi(\mathcal{S})} \to T_{\Upsilon(\mathcal{S}')}, z_{\Upsilon(\mathcal{S}')})$.

By Lemma 11.50 we get graph isomorphisms $\gamma_\Psi : G_{\mathcal{S}} \to G_{\Psi(\mathcal{S})}$ and $\tau_\Upsilon : T_{\mathcal{S}'} \to T_{\Upsilon(\mathcal{S}')}$. With them we get an isomorphism at the level of the models that pulls back index maps. That is, $\varphi_{\Upsilon(\mathcal{S}')} \circ \gamma_\Psi = \tau_\Upsilon \circ \varphi_{\mathcal{S}'}$ and $m_{\varphi_{\Upsilon(\mathcal{S}')}} = m_{\varphi_{\mathcal{S}'}} \circ \gamma_\Psi^{-1}$. It is straightforward to verify that these necessary conditions are also enough to specify an isometry of tropical morphisms.

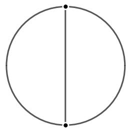
By Lemma 12.67, two distinct points of $\mathcal{G}^{\text{trop}}_{g\to 0,d}$ encode non isometric tropical morphisms.

Lemma 12.67. *Let $\Phi^{(1)}$, $\Phi^{(2)}$ be tropical morphisms, $\mathcal{S}' \subset \Delta^{(1)}$ induce vertex sets for $\Phi^{(1)}$, and $\Psi : \Gamma^{(1)} \to \Gamma^{(2)}$, $\Upsilon : \Delta^{(1)} \to \Delta^{(2)}$ be isometries. We have that $\Phi^{(2)} \circ \Psi = \Upsilon \circ \Phi^{(1)}$ if and only if $\Upsilon(\mathcal{S}') \subset \Delta^{(2)}$ induces vertex sets for $\Phi^{(2)}$, and the induced maps γ_Ψ, τ_Υ, $\varphi_{\mathcal{S}'}$ and $\varphi_{\Upsilon(\mathcal{S}')}$ satisfy that $\varphi_{\Upsilon(\mathcal{S}')} \circ \gamma_\Psi = \tau_\Upsilon \circ \varphi_{\mathcal{S}'}$, that the index maps pull back $m_{\varphi_{\Upsilon(\mathcal{S}')}} = m_{\varphi_{\mathcal{S}'}} \circ \gamma_\Psi^{-1}$, and that the lengths pull back, namely $z_{\Upsilon(\mathcal{S}')} = z_{\mathcal{S}'} \circ \tau_\Upsilon$.*

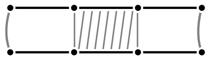
Example 12.68 (Genus 2)**.** We calculate the top-dimensional cones of $\mathcal{G}^{\text{trop}}_{2\to 0,2}$. Note that $\#(E(T)) = 2d + 2g - 5 = 3$, which in this case coincides with $3g - 3 = 3$. There are two trees with 3 edges: $T^{(1)}$ with a vertex adjacent to three leaves; and $T^{(2)}$ a path of length 3. Being change-minimal determines the fibre above a leaf and an edge leading to a leaf, so we are done if $\varphi^{(1)}$ has base tree $T^{(1)}$. If $\varphi^{(2)}$ has base tree $T^{(2)}$, then there are additionally trivalent vertices above the endpoints of the middle segment, which means an edge e with $m_\varphi(e) = 2$ above the middle segment. Computing A_φ gives diagonal matrices, so these maps are indeed full-rank and moreover implies that the structure of $\mathcal{G}^{\text{trop}}_{2\to 0,2}$ is isomorphic to that of $\mathcal{M}^{\text{trop}}_{2}$. In particular Π is bijective.



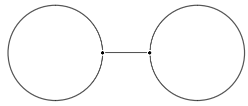
$\varphi^{(1)}$



$H^{(1)}$



$\varphi^{(2)}$



$H^{(2)}$

$A^{(1)} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ $\qquad$ $A^{(2)} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

⋆

Note that $\operatorname{Aut} H^{(1)} \cong S_3$, the symmetric group on three elements, and that all the automorphisms are induced by automorphisms of $\varphi^{(1)}$ via Lemma 12.57. A similar observation applies to $\operatorname{Aut} H^{(1)} \cong S_2$. The diagram for the category $\mathcal{G}^{\text{trop}}_{2\to 0,2}$ is shown in Figure 12.2. See Figure 3 in [Cha12] for a cone representation of $\mathcal{M}^{\text{trop}}_{2}$, which as noted coincides with $\mathcal{G}^{\text{trop}}_{2\to 0,2}$.

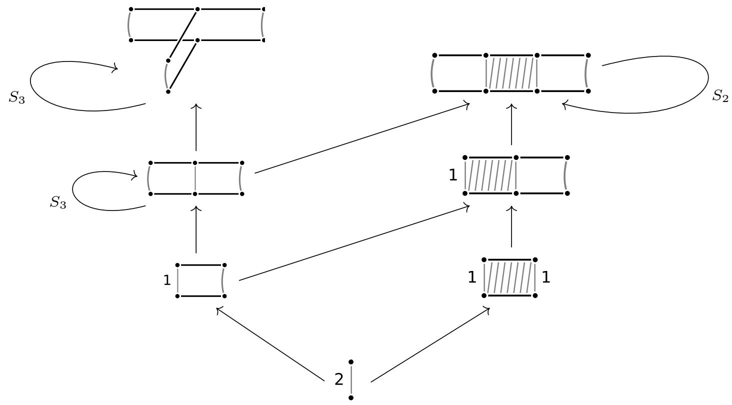


Figure 12.2: Diagram for the category $\mathcal{G}^{\text{trop}}_{2\to 0,2}$.

Chapter 13

Properties of the projection $|\Pi|$

Now consider the projection map $|\Pi| : |\mathcal{G}^{\text{trop}}_{g\to 0,d}| \to |\mathcal{M}^{\text{trop}}_g|$ that sends a tropical morphism $|\Phi| : (|\Gamma|, w_G) \to |\Delta|$ to its source, the weighted metric graph $(|\Gamma|, w_G)$. We describe a combinatorial morphism $\widetilde{\Pi}$ of polyhedral complexes such that $|\widetilde{\Pi}|$ is isomorphic to $|\Pi|$, we calculate a specific fibre that has catalan many points, and introduce an index map m_π based on a multiplicity used in toric geometry. The proof that m_π is balanced is a lengthy one and spans the remaining of this Part II. This section takes the first steps, outlining the proof, splitting it into two major cases, and doing the first case.

13.1 The projection $|\Pi|$ as a morphism $(\pi, \{\Pi_\varphi\})$ of polyhedral spaces

We begin by introducing a morphism of polyhedral spaces

$$\Pi : [C^{\text{src}}_- : \mathcal{G}^{\text{trop}}_{g\to 0,d} \to \text{Poly}^f_{\mathbb{Z}}] \to [C_- : \mathcal{M}^{\text{trop}}_g \to \text{Poly}^f_{\mathbb{Z}}],$$

and refine it to obtain a combinatorial morphism of polyhedral complexes.

Lemma 13.1. *Let d and g be positive integers. Consider the map $\pi : \mathcal{G}^{trop}_{g\to 0,d} \to \mathcal{M}^{trop}_g$ that sends a DT-morphism φ to the combinatorial type $H(\varphi)$, and a specialization $\rho : \varphi \to \varphi_0$ to the map $\rho_H : H(\varphi) \to H(\varphi_0)$ given by Construction 12.55. The map π is a functor.*

Proof. By Remark 12.58 we have that $\pi(\text{id}_\varphi) = \text{id}_{H(\varphi)}$. Also $\pi(\rho_1 \circ \rho_2) = \pi(\rho_1) \circ \pi(\rho_2)$ since composition of specializations is a specialization. □

Lemma 13.2. *Let d and g be positive integers. Consider the pair $\Pi = (\pi, \{\Pi_\varphi\}_{\varphi \in \mathcal{G}^{trop}_{g\to 0,d}})$ given by π from Lemma 13.1 and the inclusions $\Pi_\varphi : C^{src}_\varphi \to C_{H(\varphi)}$. The pair Π is a natural transformation, in fact it is a morphism of polyhedral space of cones.*

Proof. This follows from the fact that Diagram 12.61 can be composed, since composition of specializations is a specialization. □

Now assume that for some positive integer g' we have $g = 2g'$ and $d = g' + 1$. Our aim is to show that the topological realization $|\Pi| : |\mathcal{G}^{\text{trop}}_{g\to 0,d}| \to |\mathcal{M}^{\text{trop}}_g|$ admits an index map which makes it an indexed branched cover. We give the index map in Subsection 13.3, it is of the form $m_\pi \circ \text{poly}_{\mathcal{G}^{\text{trop}}_{g\to 0,d}}$, with $m_\pi : \mathcal{G}^{\text{trop}}_{g\to 0,d} \to \mathbb{Z}_{\geq 1}$ an index map. It has the property that if C^{src}_φ is the positive orthant, then m_π equals one. In Subsection 13.2 we calculate the fibre for a specific point and show the count is a catalan number.

By Conjecture 9.46 we have refinements $\text{bcs}(C^{\text{src}}_- : \mathcal{G}^{\text{trop}}_{g\to 0,d} \to \text{Poly}^f_{\mathbb{Z}})$ and $\text{bcs}(C_- : \mathcal{M}^{\text{trop}}_g \to \text{Poly}^f_{\mathbb{Z}})$ that are polyhedral complexes of simplicial cones. Moreover, by Conjecture 10.77 there is a refinement $\zeta^d_g : [\widetilde{C^{\text{src}}_-} : \widetilde{\mathcal{G}}^{\text{trop}}_{g\to 0,d} \to \text{Poly}^f_{\mathbb{Z}}] \to [C^{\text{src}}_- : \mathcal{G}^{\text{trop}}_{g\to 0,d} \to \text{Poly}^f_{\mathbb{Z}}]$ and a refinement

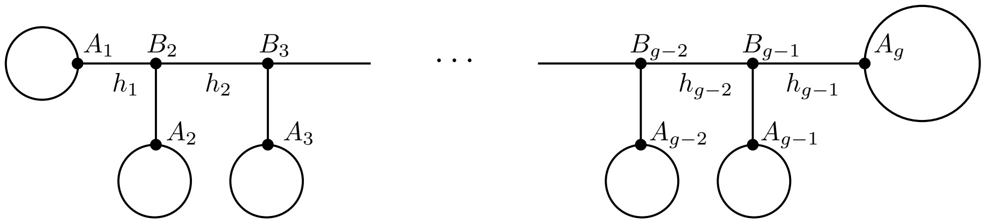


Figure 13.1: A caterpillar of loops

$\xi_g : [\widetilde{C_-} : \widetilde{\mathcal{M}}_g^{\text{trop}} \to \text{POLY}^f_{\mathbb{Z}}] \to [C_- : \mathcal{M}_g^{\text{trop}} \to \text{POLY}^f_{\mathbb{Z}}]$ such that Π induces a map $\widetilde{\Pi} : [\widetilde{C_-^{\text{src}}} : \widetilde{\mathcal{G}}_{g\to 0,d}^{\text{trop}} \to \text{POLY}^f_{\mathbb{Z}}] \to [\widetilde{C_-} : \widetilde{\mathcal{M}}_g^{\text{trop}} \to \text{POLY}^f_{\mathbb{Z}}]$ which is a combinatorial morphism of polyhedral spaces. Thus, by Theorem 10.46 we have:

Lemma 13.3. *Let $m_{\widetilde{\pi}} : \widetilde{\mathcal{G}}_{g\to 0,d}^{trop} \to \mathbb{Z}_{\geq 1}$ be an index map. If $\widetilde{\pi} : \widetilde{\mathcal{G}}_{g\to 0,d}^{trop} \to \widetilde{\mathcal{M}}_g^{trop}$ is an indexed branched cover with index map $m_{\widetilde{\pi}}$, then $|\widetilde{\Pi}| : |\widetilde{\mathcal{G}}_{g\to 0,d}^{trop}| \to |\widetilde{\mathcal{M}}_g^{trop}|$ is an indexed branched cover with index map $m_{\widetilde{\pi}} \circ \text{poly}_{\widetilde{\mathcal{G}}_{g\to 0,d}^{trop}}$.*

We have that Diagram 13.4 commutes, and the horizontal maps are homeomorphisms.

So if the vertical map on the left is an indexed branched cover, so is the one on the right, which is what we want. Lemma 13.3 proves so, that the map on the left is an indexed branched cover, and we initiate the proof on Subsection 13.3.

13.2 A fibre $|\Pi|^{-1}(\Gamma)$ with Catalan-many points

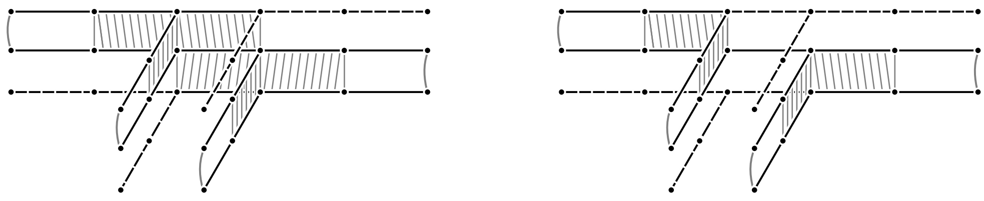
Meanwhile, we establish Theorem C for one class of metric graphs.

Definition 13.5 (caterpillar of loops)**.** The genus-g *caterpillar of loops* H_g^{CL} is obtained by taking a length-$(g-1)$ path $\langle A_1, h_1, B_2, h_2, \ldots, B_{g-2}, h_{g-2}, B_{g-1}, h_{g-1}, A_g\rangle$ and attaching loops to its ends and lollipops to its interior vertices; namely, for $i = 2, \ldots, g-1$ add a vertex A_i with a loop, and join A_i to B_i via a bridge. See Figure 13.1.

Remark 13.6. Note that H_g^{CL} is trivalent, therefore a maximal element of $(\mathcal{M}_g^{\text{trop}})^{\text{op}}$. △

We study $\pi^{-1}(H_g^{CL})$, i.e. the subset of those φ in $\mathcal{G}_{g\to 0,d}^{\text{trop}}$ for which $H(\varphi)$ is isomorphic to H_g^{CL}. We show that the properties of those φ in $\pi^{-1}(H_g^{CL})$ are the best we can hope for: C_φ^{src} is the positive orthant; there is a constructive bijection between these φ and certain combinatorial sequences counted by Catalan numbers; and the subposet $\downarrow\pi^{-1}(H_g^{CL}) \subset \mathcal{G}_{g\to 0,d}^{\text{trop}}$ is connected in codimension-1. Putting these facts together proves Theorem C for the case of caterpillars of loops.

Example 13.7. See the figures below for two DT-morphisms such that the combinatorial type of the source is the genus-4 caterpillar of loops. By Proposition 13.14 these are all such maps.



M_1 M_2

⋆

Remark 13.8 (values for other versions of gonality)**.** The *divisorial gonality* of any caterpillar of loops is 2, since the divisor $D = 2(A_1)$ has rank-1. The *harmonic-maps-to-trees gonality* from [Cha13] is also 2. That is, for every H_g^{CL} there is a degree-2 indexed branched cover Φ to a tree; see Figure 13.2. This Φ does not typically satisfy the Riemann-Hurwitz inequality. Therefore, that tree gonality is generically $g' + 1$ is a consequence of the Riemann-Hurwitz inequality. Moreover, this example shows that the difference between tree gonality and divisorial gonality cannot be bounded. △

The main reason behind the favourable combinatorial properties of caterpillars of loops is that, by a result from Part I, a lollipop in $H(\varphi)$ is above a length-2 path in T leading to a leaf, and the fibre above this path is uniquely determined. Since a caterpillar of loops is a bunch of lollipops strung together, this is enough to determine φ on the non-dangling elements of G.

Lemma 13.9. *Let (φ, m_φ) be a DT-morphism that is change-minimal and has full rank. If $A \in V(H(\varphi))$ is trivalent and incident to a bridge h_b and a loop h_l, then $h_b = \langle A, e_b, B\rangle$, $h_l = \langle A, e_1, C, e_2, A\rangle$, $\varphi(C)$ is a leaf, $\varphi(A)$ is divalent, $r_\varphi(A) = 1$, $m_\varphi(e_b) = 2$; and e_b, A, e_1, e_2, C are the only non-dangling elements in the fibres of $\varphi(e_b)$, $\varphi(A)$, $\varphi(e_1)$ and $\varphi(C)$.*

Proof. See Lemma 6.23 in Part I. □

Lemma 13.10. *Let $\varphi : G \to T$ be a degree-$(g' + 1)$ DT-morphism in $\pi^{-1}(H_{2g'}^{CL})$ and $\tilde{G}$ the deletion of dangling elements of G. The cone C_φ^{src} equals $\mathbb{R}_{\geq 0}^{G} \cap \{y(A) = 0$ for $A \in V(G)\}$, and the restriction $\gamma_{2g'}^{CL} = \varphi|_{\tilde{G}}$ depends only on $2g'$. See Figure 13.2.*

Proof. Fix an isomorphism $L : H_g^{CL} \to H(\varphi)$, and let $g = 2g'$. By Lemma 13.9 the image $\varphi(E(L(A_i)))$ is a length-2 path with interior vertex u_i and one of its ends is a leaf v_i. These paths account for $2g$ distinct edges of T and their fibres are determined. Since φ is change-minimal, $\operatorname{ch} u_i = 1$ and $\operatorname{ch} v_i = 2$, and by Lemma 12.2 this accounts for the total change $3g$ of φ. So any vertex of T distinct from u_i, v_i has ch-value equal to 0.

Consider the unique path P in G whose ends are $L(B_2)$ and $L(B_{g-1})$. The length of P is at least $g - 3$ since it contains at least $g - 4$ inner vertices (the B_i for $i = 3, \dots, g - 2$). The image $\varphi(P)$ is disjoint from $\varphi(E(L(A_i)))$, for all i; since T has $3g - 3$ edges, $\varphi(P)$ contains at most $(3g - 3) - 2g = g - 3$ edges. Since none of the interior vertices of P are above u_i, v_i, they have r_φ-value 0; applying Case-(r0) of Proposition 12.51 to them gives that consecutive edges in P have distinct images. Since T is a tree, this implies that φ is injective on P. We conclude that $\varphi(P)$ is a length-$(g - 3)$ path in T; that h_i contains a single edge of G for $i = 1, \dots, g - 1$, so each row of A_φ has a single non-zero entry; and that each $\varphi(E(L(A_i)))$ attaches to $\varphi(P)$ giving rise to $\gamma_g^{CL} : G_g^{CL} \to T_g^{CL}$ of Figure 13.2. □

Now let $\widetilde{\Gamma}$ be the metric graph (H_g^{CL}, y). Lemma 13.10 reduces the construction of a top-dimensional tropical morphism Φ in $\Pi^{-1}(\widetilde{\Gamma})$ to choosing the slopes of Φ at the edges h_i

; a priori it appears we can choose how to place the dangling elements needed to fullfill the balancing condition. However, the slopes have to satisfy a strong combinatorial condition; and by the dangling-no-glue property and Case-(d) of the local properties the dangling trees are determined by the choice of slopes.

Lemma 13.11. *Let $\Phi : \Gamma \to \Delta$ be a tropical morphism and $\widetilde{\Gamma} = (H, \ell_H)$ its deletion of dangling trees, such that there is an isomorphism $L : H_g^{CL} \to H$. If s_i is the slope at the edge $L(h_i)$, then $s_i - s_{i-1} = \pm 1$ for $i \in \{1, \dots, g - 1\}$.*

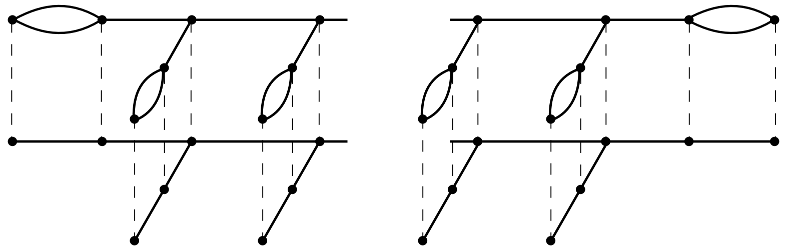


Figure 13.2: The graph morphism $\gamma_g^{CL} : G_g^{CL} \to T_g^{CL}$.

Proof. Let φ be the combinatorial type of Φ. By Lemma 13.9 the index at the bridges leading to a loop is 2. Since $\text{nd-val}\, L(B_i) = 3$ and $\text{val}\, \varphi(L(B_i)) = 3$, Proposition 12.51 implies that $r_\varphi(L(B_i)) = 0$. Lemma 12.50 gives that $r_\varphi(L(B_i)) = 2m_\varphi(L(B_i)) + 3 - 2 - (s_i + s_{i-1} + 2)$, so $s_{i-1} + s_i = 2m_\varphi(L(B_i)) - 1$. We are done since $m_\varphi(L(B_i)) \geq \max(s_i, s_{i-1})$. □

Lemma 13.12. *Let $\widetilde{\Gamma} = (H_g^{CL}, y)$, and s a sequence $(s_i)_{i=1}^{g-1} \subset \mathbb{Z}_{\geq 1}$ such that $s_1 = s_{g-1} = 2$ and $s_i - s_{i-1} = \pm 1$ for all i. There is exactly one tropical morphism $\Phi_s : \Gamma \to (T_g^{CL}, z)$ such that $\Gamma \cong_{trop} \widetilde{\Gamma}$, the slope at h_i equals s_i, and its combinatorial type is in $\pi^{-1}(H_g^{CL})$.*

Proof. Lemma 13.10 gives that to get Φ_s we have to extend γ_g^{CL} to a DT-morphism φ_s by prescribing the values of m_φ and attaching dangling trees. The index map m_φ is determined at loops and at their bridges by Lemma 13.9, at the remaining bridges by the s_i, at dangling elements by the dangling-no-glue condition, and at B_i by $s_{i-1} + s_i = 2m_\varphi(B_i) - 1$, which implies that $m_\varphi(B_i) = \max(s_i, s_{i-1})$. Note that only at B_i balancing is not satisfied yet. So we attach $m_\varphi(B_i) - 2$ paths of length-2 to B_i, and map them down to the image of the lollipop incident to B_i. If $s_i - s_{i-1} = 1$, then we attach a copy of the portion of T_g^{CL} left of the vertex $\gamma_g^{CL}(B_i)$ in Figure 13.2, mapping it down to the aforementioned left portion. If $s_i - s_{i-1} = -1$, we attach a copy of, and map down to, the right portion of T_g^{CL}. By Case-(d) of the local properties this is the only way to go. Therefore, we get a change-minimal DT-morphism φ_s with index map m_φ, and with an edge-length matrix A_{φ_s} that in each row and in each column has exactly one non-zero element, hence φ_s has full rank as desired. □

So there are as many tropical morphisms as choices of sequences s. To count these, define the sequence b with $b_1 = 1$, $b_g = -1$, and $(b_i = s_i - s_{i-1})_{i=2}^{g-1}$. We get three properties: each entry b_i is either 1 or -1, all partial sums are non-negative since the i-th partial sum equals $s_i - 1$, and 1 appears as many times as -1 since $s_2 = s_{g-1}$. A sequence with those three properties is called a *length-g ballot sequence*. It is a classical combinatorial passtime to prove these are counted by catalan numbers $C(g') = \frac{1}{g'+1}\binom{2g'}{g'}$.

Lemma 13.13. *Let g' be a positive integer. The number of length-$(2g')$ ballot sequences equals the g'-th Catalan number $C(g')$.*

Proof. See e.g. [Sta15][Theorem 1.5.1]. □

Putting everything together, we get a special case of Theorem C:

Proposition 13.14. *Let g' be a positive integer, $y \in C_{H_{2g'}^{CL}}$, and $\widetilde{\Gamma} = (H_{2g'}^{CL}, y)$. If all the lengths encoded by y are distinct, then the fibre $\Pi^{-1}(\widetilde{\Gamma})$ has $C(2g')$ points in $\mathcal{G}_{g\to 0,d}^{trop}$, and these points are pairwise connected by paths going through codimension 1 in $\mathcal{G}_{2g'\to 0,g'+1}^{trop}$.*

Proof. Let $\Phi^{(1)} : \Gamma^{(1)} \to \Delta^{(1)}$ and $\Phi^{(2)} : \Gamma^{(2)} \to \Delta^{(2)}$ be tropical morphisms, $\widetilde{\Gamma}^{(1)}$ and $\widetilde{\Gamma}^{(2)}$ the deletion of dangling trees of $\Gamma^{(1)}$ and $\Gamma^{(2)}$, respectively, such that there are isometries $L^{(1)} : \widetilde{\Gamma} \to \widetilde{\Gamma}^{(1)}$ and $L^{(2)} : \widetilde{\Gamma} \to \widetilde{\Gamma}^{(2)}$. Let $b^{(1)}$ and $b^{(2)}$ be the ballot sequences obtained from $\Phi^{(1)}$ and $\Phi^{(2)}$, respectively, using Lemma 13.12.

From the construction in Lemma 13.12 we have that if $b^{(1)}$ and $b^{(2)}$ are equal, then $\Phi^{(1)}$ and $\Phi^{(2)}$ are isomorphic. We prove the converse to conclude from Lemma 13.13 that $\Pi^{-1}(\widetilde{\Gamma})$ has $C(2g')$ points. If $\Phi^{(1)}$ and $\Phi^{(2)}$ are isomorphic, then there are isometries Ψ, Υ such that $\Phi^{(2)} \circ \Psi = \Upsilon \circ \Phi^{(1)}$. The isometry Ψ descends to an isometry $\tilde{\Psi} : \widetilde{\Gamma}^{(1)} \to \widetilde{\Gamma}^{(2)}$. So $\left(L^{(2)}\right)^{-1} \circ \tilde{\Psi} \circ L^{(1)}$ is an automorphism of $\widetilde{\Gamma}$. Since all the lengths encoded by y are distinct, the automorphism group of $\widetilde{\Gamma}$ is trivial, so $L^{(2)} = \tilde{\Psi} \circ L^{(1)}$. We get that

$$s_i^{(2)} = \frac{\ell(L^{(2)}(h_i))}{\ell(\Phi^{(2)} \circ L^{(2)}(h_i))} = \frac{\ell(\tilde{\Psi} \circ L^{(1)}(h_i))}{\ell(\Phi^{(2)} \circ \tilde{\Psi} \circ L^{(1)}(h_i))} = \frac{\ell(L^{(1)}(h_i))}{\ell(\Upsilon \circ \Phi^{(1)} \circ L^{(1)}(h_i))} = s^{(1)},$$

where we denote by $\ell(\cdot)$ the length of a real interval, and we have used the fact that Ψ and Υ are isometries.

To prove connectivity suppose that for some i we have $b_i^{(1)} = 1$, $b_{i+1}^{(1)} = -1$ and $b_i^{(2)} = -1$, $b_{i+1}^{(2)} = 1$, and at the remaining entries the sequences coincide. In short, $b^{(2)}$ arises from $b^{(1)}$ by swapping $b_i^{(1)} = 1$ with $b_{i+1}^{(1)} = -1$. We get that $m^{(1)}(L^{(1)}(h_j)) = m^{(2)}(L^{(2)}(h_j))$ for all $j \neq i$ (here we use $L^{(1)}$ and $L^{(2)}$ to refer to the underlying graph morphisms). Thus, contracting $L^{(1)}(h_i)$ in $\widetilde{\Gamma}^{(1)} \subset \Gamma^{(1)}$, and $L^{(2)}(h_i)$ in $\widetilde{\Gamma}^{(2)} \subset \Gamma^{(2)}$, yields isomorphic specializations. The set of ballot sequences is connected by this swapping operation since from any ballot sequence it is possible to reach the sequence where the first half of elements are 1's, and the second half are -1's, using swaps. Thus, all the cones are connected in codimension 1. □

Remark 13.15. Theorem 1 of [EH87] follows from studying a particular family of curves C_∞ with genus g, depicted in Figure 2 of the cited work. The dual graph of C_∞ is:

G_∞

The labelling reflects the irreducible components of C_∞, where Y_q are genus-0 curves, and E_q are genus-1 curves. Strikingly, if one imagines infinitesimal loops at E_q, then the resulting graph is tropically equivalent to a caterpillar of loops. Amusingly, we stumbled upon the family of caterpillars of loops not through this observation, but rather by initially trying to work with chains of loops as in [CDPR12]. △

13.3 An index map m_π for π

Now we introduce an index map for $\widetilde{\pi} : \widetilde{\mathcal{G}}^{\text{trop}}_{g \to 0,d} \to \widetilde{\mathcal{M}}^{\text{trop}}_g$, and sketch how to prove that $\widetilde{\pi}$ is an indexed branched cover when we have $g = 2g'$ and $d = g' + 1$ for some positive integer g'. The index map is based on a multiplicity used in the context of toric geometry. Despite verifying that such index map works with constructions and calculations, at the moment a deeper philosophical reason escapes us and we believe it to be related to a tropicalization result that warrants further investigations, as outlined in Subsection 8.3.7.

Definition 13.16. Let $\sigma = (C, N) \in \text{Cone}_{\mathbb{Z}}$ be a cone generated by the rays $\theta_1, \ldots, \theta_d$. We denote by v_i the primitive generator of θ_i, i.e. the generator for the semigroup $\theta_i \cap N$. The *multiplicity* $\text{mult}(\sigma)$ of σ equals the index

$$[N : (\mathbb{Z}_1 v_1 + \ldots \mathbb{Z}_d v_d)].$$

Definition 13.17. Consider a DT-morphism $\varphi : (G, w) \to T$ in $\mathcal{G}^{\text{trop}}_{g \to 0,d}$. By Lemma 12.52 the columns of the edge-length matrix that correspond to leaves of T have a single non-zero entry, and it equals 1/2. The reduced edge-length matrix $\widehat{A}_\varphi$ is the result of multiplying said columns by 2.

Definition 13.18. Let φ be a DT-morphism in $\mathcal{G}^{\text{trop}}_{g\to 0,d}$, and $\widehat{A}_\varphi$ its reduced edge-length matrix. Let $m_\pi : \mathcal{G}^{\text{trop}}_{g\to 0\to 0,d}$ be an index map given by

$$m_\pi(\varphi) = \text{mult}(\text{col-span}(\widehat{A}_\varphi^\top)).$$

We get an index map on $\widetilde{\mathcal{G}}^{\text{trop}}_{g\to 0,d}$ by pulling back m_π via $\zeta_g^d : \widetilde{\mathcal{G}}^{\text{trop}}_{g\to 0,d} \to \mathcal{G}^{\text{trop}}_{g\to 0,d}$, that is we set $m_{\widetilde{\pi}}$ to be $m_\pi \circ \zeta_g^d$. Our main aim for the remainder is to prove the following result.

Proposition 13.19. *Let $g = 2g'$ and $d = g'+1$, where g' is a positive integer. Consider the map $\widetilde{\pi} : \widetilde{\mathcal{G}}^{trop}_{g\to 0,d} \to \widetilde{\mathcal{M}}^{trop}_g$ from Subsection 13.1, and $m_{\widetilde{\pi}} = m_\pi \circ \zeta_g^d$. The pair $(\widetilde{\pi}, m_{\widetilde{\pi}})$ is balanced in codimension-1.*

Proof. Let $\alpha \in \widetilde{\mathcal{G}}^{\text{trop}}_{2g'\to 0,g'+1}$ be a codimension-1 element. Consider $\varphi_0 = \zeta_g^d(\alpha)$.
We distinguish three cases:
(1) $\varphi_0 = \zeta_g^d(\alpha)$ is top-dimensional in $\mathcal{G}^{\text{trop}}_{2g'\to 0,g'+1}$,
(2) $\varphi_0 = \zeta_g^d(\alpha)$ has codimension-1 and a trivalent $H(\varphi_0)$; otherwise
(3) $\varphi_0 = \zeta_g^d(\alpha)$ has codimension-1 and a non-trivalent $H(\varphi_0)$.

The first case can be shown to follow from general considerations about subdivisions. That is, this is the interior of some top-dimensional cone C^{src}_φ that was subdivided in the process of making Π a combinatorial morphism of polyhedral complexes, hence balancing is straightfowrard.

The second case we defer to Proposition 13.25. There the point is to take the work done in Part I and reformulate it to prove Equation (8.9). After such equation is proven, some remaining work is needed to argue that each one of the constructions count, that we do not have isomorphisms. This is indeed relevant because, for example in Subsection 13.2, at the combinatorial level we are not constructing catalan many DT-morphisms. Some of them are isomorphic, by a symmetry argument, but those symmetric have two distinct points realizing a given metric graph. See 13.26 for the result.

The third case we defer to Section 14. Here it is enough to make the constructions, count them, and show that no matter how $H(\varphi_0)$ regrows the missing edge, the number of DT-morphisms is the same. See 14.1 for the result. ☐

Observe that such result implies Theorem C, which we reformulate as follows:

Theorem 13.20. *Let g' be a positive integer, and $g = 2g'$ and $d = g'+1$. The projection $|\Pi| : |\mathcal{G}^{trop}_{g\to 0,d}| \to |\mathcal{M}^{trop}_g|$ given by $[\Phi : \Gamma \to \Delta] \mapsto \Gamma$, with index map $m_\pi(\varphi) = \text{mult}(\widehat{A}_\varphi^\top)$ composed with* $\text{poly}_{\mathcal{G}^{trop}_{g\to 0,d}}$*, is a surjective indexed branched cover of cone spaces. The degree* $\deg \Pi$ *equals the g'-th Catalan number. The space $|\mathcal{G}^{trop}_{g\to 0,d}|$ is connected through codimension-1.*

Proof. By Proposition 13.19, the pair $(\widetilde{\pi} : \widetilde{\mathcal{G}}^{\text{trop}}_{g\to 0,d} \to \widetilde{\mathcal{M}}^{\text{trop}}_g, m_\pi \circ \zeta_g^d)$ is a map of posets that is balanced in codimension 1. By Proposition 11.49, the poset $((\mathcal{M}^{\text{trop}}_g)^{\text{op}}, \preceq)$ is strongly connected. By Lemma 10.69 every refinement of a strongly connected poset is strongly connected, so in particular $(\widetilde{\mathcal{M}}^{\text{trop}}_g)^{\text{op}}$ is strongly connected. Therefore, we can iterate Proposition 10.73 to extend $m_{\widetilde{\pi}} = m_\pi \circ \zeta_g^d$ to a balanced map $\hat{m}$ defined on $\widetilde{\mathcal{G}}^{\text{trop}}_{g\to 0,d}$: first extend from codimension-1 to codimension-2, then to codimension-3, and so on. This is possible thanks to Lemma 10.74. Since $\widetilde{\pi}$ is combinatorial, by Proposition 10.54 we get that $(\widetilde{\pi}, \hat{m})$ is an indexed branched cover. By Lemma 13.3 and the discussion afterwards, we are done in getting that $(|\Pi|, \hat{m}\circ\text{poly})$ is an indexed branched cover. This uses Proposition 10.44 that reduced our metric problem to a combinatorial problem.

Moreover, from Lemma 10.64 and the fact from Proposition 13.14 that the fibre above caterpillar of loops is connected we get that $\mathcal{G}^{\text{trop}}_{g\to 0,d}$ is connected through codimension-1. This is possible thanks to Lemma 10.65. Finally, the count of the fibre was achieved in Proposition 13.14. All the points calculated in said proposition have multiplicity 1. Thus $\deg|\Pi|$ is the g'-th Catalan number. ☐

13.4 The map m_π is balanced on $\uparrow\varphi_0$ when $H(\varphi_0)$ is trivalent

We introduce a signed multiplicity, and use it to prove Equation (8.9) from Page 69. In turn, this implies balancing in the second case stated in the proof of Theorem C in Subsection 13.3.

Definition 13.21. Let $\varphi : (G, w) \to T$ be a DT-morphism such that $\widehat{A}_\varphi$ is a square matrix. Let d_i be the minimum positive integer such that the i-th row of $\widehat{A}_\varphi$ multiplied by d_i is an integral vector, and D_φ be the product of all the d_i. The *signed multiplicity* of φ is

$$\mathrm{Mult}_\pm(\varphi) = D_\varphi \det \widehat{A}_\varphi.$$

Remark 13.22. The sign of $\mathrm{Mult}_\pm(\varphi)$ depends on the ordering of the rows and columns of $\widehat{A}_\varphi$, namely on the ordering of $E(H(\varphi))$ and $E(T)$. The balancing condition is not affected by this, as once the choice is done for a DT-morphism φ, it induces the order on its codimension-1 specializations, this in turn carries to DT-morphisms that share codimension-1 specializations with φ, and so on. Ultimately, the choice of order has to be done once for each connected component of $\mathcal{G}^{\mathrm{trop}}_{g\to 0,d}$, thus just once by Theorem C. △

Lemma 13.23. *Let φ be a DT-morphism in $\mathcal{G}^{trop}_{g\to 0,d}$. We have that*

$$m_\pi(\varphi) = |\mathrm{Mult}_\pm(\varphi)|.$$

Proof. This is a standard fact about free abelian groups, see e.g. (!!insert ref). □

We characterize the numbers d_i when φ is top-dimensional.

Lemma 13.24. *Let φ be a top-dimensional DT-morphism, $\widehat{A}_\varphi$ its reduced edge-length matrix, h_i in $E(H)$, and d_i as in Definition 13.21. We have that*

- *(a) if h_i passes above a leaf, then $d_i = 1$;*
- *(b) if $r_\varphi(A) = 0$ for all $A \in h \cap V(G)$, then $m_\varphi(h_i \cap E(G)) = \{d_i\}$;*
- *(c) otherwise h_i decomposes into two paths P and P' such that $m_\varphi(P \cap E(G)) = \{k\}$ and $m_\varphi(P' \cap E(G)) = \{k + 1\}$, thus $d_i = k(k + 1)$.*

Proof. Observation I: By the cases (r0) and (r1) of Proposition 12.51 (local properties) if e and e' in h are adjacent, then $|m_\varphi(e) - m_\varphi(e')| \leq 1$.

Proof of (a): Choose $\widetilde{e} \in h$ such that $\varphi(\widetilde{e})$ is incident to a leaf. Assume there is $\hat{e} \in h$ with $m_\varphi(\hat{e}) \neq 1$. Let $t = \varphi(\widetilde{e})$. Remark 4.15 implies that $m_\varphi(\tilde{e}) = 1$. Thus, when going from $\hat{e}$ to $\tilde{e}$ in h, by Observation I the cardinalities of the edges change at each step by a difference of at most 1. Hence, there are two adjacent edges e, e' with $|e| = 2$ and $|e'| = 1$. Let A in $V(G)$ be the vertex incident to these two edges. Since M is full-dimensional, if $r_\varphi(A)$ were 0, it would contradict the case (r0-nd2) of the local properties. Thus, $r_\varphi(A) = 1$ and the other vertices above $\varphi(A)$ have r_φ-value equal to zero. Grow and shrink the lengths $\ell_T(\varphi(e))$ and $\ell_T(\varphi(e'))$, respectively, by the same length z. The case (r0-nd2) implies that only the length of $\ell_G(h)$ changes. It shrinks by $z/2$. Grow $\ell_T(t)$ by $z/4$, so $\ell_G(h)$ grows $z/2$. As t leads to a leaf and M is full-dimensional, this change only affects h. Thus all the lengths are the same as the starting ones, contradicting that M is full-dimensional.

Item II and III: Restating the claim: if h is partitioned into a sequence $P_1, \ldots, P_q$ of adjacent paths, where for each P_i all its edges are equipotent, and these cardinalities are different for adjacent paths, then $q \leq 2$. Assume $q \geq 3$. Recall that the deformation case {v2-r1} has two glueing datums, if one of them is full-dimensional then the other is as well. Successively apply this case to the edges in P_2, to obtain a sequence of full-dimensional glueing datums. In each step, the number of edges in P_2 decreases by one, and in P_3 increases by one. Let M' be the last glueing datum of this sequence, where P_2 has a single edge. By Observation I, the local part around P_2 is forbidden by the deformation cases in {v2-r2-nd2}. Hence M' is not full-dimensional, a contradiction. □

Using Lemma 13.24 we transform Equation ($\star$) into relations for the multiplicity.

Proposition 13.25 (balancing condition)**.** *Let g' be a positive integer, and α an element of $\widetilde{\mathcal{G}}^{trop}_{2g'\to 0,g'+1}$ such that $\varphi_0 = \iota_{\mathcal{G}}(\alpha)$ has codimension-1 in $\mathcal{G}^{trop}_{2g'\to 0,g'+1}$.*

1. *If $H(\varphi_0)$ is trivalent, then*
$$\sum_{\gamma\in\uparrow\alpha} \mathrm{Mult}_{\pm}(\iota_{\mathcal{G}}(\gamma)) = 0.$$

2. *Otherwise,* $\mathrm{Mult}_{\pm}(\iota_{\mathcal{G}}(\gamma)) = \mathrm{Mult}_{\pm}(\iota_{\mathcal{G}}(\gamma'))$ *for every pair γ, γ' in $\uparrow\alpha$.*

Proof. For the first statement, we apply Lemma 13.24 to transform Equations (14.4), (14.5), (14.7), (14.8), (14.9), (14.10), (14.11), (14.12), (14.13), and (14.14) in Section 14.4. We illustrate one case, Equation (7.7) from the deformation case {v2-r2-nd3-M-1k}. Let $M^{(1)}$, $M^{(2)}$, $M^{(3)}$ be the elements of $\uparrow M_0$. We use the notation of Section 14.4. In particular, $c^{(q)} = \det A^{(q)}$.

Note that $l(T^{(2)}) = l(T^{(3)}) = l(T_0)$, and $l(T^{(1)}) = l(T_0) + 1$. Write down the rows h_1, h_2, h_3 and columns t_1, t_2, t_3 of $A^{(q)}$.

$$\underbrace{\begin{pmatrix} 2 & 1 & 0 & \\ 0 & \frac{1}{k} & 0 & \cdots \\ 0 & 0 & \frac{1}{k} & \\ 0 & a^{(1)}_{i2} & \cdots & \\ & \vdots & & \ddots \end{pmatrix}}_{A^{(1)}} \qquad \underbrace{\begin{pmatrix} 1 & 1 & 0 & \\ \frac{1}{k-1} & \frac{1}{k} & 0 & \cdots \\ 0 & 0 & \frac{1}{k} & \\ a^{(2)}_{i1} & a^{(2)}_{i2} & \cdots & \\ & \vdots & & \ddots \end{pmatrix}}_{A^{(2)}} \qquad \underbrace{\begin{pmatrix} 0 & 1 & 0 & \\ 0 & \frac{1}{k} & 0 & \cdots \\ \frac{1}{k+1} & 0 & \frac{1}{k} & \\ a^{(3)}_{i1} & a^{(3)}_{i2} & \cdots & \\ & \vdots & & \ddots \end{pmatrix}}_{A^{(3)}}$$

$a^{(2)}_{i1} = a^{(2)}_{i2}$ for $i \geq 4$. $\qquad$ $a^{(3)}_{i1} = a^{(3)}_{i2}$ for $i \geq 4$.

Let $d_{0,i}$ be the minimum integer such that multiplying the i-th row of A_0 by $d_{0,i}$ gives an integral vector, and let $D_0 = d_{0,1} \ldots d_{0,3g-3}$. Observe that $d^{(q)}_i = d_{0,i}$ for $i \geq 4$. Thus,

$$\mathrm{Mult}_{\pm}(M^{(1)}) + \mathrm{Mult}_{\pm}(M^{(2)}) + \mathrm{Mult}_{\pm}(M^{(3)}) =$$
$$\frac{D_0}{2^{l(T_0)}} \left(\frac{d^{(1)}_1 d^{(1)}_2 d^{(1)}_3}{2 d_1 d_2 d_3} c^{(1)} + \frac{d^{(2)}_1 d^{(2)}_2 d^{(2)}_3}{d_1 d_2 d_3} c^{(2)} + \frac{d^{(3)}_1 d^{(3)}_2 d^{(3)}_3}{d_1 d_2 d_3} c^{(3)} \right)$$

.

We claim that the term in the parenthesis is zero. Let $b^{(1)} = 1$, $b^{(2)} = (k-1)$ and $b^{(3)} = (k+1)$. Equation (7.7) says that $(1/2)b^{(1)}c^{(1)} + b^{(2)}c^{(2)} + b^{(3)}c^{(3)} = 0$. So we are done if either $d^{(q)}_1 d^{(q)}_2 d^{(q)}_3 / d_1 d_2 d_3$ equals $b^{(q)}$, or $c^{(q)}$ equals zero.

Consider $A^{(1)}$ and suppose $c^{(1)} \neq 0$. As h_1 passes above a leaf, Lemma 13.24 item (a) implies that $d^{(1)}_1 = 1$. We also have that $d^{(1)}_2 = d_{0,2}$, $d^{(1)}_3 = d_{0,3}$, proving the claim. Consider $A^{(2)}$ and assume $c^{(2)} \neq 0$. Clearly $d^{(2)}_1 = d_{0,1}$ and $d^{(2)}_3 = d_{0,3}$. Observe that h_2 does not pass above a leaf. Thus, the pass-once condition allows us to assume that $\overline{(t_1, 2)}$ and $\overline{(t_2, 2)}$ are in h_2, with $|\overline{(t_1, 2)}| = k-1$ and $|\overline{(t_2, 2)}| = k$. Lemma 13.24 item (c) implies that the only class with cardinality $k-1$ in h_2 is $\overline{(t_1, 2)}$, all the others have cardinality k. So $d^{(2)}_2 = k(k-1)$ and $d_{0,2} = k$. The claim follows. A similar reasoning applies to $M^{(3)}$.

In essence: consider $M^{(q)}$ and suppose $c^{(q)} \neq 0$, so the conditions for Proposition 4.25 and Lemma 13.24 are fulfilled. If $a^{(q)}_{i1} = 2$, then h_i passes above a leaf and Lemma 13.24 item (a) applies to give $d^{(q)}_i = 1$. If $a^{(q)}_{i1} \neq a^{(q)}_{i2}$, then Lemma 13.24 item (c) applies to give that $d^{(q)}_i / d_{0,i} = |a^{(q)}_{i1}|$. Otherwise, $d^{(q)}_i / d_{0,i} = 1$. This proves the first statement.

The second statement follows from the definition of multiplicity. □

From the previous lemma we can conclude that:

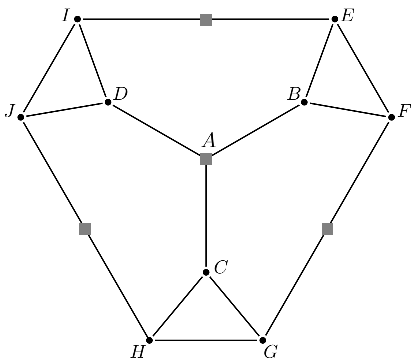
Proposition 13.26. *Let $g = 2g'$ and $d = g'+1$, where g' is a positive integer. Consider the map $\widetilde{\pi} : \widetilde{\mathcal{G}}^{trop}_{g\to 0,d} \to \widetilde{\mathcal{M}}^{trop}_g$ from Subsection 13.1, and $m_{\widetilde{\pi}} = m_\pi \circ \zeta^d_g$. Choose $\alpha \in \widetilde{\mathcal{G}}^{trop}_{g\to 0,d}$. If α has codimension-1 and $H(\zeta^d_g(\alpha))$ is trivalent, then the pair $(\widetilde{\pi}, m_{\widetilde{\pi}})$ is balanced on $\uparrow\alpha$.*

13.5 Graphs with integral edges

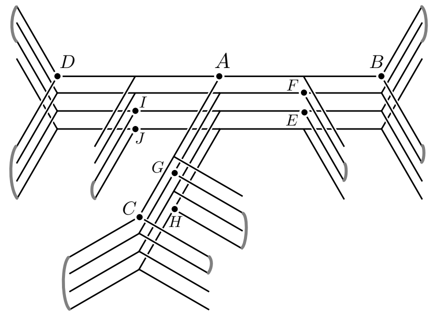
We conclude with a further insight on the connection between the divisor theory of a finite graph H and of the metric graph $\widetilde{\Gamma} = (H, \mathbf{1})$. Given a divisor D in $\mathrm{Div}(\widetilde{\Gamma})$, with the desired degree and rank, one may hope to deform D into a divisor supported on integral points, keeping the same rank and degree. A simple way for carrying out this deformation is to move the chips not supported on integral points towards the endpoints of their respective edges. The following example witnesses that this construction fails. It is referenced as *private communication* in Remark 17 of [CDJP17], where this approximation idea is also reproduced. The example itself was found with a program that randomly sampled the space of genus 6 graphs under certain constraints. The program is available as an appendix of [Var16], and its internal functioning explained in page 62.

Example 13.27 (Integral edges, non-integral divisor)**.** Let H be the genus 6 finite graph shown below, on the left. It has three rank 1 and degree 4 divisors: $D_1 = 2A + I + J$, and D_2, D_3 the two other symmetrical divisors. Consider $\widetilde{\Gamma} = (H, \mathbf{1})$. There is another divisor of interest, besides the already listed ones: $D_4 = A + m_{EI} + m_{FG} + m_{JH}$, where m_{EI} is the midpoint of EI, and so on. It can be verified by testing the 8 cases, that all possible ways to approximate the chip on m_{EI} to either E or I, and likewise for the two other chips on non integral points, produces a rank 0 divisor.

Interestingly, D_4 comes from a multiplicity 2 tropical morphism to a tree. It is shown on the right. It would be quite enlightening to find a metric graph $\widetilde{\Gamma}_0 = (H_0, \ell_0)$ such that all the points in the fibre $\Pi^{-1}(\widetilde{\Gamma}_0)$ have multiplicities greater than 1, in order to check if the induced divisors share the non-approximation property that D_4 exhibits; and study the divisors that H_0 has not coming from tropical morphisms, if any.



H



φ

⋆

Chapter 14

Constructing φ that specialize to φ_0

This section deals with constructions. In particular, the missing ones necessary to prove the balancing condition for $\widetilde{\pi}$ when φ_0 has codimension 1 and $H(\varphi_0)$ is non-trivalent. Our approach is shaped by two opposing drives. Let $\varphi : G \to T$ be a DT-morphism to a tree. On the one hand, a deformation procedure on φ should act locally. On the other hand, since we wish to construct elements in $\mathcal{G}^{\mathrm{trop}}_{g\to 0,d}$, the resulting objects should have full rank, a global condition condition on A_φ. We reconcile these opposing ends by using local conditions to filter out candidates in the constructions, and ending with a manageable case-work.

Proposition 14.1. *Let g' be a positive integer. Consider the map $\widetilde{\pi} : \widetilde{\mathcal{G}}^{trop}_{2g'\to 0,g'+1} \to \widetilde{\mathcal{M}}^{trop}_{2g'}$ from Subsection 13.1, sand $m_{\widetilde{\pi}} = m_\pi \circ \iota_{\mathcal{G}}$. Choose $\alpha \in \widetilde{\mathcal{G}}^{trop}_{g\to 0,2g'}$. If α has codimension-1 and $H(\iota_{\mathcal{G}}(\alpha))$ is not trivalent, then the pair $(\widetilde{\pi}, m_{\widetilde{\pi}})$ is balanced on $\uparrow\alpha$.*

By Proposition 13.25 Item 2, we just have to construct the DT-morphisms, and show that the number for each combinatorial type in $\uparrow H_0$ is the same. These constructions pick up the thread started in the Section 7 of Part I.

14.1 Preliminaries: making constructions manageable

Let $\varphi : G \to T$ be a DT-morphism. Consider the following proxy for having full rank.

Definition 14.2. A DT-morphism is

- *quasi full-rank* if it satisfies the dangling-no-glue and the no-return conditions.
- *quasi top-dimensional* if it is change-minimal and quasi full-rank.

If φ has full rank, then it is quasi full-rank, as expected. Moreover, the advantage of the quasi full-rank condition is that it can be checked locally. Also, as we are about to see in this subsection, being quasi full-rank already implies several of the consequences that being full-rank implies.

Remark 14.3. Let g and d be integers with $d \leq g/2 + 1$. The property of being quasi full-rank is closed under taking specializations. Per the discussion preceding Lemma 12.44, if we consider the subset of DT-morphisms in $\mathrm{DTM}^d_{g\to 0}$ that are quasi full-rank, then the maximal elements of this subset are the quasi top-dimensional DT-morphisms. △

Definition 14.4. Given a DT-morphism φ_0, we denote by star-quasi(φ_0) the set of φ in $\mathrm{DTM}^d_{g\to 0}$ that have quasi full-rank and for which there is a specialization $\varphi \to \varphi_0$.

Note that star-quasi(φ_0) is a subset of $\uparrow\varphi_0$ in $(\mathrm{DTM}^d_{g\to 0})^{\mathrm{op}}$. We index the elements of star-quasi(φ_0) as $\varphi^{(q)} : G^{(q)} \to T^{(q)}$, and their specializations as $\rho^{(q)} : (\varphi^{(q)}, m^{(q)}) \to (\varphi_0, m_0)$. In Section 7 of Part I we computed star-quasi(φ_0) for codimension-1 φ_0 in $\mathcal{G}^{\mathrm{trop}}_{2g'\to 0,g'+1}$ such that $H(\varphi_0)$ is trivalent.

14.1.1 Contracting one edge t_1 of the target T down to v_0

Let (G_0, w_0) be a weighted graph and A_0 a fixed vertex with $w_0(A_0) = 0$. We construct all the specializations $\rho : (G, w) \to (G_0, w_0)$ such that $\operatorname{contr}(\rho) = \{e\}$, with $e \in E(G)$, and $\rho(e) = A_0$. Since $w_0(A_0) = 0$, the edge e is not a loop, so its ends B and C are distinct. Assume that $\operatorname{val} B \leq \operatorname{val} C$, and label the edges incident to A_0 as $e_2, e_3, \ldots, e_{\operatorname{val} A_0+1}$. By Subsection 11.6, the graph G is determined by which subset of $\{\rho^{-1}(e_2), \rho^{-1}(e_3), \ldots, \rho^{-1}(e_{\operatorname{val} A_0+1})\}$ is incident to B and which one to C. This gives a one-to-one correspondence between unordered partitions of $\{2, \ldots, \deg A_0 + 1\}$ into two parts S and $\{2, \ldots, \deg A_0 + 1\} \setminus S$ and graphs G_S contracting to G_0.

Now suppose we have a DT-morphism $\varphi_0 : G_0 \to T_0$. The following results are useful to construct φ that specialize to φ_0 by contracting one edge of the target of φ.

Lemma 14.5. *Let φ be a quasi full-rank DT-morphism, u and v vertices and t and t' edges of T such that $E(u) = \{t', t\}$ and $E(v) = \{t\}$. If $H(\varphi)$ has trivial weights, then there is exactly one edge of $H(\varphi)$ above t and one above t', and these are a loop and a bridge.*

Proof. This proof uses the glueing datum notation from Part I. Swap trees so that $1 \sim_v 2$. Remark 4.15 (change-minimal leaves) yields that $e_1 = \overline{(t,1)}$ and $e_2 = \overline{(t,2)}$ are two distinct edges of G, with $|e_1| = |e_2| = 1$, and are the only non-dangling edges above t. This implies, by the no-return condition, that $\overline{(u,1)}$ and $\overline{(u,2)}$ are the only non-dangling classes above u (it is possible that $\overline{(u,1)} = \overline{(u,2)}$). Let h be the edge of H containing e_1, e_2. If both $\overline{(u,1)}$ and $\overline{(u,2)}$ have non-dangling valency equal to 2, then the only edge of H passing above t, and t' is h. Thus, a_{ht} and $a_{ht'}$ are the only non-zero entries in the columns of A_M corresponding to t and t', respectively, a contradiction. Therefore, assume without loss of generality that $\operatorname{nd-val} \overline{(u,1)} = 3$. As $\operatorname{val} u = 2$ and $|e_1| = 1$, by Case (r1-nd3) of the local properties we must have that $\operatorname{nd-}E(\overline{(u,1)})$ has two edges above t; namely e_1, e_2, that make the loop; and one edge above t', the bridge. □

Lemma 14.5 is used to handle the case where $\operatorname{val} v_0 = 1$. The following Lemma is used in the other cases.

Lemma 14.6. *Let $(\rho_G, \rho_T) : [\varphi : G \to T] \to [\varphi_0 : G_0 \to T_0]$ be a specialization such that φ is quasi top-dimensional with trivalent $H(\varphi)$ and $\operatorname{contr}(\rho_T) = \{t\}$ with $t \in E(T)$. If $A_0 \in {\varphi_0}^{-1}(\rho_T(t))$ satisfies that $r_0(A_0) = 0$ and $w_0(A_0) = 0$, then*

$$\#(\operatorname{nd-}E({\rho_G}^{-1}(A_0))) \leq r_0(A_0) + 1. \tag{14.1}$$

Proof. Since $w_0(A_0)$ equals 0, the graph ${\rho_G}^{-1}(A_0)$ is a tree. Let A be in $\operatorname{nd}({\rho_G}^{-1}(A_0))$. The non-dangling valency of A is at most 3, because $H(\varphi)$ is trivalent. We claim that at most 2 of these edges are in $\varphi^{-1}(t)$. This follows from the no-return condition, which φ satisfies because it is quasi top-dimensional, when t is not adjacent to a leaf. If t is adjacent to a leaf, the claim follows from Proposition 12.51. Hence, $\operatorname{nd}({\rho_G}^{-1}(A_0))$ is a path. Moreover, if A is not an end of the path $\operatorname{nd}({\rho_G}^{-1}(A_0))$, then Proposition 12.51 gives that $r_\varphi(A) \geq 1$. Therefore, Equation (14.1) follows from the fact that the number of edges in a path is one more than the number of interior vertices. □

14.1.2 Vertices in the fibre $\varphi^{-1}(v_0)$

We say a few words about the vertices in $\varphi_0^{-1}(v_0)$.

Lemma 14.7. *With the notation of Remark 14.9 we have that $r_0(A_0) = \operatorname{ch} v_0$.*

Proof. If $\operatorname{val} v_0 = 4$, then $\operatorname{ch} v_0 = 0$ and the claim is clear. If $\operatorname{val} v_0 = 3$, then $\operatorname{ch} v_0 = 1$, $\operatorname{val} u = 2$, and Proposition 12.51 (local properties) on $A_1^{(q)}$ implies that $r_{\varphi^{(q)}}(A_1^{(q)}) = 1$, giving that $r_0(A_0) = 1$. Assume that $\operatorname{val} v_0$ is 2. If $\operatorname{val} u = 1$ and $\operatorname{val} v = 3$, then the edge going from $A_1^{(q)}$ to $A_2^{(q)}$ passes above u, so there is a vertex with $r_{\varphi^{(q)}}$-value equal to 2 contracting to A_0. If $\operatorname{val} u$, $\operatorname{val} v$ are both 2, then the local properties imply that $r_{\varphi^{(q)}}(A_1^{(q)})$ and $r_{\varphi^{(q)}}(A_2^{(q)})$ are both 1, proving the claim. □

By Lemma 14.7, any other vertex B above v_0 has $r_0(B) = 0$ and case {aux-r0} applies to it. In particular, the edges $G^{(q)}$, and their indices, that contract to B are determined by the target tree $T^{(q)}$. Hence, we would only need to look at the local part of $G^{(q)}$ contracting to A_0, namely $(\rho^{(q)})^{-1}(\uparrow A_0)$.

14.1.3 The elements of star-quasi(φ_0) and balancing on them

Again, as in Subsection 7.7 of Part I, we list all the possibilities for the local structure of φ_0 above v_0, and the elements in star-quasi(φ_0). We show that the counts of Type I, Type II, and Type III DT-morphisms in $\uparrow\varphi_0$ gives the same number. Again, we look at whether $\operatorname{val} v_0$ is 4, 3 or 2. By Subsection 14.1.2, we focus on the local part around A_0. The relations for the determinants are easier here, as the entry corresponding to $h_1^{(q)}$ and $t_1^{(q)}$ is the only non-null entry in its row and column:

Lemma 14.8 (equal multiplicity)**.** *Let φ_0 be in $\mathcal{G}^{trop}_{2g'\to 0,g'+1}$ with codimension 1, and $\varphi^{(q)}$ and $\varphi^{(q')}$ in* star-quasi(φ_0)*. If $H(\varphi_0)$ is non-trivalent, then*

$$m_\pi(\varphi^{(q)}) = m_\pi(\varphi^{(q')}).$$

Proof. We show that

$$k^{(q)} \det(A_{\varphi^{(q)}}) = k^{(q')} \det(A_{M^{(q')}}),$$

where $k^{(q)}$ is $\frac{1}{2}$ if t_1 leads to a leaf, else it is the cardinality of the class $e_1^{(q)}$ through which the contracted edge $h_1^{(q)}$ of $H^{(q)}$ passes. Similarly for $k^{(q')}$.

This calculation follows from Proposition 4.25 (edge-length map is local), a cofactor expansion of the row of $A_{M^{(q)}}$ corresponding to the contracted edge of $H^{(q)}$, and the fact that the only non-zero entry of that row is $1/k^{(q)}$, with $k^{(q)}$ as in the theorem statement. Similarly for q'. ☐

Hence, $\det(A_{M^{(q)}}) \neq 0$ for all q, because we assumed that at least one of the gluing datums around M_0 is full-dimensional.

14.1.4 What happens with $A_0 \in \varphi^{-1}(v_0)$ such that $r_\varphi(A) = 0$

Let (ρ_G, ρ_T) be a specialization from $\varphi : G \to T$ to $\varphi_0 : G_0 \to T_0$ such that contr(ρ_T) is a singleton $\{t\}$, and $A_0 \in G_0$ a vertex such that $r_0(A_0) = 0$ and $\varphi_0(A_0) = \rho_T(t)$. We continue studying the graph $\rho_G^{-1}(A_0)$.

We begin by showing that $T^{(q)} = T_{S^{(q)}}$ determines nd($\rho_G^{-1}(A_0)$) when $r_0(A_0)$ is 0. Since $\operatorname{val} v_0 \geq 2$, by Lemma 4.19 (r1 implies no-return) we have that A_0 satisfies the no-return condition. Thus, if nd-val $A_0 = 2$, then φ_0 is injective on nd-$E(A_0)$. If nd-val $A_0 = 3$, suppose that φ_0 is not injective on nd-$E(A_0)$, so $\sum_{e\in \text{nd-}E(A_0)} m_0(e) \leq 2m_0(A_0)$. Hence, $r_0(A_0) = \text{nd-val}\, A_0 - 2 + 2(|A_0| - \sum_{e\in\text{nd-}E(A_0)} |e| \geq 1$, a contradiction.

- Case {aux-r0-nd2}: Assume that nd-val A_0 is 2. Let e_α, e_β be the edges in nd-$E(A_0)$, above t_α and t_β respectively. By Lemma 7.6 the vertices of nd($\rho_G^{-1}(A_0)$) are the ends of $e_\alpha^{(q)}$ and $e_\beta^{(q)}$ above u or v. If $\{\alpha,\beta\} \subset S^{(q)}$, the ends of $e_\alpha^{(q)}$, $e_\beta^{(q)}$ are above u; since $\rho_G^{-1}(A_0)$ is connected, they equal one vertex A in $G^{(q)}$, which equals A_0 as subsets of $[d]$ by Lemma 6.16. Similarly if $\{\alpha,\beta\} \subset S'^{(q)}$. Otherwise, one end A_u is above u, the other end A_v above v, so they are distinct. By connectivity of nd($\rho_G^{-1}(A_0)$) there is one edge e' joining A_u, A_v. By Lemma 7.4 and since $r_0(A_0) = 0$, the vertices A_u and A_v belong to Case (r0-nd2) of the local properties. So, as subsets of $[d]$, the classes $e_\alpha^{(q)}$, A_u, e', A_v, $e_\beta^{(q)}$ are equal; and $h(e_\alpha^{(q)}) = h(e') = h(e_\beta^{(q)})$.
- Case {aux-r0-nd3}: Assume that nd-val A_0 is 3. Let $e_\alpha, e_\beta, e_\gamma$ be the edges in nd-$E(A_0)$, above t_α, t_β, t_γ, respectively. Since $|\varphi_0(\text{nd-}E(A_0))| = 3$ and $\max(|S^{(q)}|, |S'^{(q)}|) \leq 2$, both intersections $S^{(q)} \cap \varphi_0(\text{nd-}E(A_0))$ and $S'^{(q)} \cap \varphi_0(\text{nd-}E(A_0))$ are non-empty. One of these intersections is a singleton. Assume without loss of generality that the singleton is $\{\alpha\}$. By Lemma 7.5 there is

at most one edge in $\mathrm{nd}(\rho_G^{-1}(A_0))$, therefore at most two vertices. By Lemma 7.6 the vertices of $\mathrm{nd}(\rho_G^{-1}(A_0))$ are the end A_2 of $e_\alpha^{(q)}$ above u, and the vertex A_3 that is the end of both $e_\beta^{(q)}$ and $e_\gamma^{(q)}$ above v. So A_2 and A_3 are distinct, joined by a non-dangling edge e', and $\text{nd-val}\, A_2 = 2$, $\text{nd-val}\, A_3 = 3$. By the Case (r0-nd2) of the local properties, as subsets of $[d]$, we get that $e_\alpha^{(q)}$, A_2, and e' are equal, and $h(e_\alpha^{(q)}) = h(e')$. Note that they are also a subset of A_3. Finally Lemma 6.16 implies that $A_0 = A_3 \cup A_2 = A_3$.

14.1.5 A list of cases

In Figure 14.1 we show the hierarchy of all the cases

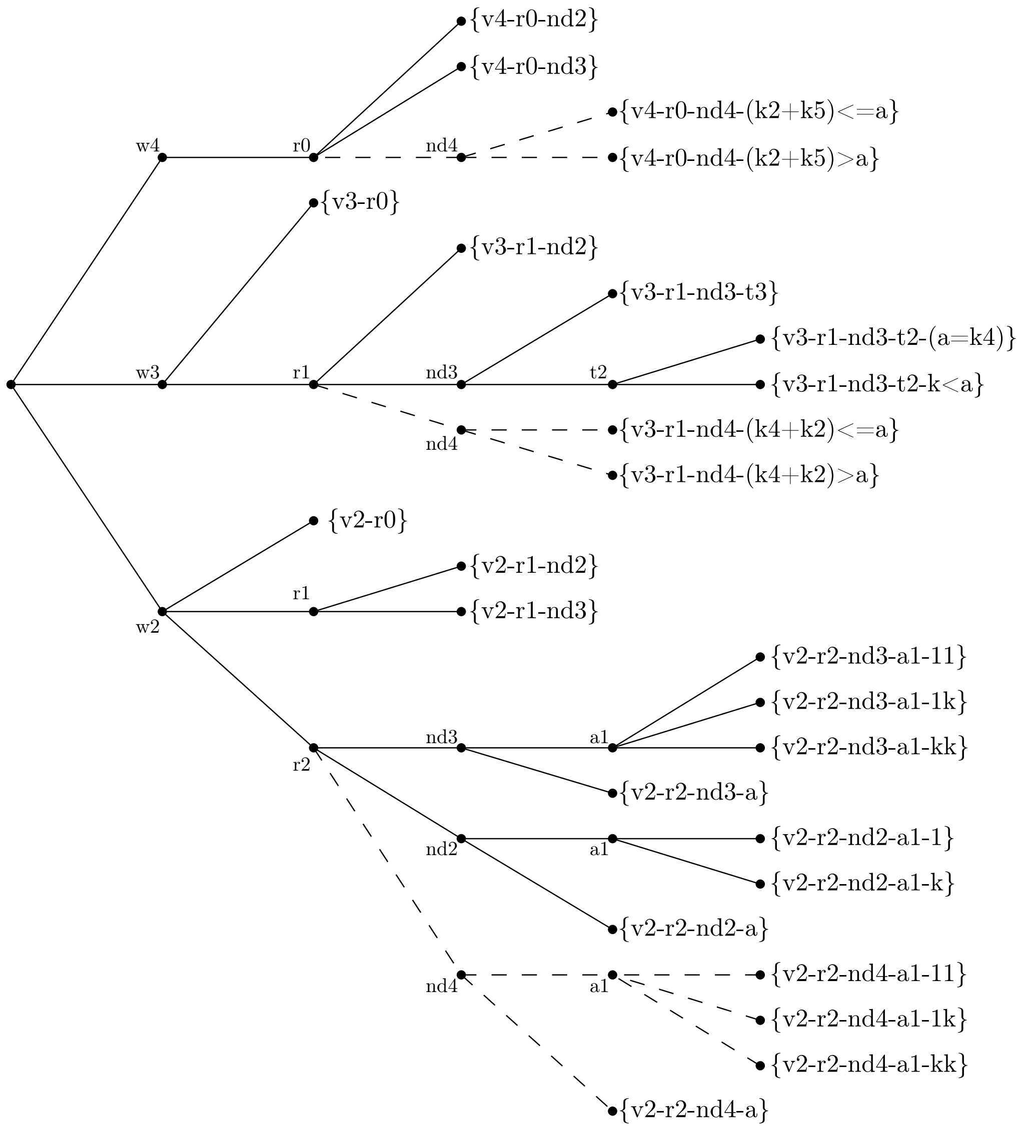


Figure 14.1: Logical flow of cases to regrow v_0. Cases with non-trivalent combinatorial types are shown dotted.

14.1.6 The case when $H(\varphi_0)$ is not trivalent

We investigate star-quasi(φ_0) in the case when $H(\varphi_0)$ is not trivalent. We have that $\dim C_\varphi^{\mathrm{src}} = \#(E(T_0)) = 3g-4$, which implies that $\#(E(H(\varphi_0))) = 3g-4$, so there is a unique vertex A_0 of $H(dtmor_0)$ that is not trivalent. Moreover, $\operatorname{val} A_0 = 4$ since $H(\varphi_0)$ arises by specializing one edge $E(\rho_H^{-1}(A_0))$ from $H(\varphi)$, which is trivalent.

We denote by $A_1^{(q)}$ and $A_2^{(q)}$ the endpoints of $h^{(q)}$, and u and v the endpoints of t.

14.1.7 Graphs contracting to $H(\varphi_0)$

Label the edges of $H(\varphi_0)$ incident to A_0 with 2, 3, 4, and 5. Construction a specialization $\rho^{(q)} : \rho^{(q)} \to \rho_0$ implies a specialization $\rho : H(\varphi^{(q)}) \to H(\varphi_0)$. By Remark 12.46, we desire $H(\varphi^{(q)})$ to be trivalent. Moreover, by our previous discussion we want $\operatorname{contr}(H(\varphi^{(q)}) \to H(\varphi_0)) = \{e_1\}$ and this e_1 is not a loop. So let $q^{(1)}$ and $q^{(2)}$ be the ends of e_1. By Subsection 11.6, to construct a trivalent graph which contracts to H_0, choose distinct indices α, β, γ, and δ in $\{2, .., 5\}$, to have $h_\alpha^{(q)}$, $h_\beta^{(q)}$ incident to $A_1^{(q)}$, and $h_\gamma^{(q)}$, $h_\delta^{(q)}$ incident to $A_2^{(q)}$. Denote this combinatorial type by $H_{\alpha,\beta}$, or H_S with $S = \{\alpha, \beta\}$. Furthermore, assume without loss of generality that $h_2^{(q)}$ is incident to $A_1^{(q)}$, that is $2 \in S$. We call a DT-morphism in star-quasi(φ_0) type I, II, or III if its combinatorial type is $H_{2,3}$, $H_{2,4}$, or $H_{2,5}$, respectively. See Figure 14.2 below.

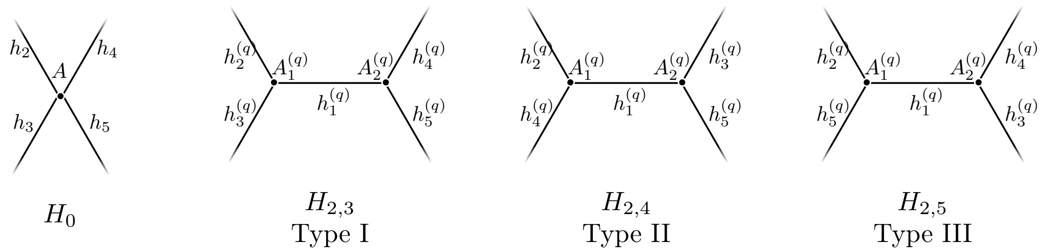


Figure 14.2

14.2 The case when $H(\varphi_0)$ is not trivalent

To summarize Subsection 14.1, the setup of our case-work is as follows:

Remark 14.9. We adopt the following notation.

1. g' is a positive integer
2. $\varphi_0 : G_0 \to T_0$, with index map m_0, is in $\mathcal{G}^{\text{trop}}_{g\to 0,g'+1}$ and has codimension-1
3. $v_0 \in V(T_0)$ is the unique vertex such that $\operatorname{ch} v_0 + \operatorname{val} v_0 - 3 > 0$
4. In fact, $\operatorname{ch} v_0 + \operatorname{val} v_0 - 3 = 1$
5. The possible values for v_0 are 2, 3 and 4
6. $A_0 \in V(H(\varphi_0))$ the unique vertex such that $\operatorname{val} A_0 > 3$
7. In fact, $\operatorname{val} A_0 = 4$
8. $a_0 = m_0(A_0)$
9. $\rho^{(q)} : (\varphi^{(q)} : G^{(q)} \to T^{(q)}, m^{(q)}) \to (\varphi_0, m_0)$ the specializations from quasi top-dimensional DT-morphisms to φ_0, indexed by q
10. For every $\rho^{(q)}$ we have that $\operatorname{contr}(\rho^{(q)})$ is a singleton $\{t_1\}$.
11. We have that $\rho^{(q)}(t_1) = v_0$
12. $r^{(q)}$ encodes the Riemann-Hurwitz inequalities of $\varphi^{(q)}$, or alternatively gives the coefficients of the ramification divisor of $\varphi^{(q)}$, see Equation (12.3) on Page 122
13. $h_1^{(q)} \in E((\rho_H^{(q)})^{-1}(A_0))$ the unique edge of $H(\varphi^{(q)})$ contracted by $\rho^{(q)}$
14. $A_1^{(q)}$ and $A_2^{(q)}$ the ends of $h_1^{(q)}$
15. $e_1^{(q)}$ the unique edge in $h_1^{(q)}$ in the cases $\operatorname{val} v_0 = 4$ and $\operatorname{val} v_0 = 3$
16. u, v the vertices of a given $T^{(q)}$ that contract to v_0
17. h_1, h_2, h_3 and h_4 the edges of $H(\varphi_0)$ incident to A_0.
18. e_2, e_3, e_4 and e_5 the non-dangling edges of G_0 incident to A_0
19. $k_i = m_{\varphi_0}(e_i)$
20. $e_i^{(q)}$ the only edge such that $\rho^{(q)}(e_i^{(q)}) = e_i$
21. $k_1^{(q)} = m_{\varphi_0}(e_1^{(q)})$
22. Goal: the construct all the possible $\rho^{(q)}$.

Throughout this subsection we refer to these facts, e.g. by Item 4 we have $\operatorname{ch} v = 4 - \operatorname{val} v_0$. △

14.2.1 The case where $\operatorname{val} v_0 = 4$ and $\text{nd-val}\, H_0 = 4$

Assume that $\operatorname{val} v_0 = 4$. By Item 4, we have that $\operatorname{ch} v_0 = 1 - (4-3) = 0$, so $r_0(A_0) = 0$ since $A_0 \in \varphi_0^{-1}(v_0)$. Thus, $r^{(q)}(A_1^{(q)}) = r^{(q)}(A_2^{(q)}) = 0$, because $r^{(q)}(A_i^{(q)}) \leq r_0(A_0)$. In this case u, v have to be trivalent to yield $\operatorname{val} v_0 = 4$. So case (r0) of the local properties applied on $A_1^{(q)}$, $A_2^{(q)}$ implies that the non-dangling edges of G_0 incident to A_0 are above distinct edges of T_0. Also that $e_1^{(q)}$ is the unique edge of $G^{(q)}$ above t_1 which contracts to A_0.

Label the edges of G_0 and T_0 such that e_i is above t_i and $m_0(e_i) \leq m_0(e_j)$ when $i < j$. There are three trees that contract to T_0, these are $T_{2,3}, T_{2,4}$, and $T_{2,5}$. Fix S and let $T^{(q)} = T_S$, so the combinatorial type is H_S. Label the edges of H_0 so h_i passes above t_i by going through e_i. Lemma 12.50 (nd. r_φ formula) applied to A implies that:

$$k_2 + k_3 + k_4 + k_5 = 2a_0 + 2. \tag{14.2}$$

Observe $(\rho^{(q)})^{-1}(A_0)$ is determined by the indices of $A_1^{(q)}$, $A_2^{(q)}$, $k_1^{(q)}$ and the target tree. It turns out to be convenient, for each $\varphi^{(q)}$, to introduce indices $\alpha, \beta, \gamma, \delta$ and relabel $A_1^{(q)}$ and $A_2^{(q)}$, with $A_+^{(q)}$, $A_-^{(q)}$, such that the following conditions are fulfilled:

(a) $e_\alpha^{(q)}$ and $e_\beta^{(q)}$ are incident to $A_-^{(q)}$, and $e_\gamma^{(q)}$ and $e_\delta^{(q)}$ are incident to $A_+^{(q)}$;

(b) $k_\alpha \leq k_\beta$, and $k_\gamma \leq k_\delta$;

(c) $k_\alpha + k_\beta \leq k_\gamma + k_\delta$, and if $k_\alpha + k_\beta = k_\gamma + k_\delta$, then $\alpha = 2$;

(d) if $2 \in \{\alpha, \beta\}$, then $\alpha = 2$,

(e) if $5 \in \{\gamma, \delta\}$ then $\delta = 5$.

These conditions can always be met, and the way is unique. The motivation for this notation is that, since both $A_-^{(q)}$ and $A_+^{(q)}$ are incident to $e_1^{(q)}$, Lemma 12.50 (nd. r_φ formula) implies that $m^{(q)}(A_-^{(q)}) \leq m^{(q)}(A_+^{(q)})$.

Observe that $\max(k_\gamma, k_\delta) = k_\delta \leq m^{(q)}(A_+^{(q)}) \leq a_0$, and that fixing a value for $m^{(q)}(A_+^{(q)})$ determines the values of $k_1^{(q)}$ and $m^{(q)}(A_-^{(q)})$. So let $m^{(q)}(A_+^{(q)}) = a_0 - K$, for some non-negative K. By Lemma 12.50 we have that:

$$\begin{aligned} m^{(q)}(A_+^{(q)}) &= a_0 - K, \\ k_1^{(q)} &= k_\alpha + k_\beta - 1 - 2K, \\ m^{(q)}(A_-^{(q)}) &= k_\alpha + k_\beta - 1 - K. \end{aligned}$$

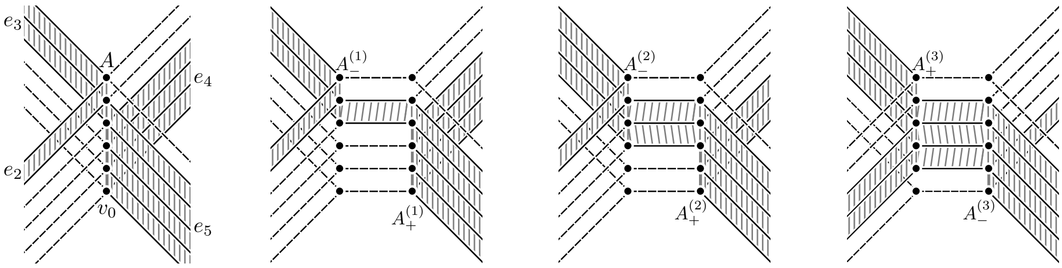
With $K = 0$ we get $m^{(q)}(A_+^{(q)}) = a_0$ and it is always possible to construct a quasi top-dimensional DT-morphism, independently of the choice of target tree. See constructions below. From there, the value of K increases, thus decreasing the value of $m^{(q)}(A_+^{(q)})$ until either $m^{(q)}(A_+^{(q)})$ equals $k_\delta = \max(k_\gamma, k_\delta)$ or $m^{(q)}(A_-^{(q)})$ equals $k_\beta = \max(k_\alpha, k_\beta)$.

Now we give a numeric interval such that K is in this interval if and only if the necessary upper bounds $a_0 \geq m^{(q)}(A_+^{(q)}), m^{(q)}(A_-^{(q)}), k_1^{(q)}$, and the necessary lower bounds $m^{(q)}(A_+^{(q)}) \geq k_\delta$, $m^{(q)}(A_-^{(q)}) \geq k_\beta$, $k_1^{(q)} \geq 1$, are satisfied. The key point is that this interval only depends on the values k_i, not on the specific target tree. Moreover, we show how to construct $\varphi^{(q)}$ when these bounds are satisfied. Thus, for each choice of target tree, and hence of combinatorial type, there is the same number of quasi top-dimensional

The upper bounds follow from substituting the inequality $a_0 + 1 \geq k_\alpha + k_\beta$ in the expressions depending on K. This inequality follows from Equation (14.2) and the fact that $k_\alpha + k_\beta \leq k_\gamma + k_\delta$. Establishing the lower bounds requires two cases. The interval to which K belongs depends on these cases as well. The main question is whether $k_\delta = k_5$ or $k_\alpha = k_2$. This is dictated by whether $k_5 + k_2 \leq a_0 + 1$ or $k_5 + k_2 > a_0 + 1$.

- Case {v4-nd4-1}: Assume that $k_5+k_2 \le a_0+1$. We get $k_2+k_* \le k_*+k_*$ for all possible choices to fill the $*$, with strict inequality if $k_5+k_2 < a_0+1$. The strict inequality implies $k_\alpha = k_2$, and for equality we have chosen already $k_\alpha = k_2$. It follows that $m^{(q)}(A_-^{(q)}) = k_2+k_\beta-1-K \ge k_\beta$ if and only if $K \le k_2-1$. Now suppose that $K \le k_2-1$, so the second lower bound is satisfied. Also, it gives $m^{(q)}(A_+^{(q)}) = a_0-K \ge a_0-k_2+1$. Observe that $a_0-k_2+1 \ge k_\delta$ if and only if $k_2+k_\delta \le a_0+1$, so the first lower bound is satisfied. For the remaining one, writing $k_1^{(q)} = m^{(q)}(A_-^{(q)})-K$ gives $k_1^{(q)} \ge 1$, because $m^{(q)}(A_-^{(q)})-K \ge k_\alpha-(k_\alpha-1) \ge 1$.

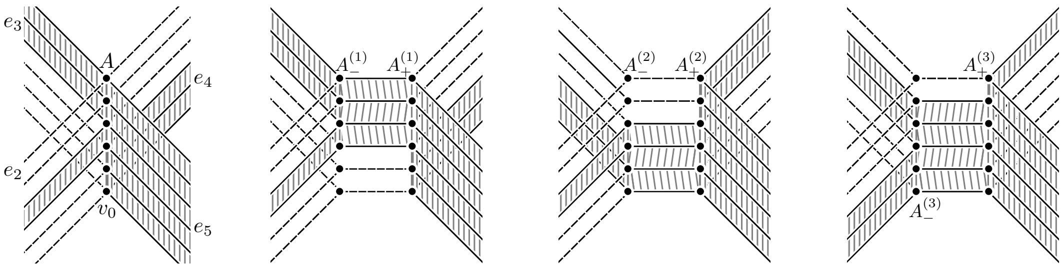
  There are k_2 DT-morphisms for each target tree. See figures below, where $K=1$.



φ_0 | $\varphi^{(1)}$ Type I | $\varphi^{(2)}$ Type II | $\varphi^{(3)}$ Type III

- Case {v4-nd4-2}: Assume that $k_5+k_2 > a_0+1$. We get $k_5+k_* > k_*+k_*$ for all possible choices. Thus $k_\delta = k_5$. It follows that $m^{(q)}(A_+^{(q)}) = a_0-K \ge k_5$ if and only if $K \le a_0-k_5$. Now suppose $K \le a_0-k_5$, so the first lower bound is satisfied. It also gives $m^{(q)}(A_-^{(q)}) = k_\alpha+k_\beta-1-K \ge k_\alpha+k_\beta+k_5-1-a_0$. Replacing with Equation (14.2) gives $m^{(q)}(A_-^{(q)}) \ge a_0-k_\gamma$. Observe that $a_0-k_\gamma \ge k_\beta$ if and only if $k_\beta+k_\gamma \le a_0$, true because otherwise by Equation (14.2) it would happen that $k_\alpha+k_5 \le a_0+1$, a contradiction. This gives the second lower bound. For the remaining one, writing $k_1^{(q)} = m^{(q)}(A_-^{(q)})-K$ gives $k_1^{(q)} \ge 1$, because $m^{(q)}(A_-^{(q)})-K \ge k_5-(a_0-k_5) > 1$, since $2k_5 \ge k_5+k_2 > a_0+1$.

  There are a_0-k_5 DT-morphisms for each target tree. See figures below, where $K=0$.



φ_0 | $\varphi^{(1)}$ Type I | $\varphi^{(2)}$ Type II | $\varphi^{(3)}$ Type III

If $k_5+k_2 \le a_0+1$, then $k_2-1 = \min(k_2-1, a_0-k_5)$. Otherwise, $\min(k_2-1, a_0-k_5)$ is a_0-k_5, so $0 \le K \le \min(k_2-1, a_0-k_5)$ and each possible value in the range gives a different DT-morphism with target tree T_S. As different trees give different combinatorial types H_S, we obtain the three possible types.

14.2.2 The case where $\operatorname{val} v_0 = 3$ and $\operatorname{nd-val} H_0 = 4$

Assume that $\operatorname{val} v_0 = 3$. Thus, $\operatorname{val} u$ is 2 and $\operatorname{val} v$ is 3, so $A_1^{(q)}$ and $A_2^{(q)}$ are above u and v, respectively. For convenience relabel $A_1^{(q)}$, $A_2^{(q)}$, with $A_v^{(q)}$, $A_u^{(q)}$, respectively.

So we have that $r_0(A) = 1$, $r_{\varphi^{(q)}}(A_u^{(q)}) = 1$ and $r_{\varphi^{(q)}}(A_v^{(q)}) = 0$. Lemma 12.50 (nd. r_φ formula) gives that $r_0(A) = 2a_0 + 4 - 2 - (k_2 + k_3 + k_4 + k_5)$. By the no-return condition, the e_i are above at least two edges of T_0. If they were above exactly two edges of T_0, then the last parenthesis in the expression for $r_0(A)$ would be at most $2a_0$, contradicting that $r_0(A) = 1$. Thus, in G_0 exactly two of the edges incident to A are above the same edge of T_0, and the other two are each above distinct edges of T_0. Hence label the edges of T_0 and G_0 such that:

(a) e_2 and e_5 are above t_2

(b) e_3 is above t_3

(c) e_4 is above t_4

(d) $k_2 \le k_5$ and $k_3 \le k_4$

As $r_0(A_0) = 1$, Lemma 12.50 (nd. r_φ formula) gives:

$$k_2 + k_3 + k_4 + k_5 = 2a_0 + 1. \tag{14.3}$$

This implies an important observation: that $k_4 + k_5 \ge a_0 + 1$, since $k_5 \ge k_2$ and $k_4 \ge k_3$. Another one is that $a_0 \ge k_2 + k_5$, because e_2 and e_5 are above the same edge of T_0.

There are three options for the singleton $S^{(q)}$, which determine the target tree. We show that the target tree and the values k_i determine $m^{(q)}(A_v^{(q)})$, $k_1^{(q)}$, and $m^{(q)}(A_u^{(q)})$. First let $T^{(q)}$ be T_2. The case (r1-nd3) applied to $A_u^{(q)}$ implies that $m^{(q)}(A_u^{(q)}) = k_1^{(q)} = k_2 + k_5$. The case (r0-nd3) applied to $A_v^{(q)}$ implies that $2m^{(q)}(A_v^{(q)}) + 1 = k_2 + k_3 + k_1^{(q)}$, and it follows from Equation (14.3) that $m^{(q)}(A_v^{(q)}) = a_0$. This gluing datum has type III. See the figures below.

Second, let $T^{(q)}$ be T_α with α equal to 3 or 4. The case (r1-nd3) applied to $A_u^{(q)}$ implies that there are two non-dangling edges of $G^{(q)}$ above t_1 incident to $A_u^{(q)}$, namely $e_1^{(q)}$ and some other e' in $G^{(q)}$. Also that $k_\alpha = k_1^{(q)} + m^{(q)}(e')$. So e' is incident to $A_u^{(q)}$ and some A' above v. Note that nd-val A' is 2, so either A' is incident to $e_2^{(q)}$ or $e_5^{(q)}$.

To explore the two options, let $e_\delta^{(q)}$ be the edge incident to A'. So e_α is above t_α and e_δ is above t_2. Let β be such that e_β is above t_3 or t_4, γ such that e_γ is above t_2, and $\{\alpha, \beta, \gamma, \delta\}$ equals $\{2, 3, 4, 5\}$. The pair (α, δ) implies some inequalities. The case (r0-nd2) applied to A' gives that $m^{(q)}(e') = m^{(q)}(e_\delta^{(q)}) = k_\delta$, so $k_1^{(q)} = k_\alpha - k_\delta$. In particular:

$$k_\alpha > k_\delta.$$

The case (r0-nd3) applied to $A_v^{(q)}$ gives $2m^{(q)}(A_v^{(q)}) + 1 = k_\beta + k_\gamma + k_1^{(q)}$. Equation (14.3) gives $2m^{(q)}(A_v^{(q)}) + 1 = 2a_0 + 1 - 2k_\delta$, so $m^{(q)}(A_v^{(q)}) = a_0 - k_\delta$.

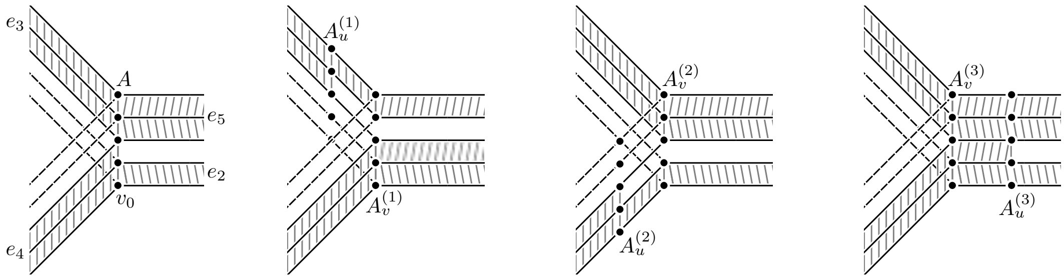
As $m^{(q)}(A_v^{(q)}) = a_0 - k_\delta$ is greater than $e_1^{(q)} = k_\alpha - k_\delta$, k_γ, k_β, we get three inequalities: first $a_0 \ge k_\alpha$; second $a_0 \ge k_\gamma + k_\delta = k_2 + k_5$; and third $a_0 \ge k_\beta + k_\delta$. The first two are always true. It follows that we can get a DT-morphism with an associated pair (α, δ) and with target tree T_α if and only if $k_\alpha > k_\delta$ and $a_0 \ge k_\beta + k_\delta$. Moreover, observe that $a_0 \ge k_\beta + k_\delta$ is equivalent to $a_0 + 1 \le k_\alpha + k_\gamma$ by Equation (14.3). If $a_0 + 1 \le k_\alpha + k_\gamma$, then $a_0 \ge k_\gamma + k_\delta = k_2 + k_5$ implies that $k_\alpha \ge a_0 + 1 - k_\gamma \ge k_\delta + 1 > k_\delta$. Thus, only

$$a_0 \ge k_\beta + k_\delta, \quad \text{or equivalently,} \quad a_0 + 1 \le k_\alpha + k_\gamma,$$

is needed as a condition. The resulting combinatorial type is $H_{\alpha\delta}$ if $\delta = 2$, or $H_{\beta\gamma}$ if $\delta \ne 2$.

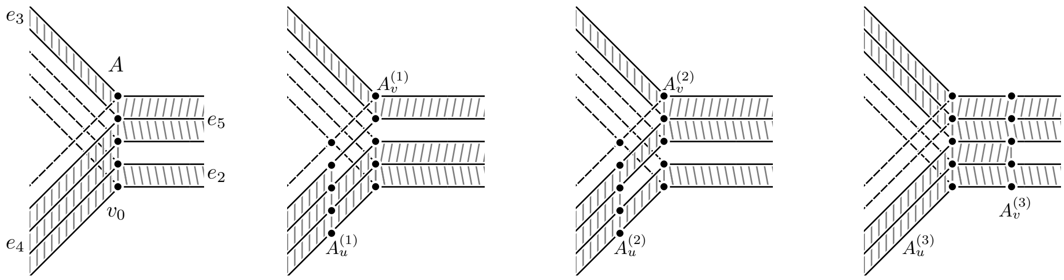
The four cases for (α, δ) are $(4, 2)$, $(3, 5)$, $(3, 2)$, and $(4, 5)$. The first two cases have combinatorial type II, and the last two cases combinatorial type I. To the first case corresponds the inequality $a_0 + 1 \le k_4 + k_5$, which has been proven to be always true. The second reverses the inequalities, so this case never happens. The third case needs $k_4 + k_2 \le a_0$, and the fourth needs $a_0 + 1 \le k_4 + k_2$, so they are mutually exclusive. Hence there is exactly one gluing datum of each type.

- Case {v3-nd4-1}: Assume that $k_4 + k_2 \leq a_0$. Type I is realised by the pair $(3, 2)$. See figures below.



φ_0 | $\varphi^{(1)}$ Type I | $\varphi^{(2)}$ Type II | $\varphi^{(3)}$ Type III

- Case {v3-nd4-2}: Assume that $k_4 + k_2 \geq a_0 + 1$. Type I is realised by the pair $(4, 5)$. See figures below.



φ_0 | $\varphi^{(1)}$ Type I | $\varphi^{(2)}$ Type II | $\varphi^{(3)}$ Type III

14.2.3 The case where $\operatorname{val} v_0 = 2$ and $\operatorname{nd-val} H_0 = 4$

Assume that $\operatorname{val} v_0 = 2$. We say that $\varphi^{(q)}$ has target I or II when its target tree is $T_\varnothing$ or T_2, respectively. By a similar reasoning as done in Subsection 7.7.4 we have that target I gives a quadruple $[e_\alpha, e_\beta; e_\gamma, e_\delta]$ of the edges incident to A where $k_\alpha = k_\beta$ and $k_\gamma = k_\delta$. Target II can be regarded as a transition of the relations above t_2 to those above t_3 via two changes. A change is either splitting a class, or merging two classes. In both targets we have $r_0(A_0) = 2$.

By the no-return condition there are two cases: two of the non-dangling edges incident to A_0 are above the same edge of T_0, and the other two are above a different edge of T_0; or there are three above one and one above the other. Label the edges of G_0 and T_0 such that: $\min(k_2, k_3, k_4, k_5) = k_2$; e_3, e_4 are above t_2 with $k_3 \leq k_4$; e_5 is above t_3.

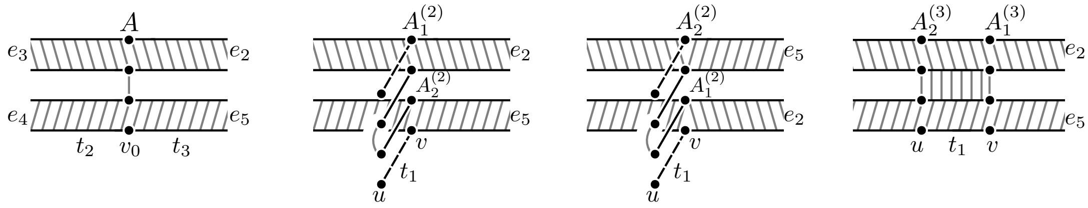
- Case {v2-nd4-t3}: Assume that $\varphi(e_2) = t_3$. Lemma 12.50 (nd. r_φ formula) applied to A_0 gives

$$r_\varphi(A_0) = 2 = 2 + 2a_0 - (k_2 + k_3 + k_4 + k_5).$$

  As $k_2 + k_5 \leq a_0$ and $k_3 + k_4 \leq a_0$, there is in fact equality. This gives $k_2 \leq k_3 \leq k_4 = a_0 - k_3 \leq a_0 - k_2 = k_5$. These inequalities are relevant for the $\varphi^{(q)}$ with target II, because if a class e_α splits above u then it must produce one class e_δ that is above t_3, namely $k_\alpha > k_\delta$, for some α equal to 2 or 5, and δ equal to 3 or 4. On the other hand, $\varphi^{(q)}$ can have target I if k_2 is equal to both k_3 and k_4, or to one of them. This divides into the following cases.

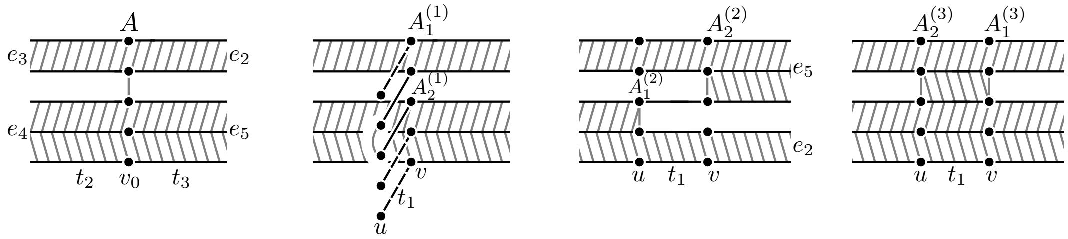
- Case {v2-nd4-t3-k2=k4}: Assume that $k_2 = k_4$. The indices k_2, k_3, k_4, and k_5 are equal, since $k_2 \leq k_3 \leq k_4 \leq k_5 = a_0 - k_2$ and $a_0 = k_3 + k_4$. There are two $\varphi^{(q)}$ with target I, corresponding to

$[e_3, e_2; e_4, e_5]$ and $[e_3, e_5; e_4, e_2]$. These are $\varphi^{(1)}$ and $\varphi^{(2)}$ respectively. There is just one possibility for target II: two classes merge above v. This is $\varphi^{(3)}$. See figures below.



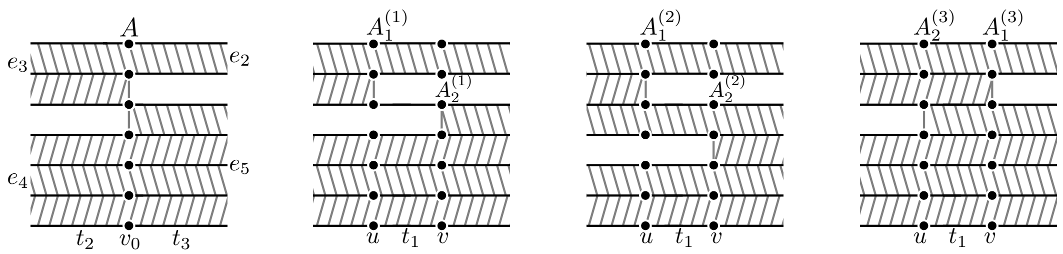
φ_0 | $\varphi^{(1)}$ Type I | $\varphi^{(2)}$ Type II | $\varphi^{(3)}$ Type III

- Case {v2-nd4-t3-k2=k3}: Assume that $k_2 = k_3 \neq k_4$. We have that $k_2 = k_3 < k_4 = k_5$. There is one $\varphi^{(q)}$ with target I, corresponding to $[e_3, e_2; e_4, e_5]$. It is $\varphi^{(1)}$. There are two possibilities for target II: the class e_4 splits above u because $k_4 > k_2$; or two classes merge above u. They are $\varphi^{(2)}$ and $\varphi^{(3)}$ respectively. See figures below.



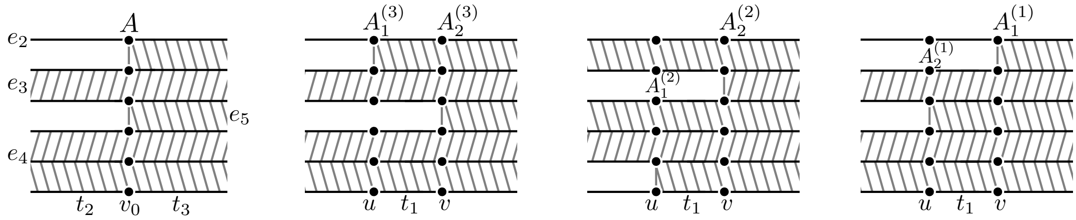
φ_0 | $\varphi^{(1)}$ Type I | $\varphi^{(2)}$ Type II | $\varphi^{(3)}$ Type III

- Case {v2-nd4-t3-k2<k3}: Assume that k_2 is distinct from k_3 or k_4. We have that $k_2 < k_3 < k_4 < k_5$. There are no $\varphi^{(q)}$ with target I. There are three possibilities for target II: one is a class split above u because $k_4 > k_2$; the second is similar because $k_3 > k_2$; and last, two classes merge above u. These are $\varphi^{(1)}$, $\varphi^{(2)}$ and $\varphi^{(3)}$ respectively. See figures below.



φ_0 | $\varphi^{(1)}$ Type I | $\varphi^{(2)}$ Type II | $\varphi^{(3)}$ Type III

- Case {v2-nd4-t2}: Assume that $\varphi(e_2) = t_2$. Note that $a_0 = k_5$, so in fact $k_2 \leq k_3 \leq k_4 < k_5$. Lemma 12.50 (nd. r_φ formula) applied to A_0 implies that $a_0 = k_5 = k_2 + k_3 + k_4$. There is no $\varphi^{(q)}$ with target I. There are three with target II. Observe that above u two classes merge, and there are three possibilities: e_2 and e_3 merge; e_2 and e_4 merge; and e_3 and e_4 merge. Above v another merge happens, to produce e_5. These are $\varphi^{(1)}$, $\varphi^{(2)}$ and $\varphi^{(3)}$, respectively. See figures below.



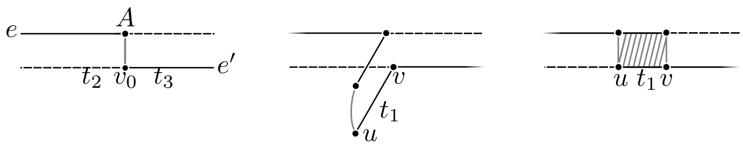
φ_0 $\varphi^{(1)}$ Type I $\varphi^{(2)}$ Type II $\varphi^{(3)}$ Type III

14.3 Forbidden local parts

Many results in graph theory concern forbidden minors, that is, given a graph theoretic property, a description of local structures that cannot appear in a graph with said property. In this subsection we derive several results in such style regarding the property of being full rank, and we use them to prove Proposition 14.10. This proposition was used in the proof of several formulas for $m_{\widetilde{\pi}}$.

We group them together in function of common limits. These limits appeared already in Subsection 7.7 of Part I, where it was argued that they cannot arise as a limit of a quasi top-dimensional DT-morphism. We keep the same naming for these limits as in Part I and reproduce the arguments that make them a forbidden part. Let $\mathbf{v}_i$ be the column of A_φ corresponding to the edge t_i.

- Case {v2-r2-nd2-ka-1}: Let $\varphi^{(1)}$ and $\varphi^{(2)}$ be gluing datums with local parts as shown below, and let φ_0 arise from contracting t_1.

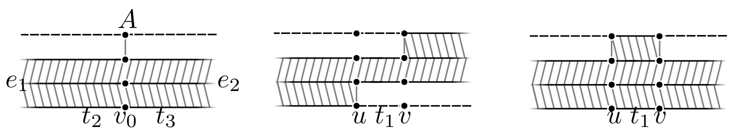


$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_1^{(1)}$	$\mathbf{v}_1^{(2)}$
1	1	2	$\frac{1}{2}$
a_i	a_i	0	a_i

φ_0 $\varphi^{(1)}$ $\varphi^{(2)}$

Since the columns $\mathbf{v}_2$ and $\mathbf{v}_3$ are equal, these local parts imply that $\varphi^{(q)}$ is not full-rank.

- Case {v2-r2-nd2-ka-2}: Let $\varphi^{(1)}$ and $\varphi^{(2)}$ be gluing datums with local parts as shown below, and let φ_0 arise from contracting t.

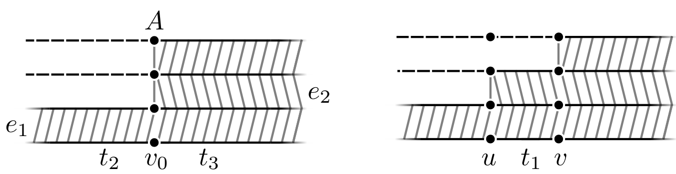


$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_1^{(1)}$	$\mathbf{v}_1^{(2)}$
$\frac{1}{k}$	$\frac{1}{k}$	$\frac{1}{k-1}$	$\frac{1}{k+1}$
a_i	a_i	a_i	a_i

φ_0 $\varphi^{(1)}$ $\varphi^{(2)}$

Since the columns $\mathbf{v}_2$ and $\mathbf{v}_3$ are equal, these local parts imply that $\varphi^{(q)}$ is not full-rank.

- Case {v2-r2-nd2-kb}: Let $\varphi^{(1)}$ be the gluing datum with a local part as shown below, and let φ_0 arise from contracting t.



$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_1^{(1)}$
$\frac{1}{k}$	$\frac{1}{k+2}$	$\frac{1}{k+1}$
a_i	a_i	a_i

φ_0 $\varphi^{(1)}$

Since $2(k+1)\mathbf{v}_1^{(1)} = k\mathbf{v}_1^{(1)} + (k+2)\mathbf{v}_1^{(1)}$, this local part implies that $\varphi^{(q)}$ is not full-rank.

Now we are ready to show:

Proposition 14.10. *Let d and g be integers, φ in $\mathrm{DTM}^d_{g\to 0}$, and h be in $H(\varphi)$. If we write $h = \langle A_0, e_1, A_1, \ldots, A_{\nu-1}, e_\nu, A_\nu\rangle$, we have that*

- *If h passes above a leaf, then $k_i = 1$ and either $\varphi(A_1)$ or $\varphi(A_{\nu-1})$ lead to a leaf.*
- *Otherwise, there are constants k and μ such that $1 \leq \mu \leq \nu$ and $k_i = k$ for $i \leq \mu$, and $k_j = k+1$ for $\mu < k$.*

Proof. It is enough to prove the proposition for full-rank change-minimal φ. Let e, e' be adjacent edges in h. By Cases (r0), (r1) and (r2) of Proposition 4.21 (local properties) we have that $m_\varphi(e)$, $m_\varphi(e')$ differ by at most 1.

Assume that h passes above a leaf. Let e_1 in h be above t_1 with ends u, v such that v is a leaf. If $\operatorname{val} u = 2$ we are done by Lemma 13.9, so assume that $\operatorname{val} u = 3$ and suppose there is e_2 in h such that $m_\varphi(e_2) > 1$. Since $m_\varphi(e_1) = 1$, we may assume that e_2 is adjacent to e_3 with $m_\varphi(e_2) \neq m_\varphi(e_3)$. By the local properties this implies that, if A is the common end of e_2, e_3, then $\operatorname{val}\varphi(A) = 2$, so $t_1 = \varphi(e_1)$, $t_2 = \varphi(e_2)$, and $t_3 = \varphi(e_3)$ are three distinct edges. Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ be the columns in A_φ corresponding to t_1, t_2, t_3, respectively. Since φ is change-minimal and by the local properties, we have that $\mathbf{v}_1$ has a single non-zero entry corresponding to the edge h, and that the entries of $\mathbf{v}_2$, $\mathbf{v}_3$ coincide outside of the entry corresponding to h. Thus $\mathbf{v}_2 - \mathbf{v}_3$ is a scalar multiple of $\mathbf{v}_1$, contradicting that φ is full-rank. Also, if neither $\varphi(e_1)$ nor $\varphi(e_\nu)$ lead to a leaf, then we would have a local part like in Case {v2-r2-nd2-M-1}, which also contraditcs that φ is full-rank.

Assume that h does not pass above a leaf. Suppose that there are indices $\alpha < \beta < \gamma$ such that $m_\varphi(e_\alpha) \neq m_\varphi(e_\beta)$ and $m_\varphi(e_\beta) \neq m_\varphi(e_\gamma)$. We may assume that $\beta = \alpha + 1$, and that $m_\varphi(e_i) = m_\varphi(e_\beta)$ for $\beta < i < \gamma$. By contracting $\varphi(e_{\gamma-1})$ and applying either Case w3-r1-nd2 or Case w2-r1-nd2 of the deformation to regrow it (see Appendix 14.4 for a Summary), we construct φ_1 such that $m_\varphi(e_{\gamma-1}) = m_\varphi(e_\gamma)$ and φ_1 is top-dimensional in $\mathrm{DTM}^d_{g\to 0}$, because φ is. Iterating, contracting $\varphi(e_{\gamma-i})$ and regrowing to a different DT-morphism, we obtain φ_i in $\max \mathrm{DTM}^d_{g\to 0}$ (because φ_{i-1} is), with $m_\varphi(e_{\gamma-i}) = m_\varphi(e_\gamma)$. Finally in $\varphi_{\gamma-\beta-1}$ we have that $m_\varphi(e_\alpha) \neq m_\varphi(e_{\alpha+1})$ and $m_\varphi(e_{\alpha+1}) \neq m_\varphi(e_{\alpha+2})$. But this implies a forbidden local part, contradicting that φ_i is in $\max \mathrm{DTM}^d_{g\to 0}$. □

14.4 Trivalent deformation

We review the constructions necessary to prove the balancing condition for Π when φ_0 is codimension-1 and $H(\varphi_0)$ is trivalent. We present the diagrams, the local part above v_0, the relevant columns of A_0 and $A^{(q)}$, the wall-relation and say a few remarks. For the proof that these cases are exhaustive see Section 7 of Part I. Let A_0 in G_0 be a vertex above v_0. Set $k_i = m_{\varphi_0}(e_i)$. From Case {v3-r1-nd3-t2-(a=k4)} onwards, A_0 is the unique vertex with r_0-value greater or equal than 1.

- Case {v4-r0}: The wall-relation in this case is:

$$\mathbf{v}_1^{(1)} + \mathbf{v}_1^{(2)} + \mathbf{v}_1^{(3)} = \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 + \mathbf{v}_5. \tag{14.4}$$

  It is obtainted by applying Cases {v4-r0-nd2} and {v4-r0-nd3}, detailed below, to the vertices above v_0.

- Case {v4-r0-nd2}: Assume that nd-$E(A_0) = \{e_\alpha, e_\beta\}$ with $\varphi_0(e_\alpha) = t_\alpha$, $\varphi_0(e_\beta) = t_\beta$.

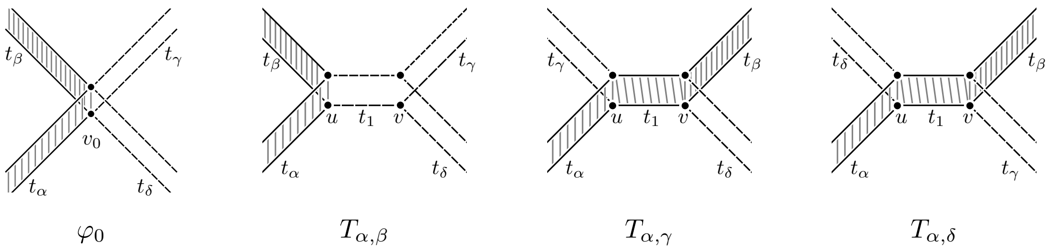


Figure 14.3

The local parts of A_0, first four columns, and of $A^{(q)}$, last three columns, are:

$\mathbf{v}_\alpha$	$\mathbf{v}_\beta$	$\mathbf{v}_\gamma$	$\mathbf{v}_\delta$	$\mathbf{v}_1^{(1)}$	$\mathbf{v}_1^{(2)}$	$\mathbf{v}_1^{(3)}$
$\frac{1}{k}$	$\frac{1}{k}$	0	0	0	$\frac{1}{k}$	$\frac{1}{k}$

- Case {v4-r0-nd3}:

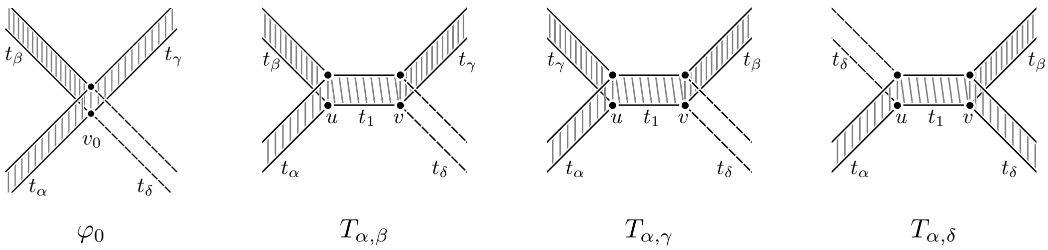


Figure 14.4

The local parts of A_0, first four columns, and of $A^{(q)}$, last three columns, are:

$\mathbf{v}_\alpha$	$\mathbf{v}_\beta$	$\mathbf{v}_\gamma$	$\mathbf{v}_\delta$	$\mathbf{v}_1^{(1)}$	$\mathbf{v}_1^{(2)}$	$\mathbf{v}_1^{(3)}$
$\frac{1}{k_\alpha}$	0	0	0	0	0	$\frac{1}{k_\alpha}$
0	$\frac{1}{k_\beta}$	0	0	0	$\frac{1}{k_\beta}$	0
0	0	$\frac{1}{k_\gamma}$	0	$\frac{1}{k_\gamma}$	0	0

- Case {v3-r1-nd3-t2-(a=k4)}: Here A_0 is the unique vertex in $\varphi_0^{-1}(v_0)$ such that $r_0(A_0) = 1$, we have that φ is injective on nd-$E(A_0)$, and $m_0(A_0) = m_0(e_4)$.

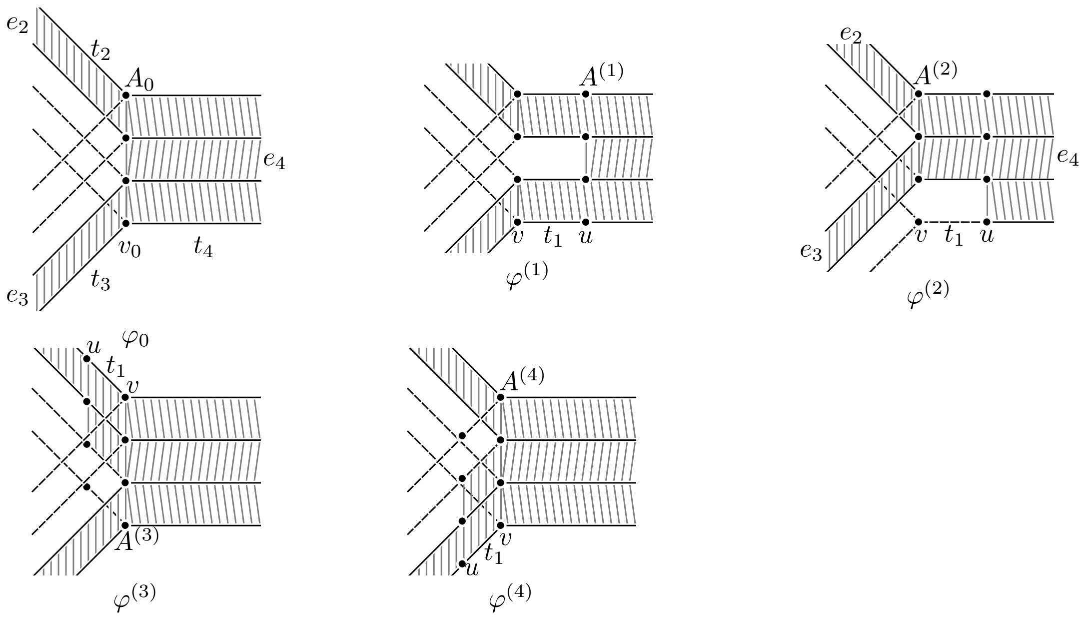


Figure 14.5

The local parts of A_0, first three columns, and of $A^{(q)}$, last four columns, are:

$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_1^{(1)}$	$\mathbf{v}_1^{(2)}$	$\mathbf{v}_1^{(3)}$	$\mathbf{v}_1^{(4)}$
$\frac{1}{k_2}$	0	0	$\frac{1}{k_2}$	0	$\frac{1}{k_2+1}$	0
0	$\frac{1}{k_3}$	0	$\frac{1}{k_3}$	0	0	$\frac{1}{k_3+1}$
0	0	$\frac{1}{k_2+k_3}$	0	$\frac{1}{k_2+k_3-1}$	0	0
a_i	b_i	c_i	c_i	c_i	a_i	b_i

The wall-relation is:

$$\mathbf{v}_1^{(1)} + (k_1 + k_2 - 1)\mathbf{v}_1^{(2)} + (k_2 + 1)\mathbf{v}_1^{(3)} + (k_3 + 1)\mathbf{v}_1^{(4)} \tag{14.5}$$
$$= (k_2 + 1)\mathbf{v}_2 + (k_3 + 1)\mathbf{v}_3 + (k_2 + k_3)\mathbf{v}_4. \tag{14.6}$$

- Case {v3-r1-nd3-t2-(a>k4)}: Here $m_0(A_0) > m_0(e_4)$.

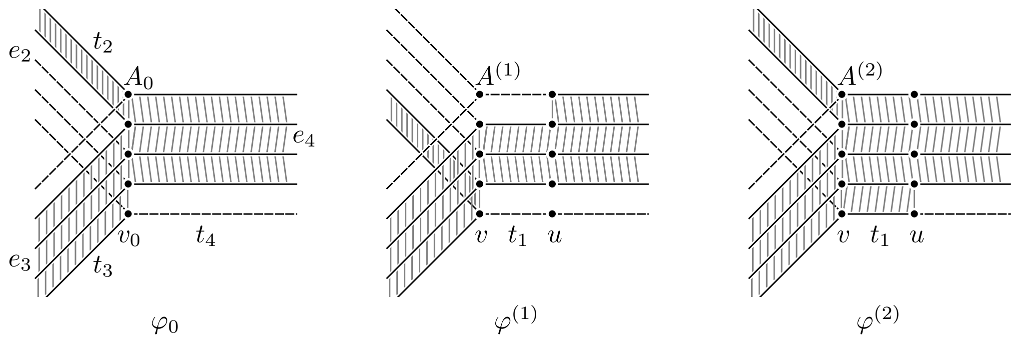


Figure 14.6

The local parts of A_0, first column, and of $A^{(q)}$, last two columns, are:

$\mathbf{v}_4$	$\mathbf{v}_1^{(1)}$	$\mathbf{v}_1^{(2)}$
0	0	0
0	0	0
$\frac{1}{k_4}$	$\frac{1}{k_4-1}$	$\frac{1}{k_4+1}$
a_i	a_i	a_i

The wall-relation is:

$$(k_4 - 1)\mathbf{v}_1^{(1)} + (k_4 + 1)\mathbf{v}_1^{(2)} = 2k_4\mathbf{v}_4. \tag{14.7}$$

- Case {v3-r1-nd3-t3}: Here A_0 is the unique vertex in $\varphi_0^{-1}(v_0)$ such that $r_0(A_0) = 1$, we have that φ is not injective on nd-$E(A_0)$, and nd-val $A_0 = 3$.

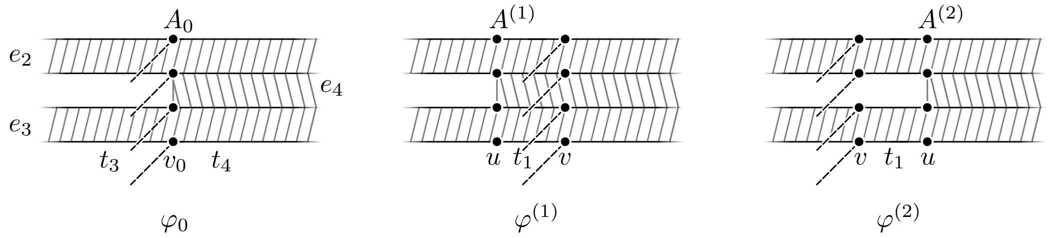


Figure 14.7

The local parts of A_0, first two columns, and of $A^{(q)}$, last two columns, are:

$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_1^{(1)}$	$\mathbf{v}_1^{(2)}$
$\frac{1}{k_2}$	0	0	$\frac{1}{k_2}$
$\frac{1}{k_3}$	0	0	$\frac{1}{k_2}$
0	$\frac{1}{k_2+k_3}$	$\frac{1}{k_2+k_3}$	0
a_i	b_i	a_i	b_i

The wall-relation is:

$$\mathbf{v}_1^{(1)} + \mathbf{v}_1^{(2)} = \mathbf{v}_3 + \mathbf{v}_4. \tag{14.8}$$

- Case {v3-r1-nd2}: Here nd-val $A_0 = 2$.

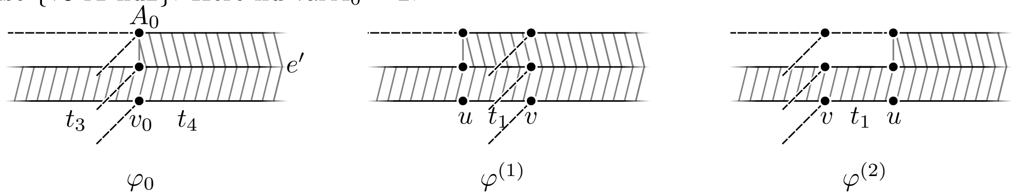


Figure 14.8

The local parts of A_0, first two columns, and of $A^{(q)}$, last two columns, are:

$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_1^{(1)}$	$\mathbf{v}_1^{(2)}$
$\frac{1}{k}$	$\frac{1}{k+1}$	$\frac{1}{k+1}$	$\frac{1}{k}$
a_i	b_i	a_i	b_i

The wall-relation is:

$$\mathbf{v}_1^{(1)} + \mathbf{v}_1^{(2)} = \mathbf{v}_3 + \mathbf{v}_4. \tag{14.9}$$

- Case {v2-r2-nd3-M-11}: Here there are 2 edges in $E(A_0)$ mapping to t_3, and m_0 equals 1 for all edges in nd-$E(A_0)$.

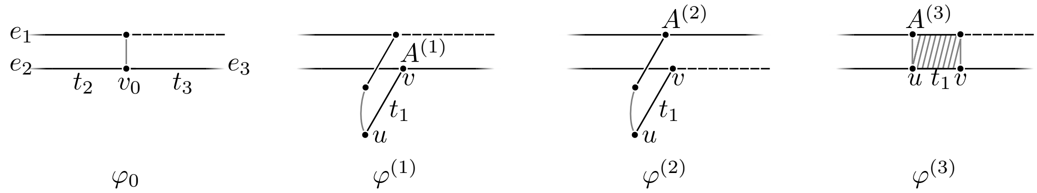


Figure 14.9

The local parts of A_0, first two columns, and of $A^{(q)}$, last three columns, are:

$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_1^{(1)}$	$\mathbf{v}_1^{(2)}$	$\mathbf{v}_1^{(3)}$
1	0	1	0	0
1	0	0	1	0
0	1	0	0	$\frac{1}{2}$
a_i	a_i	0	0	a_i

The wall-relation is:

$$\mathbf{v}_1^{(1)} + \mathbf{v}_1^{(2)} + 2\mathbf{v}_1^{(3)} = \mathbf{v}_2 + \mathbf{v}_3. \tag{14.10}$$

- Case {v2-r2-nd3-M-1k}: Here m_0 equals 1 for exactly one edge in nd-$E(A_0)$.

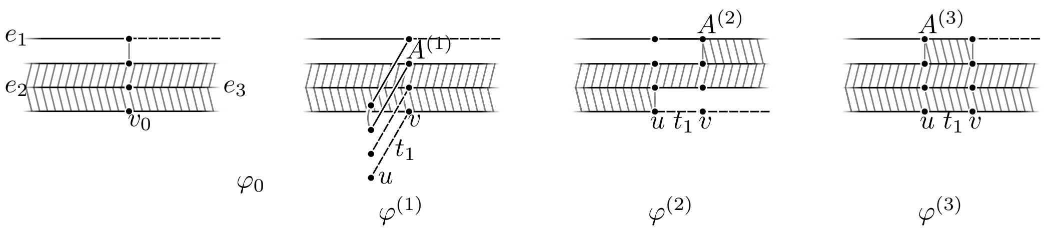


Figure 14.10

The local parts of A_0, first two columns, and of $A^{(q)}$, last three columns, are:

$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_1^{(1)}$	$\mathbf{v}_1^{(2)}$	$\mathbf{v}_1^{(3)}$
1	0	1	1	0
$\frac{1}{k}$	0	0	$\frac{1}{k-1}$	0
0	$\frac{1}{k}$	0	0	$\frac{1}{k+1}$
a_i	a_i	0	a_i	a_i

The wall-relation is:

$$\mathbf{v}_1^{(1)} + (k-1)\mathbf{v}_1^{(2)} + (k+1)\mathbf{v}_1^{(3)} = k\mathbf{v}_2 + k\mathbf{v}_3. \tag{14.11}$$

- Case {v2-r2-nd3-M-kk}: Here m_0 is strictly greater than 1 for all elements in nd-$E(A_0)$.

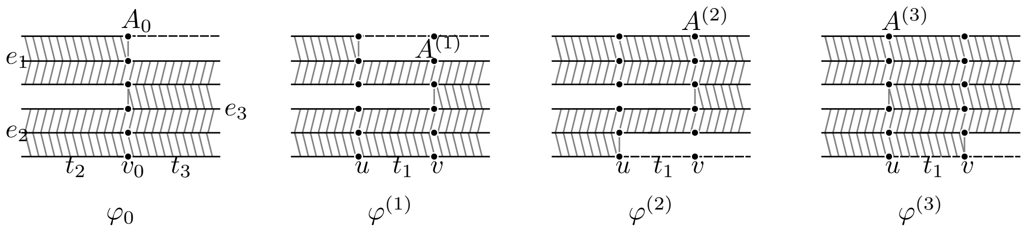


Figure 14.11

The local parts of A_0, first two columns, and of $A^{(q)}$, last three columns, are:

$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_1^{(1)}$	$\mathbf{v}_1^{(2)}$	$\mathbf{v}_1^{(3)}$
$\frac{1}{k_1}$	0	$\frac{1}{k_1-1}$	$\frac{1}{k_1}$	0
$\frac{1}{k_2}$	0	$\frac{1}{k_2}$	$\frac{1}{k_2-1}$	0
0	$\frac{1}{k_1+k_2-1}$	0	0	$\frac{1}{k_1+k_2}$
a_i	a_i	a_i	a_i	a_i

The wall-relation is:

$$(k_1-1)\mathbf{v}_1^{(1)} + (k_2-1)\mathbf{v}_1^{(2)} + (k_1+k_2)\mathbf{v}_1^{(3)} = (k_1+k_2-1)\mathbf{v}_2 + (k_1+k_2-1)\mathbf{v}_3. \tag{14.12}$$

- Case {v2-r2-nd3-P}: Here there is exactly one edge in $E(A_0)$ mapping to t_3.

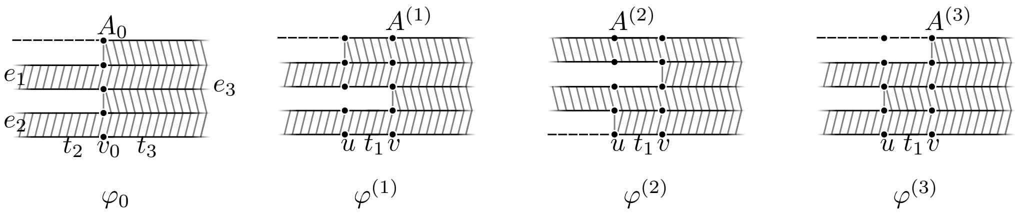


Figure 14.12

The local parts of A_0, first two columns, and of $A^{(q)}$, last three columns, are:

$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_1^{(1)}$	$\mathbf{v}_1^{(2)}$	$\mathbf{v}_1^{(3)}$
$\frac{1}{k_1}$	0	$\frac{1}{k_1+1}$	$\frac{1}{k_1}$	0
$\frac{1}{k_2}$	0	$\frac{1}{k_2}$	$\frac{1}{k_2+1}$	0
0	$\frac{1}{k_1+k_2+1}$	0	0	$\frac{1}{k_1+k_2}$
a_i	a_i	a_i	a_i	a_i

The wall-relation is:

$$(k_1+1)\mathbf{v}_1^{(1)} + (k_2+1)\mathbf{v}_1^{(2)} + (k_1+k_2)\mathbf{v}_1^{(3)} = (k_1+k_2+1)\mathbf{v}_2 + (k_1+k_2+1)\mathbf{v}_3. \tag{14.13}$$

- Case {v2-r1}: Here $r_0(A_0) = 1$ and $\operatorname{val} A_0 = 3$. There is at least one edge of nd-$E(A_0)$ above t_2 and above t_3, and there are two further cases, either nd-val $A_0 = 3$ or nd-val $A_0 = 2$. These two cases give the wall-relation:

$$\mathbf{v}_1^{(1)} + \mathbf{v}_1^{(2)} = \mathbf{v}_1 + \mathbf{v}_2. \tag{14.14}$$

- Case {v2-r1-nd3}: Here nd-val A_0 is 3.

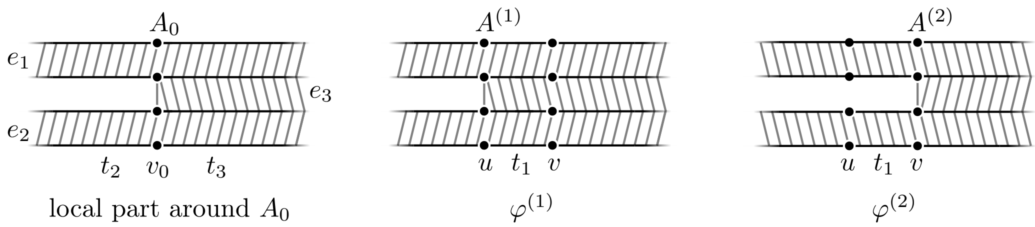


Figure 14.13

The local parts of A_0, first two columns, and of $A^{(q)}$, last three columns, are:

$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_1^{(1)}$	$\mathbf{v}_1^{(2)}$
$\frac{1}{k_1}$	0	0	$\frac{1}{k_1}$
$\frac{1}{k_2}$	0	0	$\frac{1}{k_2}$
0	$\frac{1}{k_1+k_2}$	$\frac{1}{k_1+k_2}$	0

- Case {v2-r1-nd2}: Here nd-val A_0 is 2.

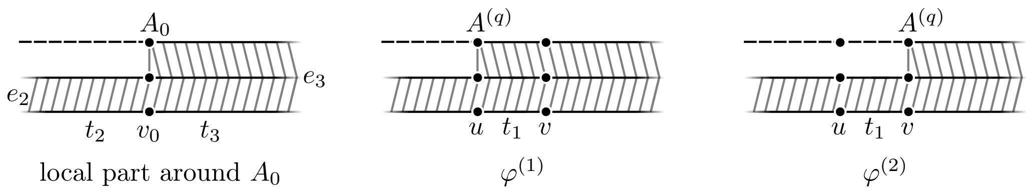


Figure 14.14

The local parts of A_0, first two columns, and of $A^{(q)}$, last three columns, are:

$$\begin{array}{cccc} \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_1^{(1)} & \mathbf{v}_1^{(2)} \\ \frac{1}{k} & \frac{1}{k+1} & \frac{1}{k+1} & \frac{1}{k} \end{array}$$

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Declaration of consent

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